

# Black-Litterman, Bayesian Shrinkage, and Factor Models in Portfolio Selection: You Can Have It All

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## Abstract

Mean-variance analysis is widely used in portfolio management to identify the best portfolio that makes an optimal trade-off between expected return and volatility. Yet, this method has its limitations, notably its vulnerability to estimation errors and its reliance on historical data. While shrinkage estimators and factor models have been introduced to improve estimation accuracy through bias-variance trade-offs, and the Black-Litterman model has been developed to integrate investor opinions, a unified framework combining three approaches has been lacking. Our study debuts a Bayesian blueprint that fuses shrinkage estimation with view inclusion, conceptualizing both as Bayesian updates. This model is then applied within the context of the Fama-French approach factor models, thereby integrating the advantages of each methodology. Finally, through a comprehensive empirical study in the US equity market spanning a decade, we show that the model outperforms both the simple  $1/N$  portfolio and the optimal portfolios based on sample estimators.

## 1 Introduction

The foundation of Modern Portfolio Theory (MPT) was laid by Markowitz in the 1950s when he introduced mean-variance analysis (MV), which characterizes assets solely by their expected return and variance. The key insight is that the risk of an asset should be assessed by its contribution to the risk of the portfolio as a whole, but not individually. One appeal of this framework for "diversification" is that it requires only two inputs: the expected return  $\mu$  and covariance  $\Sigma$  of assets in the future, which could be replaced by the sample estimators based on historical data.

Despite the elegance of the MV framework, it hasn't always translated well in practical applications. The root of the issue lies in the reliance on historical data. Past performance doesn't necessarily forecast future outcomes. Relying on in-sample optimal portfolios often leads to over-investment in historically successful assets, which doesn't always translate to success in new market conditions. Moreover, sample estimators of the two inputs are subject to high estimation errors. Given the susceptibility of the framework to errors, the estimated optimal portfolios are far from the true optimal portfolios. This observation has been extensively studied and documented by scholars such as Jobson and Korkie (1981), Best and Grauer (1991), Broadie (1993), Britten-Jones (1999), and DeMiguel et al. (2009).

To counteract these pitfalls, three primary improvements to the MV framework have emerged: the use of shrinkage estimators, factor models and the Black-Litterman model. Shrinkage estimators are designed to reduce estimation errors by optimizing the balance between bias and variance. Their foundation is closely linked to Bayesian methodologies that pull posterior mean estimates toward prior values. Notable Bayesian shrinkage approaches include Jorion’s Bayes-Stein technique (1986), the empirical Bayesian methodology by Frost and Savarino (1986), and MacKinlay and Pastor’s asset pricing model-based method (2000).

On the other hand, the Black-Litterman model offers a unique approach, focusing on blending implied equilibrium returns with investors’ predictions regarding asset performance. Yet, as Schöttle et al. (2010) argued, the Black-Litterman model can be seen as a specialized Bayesian approach, which integrates investor forecasts to update prior beliefs which are based on equilibrium.

Thus, at their core, both the shrinkage and Black-Litterman models utilize Bayesian updating to refine data-based estimates. While both methods have been well-explored individually, a general framework for combining both has been lacking. In fact, by formalizing both as Bayesian updating, one can synergize shrinkage and the inclusion of views under a single model.

In parallel, factor models have emerged as an indispensable tool in both academic research and practical applications within finance. Their utility is two-fold: they offer a streamlined model for asset returns, adeptly mitigating the idiosyncratic noise in historical data, and enhancing interpretability by elucidating the specific sources of returns and risks. By decomposing asset returns into systematic components (attributed to a select few factors) and idiosyncratic noise, they not only clarify where returns and risks originate but also furnish a more transparent structure for both the mean and covariance of asset returns. This decomposition facilitates a refined bias-variance trade-off, improving estimation accuracy.

While each of the methodologies —shrinkage estimators, the Black-Litterman model, and factor models— stand as stalwarts in their own right, their potential symbiosis remains largely uncharted. This research aims to navigate this interdisciplinary nexus. We set forth a comprehensive framework that encapsulates shrinkage and view inclusion within the Bayesian paradigm and subsequently adapts this methodology to seamlessly interface with factor models. As a result, we derive portfolios that not only surpass the equal-weight benchmark but also demonstrate resilience against daily data fluctuations, offer enhanced diversification, provide transparency in terms of risk and return origins, and effectively incorporate investor forecasts.

The remaining section is structured as follows:

In Section 2, we introduce a framework that merges shrinkage and view inclusion, conceptualizing both as Bayesian updating.

In Section 3, we integrate this framework within factor models, culminating in a fusion of the three key approaches that form the cornerstone of this research.

In Section 4, we analyze the empirical performance of the model when applied to the 200 largest stocks in the US equity market spanning a decade.

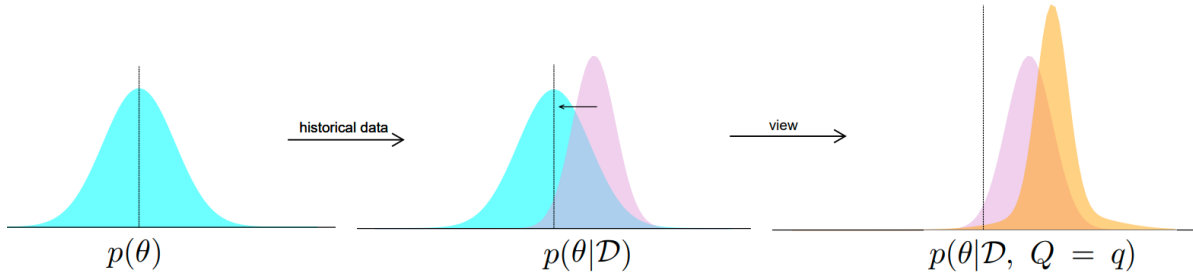
## 2 A Unified Bayesian Framework for Shrinkage and Views

To get started, one first specifies a parametric model for asset returns  $p(r_t|\theta)$ , with  $\theta \in \Theta$  being the unknown parameter of interest. Then, he specifies a prior distribution,  $p(\theta)$ , which incorporates shrinkage on these unknown parameters. Once historical asset returns ( $\mathcal{D}$ ) are observed, beliefs are updated to give the posterior distribution,  $p(\theta|\mathcal{D})$ . For individuals with future market predictions that can be represented by random variables  $Q$  contingent on  $\theta$ , expressed as  $p(q|\theta, \mathcal{D})$ , beliefs undergo a further update to acquire the posterior distribution  $p(\theta|\mathcal{D}, Q = q)$ . Optimal portfolios are then computed using the posterior predictive moments of asset returns,  $\mathbb{E}[r_t|\mathcal{D}, Q = q]$  and  $\text{Var}[r_t|\mathcal{D}, Q = q]$ . The framework is summarized as follows:

1. Asset return's parametric model:  $p(r_t|\theta)$  where  $r_t \in \mathbb{R}^M$  is the asset returns of  $M$  assets at time  $t$ .
2. Introducing a shrinkage prior:  $p(\theta)$ . The prior mean typically represents the shrinkage target while the prior variance indicates shrinkage intensity.
3. First Bayesian updating (shrinkage):  $p(\theta|\mathcal{D}) \propto p(\mathcal{D}|\theta)p(\theta)$ , taking into account historical asset returns,  $\mathcal{D}$ .
4. Second Bayesian updating (view inclusion):  $p(\theta|\mathcal{D}, Q = q) \propto p(\theta|\mathcal{D})p(q|\theta, \mathcal{D})$ .
5. Computing optimal portfolios using the posterior predictive moments:  $\mathbb{E}[r_t|\mathcal{D}, Q = q]$  and  $\text{Var}[r_t|\mathcal{D}, Q = q], t > T$ .

The concluding posterior distribution,  $p(r_t|\mathcal{D}, Q = q)$ , reflects an individual's uncertainty regarding asset returns following belief updates from historical returns and forecasts (views), illustrated in Figure 1.

Figure 1: Dual Bayesian Updating of Parameters



While shrinkage and the incorporation of views both functionally update distributions, they serve distinct roles. Shrinkage aims to reduce estimators' variance by integrating a constant zero-variance target, improving the estimator through bias-variance trade-offs. In contrast, the incorporation of

views gravitates the estimate towards better-aligned future market predictions. Both shrinkage and view inclusion can harmoniously function together, as discussed further in subsequent sections.

Contrasting this with the Black-Litterman-Bayes (BLB) framework presented by Kolm et al. (2021), we spot two primary differences:

1. Our framework's introduction of a shrinkage prior,  $p(\theta)$ , absent in BLB.
2. An overt Bayesian updating based on historical returns, emphasizing Bayesian principles.

Following this, we integrate this framework with the Fama-French approach factor models for improved portfolio optimization.

### 3 Enhancing Factor Models

#### 3.1 Fama-French approach factor model

Each asset's return, spanning  $m = 1, \dots, M$ , is modeled by a linear time-series model:

$$r_m = F\beta_m + \epsilon_m, \quad m = 1, \dots, M$$

Here,  $r_m = [r_{m,1}, \dots, r_{m,T}]^T \in \mathbb{R}^T$  denotes the return of asset  $m$  over a span of  $T$  periods, while  $F = [f_1, \dots, f_T]^T \in \mathbb{R}^{T \times K}$  represents the return of the  $K$  observable portfolio factors. The residual is denoted by  $\epsilon_m \in \mathbb{R}^T$ . As with the common linear regression model, assume  $\epsilon_m \sim \mathcal{N}(0, \sigma_m^2 \mathbb{I}_T)$ . Furthermore, the portfolio factors' return is modeled as:

$$f_t \sim^{iid} \mathcal{N}(\mu_f, \Lambda_f)$$

The primary parameters of interest in this setup are  $\theta = \{\{\beta_m\}, \{\sigma_m\}, \mu_f, \Lambda_f\}$ .

Factor models of this kind are often called the Fama-French approach factor models. To construct a factor represented by a portfolio, one ranks stocks based on a given attribute, defines the corresponding quintile portfolios, and forms a hedge portfolio that long the top quintile assets and shorts the bottom quintile assets. The expected excess return of the resulting hedge portfolio represents a risk premium from investing in that particular source of risk. Given the factors' interpretability, views regarding them are more straightforwardly formulated.

For Bayesian inference, one specifies a prior on the unknown variables. For shrinkage implementation, a prior with a constant prior mean (serving as the shrinkage target) is ideal. As we condition on the observed data (i.e., the first Bayesian update), yielding  $p(\theta|\mathcal{D})$ , the parameters' posterior mean becomes a weighted average of the prior mean and MLE estimates. This means that MLE estimates are shrunk towards the designated shrinkage target. Following this, if any supplementary data or views about the unknown parameters emerge, one can augment their beliefs through additional Bayesian updates. It's possible to undergo numerous belief revisions, and under appropriate assumptions, the sequence of these updates remains inconsequential. An investor's ultimate portfolio choices depend on their final beliefs. In mean-variance (MV) analysis, the two requisite moments of asset returns under the Bayesian factor model can be deduced as follows:

**Proposition 1.** Let  $\mathbb{E}[\cdot] = \mathbb{E}[\cdot|\mathcal{D}, Q = q]$ ,  $\text{Var}[\cdot] = \text{Var}[\cdot|\mathcal{D}, Q = q]$  and  $\text{Cov}(x, y) = \text{Cov}(x, y|\mathcal{D}, Q = q)$ . The posterior predictive moments under the set of prior assumptions is

$$\begin{aligned}\mathbb{E}[r_m] &= \mathbb{E}[\beta_m]^T \mathbb{E}[\mu_f] \\ \text{Var}[r_m] &= \mathbb{E}[\sigma_m^2] + \text{Tr}(\mathbb{E}[f f^T] \text{Var}[\beta_m]) + \mathbb{E}[\beta_m]^T \text{Var}[f] \mathbb{E}[\beta_m] \\ \text{Cov}(r_i, r_j) &= \mathbb{E}[\beta_i]^T \text{Var}[f] \mathbb{E}[\beta_j]\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[f f^T] &= \mathbb{E}[\Lambda_f] + \text{Var}[\mu_f] + \mathbb{E}[\mu_f] \mathbb{E}[\mu_f]^T \\ \text{Var}[f] &= \mathbb{E}[\Lambda_f] + \text{Var}[\mu_f]\end{aligned}$$

$\mathbb{E}[\mu_f]$ ,  $\mathbb{E}[\Lambda_f]$ ,  $\mathbb{E}[\beta_m]$ ,  $\mathbb{E}[\sigma_m^2]$ ,  $\text{Var}[\beta_m]$ ,  $\text{Var}[\mu_f]$  are obtained from the final posterior distributions after Bayesian updates

Once the final posterior predictive moments are available, the optimal portfolios with various constraints can be solved either analytically or numerically. In particular, the minimum variance portfolio ("MV"), and efficient portfolios with a target expected return  $m$  and budget constraint ("WB(m)"), has the well-known closed-form solutions stated below:

**Proposition 2.** Let  $\mu = \mathbb{E}[r]$  be the expected return of assets,  $\Sigma = \text{Var}[r]$  be the covariance. The minimum variance portfolio,  $w^{MV}$ , is the solution to the problem:

$$\begin{aligned} & \min_w w^T \Sigma w \\ \text{s.t.} \quad & w^T \mathbf{1} = 1 \end{aligned}$$

and the solution is

$$w^{MV} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

**Proposition 3.** Let  $\mu = \mathbb{E}[r]$  be the expected return of assets,  $\Sigma = \text{Var}[r]$  be the covariance. The efficient portfolio with target expected return  $m$  and budget constraint,  $w^{(m)}$ , is the solution to the problem:

$$\begin{aligned} & \min_w w^T \Sigma w \\ \text{s.t.} \quad & w^T \mathbf{1} = 1 \\ & w^T \mu = m \end{aligned}$$

and the solution is

$$w^{(m)} = \Sigma^{-1}(a\mu + b1)$$

where

$$a = 1^T \Sigma^{-1} e$$

$$b = -\mu^T \Sigma^{-1} e$$

$$e = \frac{m1 - \mu}{d}$$

$$d = (1^T \Sigma^{-1} 1)(\mu^T \Sigma^{-1} \mu) - (1^T \Sigma^{-1} \mu)^2$$

The aforementioned portfolios, with their closed-form solutions, serve as our primary analytical tools. The minimum-variance portfolio, reliant solely on parameter  $\Sigma$ , effectively assesses covariance estimators. In contrast, the efficient portfolios utilize both  $\mu$  and  $\Sigma$ , permitting a comprehensive evaluation of mean and covariance estimators together

### 3.2 Shrinkage through Bayesian updating

We now proceed to define the shrinkage prior for the initial Bayesian update. To maintain closed-form solutions in MV analysis, it's advantageous to employ a fully conjugate prior, thereby circumventing estimation errors due to sampling. The two priors well-suited for this context are Zellner's g-prior and the Normal-Inverse-Wishart prior, presented as follows:

$$\beta_m | \sigma_m^2 \sim \mathcal{N}(\beta_{m,0}, g\sigma_m^2 (F^T F)^{-1}), \quad m = 1, \dots, M$$

$$p(\sigma_m^2) \propto \frac{1}{\sigma_m^2}, \quad m = 1, \dots, M$$

$$p(\mu_f, \Lambda_f) \propto |\Lambda_f|^{-\left(\frac{K+1}{2}\right)}$$

The Zellner's g-prior for the regression coefficients,  $\beta_m$ , is favored in Bayesian analyses, especially since its marginal likelihood has a closed-form. This characteristic proves advantageous for tasks like variable selection and model averaging. The priors for  $\sigma_m^2$  and  $(\mu_f, \Lambda_f)$ , known as Jeffrey's priors, are essentially uninformative. Such priors signify a lack of prior information, allowing the sample data to largely drive the posterior.

The coefficients  $\beta_m$  significantly affect both expected returns, contingent on  $f_t$ , and the covariance structure. Using  $\beta_{m,0} = 0$  induces regularization similar to ridge regression. This form of regularization benefits estimation by striking a balance between bias and variance. Since we've incorporated shrinkage on asset returns via factor coefficients, an uninformative prior for  $p(\mu_f, \Lambda_f)$  is chosen to avoid over-shrinking.

The covariance term  $g\sigma^2(F^T F)^{-1}$  can be considered a generalization of the unit information prior  $n\sigma^2(F^T F)^{-1}$ , which represents one unit of sample information as prior observation. The smaller value of  $g$ , the larger the prior observation size, the more evidence from prior beliefs, and

consequently the stronger shrinkage towards the prior mean  $\beta_{m,0}$ . Thus,  $g$  emerges as a measure of shrinkage intensity.

Under the set of priors, the posterior distributions after the first Bayesian update are stated below:

**Proposition 4.** *Let  $r_m = [r_{m,1}, \dots, r_{m,T}]^T \in \mathbb{R}^T$  be returns of asset  $m$  over  $T$  period,  $F = [f_1, \dots, f_T]^T \in \mathbb{R}^{T \times K}$  be factor returns. The marginal posterior of  $\sigma_m^2$  and  $\beta_m$  under the set of prior assumptions is*

$$\sigma_m^2 | \mathcal{D} \sim IG\left(\frac{T}{2}, \frac{SSR_{g,m}}{2}\right)$$

$$\beta_m | \mathcal{D} \sim St(T, \bar{\beta}_m, \Sigma_m)$$

where

$$SSR_{g,m} = (r_m - F\hat{\beta}_m)^T(r_m - F\hat{\beta}_m) + \frac{1}{g+1}(\hat{\beta}_m - \beta_{m,0})^T F^T F(\hat{\beta}_m - \beta_{m,0})$$

$$\bar{\beta}_m = \frac{1}{g+1}\beta_{m,0} + \frac{g}{g+1}\hat{\beta}_m$$

$$\hat{\beta}_m = (F^T F)^{-1} F^T r_m$$

$$\Sigma_m = \frac{g}{g+1} (F^T F)^{-1} \frac{SSR_{g,m}}{T}$$

where  $St$  denotes multivariate student's t distribution. The posterior mean  $\bar{\beta}_m$  is a weighted average of the prior mean and OLS estimate, weighted by  $g$ . Such shrinkage of MLE estimates toward the prior mean is very common in Bayesian models, and it is the reason why shrinkage methods are connected with Bayesian principles. The parameter  $g$  has the interpretation as the fraction of information available in the prior relative to the sample. As  $g \rightarrow \infty$ , the posterior mean of  $\beta_m$  reduces to the OLS estimate, so is the residual sum of square  $SSR_{g,m}$ . As  $g \rightarrow 0$ , prior information dominates, and the sample has negligible effects on the posterior.

As for  $\mu_f$  and  $\Lambda_f$ , the posterior distribution of  $\mu_f$  and  $\Lambda_f$  depends only on  $F$ . The marginal posterior distribution following the first Bayesian update is:

**Proposition 5.** *The marginal posterior of  $\mu_f$  and  $\Lambda_f$  under the set of prior assumptions is*

$$\mu_f | \mathcal{D} \sim St(T - K, \bar{f}, \frac{\Lambda_n}{(T)(T - K)})$$

$$\Lambda_f | \mathcal{D} \sim IW(T - 1, \Lambda_n)$$

where

$$\Lambda_n = \sum_{t=1}^T (f_t - \bar{f})(f_t - \bar{f})^T$$

$$\bar{f} = \frac{1}{T} \sum_{t=1}^T f_t$$

where  $\mathcal{IW}$  denotes Inverse-Wishart distribution. As an uninformative prior is used, the posterior mean of  $\mu_f$  is simply the sample mean, and the posterior mean of  $\Lambda_f$  is the re-scaled sample covariance.

When an individual lacks specific views or forecasts, negating the need for a second Bayesian update, they can directly utilize the posterior moments from the aforementioned distributions for portfolio optimization as outlined in Proposition 1.

### 3.3 Determining shrinkage intensity

The degree of shrinkage, represented by the hyper-parameter  $g$ , can be thought of as a measure of the relative weight we give to prior beliefs compared to observed data. While one might simply choose a fixed value for  $g$ , reflecting a constant prior belief strength relative to sample data, there are more nuanced methods that derive  $g$  from the observed data itself.

This adjustment of  $g$  is made possible by utilizing the marginal likelihood,  $p(r_m|g)$ , which expresses the likelihood of observed data under a certain model, which can be obtained by integrating the unknown variables from the full joint distributions. Through the marginal likelihoods, one can determine hyper-parameter values that most align with the observed data. For Zellner's  $g$ -prior with  $\beta_{m,0} = 0$ , the marginal likelihood  $p(r_m|g)$  has a well-known explicit form:

**Proposition 6.** *The marginal likelihood  $p(r_m|g)$  for asset  $m$  under the given set of priors is:*

$$p(r_m|g) = \Gamma\left(\frac{T-1}{2}\right) \pi^{-\frac{T-1}{2}} T^{-\frac{1}{2}} \|r_m - \bar{r}_m\|^{-(T-1)} \frac{(1+g)^{(T-K-1)/2}}{(1+g(1-R^2))^{(T-1)/2}}$$

where  $\|\cdot\|$  is a vector norm,  $R^2 = 1 - \frac{(r_m - F\hat{\beta})^T (r_m - F\hat{\beta})}{(r_m - \bar{r}_m)^T (r_m - \bar{r}_m)}$  is the coefficient of determination.

Employing this marginal likelihood, various methodologies have been conceived to determine the optimal value of  $g$ . One such method proposed by Liang et al. (2008) introduces a hyper  $g$ -prior on  $g$  itself, enabling the calculation of  $g$ 's posterior distribution via  $p(g|y) \propto p(y|g)p(g)$ . This model then averages out to get a mix of different  $g$ -values. Due to its computational intensity, we opt for



the empirical Bayes estimate  $g^*$ , which maximizes the marginal likelihood:

$$\begin{aligned}
g^* &= \max_g \prod_{m=1}^M p(r_m|g) \\
&= \max_g \sum_{m=1}^M \ln p(r_m|g) \\
&= \min_g \sum_{m=1}^M \left[ -\frac{T-K-1}{2} \ln(1+g) + \frac{T-1}{2} \ln(1+g(1-R^2)) \right]
\end{aligned}$$

Note that we use a single  $g$  value for all the  $M$  assets, indicating a consistent degree of shrinkage for all assets.

### 3.4 Effect of shrinkage

By specifying  $\beta_{m,0} = 0$ , we are essentially driving the posterior predictive moments toward a simpler structure. This is shown below:

#### 1. Expected Returns Towards Zero:

As  $g$  tend to 0, we see that:

$$\mathbb{E}[\beta_m|\mathcal{D}] \xrightarrow{g \rightarrow 0} \beta_{m,0} = 0, \quad m = 1, \dots, M$$

This, in turn, means that our expected returns, given by:

$$\mathbb{E}[r_{m,t}|\mathcal{D}] = \mathbb{E}[\beta_m|\mathcal{D}]^T \mathbb{E}[\mu_f]$$

will also converge to 0 for  $m = 1, \dots, M$  and  $t > T$ .

#### 2. Diagonal Predictive Covariance:

The predictive covariance matrix evolves into a diagonal structure when  $g$  approaches 0. Specifically:

$$\text{Cov}(r_{i,t}, r_{m,t}|\mathcal{D}) = \mathbb{E}[\beta_i|\mathcal{D}]^T \text{Var}[f|\mathcal{D}] \mathbb{E}[\beta_m|\mathcal{D}] \xrightarrow{g \rightarrow 0} 0, \quad \forall i \neq j, t > T$$

This suggests that off-diagonal elements of the covariance matrix tend to be zero, de-correlating assets as  $g$  goes to zero.

#### 3. Variance and Second Moments:

For the predictive returns  $r = [r_{1,t}, \dots, r_{M,t}]^T, t > T$ , the variance simplifies to:

$$\text{Var}[r|\mathcal{D}] \xrightarrow{g \rightarrow 0} b * \text{diag}(\kappa_1, \dots, \kappa_M)$$

where  $\kappa_m$  is a multiple of the sample's second moment of asset returns, given by:

$$\kappa_m = \frac{r_m^T r_m}{T - 2}$$

#### 4. Efficient Portfolios and Shrinkage:

In the scenario where  $g$  approaches 0, constructing efficient portfolios with a targeted expected return that is non-zero becomes infeasible. However, the minimum-variance portfolio remains attainable. This portfolio's weights become inversely proportional to each asset's sample second moment. Specifically, the weight vector  $w^{MV}$  becomes:

$$w^{MV} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} = \left[ \frac{b_1}{\sum_{m=1}^M b_i}, \dots, \frac{b_M}{\sum_{m=1}^M b_i} \right]^T$$

where  $b_m = \frac{1}{\kappa_m}$

In essence, as shrinkage is applied, the posterior moments of asset returns are guided toward a more straightforward structure. The covariance matrix becomes predominantly diagonal, emphasizing the lack of correlations between different assets, and the mean vector evolves toward zero. This process underscores the power of shrinkage in Bayesian contexts, driving model outcomes towards simpler, often better results.

### 3.5 View update: incorporating investor beliefs

In a diverse asset portfolio, investor views often manifest as expectations on return and covariance of the underlying factors. Such views, grounded in forward-looking analyses or insights not necessarily captured by historical data, can be strategically integrated into the statistical model. This integration, carried out via a second Bayesian update, influences the ultimate asset allocation. Notably, while the first update (shrinkage) governs the model's complexity (like reducing trivial covariances), the second (view inclusion) can redirect allocations, perhaps emphasizing certain factors over others.

For effective view incorporation, these beliefs must be quantitatively represented. For views on expected returns, a linear formulation similar to the Black-Litterman model can be employed. Let's consider an example: based on upcoming governmental policies, an investor predicts that the tech industry will outperform the healthcare sector by a monthly return of 5%. Such subjective, "forward-looking" beliefs can be integrated into the statistical model, assisting in devising an allocation strategy poised to capitalize on these insights.

Let  $P \in \mathbb{R}^{N \times K}$  denote a full-ranked pick matrix,  $Q \in \mathbb{R}^N$  represent the view vector, where  $N$  is the number of views. If there are three factors, and an investor believes that the first factor will outperform the second by  $x\%$  and the second factor will yield  $y\%$ , the view can be mathematically represented as:

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} x\% \\ y\% \end{bmatrix}$$

With the formulation

$$Q = P\mu_f + \epsilon$$

the view  $Q = q$  is seen as a perturbed observation of the linear transformation of the expected factor return. For this inclusion to be coherent, assume that the noise  $\epsilon$  and  $\mu_f$  have a joint multivariate Student-t distribution conditioned  $\mathcal{D}$ , where  $\text{Var}(\epsilon|\mathcal{D}) = \Omega$ , and  $\text{Cov}(\epsilon, \mu_f|\mathcal{D}) = 0$ . This noise aligns with the posterior distribution's degrees of freedom of  $\mu_f$ . The covariance matrix,  $\Omega$ , reflects one's confidence in a particular view. A larger variance indicates uncertainty, dampening its influence on the updated belief. In the Black-Litterman model, it's commonly assumed that  $\Omega$  is diagonal.

Let  $\Sigma_n = \frac{\Lambda_n}{(T)(T-K)}$ . Schöttle et al. (2010) shows that the updated posterior follows multivariate Student-t distribution. Hence, the posterior distribution of  $\mu_f$  after the second Bayesian update is:

$$\begin{aligned} & \begin{bmatrix} \mu_f \\ Q \end{bmatrix} | \mathcal{D} \sim St(T-K, \begin{bmatrix} \bar{f} \\ P\bar{f} \end{bmatrix}, \begin{bmatrix} \Sigma_n & \Sigma_n P^T \\ P\Sigma_n & P\Sigma_n P^T + \Omega \end{bmatrix}) \\ \mu_f | \mathcal{D}, Q = q & \sim St(T-K, \bar{f} + \Sigma_n P^T (P\Sigma_n P^T + \Omega)^{-1} (q - P\bar{f}), \Sigma_n - \Sigma_n P^T (P\Sigma_n P^T + \Omega)^{-1} P\Sigma_n) \end{aligned}$$

This representation reveals that a viewpoint on a subset of factors disperses to the other factors via the covariance matrix  $\Sigma_n$ . Thus, taking a stance on a singular factor indirectly modifies other factors through the differences  $\Sigma_n P^T (P\Sigma_n P^T + \Omega)^{-1} (q - P\bar{f})$ .

Building on the analysis by Schöttle et al. (2010), it's plausible to infer that the error  $\epsilon$  in  $Q$  arises from a lifted uncertainty in  $\mu_f$ . This naturally results in:

$$\Omega = cP\Sigma_n P^T$$

for some constant  $c > 0$ .

Subsequent to the Bayesian update, the posterior for  $\mu_f$  is:

$$\mu_f | \mathcal{D}, Q = q \sim St(T-K, \bar{f} + (\frac{1}{c+1})\Sigma_n P^T (P\Sigma_n P^T)^{-1} (q - P\bar{f}), \Sigma_n - (\frac{1}{c+1})\Sigma_n P^T (P\Sigma_n P^T)^{-1} P\Sigma_n)$$

In scenarios where  $c$  approaches zero,  $Q$  accurately mirrors  $P\mu_f$ . Conversely, as  $c \rightarrow \infty$ ,  $Q = q$  loses its significance, defaulting back to the original posterior distribution.

Addressing views on factor covariances,  $\Lambda_f$ , they can be integrated by conceptualizing them as a Wishart random variable, centered at  $\Lambda_f$ . If  $Z$  represents this view and  $\mathcal{W}$  denotes the Wishart distribution, then for a positive constant  $d > 0$ :

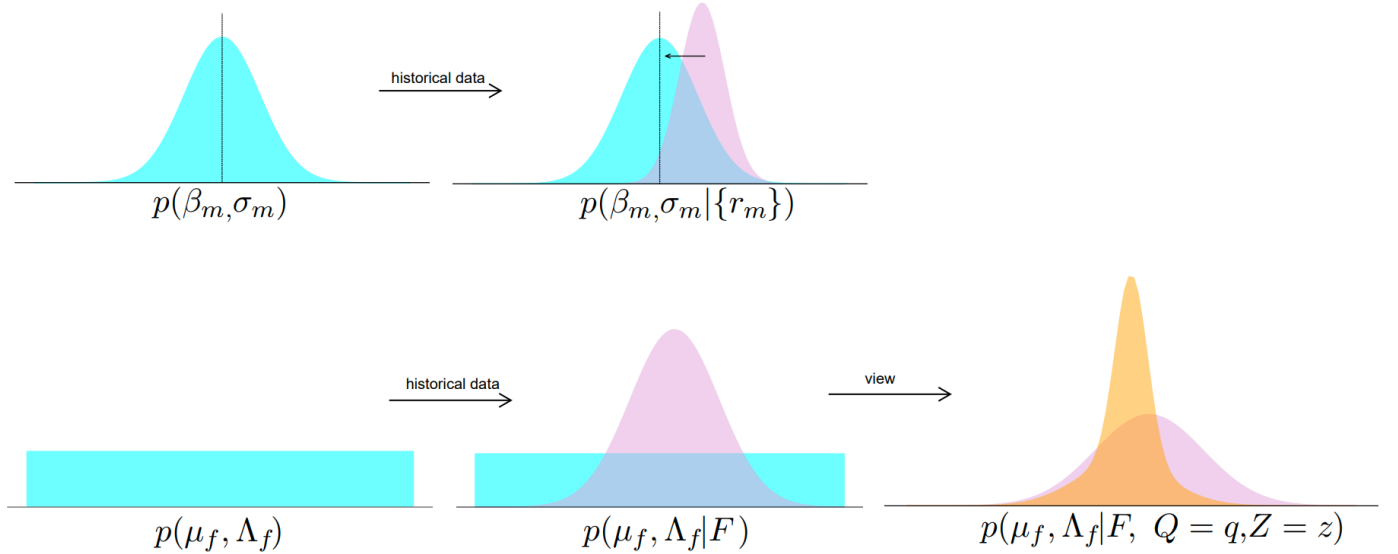
$$Z | \Lambda_f, \mathcal{D} \sim \mathcal{W}(d, \Lambda_f/d)$$

This results in a formulation where  $d$  serves as a confidence metric. It can be understood as the equivalent sample size (or the observation period) related to the views. A larger  $d$  signifies a greater variance of view elements, implying a lower confidence in the view, which in turn lessens its impact on the subsequent posterior. After executing the second Bayesian update, the posterior distribution for  $\Lambda_f$  evolves into:

$$\Lambda_f | \mathcal{D}, Z = z \sim \mathcal{IW}(T - 1 + d, \Lambda_n + d \times z)$$

To summarize, the two Bayesian updating under the model can be illustrated as follow:

Figure 2: Dual Bayesian Updating under Model



## 4 Empirical study

### 4.1 Performance Analysis without Views

In the subsequent empirical analysis, we assess the performance of three distinct models in the context of the US equity market. The first is rooted in the traditional approach, utilizing sample mean and sample covariance matrices for portfolio construction, henceforth referred to as the 'Sample' model. In contrast, the second and the third, known as the 'Model(S)' and 'Model(NS)', employ the techniques delineated in Section 3, though it excludes any views. For the 'Model(S)', the chosen parameters are given by  $\beta_{m,0} = 0$  and  $g = g^*$ . For the 'Model(NS)',  $g$  is chosen arbitrarily large to impose negligible shrinkage. Our primary aim is to highlight the enhancements in performance stemming from the Bayesian factor model and the adaptive shrinkage.

For our evaluation, we resort to the ten industry portfolios as defined by Fama and French. These portfolios can be accessed through Fama and French's data library. Such portfolios emerge from

categorizing stocks into ten distinct industries based on their Compustat SIC codes, subsequently forming a value-weighted portfolio for each industry grouping.

From this dataset, we focus on the top 200 US equities as of the start of 2012. The analysis extends from the following day, 2013-01-04, through to 2023-02-28. For each trading day within this timeframe, the preceding 251 days' returns form the basis for parameter estimation. This encompasses the calculation of the optimal shrinkage,  $g^*$ , as well as other relevant parameters in the posterior distributions.

Our evaluation then zeros in on two portfolio types: the minimum-variance portfolios (abbreviated as "MV") and the efficient portfolios, denoted as "WB(m)". The latter targets specific daily returns, with m ranging from [0.03%, 0.05%, 0.07%, 0.09%, 0.12%]. Once constructed based on each model, these portfolios are held for a single day.

The analysis utilizes adjusted closing prices, accounting for potential dividends and stock splits. The ensuing performance of each portfolio type, based on each model, is quantified using the Sharpe ratio (exclusive of transaction costs) and annualized volatility.

Our findings are succinctly presented in Tables 1 and 2 below.

Table 1: Sharpe Ratio

Portfolios	Model(S)	Model(NS)	Sample
MV	0.832	0.824	0.200
WB(0.0003)	0.822	0.816	0.211
WB(0.0005)	0.818	0.812	0.179
WB(0.0007)	0.797	0.791	0.147
WB(0.0009)	0.763	0.760	0.115
WB(0.0012)	0.703	0.702	0.067

Table 2: Daily Turnover

Portfolios	Model(S)	Model(NS)	Sample
MV	0.167	0.174	2.299
WB(0.0003)	0.174	0.180	2.254
WB(0.0005)	0.193	0.197	2.250
WB(0.0007)	0.223	0.226	2.261
WB(0.0009)	0.260	0.261	2.285
WB(0.0012)	0.325	0.322	2.347

### Performance Metrics and Portfolio Turnover

The tables illustrate a glaring weakness of the sample-estimator-based model, namely its subpar out-of-sample performance coupled with an exceptionally high turnover rate. After accounting for standard transaction costs, this high churn effectively decimates the portfolio's returns, resulting in

negative Sharpe ratios. This observation aligns well with numerous empirical studies which highlight the pitfalls of relying excessively on sample estimators for portfolio optimization.

In comparison, the 'Model(NS)' method consistently achieves higher Sharpe ratios and significantly reduces turnover. This reduction can be linked to the factor model's simplicity and the Bayesian approach's uncertainty quantification. This makes the resulting optimal portfolios less sensitive to daily data fluctuations. The introduction of adaptive shrinkage in 'Model(S)' slightly enhances the portfolio performance. As a reference point, an equal-weighted  $1/N$  portfolio over the same timeframe yields a Sharpe ratio of 0.791. This metric is eclipsed by the Sharpe ratios of most portfolios formulated using the Bayesian model.

Additionally, portfolio performance was also assessed using a consistent shrinkage strength, with  $g$  values ranging from 5 to 50. The data indicates that fine-tuning the shrinkage strength could potentially boost performance. However, this also introduces the potential for estimation errors related to data mining. The result is shown in Table 3 below.

Table 3: Sharpe Ratio with Constant Shrinkage Intensity

Portfolios	g=50	g=40	g=30	g=20	g=10	g=5	Model(S)	Model(NS)
MV	0.832	0.834	0.837	0.842	0.853	0.864	0.832	0.824
WB(0.0003)	0.823	0.825	0.827	0.831	0.841	0.848	0.822	0.816
WB(0.0005)	0.818	0.820	0.822	0.826	0.834	0.839	0.818	0.812
WB(0.0007)	0.796	0.797	0.799	0.801	0.806	0.805	0.797	0.791
WB(0.0009)	0.763	0.763	0.764	0.765	0.766	0.759	0.763	0.760
WB(0.0012)	0.703	0.702	0.702	0.702	0.698	0.685	0.703	0.702

### Portfolio Diversification Analysis

Diversification remains a cornerstone of portfolio optimization, aiming to maximize returns while minimizing risk through a balanced asset allocation. To quantify the degree of diversification in the portfolios produced by the two models, we leveraged the Gini impurity measure applied to the weight vector of each portfolio daily. This measure is given by:

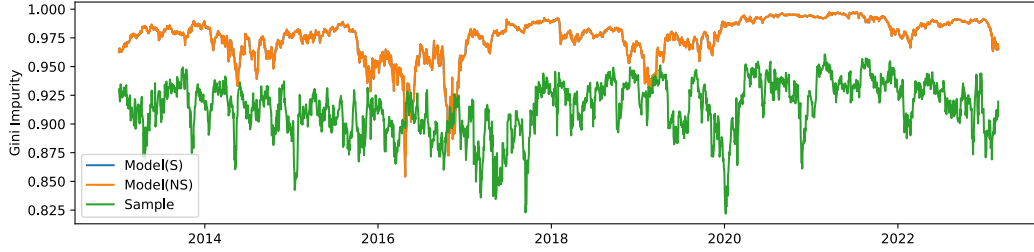
$$\text{Gini Impurity} = 1 - \sum_{m=1}^M w_n^2$$

Higher Gini impurity values signify enhanced diversification, indicating a more even distribution of weights across a larger number of assets rather than a concentration of just a few.

For a comparative analysis, we zeroed in on the "WB(0.0007)" portfolio. As illustrated in Figure 3, portfolios formulated using sample estimators suffer from poor diversification. They display a pronounced bias towards assets that have historically performed well, leading to a heavy concentration in a handful of stocks. In stark contrast, portfolios derived from the 'Model(S)' and 'Model(NS)' exhibit superior diversification for the majority of the observed period. This trend of enhanced diversification holds true across all portfolios produced using the Bayesian approach,

rendering it a more robust and effective tool for portfolio optimization than traditional sample estimators. The enhanced diversification appears to be primarily attributed to the Bayesian factor model rather than the adaptive shrinkage, as indicated by the similar gini impurity between 'Model(S)' and 'Model(NS)'.

Figure 3: Gini Impurity of Weight Vector



### Distance Metrics for Estimates' Differences

Portfolio performance measures, such as the Sharpe ratio, provide a holistic view of the effectiveness of our model but remain silent on its granular impact on the mean and covariance estimates. To delineate this effect, we employ two distance metrics to contrast the mean and covariance estimates drawn from different models. The distance metrics are defined as:

$$d_1(A, B) = \sum_{i=1}^M |a_i - b_i|$$

$$d_2(A, B) = \sum_{i=1}^M \sum_{j=1}^i |a_{ij} - b_{ij}|$$

Here,  $d_1$  gauges the aggregate absolute difference in mean estimates between the two models, while  $d_2$  captures the cumulative absolute differences in the covariance estimates.

Figures 4 and 5 detail the daily discrepancies between the estimates of the two models. Both 'Model(NS)' and 'Model(S)' regularly yield estimates that markedly differ from the sample estimators. Specifically, the mean estimate variation is roughly 10% of the total absolute daily returns. While differences in mean estimates from adaptive shrinkage appear minimal, the variations in covariance estimates are pronounced. This significant variance in covariance due to shrinkage likely drives the previously noted enhancement in portfolio performance, indicating that shrinkage more profoundly impacts covariance than mean.

The temporal variation in this difference is noteworthy. Periods rife with market volatility, such as the tumultuous phase during the Covid-19 market crash in 2020, witnessed the discrepancy escalating to approximately 12%. This divergence further intensified, reaching 14%, during the market's subsequent recovery. An analogous trend is discerned in the covariance estimates.

An intriguing observation emerges when examining the role of the shrinkage parameter  $g^*$ . This parameter, which dictates the degree of shrinkage, is instrumental in shaping the differences in the estimates. The value of  $g^*$  during the observation period is shown in Figure 6.

Figure 4: Distance In Mean Estimate

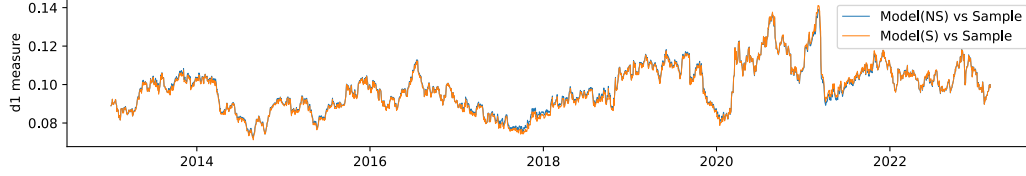


Figure 5: Distance In Covariance Estimate

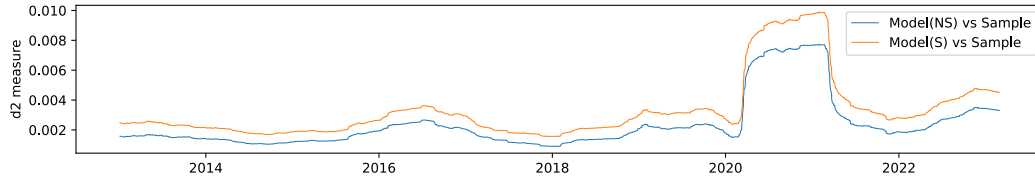
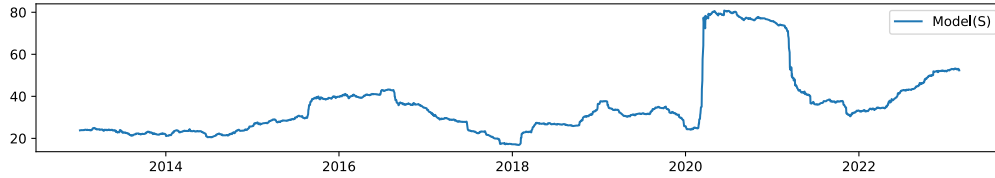


Figure 6: Strength Of Shrinkage



The ultimate measure of any portfolio model's effectiveness is the cumulative returns it provides. While tables and metrics shed light on different dimensions of the model's performance, nothing speaks louder than the actual returns. In Figure 7, the cumulative return of "WB(0.0007)" illustrates the pronounced advantage of 'Model(S)' over 'Sample'. A discernible feature from the graph is 'Model's' resilience during tumultuous periods, most notably during the 2020 Covid crash. While both portfolios experience declines, the Bayesian model manifests markedly diminished drawdowns. Such robustness is not exclusive to "WB(0.0007)"; a similar trend is echoed across all other portfolios constructed using 'Model'. The improvement is attributable to both the parsimony of factor model, uncertainty quantification of the Bayesian approach, and the regularization from the adaptive shrinkage.



Figure 7: Cumulative Return of WB(0.0007)



The superiority of a Bayesian factor model incorporating adaptive shrinkage on factor loadings through the first Bayesian update becomes evident. By consistently offering enhanced estimates, it paves the way for improved portfolio performance, delivering both higher returns and minimized drawdowns. The findings substantiate the value of our model as a potent tool for portfolio management, especially in unpredictable market terrains.

## 4.2 Prophetic Views: Expected Return Simulation

Given the experimental nature of this section, we deploy a simulation-based approach, leveraging the unique characteristic of "prophetic views" - these views provide perfect foresight into the next day's factor portfolios. Such a setup is intended to capture the extreme end of the spectrum where an investor possesses flawless predictive abilities. While in practice such precision is unattainable, this exercise is illustrative of the maximal potential benefits from view incorporation. In this context, the parameters are set to:  $\beta_{m,0} = 0, g = g^*$ .

For relative views, the investor predicts how each industry factor portfolio will perform relative to the others. For absolute views, the investor prophesies the exact return of each factor portfolio. In both cases, the parameter  $c$  modulates the confidence in these views.

For relative views, the pick matrix  $P$  and view matrix  $Q$  are defined as follows. The pick matrix is full-ranked, with the last view being an absolute view.

$$P_t = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q_t = \begin{bmatrix} f_{t+1,1} - f_{t+1,2} \\ f_{t+1,2} - f_{t+1,3} \\ \vdots \\ f_{t+1,K-1} - f_{t+1,K} \\ f_{t+1,K} \end{bmatrix}$$

For absolute view, the pick matrix  $P$  and view matrix  $Q_t$  are

$$P_t = \mathbb{I}_K, \quad Q_t = f_{t+1}$$

The performance difference between the two types of views—absolute and relative—is intriguing. While both types of views are full-ranked, their distinct characteristics render different advantages to the Bayesian model. Results are shown in Table 4 and 5 below.

### Absolute vs. Relative Views:

Absolute views provide direct forecasts about the expected returns of specific assets or portfolios, thus offering clear and unequivocal information for the model. Relative views, in contrast, only provide information about the order or relative performance of assets. Even if this relative order is predicted correctly, the magnitude of the returns remains unknown. This can be a significant limitation, especially if the model needs quantitative values to optimize against. It is this clarity and directness of absolute views that make them more beneficial for the Bayesian model.

### Optimal Confidence Level $c$ :

The observation that there exists an optimal confidence level is quite fascinating. While one might naively assume that having more confidence in perfect predictions should always be better, this is not the case. A plausible explanation is that overconfidence can lead to overfitting or extreme portfolio weights, which might amplify the impact of other estimation errors in the model.

Table 4: Relative view: Sharpe ratio

Portfolios	c=27	c=23	c=19	c=15	c=13	c=9	c=5	c=1
MV	0.81	0.81	0.81	0.81	0.81	0.81	0.81	0.81
WB(0.0003)	1.61	1.67	1.72	1.75	1.74	1.64	1.41	1.07
WB(0.0005)	2.22	2.34	2.46	2.51	2.50	2.32	1.91	1.26
WB(0.0007)	2.74	2.93	3.11	3.22	3.21	2.98	2.39	1.45
WB(0.0009)	3.15	3.41	3.67	3.85	3.85	3.61	2.86	1.63
WB(0.0012)	3.60	3.96	4.34	4.63	4.68	4.45	3.54	1.91
WB(0.0015)	3.88	4.32	4.81	5.23	5.34	5.18	4.17	2.19

Table 5: Absolute view: Sharpe ratio

Portfolios	c=27	c=23	c=19	c=15	c=13	c=9	c=5	c=1
MV	0.81	0.81	0.81	0.81	0.81	0.81	0.81	0.81
WB(0.0003)	3.22	3.42	3.59	3.67	3.64	3.38	2.74	1.59
WB(0.0005)	5.10	5.49	5.82	6.00	5.98	5.55	4.37	2.21
WB(0.0007)	6.71	7.30	7.81	8.15	8.17	7.66	5.99	2.83
WB(0.0009)	8.01	8.78	9.48	10.04	10.13	9.66	7.58	3.44
WB(0.0012)	9.40	10.43	11.40	12.33	12.59	12.39	9.91	4.37
WB(0.0015)	10.29	11.52	12.74	14.02	14.48	14.73	12.12	5.28

In summary, the simulation analysis reveals that the Bayesian model can derive considerable advantages from precise forward-looking perspectives offered by an investor. The enhancement is particularly pronounced when these perspectives are conveyed as absolute views. While in real-world scenarios, possessing prophetic insights is unattainable, this suggests that the model has the capacity to harness accurate views to bolster portfolio performance.

### 4.3 Prophetic Views: Covariance Simulation

Parallel to our earlier exploration with means, here we delve into the scenario where investors are armed with 'prophetic insights' regarding factor covariance. This grants them precise foresight into the next day's factor covariance, denoted by  $f_{t+1}f_{t+1}^T$ . This paradigm underscores the apex of potential advantages when imbibing views on factor covariance.

The ensuing data, captured in Table 6 and 7, elucidates the Sharpe ratio and annualized volatility over a varied confidence spectrum, represented by  $d = [10, 20, 30, 40, 50]$ . Precise forecast of factor covariance invariably reduces portfolio volatility, escalating with an increasing confidence parameter  $d$ . Nevertheless, this volatility reduction doesn't consistently result in superior Sharpe ratios. This aligns with our expectations: while the MV portfolio minimizes volatility, it doesn't inherently maximize the Sharpe ratio. When exploring efficient portfolios, Sharpe ratio enhancement is restrained, becoming greater when the target expected returns increase. A plausible explanation might be that higher expected returns allocate more weight to assets exhibiting greater volatility, yet potentially offering a favorable Sharpe ratio.

Table 6: View on Factor Covariance: Annualized Volatility

Portfolios	d=50	d=40	d=30	d=20	d=10
MV	0.112	0.113	0.114	0.116	0.119
WB(0.0003)	0.113	0.114	0.116	0.118	0.121
WB(0.0005)	0.113	0.115	0.117	0.119	0.123
WB(0.0007)	0.115	0.117	0.119	0.122	0.127
WB(0.0009)	0.118	0.121	0.124	0.127	0.134
WB(0.0012)	0.126	0.129	0.132	0.138	0.147

Table 7: View on Factor Covariance: Sharpe Ratio

Portfolios	d=50	d=40	d=30	d=20	d=10
MV	0.890	0.893	0.896	0.898	0.895
WB(0.0003)	0.905	0.908	0.911	0.912	0.902
WB(0.0005)	0.925	0.931	0.935	0.935	0.920
WB(0.0007)	0.934	0.941	0.945	0.944	0.922
WB(0.0009)	0.932	0.939	0.943	0.939	0.910
WB(0.0012)	0.912	0.919	0.922	0.914	0.875

In short, having a prophetic forecast of factor covariance does indeed confer improvements to efficient portfolios. However, the magnitude of this improvement doesn't parallel the gains derived from views on expected return. This reaffirms the MV framework's sensitivity to errors in mean estimations compared to covariance estimations, a finding consistent with many empirical studies.

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