Forward At The Money Forward Implied Volatility and forward underlying move estimations

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Abstract

Ahead of the 23^{rd} June UK referendum on "Brexit", this note provides a technique for estimating the Forward (at referendum date) At The Money Forward (ATMF) implied volatility for equity or FX Indexes. We provide a closed form formula for the forward underlying expected moves (for short terms maturities) post the referendum date. We provide a closed form formula for the forward underlying expected moves conditional to the adverve event (vote in favour of 'leaving' the European Union (EU) area) happening. We finally provide a closed form formula for the forward underlying expected moves conditional to the adverve event not happening . More genarally the framework here can be used to estimate forward implied volatility and forward asset price moves post a potentially adverse event to come in the future.

1 Introduction

In the attempt to imply what market participants think will be the behaviour (moves) of some market assets, especially GBP/USD and GBP/EUR foreign exchange rates, post the 23^{rd} June 2016 referendum, we provide here a technique based on a parametrisation of the ATMF implied volatility, for the forward volatility estimation. The parametrisation is performed in two steps: the first step consists in assuming a Stochastic Diffusion Equations (SDEs) for both the asset price and its volatility. And for this diffusion to calculate the closed form formula for the price of a Variance Swap contract, on the asset price. The second step consists in using the Gatheral's approximation in [1] to switch from variance swap price to the square of the ATMF implied volatility. This gives us a parametric function for the spot (as of today) ATMF implied volatility. We then calibrate the diffusion model parameters to fit the market quotes of

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the ATMF implied volatility (as of today). These calibrated parameters are finally used, along with the parametric function representing the spot ATMF implied volatility, for estimating the forward ATMF implied volatility. Our approach for estimating the forward volatility is based on the additive property of the forward instantaneous variance. Having the forward ATMF volatilities we finally estimate the asset's price forward expected moves.

Our approach for estimating the forward asset's price expected moves is based on that the probability for the adverse event to happen is the probability that the asset price decreases, below the level of the "risk-neutral" expected move, the day following the event (here the referendum). By doing so we come out with a closed form formula for the forward expected move, consistent with short interest rates, consistent with the forward implied volatility estimated, and consistent with the market quote (or poll results) of the probability for the adverse event to happen. We finally approximate the price of a straddle option, for short maturities from the forward start date, to get an estimation of both the expected moves conditional to the adverse event happening and conditional to the adverse event not happening.

2 The Dynamics

Let X be the value of a foreign exchange rate[1] between a given domestic and a given foreign currency: X_t is the value at time t of 1 unit of foreign currency, expressed in domestic currency. let r^d be the short interest rate in the domestic economy; r^f be the short interest rate in the foreign economy; and q the collateral basis spread between the two economies. Let V denote the instantaneous variance of X's returns. Then we will postulate the following diffusion equations for both the asset and its variance processes:

$$\begin{aligned} \frac{dX_t}{X_t} &= \left(r^d(t) - r^f(t) + q(t)\right) dt + \sqrt{V_t} dW_t^X \\ dV_t &= \kappa \left(V_\infty - V_t\right) dt + \gamma V_t dW_t^V \\ d \prec W^X, W^V \succ_t &= \rho(t) dt \end{aligned} \tag{1}$$

Where:

- $\left(W^X\right)_{t\geq 0}$ and $\left(W^V\right)_{t\geq 0}$are two brownian motions driving the uncertainties on the asset price and its variance processes respectively. ρ is the instantaneous correlation between these two brownian motions;
- κ; V_∞ *and* γ are parameters of the variance process equation, and should be calibrated on well chosen market instruments;
- κ is the mean reversion parameter. It drives how quick the variance process reverts towards its long term level V_∞. The bigger is κ, the quicker the variance process reverts towards V_∞;

[1] Could be any equity underlying as well.

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- V_∞ is the long term level towards which the variance process will converge (in expectation) after a period of diffusion;
- γ is a parameter driving the volatility of the variance process: volatility of volatility;
- Dynamics in equation 1 is referred to as log normal model or Gatheral (Jim) model, and is widely used within the industry.

3 Variance Swap price

What we are trying to achieve, at this stage, is to calibrate the SDE 1 parameters on market ATMF implied volatility term structure. But because it is not obvious (on our knowledge) to derive a closed form formula for the ATMF implied volatility in dynamics 1, we will get around the obstacle by using Variance Swap contracts. But first let's define what we mean by Variance Swap.

We call Variance Swap of a given maturity T the strike, today, of a swap on the asset price's realised variance (between today and the maturity date) such that the contract is at par; i.e the value of the swap is zero. An approximation of the variance swap price is:

$$\begin{aligned} VS(t,T) &= \mathbb{E}_t\left[\frac{1}{T-t}\int_t^T V_s ds\right] \\ &= \frac{1}{T-t}\int_t^T \mathbb{E}_t\left[V_s\right] ds \\ &= \frac{1}{T-t}\int_t^T \sigma(t,s)^2 ds \end{aligned} \tag{2}$$

Where $\sigma(t,s)^2 := \mathbb{E}_t\left[V_s\right]\ \ \forall\ t \le s \le T$ is the forward instantaneous variance.

For the chosen diffusion 1, and assuming that the square of variance process does not diverge between today and maturity date, actually $\int_t^T \mathbb{E}_t\left[V_s^2 e^{-2\kappa(T-s)}\right] ds < +\infty$, then we have the following closed form formula for the forward instantaneous variance process:

$$\sigma(t,s)^2 = V_\infty + e^{-\kappa(s-t)}\left(V_t - V_\infty\right)$$

Inserting in equation 2 we get the following closed form formula for the Variance Swap:

$$VS(t,T) = V_\infty + \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa\left(T-t\right)}\left(V_t - V_\infty\right) \tag{3}$$

4 From Variance Swap to the ATMF implied volatility

Now we have Closed form form formula for Variance Swap price, we will use Gatheral statements in [1] to obtain the Black and Scholes ATMF implied volatility. Gatheral suggests the following relationship between variance swap price and implied volatility on call and put prices:

$$VS(t,T) = \int_{-\infty}^{+\infty} \sigma_{BS}\left(t,T,z\right)^2 \mathcal{N}^{'}\left(z\right) dz \tag{4}$$

Where $\mathcal{N}$ is the normal distribution cumulative function; $\sigma_{BS}\left(t,T,z\right)$ is the Black and Scholes implied volatility of maturity T , and z is the log moneyness. If we assume the following parametrisation for the Black and Scholes implied volatility:

$$\sigma_{BS}\left(t,T,z\right) = \sigma_{ATMF}\left(t,T\right) + skew(t,T)z + \frac{1}{2}curve(t,T)z^2$$

Then from equation 4 we get that

$$VS(t,T) = \sigma_{ATMF}\left(t,T\right)^2 + \beta(t,T) \tag{5}$$

With

$$\beta(t,T) := \sigma_{ATMF}\left(t,T\right) curve(t,T) + skew(t,T)^2 + \frac{3}{4}curve(t,T)^2 \tag{6}$$

We then get the following formula for the ATMF implied volatility

$$\sigma_{ATMF}\left(t,T\right) = \sqrt{V_\infty + \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa\left(T-t\right)}\left(V_t - V_\infty\right) - \beta(t,T)} \tag{7}$$

For each maturity date $\beta(t,T)$ can be calculated using market data, and following equation 6. We will use the 25 risk reversal for approximating the skew in FX; and we will use the 25 butterfly for estimating the curve.

Equation 7 can be developped to get the final parametric function for the ATMF implied volatility:

$$\sigma_{ATMF}\left(t,T\right) = \sqrt{V_\infty + \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa\left(T-t\right)}\left(V_t - V_\infty\right) - skew(t,T)^2 - \frac{1}{2}curve(t,T)^2} - \frac{1}{2}curve(t,T) \tag{8}$$

5 Adding Jumps

Ahead of the 23rd June, 2016 referendum on UK exiting (or not) the Euro zone, we observe a jump, around the referendum date, on the GBP/USD ATMF implied volatility term structure, as shown on the following chart:

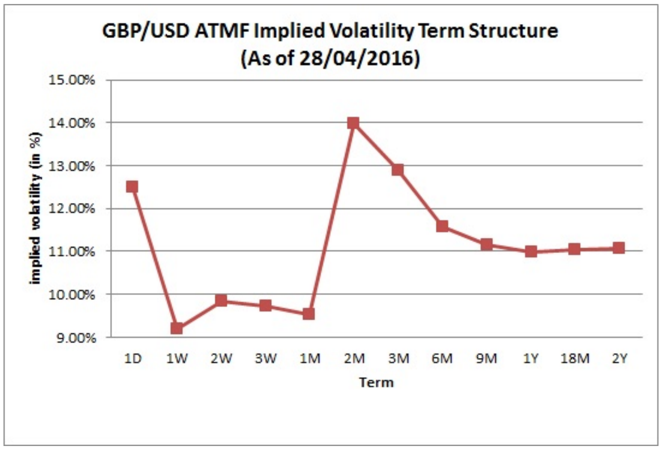

Figure 1: GBP/USD ATMF implied Volatility term structure. Source: Bloomberg

A time continuous model as described in the previous sections cannot capture jumps such as the one we observe between the 1 month and the 2 month terms on figure 1. In order to do this we will add the possibility for the model to jump within a short window of time around the referendum date. We actually assume that the instantaneous volatility will jump within a short period of time (typically 1 day) before the referendum date. We will model this jump as following:

$$\sigma^2_{Jump}(t,s) = \lambda J \mathbf{1}_{\{T_F - \varepsilon < s \leq T_F\}} \tag{9}$$

Equation 9 suggests the forward instantaneous variance jumps within the window of time $[T_F - \varepsilon, T_F]$. J can be seen as the size of the jump; and λ is the Poisson intensity: the number of jumps per unit of time. Typically ε will be equal to 1 day (before the referendum date).

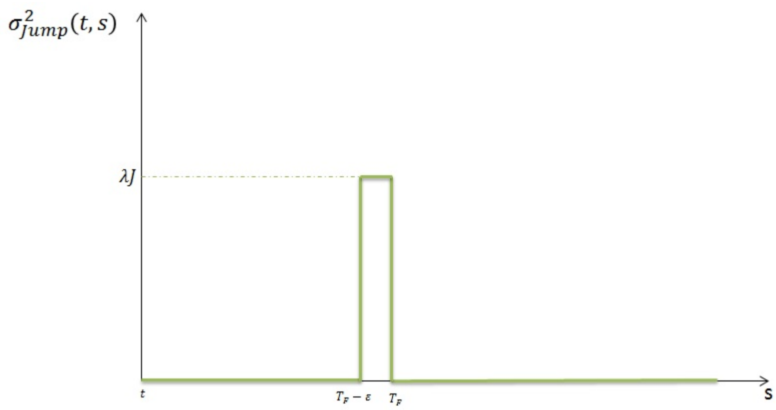

Figure 2: forward instantaneous volatility jump modelling

We then get the following approximation for the jump component of the ATMF variance, for a given maturity:

$$\sigma^2_{ATMF-Jump}\left(t,T\right)=\lambda J\frac{\varepsilon\wedge\left(T-T_F+\varepsilon\right)}{T-t}\mathbf{1}_{\{T>T_F-\varepsilon\}}$$

With $x\wedge y:=min(x,y)$

We therefore modify the ATMF volatility expression as follow

$$\sigma_{ATMF}\left(t,T\right)=\sqrt{V_\infty+\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa\left(T-t\right)}\left(V_t-V_\infty\right)-\beta(t,T)+\sigma^2_{ATMF-Jump}\left(t,T\right)} \tag{10}$$

Which gives the following Final parametric function for the ATMF implied volatility:

$$\sigma_{ATMF}\left(t,T\right)=\sqrt{V_\infty+\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa\left(T-t\right)}\left(V_t-V_\infty\right)+\sigma^2_{ATMF-Jump}\left(t,T\right)-skew(t,T)^2-\frac{1}{2}curve(t,T)^2}-\frac{1}{2}curve(t,T) \tag{11}$$

6 Calibration

The calibration will consist in getting the values of parameters κ , V_t , , V_∞ , *and* λJ such that analytical formula in equation 11 matches market values in Figure 1. Results are given in appendix section.

7 The forward ATMF implied volatility

Now that we have calibrated the model parameters as described in previous section, we will approximate the forward ATMF implied volatility, using the additive property of the forward instantaneous variance:

$$\int_t^T \sigma^2(t,s)ds = \int_t^{T_F} \sigma^2(t,s)ds + \int_{T_F}^T \sigma^2(t,s)ds$$

We get the following approximation for the forward ATMF implied volatility

$$\sigma_{ATMF}(t,T_F,T) \approxeq \sqrt{\frac{(T-t)\,\sigma_{ATMF}(t,T)^2 - (T_F-t)\,\sigma_{ATMF}(t,T_F)^2}{T-T_F}}$$

Using equation 8 (here we are assuming no jumps) we get

$$\sigma_{ATMF}(t,T_F,T) \approxeq \sqrt{V_\infty + \frac{\left(e^{-\kappa(T_F-t)} - e^{-\kappa(T-t)}\right)}{\kappa(T-T_F)}(V_t - V_\infty) - \frac{(T-t)\,\beta(t,T) - (T_F-t)\,\beta(t,T_F)}{T-T_F}}$$

With the possibility of jumps we will use the following formula for the forward ATMF implied volatility

$$\sigma_{ATMF}(t,T_F,T) \approxeq \sqrt{V_\infty + \frac{\left(e^{-\kappa(T_F-t)} - e^{-\kappa(T-t)}\right)}{\kappa(T-T_F)}(V_t - V_\infty) - \frac{(T-t)\,\beta(t,T) - (T_F-t)\,\beta(t,T_F)}{T-T_F} + \sigma^2_{ATMF-Jump}(t,T_F,T)} \tag{12}$$

Where $\sigma^2_{ATMF-Jump}(t,T_F,T)$ is calculated using the additive property of the forward instantaneous variance, on the jump component:

$$\sigma^2_{ATMF-Jump}(t,T_F,T) = \lambda J \frac{\varepsilon \wedge (T-T_F+\varepsilon)}{T-T_F}\mathbf{1}_{\{T>T_F\}}$$

β is calculated following equation 6; and the other parameters are calibrated as explained in section 6.

Remark: If the date when the jump occurs is T_{Jump} different from the forward starting date T_F then the jump component in the forward volatility expression (equation 12) will be changed as follow:

$$\sigma^2_{ATMF-Jump}\left(t,T_{Jump},T_F,T\right) = \lambda J\left(\frac{\varepsilon \wedge \left(T-T_{Jump}+\varepsilon\right)}{T-T_F}\mathbf{1}_{\left\{T\geq T_{Jump}-\varepsilon\right\}} - \frac{\varepsilon \wedge \left(T_F-T_{Jump}+\varepsilon\right)}{T-T_F}\mathbf{1}_{\left\{T_F\geq T_{Jump}-\varepsilon\right\}}\right)\mathbf{1}_{\{T>T_F\}}$$

everything else will still hold.

8 The case of a basket made of two indexes

For some reasons we may want to estimate the forward ATMF implied volatility term structure for a basket made of two traded indexes. It is possible to achieve that using the same approch as described in the previous sections. In this section We will give the expressions to use for the "market" volatility, skew and curve for the basket index.

Let say we make a basket of $\omega \in [0,1]$ proportion of asset X_1 and $1-\omega$ proportion of asset X_2. We therefore make a basket with value

$$X(t) = \omega X_1(t) + (1-\omega)\, X_2(t)$$

And we want to estimate the forward ATMF implied volatility for basket X. We will use the same approach as described in previous section, with the following for basket X's "market" volatility:

$$\sigma_X(t,T) = \sqrt{\omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho_{12}\sigma_1\sigma_2}$$

Where $\sigma_i := \sigma_{X_i}(t,T)$ i$\in \{1,2\}$ is the market ATMF implied volatility for index X_i ; and ρ_{12} is the correlation between index X_1 and index X_2, that can be estimated from historical data.

We will use the followings for X's "market" skew and curve:

$$skew_X(t,T) = \frac{1}{\sigma_X(t,T)}\left[\left(\omega^2\sigma_1 + \rho_{12}\omega(1-\omega)\sigma_2\right) skew_1 + \left((1-\omega)^2\sigma_2 + \rho_{12}\omega(1-\omega)\sigma_1\right) skew_2\right]$$

And

$$curve_X(t,T) = \frac{1}{\sigma_X(t,T)}\left[-skew_X(t,T)^2 + \hat{skew}^2 + \left(\omega^2\sigma_1 + \rho_{12}\omega(1-\omega)\sigma_2\right) curve_1 + \left((1-\omega)^2\sigma_2 + \rho_{12}\omega(1-\omega)\sigma_1\right) curve_2\right]$$

with

$$\hat{skew}^2 := \omega^2 skew_1^2 + (1-\omega)^2 skew_2^2 + 2\omega(1-\omega)\rho_{12} skew_1 skew_2$$

Where $skew_i,\ curve_i$ i$\in \{1,2\}$ are market ATMF skew and curve for index X_i.

9 Underlying forward moves

In this section we will compute the expected forward moves of the asset price(estimation, today, of the underlying's moves from the forward starting date), consistent with the forward volatilities calculated in previous sections. The idea here is to assess how big the adverse event to come (here the referendum) will impact some assets' prices.

Given the forward volatilities calculated in previous section, we will make no assumption on the direction (up or down) of the underlying move. Our approach is based on the Black and Scholes valuation of call and put options prices. Let's recall Black and Scholes results.

9.1 Recalls on call and put prices in Black and Scholes model

Black and Scholes model is one where we assume a flat (constant) volatility for the underlying. With this respect, the price of a call [2] option with maturity T

[2] pay off: $max(X_T - K, 0)$

and strike K is

$$Call\,(t,T,K) = P(t,T)\,(F_{t,T}\mathcal{N}\,(d_+) - K\mathcal{N}\,(d_-)) \tag{13}$$

With

$$d_+ = \frac{\ln\left(\frac{F_{t,T}}{K}\right) + \frac{1}{2}\sigma^2\,(T-t)}{\sigma\sqrt{T-t}}$$

$$d_- = d_+ - \sigma\sqrt{T-t}$$

$$\mathcal{N}\,(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx$$

$P(t,T)$ is the zero-coupon bond of maturity T; $F_{t,T}$ is the forward of maturity T; and $\sigma := \sigma_{implied}\,(T,K)$ is the so called Black and Scholes implied volatility. $\mathcal{N}$ is the normal-distribution cumulative density function.

For the put[3] price the formula is

$$Put\,(t,T,K) = -P(t,T)\,(F_{t,T}\mathcal{N}\,(-d_+) - K\mathcal{N}\,(-d_-)) \tag{14}$$

9.2 Forward expected move estimation [4]

The results we will derive here will only be consistent with short maturities from the forward starting date: $0 \leq T - T_F$ small. The idea is to assume that for very short maturities the probability for the adverse event (vote for leaving the EU area in the case of the "Brexit" referendum) to happen is the probability that the asset price decreases, more than the "risk-neutral" expected move, compared to its value on the event date. Let p be the probability (as of today) that the adverse event will happen; let assume that T_F is the event (referendum) date. Then, for any maturity date T reasonably close to the event date T_F we suggest that

$$\begin{aligned} p(t) &= \mathbb{P}_t\left[X_T < X_{T_F} e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds}\right] \\ &= \mathbb{E}_t\left[\mathbf{1}_{\left\{X_T < X_{T_F} e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds}\right\}}\right] \end{aligned} \tag{15}$$

We think one week is a reasonable valid maturity maximum horizon, for equation 15.

Let x be the relative move of the asset price from the event date:

[3] pay off: $max(K - X_T, 0)$

[4] Analysis here is consistent with a risky asset subject to depreciation in case of adverse event happening. The approach here can be easily adapted for the case of assets subject to appreciation in case of adverse event happening.

$$x := \frac{X_T - X_{T_F}}{X_{T_F}}$$

And let $\bar{x}$ be the expectation of x:

$$\bar{x} = \mathbb{E}_t\left[x\right]$$

Then $\bar{x}$ is what we are trying to estimate in this section, and is what we refer to as the expected forward move.

In "risk-neutral" framework, without jumps and without dividends (for equity index) it is known that (assuming non stochastic interest rates)

$$\bar{x} = \begin{cases} e^{\int_{T_F}^{T} r^d(s)ds} - 1 & for\ \ Equity\ \ Index \\ e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds} - 1 & for\ \ FX\ \ Index \end{cases}$$

In this framework we are not assuming a "risk-neutral" environment. We actually estimate the implied expected move, consistent with the market probability p, and consistent with the forward implied volatility calculated in previous sections. Assuming non-stochastic interest rates, the right hand of equation 15 can be expressed in terms of forward starting options prices:

$$\begin{aligned} p(t) &= \frac{1}{P(t, T_F, T)} DigitalPut_t\left(T_F, T, X_{T_F} e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds}\right) \\ &= \frac{1}{P(t, T_F, T)} \frac{\partial Put_t}{\partial K}\left(T_F, T, X_{T_F} e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds}\right) \end{aligned} \tag{16}$$

In the Black and Scholes model, expression 16 can be simplified and we get

$$p(t) = \mathcal{N}\left(-d_-\left(\bar{x}\right) + \frac{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds}{\sigma\sqrt{T - T_F}}\right) \tag{17}$$

With

$$d_-\left(\bar{x}\right) = \frac{\ln\left(1 + \bar{x}\right) - \frac{1}{2}\sigma^2\left(T - T_F\right)}{\sigma\sqrt{T - T_F}}$$

And $\sigma \equiv \sigma_{ATMF}\left(t, T_F, T\right)$: is the forward ATMF implied volatility calculated in section 7.

Given p from market quotes or polls results, and given the forward implied volatility σ, we solve 17 to finally get the following expression for the implied forward expected move:

$$\bar{x} = e^{\int_{T_F}^{T}\left(r^d(s) - r^f(s)\right)ds} e^{\sigma\sqrt{T - T_F}\left(\frac{1}{2}\sigma\sqrt{T - T_F} - \mathcal{N}^{-1}(p)\right)} - 1 \tag{18}$$

For constant interest rates the forward expected move can be simplified as follow:

$$\bar{x} = e^{\left(r^d - r^f\right)(T - T_F)} e^{\sigma\sqrt{T - T_F}\left(\frac{1}{2}\sigma\sqrt{T - T_F} - \mathcal{N}^{-1}(p)\right)} - 1$$

Remarks:

The expected move we derived in expression 18 is consistent with short interest rates differential, it is consistent with the level of forward implied volatility, and it is consistent with the market (or polls) probability that the adverse event will happens. The expression we derived is easy to use, to implement and to memorise. In particular if we assume a zero flat volatility, then formula 18 gives that the move is just the interest rate differential, as expected.

If we read p as the probability that the asset price drops shortly after the event (here the referendum) date, then a "risk neutral" environment should be consistent with $p = \mathcal{N}\left(\frac{1}{2}\sigma\sqrt{T - T_F}\right)$, which is close to $\frac{1}{2}$ when volatility is small (normal market conditions) and means there is as much chance for the asset price to move up or down, compared to "the risk-neutral" expected move, from that specific date. In this case the expected move would only be the interest rate differential, as expected.

In the case of Brexit referendum p will also be equal to the probabiliy that UK votes to leave Euro area (if X is the GBP/USD FX rate for instance). We will get this probability from Betfair (a betting agency). As of 28/04/2016 it was quoting at 29.63%. The fact that p is significantly less than $\frac{1}{2}$ means that market thinks the asset price distribution for the day following the event is not symmetric around the interest rate differential (almost zero), but is skewed in the up direction. It also means that the real environment around the event is not "risk-neutral", and that the asset price may move up with higher probability. Formula 18 captures this expected move.

Expression 18 gives a guideline how to adjust the short interest rates differential, if we want to control the asset's move. Assume we want to set the asset's move at a fixed level $\bar{x}$, by adjusting the short interest rates differential. Then, by inverting equation 18, we get that the level of rates differential should be

$$r^d - r^f = \frac{\ln(1+\bar{x}) - \sigma\sqrt{T - T_F}\left(\frac{1}{2}\sigma\sqrt{T - T_F} - \mathcal{N}^{-1}(p)\right)}{T - T_F}$$

In Particular if we want to prevent any move: $\bar{x} = 0$, then we should monitor the short interest rate differential such that $r^d - r^f = \frac{\sigma\sqrt{T - T_F}}{T - T_F}\left(\mathcal{N}^{-1}(p) - \frac{1}{2}\sigma\sqrt{T - T_F}\right)$. Calculated as of 28/04/2016, with $T - T_F = \frac{1}{256}$ (one day); $\sigma = 71.80\%$ (estimated as in section 7), and $p = 29.63\%$ (From Befair market quote) we get $r^d - r^f = -640.47\%$. Which means we should set the SONIA rate +640.47% (absolute value) higher than US short rate. The interpretation of this is that it is practically impossible to adjust interest rates or use rates policy in order to monitor events like the 'Brexit' referendum.

9.3 Forward expected moves conditional to the adverse event

In this sub-section we will use the expected move computed in the previous sub-section to determine a rough approximation of both the forward expected move conditional to the adverse event happening, and the forward expected move conditional to the adverse event not happening. We denote $\bar{x}^-$the forward expected move conditional to the adverse event happening, and $\bar{x}^+$ the forward expected move conditional to the adverse event not happening. We have that

$$x^+ := max(x, 0)$$

$$\begin{aligned} \bar{x}^+ : &= \mathbb{E}_t \left[x \,|\, x > 0 \right] \\ &= \frac{\mathbb{E}_t \left[x^+ \right]}{1-p} \end{aligned}$$

And

$$x^- := min(x, 0)$$

$$\begin{aligned} \bar{x}^- : &= \mathbb{E}_t \left[x \,|\, x < 0 \right] \\ &= \frac{\mathbb{E}_t \left[x^- \right]}{p} \end{aligned}$$

The following relation holds

$$p\bar{x}^- + (1-p)\bar{x}^+ = \bar{x} \tag{19}$$

We will use forward starting Straddle price to get another equation involving $\bar{x}^-$ and $\bar{x}^+$. Straddle contract is the sum of a put and call options with same strike prices and same maturity. The Straddle's payoff is the absolute value of the spread between the underlying asset's price at maturity, and the strike price (see Figure 3)

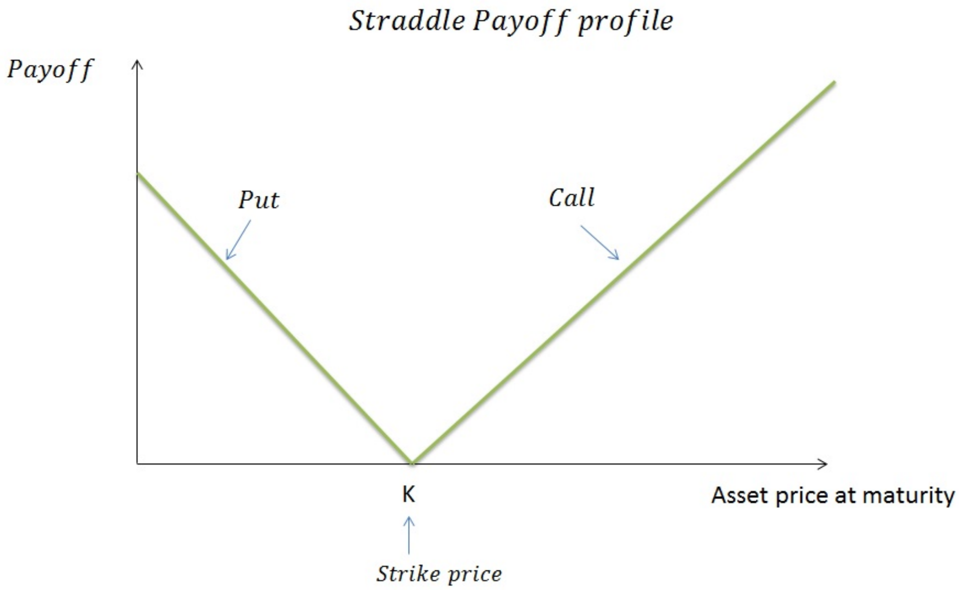

Figure 3: Straddle Payoff profile

Let consider a Straddle option starting on the event date T_F, with maturity T after the event date, with strike X_{T_F}. The value of this ATM spot forward starting option is

$$Straddle_t(T_F, T, X_{T_F}) = Put_t(T_F, T, X_{T_F}) + Call_t(T_F, T, X_{T_F})$$

Under the p-probability as developped in the previous sub-section we have that the value of this straddle is (using the same notations as in the previous section)

$$Straddle_t(T_F, T, X_{T_F}) = P(t,T)\mathbb{E}_t^T\left[X_{T_F}\right]\left[(1+\bar{x})\left(2\mathcal{N}\left(d_+\left(\bar{x}\right)\right)-1\right)-\left(2\mathcal{N}\left(d_-\left(\bar{x}\right)\right)-1\right)\right] \tag{20}$$

Finally we make the rough approximation that

$$Straddle_t(T_F, T, X_{T_F}) = P(t,T)\mathbb{E}_t^T\left[X_{T_F}\right]\left((1-p)\bar{x}^+ - p\bar{x}^-\right) \tag{21}$$

Matching equations 20 and 21 give the following equation

$$(1-p)\bar{x}^+ - p\bar{x}^- = (1+\bar{x})\left(2\mathcal{N}\left(d_+\left(\bar{x}\right)\right)-1\right)-\left(2\mathcal{N}\left(d_-\left(\bar{x}\right)\right)-1\right) \tag{22}$$

With

$$d_+\left(\bar{x}\right) = \frac{\ln\left(1+\bar{x}\right)+\frac{1}{2}\sigma^2\left(T-T_F\right)}{\sigma\sqrt{T-T_F}}$$

And

$$d_-\left(\bar{x}\right) = d_+\left(\bar{x}\right) - \sigma\sqrt{T-T_F}$$

Equations 19 and 22 finally give us the estimations for both expected appreciation conditional to the adverse event not happening, and the expected depreciation conditional to the adverse event happening:

$$\bar{x}^{+} = \frac{(1+\bar{x})\mathcal{N}\left(d_{+}\left(\bar{x}\right)\right) - \mathcal{N}\left(d_{-}\left(\bar{x}\right)\right)}{1-p}$$

And

$$\bar{x}^{-} = \frac{(1+\bar{x})\mathcal{N}\left(-d_{+}\left(\bar{x}\right)\right) - \mathcal{N}\left(-d_{-}\left(\bar{x}\right)\right)}{p}$$

10 Conclusion

In this paper we have explained how to estimate the forward ATMF implied volatility, when we only know the spot (as of today) implied volatility surface. We have implemented a model that is calibrated on the spot implied volatility surface, and can generate the forward ATMF implied volatility. We have added the possibility of jump for the forward instantaneous volatility. In the case of the 'Brexit' british referendum, regarding the GBP/USD and the GBP/EUR FX rates, the model has succeeded predicting the market implied volatility term structure on 23/06/2016 (see figures 5 and 8 in appendix).

Finally we have used the calculated forward ATMF implied volatilities to estimate the underlying expected moves post the referendum. We derived a closed form formula for the forward expected move. The expected move calculated is consistent with the rates differential, the forward volatility and the probability for the adverse event to happen (here UK voting to 'leave' the EU area). By approximating the price of a 1-day straddle option, we have been able to obtain closed form formulas for both the 1-day (move between the 23/06/2016 and 24/06/2016) expected moves conditional to 'stay' and to 'leave', respectively. It happened that the real moves in the market between 23/06/2016 and 24/06/2016 were in the range predicted by the model, for both the GBP/USD and the GBP/EUR.

The work in this paper could be used to estimate the impact, on assets in FX or Equity classes, for any future potentially adverse event to come, at a fixed and known date. The model can easily be extended to monitor more than one adverse events to come, by just adding as much jumps as the number of additional events, and to sum up all the jumps for different events dates.

References

[1] Gatheral J. (2004). Lecture 7: Quadratic Variation-based Payoffs, Case Studies in Financial Modeling Course Notes. Courant Institute of Mathematical Sciences, Fall Term, p. 125.

[2] Kouokap Youmbi D. (2013). A Short Note on Volatility Models. Working paper. SSRN

Appendix: Numerical tests and results

Calibration

Parameter	Value
$\sqrt{V_0}$	13.67%
$\sqrt{V_\infty}$	8.84%
k	130.88%
λJ	122.77%

Table 1: Calibrated model Parameters (as of 17/06/2016)

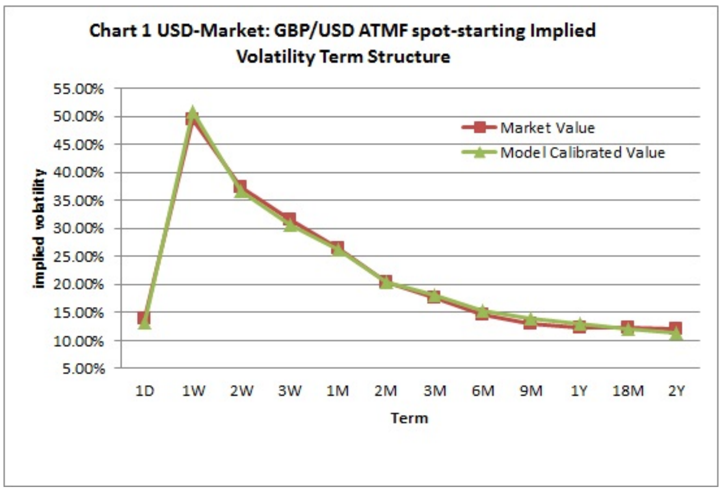

Figure 4: Calibration accuracy (as of 17/06/2016). Source (for market value): Bloomberg

Computation Results

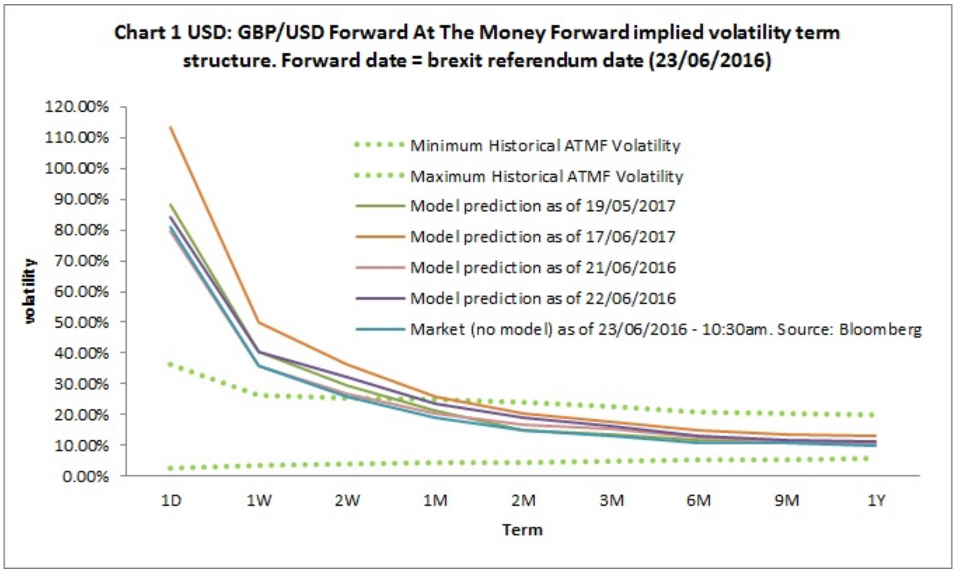

Figure 5: GBP/USD implied volatility Term structure. Model predictions were quite accurate

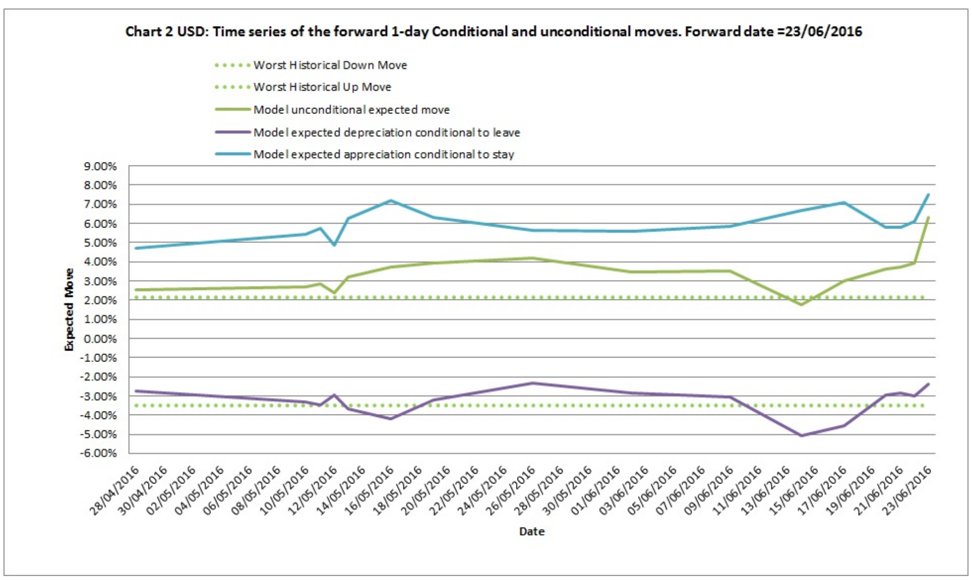

Figure 6: GBP/USD model expected moves between 23/06/2016 and 24/06/2016.

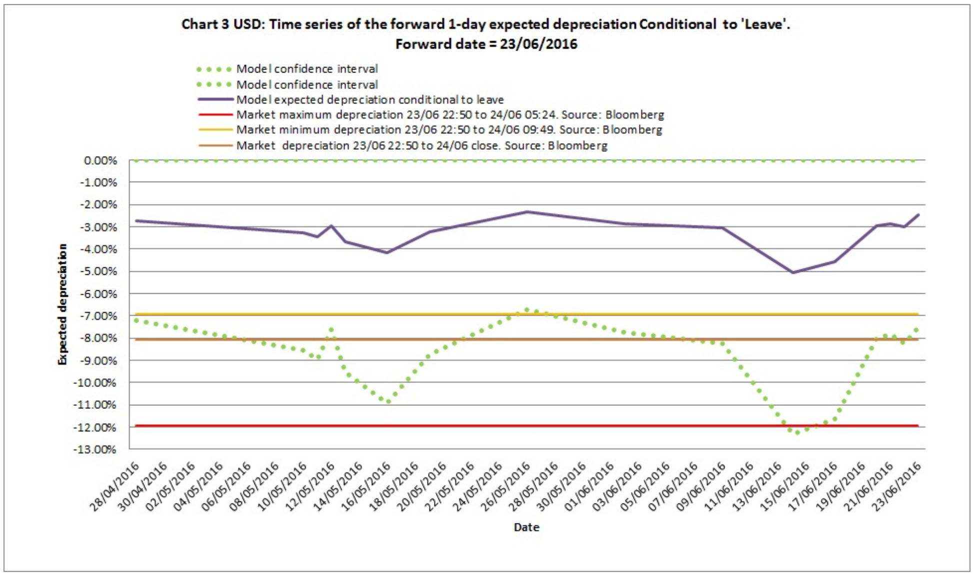

Figure 7: GBP/USD model expected depreciation between 23/06/2016 and 24/06/2016, conditional to 'Leave' vote. We can see model predictions were quite accurate, compared to the GBP/USD market moves on 24/06/2016

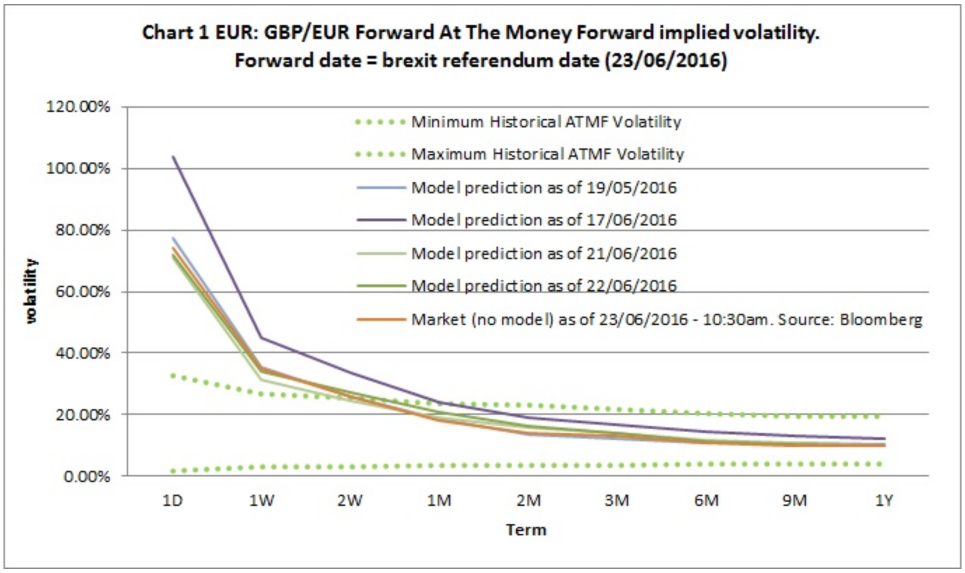

Figure 8: EUR/GBP implied volatility Term structure. Model predictions were quite accurate

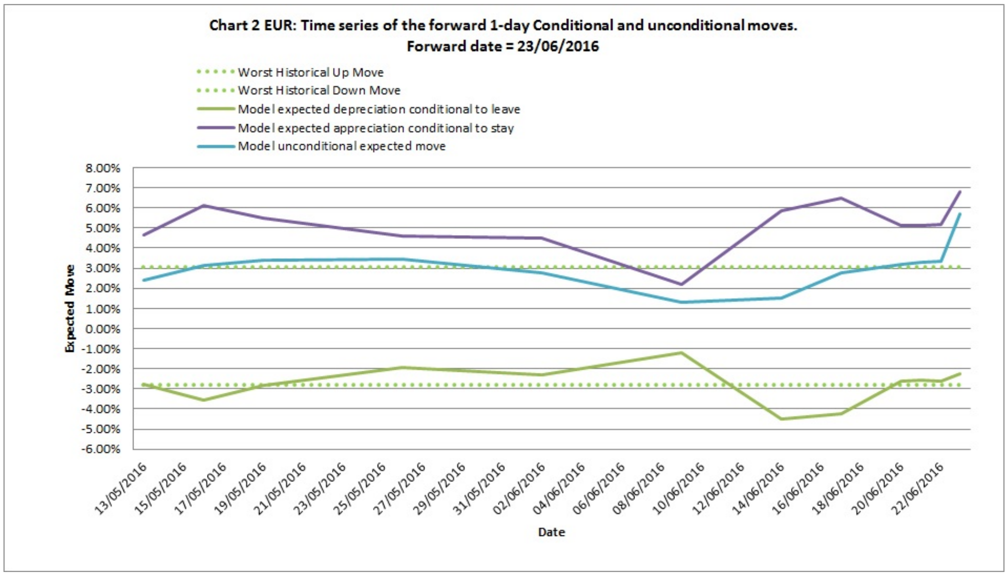

Figure 9: GBP/EUR model expected moves between 23/06/2016 and 24/06/2016.

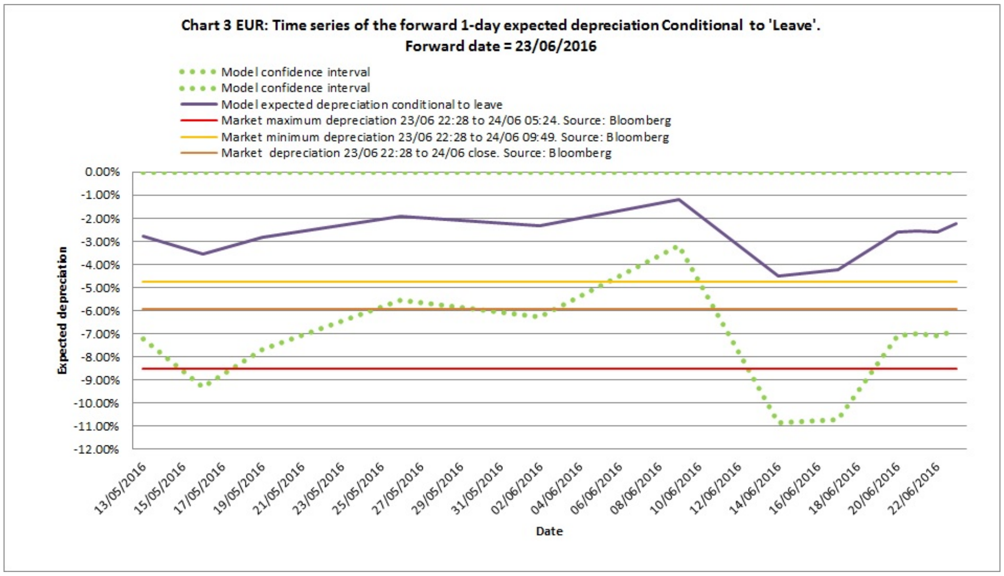

Figure 10: GBP/EUR model expected depreciation between 23/06/2016 and 24/06/2016, conditional to 'Leave' vote. We can see model predictions were quite accurate, compared to the GBP/EUR market moves on 24/06/2016