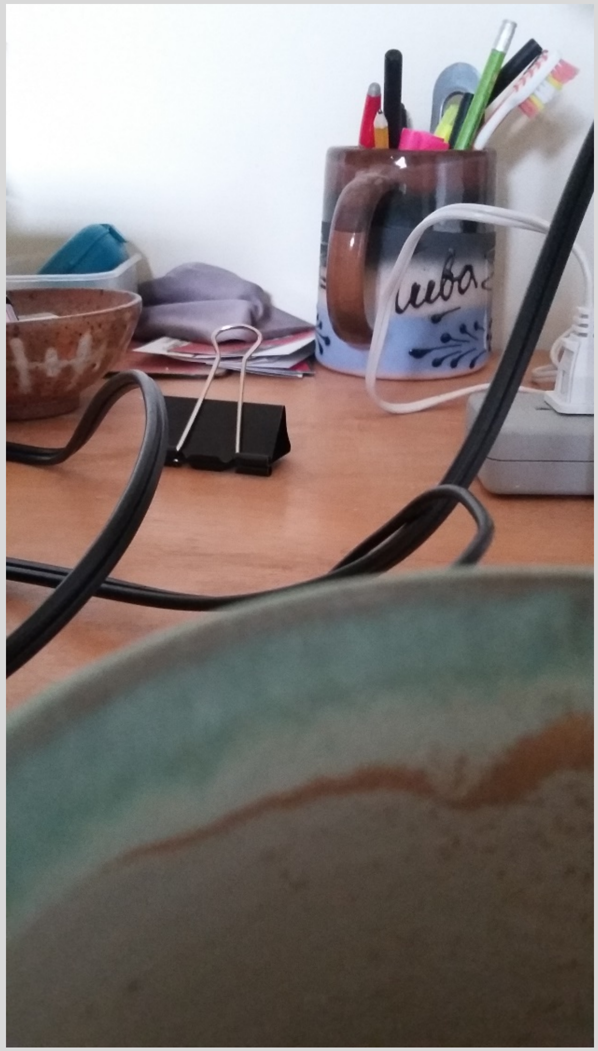
COMP 558

Lecture 3

Image Filtering

Example of image (cell phone)



real desk top

Tea mug

R component

G component

B component

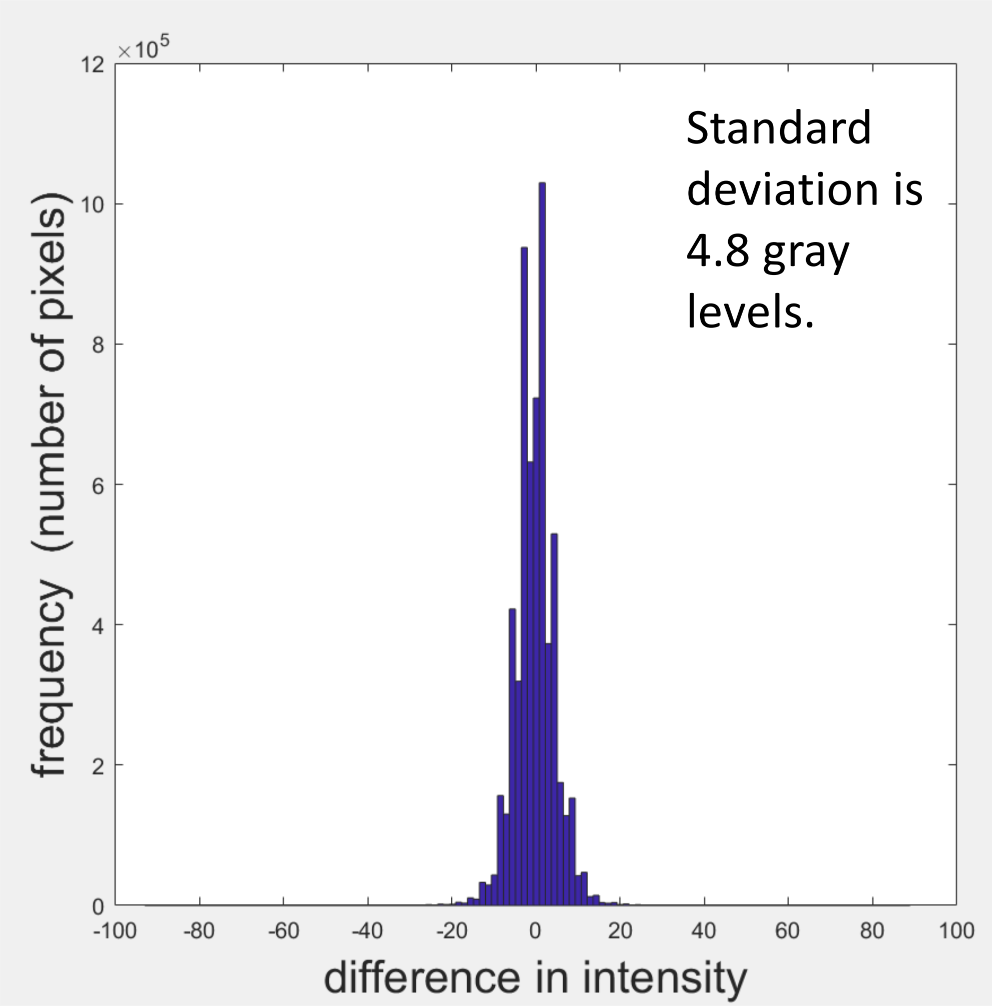
Although the image on the previous slide has noticeable color, the RGB values at each pixel are correlated.

Unless one is dealing with color issues specifically, it is common in computer vision to present models/methods using gray level only. That is what we do too.

Image noise

Here I take a second photo from the same position, and make an image (right) showing the *(enhanced)* difference between the green channels.   White is positive, black is negative.

Image noise



Difference image
(between the green channels)

Image smoothing

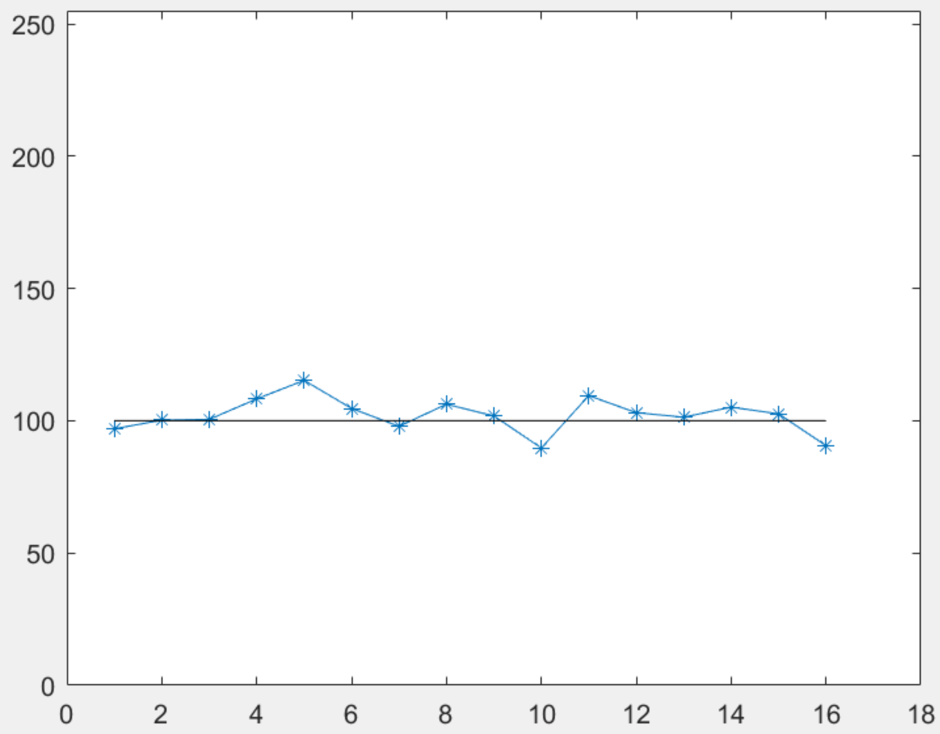
To reduce noise, one often smooths the image by averaging intensities (RGBs).

Local average (1D for now)

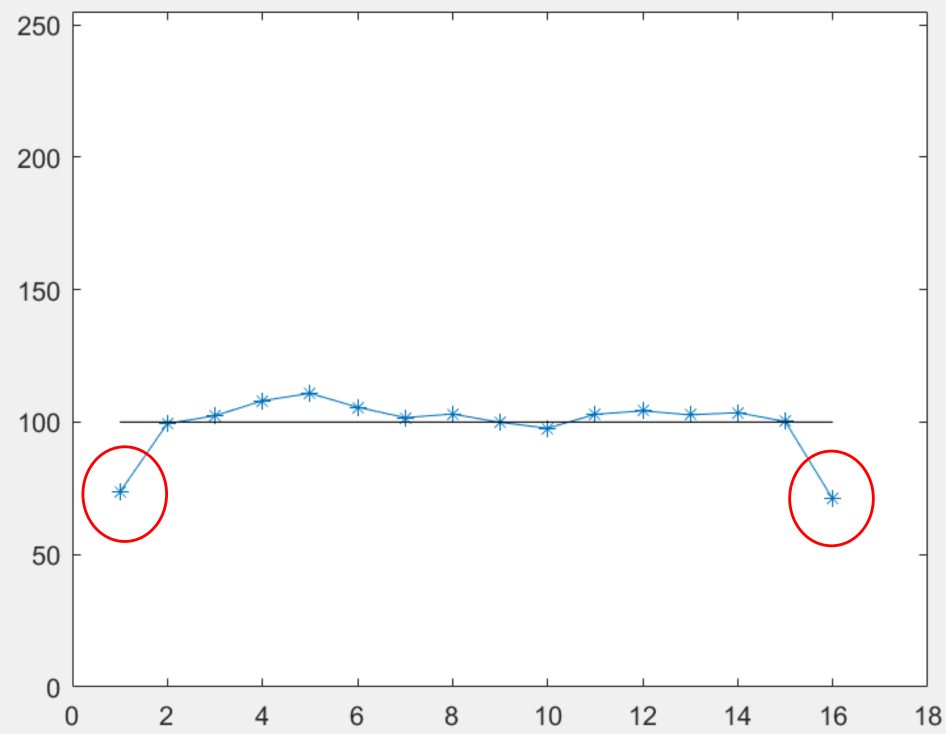
$$I_{smooth}(x) \equiv \frac{1}{4}I(x+1) + \frac{1}{2}I(x) + \frac{1}{4}I(x-1)$$

Example (noise)

$I(x)$ = 100 + independent noise



$$I_{smooth}(x) \equiv \frac{1}{4}I(x+1) + \frac{1}{2}I(x) + \frac{1}{4}I(x-1)$$



This happens because Matlab assumes value of function is 0 beyond the points where it is defined.

Image smoothing

To reduce noise, one often smooths the image by averaging intensities (RGBs).

This also smooths the “signal”, however.

Example (edge with no noise)

$I(x)$

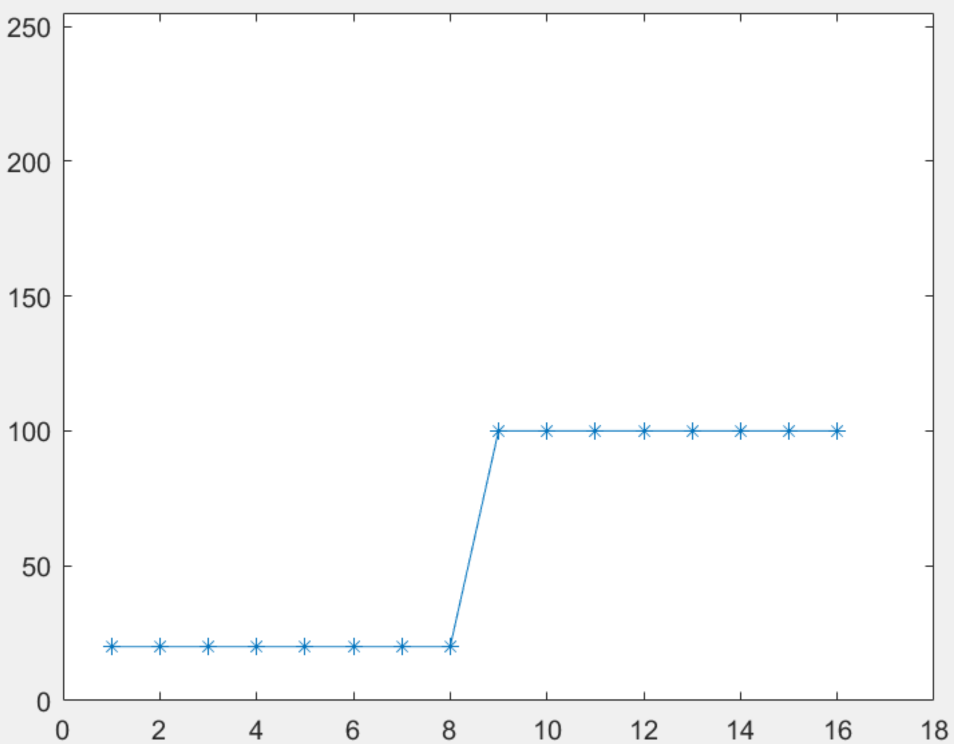


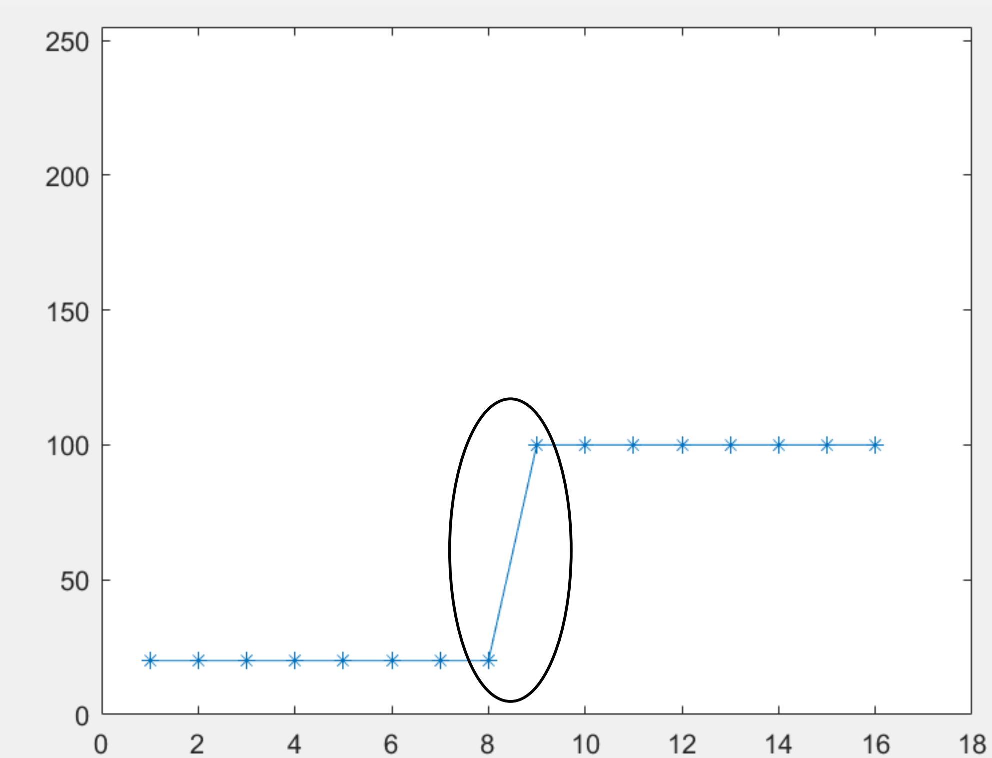
Image is defined on 16 pixels and takes gray level values 20 and 100.

There is an intensity “step edge” from pixel x=8 to x=9.

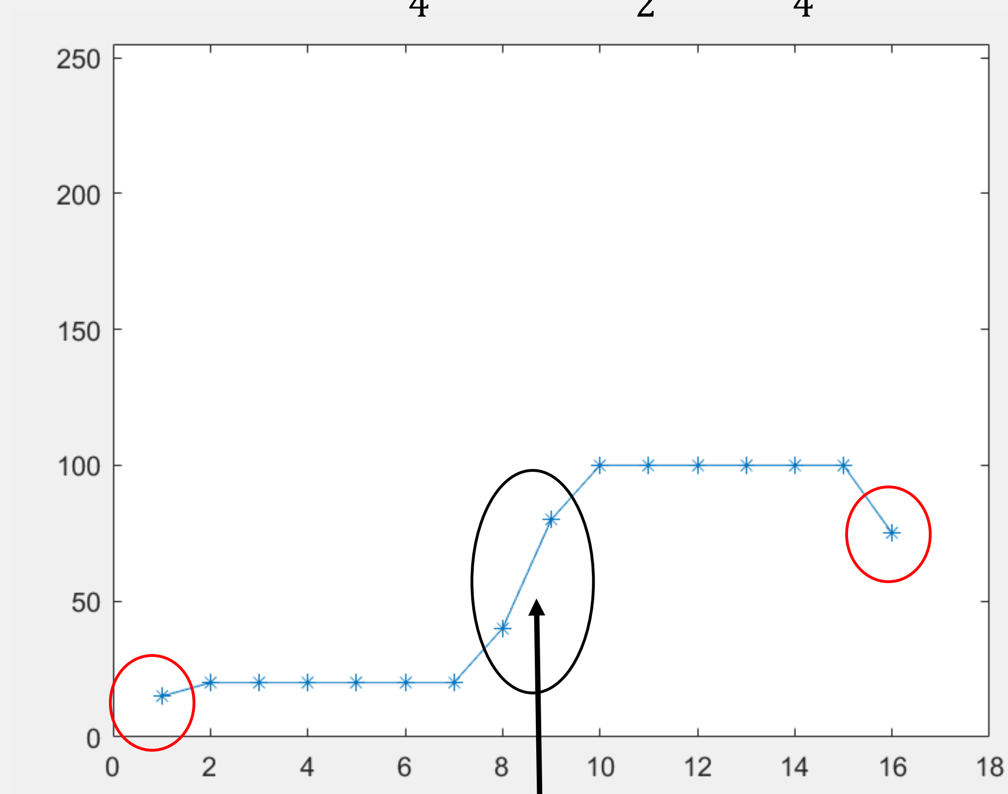
What happens if we take the local average?

Example (edge with no noise)

$I(x)$



$$I_{smooth}(x) \equiv \frac{1}{4}I(x+1) + \frac{1}{2}I(x) + \frac{1}{4}I(x-1)$$



The step edge is smoothed out.

We can also smooth over a larger neighborhood.

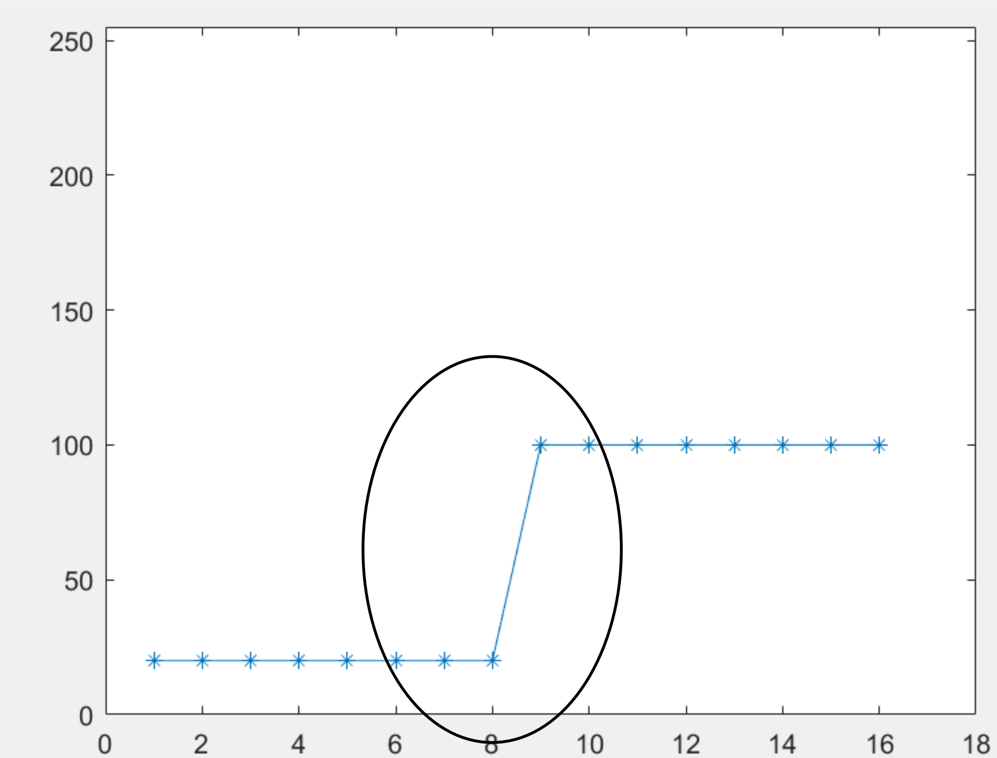
$$I_{smooth}(x) \equiv \frac{3}{16} I(x+1) + \frac{1}{2} I(x) + \frac{3}{16} I(x-1)$$

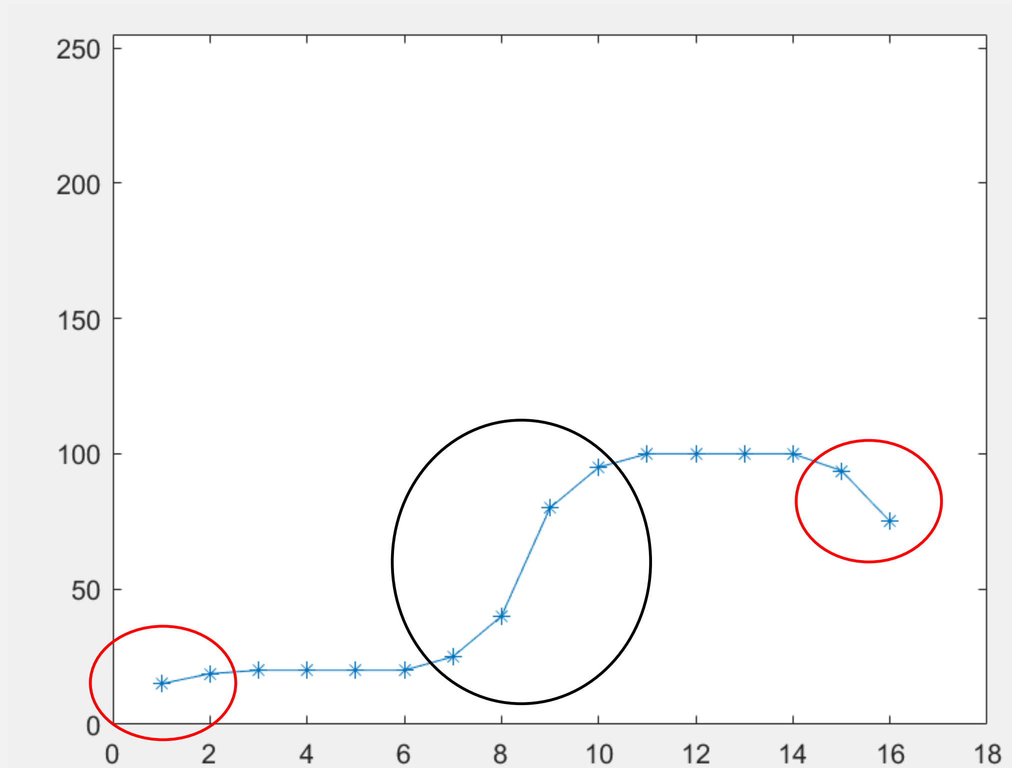
$$+ \frac{1}{16} I(x+2) + \frac{1}{16} I(x-2)$$

Example (edge with no noise)

$$I_{smooth}(x) \equiv \frac{3}{16}I(x+1) + \frac{1}{2}I(x) + \frac{3}{16}I(x-1) + \frac{1}{16}I(x+2) + \frac{1}{16}I(x-2)$$

$I(x)$





How to choose the weights of the smoothing function?

Gaussian function

$$G(x) \quad = \quad \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Gaussian function

$$G(x) \;=\; \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

It integrates to 1 (because of normalization constant).

It has some remarkable mathematical properties. (Next slide)

It is often called a “normal” distribution in probability and statistics.

We can use it for the smoothing weights.

We shall also see it later when we discuss Gaussian scale spaces.

ASIDE: Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2

Then $\frac{1}{\sqrt{n}}\,(X_1 + X_2 + \ldots + X_n)$ converges in distribution to

$$G(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Gaussian smoothing

As we use a bigger scale σ,

- we reduce the noise
- we average together more pixels,   blurring the signal more.

Local difference

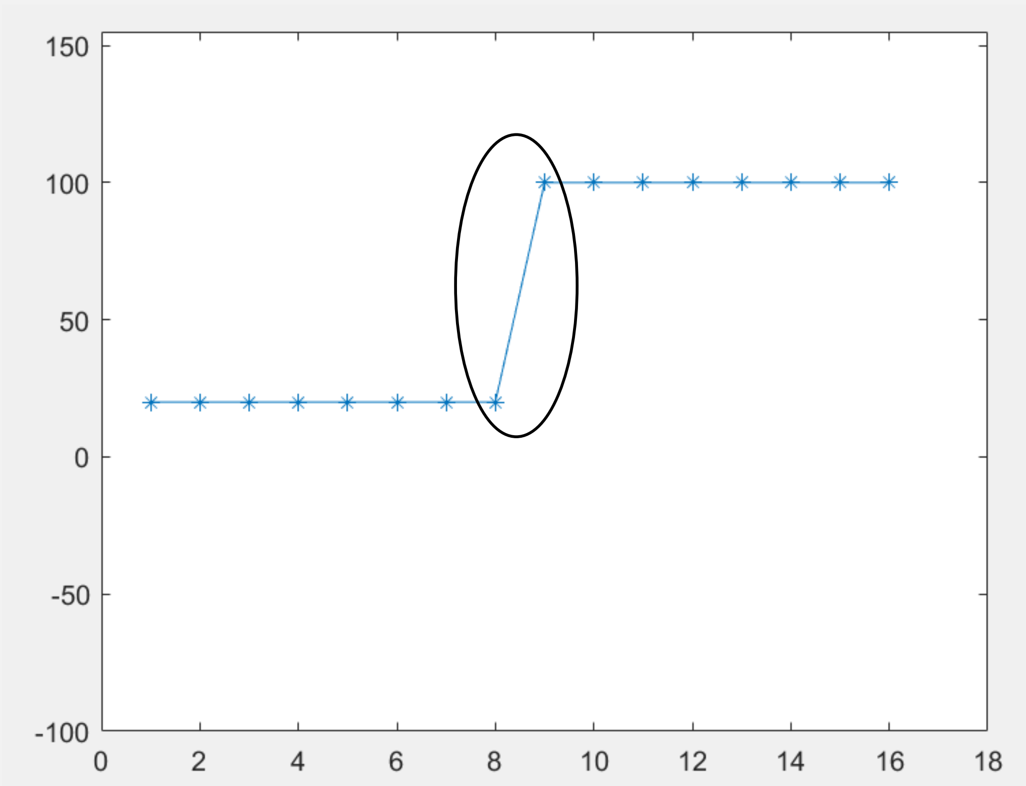
Often one would like to *enhance* local differences in intensity, since these differences indicate interesting events such as *edges*.

$$\frac{d\, I(x)}{d\, x} \approx \frac{1}{2} I(x+1) - \frac{1}{2} I(x-1)$$

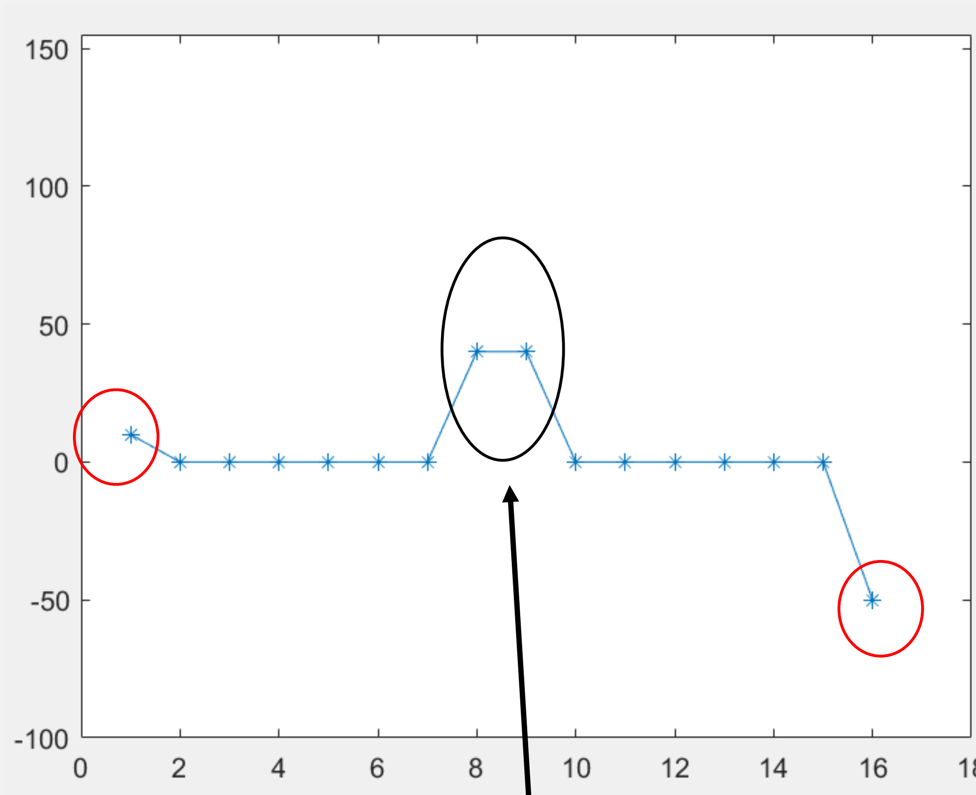
"central difference"
i.e. centered at $x = 0$

Example (edge)

$I(x)$



$\frac{1}{2}I(x+1) - \frac{1}{2}I(x-1)$



Positive derivative (slope)

Image Filtering

More generally:

- We can take higher order derivatives (2^{nd} & beyond)
- We can combine local averaging and differences
  e.g. smooth out noise + take derivatives.
- We can take *arbitrary local linear combinations*.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u-x)\, I(u)$$

Place a copy of the function f( ) at position x, and then sum up the products of this shifted f( ) weighted by I(x).

Think of f( ) as a *template* that we match to the image (by an inner product).

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u-x)\, I(u)$$

Example: what is the function $f(x)$ in this case ?

$$f(x) \otimes I(x) \equiv \frac{1}{4} I(x+1) + \frac{1}{2} I(x) + \frac{1}{4} I(x-1)$$

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u-x)\, I(u)$$

Example: what is the function $f(x)$ in this case ?

$$f(x) \otimes I(x) \equiv \frac{1}{4} I(x+1) + \frac{1}{2} I(x) + \frac{1}{4} I(x-1)$$

$f(x) = \frac{1}{4}$ when x = +1, -1, $f(x) = \frac{1}{2}$ when x = 0, and $f(x) = 0$ otherwise.

Convolution

$$f(x) * I(x) \equiv \sum_{u} f(x-u)\ I(u)$$

Convolution sums up copies of shifted versions of the function $f(x)$, and weights each shifted copy by a value of $I(x)$.

$$f(x) * I(x) \equiv \sum_{u} f(x-u)\, I(u)$$

$$= \quad f(x)\, I(0) + \ldots$$

$$+ \; f(x-1)\, I(1) \; + \; f(x-2)\, I(2) \; + \; \ldots$$

$$+ \; f(x+1)\, I(-1) + \; f(x+2)\, I(-2) + \; \ldots$$

Convolution sums up copies of shifted versions of the function $f(x)$, and weights each shifted copy by a value of $I(x)$.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_u f(u-x)\, I(u)$$

$$\Big\downarrow \textit{flip}$$

Convolution

$$f(x) * I(x) \equiv \sum_u f(x-u)\, I(u)$$

Any cross-correlation can be written as a convolution, and vice-versa, just by flipping the function $f(\,)$.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_u f(u-x)\, I(u)$$

Sliding a template across an image, and taking inner product.

Convolution

$$f(x) * I(x) \equiv \sum_u f(x-u)\, I(u)$$

Summing weighted, shifted versions of the function $f(x)$.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u - x)\, I(u)$$

How well does the filter match the image, when filter is placed at position x ?

Convolution

$$f(x) * I\,(x\,) \equiv \sum_{u} f(x - u)\, I(u)$$

How much does image intensity at position x contribute to the filtered output?

Some algebraic properties of convolution

For any $f_1(x),\ f_2(x), f_3(x)$:

$$f_1 * f_2 = f_2 * f_1$$

Cross correlation does not have this property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$$

$$(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$$

Convolution of an image $I(x)$ with a function $f(x)$ is called "filtering an image."

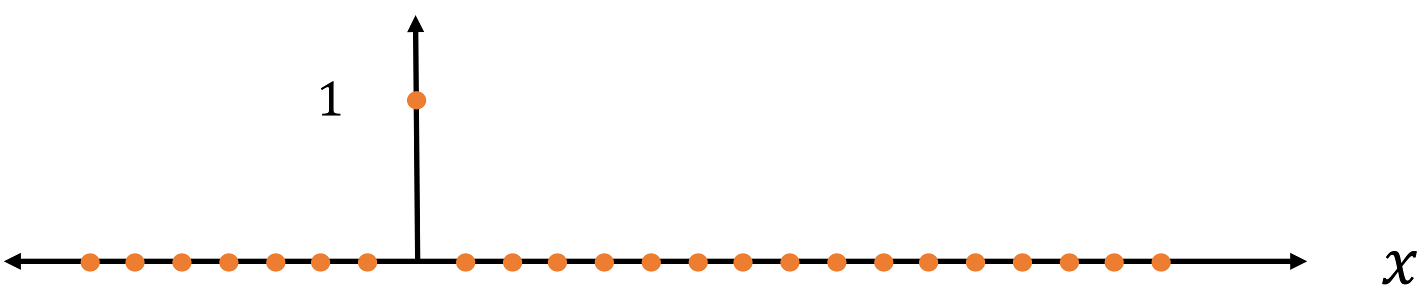
$f(x)$ is called a "filter" or "kernel".

-1	0	1	x
$\frac{1}{2}$	0	$-\frac{1}{2}$	Local difference filter
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	Local average filter

$f(x)$ has value 0 outside of domain shown.

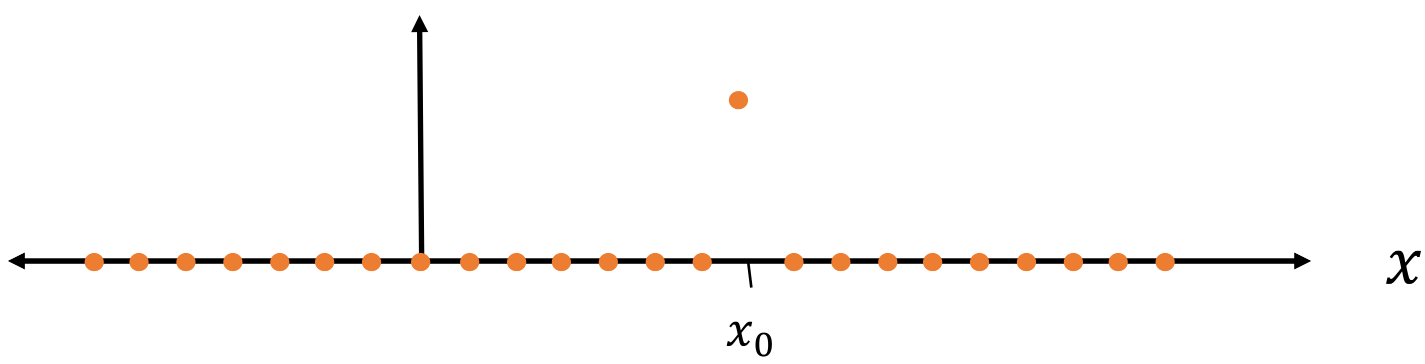
Impulse function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$



Shifted Impulse function

$$\delta(x - x_0) = \begin{cases} 1, & x = x_0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x) * I(x) = \sum_{u} \delta(x-u)\ I(u)$$

$$= \ ?$$

$$\delta(x) * I(x) = \sum_{u} \delta(x-u)\, I(u)$$

$$= I(x)$$

Any image $I(x)$ can be written of as a sum of delta functions.

Impulse Response function

Filters (kernels) are often referred to as “impulse response” functions.

Why?

$$\delta(x) * f(x) \;=\; f(x)$$

impulse image

impulse response

Boundary conditions

$$f(x) * I\,(x\,) \quad = \sum_{u=0}^{N-1} f(x-u)\, I(u)$$

Assume image $I(x)$ has value 0 outside range x in 0 to N-1.

However, $f(x)$ is defined on positive and negative x.

This leads to some technical issues (boundary conditions) that arise in practice. You might have to deal with these issues in Assignments.

ASIDE: Continuous Convolution

$$f(x) * I\,(x\,) \;\equiv \int f(x-u)\, I(u)\, du$$

The notion of an impulse function is more subtle in the continuous case. We don't need to use it.

2D cross correlation

$$f(x,y) \otimes I(x,y) \quad \equiv \sum_{u,v} f(u-x, v-y)\, I(u,v)$$

Sliding a template across an image, and taking inner product.

2D convolution

$$f(x,y) * I(x,y) \quad \equiv \sum_{u,v} f(x-u, y-v)\, I(u,v)$$

Summing the impulse responses from all the pixels.

2D Gaussian

Here we simplify notation and assume $\mu = 0$ and σ is the same for x and y.

$$G(x, y, \sigma) \equiv G(x, \sigma)\, G(y, \sigma) \qquad \text{i.e. separable}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

2D Gaussian

$$G(x, y) = \frac{1}{2\pi\sigma^2} \, e^{-\frac{x^2 + y^2}{2\sigma^2}}$$

It is radially symmetric.

It integrates to 1, because it is separable and 1D Gaussian integrates to 1.

$$\iint G(x, y)\, dx\, dy = \int G(x) dx \int G(y) dy$$

2D Gaussian smoothing

$$I_{smooth}(x, y) = G(x, y, \sigma) * I(x, y)$$

We are taking a local weighted average of points, where weights sum to 1.

ASIDE: Fast 2D smoothing

Using the separability of the Gaussian makes smoothing $O(N\ M^2)$ instead of $O(N^2 M^2)$ where

N is the width of the Gaussian kernel

M is the height and width of the image.

$$\iint G(x-u, y-v)\, I(u,v)\, du\, dv \;=\; \int G(y-v) \int G(x-u) I(u,v) du \;\; dv$$

Partial Derivatives

$$\frac{\partial}{\partial x} I(x, y) \approx \frac{1}{2} I(x+1, y) - \frac{1}{2} I(x-1, y)$$

$\frac{1}{2}$	0	- $\frac{1}{2}$

$$\frac{\partial}{\partial y} I(x, y) \approx \frac{1}{2} I(x, y+1) - \frac{1}{2} I(x, y-1)$$

- $\frac{1}{2}$
0
$\frac{1}{2}$

Image gradient

$$\nabla I(x, y) \equiv \left(\frac{\partial}{\partial x} I(x, y), \frac{\partial}{\partial y} I(x, y) \right)$$

What is the direction in which the image intensity is changing at the fastest rate, and what is that rate?

Image gradient

$$\nabla I(x, y) \equiv \left(\frac{\partial}{\partial x} I(x, y), \frac{\partial}{\partial y} I(x, y), \right)$$

$$\approx \left(\frac{1}{2} I(x+1, y) - \frac{1}{2} I(x-1, y), \quad \frac{1}{2} I(x, y+1) - \frac{1}{2} I(x, y-1) \right)$$

Magnitude of image gradient

$$\| \nabla I(x,y) \| \equiv \sqrt{\left(\frac{\partial}{\partial x} I(x,y)\right)^2 + \left(\frac{\partial}{\partial y} I(x,y)\right)^2}$$

$$\approx \frac{1}{2}\sqrt{\begin{array}{l}\big(I(x+1,y) - I(x-1,y)\big)^2 \\ \quad +\big(I(x,y+1) - I(x,y-1)\big)^2\end{array}}$$

Summary

- image noise and smoothing
- local differences
- cross-correlation versus convolution
- impulse functions
- 1D vs. 2D
- Gaussians