Lecture 20

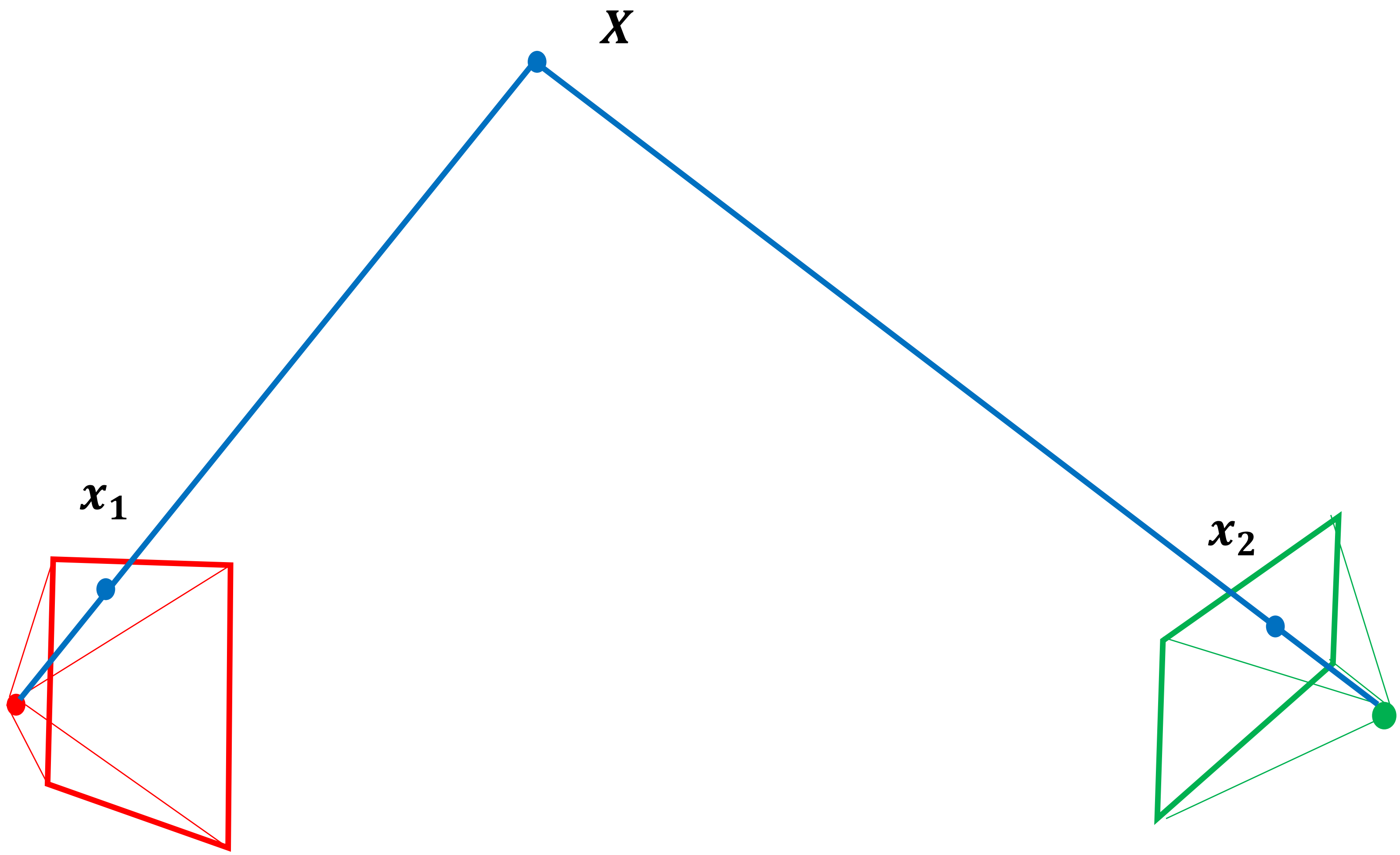
Epipolar Geometry

(Slides courtesy Mike Langer)

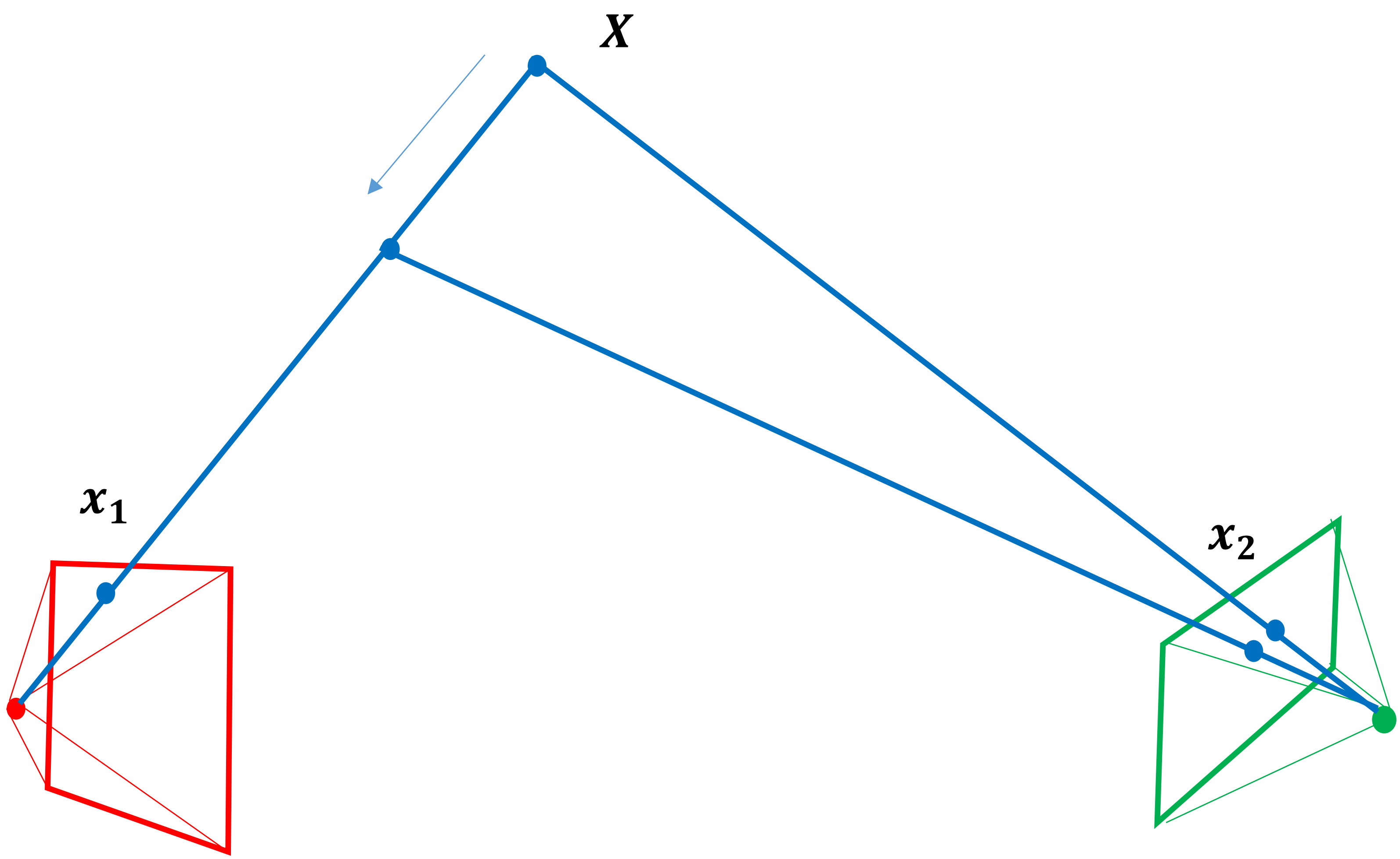
"Binocular" vision – two cameras

Camera i has a $\boldsymbol{P_i}$ matrix with internals $\boldsymbol{K_i}$ and externals $\boldsymbol{R_i}$ and $\boldsymbol{C_i}$. Today we assume that these are all *unknown.*

A 3D point $\boldsymbol{X}$ at depth $Z_i > 0$ in camera i will project to camera i's image plane at position $\boldsymbol{x_i}$. This point might not lie within the camera's field of view (as determined by the camera matrix $\boldsymbol{K_i}$ and the range of pixel indices).



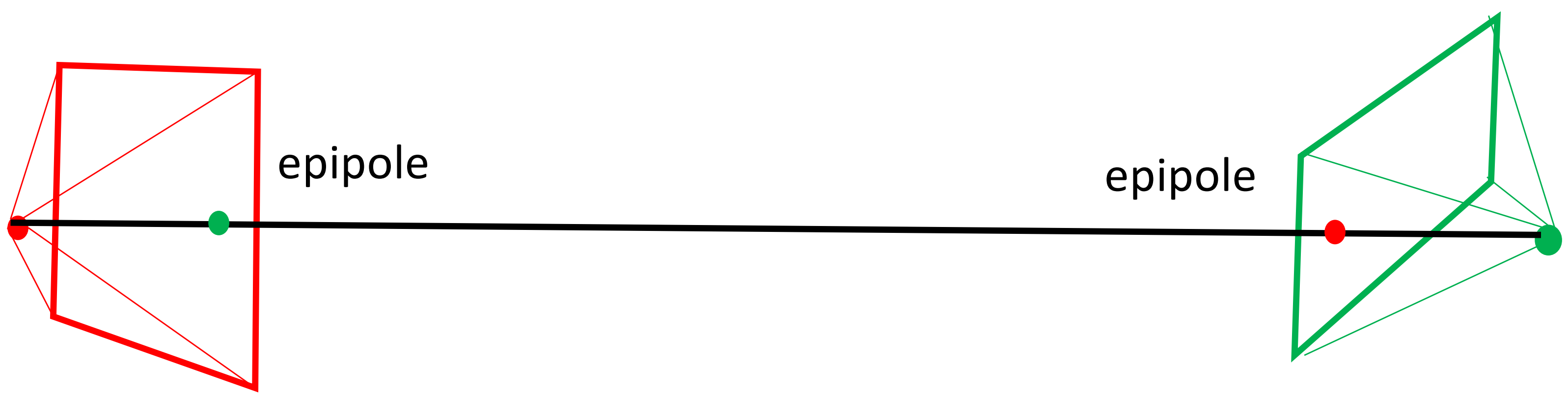
Moving the 3D point $\boldsymbol{X}$ along the projection ray toward the left camera doesn't change its position in the left image. But it does change its position in the right image.



Epipole

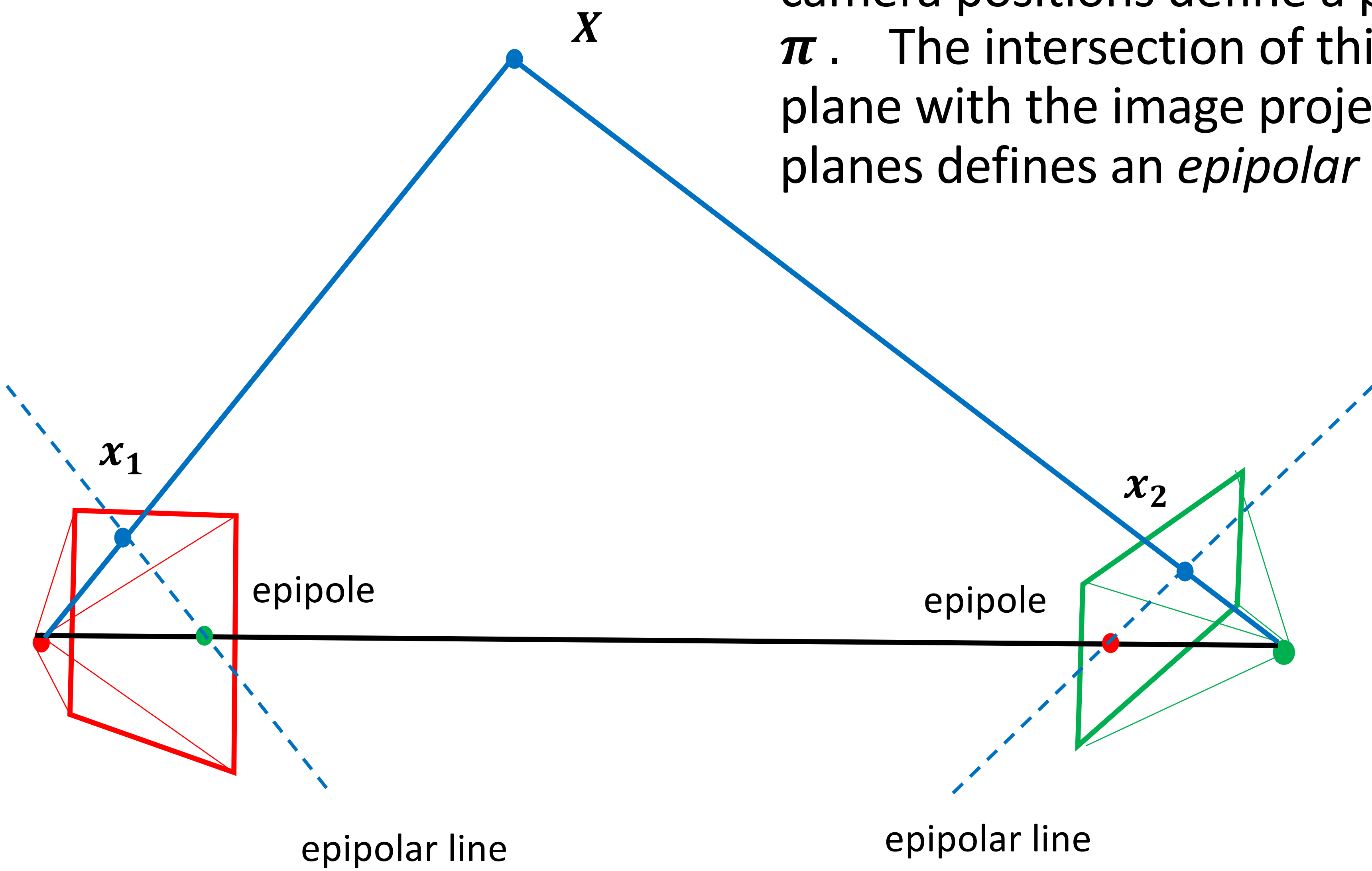
Each camera position is a 3D point. So each camera position defines an image position (possibly at infinity) in the other camera's projection plane. Such a position in called an *epipole*.

The cameras typically do not see each other. So the epipoles typically lie outside the field of view (unlike what is shown below):

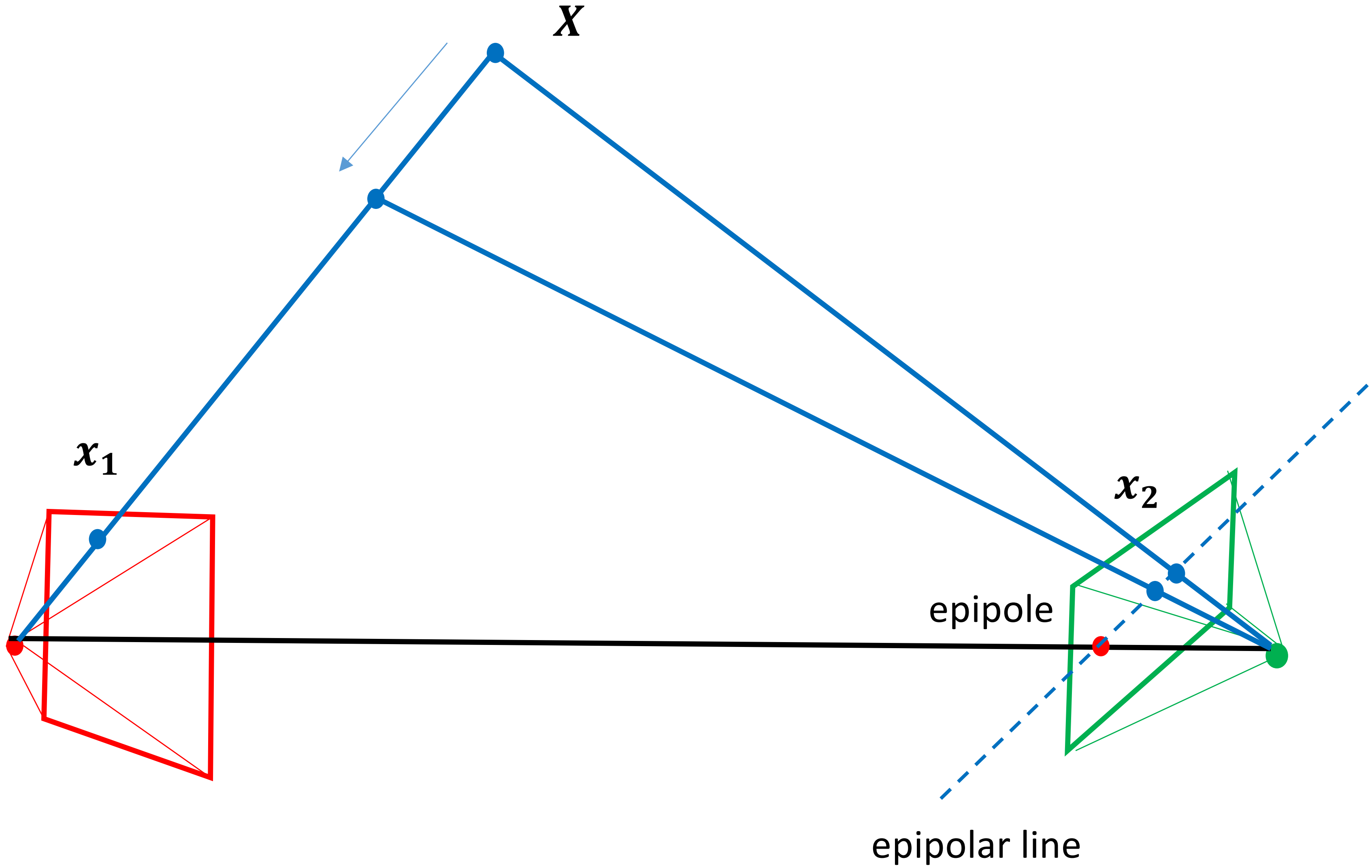


Epipolar Line

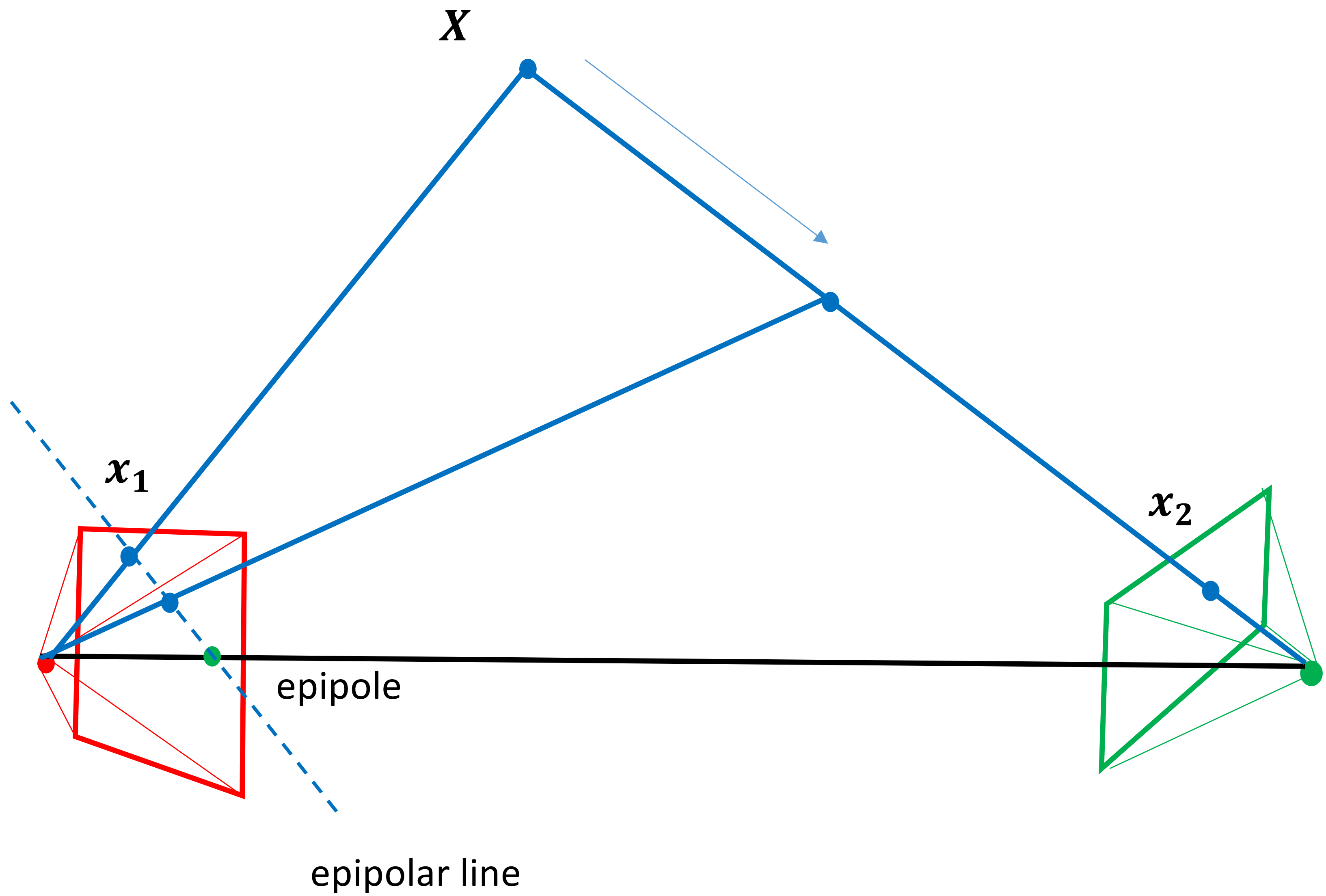
A 3D point $\boldsymbol{X}$ and the two camera positions define a plane $\boldsymbol{\pi}$. The intersection of this plane with the image projection planes defines an *epipolar line*.



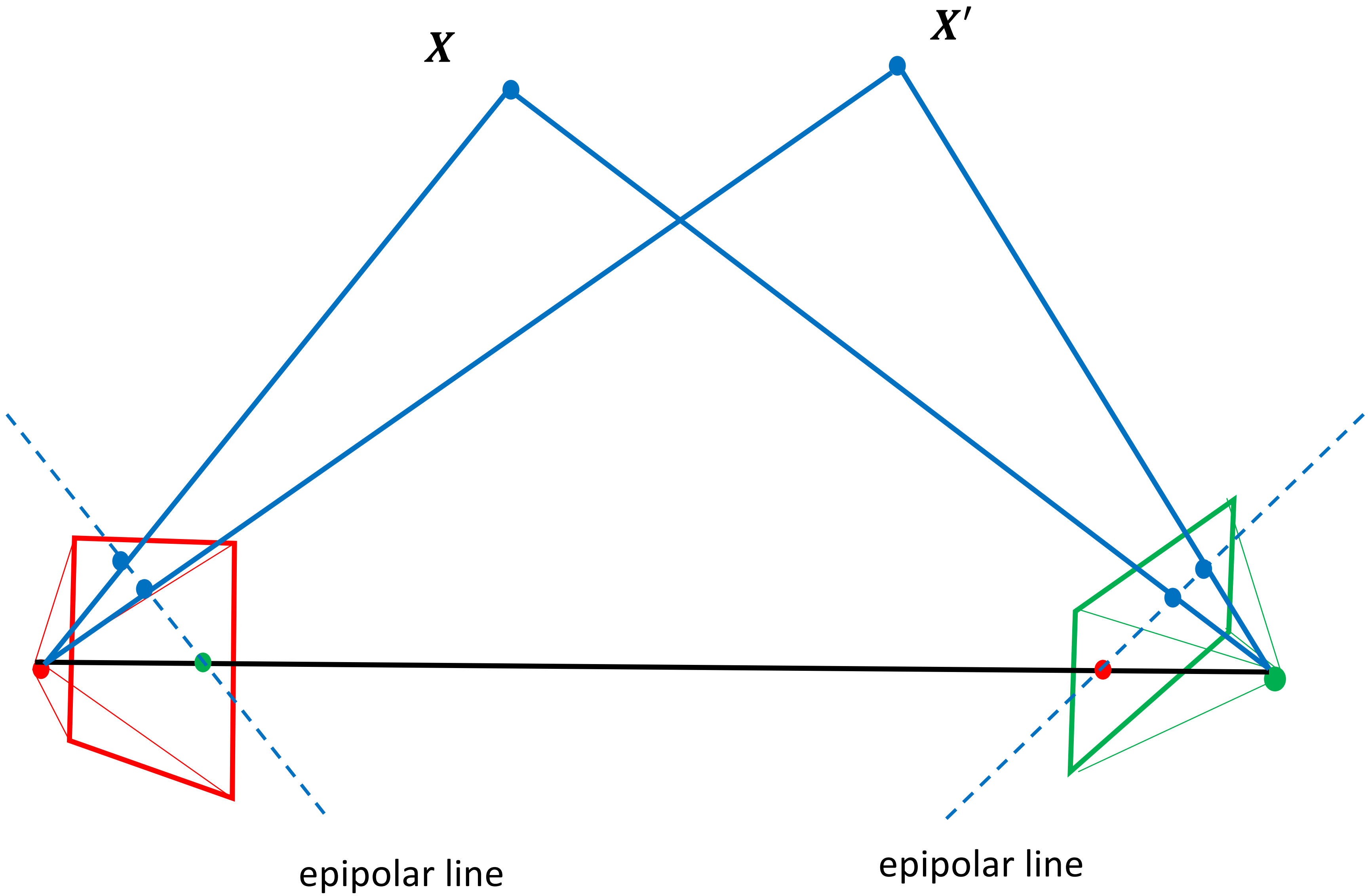
Moving the 3D point $\boldsymbol{X}$ along the projection ray toward the left camera doesn't change its position in the left image. In the right image, the point moves along the epipolar line.



Moving the 3D point $\boldsymbol{X}$ along the projection ray toward the right camera doesn't change its position in the right image. In the left image the point moves along the epipolar line.

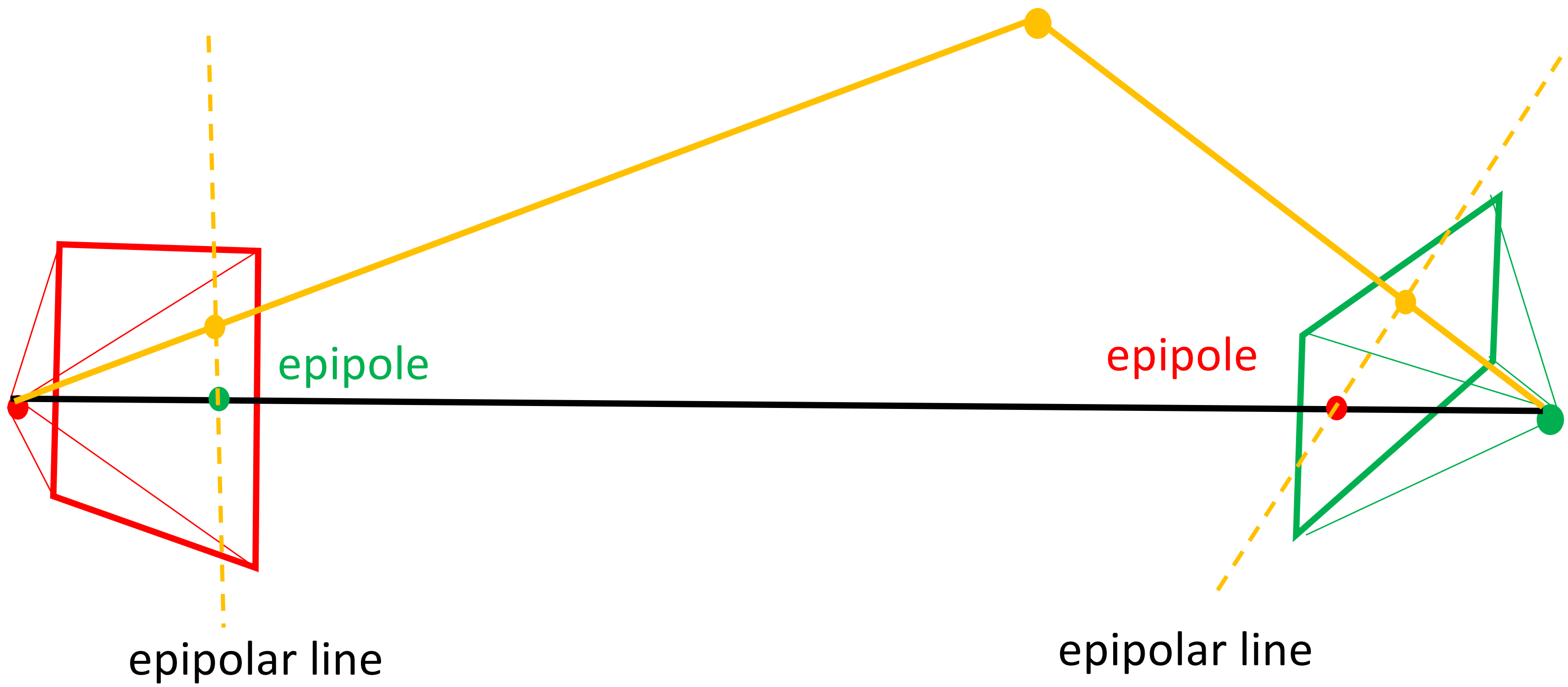


Any 3D point within the same plane $\boldsymbol{\pi}$ will project to the same epipolar lines.

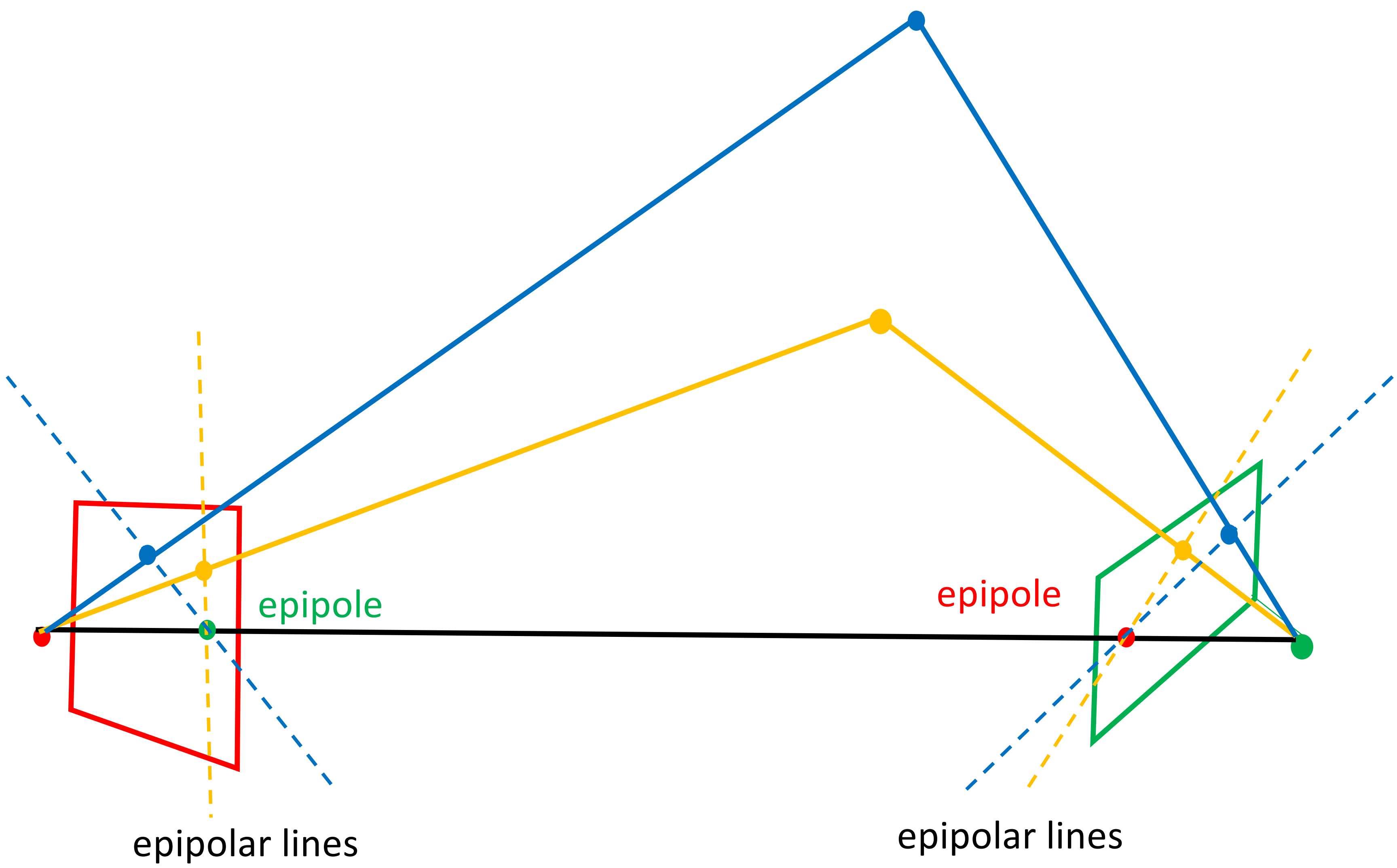


A 3D point within a *different* plane will project to different epipolar lines.

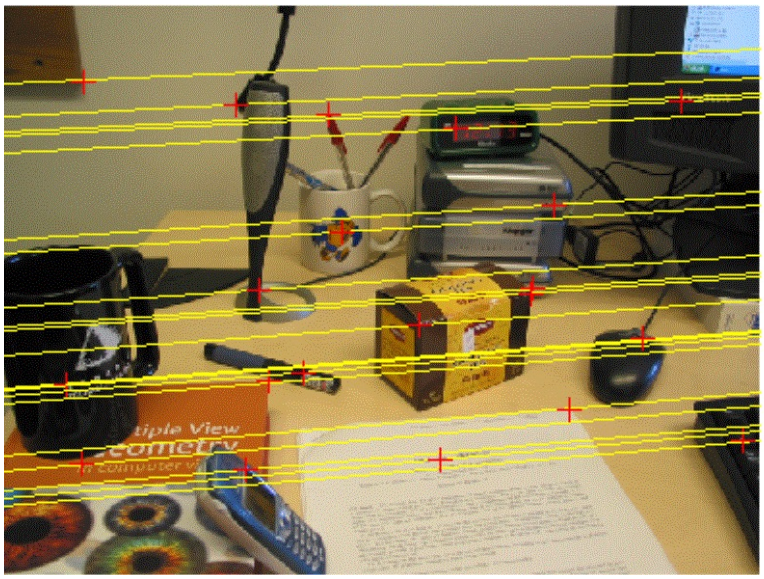
The epipoles do not change since they depend on cameras only.

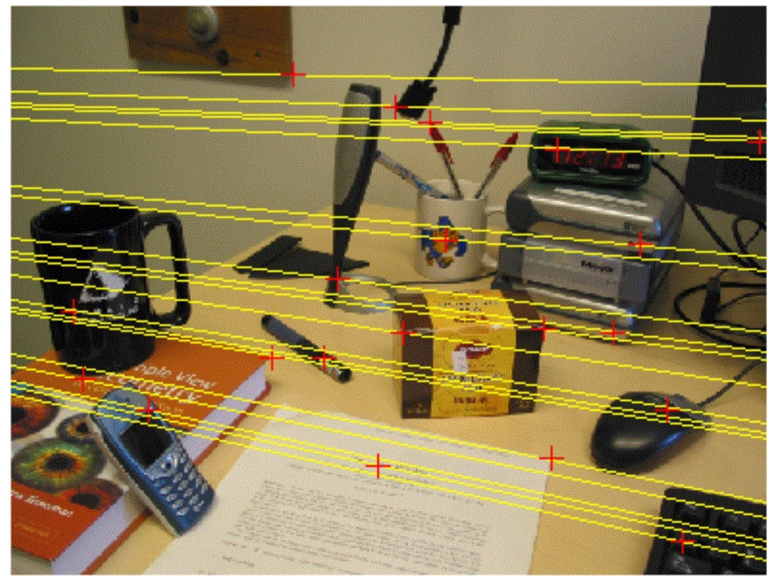


Epipolar lines always contain the epipoles (hence, the name).



Example

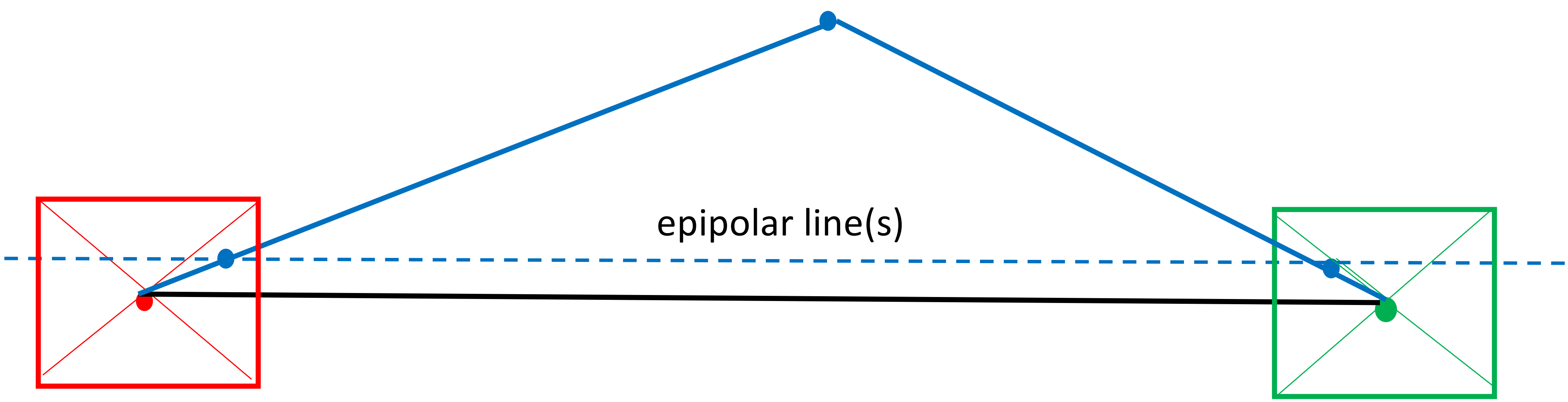




Some epipolar lines are shown.
In this example, the epipoles are outside the field of view.

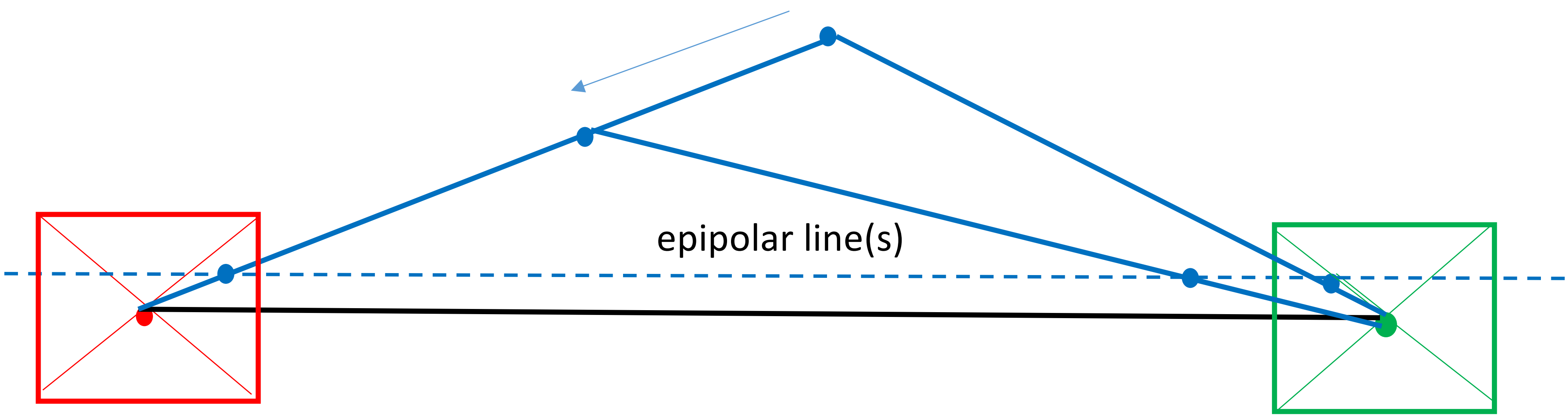
Special case: rectified cameras

The cameras are rectified if their $\boldsymbol{XYZ}$ axes are parallel to each other, and the epipoles are in the direction of the $\boldsymbol{X}$ axes. In this case, the epipolar line(s) are the same in the two images.



Special case: rectified cameras

Moving the scene point toward the left camera along a line does not affect the position in the left image. But it does affect the position in the right image: the point leaves the right image!



Overview of today

- Epipolar geometry
- Essential matrix (algebra)

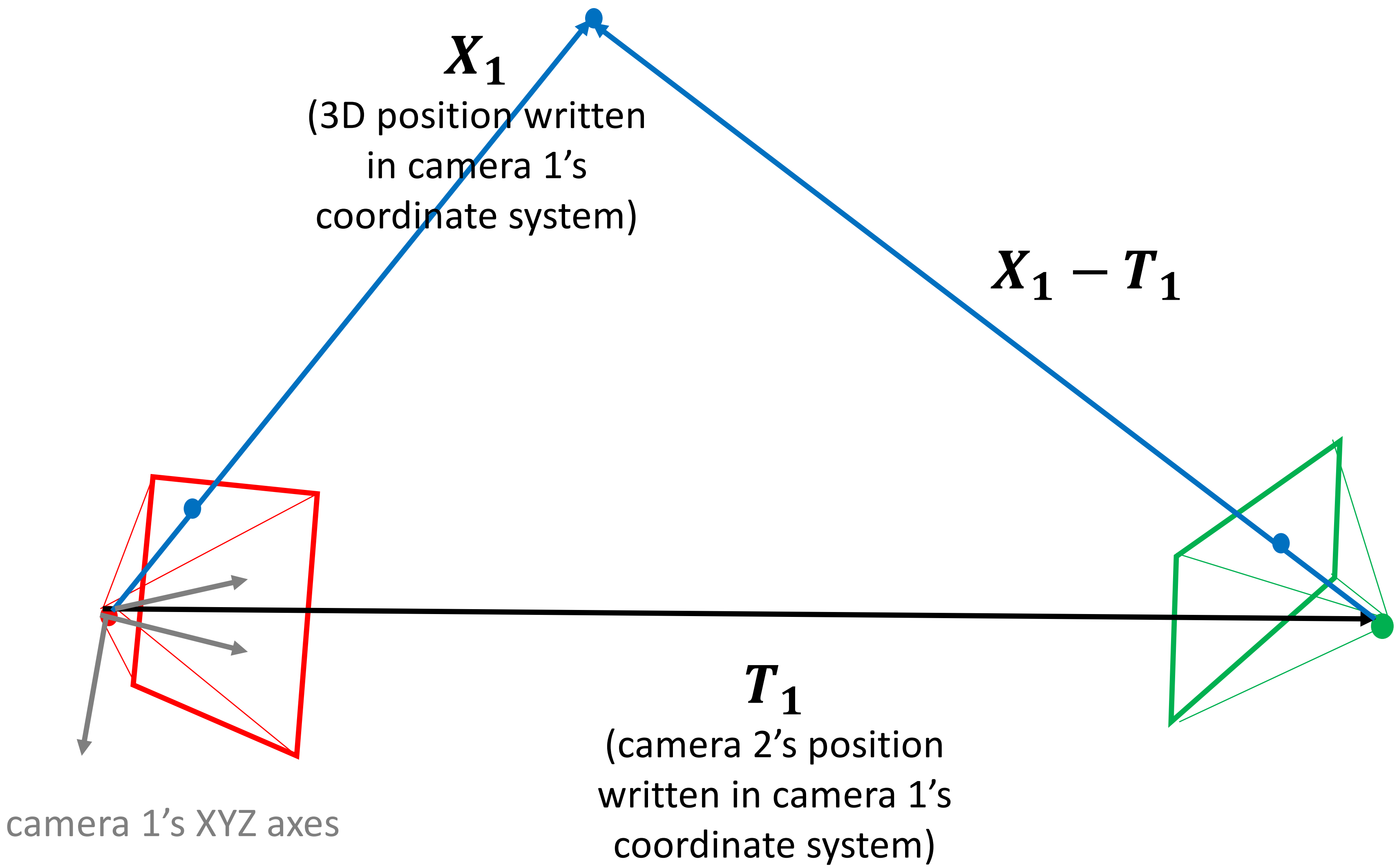
  Uses projection plane coordinates, not pixel coordinates.

- Fundamental matrix (algebra)

  Uses pixel coordinates.

The three vectors $\boldsymbol{X_1}, \boldsymbol{T_1}, \boldsymbol{X_1} - \boldsymbol{T_1}$ form a triangle in the figure.

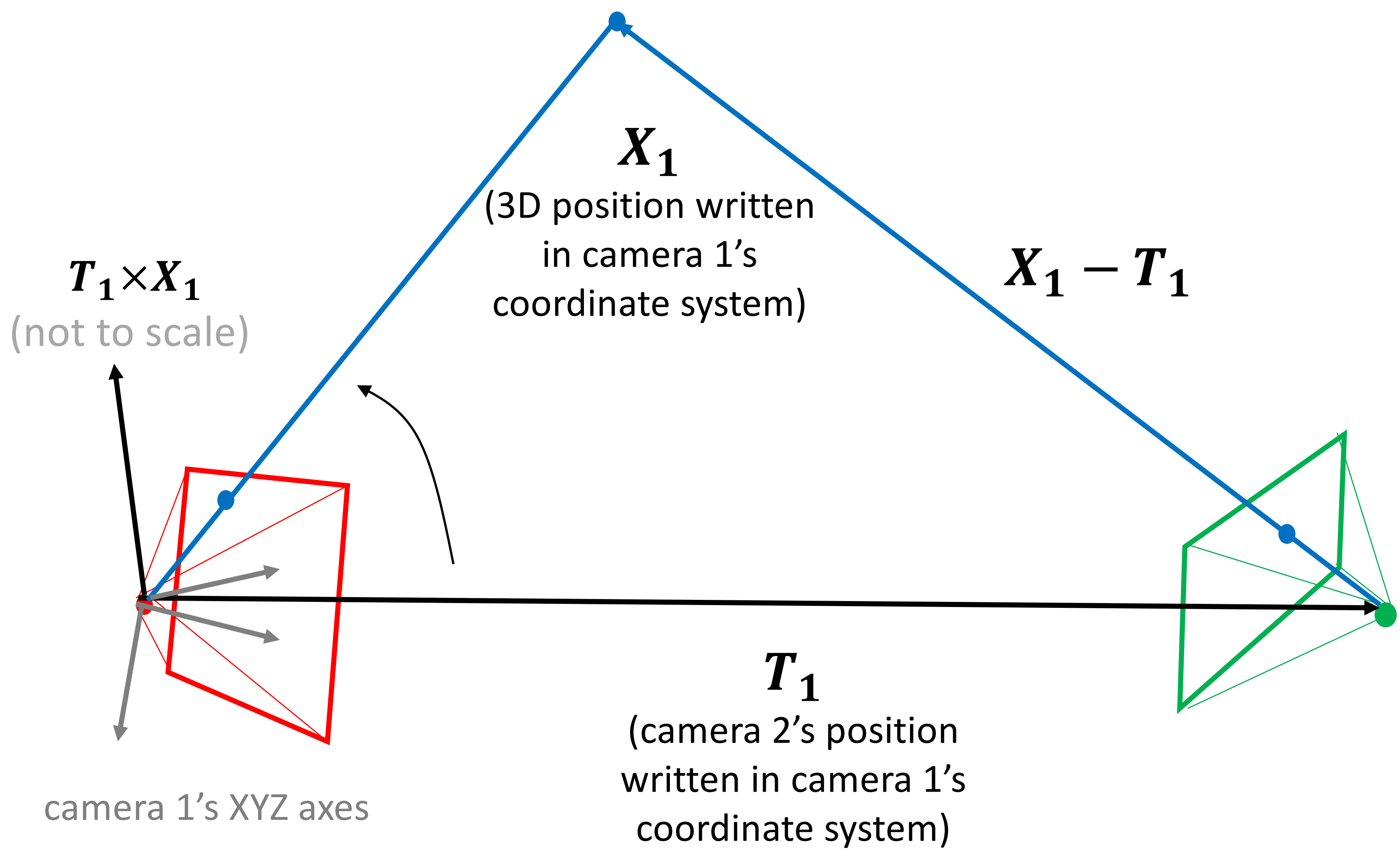
These vectors are all expressed in camera 1's coordinate system.



The three vectors $\boldsymbol{X_1}, \boldsymbol{T_1}, \boldsymbol{X_1} - \boldsymbol{T_1}$ form a triangle in the figure.

$\boldsymbol{T_1} \times \boldsymbol{X_1}$ is perpendicular to this triangle.

In particular, $(\boldsymbol{X_1} - \boldsymbol{T_1}) \cdot (\boldsymbol{T_1} \times \boldsymbol{X_1}) = \boldsymbol{0}$. ← main equation of today



Recall: Cross Product

lecture 13, slide 26

$$\boldsymbol{T_1} \times \boldsymbol{X_1} = \begin{bmatrix} 0 & -T_Z & T_Y \\ T_Z & 0 & -T_X \\ -T_Y & T_X & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

We will write $\boldsymbol{T_1} \times \boldsymbol{X_1}$ as $[\boldsymbol{T_1}]_\times \boldsymbol{X_1}$.

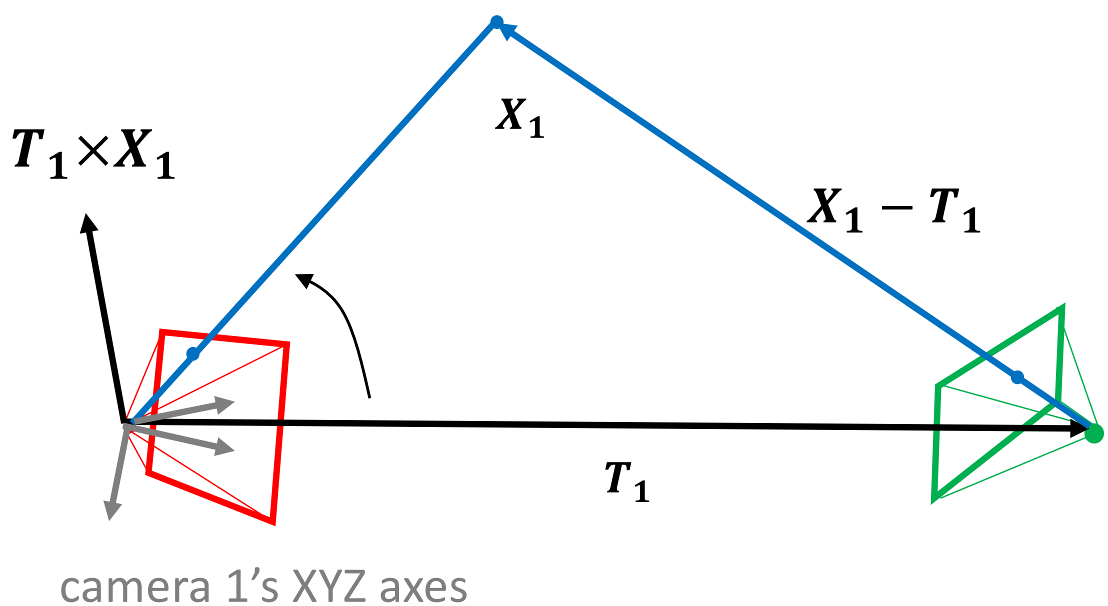
This expresses the cross product as a linear transformation defined by $\boldsymbol{T_1}$ and applied to vector $\boldsymbol{X_1}$.

$$(\boldsymbol{X_1} - \boldsymbol{T_1}) \cdot (\boldsymbol{T_1} \times \boldsymbol{X_1}) = \mathbf{0}$$

$$\boldsymbol{T_1} \times \boldsymbol{X_1} \equiv [\boldsymbol{T_1}]_\times \ \boldsymbol{X_1}$$

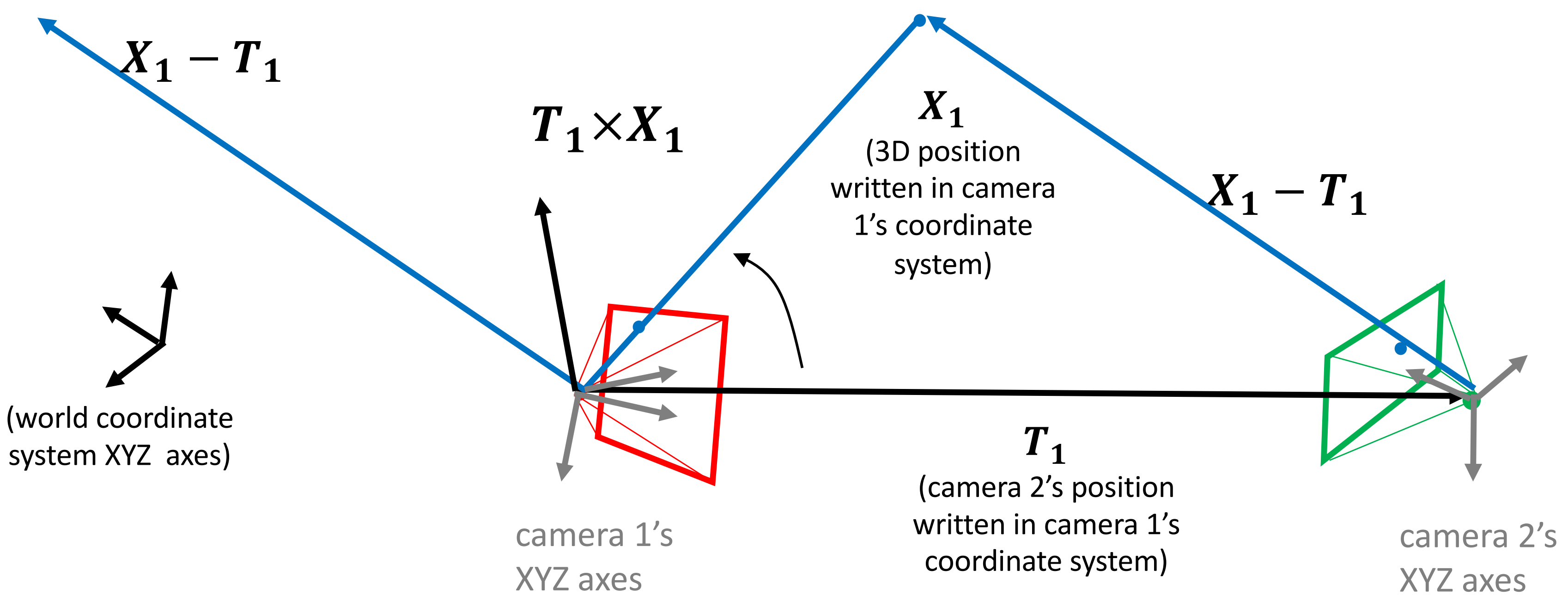
So, $(\boldsymbol{X_1} - \boldsymbol{T_1})^T \, [\boldsymbol{T_1}]_\times \, \boldsymbol{X_1} = \mathbf{0}$

This just says again that $\boldsymbol{T_1} \times \boldsymbol{X_1}$ is perpendicular to the triangle.



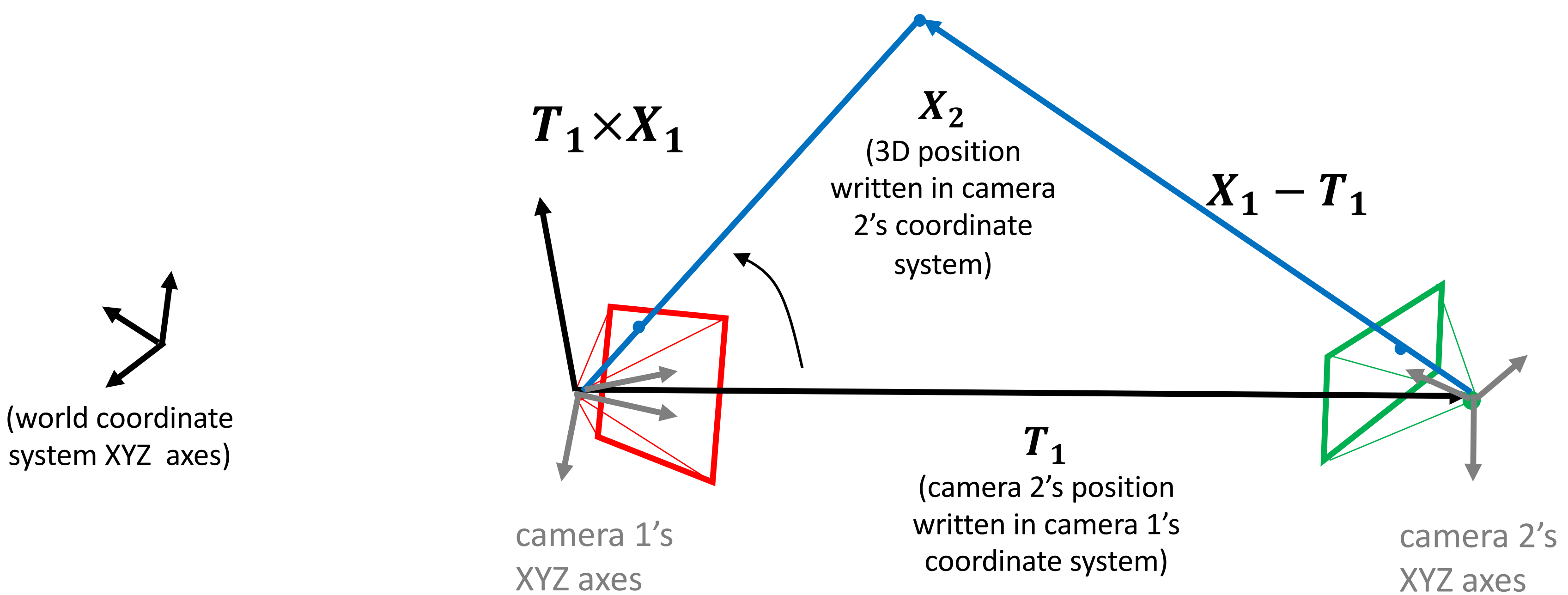
We would like to write $\boldsymbol{X_1} - \boldsymbol{T_1}$ in camera 2's coordinate system, so that we can say where the 3D point in question appears in camera 2's image plane.

How ?



We would like to write $\boldsymbol{X_1} - \boldsymbol{T_1}$ in camera 2's coordinate system, so that we can say where the 3D point in question appears in camera 2's image plane.

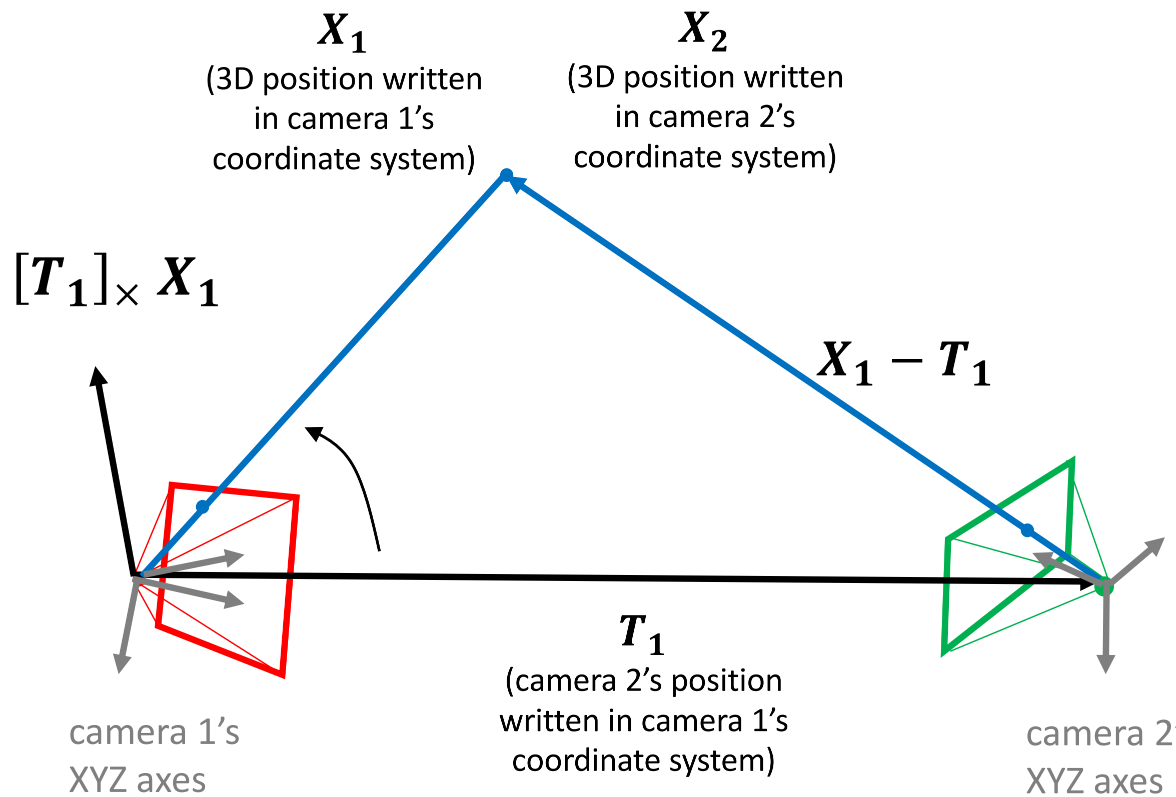
$\boldsymbol{X_2} = \boldsymbol{R_2}\,\boldsymbol{R_1^{\mathrm{T}}}\,(\boldsymbol{X_1} - \boldsymbol{T_1})$ or equivalently $\boldsymbol{R_1}\,\boldsymbol{R_2^{\mathrm{T}}}\,\boldsymbol{X_2} = \boldsymbol{X_1} - \boldsymbol{T_1}$



Substituting into $(\boldsymbol{X_1} - \boldsymbol{T_1})^{T}\, [\boldsymbol{T_1}]_\times\, \boldsymbol{X_1} = \mathbf{0},$ we get

$$\left(\boldsymbol{R_1}\, \boldsymbol{R_2^{\mathrm{T}}}\ \boldsymbol{X_2}\right)^{T} [\boldsymbol{T_1}]_\times\, \boldsymbol{X_1} = \mathbf{0}\ .$$

This gives us a constraint on the image position of this 3D point in the left and right camera's coordinate systems.

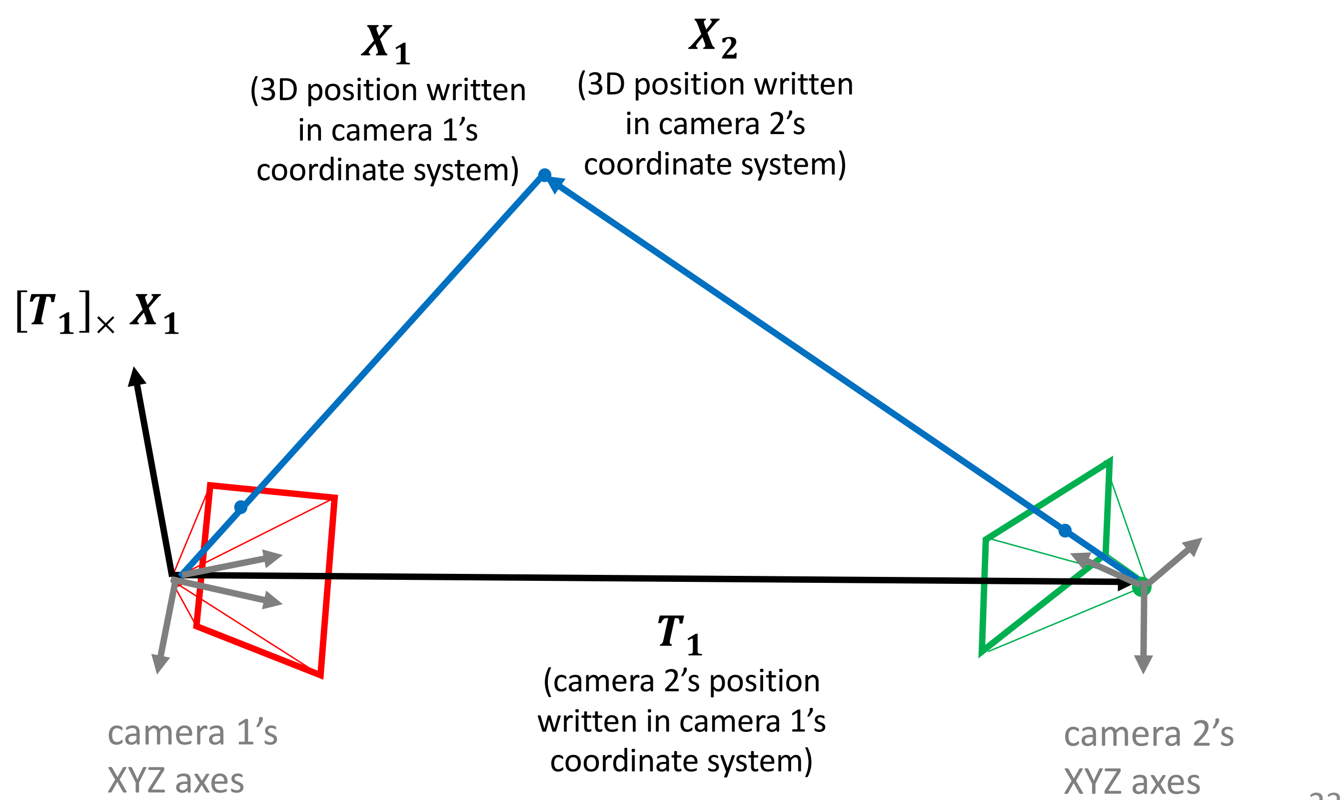


Essential Matrix $\boldsymbol{E}$ (definition)

$$\left(R_1\, R_2^{\mathrm{T}}\ X_2\right)^T [T_1]_\times X_1 = 0$$

$$\boldsymbol{X_2^{\mathrm{T}} E\, X_1 = 0}$$

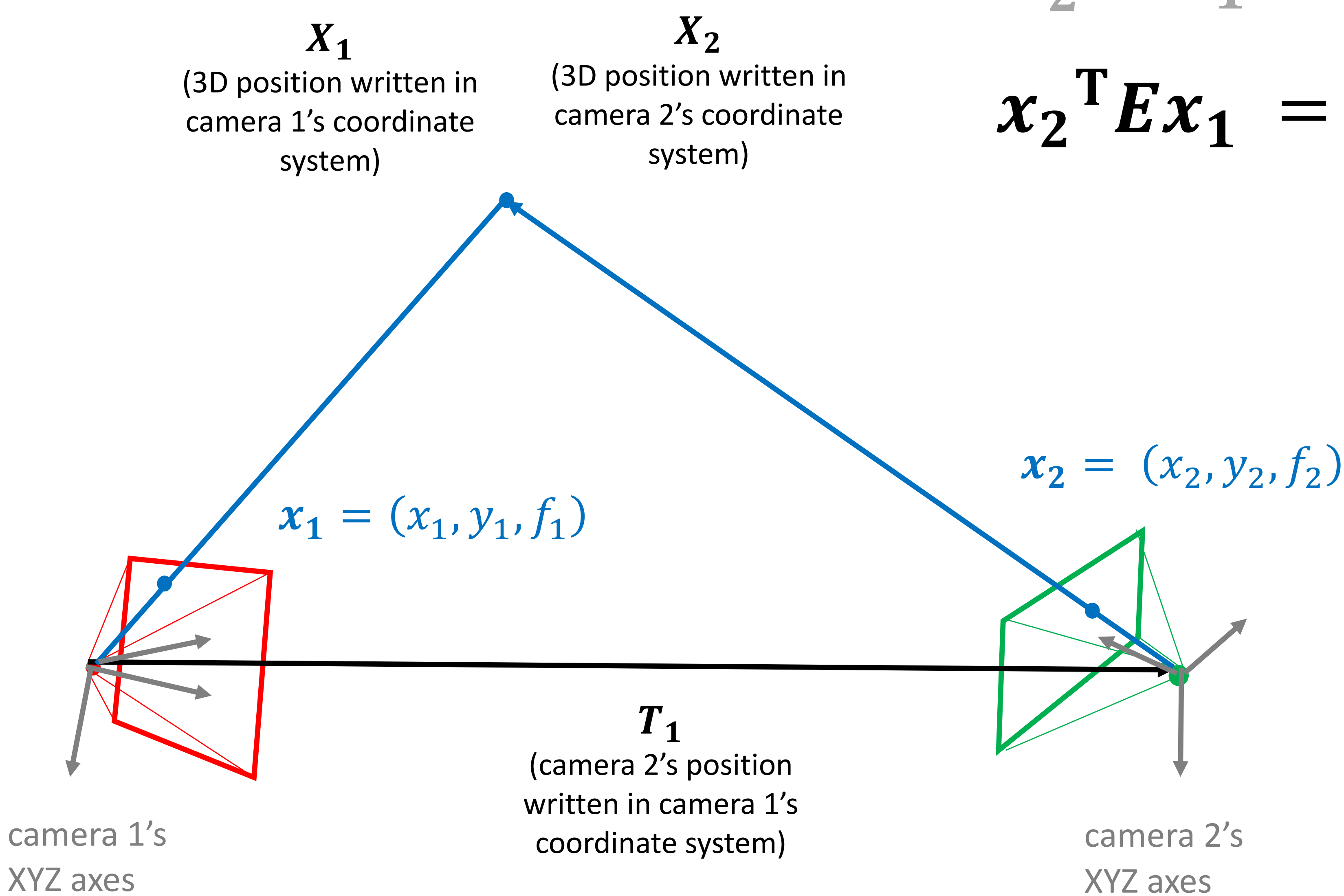
$$\boldsymbol{X_2^{\mathrm{T}}\ \underbrace{R_2\, R_1^{\mathrm{T}}\ [T_1]_\times}_{E}\ X_1 = 0}$$



We can multiply $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ by constants and the equation still holds. Thus.....

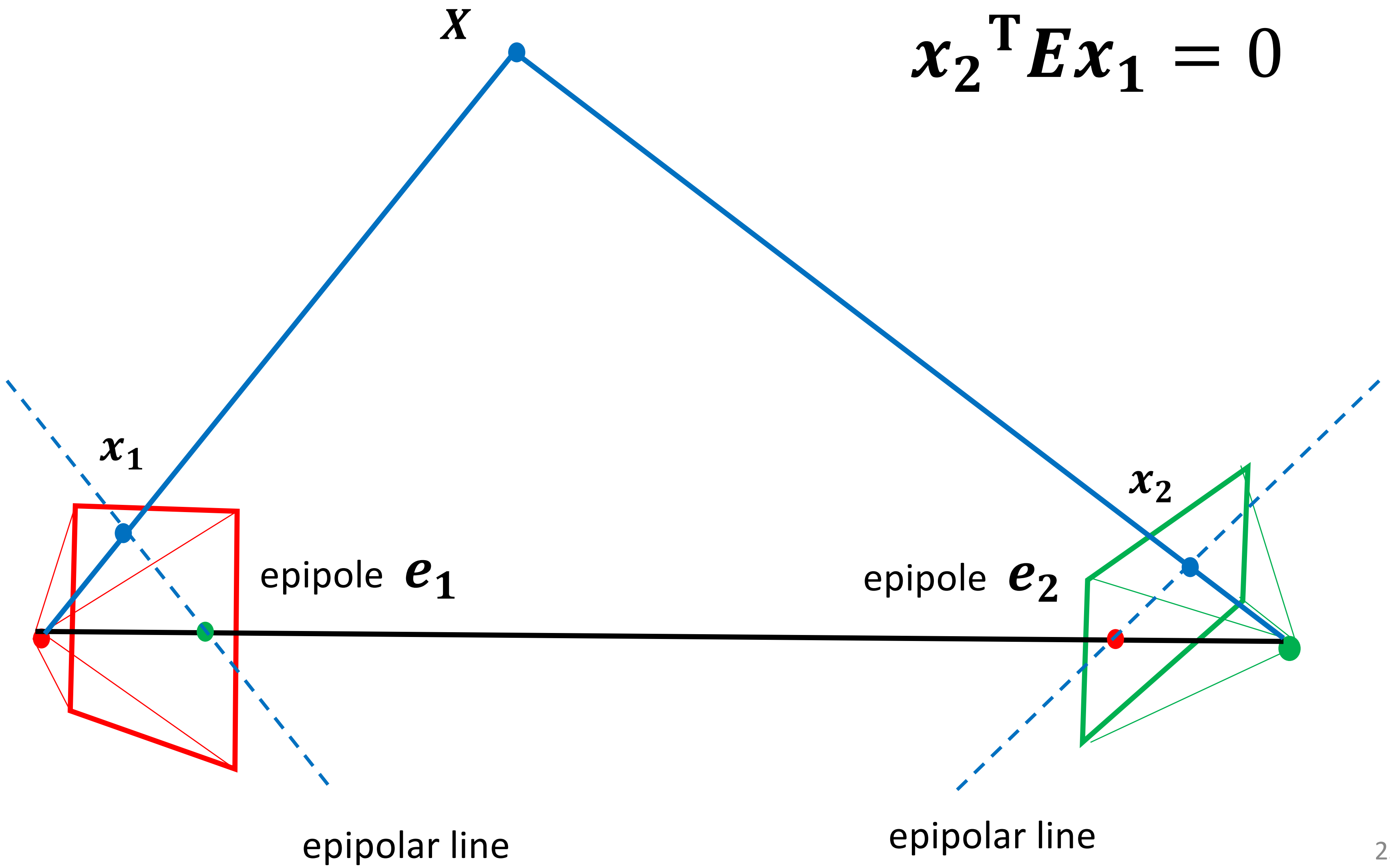
$$X_2^{\mathrm{T}} E\, X_1 = 0$$

$$\boldsymbol{x_2}^{\mathrm{T}} \boldsymbol{E} \boldsymbol{x_1} = 0$$



How to express epipolar lines and epipoles using the essential matrix?

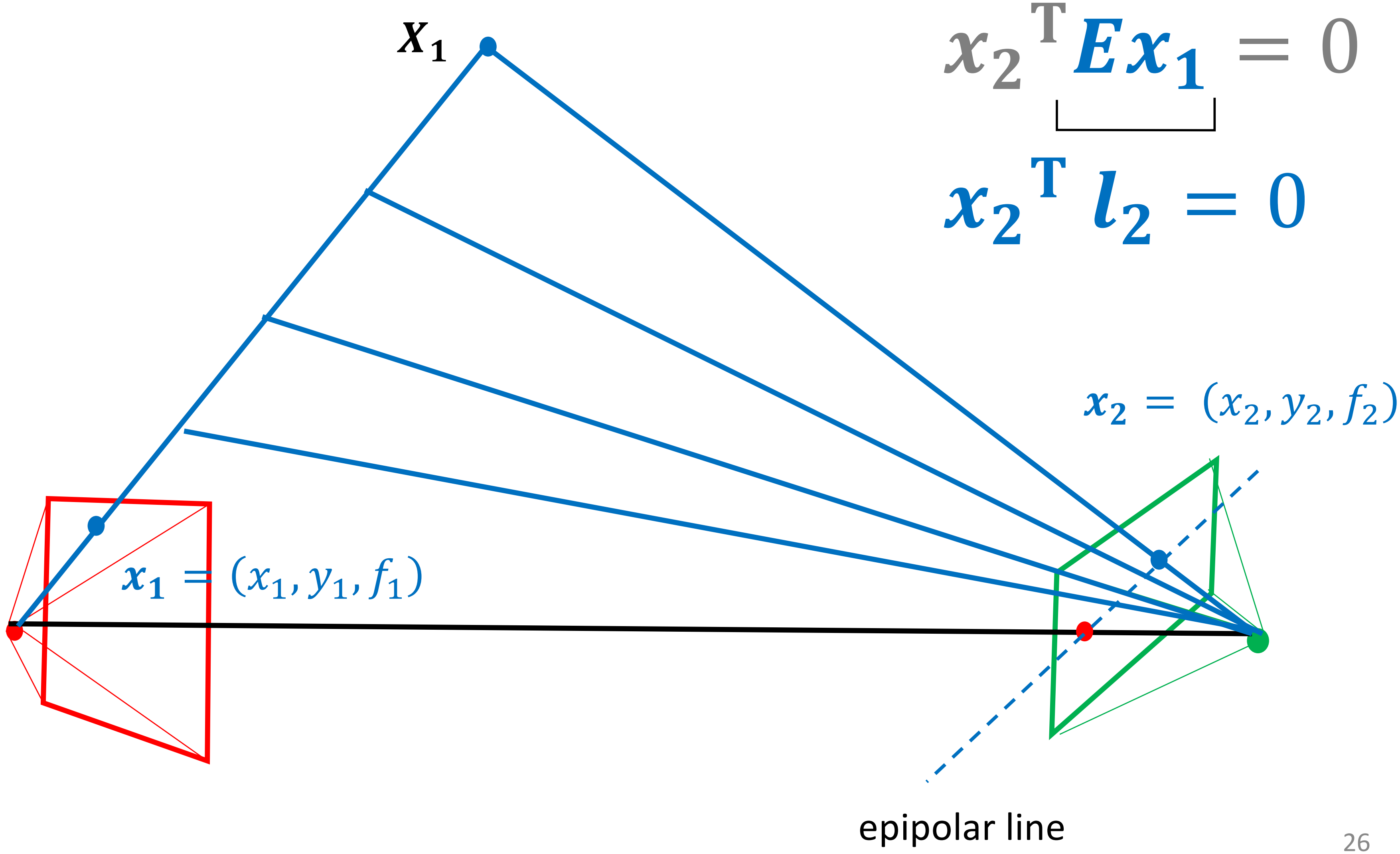
$$\boldsymbol{x_2}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{x_1} = 0$$



If we choose a point $\boldsymbol{x_1}$ in image 1, then this defines a line in image 2, namely the image of the ray from camera 1 through $\boldsymbol{x_1}$. This gives an epipolar line in image 2.

$$x_2^{\mathrm{T}} \underbrace{\boldsymbol{E}\boldsymbol{x_1}} = 0$$

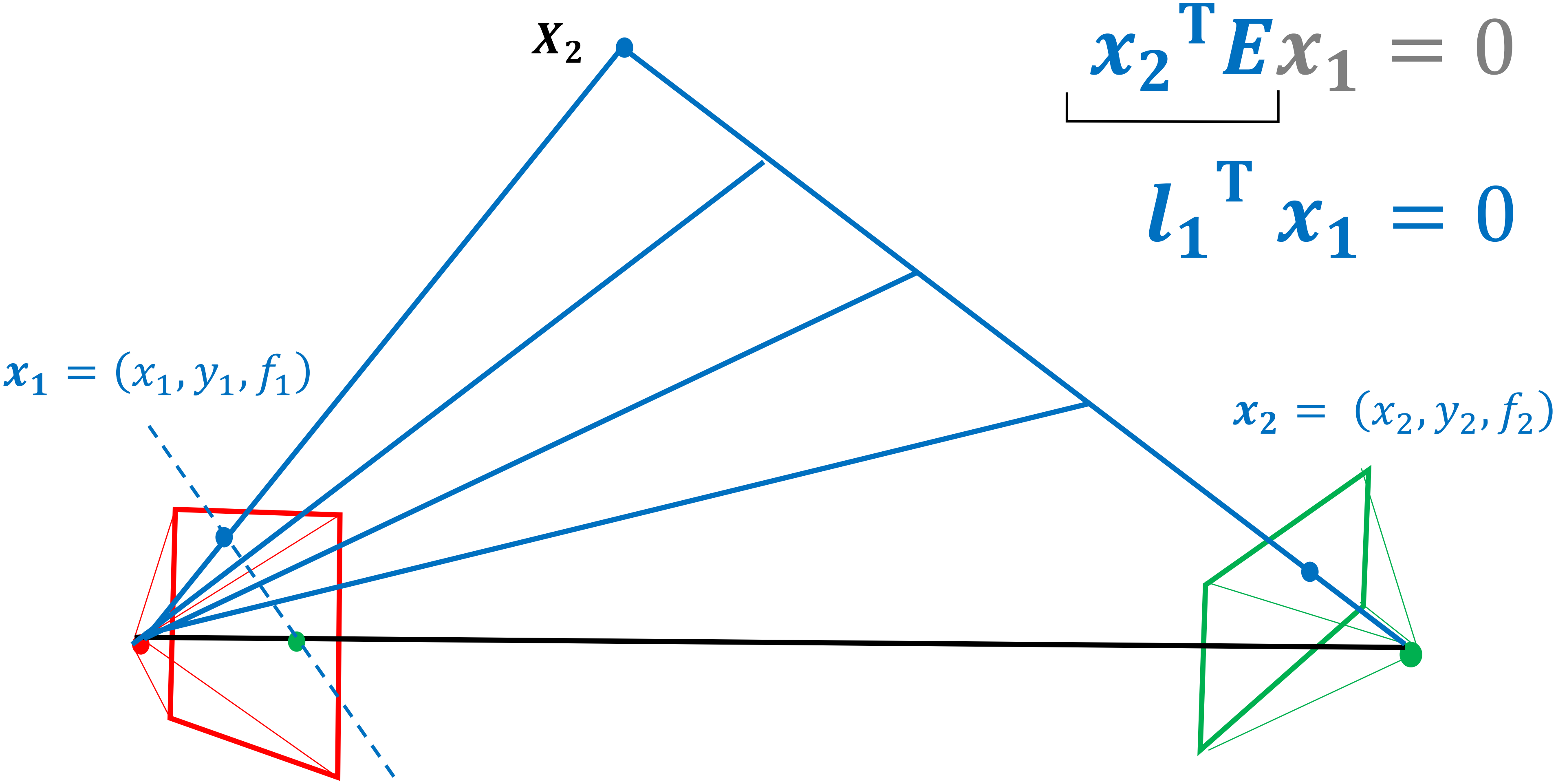
$$\boldsymbol{x_2}^{\mathrm{T}} \boldsymbol{l_2} = 0$$



If we choose a point $\boldsymbol{x_2}$ in image 2, then this defines a line in image 1, namely the image of the ray from camera 2 through $\boldsymbol{x_2}$. This gives an epipolar line in image 1.

$$\underbrace{\mathbf{x_2}^{\mathrm{T}} \boldsymbol{E}}\,\mathbf{x_1} = 0$$

$$\boldsymbol{l_1}^{\mathrm{T}}\,\mathbf{x_1} = 0$$



epipolar line

Properties of Essential Matrix $\boldsymbol{E}$

$\boldsymbol{E} \equiv \boldsymbol{R_2}\, \boldsymbol{R_1^{\mathrm{T}}}\, [\boldsymbol{T_1}]_{\times}$ where $[\boldsymbol{T_1}]_{\times}\, \boldsymbol{X_1} \equiv \boldsymbol{T_1} \times \boldsymbol{X_1}$

$\boldsymbol{E}$ is a 3x3 matrix. It has rank 2. Why?

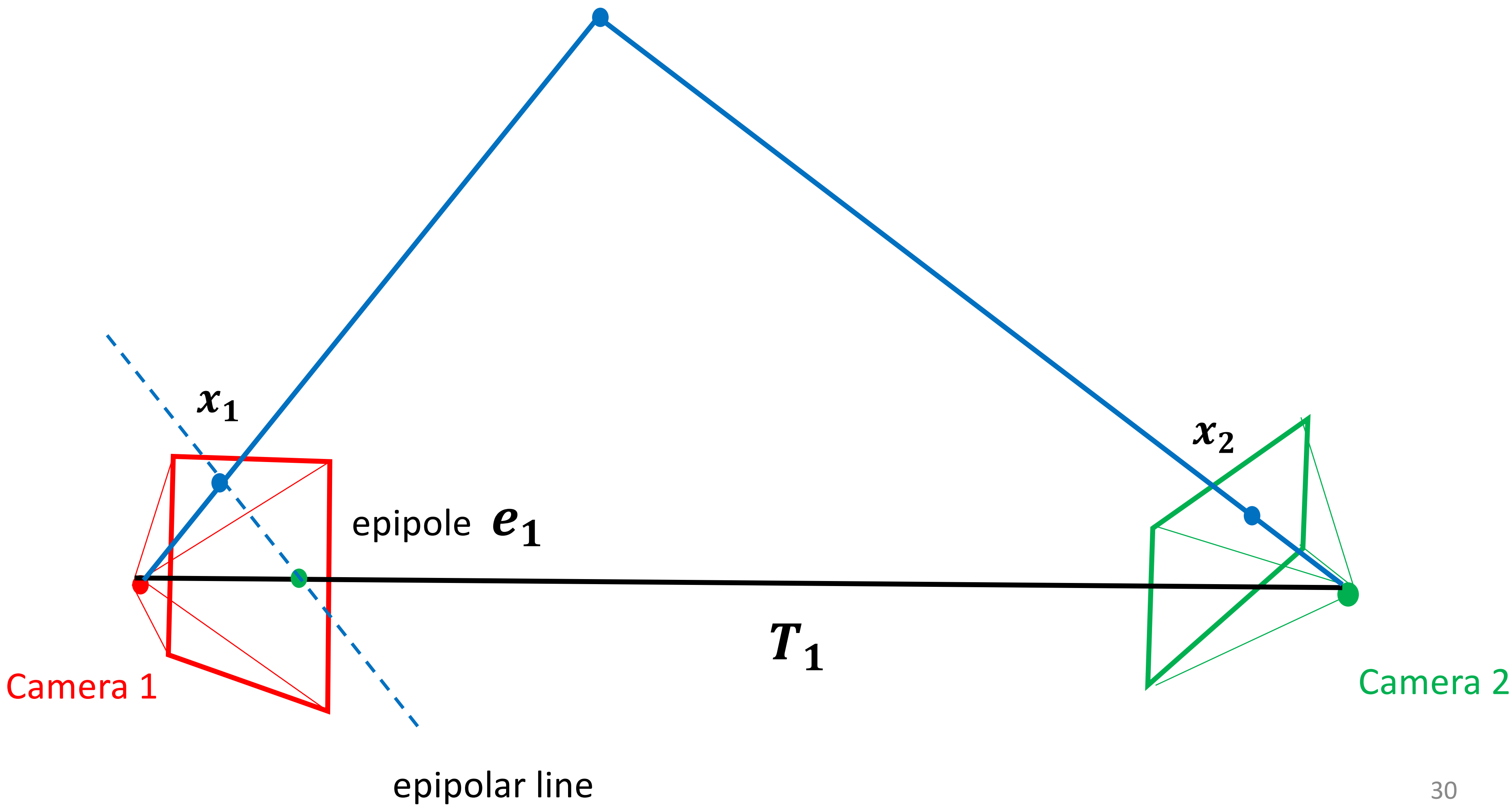
Properties of Essential Matrix $\boldsymbol{E}$

$\boldsymbol{E} \equiv \boldsymbol{R_2}\, \boldsymbol{R_1^{\mathrm{T}}}\, [\boldsymbol{T_1}]_\times$ where $[\boldsymbol{T_1}]_\times \boldsymbol{X_1} \equiv \boldsymbol{T_1} \times \boldsymbol{X_1}$

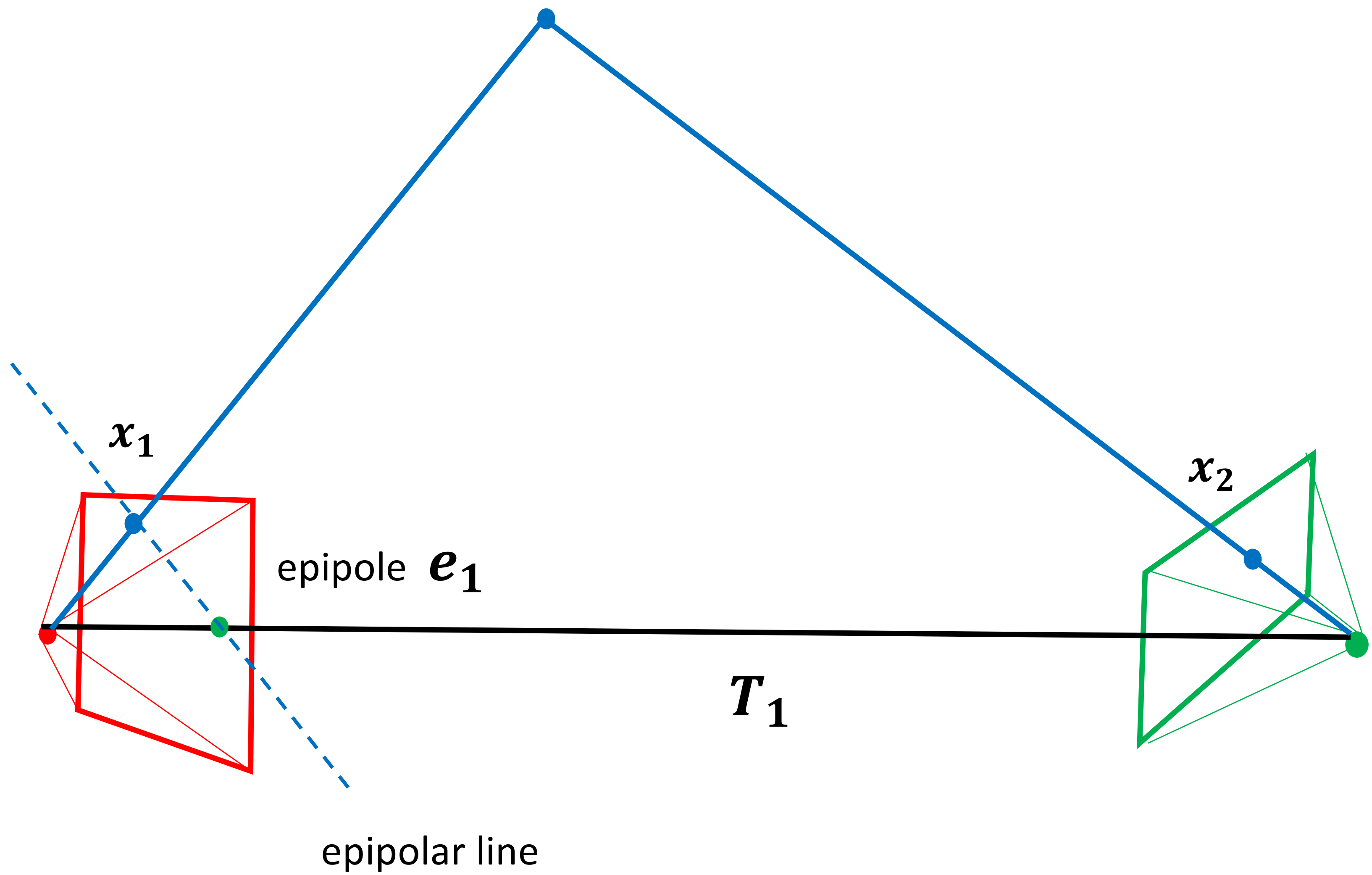
$\boldsymbol{E}$ is a 3x3 matrix. It has rank 2. Why?

$\boldsymbol{E}\ \boldsymbol{T_1} = 0$, and by inspection $\boldsymbol{E}\, \boldsymbol{X_1} \neq \boldsymbol{0}$ for any vector $\boldsymbol{X_1}$ that isn't parallel to $\boldsymbol{T_1}$.

The epipole $\boldsymbol{e_1}$ is the position of camera 2 in camera 1 image position. It has the same direction as $\boldsymbol{T_1}$. Thus $\boldsymbol{Ee_1} = 0$.



The epipole $\boldsymbol{e_1}$ is the position of camera 2 in camera 1 image position. It has the same direction as $\boldsymbol{T_1}$. Thus $\boldsymbol{Ee_1} = 0$. Thus, for any $\boldsymbol{x_2}$ in image 2, $\boldsymbol{x_2}^{\mathbf{T}}\boldsymbol{E}\boldsymbol{e_1} = 0$. Equivalently, the epipole $\boldsymbol{e_1}$ lies on all epipolar lines $\boldsymbol{l_1}^{\mathbf{T}}\boldsymbol{x_1} = 0$.

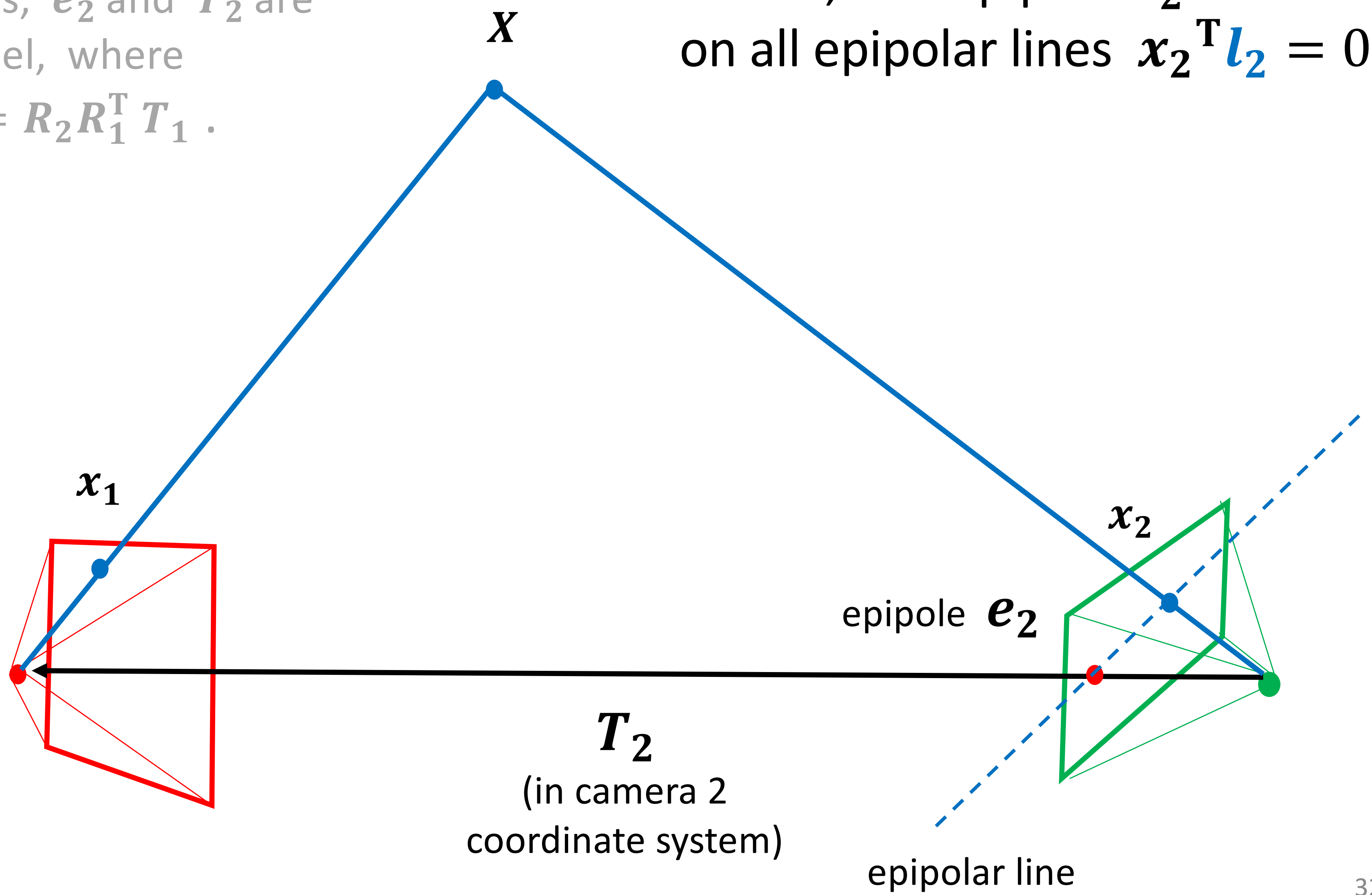


See lecture notes for the calculation. Use the fact that $\boldsymbol{R}_1 \boldsymbol{R}_2^{\mathrm{T}} \boldsymbol{e}_2$ is parallel to $\boldsymbol{T}_1$.

That is, $\boldsymbol{e}_2$ and $\boldsymbol{T}_2$ are parallel, where $\boldsymbol{T}_2 = \boldsymbol{R}_2 \boldsymbol{R}_1^{\mathrm{T}} \boldsymbol{T}_1$.

One can similarly show that $\boldsymbol{e}_2{}^{\mathbf{T}} \boldsymbol{E} \boldsymbol{x}_1 = 0$ for any $\boldsymbol{x}_1$ in image 1.

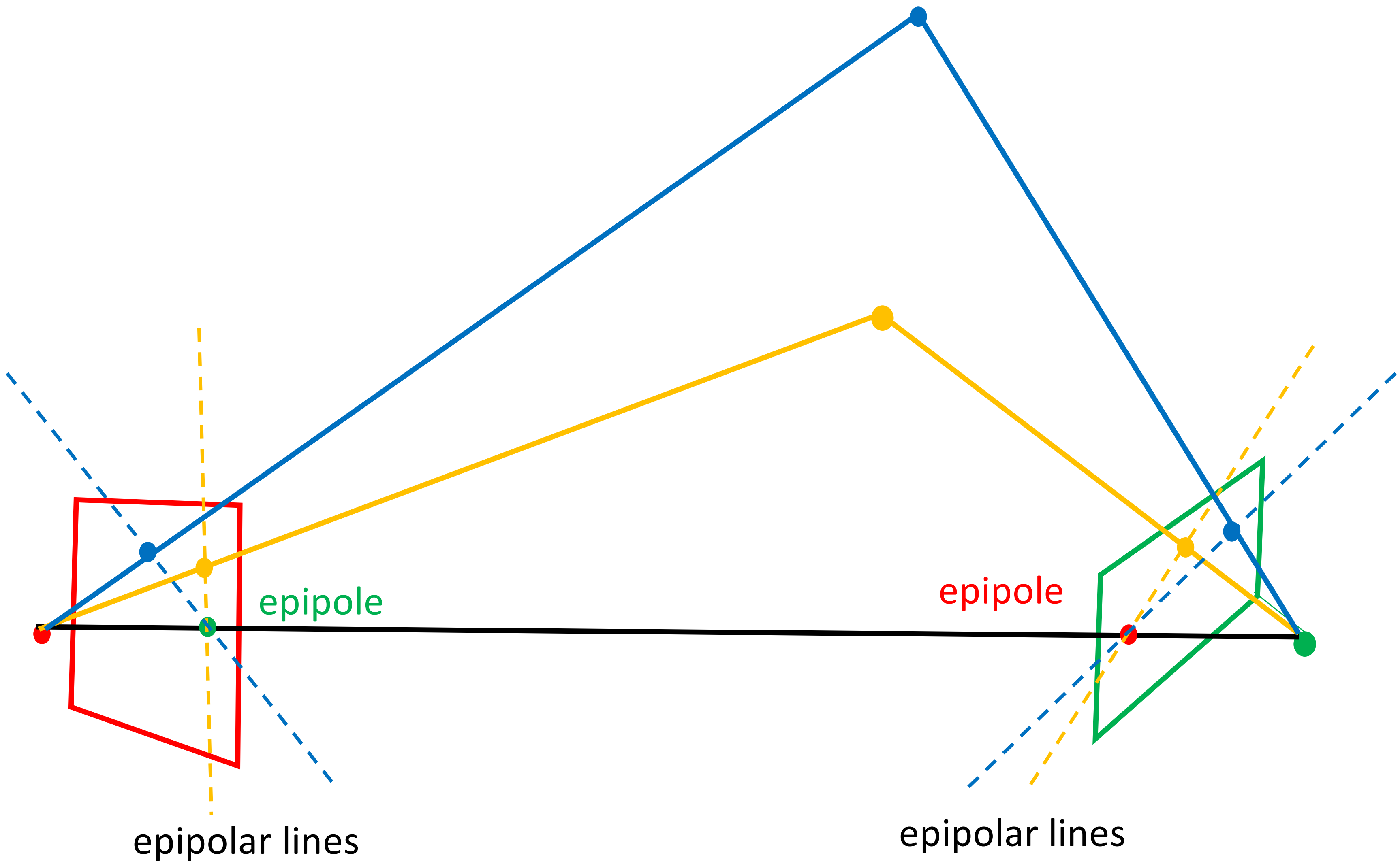
That is, the epipole $\boldsymbol{e}_2$ lies on all epipolar lines $\boldsymbol{x}_2{}^{\mathbf{T}} \boldsymbol{l}_2 = 0$.



Summary: All epipolar lines pass through the epipoles.

Equivalently, the epipoles are where the epipolar lines intersect.

We have given both geometric and algebraic arguments.



Overview of today

- Epipolar geometry
- Essential matrix (algebra)

  Uses projection plane coordinates, not pixel coordinates.

- Fundamental matrix (algebra)

  Uses pixel coordinates.

$$\mathbf{X}_2^{\mathrm{T}} \boldsymbol{E}\, \mathbf{X}_1 = \mathbf{0}$$

$$\boldsymbol{x}_2^{\mathrm{T}} \boldsymbol{E}\, \boldsymbol{x}_1 = \mathbf{0}$$

Recall that we can write either since $\boldsymbol{X}_i$ and $\boldsymbol{x}_i$ differ only by multiplicative constants.

How can we write out this constraint in terms of *pixel* positions $\widetilde{\boldsymbol{x}}_1$ and $\widetilde{\boldsymbol{x}}_2$?

Use this trick:

$$\left(\boldsymbol{K}_2^{-1} \boldsymbol{K}_2 \boldsymbol{x}_2 \right)^{\mathrm{T}} \boldsymbol{E}\, \boldsymbol{K}_1^{-1} \boldsymbol{K}_1 \boldsymbol{x}_1 = \mathbf{0}$$

Fundamental Matrix $\boldsymbol{F}$

$$\boldsymbol{x}_2^{\mathrm{T}} \boldsymbol{E}\, \boldsymbol{x}_1 = \mathbf{0}$$

$$\left(\boldsymbol{K}_2^{-1}\boldsymbol{K}_2\boldsymbol{x}_2\right)^{\mathrm{T}} \boldsymbol{E}\, \boldsymbol{K}_1^{-1}\boldsymbol{K}_1\boldsymbol{x}_1 = \mathbf{0}$$

$$\widetilde{\boldsymbol{x}}_2^{\mathrm{T}} \underbrace{\left(\boldsymbol{K}_2^{-1}\right)^{\mathrm{T}} \boldsymbol{E}\ \ \boldsymbol{K}_1^{-1}}_{\boldsymbol{F}} \widetilde{\boldsymbol{x}}_1 = \mathbf{0}$$

$$\widetilde{\boldsymbol{x}}_1 = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ 1 \end{bmatrix}$$

pixel position
in image 1

$$\widetilde{\boldsymbol{x}}_2 = \begin{bmatrix} \tilde{x}_2 \\ \tilde{y}_2 \\ 1 \end{bmatrix}$$

pixel position
in image 2

Fundamental Matrix $\boldsymbol{F}$

$$\boldsymbol{x_2}^{\mathrm{T}} \boldsymbol{E}\, \boldsymbol{x_1} = \mathbf{0}$$

$$\widetilde{\boldsymbol{x}}_{\mathbf{2}}^{\mathrm{T}} \boldsymbol{F}\, \widetilde{\boldsymbol{x}}_{\mathbf{1}} = \mathbf{0}$$

$\widetilde{\boldsymbol{x}}_{\mathbf{2}} = \begin{bmatrix} \tilde{x}_2 \\ \tilde{y}_2 \\ 1 \end{bmatrix}$

pixel position in image 2

pixel position in image 1

$\widetilde{\boldsymbol{x}}_{\mathbf{1}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ 1 \end{bmatrix}$

Epipolar lines in pixel space

If we choose a pixel $\tilde{\boldsymbol{x}}_2$ in image 2, then this defines an epipolar line in image 1, namely the image of the ray from camera 2 through $\tilde{\boldsymbol{x}}_2$.

$$\underbrace{\tilde{\boldsymbol{x}}_2^{\mathrm{T}}\,\boldsymbol{F}}\,\tilde{\boldsymbol{x}}_1 = \mathbf{0}$$

$$\boldsymbol{l}_1^{\mathrm{T}}\,\tilde{\boldsymbol{x}}_1 = \mathbf{0}$$

$$\tilde{\boldsymbol{x}}_1 = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \qquad \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} \tilde{x}_2 \\ \tilde{y}_2 \\ 1 \end{bmatrix}$$

If we choose a pixel $\tilde{\boldsymbol{x}}_1$ in image 1, then this defines an epipolar line in image 2, namely the image of the ray from camera 1 through $\tilde{\boldsymbol{x}}_1$.

$$\tilde{\boldsymbol{x}}_2^{\mathrm{T}}\,\underbrace{\boldsymbol{F}\,\tilde{\boldsymbol{x}}_1} = \mathbf{0}$$

$$\tilde{\boldsymbol{x}}_2^{\mathrm{T}}\,\boldsymbol{l}_2 = \mathbf{0}$$

Epipoles in pixel space

$\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ are the epipoles in the image projection plane of camera's 1 and 2.

By definition, the epipoles in pixel units are
$\tilde{\boldsymbol{e}}_1 = \boldsymbol{K}_1 \boldsymbol{e}_1$ and $\tilde{\boldsymbol{e}}_2 = \boldsymbol{K}_2 \boldsymbol{e}_2$.

Verify for yourself that $\boldsymbol{F}\tilde{\boldsymbol{e}}_1 = \boldsymbol{0}$ and $\tilde{\boldsymbol{e}}_2^{\mathrm{T}} \boldsymbol{F} = \boldsymbol{0}$.

Again, the epipoles lie on all epipolar lines. They are the intersection of the epipolar lines.

Next two lectures

- Given two images, how can one estimate the fundamental matrix $\boldsymbol{F}$?
- How can one estimate depth from two images?

Reminders

- Midterm 2 is next week on November 18th. It will be in-class at class time. Please bring your laptop and please make sure it is charged!
- Assignment 3 in preparation (should be ready by this weekend)