JOINT INSTITUTE
交大密西根学院

VV 285
RC 1

SUN YAN

Content

- Systems of Linear Equations
- Finite-Dimensional Vector Spaces
- Inner Product Spaces

Linear System

A ***linear system*** of m (algebraic) equations in n unknowns $x_1, \dots, x_n \in V$ is a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1.1.1}$$

where $b_1, \dots, b_m \in V$ and $a_{ij} \in \mathbb{F}$, $i = 1, \dots, m$, $j = 1, \dots, n$.

- When $b_1 = b_2 = \dots = b_m = 0$, it is called a **homogeneous system**. Otherwise, it is called an **inhomogeneous system**.
- If $m < n$, the system is called **underdetermined**. If $m > n$ it is called **overdetermined**.

Linear System

- $x_1 = x_2 = \ldots = x_n = 0$ is called the **trivial** solution to the system.

- Inhomogeneous system may have either:
  - a unique solution or
  - No solution
  - An infinite number of solutions

- Homogeneous system always has trivial solution and either:
  - No non-trivial solution or
  - An infinite number of non-trivial solutions

The Gauß-Jordan Algorithm

- The goal of the the Gauß-Jordan Algorithm is to transform a system first into the upper triangular form and subsequently into the diagonal form by elementary row manipulations.

$$\left(\begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array}\right)$$

Upper triangular form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array}\right)$$

Diagonal form

The Gauß-Jordan Algorithm

- **Elementary row manipulations:**
- 1. Swapping (interchanging) two rows,
- 2. Multiplying each element in a row with a number,
- 3. Adding a multiple of one row to another row.

Solution Set

- The **solution set** S is the set of all n-tuples of numbers $x_1, \cdots, x_n$ that satisfy the system of equations.

  - If a linear system has a unique solution, the set S contains a single point.

  - If there is no solution, $S = \emptyset$.

  - If there is more than one solutions, S is an infinite set.

Existence of Solutions

- Lemma 1.1.8

  Fundamental lemma for homogeneous equations

The homogeneous system

$$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=0$$

$$\ldots\ldots$$

$$a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=0$$

of m equations in n real or complex unknowns x_1, ... , x_n has a non-trivial solution if $n > m$.

Proved by induction

Uniqueness of Solutions

- A system of m equations with n unknowns will have a unique solution iff it is **diagonizable**.

- Equivalently, iff the system can be transformed into an **upper triangular form**, a unique solution exists

Exercise

- Judge whether the following is a linear system, and whether they are homogeneous. If it is a linear system, solve it with Gauß-Jordan Algorithm.

$$x_1+x_3+6x_4=26$$
$$9x_1+8x_2+9x_3+4x_4=58$$
$$2x_1+x_2+8x_3+6x_4=37$$
$$6x_1+7x_2+9x_3+4x_4=52$$

Content

- Systems of Linear Equations
- Finite-Dimensional Vector Spaces
- Inner Product Spaces

Linear Independence

1.2.1. Definition. Let V be a real or complex vector space and $v_1, \dots, v_n \in V$. Then the vectors $v_1, \dots, v_n$ are said to be ***independent*** if for all $\lambda_1, \dots, \lambda_n \in \mathbb{F}$

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

A finite set $M \subset V$ is called an ***independent set*** if its elements are independent.

Linear Independence

1.2.4. Definition. Let $v_1, \dots, v_n \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. Then the expression

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a ***linear combination*** of the vectors $v_1, \dots, v_n$.

Span

The set

$$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ y \in V : y = \sum_{k=1}^{n} \lambda_k v_k,\ \ \lambda_1, \ldots, \lambda_n \in \mathbb{F} \right\}$$

is called the (***linear***) ***span*** or the ***linear hull*** of the vectors $v_1, \ldots, v_n$.

Linear Independence

1.2.6. Lemma. The vectors $v_1, \dots, v_n \in V$ are independent if and only if none of them is contained in the span of all the others.

Proof:

$$\underset{k\in\{1,\dots,n\}}{\exists} \quad v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, v_n\}$$

$$\Leftrightarrow \quad \underset{k\in\{1,\dots,n\}}{\exists} \quad \underset{\substack{\lambda_i\in\mathbb{F} \\ i\in\{1,\dots,n\}\setminus\{k\} \\ \sum|\lambda_i|\neq 0}}{\exists} \quad v_k = \sum_i \lambda_i v_i$$

$$\Leftrightarrow \quad \underset{\substack{\lambda_i\in\mathbb{F} \\ i\in\{1,\dots,n\} \\ \sum|\lambda_i|\neq 0}}{\exists} \quad \sum_i \lambda_i v_i = 0$$

Basis

1.2.8. Definition. Let V be a real or complex vector space. An n-tuple $\mathscr{B} = (b_1, \dots, b_n) \in V^n$ is called an ***(ordered and finite) basis*** of V if every vector v has a unique representation

$$v = \sum_{i=1}^{n} \lambda_i b_i, \qquad \lambda_i \in \mathbb{F}.$$

The numbers λ_i are called the ***coordinates*** of v with respect to $\mathscr{B}$.

Basis

1.2.10. Theorem. Let V be a real or complex vector space. An n-tuple $\mathscr{B} = (b_1, \dots, b_n) \in V^n$ is a basis of V if and only if

(i) the vectors $b_1, \dots, b_n$ are linearly independent, i.e., $\mathscr{B}$ is an independent set, and

(ii) $V = \operatorname{span} \mathscr{B}$.

Proof.

(->) 1. V ⊂ span B and span B ⊂ V -> (ii)

2. Since B is a basis, the representation of 0 is unique with all λ equals to 0. We can deduce vectors are linear independent with definition.

(<-) 1. every v ∈ V is an element of the span of B. (can be represented by B)

2. Suppose the representation is not unique, contradiction with b_i are all linear independent.

Basis

1.2.11. Definition. Let V be a real or complex vector space. Then V is called ***finite-dimensional*** if either

- $V = \{0\}$ or
- V possesses a finite basis.

If V is not finite-dimensional, we say that it is ***infinite-dimensional***.

Basis

1.2.13. Theorem. Let V be a real or complex finite-dimensional vector space, $V \neq \{0\}$. Then any basis of V has the same length (number of elements) n.

Dimension

1.2.14. Definition. Let V be a finite-dimensional real or complex vector space. We define the ***dimension*** of V, denoted $\dim V$, as follows:

(i) If $V = \{0\}$, $\dim V = 0$.

(ii) If $V \neq \{0\}$, $\dim V = n$, where n is the length of any basis of V.

If V is an infinite-dimensional vector space we write $\dim V = \infty$.

Basis Extension Theorem

1.2.17. Lemma. Let $a_1, \dots, a_{n+1} \in V$ and assume that $a_1, \dots, a_n$ are independent and that $a_1, \dots, a_{n+1}$ are dependent. Then a_{n+1} is a linear combination of (some of) $a_1, \dots, a_n$.

1.2.18. Definition. Let V be a real or complex vector space and $A \subset V$ a finite set. An independent subset $F \subset A$ is called ***maximal*** if every $x \in A$ is a linear combination of elements of F.

1.2.20. Theorem. Let V be a vector space and $A \subset V$ a finite set. Then every independent subset $A' \subset A$ lies in some maximal subset $F \subset A$.

Basis Extension Theorem

1.2.21. Basis Extension Theorem. Let V be a finite-dimensional vector space and $A' \subset V$ an independent set. Then there exists a basis of V containing A'.

1.2.22. Corollary. Let V be an n-dimensional vector space, $n \in \mathbb{N}$. Then any independent set A with n elements is a basis of V.

1.2.23. Corollary. Let V be an n-dimensional vector space, $n \in \mathbb{N}$. Then an independent set A may have at most n elements.

Sum of Vector Space

1.2.24. Definition. Let V be a real or complex vector space and U, W be sets in V.

(i) We define the ***sum of*** U ***and*** W by

$$U + W := \left\{ v \in V : \underset{u \in U}{\exists} \; \underset{w \in W}{\exists} : v = u + w \right\}.$$

(ii) If U and W are subspaces of V with $U \cap W = \{0\}$, the sum $U + W$ is called ***direct***, and we denote it by $U \oplus W$.

Sum of Vector Space

1.2.27. Lemma. The sum $U + W$ of vector spaces U, W is direct if and only if all $x \in U + W$, $x \neq 0$, have a unique representation $x = u + w$, $u \in U$, $w \in W$.

Sum of Vector Space

1.2.28. Theorem. Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W.$$

Exercise 1

1. $V = \text{span}\{v_1, v_2, \ldots v_n\}$ prove that dim V ≤ n

2. $\{a_1, a_2, \ldots, a_n\}$ is a basis of V. $b_1, b_2 \ldots b_n$ are vectors in V. If each a_i can be expressed as linear combination of $b_1, b_2 \ldots b_n$

   Prove that $\{b_1, b_2 \ldots b_n\}$ is also a basis of V.

Exercise 2

1.Find all vector spaces that have exactly one basis.

2 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Exercise 3

You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)& \\ &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Content

- Systems of Linear Equations
- Finite-Dimensional Vector Spaces
- Inner Product Spaces

Inner product space

1.3.1. Definition. Let V be a real or complex vector space. Then a map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{F}$ is called a ***scalar product*** or ***inner product*** if for all $u, v, w \in V$ and all $\lambda \in \mathbb{F}$

(i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$,

(ii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,

(iii) $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$,

(iv) $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is called an ***inner product space***.

Example of Inner Product

- In $\mathbb{R}^n$ we define the ***canonical*** or ***standard scalar product***

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i, \qquad x, y \in \mathbb{R}^n. \qquad (1.3.1)$$

- In $\mathbb{C}^n$ we can define the inner product

$$\langle x, y \rangle := \sum_{i=1}^{n} \overline{x_i} y_i, \qquad x, y \in \mathbb{C}^n.$$

- In $C([a, b])$, the space of complex-valued, continuous functions on the interval $[a, b]$, we can define an inner product by

$$\langle f, g \rangle := \int_a^b \overline{f(x)} g(x)\, dx, \qquad f, g \in C([a, b]).$$

The Induced Norm

1.3.4. Definition. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The map

$$\| \cdot \| \colon V \to \mathbb{R}, \qquad \|v\| = \sqrt{\langle v, v \rangle}$$

is called the ***induced norm*** on V.

1.3.5. Examples.

- The induced norm in $\mathbb{R}^n$ and $\mathbb{C}^n$ is given by

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2, \tag{1.3.2}$$

  which is the usual euclidean norm.
- The induced norm on $C([a, b])$ is

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b |f(x)|^2 \, dx} = \|f\|_2$$

  which is just the 2-norm.

The Induced Norm

1.3.6. Cauchy-Schwarz Inequality. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space. Then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\| \qquad \text{for all } u, v \in V$$

where $\|\cdot\|$ is the induced norm.

6.17 Example ***examples of the Cauchy–Schwarz Inequality***

(a) If $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbf{R}$, then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1{}^2 + \cdots + x_n{}^2)(y_1{}^2 + \cdots + y_n{}^2).$$

(b) If f, g are continuous real-valued functions on $[-1, 1]$, then

$$\left| \int_{-1}^{1} f(x) g(x)\, dx \right|^2 \leq \left(\int_{-1}^{1} \big(f(x)\big)^2\, dx \right) \left(\int_{-1}^{1} \big(g(x)\big)^2\, dx \right).$$

The Induced Norm

1.3.7. Corollary. The induced norm is actually a norm, i.e., it satisfies

(i) $\|v\| \geq 0$, $\|v\| = 0 \Leftrightarrow v = 0$,

(ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$,

(iii) $\|u + v\| \leq \|u\| + \|v\|$

for all $u, v \in V$ and $\lambda \in \mathbb{F}$.

Angle Between Vectors

1.3.9. Definition. Let V be a real inner product space and $u, v \in V$. We define the ***angle*** $\alpha(u, v) \in [0, \pi]$ ***between*** u ***and*** v by

$$\cos\alpha(u, v) = \frac{\langle u, v\rangle}{\|u\|\|v\|}. \tag{1.3.3}$$

Orthogonality

1.3.11. Definition. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space.

(i) Two vectors $u, v \in V$ are called ***orthogonal*** or ***perpendicular*** if $\langle u, v \rangle = 0$. We then write $u \perp v$.

(ii) We call
$$M^{\perp} := \Big\{ v \in V : \underset{m \in M}{\forall} \langle m, v \rangle = 0 \Big\}$$
the ***orthogonal complement*** of a set $M \subset V$.

For short, we sometimes write $v \perp M$ instead of $v \in M^{\perp}$ or $v \perp m$ for all $m \in M$.

Orthogonality

1.3.12. Lemma. The orthogonal complement $M^\perp$ is a subspace of V.

1.3.13. Pythagoras's Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and M some subset of V. Let $z = x + y$, where $x \in M$ and $y \in M^\perp$. Then

$$\|z\|^2 = \|x\|^2 + \|y\|^2.$$

Orthonormal Systems

1.3.14. Definition. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space. A tuple of vectors $(v_1, \dots, v_r) \subset V$ is called a ***(finite) orthonormal system*** if

$$\langle v_j, v_k \rangle = \delta_{jk} := \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \qquad j, k = 1, \dots, r,$$

i.e., if $\|v_k\| = 1$ and $v_j \perp v_k$ for $j \neq k$.

Orthonormal Systems

1.3.16. Lemma. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space and $\mathscr{F} = (v_1, \dots, v_r) \subset V$ an orthonormal system. Then the elements of $\mathscr{F}$ are linearly independent.

Orthonormal Bases

1.3.17. Definition. Let $(V, \langle \cdot\,, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathscr{B} = (e_1, \dots, e_n)$ a basis of V. If $\mathscr{B}$ is also an orthonormal system, we say that $\mathscr{B}$ is an ***orthonormal basis*** (ONB).

1.3.18. Theorem. Let $(V, \langle \cdot\,, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathscr{B} = (e_1, \dots, e_n)$ an orthonormal basis of V. Then every $v \in V$ has the basis representation

$$v = \sum_{j=1}^{n} \langle e_j, v \rangle e_j .$$

1.3.19. Definition. The numbers $\langle e_j, v \rangle$ are called ***Fourier coefficients*** of v with respect to the basis $\mathscr{B}$. The vector

$$\pi_{e_i} v := \langle e_i, v \rangle e_i$$

is called the ***projection of*** v ***onto*** e_i

Orthonormal Bases

1.3.20. Parseval's Theorem. Let $(V, \langle \cdot\,, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathscr{B} = \{e_1, \dots, e_n\}$ an orthonormal basis of V. Then

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$$

for any $v \in V$.

The Projection Theorem

1.3.21. Projection Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be a (possibly infinite-dimensional) inner product vector space and $(e_1, \dots, e_r)$, $r \in \mathbb{N}$, be an orthonormal system in V. Denote $U := \operatorname{span}\{e_1, \dots, e_r\}$.

Then for every $v \in V$ there exists a unique representation

$$v = u + w \qquad\qquad \text{where } u \in U \text{ and } w \in U^{\perp}$$

and $u = \sum_{i=1}^{r} \langle e_i, v \rangle e_i$, $w := v - u$.

The Projection Theorem

1.3.22. Definition. The vector

$$\pi_U v := \sum_{i=1}^{r} \langle e_i, v \rangle e_i$$

is called the ***orthogonal projection of v onto U***. The projection theorem essentially states that $\pi_U v$ always exists and is independent of the choice of the orthonormal system (it depends only on the span U of the system).

Orthogonal Subspaces

1.3.24. Corollary. Let $(V, \langle \cdot , \cdot \rangle)$ be a (possibly infinite-dimensional) inner product vector space and let $U \subset V$ be a finite-dimensional subspace. Then

$$V = U \oplus U^{\perp}$$

If V is finite-dimensional, then

$$\dim V = \dim U + \dim U^{\perp}.$$

This follows directly from the Projection Theorem with Lemma 1.2.27 and Theorem 1.2.28.

Bessel's Inequality

1.3.25. Bessel Inequality. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product space and $(e_1, \dots, e_n)$ an orthonormal system in V. Then, for any $v \in V$ and any $r \leq n$,

$$\sum_{k=1}^{r} |\langle e_k, v\rangle|^2 \leq \|v\|^2. \tag{1.3.5}$$

Gram-Schmidt Orthonormalization

We set

$$w_1 := \frac{v_1}{\|v_1\|}$$

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\|}, \qquad k = 2, \ldots, n,$$

And hence obtain an orthonormal system as desired

Exercise 1

- Which of following can be a inner product of $C[-1,1]$
- 1. <f,g>= $\int_{-1}^{1} \overline{f^2(x)} g^2(x) \mathrm{d}x$
- 2. <f,g>= $\int_{-1}^{1} x\overline{f(x)} g(x) \mathrm{d}x$.
- 3. <f,g>= $\int_{-1}^{1} x^2\overline{f(x)} g(x) \mathrm{d}x$
- 4. <f,g>= $\int_{-1}^{1} e^x\, \overline{f(x)} g(x) \mathrm{d}x$

Exercise 2

6.22 Parallelogram Equality

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

27 Suppose $u, v, w \in V$. Prove that

$$\|w - \tfrac{1}{2}(u + v)\|^2 = \frac{\|w - u\|^2 + \|w - v\|^2}{2} - \frac{\|u - v\|^2}{4}.$$

Exercise 3

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx.$$

Exercise 4

Suppose $v_1 = (2,1,1)^T, v_2 = (5,2,0)^T, v_3 = (9,8,5)^T$.

Find an orthonormal basis for the vector space span$\{v_1, v_2, v_3\}$.

Reference

Hohberger, Horst. "VV285_main.pdf"

Axler, Sheldon. *Linear Algebra Done Right*. Springer, 2017.

Thank You