JOINT INSTITUTE
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VV 285
RC 4

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Comment on MID 1

Exercise 4

On the space $V = \mathrm{Mat}(2\times 2;\mathbb{R})$ define the scalar product $\langle A, B\rangle_{\mathrm{tr}} := \mathrm{tr}(A^T B)$. Find the matrix elements and the adjoint of the map

$$L\colon\ \mathrm{Mat}(2\times 2;\mathbb{R}) \to \mathrm{Mat}(2\times 2;\mathbb{R}), \qquad\qquad LX = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} X.$$

(6 Marks)

Content

- Convergence and Continuity
- Functions and Derivatives
- Exercise

Defi: Interior, Exterior and Boundary Point

2.1.16. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$.

(i) A point $x \in M$ is called an ***interior point of*** M if there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset M$.

(ii) The set of interior points of M is denoted by $\operatorname{int} M$.

(iii) A point $x \in V$ is called a ***boundary point of*** M if for every $\varepsilon > 0$ $B_\varepsilon(x) \cap M \neq \emptyset$ and $B_\varepsilon(x) \cap (V \setminus M) \neq \emptyset$.

(iv) The set of boundary points of M is denoted by ∂M.

(v) A point that is neither a boundary nor an interior point of M is called an ***exterior point of*** M.

2.1.17. Remarks.

(i) An exterior point of M is an interior point of $V \setminus M$. (Check this!)

(ii) For given M, any point of V is either an interior, boundary or exterior point of M.

Defi: open and closed set

2.1.1. Definition. Let $(V, \|\cdot\|)$ be a normed vector space. Then

$$B_\varepsilon(a) := \{x \in V : \|x - a\| < \varepsilon\}, \qquad a \in V,\ \varepsilon > 0, \tag{2.1.1}$$

is called an ***open ball*** of radius ε about a.

2.1.2. Definition. Let $(V, \|\cdot\|)$ be a normed vector space. A set $U \subset V$ is called ***open*** if for every $a \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.

2.1.18. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then M is said to be ***closed*** if its complement $V \setminus M$ is open.

Defi: open and closed set

2.1.19. Remark. Of course, a set M does not need to be either open or closed. Some sets are open and closed at the same time.

2.1.20. Examples.

(i) A set consisting of a single point, $M = \{a\} \subset V$, is a closed set.

(ii) The empty set $\emptyset \subset V$ is closed.

(iii) The entire space V is a closed set in V.

2.1.21. Lemma. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$.

(i) The set M is open if and only if $M = \operatorname{int} M$.

(ii) The set M is closed if and only if $\partial M \subset M$.

Fun Problem: Could your design a set that is neither open nor closed?

Defi: closure

2.1.22. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then

$$\overline{M} := M \cup \partial M$$

is called the ***closure*** of M.

2.1.23. Remark. It is not hard to show that the closure of a set M is a closed set. In fact, it is the smallest set that both contains M and is closed.

The closure of a set may also be characterized in terms of sequences:

2.1.24. Lemma. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then

$$\overline{M} = \Big\{ x \in V : \underset{(x_n)_{n\in\mathbb{N}}}{\exists} x_n \in M \text{ and } x_n \to x \Big\} \tag{2.1.8}$$

Defi: Equivalence of Norm

2.1.5. Definition. Let V be a vector space on which we may define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then the two norms are called ***equivalent*** if there exists two constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1 \qquad \text{for all } x \in V. \tag{2.1.3}$$

2.1.11. Theorem. In a finite-dimensional vector space, all norms are equivalent.

Proof: Equivalence of Norm

2.1.11. Theorem. In a finite-dimensional vector space, all norms are equivalent.

Proof. We want to show that for any norm in V, $\|\cdot\|$,
$\exists C_1, C_2$ such that $C_1\|v\|_0 \le \|v\| \le C_2\|v\|_0$
Where we define $\|\cdot\|_0$ as:

$$\|v\|_0=\sum_{i=1}^{n} |\lambda_i| \text{ with } v=\sum_{i=1}^{n} \lambda_i v_i$$

This is realized by

1. $\exists C_1, \|v\| \ge C_1 \sum_{i=1}^{n} |\lambda_i|$
   1.1 Theorem of Bolzano-Weierstrass in R^n.
   1.2 Norm inequality.
2. $\exists C_2, \|v\| \le C_2 \sum_{i=1}^{n} |\lambda_i|$: Triangle inequality.

Defi: Bounded and Compact Set

2.1.30. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then M is said to be ***bounded*** if there exists some $R > 0$ such that $M \subset B_R(0)$.

2.1.31. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $K \subset V$. Then K is said to be ***compact*** if every sequence in K has a convergent subsequence with limit contained in K.

2.1.32. Theorem. Let $(V, \|\cdot\|)$ be a (possibly infinite-dimensional) normed vector space and $K \subset V$ be compact. Then K is closed and bounded.

2.1.33. Theorem. Let $(V, \|\cdot\|)$ be a finite-dimensional vector space and let $K \subset V$ be closed and bounded. Then K is compact.

Every Linear Map in a finite-dimensional vector space is bounded

Defi: Continuous function

2.1.25. Definition. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f\colon U \to V$ a function. Then f is ***continuous at*** $a \in U$ if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall} \quad \|x-a\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(a)\|_2 < \varepsilon. \tag{2.1.9}$$

Iff

$$\underset{\substack{(x_n)_{n\in\mathbb{N}} \\ x_n\in U}}{\forall} \quad x_n \to a \quad \Rightarrow \quad f(x_n) \to f(a). \tag{2.1.10}$$

Suppose that $f\colon M \to N$, where M, N are any sets. Let $A \subset M$. Then we define the ***image of*** A by

$$f(A) := \{y \in N\colon y = f(x) \text{ for some } x \in A\}.$$

In particular, we can write

$$\operatorname{ran} f = f(M).$$

Similarly, for $B \subset N$ we define the ***pre-image of*** B by

$$f^{-1}(B) := \{x \in M\colon f(x) = y \text{ for some } y \in B\}. \tag{2.1.11}$$

Defi: Continuous function

2.1.37. Definition. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces, $\Omega \subset U$ and $f\colon \Omega \to V$ a function. Then f is ***uniformly continuous in Ω*** if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in\Omega}{\forall} \quad \|x-y\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(y)\|_2 < \varepsilon. \tag{2.1.13}$$

2.1.35. Proposition. Let $(U, \|\cdot\|_1)$, $(V, \|\cdot\|_2)$ be normed vector spaces and $K \subset U$ compact. Let $f\colon K \to V$ be continuous. Then $\operatorname{ran} f = f(K)$ is compact in V.

2.1.36. Theorem. Let $(V, \|\cdot\|)$ be a normed vector space and $K \subset V$ compact. Let $f\colon K \to \mathbb{R}$ be continuous. Then f has a maximum in K, i.e., there exists an $x \in K$ such that $f(y) \le f(x)$ for all $y \in K$.

2.1.38. Theorem. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces, $K \subset U$ a compact set and $f\colon K \to V$ continuous on K. Then f is uniformly continuous on K.

Content

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Defi-The Derivative of Fucntion

- The **derivative** of $f : \Omega \to V$ at x:
  $L_x = Df|_x$
  such that $f(x+h) = f(x) + L_x h + o(h)$ as $h \to 0$

- The **derivative** as a map D
  $D : C^1(\Omega, V) \to C(\Omega, L(X, V)),\ f \to Df$

- The partial derivative with respect to x_j at $x \in \Omega$

$$\frac{dy}{dx_j}\Big|_x := \lim_{h\to 0} \frac{(f(x+he_j) - f(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x_1, x_2, \cdots, x_j + h, \cdots, x_n) - f(x))}{h}$$

Defi-The Jacobian

Definition. Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$. Assume that all partial derivatives $\partial_{x_j} f_i$ of f exist at $x \in \Omega$. The matrix

$$J_f(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \Bigg|_x$$

is called the ***Jacobian*** of f.

- Note: The existences of all partial derivatives (and thus the Jacobian) do not guarantee the existence of derivative of the original function.

Defi-The Jacobian

e.g. $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0, 0) \\ 0 & (x_1, x_2) = (0, 0) \end{cases}$$

Defi-The Jacobian

- 2.2.18. Theorem. Let $\Omega \subset \mathrm{R}_n$ be an open set and $f :\rightarrow \mathrm{R}^m$ such that all partial derivatives $\partial x_j\, f_i$ exist on Ω.
- 1. all partial derivatives $\partial x_j\, f_i$ are bounded on $\Omega \Rightarrow f \in C(\Omega, \mathrm{R}_m)$.
  2. all partial derivatives $\partial x_j\, f_i$ are continuous on $\Omega \Rightarrow f \in C^1(\Omega, \mathrm{R}_m)$ with derivative given by the Jacobian.

- Question: $\partial x_j\, f_i$ are not continuous, can we say that f is not differentiable?

Defi-The Jacobian

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Defi- Product Rule

2.2.20. Definition. Let X_1, X_2, V be normed vector spaces. A map $\odot\colon X_1 \times X_2 \to V$ is called a ***(generalized) product*** if

1. $\odot$ is bilinear, i.e., linear in each entry and
2. $\|u \odot v\|_V \leq \|u\|_{X_1} \|v\|_{X_2}$ for all $u \in X_1$, $v \in X_2$.

2.2.22. Product Rule. Let U, X_1, X_2, V be finite-dimensional vector spaces and $\Omega \subset U$ an open set. Let $f\colon \Omega \to X_1$ and $g\colon \Omega \to X_2$ be differentiable maps and $\odot\colon X_1 \times X_2 \to V$ a generalized product. Then $f \odot g\colon \Omega \to V$ is also differentiable and

$$D(f \odot g) = (Df) \odot g + f \odot (Dg). \qquad (2.2.2)$$

Defi- Chain Rule

2.2.23. Chain Rule. Let U, X, V be finite-dimensional vector spaces and $\Omega \subset U$, $\Sigma \subset X$ open sets. Let $g\colon \Omega \to \Sigma$ and $f\colon \Sigma \to V$ be differentiable maps. Then the composition $f \circ g\colon \Omega \to V$ is also differentiable and for all $x \in \Omega$

$$D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x, \qquad (2.2.4)$$

where the right-hand side is a composition of linear maps.

2.2.24. Example. Consider the polar coordinates $(r, \phi) \in (0, \infty) \times [0, 2\pi)$, defined through the map

$$\Phi(r, \phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}.$$

Content

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Exercise 1-Derivative

- Example 1. Calculate the derivative of inverse of matrices A^{-1} of A in $GL(n; \mathbb{R})$. Namely, the derivative of $F : A \to A^{-1}$.

Exercise 2-Jacobian

7. Let $F:\mathbf{R}^3 \to \mathbf{R}^3$ be the mapping defined by

$$F(r, \theta, \varphi) = (r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi)$$

or in other words

$$x = r \sin \varphi \cos \theta, \qquad y = r \sin \varphi \sin \theta, \qquad z = r \cos \varphi.$$

Find the Jacobian matrix and Jacobian determinant of this mapping.

Exercise 3-Chain Rule

- Exercise 3. Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinates: $(r, \varphi, \theta) \in (0, \infty) \times [0, 2\pi) \times [0, \pi]$ and

$$\Phi(r, \varphi, \theta) = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

Reference

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Thank You

Have a nice day!