JOINT INSTITUTE
交大密西根学院

VV 285
RC 7

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Content

- Curves in Vector Spaces
- Potential Functions
- The second derivative
- Exercise

Curvature

2.3.29. Definition. The ***curvature*** of a smooth C^2-curve $\mathcal{C} \subset V$ is

$$\kappa : \mathcal{C} \to \mathbb{R}, \qquad \kappa \circ \ell^{-1}(s) := \left\| \frac{d}{ds}\left(T \circ \ell^{-1}(s)\right) \right\|$$

where T is the unit tangent vector and $\ell^{-1} : I \to \mathcal{C}$ is the curve length parametrization of $\mathcal{C}$.

$$\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)\Big|_{s=\ell\circ\gamma(t)} = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}. \qquad (2.3.10)$$

- ***Note.*** The curvature κ **does not** depend on the orientation of C.

Try with the example

1. $t \in (0, 2\pi)$ $\quad \gamma(t) = \begin{pmatrix} Sin(t) \\ \cos(t) \\ t \end{pmatrix}$

Curvature in R^3

2.3.31. Lemma. Let $\mathcal{C} \subset \mathbb{R}^3$ be a smooth C^2-curve with parametrization $\gamma : I \to \mathcal{C}$, then

$$\kappa \circ \gamma(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

- Try with the example before:

$$t \in (0, 2\pi) \quad \gamma(t) = \begin{pmatrix} Sin(t) \\ \cos(t) \\ t \end{pmatrix}$$

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The directional derivative

- Definition. Let $\Omega \in \mathbb{R}n$ be an open set, $f : \Omega \to \mathbb{R}$ continuous and $h \in \mathbb{R}n$, **$\|h\| = 1$** be a unit vector. Then the ***directional derivative*** $D_h f$ in the direction h is defined by

$$D_h f|_x := \frac{d}{dt} f(x+th)\,|_{t=0}$$

- if the right-hand side exists.
- ***Note:***
- The directional derivative is the derivative of f at x along the line segment joining x and $x + h$. It gives the slope of the tangent line of f at x in the direction of h.
- The directional derivative is a number.

The directional derivative

- Results.
- The tangent line of $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction h:
- $$t_{f,x;h}(s) = \begin{pmatrix} x + sh \\ f(x) + D_h f|_x s \end{pmatrix}, \qquad s \in \mathbb{R},$$
- If f is differentiable and the line segment is parametrized by $\gamma(t) = x + th$,

$$D_h f|_x = Df|_x h = \langle \nabla f(x), h \rangle.$$

$$\nabla f(x) := (J_f(x))^T = \begin{pmatrix} \frac{\partial f}{\partial x_1}\Big|_x \\ \vdots \\ \frac{\partial f}{\partial x_n}\Big|_x \end{pmatrix}$$

Exercise

- Find the directional derivative of the function $f(x,y)=\ln(x^2+2y^2)^{1/2}$

  at (1, 1) along the direction (2, 1).

The normal Derivative in R^2

Definition. Let $\Omega \subset \mathbb{R}^2$ be an open set, $f : \Omega \to \mathbb{R}$ and $\mathcal{C}$ a simple smooth C^2-curve in Ω. Let $p \in \mathcal{C}$ and $N(p)$ denote the normal vector at p. Then

$$\left.\frac{\partial f}{\partial n}\right|_p := D_{N(p)} f|_p$$

is called the ***normal derivative of f at p*** with respect to the curve $\mathcal{C}$.

Small Exercise

- Find the directional derivative of the function $f(x,y)=\ln(x^2+2y^2)^{1/2}$ along the parametrization $\gamma(t)=(t,t^2)$

The gradient

The *gradient* $\nabla f(x)$ is the transpose of the Jacobian.

- $\nabla f(x)$ points in the direction of the greatest directional derivative of f at x.

$$D_h f(x) = \langle \nabla f(x), h \rangle = |\nabla f(x)| \cos \angle(\nabla f(x), h)$$

- $\nabla f(x)$ is perpendicular to the contour line of f at x.

$$\langle \nabla f(x), h_0 \rangle = 0$$

The tangent plane

2.4.6. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $f\colon \Omega \to \mathbb{R}$ differentiable at $x_0 \in \Omega$. Then the equation

$$x_{n+1} = Tf(x; x_0), \qquad\qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

defines the ***tangent plane*** to the graph $\Gamma(f) \in \mathbb{R}^n \times \mathbb{R}$ of f at the point $(x_0, f(x_0)) \in \mathbb{R}^{n+1}$.

The tangent plane in R^3

- In R^3, the tangent plane is found by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

- We have two tangent vectors defined by

$$t_1 := \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix} \quad \text{and} \quad t_2 := \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

- We have a normal vector defined by

$$n = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix}$$

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The second Derivative

2.5.1. Definition. Let X, V be finite-dimensional normed vector spaces and $\Omega \subset X$ an open set. A function $f: \Omega \to V$ is said to be ***twice differentiable at*** $x \in \Omega$ if

- f is differentiable in an open ball $B_\varepsilon(x)$ around x and
- the function $Df: B_\varepsilon(x) \to \mathscr{L}(X, V)$ is differentiable at x.

We say that f is twice differentiable on Ω if f is twice differentiable at every $x \in \Omega$.

Actually, the second derivative is a map:

$$D(Df) =: D^2 f: \Omega \to \mathscr{L}(X, \mathscr{L}(X, V)).$$

And is found by:

$$Df|_{x+h} = Df|_x + D^2 f|_x h + o(h) \qquad \text{as } h \to 0.$$

Example

- Calculate the second, third and forth derivatives of the map,
- $\Phi : \mathrm{Mat}(n \times n; \mathrm{R}) \to \mathrm{Mat}(n \times n; \mathrm{R})$, $\Phi(A) = A^3$

The Hessian

The Hessian. For a differentiable potential function $f : \mathbb{R}^n \to \mathbb{R}$. The derivative is given by the Jacobian:

$$Df|_x = \begin{pmatrix} \left.\frac{\partial f}{\partial x_1}\right|_x & \cdots & \left.\frac{\partial f}{\partial x_n}\right|_x \end{pmatrix} \quad , \qquad Df|_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$$

and the second derivative is found by ***Hessian*** where

$$\mathrm{Hess}\, f(x) = D(\nabla f)|_x = \begin{pmatrix} \left.\frac{\partial^2 f}{\partial x_1 \partial x_1}\right|_x & \left.\frac{\partial^2 f}{\partial x_2 \partial x_1}\right|_x & \cdots & \left.\frac{\partial^2 f}{\partial x_n \partial x_1}\right|_x \\ \vdots & \vdots & & \vdots \\ \left.\frac{\partial^2 f}{\partial x_1 \partial x_n}\right|_x & \left.\frac{\partial^2 f}{\partial x_2 \partial x_n}\right|_x & \cdots & \left.\frac{\partial^2 f}{\partial x_n \partial x_n}\right|_x \end{pmatrix}$$

$$D^2 f|_x h = \mathrm{Hess}\, f(x) h$$

Bilinear Maps

Note that the expression $\langle \text{Hess}\, f(x)h, \tilde{h}\rangle$ is linear in both h and $\tilde{h}$; hence we can also regard the second derivative as a ***bilinear map***

$$D^2 f|_x \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (h, \tilde{h}) \mapsto \langle \text{Hess}\, f(x)h, \tilde{h}\rangle.$$

2.5.3. Definition. Let X, V be finite-dimensional normed vector spaces. The set of multilinear maps from X to V is denoted by

$$\mathscr{L}^{(n)}(X, V) := \Big\{ L \colon X \times \cdots \times X \to V \colon L \text{ linear in each component} \Big\}.$$

In the special case $V = \mathbb{R}$ an element of $\mathscr{L}^{(n)}(X, V)$ is called a ***multilinear form***.

Bilinear form in R^n

- Every linear map $L \in (\mathrm{R}^n)^*$ has the form $L = \langle z, \cdot\rangle$ for some $z \in \mathrm{R}^n$.
- Interpret an element $\boldsymbol{A} \in \mathbf{L}(\mathbf{R}^n, \mathbf{L}(\mathbf{R}^n, \mathbf{R}))$ as a linear map, $A: y \rightarrow L_y := \langle z_y, \cdot \rangle, z_y = A(y)$.
- A is actually a matrix A: $y \rightarrow z_y$.
- For every $y \in \mathrm{R}^n$ we obtain a linear map $\langle Ay, \cdot\rangle \in \mathrm{L}(\mathrm{R}^n, \mathrm{R})$.
- Then $L_y x = \langle Ay, x\rangle = L(x,y)$.

Schwarz's Theorem

2.5.5. Schwarz's Theorem. Let X, V be normed vector spaces and $\Omega \subset X$ an open set. Let $f \in C^2(\Omega, V)$. Then $D^2 f|_x \in \mathscr{L}^{(2)}(X \times X, V)$ is symmetric for all $x \in \Omega$, i.e.,

$$D^2 f(u, v) = D^2 f(v, u), \qquad \text{for all } u, v \in X.$$

- This implies that if *f* is **twice continuously differentiable**, the Hessian of *f* at *x* is symmetric.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

In other words, if f is twice continuously differentiable, the order of differentiation in the second-order partial derivatives does not matter.

Schwarz's Theorem

- Note: The Schwarz's theorem will hold if all the second-order partial derivatives **are continuous**.
- It will break if the function fails to differentiable partial derivative.
- Example:
- f(x,y)= $\begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}. & for(x,y)! = (0,0) \\ 0 & for(x,y) = (0,0) \end{cases}$
- Since the second partial derivative is not continuous at (0,0)
- Try to see that it is not symmetric

Schwarz's Theorem

- f(x,y)= $\begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}. & for(x,y)! = (0,0) \\ 0 & for(x,y) = (0,0) \end{cases}$

- Try to calculate

$$\frac{\partial^2}{\partial x \partial y} f|(_{0,0})$$

$$\frac{\partial^2}{\partial y \partial x} f|(_{0,0})$$

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Exercise 1

- Find the first and second derivatives of the following functions:
- 1. f :Mat($n \times n$;R)→R, f(A)=tr(A^2)
- 2. g :Mat($n \times n$;R)→R, f(A)=$(\text{tr}A)^2$

- Find the second derivative of the determinant of an invertible matrix A.

Det: Mat(n x n;R)->R

The Frenet–Serret formulas (in R^3)

$$
\begin{aligned}
\frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N}, \\
\frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\
\frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N},
\end{aligned}
$$

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \qquad \kappa = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

$$N(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|} \qquad \tau = \frac{\det(\gamma'(t), \gamma''(t), \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

$$B(t) = T \times N$$

Reference

Hohberger, Horst. “VV285_main.pdf”

Serge Lang, Calculus of Several Variables, 1987

Chen, Xiwen “RC_7.pdf”,2018

Yuan, Jian “RC_7.pdf”,2018

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Thank You

Have a nice day!