JOINT INSTITUTE
交大密西根学院

VV 285
RC 9

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Content

- The Riemann Integral and Measurable Sets
- Integration in Practice
- Exercise

N-cuboid

3.3.1. Definition. Let a_k, b_k, $k = 1, \ldots, n$ be pairs of numbers with $a_k < b_k$. Then the set $Q \subset \mathbb{R}^n$ given by

$$\begin{aligned} Q &= [a_1, b_1] \times \cdots \times [a_n, b_n] \\ &= \{x \in \mathbb{R}^n \colon x_k \in [a_k, b_k],\ k = 1, \ldots, n\} \end{aligned}$$

is called an ***n-cuboid***. We define the volume of Q to be

$$|Q| := \prod_{k=1}^{n} (b_k - a_k).$$

We will denote the set of all n-cuboids by $\mathcal{Q}_n$.

Upper and Lower Volumes of Sets

3.3.3. Definition. Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty set. We define the ***outer*** and ***inner volume*** of Ω by

$$\overline{V}(\Omega) := \inf\left\{\sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\ Q_0, \dots, Q_r \in \mathcal{Q}_n,\ \Omega \subset \bigcup_{k=1}^{r} Q_k\right\},$$

$$\underline{V}(\Omega) := \sup\left\{\sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\ Q_0, \dots, Q_r \in \mathcal{Q}_n,\ \Omega \supset \bigcup_{k=1}^{r} Q_k,\ \bigcap_{k=1}^{r} Q_k = \emptyset\right\}.$$

It is easy to see that $0 \le \underline{V}(\Omega) \le \overline{V}(\Omega)$.

Measurable Sets

3.3.4. Definition. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is said to be ***(Jordan) measurable*** if either

(i) $\overline{V}(\Omega) = 0$ or

(ii) $\overline{V}(\Omega) = \underline{V}(\Omega)$.

In the first case, we say that Ω has (Jordan) ***measure zero***, in the second case we say that

$$|\Omega| := \overline{V}(\Omega) = \underline{V}(\Omega)$$

is the Jordan measure of Ω.

Measurable Sets

3.3.5. Examples.

(i) A set $\{x\}$ consisting of a single point $x \in \mathbb{R}^n$ is a set of measure zero.

(ii) A subset of $\mathbb{R}^n$ consisting of a finite number of single points is a set of measure zero.

(iii) A curve of finite length $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, is a set of measure zero.

(iv) A bounded section of a plane in $\mathbb{R}^3$ is a set of measure zero.

(v) The set of rational numbers in the interval $[0, 1]$ has measure zero.

(vi) The set of irrational numbers in the interval $[0, 1]$ is not (Jordan) measurable.

Step Functions of Cuboids

3.3.8. Definition. A ***partition*** P of an n-cuboid $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a tuple $P = (P_1, \dots, P_n)$ such that $P_k = (a_{k0}, \dots, a_{km_k})$ is a partition of the interval $[a_k, b_k]$.

The partition P of Q induces cuboids of the form

$$Q_{j_1 j_2 \dots j_n} := [a_{1(j_1-1)}, a_{1j_1}] \times \cdots \times [a_{n(j_n-1)}, a_{nj_n}] \subset Q.$$

3.3.9. Definition. Let $Q \subset \mathbb{R}^n$ be an n-cuboid. A function $f\colon Q \to \mathbb{R}$ is called a ***step function with respect to a partition*** P if there exist numbers $y_{j_1 j_2 \dots j_n} \in \mathbb{R}$ such that $f(x) = y_{j_1 j_2 \dots j_n}$ whenever $x \in \operatorname{int} Q_{j_1 j_2 \dots j_n}$, $j_k = 1, \dots, m_k$, $k = 1, \dots, n$.

Step Functions of Cuboids

3.3.11. Theorem. Let $Q \subset \mathbb{R}^n$ be a cuboid and $f : Q \to \mathbb{R}$ a step function with respect to some partition P of Q, then the ***integral*** is

$$\int_Q f := I_P(f) = \sum_{\substack{j_1=1,\ldots,m_1 \\ \vdots \\ j_n=1,\ldots,m_n}} |Q_{j_1\ldots j_n}| \cdot y_{j_1\ldots j_n}$$

Darboux Integral

3.3.12. Definition. Let $Q \subset \mathbb{R}^n$ be an n-cuboid and f a bounded real function on Q. let $\mathscr{U}_f$ denote the set of all step functions u on Q such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on Q such that $v \leq f$. The function f is then said to be ***(Darboux)-integrable*** if

$$\sup_{v \in \mathscr{L}_f} \int_Q v = \inf_{u \in \mathscr{U}_f} \int_Q u.$$

In this case, the ***(Darboux) integral of f over Q***, $\int_Q f$, is defined to be this common value.

Rieman Integral

3.3.13. Theorem. A bounded function $f\colon Q \to \mathbb{R}$ is Riemann-integrable if and only if for every $\varepsilon > 0$ there exist step functions u_ε and v_ε such that $v_\varepsilon \le f \le u_\varepsilon$ and

$$\int_Q u_\varepsilon - \int_Q v_\varepsilon \le \varepsilon.$$

3.3.14. Proposition. Let $Q \subset \mathbb{R}^n$ be an n-cuboid and $f\colon Q \to \mathbb{R}$ be bounded and continuous almost everywhere. Then f is (Riemann) integrable.

Rieman Integral

3.3.16. Lemma. Let $\Omega \in \mathbb{R}^n$ be a bounded set. Then Ω is Jordan-measurable if and only if its boundary $\partial\Omega$ has Jordan measure zero.

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3.3.17. Corollary. Let $\Omega \subset \mathbb{R}^n$ be a bounded Jordan-measurable set and let $f : \Omega \to \mathbb{R}$ be continuous a.e. Then f is integrable on Ω.

Rieman Integral

3.3.18. Lemma.

(i) Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Then
$$|\Omega| = \int_\Omega 1.$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a set of measure zero and and $f\colon \Omega \to \mathbb{R}$ some function that is integrable on Ω. Then $\int_\Omega f = 0$.

(iii) Let $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \Omega$ be measurable sets and $f\colon \mathbb{R}^n \to \mathbb{R}$ integrable on Ω. Then f is also integrable on Ω'.

(iv) Let $\Omega, \Omega' \subset \mathbb{R}^n$ measurable sets and $f\colon \mathbb{R}^n \to \mathbb{R}$ integrable on both of them. Then f is integrable on $\Omega \cup \Omega'$ and
$$\int_{\Omega\cup\Omega'} f = \int_\Omega f + \int_{\Omega'} f - \int_{\Omega\cap\Omega'} f.$$

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Practical Integration over Cuboids

3.4.1. Fubini's Theorem. Let Q_1 be and n_1-cuboid and Q_2 an n_2-cuboid so that $Q := Q_1 \times Q_2 \subset \mathbb{R}^{n_1+n_2}$ is an (n_1+n_2)-cuboid. Assume that $f\colon Q \to \mathbb{R}$ is integrable on Q and that for every $x \in Q_1$ the integral

$$g(x) = \int_{Q_2} f(x, \cdot\,)$$

exists. Then

$$\int_Q f = \int_{Q_1 \times Q_2} f = \int_{Q_1} g = \int_{Q_1} \Big(\int_{Q_2} f \Big).$$

Ordinate and Simple Regions in $\mathbb{R}^2$

3.4.3. Definition. A set $D \subset \mathbb{R}^2$ is called an ***ordinate region with respect to*** x_2, if there exists an interval $I \subset \mathbb{R}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi_2 \colon I \to \mathbb{R}$ such that

$$D = \big\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \in I,\ \varphi_1(x_1) \le x_2 \le \varphi_2(x_1)\big\}.$$

If the role of x_1 and x_2 above is interchanged, we say that D is ***an ordinate region with respect to*** x_1.

If $D \subset \mathbb{R}^2$ is an ordinate region both with respect to x_1 and x_2, we say that D is a ***simple region***.

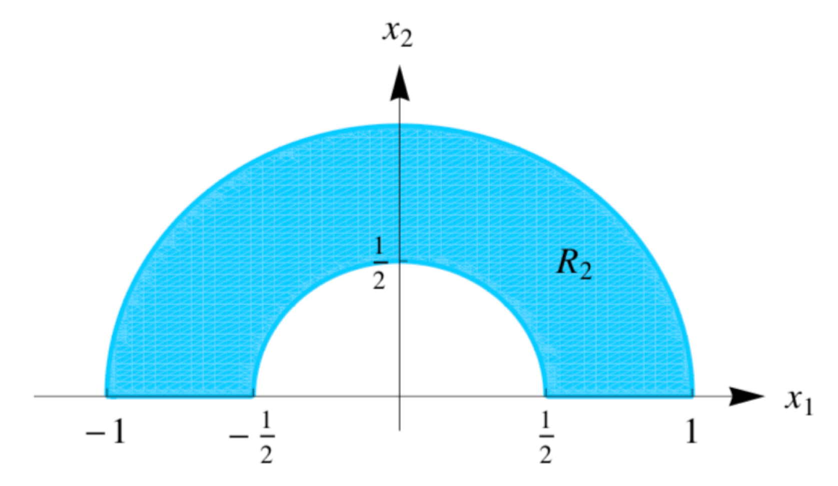
Ordinate Regions in R^n

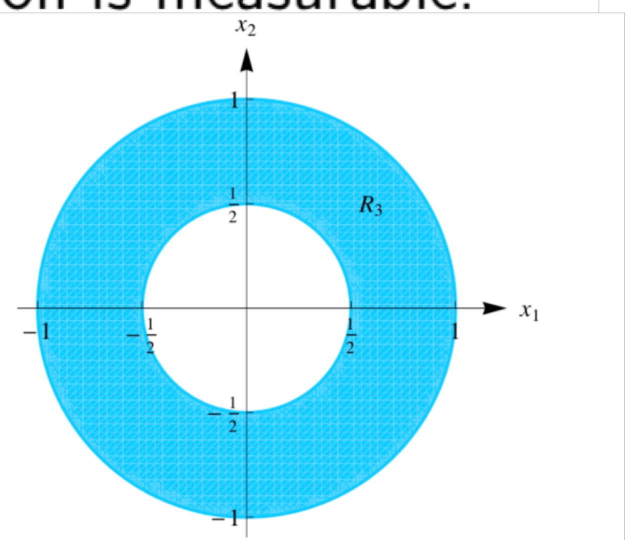
3.4.7. Definition. A subset $U \subset \mathbb{R}^n$ is said to be an ***ordinate region (with respect to x_k)*** if there exists a measurable set $\Omega \subset \mathbb{R}^{n-1}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi_2 \colon \Omega \to \mathbb{R}$, such that

$$U = \{(x \in \mathbb{R}^n : x \in \Omega,\ \varphi_1(\hat{x}^{(k)}) \le x_k \le \varphi_2(\hat{x}^{(k)})\}.$$

If U is an ordinate region with respect to each x_k, $k = 1, \dots, n$, it is said to be a ***simple region***.

3.4.8. Lemma. Any ordinate region is measurable.





Substitution Rule

3.4.12. Substitution Rule. Let $\Omega \subset \mathbb{R}^n$ be open and $g\colon \Omega \to \mathbb{R}^n$ injective and continuously differentiable. Suppose that $\det J_g(y) \neq 0$ for all $y \in \Omega$. Let K be a compact measurable subset of Ω. The $g(K)$ is compact and measurable and if $f\colon g(K) \to \mathbb{R}$ is integrable, then

$$\int_{g(K)} f(x)\, dx = \int_K f(g(y)) \cdot |\det J_g(y)|\, dy.$$

Coordinate System

(i) Polar coordinates in $\mathbb{R}^2$ are defined by a map

$$\Phi\colon (0,\infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}, \qquad (r, \phi) \mapsto (x, y)$$

where

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Note that this map is bijective and even C^∞ in the interior of its domain. An alternative (but rarely used) version of polar coordinates would map $x = r\sin\phi$, $y = r\cos\phi$. This simply corresponds to a different geometrical interpretation of the angle ϕ. In any case,

$$|\det J_{\Phi}(r,\phi)| = \left|\det \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}\right| = r$$

Coordinate System

(ii) Cylindrical coordinates in $\mathbb{R}^3$ are given through a map

$$\Phi\colon (0,\infty)\times[0,2\pi)\times\mathbb{R}\to\mathbb{R}^3\setminus\{0\}, \qquad (r,\phi,\zeta)\mapsto(x,y,z)$$

defined by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z=\zeta$$

In this case,

$$|\det J_\Phi(r,\phi,\zeta)| = \left|\det\begin{pmatrix}\cos\phi & -r\sin\phi & 0\\ \sin\phi & r\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\right| = r$$

Coordinate System

(iii) Spherical coordinates in $\mathbb{R}^3$ are often defined through a map

$$\Phi\colon (0,\infty)\times[0,2\pi)\times(0,\pi)\to\mathbb{R}^3\setminus\{0\},\quad (r,\phi,\theta)\mapsto(x,y,z),$$
$$x = r\cos\phi\sin\theta,\quad y = r\sin\phi\sin\theta,\quad z = r\cos\theta.$$

Of course, there is a certain freedom in defining θ and ϕ, so there are alternative formulations. The modulus of the determinant of the Jacobian is given by

$$\begin{aligned}|\det J_\Phi(r,\phi,\theta)| &= \left|\det\begin{pmatrix}\cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta\\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta\\ \cos\theta & 0 & -r\sin\theta\end{pmatrix}\right|\\ &= r^2\sin\theta\end{aligned}$$

Coordinate System

(iv) In $\mathbb{R}^n$, we can define spherical coordinates by

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{n-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1} \\
x_n &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

with $r > 0$ and $0 < \theta_k < \pi$, $k = 1, \ldots, n-2$, and $0 < \theta_{n-1} < 2\pi$.

Here, the determinant of the Jacobian can be shown to be

$$|\det J_\Phi(r, \theta_1, \ldots, \theta_{n-1})| = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\ldots\sin\theta_{n-1}.$$

Small Exercise-Gauss Intergral

3.4.17. Example. Our aim is to prove that the Gauß integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

Green's Theorem

3.4.18. Green's Theorem. Let $R \subset \mathbb{R}^2$ be a bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\int_{\partial R^*} F \, \mathrm{d}\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathrm{d}x$$

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- **The Riemann Integral and Measurable Sets**
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Exercise 1: Evaluate the integral

- Problem 1&2:See Black Board;

Exercise 2: Green's Theorem

- See Black Board:

Exercise 3: Thinking Problem

- Calculate the volume V(B) of the n-dimensional ball:
- $B = \{x = (x^1, x^2 \ldots x^n) : \|x\| \leq R\}$

Reference

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Thank You

Have fun in Final Exam