



JOINT INSTITUTE
交大密西根学院

VV 285
RC 8

SUN YAN

Content



- **Vector Fields and Line Integrals**
- **Circulation and Flux**
- **Exercise**

The line integral of a potential function



3.1.1. Definition. Let $\Omega \subset \mathbb{R}^n$, $f: \Omega \rightarrow \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma: I \rightarrow \mathcal{C}$. We then define the **line integral of the potential f along \mathcal{C}^*** by

$$\int_{\mathcal{C}^*} f \, ds := \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)| \, dt$$

- Note : It is independent of the parametrization you choose.
- The symbol “ds” can be interpreted geometrically as a scalar line element and one often writes
- $ds = |\gamma'(t)| \, dt$

The line integral of a potential function-Example



- Suppose a wire is in the shape of a circle, C^* :
- $x^2+y^2 = 1$. The density ρ (linear density) at point (x,y) is $\rho(x,y) = 1+xy$. Calculate its total mass.

Vector Field



3.1.4. Definition. Let $\Omega \subset \mathbb{R}^n$. Then a function $F: \Omega \rightarrow \mathbb{R}^n$,

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix}.$$

is called a **vector field** on Ω .

3.1.5. Example. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a potential function. Then the **gradient field of f** given by

$$F: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad F(x) = \nabla f(x),$$

associates to every $x \in \mathbb{R}^n$ the direction of largest slope of f .

The line integral of Vector Field



3.1.8. Definition. Let $\Omega \subset \mathbb{R}^n$, $F: \Omega \rightarrow \mathbb{R}$ be a continuous vector field and $C^* \subset \Omega$ an oriented open, smooth curve in \mathbb{R}^n . We then define the **line integral of the vector field F along C^*** by

$$\int_{C^*} F d\vec{s} := \int_{C^*} \langle F, T \rangle ds \quad (3.1.5)$$

- Note:
- The line integral of a vector field does not depend on parametrization of C^* .
- The **vectorial line element** is given by
 - $d\vec{s} = \gamma'(t) dt$
- To calculate line integral using parametrization $\gamma: I \rightarrow C$

$$\int_{C^*} F d\vec{s} = \int_I \langle F \circ \gamma(t), \gamma'(t) \rangle dt$$

The line integral of Vector Field



If we calculate the line integral using a concrete parametrization $\gamma: I \rightarrow \mathcal{C}$, we obtain

$$\begin{aligned}\int_{\mathcal{C}^*} F d\vec{s} &= \int_{\mathcal{C}^*} \langle F, T \rangle ds = \int_I \langle F \circ \gamma(t), T \circ \gamma(t) \rangle \|\gamma'(t)\| dt \\ &= \int_I \left\langle F \circ \gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right\rangle \|\gamma'(t)\| dt \\ &= \int_I \langle F \circ \gamma(t), \gamma'(t) \rangle dt\end{aligned}\tag{3.1.6}$$

The line integral of a vector field- Toy Example in your physics hw.



Calculate work done by the force fields

(A) $\mathbf{F}_1(\mathbf{r}) = (x^2z, -xy, 5), \quad$ (B) $\mathbf{F}_2(\mathbf{r}) = (-2x - yz, z - xz, y - xy),$

acting upon a particle that moves from $(-1, 0, 0)$ to $(1, 0, 0)$ along

- (a) the x axis,
- (b) an arc of the circle $x^2 + y^2 = 1$, so that $y \geq 0$,
- (c) the curve Γ defined by parametric equations $x(t) = t$, $y(t) \equiv 0$, $z(t) = t^2 - 1$, where $-1 \leq t \leq 1$.

Potential field



3.1.11. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. A vector field $F: \Omega \rightarrow \mathbb{R}^n$ is said to be a **potential field** if there exists a differentiable potential function $U: \Omega \rightarrow \mathbb{R}$ such that

$$F(x) = \nabla U(x).$$

Supposing that the initial point of the curve is p_{initial} and the final point is p_{final} , we have from the fundamental theorem of calculus

$$\int_{\mathcal{C}^*} F d\vec{s} = \int_I (U \circ \gamma)'(t) dt = U(p_{\text{final}}) - U(p_{\text{initial}}).$$

3.1.13. Lemma. Let $\Omega \subset \mathbb{R}^n$ be open, $F: \Omega \rightarrow \mathbb{R}^n$ a potential field and $\mathcal{C} \subset \Omega$ a closed curve. Then

$$\oint_{\mathcal{C}} F d\vec{s} = 0.$$

Conservative field



3.1.14. **Definition.** Let $\Omega \subset \mathbb{R}^n$ be open and $F: \Omega \rightarrow \mathbb{R}^n$ a vector field. If the integral along any open curve \mathcal{C}^* depends only on the initial and final points or, equivalently,

$$\oint_{\mathcal{C}} F \cdot d\vec{s} = 0 \quad \text{for any closed curve } \mathcal{C},$$

then F is called **conservative**.

3.1.15. **Remark.** We note explicitly that every potential field is a conservative field.

In fact, under certain conditions a conservative field is also a potential field.

3.1.16. **Definition.** Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be **(pathwise) connected** if for any two points in Ω there exists an open curve within Ω joining the two points.

Conservative field



3.1.15. Remark. We note explicitly that every potential field is a conservative field.

In fact, under certain conditions a conservative field is also a potential field.

3.1.16. Definition. Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be **(pathwise) connected** if for any two points in Ω there exists an open curve within Ω joining the two points.

3.1.17. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F: \Omega \rightarrow \mathbb{R}^n$ is a continuous, conservative field. Then F is a potential field.

■ Proof of 3.1.17

Simply Connected Set

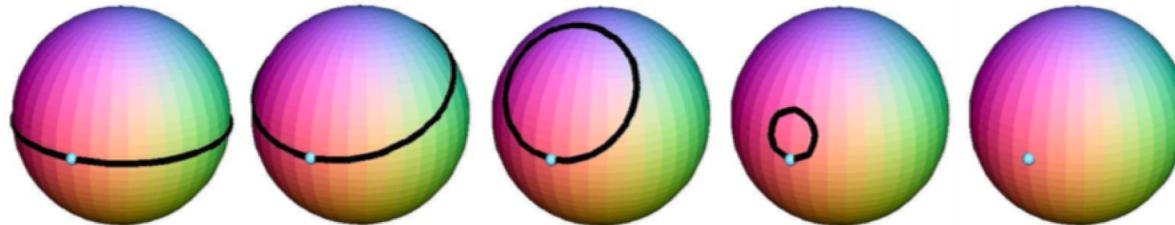


3.1.23. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set.

- (i) A closed curve $\mathcal{C} \subset \Omega$ given as the image of a map $g: S^1 \rightarrow \mathcal{C}$ is said to be **contractible to a point** if there exists a continuous function $G: D \rightarrow \Omega$ such that $G|_{S^1} = g$.
- (ii) The set Ω is said to be **simply connected** if it is connected and every closed curve in Ω is contractible to a point.

Example.

1. $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.
2. $\mathbb{R}^3 \setminus \{0\}$ is simply connected.



Salix alba. *A homotopy of a circle around a sphere can be reduced to single point.* 2006. Wikipedia. Wikimedia Foundation. Web. 12 July 2012

Simply Connected Set



3.1.18. **Lemma.** Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F: \Omega \rightarrow \mathbb{R}^n$ is continuously differentiable. Then F is a potential field only if for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}. \quad (3.1.9)$$

3.1.21. **Theorem.** Let $\Omega \subset \mathbb{R}^n$ be a **simply connected** open set and suppose that $F: \Omega \rightarrow \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

then F is a potential field.

Finding Potential



Usually We find potential in \mathbb{R}^2

Steps:

1. Check potential fields.(Continuous+Conervative)
2. Integrate with respect to x_1 .
3. Integrate with respect to x_2 .

Differential Form



3.1.25. Definition. Let $F_1, \dots, F_n: \mathbb{R}^n \rightarrow \mathbb{R}$ be scalar functions. Then

$$\alpha = F_1 dx_1 + \cdots + F_n dx_n$$

is said to be a ***differential one-form***.

Try Example



Example. Denote by $H = \{(x,y) : y > 0\} \subset \mathbb{R}^2$ the upper half-space of \mathbb{R}^2 and consider the two vector fields $F, G : H \rightarrow \mathbb{R}^2$ with $(x,y) \in H$,

$$F(x,y) = (4x^2 + 4y^2, 8xy - \ln y + e^y),$$

$$G(x,y) = (x + 2xy - y^2, -2xy + x^2)$$

Q1: Are the two fields conservative?

Q2: If so, calculate the potential field.

Content



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Circulation and Flux



3.2.2. Definition. Let $\Omega \subset \mathbb{R}^2$ be an open set, $F: \Omega \rightarrow \mathbb{R}^2$ a continuously differentiable vector field and \mathcal{C}^* a positively oriented closed curve in \mathbb{R}^2 . Then

$$\int_{\mathcal{C}^*} \langle F, T \rangle \, ds \quad (3.2.1)$$

is called the (total) **circulation** of F along \mathcal{C} and

$$\int_{\mathcal{C}^*} \langle F, N \rangle \, ds \quad (3.2.2)$$

is called the (total) **flux** of F through \mathcal{C} .

Here T denotes the usual unit tangent vector to \mathcal{C} . However, N is taken to the “unit normal vector” in the sense that

1. $\|N\| = 1$,
2. $\langle N, T \rangle = 0$,
3. N points **outwards** from the region bounded by \mathcal{C} .

3.2.1. Example. Let $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ be the unit circle, bounding the unit disc. Consider the two vector fields

$$F, G : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad G(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

The unit tangent vector to S^1 at (x_1, x_2) is given by $T = (-x_2, x_1)$, so

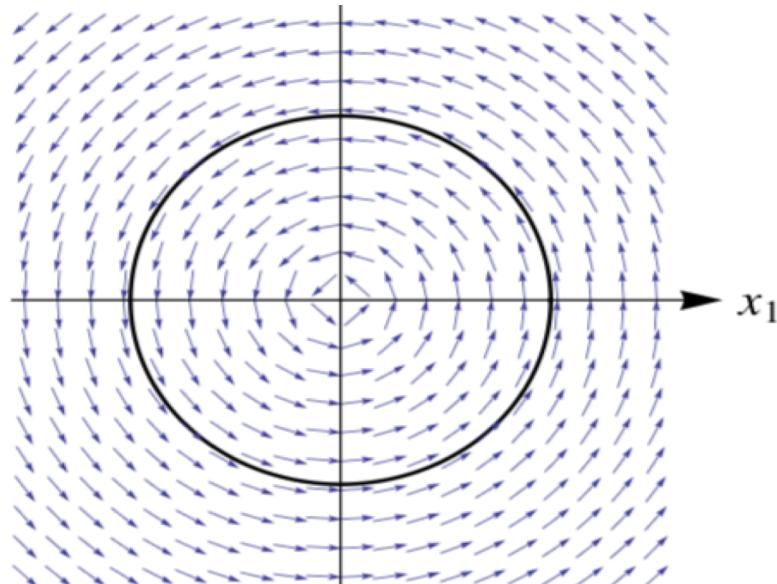
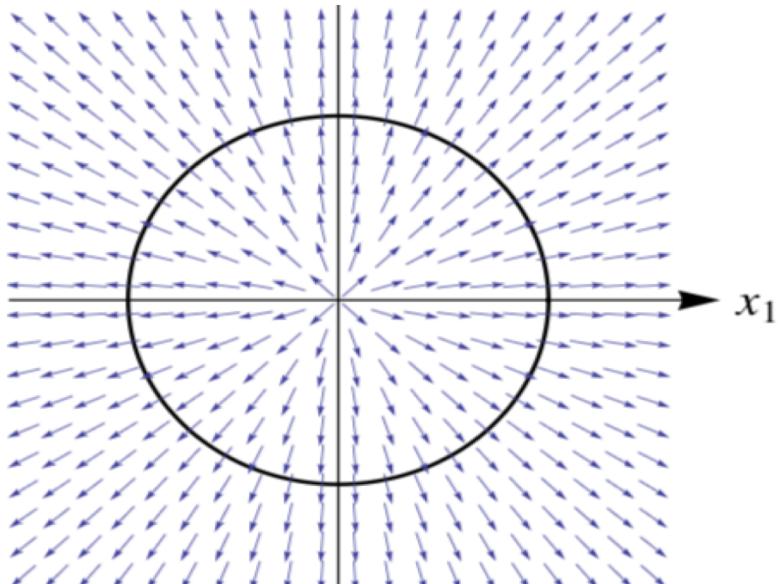
$$\langle T, F \rangle|_{S^1} = -x_2 x_1 + x_1 x_2 = 0,$$

$$\langle T, G \rangle|_{S^1} = -x_2(-x_2) + x_1 x_1 = 1.$$

The unit normal vector at $x \in S^1$ is given by $N = -x$, so

$$\langle N, F \rangle|_{S^1} = -x_1 x_1 + -x_2 x_2 = -1,$$

$$\langle N, G \rangle|_{S^1} = x_1(-x_2) + x_2 x_1 = 0.$$



Flux density and divergence



3.2.5. Definition. Let $\Omega \subset \mathbb{R}^n$ and $F: \Omega \rightarrow \mathbb{R}^n$ be a continuously differentiable vector field. Then

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

is called the ***divergence*** of F .

The Circulation Density — Rotation / Curl



3.2.6. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F: \Omega \rightarrow \mathbb{R}^n$ a continuously differentiable vector field. Then the anti-symmetric, bilinear form

$$\text{rot } F|_x: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad \text{rot } F|_x(u, v) := \langle DF|_x u, v \rangle - \langle DF|_x v, u \rangle \quad (3.2.3)$$

is called the **rotation** (in mainland Europe) or **curl** (in anglo-saxon countries) of the vector field F at $x \in \mathbb{R}^n$.

3.2.7. Theorem. Let $\Omega \subset \mathbb{R}^2$ be open and $F: \Omega \rightarrow \mathbb{R}^2$ a continuously differentiable vector field. Then there exists a uniquely defined continuous potential function $\text{rot } F: \Omega \rightarrow \mathbb{R}$ such that

$$\text{rot } F|_x(u, v) = \text{rot } F(x) \cdot \det(u, v). \quad (3.2.5)$$

The rotation in \mathbf{R}^2 and \mathbf{R}^3



In \mathbf{R}^2 , a scalar function

$$\text{rot } F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$

In \mathbf{R}^3 , a scalar function

$$\text{rot } F|_x(u, v) = \det(\text{rot } F(x), u, v) = \langle \text{rot } F(x), u \times v \rangle.$$

$$\text{rot } F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}.$$

Note: it needs to be a **continuously differentiable** vector field on a **open set**.

Sample Exercise



Given an electric field $E = c(2bxy, x^2 + ay^2)$, $a, b, c \in \mathbb{R}$, determine values for a and b such that $\operatorname{div} E = 0$ and $\operatorname{rot} E = 0$. Then, find a potential function V for E with these values of a and b .

Irrational Fields



A continuously differentiable field $F : \Omega \rightarrow \mathbb{R}^n$ such that $\text{rot } F|_x = 0$ for all $x \in \Omega$ is **irrotational**. Then

$$(DF|_x)^T = DF|_x$$

Note: A potential field is irrotational.

Fluid Statistics



For potential flows, we have that

$$F = \nabla U \quad \operatorname{div} F = 0.$$

Then, we have

$$\operatorname{div}(\nabla U) = \operatorname{div} \begin{pmatrix} \frac{\partial U}{\partial x_1} \\ \vdots \\ \frac{\partial U}{\partial x_n} \end{pmatrix} = \frac{\partial^2 U}{\partial x_1^2} + \cdots + \frac{\partial^2 U}{\partial x_n^2} = \Delta U = 0.$$

Triangle Calculus



$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

$$\text{div } F = \langle \nabla, F \rangle = \left\langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}, \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \right\rangle = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Divergence

$$\text{rot } F = \nabla \times F(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Rotation

$$\begin{aligned} \langle \nabla, \nabla \rangle &= \left(\frac{\partial}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial}{\partial x_n} \right)^2 \\ &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \Delta \end{aligned}$$

Laplace Notation

Content



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Exercise 1



- Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
- Prove that the vector field $F : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3$, $F(x) = g(\|x\|)x$ is conservative.

Exercise 2



Consider a generalized vector space with dimension n ,

$$F = \frac{cx}{\|x\|^n}$$

where $c \in \mathbb{R}$ is constant. Prove that $\operatorname{div} F = 0$, $\operatorname{rot} F = 0$.

Reference



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Thank You

Have a nice day!