VV 285
MID 1 Part II
Inner Product Space

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Overview

- Inner Product Space (slide 77-78)
- Induced Norm (slide 79-81)
- Orthogonality\Orthonormal Basis (slide 85-87)
- Projection (slide 92)
- Gram-Schmidt Orthonormalization (slide 100-103)

Definition-Inner Product

- Whether a map $\langle\cdot,\cdot\rangle : V \times V \to F$ can be a inner product:
- Verify for all $u,v,w \in V$ and all $\lambda \in F$:

1. $\langle v,v\rangle \geq 0$ and $\langle v,v\rangle = 0$ if and only if $v = 0$
2. $\langle u,v+w\rangle = \langle u,v\rangle+\langle u,w\rangle$
3. $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$
4. $\langle u,v\rangle = \overline{\langle v,u\rangle}$

We call the pair $(V,\langle\cdot,\cdot\rangle)$ inner product space;

Definition-Induced Norm

1.3.4. Definition. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The map

$$\| \cdot \| \colon V \to \mathbb{R}, \qquad \|v\| = \sqrt{\langle v, v \rangle}$$

is called the ***induced norm*** on V.

1.3.6. Cauchy-Schwarz Inequality. Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product vector space. Then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\| \qquad \text{for all } u, v \in V$$

where $\| \cdot \|$ is the induced norm.

1.3.7. Corollary. The induced norm is actually a norm, i.e., it satisfies

(i) $\|v\| \geq 0$, $\|v\| = 0 \Leftrightarrow v = 0$,
(ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$,
(iii) $\|u + v\| \leq \|u\| + \|v\|$

for all $u, v \in V$ and $\lambda \in \mathbb{F}$.

Definition-Orthogonal and Orthonormal

1.3.11. Definition. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space.

(i) Two vectors $u, v \in V$ are called ***orthogonal*** or ***perpendicular*** if $\langle u, v \rangle = 0$. We then write $u \perp v$.

(ii) We call

$$M^{\perp} := \Big\{ v \in V : \underset{m \in M}{\forall} \langle m, v \rangle = 0 \Big\}$$

the ***orthogonal complement*** of a set $M \subset V$.

For short, we sometimes write $v \perp M$ instead of $v \in M^{\perp}$ or $v \perp m$ for all $m \in M$.

iv) Let $A \in \mathrm{Mat}(n \times n, \mathbb{C})$. Show that

$$(\operatorname{ran} A)^{\perp} = \ker A^*, \qquad\qquad (\ker A)^{\perp} = \operatorname{ran} A^*.$$

(2 Marks)

Definition-Orthogonal and Orthonormal

1.3.14. Definition. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space. A tuple of vectors $(v_1, \dots, v_r) \subset V$ is called a ***(finite) orthonormal system*** if

$$\langle v_j, v_k \rangle = \delta_{jk} := \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \qquad j, k = 1, \dots, r,$$

i.e., if $\|v_k\| = 1$ and $v_j \perp v_k$ for $j \neq k$.

Exercise 2.2

Let

$$U = \{x \in \mathbb{R}^4 : x_1 + x_2 + x_3 = 0,\ x_1 + 3x_2 = x_4\}, \qquad V = \{x \in \mathbb{R}^4 : x_1 = x_4\}.$$

ii) Give orthonormal bases for U, V and $U + V$.
(2 Marks)

Definition-Projection

1.3.21. Projection Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be a (possibly infinite-dimensional) inner product vector space and $(e_1, \ldots, e_r)$, $r \in \mathbb{N}$, be an orthonormal system in V. Denote $U := \operatorname{span}\{e_1, \ldots, e_r\}$.

Then for every $v \in V$ there exists a unique representation

$$v = u + w \qquad\qquad \text{where } u \in U \text{ and } w \in U^\perp$$

and $u = \sum_{i=1}^{r} \langle e_i, v\rangle e_i$, $w := v - u$.

1.3.22. Definition. The vector

$$\pi_U v := \sum_{i=1}^{r} \langle e_i, v\rangle e_i$$

is called the ***orthogonal projection of v onto U***. The projection theorem essentially states that $\pi_U v$ always exists and is independent of the choice of the orthonormal system (it depends only on the span U of the system).

Gram-Schmidt Orthonormalization

We set

$$w_1 := \frac{v_1}{\|v_1\|}$$

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\|}, \qquad k = 2, \dots, n,$$

And hence obtain an orthonormal system as desired

Think about it.

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

Reference

Hohberger, Horst. “VV285_main.pdf”

Axler, Sheldon. *Linear Algebra Done Right*. Springer, 2017.

Lang, Serge. *Linear Algebra,*1987

Thank You