VV 285
RC 6

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Content

- Curves in Vector Spaces
- Exercise

Curves

Definition. Let V be a finite-dimensional vector space and $I \subset \mathbb{R}$ an interval.

- ***Curve:*** a set $C \subset V$ with a continuous, surjective and **locally** injective map $\gamma : I \to C$.
- ***Parametrization:*** the map γ.
- ***Parametrized curve:*** C together with its parametrization γ.
- ***Simple curve:*** γ is **globally** injective.
- ***Closed:*** $\lim_{t \to a} \gamma(t) = \lim_{t \to b} \gamma(t)$.
- ***Open:*** $x := \lim_{t \to a} \gamma(t), y := \lim_{t \to b} \gamma(t), x \neq y$. x is the ***initial point*** and y is the ***final point***.

Reparametrization of Curves

Definition. Suppose $C \subset V$ is a curve with parametrization $\gamma : I \to C$.

Reparametrization: a continuous, bijective map $r : J \to I$.

Orientation-preserving r: r is increasing.

Orientation-reversing r: r is decreasing.

Curves Examples

2.3.11. Example. Consider the unit circle S^1 of Example 2.3.5 with parametrizations

$$\gamma \colon [0, 2\pi] \to S^1, \qquad \gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix},$$

$$\widetilde{\gamma} \colon [0, 1] \to S^1, \qquad \widetilde{\gamma}(t) = \begin{pmatrix} \cos(2\pi t) \\ -\sin(2\pi t) \end{pmatrix}.$$

Reparametrization of Curves

A ***curve in polar coordinates*** is parametrized by

$$\gamma(t) = \begin{pmatrix} f(t)\cos t \\ f(t)\sin t \end{pmatrix},$$

Where $f : \mathbb{R} \to \mathbb{R}$.

Example. See Blackboard.

Smoth of Curves

Definition. A curve C $\subset V$ with parametrization $\gamma : I \to$ C is ***smooth*** if

1. γ is continuously differentiable on int I and
2. $D\gamma|_{t.}$ $!$=0 for all $t \in$ int I.

Definition. A ***smooth parametrization*** is

1. continuously differentiable and
2. the derivative is non-vanishing in the interior of its domain.

Tangent Line and Tangent Vector

In fact, if $\mathscr{C} \subset \mathbb{R}^n$ is a curve and $p = \gamma(t_0) \in \mathscr{C}$, then

$$T_p\mathscr{C} = \{x \in \mathbb{R}^n \colon x = \gamma(t_0) + \gamma'(t_0)t \colon t \in \mathbb{R}\}$$

gives the ***tangent line*** to Γ at p.

2.3.19. Definition. Let $\mathscr{C}^* \subset \mathbb{R}^n$ be an oriented smooth curve and $p \in \mathscr{C}^*$. Let $\gamma \colon I \to \mathbb{R}^n$ be a parametrization of $\mathscr{C}^*$. Then we define the ***unit tangent vector*** to $\mathscr{C}^*$ at $p = \gamma(t)$ by

$$T \circ \gamma(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \qquad t \in \operatorname{int} I. \tag{2.3.4}$$

This defines the ***tangent vector field*** $T \colon \mathscr{C}^* \to \mathbb{R}^n$ on $\mathscr{C}$.

It does not depend on the parametrization except the direction

Tangent Line and Tangent Vector

It does not depend on the parametrization except the direction

In fact, suppose $\gamma\colon I \to \mathscr{C}$, $\widetilde{\gamma}\colon J \to \mathscr{C}$ are two smooth parametrizations connected by a reparametrization $r\colon J \to I$ so that $\widetilde{\gamma} = \gamma \circ r$. Let $p \in \mathscr{C}$ satisfy $p = \gamma(t) = \widetilde{\gamma}(\tau)$, $t = r(\tau)$.

Then

$$\widetilde{\gamma}'(\tau) = \frac{d}{d\tau}\gamma(r(\tau)) = \gamma'(r(\tau))r'(\tau) = \gamma'(t)r'(\tau).$$

$$T \circ \widetilde{\gamma}(\tau) = \frac{\widetilde{\gamma}'(\tau)}{\|\widetilde{\gamma}'(\tau)\|} = \frac{r'(\tau)}{|r'(\tau)|}\frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{r'(\tau)}{|r'(\tau)|}T \circ \gamma(t).$$

The normal vector and a curve

2.3.21. Definition. Let $\mathscr{C} \subset V$ be a smooth C^2-curve. Let $\gamma\colon I \to V$ be a smooth C^2-parametrization of $\mathscr{C}$. Then the unit normal vector $N\colon \mathscr{C} \to \mathbb{R}$ is defined by

$$N \circ \gamma(t) := \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}, \qquad t \in \operatorname{int} I. \tag{2.3.8}$$

Note. The unit normal vector does not depend on γ on

magnitude

orientation

Curve Length

2.3.25. Theorem. $\mathcal{C} \subset V$ is a smooth and ***open*** curve with parametrization $\gamma : [a, b] \to \mathcal{C}$. Then $\mathcal{C}$ is rectifiable iff

$$\int_a^b \|\gamma'(t)\| \mathrm{d}t < \infty$$

and the ***curve length*** is

$$\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

which is independent of γ.

Curve Length

The ***length function*** is defined as

$$(\ell \circ \gamma)(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$$

The curve length gives the ***natural parametrization*** of an oriented curve $\mathcal{C}$.

$$\gamma = \ell : I \to \mathcal{C}, \qquad \operatorname{int} I = (0, \ell(\mathcal{C}))$$

Note. Then we also obtain

$$\|\gamma'(t)\| = \frac{\mathrm{d}\ell \circ \gamma(t)}{\mathrm{d}t}$$

Content

- **Curves in Vector Spaces**
- **Exercise**

Exercise 1

- Find the tangent vector, normal vector and curve length of the following parametrization of curves.

1. $t \in (0, 2\pi)$ $\quad \gamma(t) = \begin{pmatrix} t-\sin(t) \\ 1-\cos(t) \end{pmatrix}$

2. $t \in (0, 2\pi)$ $\quad \gamma(t) = \begin{pmatrix} Sin(t) \\ \cos(t) \\ t \end{pmatrix}$

Exercise 2

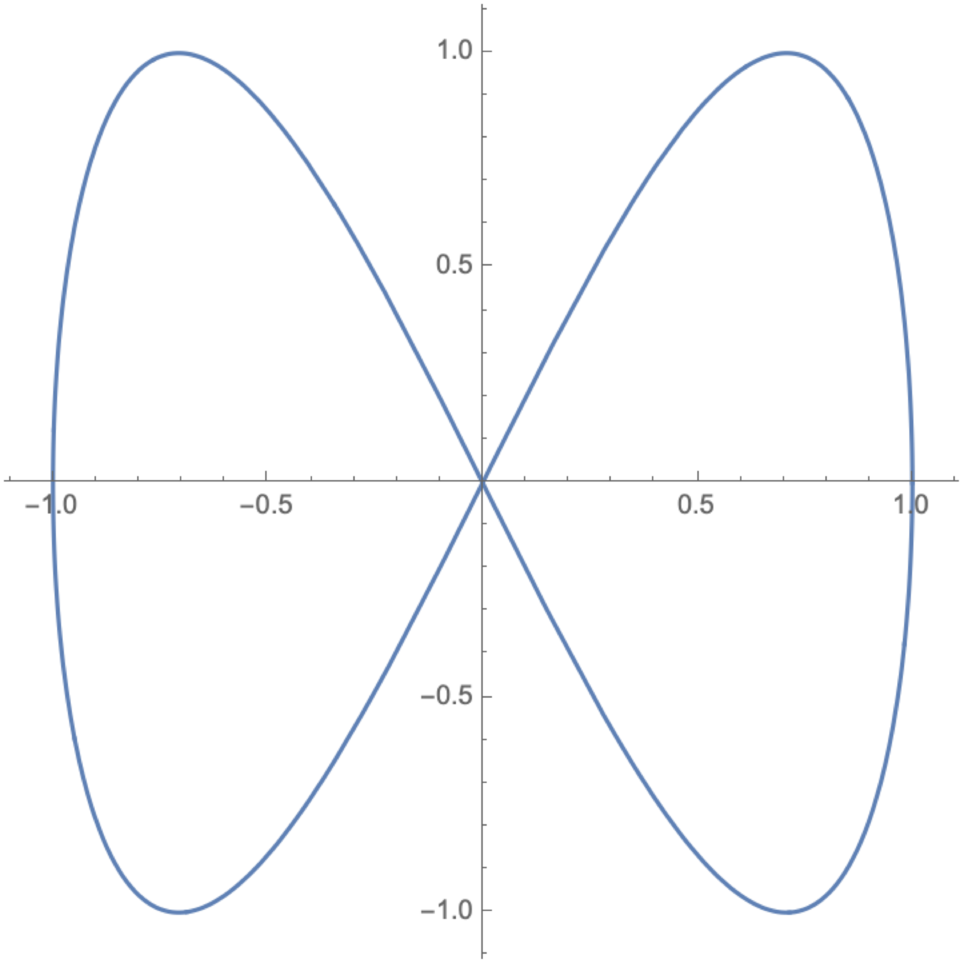
. Sketch the trace of the curve γ represented on $[0, 2\pi]$ by $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (\sin 2t)\mathbf{e}_2$. Find the tangent vectors to γ at the double point $(0, 0)$.

Is this curve closed? Is this curve simple?

Exercise 2

In[4]:= `ParametricPlot[{Cos[t], Sin[2 t]}, {t, 0, 2 Pi}]`

Out[4]=



Exercise 3

A plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by

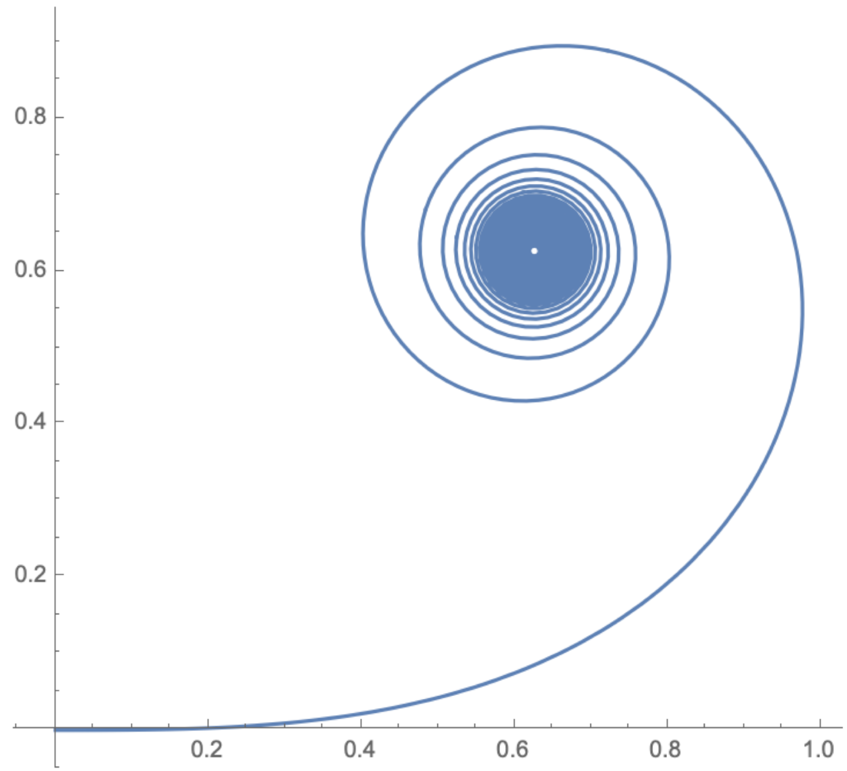
$$\gamma : [0, \infty) \to \mathbb{R}^2, \qquad \gamma(t) = \begin{pmatrix} \int_0^t \cos(s^2)\,ds \\ \int_0^t \sin(s^2)\,ds \end{pmatrix}$$

1. Sketch the curve. How does it behave as $t \to \infty$?
2. Show that γ is the curve length parametrization of $\mathcal{C}$.

Exercise 3

In[2]:=
```
ParametricPlot[{Integrate[Cos[x^2], {x, 0, t}], Integrate[Sin[x^2], {x, 0, t}]},
 {t, 0, 50}, PlotRange -> Full]
```

Out[2]=



Exercise 4

8. Let γ_0 and γ_1 be curves represented on $[a, b]$ by $\mathbf{g}_0$ and $\mathbf{g}_1$, respectively. For every $u \in [0, 1]$ let γ_u be the curve represented by $\mathbf{g}_u(t) = u\mathbf{g}_1(t) + (1 - u)\mathbf{g}_0(t)$, $a \leq t \leq b$. Let $l(u)$ be the length of γ_u. Prove that l is a convex function on $[0, 1]$. When is the convexity strict?

Reference

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Serge Lang, Calculus of Several Variables, 1987

Chen, Xiwen “RC_6.pdf”,2018

Yuan, Jian “RC_6.pdf”,2018

Thank You

Have a nice day!