VV 285
Final Review Part 1

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Content

- Vector Fields and Line Integrals

- Circulation and Flux

The line integral of a potential function

3.1.1. Definition. Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma : I \to \mathcal{C}$. We then define the ***line integral of the potential*** f ***along*** $\mathcal{C}^*$ by

$$\int_{\mathcal{C}^*} f\, ds := \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)|\, dt$$

- Note : It is independent of the paramatrization you choose.

- The symbol "ds" can be interpreted geometrically as a scalar line element and one often writes
- $ds = |\gamma'(t)|\, dt$

The line integral of a potential function-Ex

3.1.3. Example. The mass of a physical wire (interpreted as a curve; i.e., having no thickness) can be obtained by integrating its density along its path. If a wire $\mathcal{C}$ is taken to have variable density ϱ its mass is given by

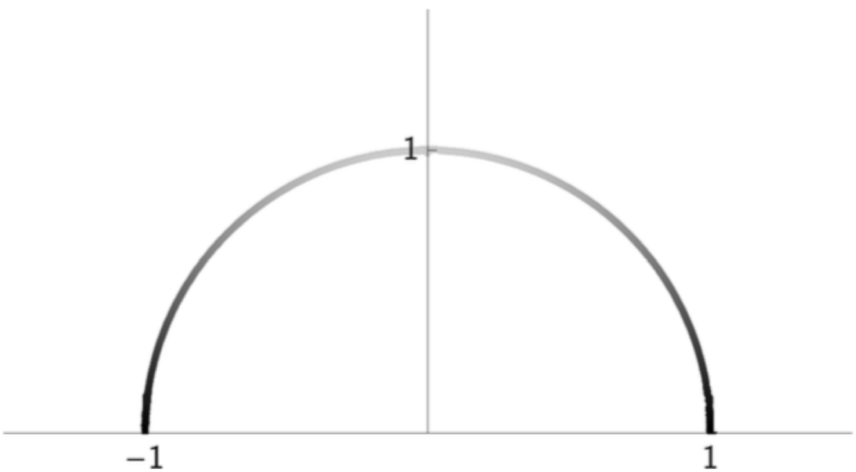
$$m = \left| \int_{\mathcal{C}} \varrho \, ds \right|.$$

As an example, consider a semi-circular wire

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \; y \geq 0\}$$

with density $\varrho(x, y) = k(1 - y)$ where $k > 0$ is a constant. (Thus the wire is denser at its base and light at the top. We might alternatively interpret the varying density as varying thickness of a uniformly dense wire.)

The line integral of a potential function-Ex



We choose the parametrization $\gamma(t) = (\cos t, \sin t)$, $I = [0, \pi]$. We have

$$\int_{\mathscr{C}} \varrho \, ds = \int_0^\pi \varrho \circ \gamma(t) \cdot \|\gamma'(t)\| \, dt = \int_0^\pi k(1 - \sin t) \cdot 1 \, dt = k(\pi - 2),$$

so $m = |k(\pi - 2)| = k(\pi - 2)$.

Vector Field

3.1.4. Definition. Let $\Omega \subset \mathbb{R}^n$. Then a function $F : \Omega \to \mathbb{R}^n$,

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix} .$$

is called a ***vector field*** on Ω.

3.1.5. Example. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a potential function. Then the ***gradient field of*** f given by

$$F : \mathbb{R}^n \to \mathbb{R}^n, \qquad\qquad F(x) = \nabla f(x),$$

associates to every $x \in \mathbb{R}^n$ the direction of largest slope of f.

The line integral of Vector Field

3.1.8. Definition. Let $\Omega \subset \mathbb{R}^n$, $F : \Omega \to \mathbb{R}$ be a continuous vector field and $\mathcal{C}^* \subset \Omega$ an oriented open, smooth curve in $\mathbb{R}^n$. We then define the ***line integral of the vector field F along $\mathcal{C}^*$*** by

$$\int_{\mathcal{C}^*} F\, d\vec{s} := \int_{\mathcal{C}^*} \langle F, T \rangle\, ds \qquad (3.1.5)$$

- Note:
- The line integral of a vector field does not depend on parametrization of C^*.
- The ***vectorial line element*** is given by
  - $d\vec{s} = \gamma'(t)\, dt$
- To calculate line integral using parametrization $\gamma : I \to C$

$$\int_{\mathcal{C}^*} F\, d\vec{s} = \int_I \langle F \circ \gamma(t), \gamma'(t) \rangle\, dt$$

The line integral of Vector Field

If we calculate the line integral using a concrete parametrization $\gamma \colon I \to \mathcal{C}$, we obtain

$$\begin{aligned}\int_{\mathcal{C}^*} F\, d\vec{s} = \int_{\mathcal{C}^*} \langle F, T \rangle\, ds &= \int_I \langle F \circ \gamma(t), T \circ \gamma(t) \rangle \|\gamma'(t)\|\, dt \\ &= \int_I \left\langle F \circ \gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right\rangle \|\gamma'(t)\|\, dt \\ &= \int_I \langle F \circ \gamma(t), \gamma'(t) \rangle\, dt \end{aligned} \tag{3.1.6}$$

The line integral of a vector field- Toy Example in your physics hw.

Calculate work done by the force fields

(A) $\mathbf{F}_1(\mathbf{r}) = (x^2 z, -xy, 5)$, (B) $\mathbf{F}_2(\mathbf{r}) = (-2x - yz, z - xz, y - xy)$,

acting upon a particle that moves from $(-1, 0, 0)$ to $(1, 0, 0)$ along

(a) the x axis,

(b) an arc of the circle $x^2 + y^2 = 1$, so that $y \geq 0$,

(c) the curve Γ defined by parametric equations $x(t) = t$, $y(t) \equiv 0$, $z(t) = t^2 - 1$, where $-1 \leq t \leq 1$.

Potential field

3.1.11. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. A vector field $F\colon \Omega \to \mathbb{R}^n$ is said to be a ***potential field*** if there exists a differentiable potential function $U\colon \Omega \to \mathbb{R}$ such that

$$F(x) = \nabla U(x).$$

Supposing that the initial point of the curve is p_{initial} and the final point is p_{final}, we have from the fundamental theorem of calculus

$$\int_{\mathcal{C}^*} F\, d\vec{s} = \int_I (U \circ \gamma)'(t)\, dt = U(p_{\text{final}}) - U(p_{\text{initial}}).$$

3.1.13. Lemma. Let $\Omega \subset \mathbb{R}^n$ be open, $F\colon \Omega \to \mathbb{R}^n$ a potential field and $\mathcal{C} \subset \Omega$ a closed curve. Then

$$\oint_{\mathcal{C}} F\, d\vec{s} = 0.$$

Conservative field

3.1.14. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F\colon \Omega \to \mathbb{R}^n$ a vector field. If the integral along any open curve $\mathcal{C}^*$ depends only on the initial and final points or, equivalently,

$$\oint_{\mathcal{C}} F\, d\vec{s} = 0 \qquad \text{for any closed curve } \mathcal{C},$$

then F is called ***conservative***.

3.1.15. Remark. We note explicitly that every potential field is a conservative field.

In fact, under certain conditions a conservative field is also a potential field.

3.1.16. Definition. Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be ***(pathwise) connected*** if for any two points in Ω there exists an open curve within Ω joining the two points.

Conservative field

3.1.17. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is a continuous, conservative field. Then F is a potential field.

- Proof of 3.1.17

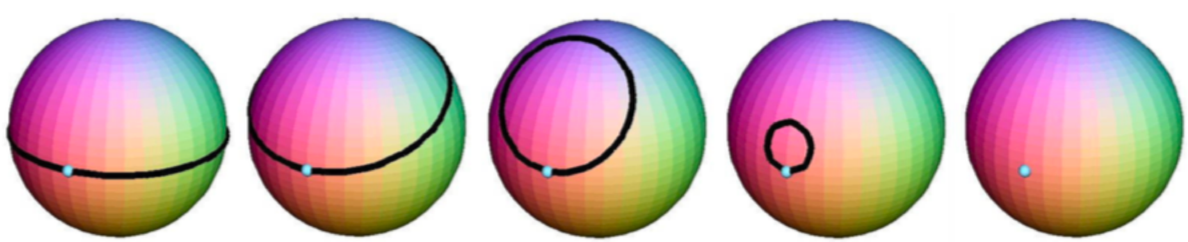
Simply Connected Set

3.1.23. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set.

(i) A closed curve $\mathcal{C} \subset \Omega$ given as the image of a map $g\colon S^1 \to \mathcal{C}$ is said to be ***contractible to a point*** if there exists a continuous function $G\colon D \to \Omega$ such that $G|_{S^1} = g$.

(ii) The set Ω is said to be ***simply connected*** if it is connected and every closed curve in Ω is contractible to a point.

Example.
1. $R^2 \setminus \{0\}$ is not simply connected.
2. $R^3 \setminus \{0\}$ is simply connected.



Salix alba. *A homotopy of a circle around a sphere can be reduced to single point.* 2006. Wikipedia. Wikimedia Foundation. Web. 12 July 2012

Simply Connected Set

3.1.18. Lemma. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. Then F is a potential field only if for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}. \tag{3.1.9}$$

3.1.21. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a ***simply connected*** open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

then F is a potential field.

Finding Potential

Usually We find potential in R^2

Steps:

1. Check potential fields.(Continuous+Convervative)
2. Integrate with respect to x_1.
3. Integrate with respect to x_2.

Differential Form

3.1.25. Definition. Let $F_1, \ldots, F_n : \mathbb{R}^n \to \mathbb{R}$ be scalar functions. Then

$$\alpha = F_1\,dx_1 + \cdots + F_n\,dx_n$$

is said to be a ***differential one-form***.

Try Example

Example Denote by H = $\{(x,y) : y > 0\} \subset$ R^2 the upper half- space of R^2 and consider the two vector fields F , G : H $\to R^2$ with $(x,y) \in$ H,

$F(x,y)=(4x^2+4y^2,8xy-\ln y+e^y)$,

$G(x,y)=(x+2xy-y^2,-2xy+x^2)$

Q1: Are the two fields conservative?

Q2: If so, calculate the potential function U;

Content

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Circulation and Flux

3.2.2. Definition. Let $\Omega \subset \mathbb{R}^2$ be an open set, $F\colon \Omega \to \mathbb{R}^2$ a continuously differentiable vector field and $\mathcal{C}^*$ a positively oriented closed curve in $\mathbb{R}^2$. Then

$$\int_{\mathcal{C}^*} \langle F, T \rangle \, ds \tag{3.2.1}$$

is called the (total) ***circulation*** of F along $\mathcal{C}$ and

$$\int_{\mathcal{C}^*} \langle F, N \rangle \, ds \tag{3.2.2}$$

is called the (total) ***flux*** of F through $\mathcal{C}$.

Here T denotes the usual unit tangent vector to $\mathcal{C}$. However, N is taken to the "unit normal vector" in the sense that

1. $\|N\| = 1$,
2. $\langle N, T \rangle = 0$,
3. N points ***outwards*** from the region bounded by $\mathcal{C}$.

Flux density and divergence

3.2.5. Definition. Let $\Omega \subset \mathbb{R}^n$ and $F\colon \Omega \to \mathbb{R}^n$ be a continuously differentiable vector field. Then

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

is called the ***divergence*** of F.

The Circulation Density — Rotation / Curl

3.2.6. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F\colon \Omega \to \mathbb{R}^n$ a continuously differentiable vector field. Then the anti-symmetric, bilinear form

$$\mathit{rot}\, F|_x\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \mathit{rot}\, F|_x(u,v) := \langle DF|_x u, v\rangle - \langle DF|_x v, u\rangle \tag{3.2.3}$$

is called the ***rotation*** (in mainland Europe) or ***curl*** (in anglo-saxon countries) of the vector field F at $x \in \mathbb{R}^n$.

3.2.7. Theorem. Let $\Omega \subset \mathbb{R}^2$ be open and $F\colon \Omega \to \mathbb{R}^2$ a continuously differentiable vector field. Then here exists a uniquely defined continuous potential function $\operatorname{rot} F\colon \Omega \to \mathbb{R}$ such that

$$\mathit{rot}\, F|_x(u,v) = \operatorname{rot} F(x) \cdot \det(u,v). \tag{3.2.5}$$

The rotation in R^2 and R^3

In R^2, a scalar function

$$\operatorname{rot} F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$

In R^3, a scalar function

$$\operatorname{rot} F|_x(u, v) = \det(\operatorname{rot} F(x), u, v) = \langle \operatorname{rot} F(x), u \times v \rangle.$$

$$\operatorname{rot} F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}.$$

Note: it needs to be a **continuously differentiable** vector field on a open set.

Sample Exercise

Given an electric field $E = c(2bxy, x^2 + ay^2)$, a, b, $c \in R$, determine values for a and b such that div $E = 0$ and rot $E = 0$. Then, find a potential function V for E with these values of a and b.

Irrotational Fields

A continuously differentiable field $F : \Omega \to R^n$ such that $\mathrm{rot}F|_x = 0$ for all $x \in \Omega$ is **irrotational**. Then

$(DF|_x)^T = DF|_x$

Note: A potential field is irrotational.

Fluid Statistics

For potential flows, we have that

$$F = \nabla U \qquad \operatorname{div} F = 0.$$

Then, we have

$$\operatorname{div}(\nabla U) = \operatorname{div} \begin{pmatrix} \frac{\partial U}{\partial x_1} \\ \vdots \\ \frac{\partial U}{\partial x_n} \end{pmatrix} = \frac{\partial^2 U}{\partial x_1^2} + \cdots + \frac{\partial^2 U}{\partial x_n^2} = \Delta U = 0.$$

Triangle Calculus

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

$$\operatorname{div} F = \langle \nabla, F \rangle = \left\langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}, \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \right\rangle = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Divergence

$$\operatorname{rot} F = \nabla \times F(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Rotation

$$\begin{aligned} \langle \nabla, \nabla \rangle &= \left(\frac{\partial}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial}{\partial x_n} \right)^2 \\ &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \Delta \end{aligned}$$

Laplace Notation

Reference

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Thank You

Have a nice day!