JOINT INSTITUTE
交大密西根学院

VV 285
RC 2

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Content

- Feedback for Assignment 1
- Linear Map
- Matrices

Ex 1.7

Exercise 1.7

For $x = (x_1, x_2, x_3.x_4) \in \mathbb{R}^4$ let

$$f_1(x) := x_1 + 2x_2 + x_3 - x_4, \qquad f_2(x) := 3x_1 + 5x_2 - x_3 - 6x_4, \qquad f_3(x) := -2x_1 - x_2 + 10x_3 + 11x_4.$$

Further set $U := \{x \in \mathbb{R}^4 \mid f_1(x) = 0 \text{ and } f_2(x) = 0 \text{ and } f_3(x) = 0\}$.

i) Show that U is a subspace of the real vector space $\mathbb{R}^4$.
(2 Marks)

ii) Find a basis of U. (Hint: U is the solution set to the system of equations $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = 0$.)
(3 Marks)

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Linear Map

1.4.1. Definition. Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be vector spaces that are either both real or both complex. Then a map $L\colon U \to V$ is said to be ***linear*** if it is both ***homogeneous***, i.e.,

$$L(\lambda \odot u) = \lambda \boxdot L(u) \tag{1.4.1a}$$

and ***additive***, i.e.,

$$L(u \oplus u') = L(u) \boxplus L(u'), \tag{1.4.1b}$$

for all $u, u' \in U$ and $\lambda \in \mathbb{F}$. The set of all linear maps $L\colon U \to V$ is denoted by $\mathscr{L}(U, V)$.

For linear maps, we often write simply Lu instead of $L(u)$.

Linear Map

Justify whether the following is a linear map:

1.A map $R \to R$, with the form $x \to 0$ for all x.

2.A map $C \to C$, if C is regarded as a real vector space, with the form $x \to \bar{x}$ for all x.

3.A map $C \to C$, if C is regarded as a complex vector space, with the form $x \to \bar{x}$ for all x.

Structure-Preserving

A linear map

$$L : U \to V$$

between vector spaces $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ is a structure-preserving map.

$$\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "\lambda\odot"'] & V \arrow[d, "\lambda\boxdot"] \\
U & V \arrow[l, "L^{-1}"']
\end{tikzcd}$$

Linear Map

1.4.4. Theorem. Let U, V be real or complex vector spaces and $(b_1, \ldots, b_n)$ a basis of U (in particular, it is assumed that $\dim U = n < \infty$). Then for every n-tuple $(v_1, \ldots, v_n) \in V^n$ there exists a unique linear map $L \colon U \to V$ such that $Lb_k = v_k$, $k = 1, \ldots, n$.

Ran and Kernel

1.4.8. Definition. Let U, V be real or complex vector spaces and $L \in \mathscr{L}(U, V)$. Then we define the range of L by

$$\operatorname{ran} L := \Big\{ v \in V : \underset{u \in U}{\exists} v = Lu \Big\}$$

and the ***kernel*** of L by

$$\ker L := \{ u \in U : \ Lu = 0 \}.$$

It is easy to see that $\operatorname{ran} L \subset V$ and $\ker L \subset U$ are subspaces.

Nomenclature

According to their properties, there are several fancy names for linear maps. A homomorphism $L \in \mathscr{L}(U, V)$ is said to be

- an ***isomorphism*** if L is bijective;
- an ***endomorphism*** if $U = V$;
- an ***automorphism*** if $U = V$ and L is bijective;
- ***epimorph*** if L is surjective;
- ***monomorph*** if L is injective.

Isomorphisms

1.4.11. Theorem. Let U, V be finite-dimensional vector spaces and $L \in \mathscr{L}(U, V)$. Then L is an isomorphism if and only if for every basis $(b_1, \dots, b_n)$ of U the tuple $(Lb_1, \dots, Lb_n)$ is a basis of V.

1.4.12. Definition. Two vector spaces U and V are called ***isomorphic***, written $U \cong V$, if there exists an isomorphism $\varphi \colon U \to V$.

1.4.13. Lemma. Two finite-dimensional vector spaces U and V are isomorphic if and only if they have the same dimension:

$$U \cong V \quad \Leftrightarrow \quad \dim U = \dim V$$

Dimension Formula

1.4.14. Dimension Formula. Let U, V be real or complex vector spaces, $\dim U < \infty$. Let $L \in \mathscr{L}(U, V)$. Then

$$\dim \operatorname{ran} L + \dim \ker L = \dim U. \tag{1.4.3}$$

1.4.15. Corollary. Let U, V be real or complex finite-dimensional vector spaces with $\dim U = \dim V$. Then a linear map $L \in \mathscr{L}(U, V)$ is injective if and only if it is surjective.

Small Exercise

Prove the following:

Suppose U and V are finite-dimensional vector spaces such that dim U > dim V. Then no linear map from U to V is injective.

Similarly, Suppose U and V are finite-dimensional vector spaces such that dim U < dim V. Then no linear map from U to V is surjective.

Normed Vector Spaces

1.4.16. Definition. Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed vector spaces. Then a linear map $L\colon U \to V$ is said to be ***bounded*** if there exists some constant $c > 0$ (called a ***bound*** for L) such that

$$\|Lu\|_V \leq c \cdot \|u\|_U \qquad \text{for all } u \in U. \tag{1.4.6}$$

The Operator Norm

1.4.19. Definition and Theorem. Let U, V be normed vector spaces. Then the set of bounded linear maps $\mathscr{L}(U, V)$ is also a vector space and

$$\|L\| := \sup_{\substack{u \in U \\ u \neq 0}} \frac{\|Lu\|_V}{\|u\|_U} = \sup_{\substack{u \in U \\ \|u\|_U = 1}} \|Lu\|_V . \tag{1.4.7}$$

defines a norm, the so-called ***operator norm*** or ***induced norm*** on $\mathscr{L}(U, V)$.

Small Exercise:

1. Prove that it is a norm. e.g. It satisfies the properties of norm.
2. Prove the property that

$\|L_2 L_1\| \leq \|L_2\| \cdot \|L_1\|$, $L_1 \in L(U,V)$, $L_2 \in L(V,W)$.

Dual Space

3.94 Definition ***dual space,*** V'

The ***dual space*** of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

3.96 Definition ***dual basis***

If $v_1, \ldots, v_n$ is a basis of V, then the ***dual basis*** of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Note: It is just different notation for V* and V'. Recommend to use V* instead of V'

What is the dual basis of the standard basis $e_1, \ldots, e_n$ of F^n?

Exercise 1

Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Exercise 2

The ***double dual space*** of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

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Matrices

1.5.1. Definition. An $m \times n$ matrix over the complex numbers is a map

$$a\colon \{1, \dots, m\} \times \{1, \dots, n\} \to \mathbb{C}, \qquad (i, j) \mapsto a_{ij}.$$

We represent the graph of a through

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

We denote the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathrm{Mat}(m \times n; \mathbb{C})$.

Matrices

1.5.3. Theorem. Each matrix $A \in \operatorname{Mat}(m \times n; \mathbb{R})$ uniquely determines a linear map $j(A) \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that the columns $a_{\cdot k}$ are the images of the standard basis vectors $e_k \in \mathbb{R}^n$; in particular,

$$j\colon \operatorname{Mat}(m \times n; \mathbb{R}) \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$$

is an isomorphism, $\operatorname{Mat}(m \times n; \mathbb{R}) \cong \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, so every map $L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ corresponds to a matrix $j^{-1}(L)$ whose columns $a_{\cdot k}$ are the images of the standard basis vectors $e_k \in \mathbb{R}^n$.

Matrix Product

1.5.4. Definition. Let $A \in \mathrm{Mat}(l \times m; \mathbb{C})$ and $B \in \mathrm{Mat}(m \times n; \mathbb{C})$. Then we define the ***product of*** $A = (a_{ik})$ ***and*** $B = (b_{kj})$ by

$$AB \in \mathrm{Mat}(l \times n; \mathbb{C}), \qquad AB := \left(\sum_{k=1}^{m} a_{ik} b_{kj}\right)_{\substack{i=1,\ldots,l \\ j=1,\ldots,n}}$$

Property:

1. (AB)C=A(BC) Associativity applies!
2. AB!=BA. Commutativity does not apply!
3. A(B+C)=AB+AC. Distributivity also applies!

Simple Practice

Small Exercise of matrix product:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 4 \\ 7 & 4 \end{pmatrix}$$

Transpose and Adjoint

For $A = (a_{ij}) \in \mathrm{Mat}(m \times n; \mathbb{F})$ we define the ***transpose*** of A by

$$A^T \in \mathrm{Mat}(n \times m; \mathbb{F}), \qquad A^T = (a_{ji}).$$

We also define the ***adjoint***

$$A^* \in \mathrm{Mat}(n \times m; \mathbb{F}), \qquad A^* = \overline{A}^T = (\overline{a_{ji}}).$$

It is not hard to see

$$\langle x, Ay \rangle = \langle A^* x, y \rangle.$$

Matrix of a linear map

Let U, V be finite-dimensional real or complex vector spaces with bases

$$\mathscr{A} = (a_1, \dots, a_n) \subset U \qquad \text{and} \qquad \mathscr{B} = (b_1, \dots, b_m) \subset V.$$

Define the isomorphisms

$$\begin{aligned} \varphi_{\mathscr{A}} &: U \xrightarrow{\cong} \mathbb{R}^n, & \varphi_{\mathscr{A}}(a_j) &= e_j, & j &= 1, \dots, n, \\ \varphi_{\mathscr{B}} &: V \xrightarrow{\cong} \mathbb{R}^m, & \varphi_{\mathscr{B}}(b_j) &= e_j, & j &= 1, \dots, m. \end{aligned}$$

Then any linear map $L \in \mathscr{L}(U, V)$ induces a matrix $A = \Phi_{\mathscr{A}}^{\mathscr{B}}(L)$ $\in \operatorname{Mat}(m \times n; \mathbb{R})$ through

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ {\scriptstyle \varphi_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathscr{B}}} \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array} \qquad\qquad \Phi_{\mathscr{A}}^{\mathscr{B}}(L) = A = \varphi_{\mathscr{B}} \circ L \circ \varphi_{\mathscr{A}}^{-1}$$

Inverse of Matrix

1.5.9. Definition. A matrix $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ is called ***invertible*** if there exists some $B \in \mathrm{Mat}(n \times n; \mathbb{R})$ such that

$$AB = BA = \mathrm{id} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \qquad (1.5.3)$$

We then write $B = A^{-1}$ and say that A^{-1} is the ***inverse*** of A.

Inverse of Matrix

1.5.13. Lemma. Let $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$. Then A is ***invertible*** if and only if there exists an elementary matrix S corresponding to elementary row operations that transform A into the unit matrix $SA = \mathrm{id}$.

Notice that invertibility is equivalent to injectivity + surjectivity.
A linear map is invertible iff it is injective and surjective.
Hence: isomorphism <->invertible<->dim U=dim V for L(U,V)
(U,V be to finite vector space over F)

Small Exercise: Try to find the inverse of the matrix

$$\begin{pmatrix} 1 & 8 \\ 9 & 4 \end{pmatrix}$$

Inverse Maps

1.5.15. Remark. We note that if $A, B \in \mathrm{Mat}(n \times n; \mathbb{R})$ are invertible, then so is their product $AB \in \mathrm{Mat}(n \times n; \mathbb{R})$ and $(AB)^{-1} = B^{-1}A^{-1}$.

We can use this procedure to find the inverse of any vector space isomorphism L:

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathscr{A}}"'] & V \arrow[d, "\varphi_{\mathscr{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad L^{-1} = \varphi_{\mathscr{A}}^{-1} \circ A^{-1} \circ \varphi_{\mathscr{B}}$$

1.5.16. Example. Let $\mathscr{P}_2$ be the space of polynomials of degree not more than 2. Consider the linear map

$$L\colon \mathscr{P}_2 \to \mathscr{P}_2, \qquad ax^2 + bx + c \mapsto \frac{a+b+c}{3}x^2 + \frac{a+b}{2}x + \frac{a-c}{2}$$

Exercise 1

Let A be an n × n matrix, and let $a_{\cdot 1},\ldots,a_{\cdot n}$ be its columns. Show that A is invertible if and only if $a_{\cdot 1}, . . . , a_{\cdot n}$ are linearly independent.

Exercise 2

9 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

11 Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

Reference

Hohberger, Horst. “VV285_main.pdf”

Axler, Sheldon. *Linear Algebra Done Right*. Springer, 2017.

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Thank You