# The Sonification of Irrational Numbers

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**Executive Summary:**

In this project, sonification will be applied in mathematics and music by representing irrational numbers as melodies by assigning their digits to musical notes. With sonification, certain patterns or structures in these irrational numbers can be put into a new perspective with their resulting melodies. The irrational numbers pi, e, and the phi will be represented as musical melodies, and research will be provided regarding what can be uncovered within these numbers through this process.

**Introduction:**

Irrational numbers are special and unique; they are used in a vast number of formulas, and they cannot be represented in the form of a fraction, like any other type of number in mathematics. This is the reason why they are called “irrational”; they cannot be expressed with a ratio of two whole numbers, or a fraction. Their digits are also random and seemingly illogically generated, which is a source of interest to use sonification. Theoretically, potentially any type of melody may eventually be produced through the infinite sonification of these numbers. This project will explore the types of different melodies produced by irrational numbers, mainly pi, e, and phi.

**Background and Literature:**

This section will cover studies about the properties of the digits of pi, e, and phi, since it is necessary to understand the properties of these numbers before sonifying them.

The irrational number pi ($\pi$) is the "ratio" of any circle's circumference to its diameter. Note that it is not a proper ratio as previously mentioned, but its origin comes from the division of two values. According to "On the Rapid Computation of Various Polylogarithmic Constants" by David Bailey, Peter Borwein and Simon Plouffe, the digits of pi appear to be random, but they follow a uniform distribution. This means that every digit in pi occurs at roughly the same frequency; there is no tendency for one digit to appear more than another.

Moving on, the irrational number e is used to describe the growth and decay of exponential functions, such as compound interest formulas. According to "On the Binary Expansions of Algebraic Numbers" written by David Bailey, Jonathan Borwein, Richard Crandall and Carl Pomerance, it is surprisingly unknown whether the digits of e follow a uniform distribution. However, they still do not appear to follow any predictable pattern, much like any other irrational number.

Finally, the irrational number phi (φ) is often referred to as the "Golden Ratio." It is defined as the ratio of two numbers such that the larger number is equal to the smaller number and phi itself. These two numbers can be any pair of numbers as long as they follow this rule, and architects favor these numbers because they exhibit aesthetically pleasing qualities due to their proportions. In addition to e, the digits of phi are also not known to be uniformly distributed. According to "The Golden Ratio: The Story of Phi, the World's Most Astonishing Number" by Mario Livio, some studies suggest that the frequency of the digit 9 is higher than expected, while the frequency of the digit 0 is lower than expected.

**Research:**

This section will explore the musical melodies produced from the sonification of pi, e, and phi. Digits 1-8 were assigned to their corresponding notes in the C major scale, where 1 was assigned to C, 2 was assigned to D, all the way to 8 being assigned to C (one octave higher than the starting C note). Since there are only 8 notes in the C major scale, digits 0 and 9 were assigned to a 3 note A minor and C major chords respectively.

A python script was used to read characters from 3 files, each containing the digits of pi, e and phi, assign music notes to each character, and convert the resulting audios into midi files. The script is shown on the next page.

The midi files were then converted into .wav files for compatibility with all machines, which are attached along with this report.

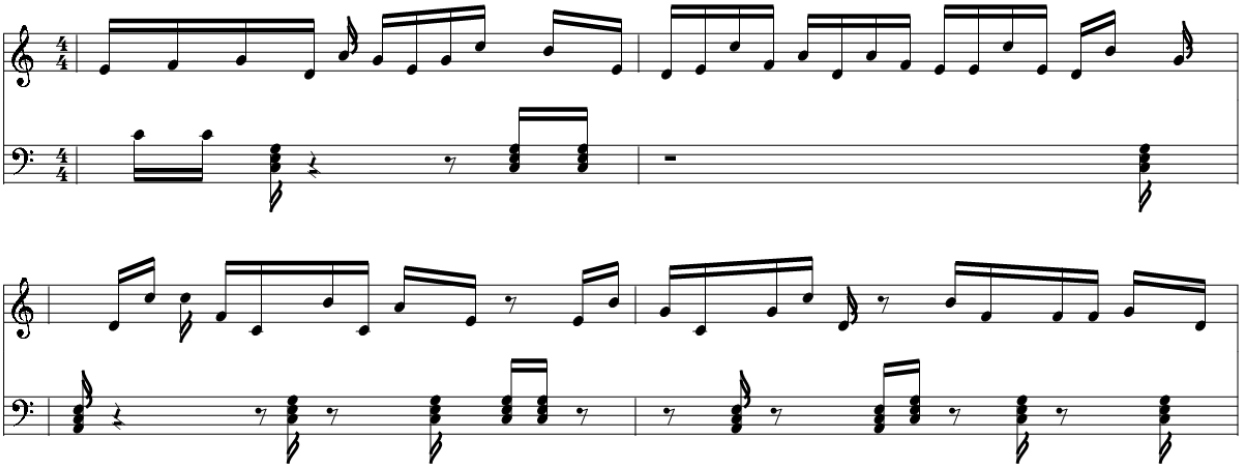
To produce the figures of sheet music below, the Midi Sheet Music Software was used to convert the midi files into presentable music notes.

The python script used to convert the irrational numbers to midi files (with comments bolded):

```
from midiutil import MIDIFile

#Define a dictionary to map digits to notes of C major
notes = {
    '1': 60,
    '2': 62,
    '3': 64,
    '4': 65,
    '5': 67,
    '6': 69,
    '7': 71,
    '8': 72,
    '9': 'chord',
    '0': 'chord'
}

#Open the text files with the numbers' digits and read them
with open('pi.txt', 'r') as f:
    pi_digits = f.read().replace('\n', '')

#Create a new MIDI file with one track
midi_file = MIDIFile(1)

#Add tempo and track name
midi_file.addTempo(0, 0, 60)
midi_file.addTrackName(0, 0, "Melody")

#Loop through each digit and add its corresponding note to the MIDI file
time = 0
for digit in pi_digits:
    note = notes[digit]

    if note == 'chord':

        #Add chords for digits 0 and 9
        duration = 0.25
        if digit == '0':
            midi_file.addNote(0, 0, 45, time, duration, 127)
            midi_file.addNote(0, 0, 48, time, duration, 127)
            midi_file.addNote(0, 0, 52, time, duration, 127)
        else:
            midi_file.addNote(0, 0, 48, time, duration, 127)
            midi_file.addNote(0, 0, 52, time, duration, 127)
            midi_file.addNote(0, 0, 55, time, duration, 127)

    else:
        duration = 0.25
        pitch = note
        midi_file.addNote(0, 0, pitch, time, duration, 127)

    time += duration

#Write the MIDI file to disk
with open("pi_melody.mid", "wb") as output_file:
    midi_file.writeFile(output_file)
```

Sonification of pi

Sheet music of the first 32 digits of pi (3.14159265358979323846264338327 9):

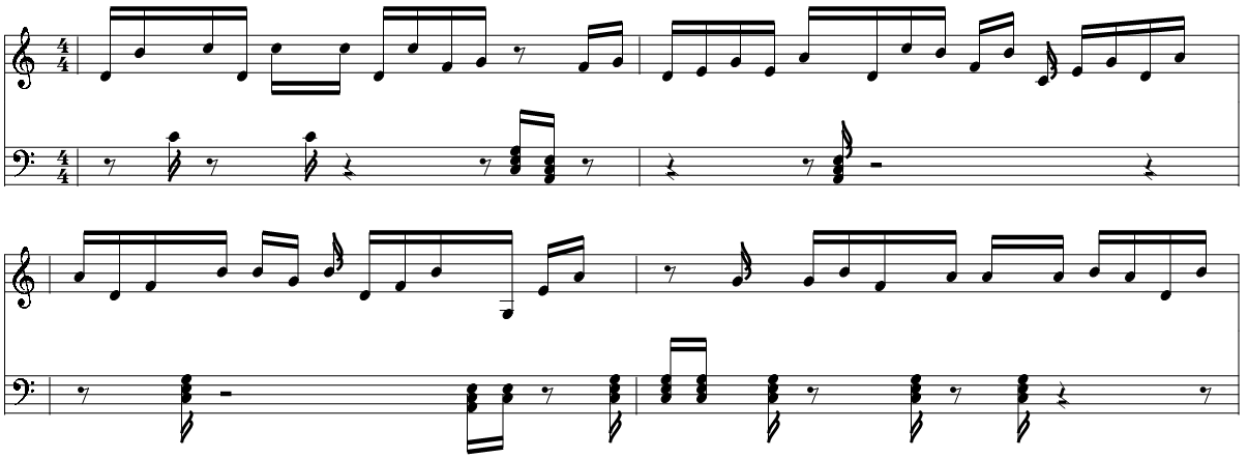


Characteristics:

- The melody seems rather joyful with a high note in the beginning.
- The high frequency of the number 9 in this snippet makes the C major chord very prevalent, which could contribute to the joyful tone.
- The number 0 is absent in this snippet, making the entire section contain only major notes and chords, also possibly contributing to the joyful tone.

Sonification of *e*

Sheet music of the first 32 digits of *e* (2.71828182845904523536028747135 2):

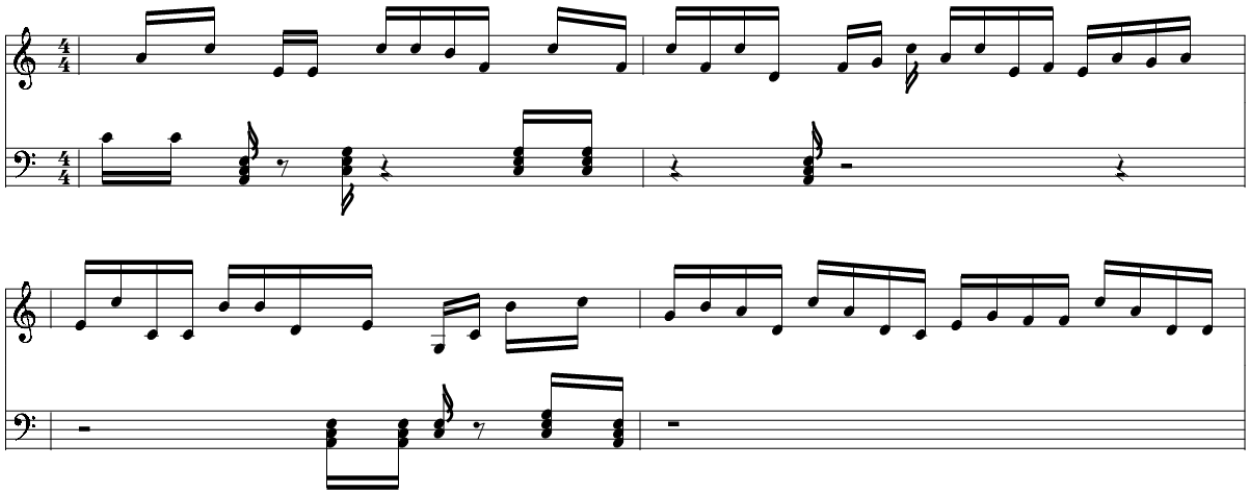


Characteristics:

- This melody seems rather suspenseful, as if it is building towards something.
- The high frequency of the number 8 in the beginning causes the melody to oscillate between low and high notes in a “stepping” manner.
- The presence of 0 in this snippet introduces the A minor chord which could contribute to the suspenseful tone.

Sonification of phi

Sheet music of the first 32 digits of phi (1.61803398874989484820458683436 5):



Characteristics:

- This melody sounds rather dissonant with the A minor chords, C major chords, and high notes playing in succession in this section.
- Since there is a higher frequency of numbers above 5 in this section, this can cause dissonance when paired with the A minor chords that are playing in place of the 0s.
- There is a note of suspense with the continuous single notes in the last measure due to the absence of chords.

Conclusion:

Although the three numbers have random digit sequences, each number produced a unique melody with their selected section of digits. The pi melody was found to be joyful, which is fascinatingly relevant to its circular applications in mathematics, since circles can represent complete structures and harmony. The e melody was also relevant to its mathematical uses, since it represents continuous growth and decay, and its melody in its selected section had a suspenseful tone as if it was building towards a certain point. The phi melody actually contrasted with its mathematical origin, since it was dissonant when phi represents aesthetically pleasing structures in mathematics.

Additional notes:

Since these numbers are infinitely random, they may contain various musical works very far into their decimal representation.

For example, using The Pi-Search Page, the first 7 notes of “Mary Had a Little Lamb” represented by numbers, 3212333, were found at the 44765416th digit of pi. Since this is an online tool, the power of this search engine is quite limited. However, this does not disprove the fact that many other songs of various lengths and complexities could possibly be found within the infinitely many digits of pi, e, phi, or any other irrational number. The whole song of “Mary Had a Little Lamb” can very easily be within pi’s digits, and the only thing limiting this finding is our searching capabilities.

**Works Cited:**

Bailey, David H., et al. “On the Binary Expansions of Algebraic Numbers.” Journal de Théorie Des Nombres de Bordeaux, vol. 16, no. 3, 2004, pp. 487–518. JSTOR, http://www.jstor.org/stable/43974296.

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“Download.” Midi Sheet Music, http://midisheetmusic.com/download.html.

Livio, M. (2002). “The Golden Ratio: The Story of Phi, the World's Most Astonishing Number.” Broadway Books.

*The Pi-Search Page*, https://www.angio.net/pi/.