Macro Modeling — Gavin Engelstad et al.

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Midterm

1 Habit Persistence

Some economists argue that consumers like living the good life and they most often get used to it. Therefore, utility not only depends on current but also on *previous* levels of consumption. Models like these are said to exhibit *habit persistence in consumption* and have been fairly successful in accounting for several features of macro-finance data and in improving the fit of DSGE models.

1.1 The Model Economy

Consider an infinitely-lived economy consisting of a representative household and a representative firm. Assume that both the consumer and the firm are price takers. The household chooses infinite sequences of consumption C_t, investment I_t, and hours worked N_t to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(C_t, C_{t-1}, N_t)$$

where $\beta \in (0,1)$ is the household's discount factor and where the instantaneous utility function u takes the form

$$u(C_t, C_{t-1}, N_t) = \ln\left(C_t - \nu C_{t-1}\right) + \phi\left(1 - N_t\right).$$

In the above, parameter $\nu \geq 0$ governs the household's habit persistence and parameter $\phi > 0$ governs the relative importance of consumption and leisure. The household's endowment consists of an initial capital stock K_0 and a (normalized) time endowment of 1 units of time every period, which it can devote either to work or leisure. Labor is paid at a rate w_t per hour. Finally, assume that the household makes investment decisions each period, following the law of motion for capital

$$K_{t+1} = (1-\delta)K_t + I_t$$

where $\delta \in [0,1]$ is the depreciation rate of physical capital. Each period, the household rents capital K_t to the representative firm and obtains a rental rate of r_t per unit.

The representative firm uses a constant returns to scale technology F to produce output, following:

$$Y_t = z_t F\left(K_{Ft}, N_{Ft}\right) = z_t K_{Ft}^{\alpha} N_{Ft}^{1-\alpha}.$$

In the above, z_t is a stochastic productivity shock, K_{Ft} denotes the firm's capital input, and N_{Ft} denotes its labor input. The process for z_t follows

$$z_t = (1-\rho)z_{ss} + \rho z_{t-1} + \varepsilon_{zt},$$

where $\rho \in (0,1)$, z_{ss} is the steady-state level of the productivity shock, and $\varepsilon_{zt} \sim \mathcal{N}(0, \sigma_z^2)$.

Exercise 1.1. What's the (economic) effect of habit persistence (alternatively, of parameter ν)? In one or two paragraphs, provide an intuitive explanation and comment on whether you believe including a specification like this in a DSGE model makes sense. You don't have to do any analytical derivation, but be sure to thoroughly justify your answer. [10]

Solution. I hate writing. I'll do this later.

Exercise 1.2. Set up the social planner problem associated to this economy. While you're at it, explain why you don't need to define a competitive equilibrium for this problem. [5]

Solution. The solution to the Social Planner Problem is the set of household allocations $\{Y_t, K_{t+1}, C_t, I_t, N_t\}$ such that the households solve

$$\begin{aligned}
\max \quad & U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln\left(C_t - \nu C_{t-1}\right) + \phi\left(1 - N_t\right)\right) \\
\text{subject to} \quad & K_{t+1} = (1-\delta)K_t + I_t \\
& C_t + I_t = Y_t \\
& Y_t = z_t K_t^{\alpha} N_t^{1-\alpha} \\
& Y_t, K_{t+1}, C_t, I_t, N_t \geq 0.
\end{aligned}$$

We can do this instead of a Competitive Equilibrium because the first and second welfare theorems hold since we have perfect competition and no distorting taxes.

Exercise 1.3. Use your answer to Exercise 1.2 to characterize the equilibrium of this economy. You'll see that the first-order condition for consumption is a bit more complicated than usual, so feel free to keep the Lagrange multiplier as an endogenous variable. [10]

Solution. To make the first order conditions easier, we'll solve

$$\begin{aligned}
\max \quad & U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln\left(C_t - \nu C_{t-1}\right) + \phi\left(1 - N_t\right)\right) \\
\text{subject to} \quad & C_t + K_{t+1} = z_t K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t \\
& K_{t+1}, C_t, N_t \geq 0.
\end{aligned}$$

This is easily shown to be equivalent to the earlier Social Planner Problem we set up earlier.

Using this setup, the Lagrangian is

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln\left(C_t - \nu C_{t-1}\right) + \phi\left(1 - N_t\right) + \lambda_t \left(z_t K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t - C_t - K_{t+1}\right)\right).$$

Solving this gets the set of first order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \beta^{t+1}\lambda_{t+1}\left(z_{t+1}\alpha K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + (1-\delta)\right) - \beta^t\lambda_t \\
\frac{\partial \mathcal{L}}{\partial C_t} &= \beta^t\left(\frac{1}{C_t - \nu C_{t-1}} - \lambda_t\right) - \beta^{t+1}\frac{\nu}{C_{t+1} - \nu C_t} \\
\frac{\partial \mathcal{L}}{\partial N_t} &= \beta^t\left((1-\alpha)\lambda_t z_t K_t^{\alpha}N_t^{-\alpha} - \phi\right) \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= \beta^t\left(z_t K_t^{\alpha}N_t^{1-\alpha} + (1-\delta)K_t - C_t - K_{t+1}\right).
\end{aligned}$$

To simplify this again, we add the conditions

$$\begin{aligned}
Y_t &= z_t K_t^{\alpha}N_t^{1-\alpha} \\
r_t &= \alpha z_t K_t^{\alpha-1}N_t^{1-\alpha} \\
w_t &= (1-\alpha)z_t K_t^{\alpha}N_t^{-\alpha} \\
I_t &= K_{t+1} - (1-\delta)K_t,
\end{aligned}$$

rearrange, and set the FOCs to zero to get the system

$$\begin{aligned}
Y_t &= z_t K_t^{\alpha}N_t^{1-\alpha} \\
r_t &= \alpha Y_t K_t^{-1} \\
w_t &= (1-\alpha)Y_t N_t^{-1} \\
I_t &= K_{t+1} - (1-\delta)K_t \\
Y_t &= C_t + I_t \\
\lambda_t &= \frac{1}{C_t - \nu C_{t-1}} - \frac{\beta\nu}{C_{t+1} - \nu C_t} \\
\phi &= \lambda_t w_t \\
\lambda_t &= \beta(\lambda_{t+1}r_{t+1} + (1-\delta)\lambda_{t+1}).
\end{aligned}$$

Exercise 1.4. Use your answer to Exercise 1.3 to find the closed-form solutions to the steady state values for all the economy's variables. You should set $z_{ss} = 1$. [10]

Solution. Setting everything in the system equal to their steady state values gets the system

$$\begin{aligned}
Y_{ss} &= z_{ss} K_{ss}^{\alpha}N_{ss}^{1-\alpha} \\
r_{ss} &= \alpha Y_{ss} K_{ss}^{-1} \\
w_{ss} &= (1-\alpha)Y_{ss} N_{ss}^{-1} \\
I_{ss} &= K_{ss} - (1-\delta)K_{ss} \\
Y_{ss} &= C_{ss} + I_{ss} \\
\lambda_{ss} &= \frac{1}{C_{ss} - \nu C_{ss}} - \frac{\beta\nu}{C_{ss} - \nu C_{ss}} \\
\phi &= \lambda_{ss} w_{ss} \\
\lambda_{ss} &= \beta(\lambda_{ss}r_{ss} + (1-\delta)\lambda_{ss}).
\end{aligned}$$

Doing some hellish algebra[1] to solve these gets

$$\begin{aligned}
r_{ss} &= \frac{1}{\beta} - 1 + \delta \\
K_{ss} &= z_{ss}^{\frac{1}{1-\alpha}} \frac{(1-\alpha)(1-\beta\nu)}{\phi(1-\nu)\left(\frac{r_{ss}}{\alpha} - \delta\right)} \left(\frac{r_{ss}}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \\
Y_{ss} &= \frac{r_{ss}}{\alpha} K_{ss} \\
C_{ss} &= \left(\frac{r_{ss}}{\alpha} - \delta\right) K_{ss} \\
I_{ss} &= \delta K_{ss} \\
N_{ss} &= \left(\frac{r_{ss}}{\alpha z_{ss}}\right)^{\frac{1}{1-\alpha}} K_{ss} \\
w_{ss} &= \frac{\phi(1-\nu)}{1-\beta\nu} \left(\frac{r_{ss}}{\alpha} - \delta\right) K_{ss} \\
\lambda_{ss} &= \frac{1-\beta\nu}{(1-\nu)C_{ss}}
\end{aligned}$$

Exercise 1.5. Use your answer to Exercise 1.3 and Uhlig's rules to log-linearize the equations that characterize the equilibrium of this economy.

Solution. The equations become[2]

$$\begin{aligned}
Y_t &= z_t K_t^{\alpha} N_t^{1-\alpha} && \Longrightarrow && \hat{Y}_t = \hat{z}_t + \alpha \hat{K}_t + (1-\alpha)\hat{N}_t \\
r_t &= \alpha Y_t K_t^{-1} && \Longrightarrow && \hat{r}_t = \hat{Y}_t - \hat{K}_t \\
w_t &= (1-\alpha) Y_t N_t^{-1} && \Longrightarrow && \hat{w}_t = \hat{Y}_t - \hat{N}_t \\
I_t &= K_{t+1} - (1-\delta)K_t && \Longrightarrow && I_{ss}\hat{I}_t = K_{ss}\hat{K}_{t+1} - (1-\delta)K_{ss}\hat{K}_t \\
Y_t &= C_t + I_t && \Longrightarrow && Y_{ss}\hat{Y}_t = C_{ss}\hat{C}_t + I_{ss}\hat{I}_t \\
\lambda_t &= \frac{1}{C_t - \nu C_{t-1}} - \frac{\beta\nu}{C_{t+1} - \nu C_t} && \Longrightarrow && \lambda_{ss}\hat{\lambda}_t = \frac{\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t-1} + \frac{\beta\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t+1} - \frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}}\hat{C}_t \\
\phi &= \lambda_t w_t && \Longrightarrow && 0 = \hat{\lambda}_t + \hat{w}_t \\
\lambda_t &= \beta(\lambda_{t+1} r_{t+1} + (1-\delta)\lambda_{t+1}) && \Longrightarrow && \lambda_{ss}\hat{\lambda}_t = \beta\lambda_{ss}(r_{ss} + 1 - \delta)\hat{\lambda}_{t+1} + \beta\lambda_{ss} r_{ss}\hat{r}_{t+1}.
\end{aligned}$$

We also have the law of motion for TFP, which becomes

$$z_t = (1-\rho)z_{ss} + \rho z_{t-1} + \varepsilon_{zt} \qquad \Longrightarrow \qquad \hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon_{zt}$$

Exercise 1.6. Use your answer to Exercise 1.5 to map the log-linearized equilibrium conditions into the system of matrix equations

$$\begin{aligned}
\vec{0} &= \mathbf{A}\vec{x}_{t+1} + \mathbf{B}\vec{x}_t + \mathbf{C}\vec{y}_t + \mathbf{D}\vec{z}_t \\
\vec{0} &= \mathbb{E}\left[\mathbf{F}\vec{x}_{t+2} + \mathbf{G}\vec{x}_{t+1} + \mathbf{H}\vec{x}_t + \mathbf{J}\vec{y}_{t+1} + \mathbf{K}\vec{y}_t + \mathbf{L}\vec{z}_{t+1} + \mathbf{M}\vec{z}_t\right] \\
\vec{z}_{t+1} &= \mathbf{N}\vec{z}_t + \vec{e}_{t+1}
\end{aligned}$$

[1] Check out Appendix A.
[2] For derivations see Appendix B

where $\vec{x}_t$ is an $(m \times 1)$ vector of endogenous state variables, z_t is a $(k \times 1)$ vector of exogenous state variables, and y_t is an $(n \times 1)$ vector of other endogenous variables. [15]

Solution. Let

$$\vec{x}_t = \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix}, \vec{y}_t = \begin{pmatrix} \hat{Y}_t \\ \hat{I}_t \\ \hat{N}_t \\ \hat{r}_t \\ \hat{w}_t \\ \hat{\lambda}_t \end{pmatrix}, \vec{z}_t = \begin{pmatrix} \hat{z}_t \end{pmatrix}.$$

The matrices are

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ K_{ss} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \alpha & 0 \\ -1 & 0 \\ 0 & 0 \\ -(1-\delta)K_{ss} & 0 \\ 0 & C_{ss} \\ 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 & 0 & (1-\alpha) & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -I_{ss} & 0 & 0 & 0 & 0 \\ -Y_{ss} & I_{ss} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in the first equation,

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{\beta\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 0 & -\frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0 & \frac{\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \\ 0 & 0 & 0 & \beta\lambda_{ss} r_{ss} & 0 & \beta\lambda_{ss}(r_{ss}+1-\delta) \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in the second equation, and

$$\mathbf{N} = \begin{pmatrix} \rho \end{pmatrix}, \vec{e}_{t+1} = \begin{pmatrix} \varepsilon_{zt} \end{pmatrix}$$

in the last one.[3]

Exercise 1.7. Let $\beta = 0.9$, $\nu = 0.85$, $\phi = 0.5$, $\delta = 0.1$, $\alpha = 0.36$,and $\rho = 0.95$. Use the `uc_xyz` function from the `macro_modeling` module and your answer to Exercise 1.6 to derive the matrices $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ that yield the policy functions

$$\begin{aligned} \vec{x}_{t+1} &= \mathbf{P}\vec{x}_t + \mathbf{Q}\vec{z}_t \\ \vec{y}_t &= \mathbf{R}\vec{x}_t + \mathbf{S}\vec{z}_t \end{aligned}$$

[3] Check out Appendix C for the setup.

Offer a brief interpretation of your findings. If you think it is useful, you may write out the policy functions for each variable in x_t and y_t. [15]

Solution. I got the system

$$\begin{pmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{pmatrix} = \begin{pmatrix} 0.77940762 & 0.05395422 \\ 0.04173628 & 0.81810069 \end{pmatrix} \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix} + \begin{pmatrix} 0.32412304 \\ 0.18995039 \end{pmatrix} \left(\hat{z}_t\right)$$

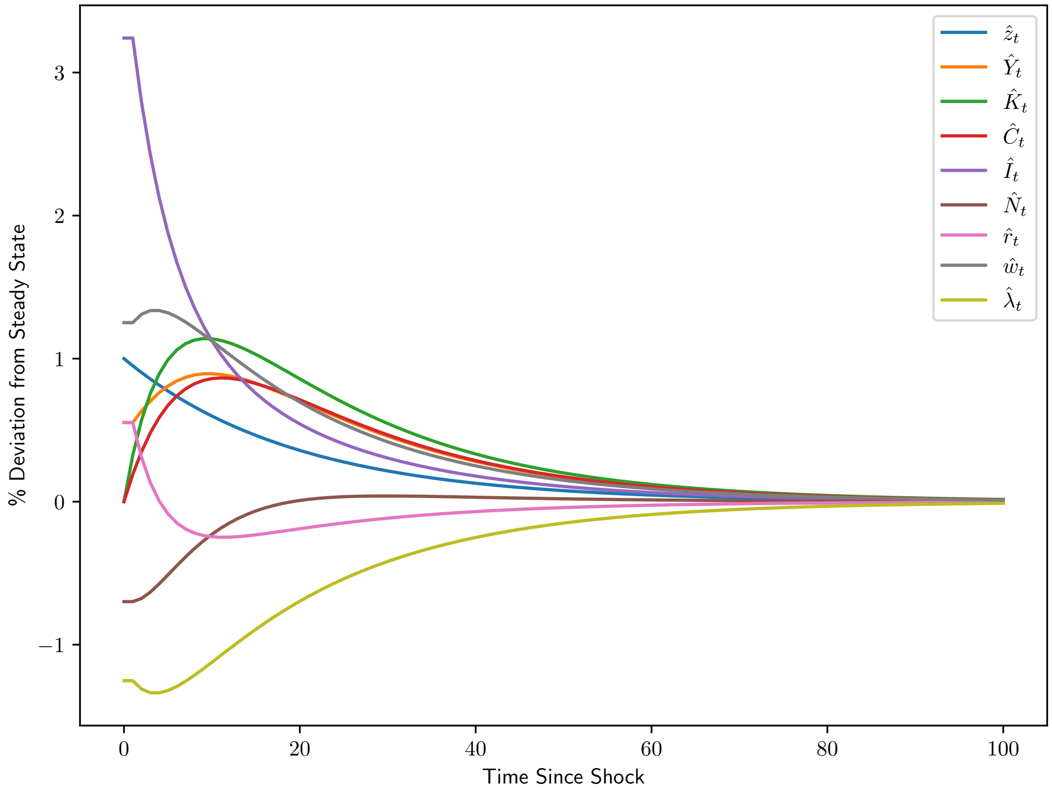
$$\begin{pmatrix} \hat{Y}_t \\ \hat{I}_t \\ \hat{N}_t \\ \hat{r}_t \\ \hat{w}_t \\ \hat{\lambda}_t \end{pmatrix} = \begin{pmatrix} -0.20564175 & 0.92147982 \\ -1.20592382 & 0.53954217 \\ -0.88381523 & 1.43981222 \\ -1.20564175 & 0.92147982 \\ 0.67817348 & -0.5183324 \\ -0.67817348 & 0.5183324 \end{pmatrix} \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix} + \begin{pmatrix} 0.55271507 \\ 3.24123037 \\ -0.6988827 \\ 0.55271507 \\ 1.25159777 \\ -1.25159777 \end{pmatrix} \left(\hat{z}_t\right).$$

This suggests that $\hat{K}_{t+1}$ and $\hat{C}_{t+1}$ are increasing with respect to $\hat{K}_t$ and $\hat{C}_t$ (since the coefficients are positive), but do regress closer to the steady state (since they are less than 1). They also are increasing with $\hat{z}_t$, suggesting that they'll behave procyclically.

Similarly, we find that $\hat{Y}_t$, $\hat{I}_t$, $\hat{r}_t$, and $\hat{w}_t$ behave procyclically, since the $\hat{z}_t$ coefficient is positive. $\hat{N}_t$ (and $\hat{\lambda}_t$) behave countercyclical, suggesting that when labor output (and output in general) is more efficient, households take advantage by having more leisure.

Exercise 1.8. Use your answer to Exercise 1.7 to plot the impulse-response functions of output, consumption, hours worked, and investment in response to a TFP shock. Are TFP shocks still consistent with the business cycle facts? *Hint*: strictly speaking, you don't need Python to answer this question. [10]

Solution. I got the below figure.



Exercise 1.9. Redo Exercise 1.8 under the assumption that $\nu = 0$ (this is, there is no habit persistence, which is the baseline we've covered in class so far). Compare the impulse-response functions of both specifications. How are they different? This is an open-ended question, but a good answer should look at how the impulse-response functions differ in amplitude (i.e., how large are the changes plotted in your graph) and persistence (i.e., for how many periods are the changes different from zero). [5]

Solution. TBD

Appendices

A Steady State Derivation

This was awful and not fun.

Start with

$$Y_{ss} = z_{ss} K_{ss}^{\alpha} N_{ss}^{1-\alpha} \tag{A.1}$$
$$r_{ss} = \alpha Y_{ss} K_{ss}^{-1} \tag{A.2}$$
$$w_{ss} = (1-\alpha) Y_{ss} N_{ss}^{-1} \tag{A.3}$$
$$I_{ss} = K_{ss} - (1-\delta) K_{ss} \tag{A.4}$$
$$Y_{ss} = C_{ss} + I_{ss} \tag{A.5}$$
$$\lambda_{ss} = \frac{1}{C_{ss} - \nu C_{ss}} - \frac{\beta\nu}{C_{ss} - \nu C_{ss}} \tag{A.6}$$
$$\phi = \lambda_{ss} w_{ss} \tag{A.7}$$
$$\lambda_{ss} = \beta(\lambda_{ss} r_{ss} + (1-\delta)\lambda_{ss}). \tag{A.8}$$

λ_{ss} cancels in A.8 to get

$$1 = \beta(r_{ss} + 1 - \delta)$$

which means

$$r_{ss} = \frac{1}{\beta} - 1 + \delta.$$

A.6 becomes

$$\lambda_{ss} = \frac{1}{C_{ss} - \nu C_{ss}} - \frac{\beta\nu}{C_{ss} - \nu C_{ss}} = \frac{1-\beta\nu}{(1-\nu)C_{ss}}.$$

For the next equations, we use the "solve everything for K and hope for the best" strategy.

Starting with A.4, that gets

$$I_{ss} = K_{ss} - (1-\delta)K_{ss} = \delta K_{ss}.$$

Then, treating r_{ss} as solved, A.2 is

$$r_{ss} = \alpha Y_{ss} K_{ss}^{-1},$$

which becomes

$$Y_{ss} = \frac{r_{ss}}{\alpha} K_{ss}.$$

Plugging the expression for Y_{ss} and $z_{ss} = 1$ into A.1 gets

$$\frac{r_{ss}}{\alpha} K_{ss} = z_{ss} K_{ss}^{\alpha} N_{ss}^{1-\alpha}.$$

Dividing this by K_{ss}^{α} gets

$$\frac{r_{ss}}{\alpha z_{ss}} K_{ss}^{1-\alpha} = N_{ss}^{1-\alpha},$$

which when raised to the $\frac{1}{1-\alpha}$ power gets

$$N_{ss} = \left(\frac{r_{ss}}{\alpha z_{ss}}\right)^{\frac{1}{1-\alpha}} K_{ss}.$$

Using the expressions for Y_{ss} and I_{ss} in A.5 gets

$$\frac{r_{ss}}{\alpha} K_{ss} = C_{ss} + \delta K_s s,$$

which simplifies to

$$C_{ss} = \left(\frac{r_{ss}}{\alpha} - \delta\right) K_{ss}.$$

Finally, plugging the expression for λ_{ss} into A.7 gets

$$\phi = \frac{1-\beta\nu}{(1-\nu)C_{ss}} w_{ss},$$

which becomes

$$w_{ss} = \frac{\phi(1-\nu)}{1-\beta\nu} C_{ss} = \frac{\phi(1-\nu)}{1-\beta\nu}\left(\frac{r_{ss}}{\alpha} - \delta\right) K_{ss}.$$

Finally, plugging everything into A.3 gets

$$\frac{\phi(1-\nu)}{1-\beta\nu}\left(\frac{r_{ss}}{\alpha} - \delta\right) K_{ss} = (1-\alpha)\left(\frac{r_{ss}}{\alpha} K_{ss}\right)\left(\left(\frac{r_{ss}}{\alpha z_{ss}}\right)^{\frac{1}{1-\alpha}} K_{ss}\right)^{-1} = (1-\alpha) z_{ss}^{\frac{1}{1-\alpha}} \left(\frac{r_{ss}}{\alpha}\right)^{\frac{\alpha}{\alpha-1}},$$

which gets the solution

$$K_{ss} = z_{ss}^{\frac{1}{1-\alpha}} \frac{(1-\alpha)(1-\beta\nu)}{\phi(1-\nu)\left(\frac{r_{ss}}{\alpha} - \delta\right)} \left(\frac{r_{ss}}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

Using the expression for r_{ss}, we can find K_{ss} and can plug in values backwards to get all the other variables.

B Log-Linearizing Everything

Mario, I've done more algebra for this than any of my math classes.

Start with the system

$$Y_t = z_t K_t^{\alpha} N_t^{1-\alpha} \tag{B.1}$$
$$r_t = \alpha Y_t K_t^{-1} \tag{B.2}$$
$$w_t = (1-\alpha) Y_t N_t^{-1} \tag{B.3}$$
$$I_t = K_{t+1} - (1-\delta) K_t \tag{B.4}$$
$$Y_t = C_t + I_t \tag{B.5}$$
$$\lambda_t = \frac{1}{C_t - \nu C_{t-1}} - \frac{\beta\nu}{C_{t+1} - \nu C_t} \tag{B.6}$$
$$\phi = \lambda_t w_t \tag{B.7}$$
$$\lambda_t = \beta(\lambda_{t+1} r_{t+1} + (1-\delta)\lambda_{t+1}). \tag{B.8}$$

Moving top to bottom, B.1 becomes

$$Y_{ss}(1+\hat{Y}_t) = z_{ss}K_{ss}^{\alpha}N_{ss}^{1-\alpha}(1+\hat{z}_t+\alpha\hat{K}_t+(1-\alpha)\hat{N}_t).$$

Canceling A.1 and subtracting 1 from both sides gets

$$\hat{Y}_t = \hat{z}_t+\alpha\hat{K}_t+(1-\alpha)\hat{N}_t.$$

B.2 becomes

$$r_{ss}(1+\hat{r}_t) = \alpha Y_{ss}K_{ss}^{-1}(1+\hat{Y}_t-\hat{K}_t).$$

Canceling A.2 and subtracting 1 from both sides gets[4]

$$\hat{r}_t = \hat{Y}_t-\hat{K}_t.$$

B.3 becomes

$$w_{ss}(1+\hat{w}_t) = (1-\alpha)Y_{ss}N_{ss}^{-1}(1+\hat{Y}_t-\hat{N}_t),$$

which when we cancel A.3 and subtract 1 gets

$$\hat{w}_t = \hat{Y}_t-\hat{N}_t.$$

B.4 becomes

$$I_{ss}(1+\hat{I}_t) = K_{ss}(1+\hat{K}_{t+1})-(1-\delta)K_{ss}(1+\hat{K}_t),$$

which means

$$I_{ss}+I_{ss}\hat{I}_t = K_{ss}+K_{ss}\hat{K}_{t+1}-(1-\delta)K_{ss}-(1-\delta)K_{ss}\hat{K}_t.$$

Canceling A.4 gets

$$I_{ss}\hat{I}_t = K_{ss}\hat{K}_{t+1}-(1-\delta)K_{ss}\hat{K}_t.$$

B.5 becomes

$$Y_s s(1+\hat{Y}_t) = C_{ss}(1+\hat{C}_t)+I_{ss}(1+\hat{I}_t),$$

which when expanded is

$$Y_s s+Y_{ss}\hat{Y}_t = C_{ss}+C_{ss}\hat{C}_t+I_{ss}+I_{ss}\hat{I}_t.$$

Canceling A.5 gets

$$Y_{ss}\hat{Y}_t = C_{ss}\hat{C}_t+I_{ss}\hat{I}_t.$$

Before we log-linearize B.6, we define

$$X_t = C_t-\nu C_{t-1}$$

[4]This is going to get really repetitive.

so that[5]

$$X_{ss}\hat{X}_t = C_{ss}\hat{C}_t - \nu C_{ss}\hat{C}_{t-1}$$

Plugging X_t into B.6 gets

$$\lambda_t = \frac{1}{X_t} - \frac{\beta\nu}{X_{t+1}} = X_t^{-1} - \beta\nu X_{t+1}^{-1}.$$

This becomes

$$\lambda_{ss}(1+\hat{\lambda}_t) = X_{ss}^{-1}(1-\hat{X}_t) - \beta\nu X_{ss}^{-1}(1-\hat{X}_{t+1}).$$

Canceling the steady state equation

$$\lambda_{ss} = X_{ss}^{-1} - \beta\nu X_{ss}^{-1}$$

gets

$$\lambda_{ss}\hat{\lambda}_t = \beta\nu X_{ss}^{-1}\hat{X}_{t+1} - X_{ss}^{-1}\hat{X}_t.$$

Plugging in

$$X_{ss}^{-1}\hat{X}_t = \frac{1}{(1-\nu)^2 C_{ss}}\hat{C}_t - \frac{\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t-1}$$

gets the final log-linearized system

$$\begin{aligned}\lambda_{ss}\hat{\lambda}_t &= \frac{\beta\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t+1} - \frac{\beta\nu^2}{(1-\nu)^2 C_{ss}}\hat{C}_t - \frac{1}{(1-\nu)^2 C_{ss}}\hat{C}_t + \frac{\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t-1} \\ &= \frac{\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t-1} + \frac{\beta\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t+1} - \frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}}\hat{C}_t.\end{aligned}$$

B.7 becomes

$$\phi = \lambda_{ss}w_{ss}(1+\hat{\lambda}_t+\hat{w}_t),$$

which canceling A.7 and subtracting 1 becomes

$$0 = \hat{\lambda}_t + \hat{w}_t.$$

Finally, B.8 becomes

$$\lambda_{ss}(1+\hat{\lambda}_t) = \beta\lambda_{ss}r_{ss}(1+\hat{\lambda}_{t+1}+\hat{r}_{t+1}) + \beta(1-\delta)\lambda_{ss}(1+\hat{\lambda}_{t+1}),$$

which means

$$\lambda_{ss} + \lambda_{ss}\hat{\lambda}_t = \beta\lambda_{ss}r_{ss} + \beta\lambda_{ss}r_{ss}\hat{\lambda}_{t+1} + \beta\lambda_{ss}r_{ss}\hat{r}_{t+1} + \beta(1-\delta)\lambda_{ss} + \beta(1-\delta)\lambda_{ss}\hat{\lambda}_{t+1}.$$

[5]Want proof? We create

$$X_{ss}(1+\hat{X}_t) = C_{ss}(1+\hat{C}_t) - \nu C_{ss}(1+\hat{C}_{t-1}),$$

which cancels with the steady state equation

$$X_{ss} = C_{ss} - \nu C_{ss}.$$

to get

$$X_{ss}\hat{X}_t = C_{ss}\hat{C}_t - \nu C_{ss}\hat{C}_{t-1}.$$

Canceling A.8 from this gets

$$\lambda_{ss}\hat{\lambda}_t = \beta\lambda_{ss}(r_{ss} + 1 - \delta)\hat{\lambda}_{t+1} + \beta\lambda_{ss}r_{ss}\hat{r}_{t+1}.$$

To log-linearize the law of motion for TFP, we start with

$$z_t = (1-\rho)z_{ss} + \rho z_{t-1} + \varepsilon_{zt}.$$

Then, we turn it into

$$z_{ss}(1+\hat{z}_t) = (1-\rho)z_{ss} + \rho z_{ss}(1+\hat{z}_{t-1}) + \varepsilon_{zt}.$$

Subtracting z_{ss} on both sides gets

$$\hat{z}_t = \rho\hat{z}_{t-1} + \varepsilon_{zt}$$

C Creating the Matrices

The first matrix equation,

$$\vec{0} = \mathbf{A}\vec{x}_{t+1} + \mathbf{B}\vec{x}_t + \mathbf{C}\vec{y}_t + \mathbf{D}\vec{z}_t,$$

will include the equations

$$\begin{aligned}
\hat{Y}_t &= \hat{z}_t + \alpha\hat{K}_t + (1-\alpha)\hat{N}_t \\
\hat{r}_t &= \hat{Y}_t - \hat{K}_t \\
\hat{w}_t &= \hat{Y}_t - \hat{N}_t \\
I_{ss}\hat{I}_t &= K_{ss}\hat{K}_{t+1} - (1-\delta)K_{ss}\hat{K}_t \\
Y_{ss}\hat{Y}_t &= C_{ss}\hat{C}_t + I_{ss}\hat{I}_t \\
0 &= \hat{\lambda}_t + \hat{w}_t
\end{aligned}$$

Therefore, it becomes

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ K_{ss} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ -1 & 0 \\ 0 & 0 \\ -(1-\delta)K_{ss} & 0 \\ 0 & C_{ss} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & 0 & (1-\alpha) & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -I_{ss} & 0 & 0 & 0 & 0 \\ -Y_{ss} & I_{ss} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{Y}_t \\ \hat{I}_t \\ \hat{N}_t \\ \hat{r}_t \\ \hat{w}_t \\ \hat{\lambda}_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\hat{z}_t).$$

Thus,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ K_{ss} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \alpha & 0 \\ -1 & 0 \\ 0 & 0 \\ -(1-\delta)K_{ss} & 0 \\ 0 & C_{ss} \\ 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 & 0 & (1-\alpha) & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -I_{ss} & 0 & 0 & 0 & 0 \\ -Y_{ss} & I_{ss} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second matrix equation,

$$\vec{0} = \mathbb{E}\left[\mathbf{F}\vec{x}_{t+2} + \mathbf{G}\vec{x}_{t+1} + \mathbf{H}\vec{x}_t + \mathbf{J}\vec{y}_{t+1} + \mathbf{K}\vec{y}_t + \mathbf{L}\vec{z}_{t+1} + \mathbf{M}\vec{z}_t\right],$$

will include the equations[6]

$$\begin{aligned} \lambda_{ss}\hat{\lambda}_{t+1} &= \frac{\nu}{(1-\nu)^2 C_{ss}}\hat{C}_t + \frac{\beta\nu}{(1-\nu)^2 C_{ss}}\hat{C}_{t+2} - \frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}}\hat{C}_{t+1} \\ \lambda_{ss}\hat{\lambda}_t &= \beta\lambda_{ss}(r_{ss}+1-\delta)\hat{\lambda}_{t+1} + \beta\lambda_{ss} r_{ss}\hat{r}_{t+1}. \end{aligned}$$

Therefore, it becomes

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} =& \mathbb{E}\left[\begin{pmatrix} 0 & \frac{\beta\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K}_{t+2} \\ \hat{C}_{t+2} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{pmatrix} + \begin{pmatrix} 0 & \frac{\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{K}_t \\ \hat{C}_t \end{pmatrix} \right. \\ & + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \\ 0 & 0 & 0 & \beta\lambda_{ss} r_{ss} & 0 & \beta\lambda_{ss}(r_{ss}+1-\delta) \end{pmatrix} \begin{pmatrix} \hat{Y}_{t+1} \\ \hat{I}_{t+1} \\ \hat{N}_{t+1} \\ \hat{r}_{t+1} \\ \hat{w}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \end{pmatrix} \begin{pmatrix} \hat{Y}_t \\ \hat{I}_t \\ \hat{N}_t \\ \hat{r}_t \\ \hat{w}_t \\ \hat{\lambda}_t \end{pmatrix} \\ & \left. + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left(\hat{z}_{t+1}\right) + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left(\hat{z}_t\right) \right] \end{aligned}$$

Thus,

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{\beta\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 0 & -\frac{1+\beta\nu^2}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0 & \frac{\nu}{(1-\nu)^2 C_{ss}} \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \\ 0 & 0 & 0 & \beta\lambda_{ss} r_{ss} & 0 & \beta\lambda_{ss}(r_{ss}+1-\delta) \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{ss} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[6]The time is shifted one period forward from the log-linearized version in the first one.

Finally, the last matrix equation,

$$\vec{z}_{t+1} = \mathbf{N}\vec{z}_t + \vec{e}_{t+1}$$

will include the law of motion for TFP,

$$\hat{z}_t = \rho\hat{z}_{t-1} + \varepsilon_{zt}.$$

Therefore, it becomes

$$\begin{pmatrix}\hat{z}_{t+1}\end{pmatrix} = \begin{pmatrix}\rho\end{pmatrix}\begin{pmatrix}\hat{z}_t\end{pmatrix} + \begin{pmatrix}\varepsilon_{zt}\end{pmatrix}.$$

Thus,

$$\mathbf{N} = \begin{pmatrix}\rho\end{pmatrix}, \vec{e}_{t+1} = \begin{pmatrix}\varepsilon_{zt}\end{pmatrix}.$$