Accuracy of Numerical Methods

1 Introduction

In this paper we will investigate methods of approximating solutions of ordinary differential equations of the form:

$$\begin{cases} \frac{dy}{dt} = f(t, y), \\ y(t_0) = y_0 \end{cases} \tag{1}$$

It is often the case that differential equations cannot be solved analytically, therefore it is important to understand and utilize numerical methods to approximate solutions to these equations. In order to study the accuracy of such methods, we implemented three different approximation methods to model a falling object under the force of gravity and the drag force, and measured the error between the approximate solutions and the exact solution.

2 The Problem

We will consider the situation of a falling object:

$$\begin{cases} \frac{dv}{dt} = g - \frac{c_d}{m} v^2, \\ v(0) = 0 \end{cases} \tag{2}$$

Where $g \approx 9.81 m/s^2$ is the free-fall acceleration, $m = 75$kg is the mass of the object, and $c_d = 0.25$kg/m is the drag coefficient. In this case, we know that the exact solution to this equation:

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(t \sqrt{\frac{gc_d}{m}} \right) \tag{3}$$

We will find approximate solutions to (2) using:

- Euler Method
- Trapezoidal Method
- Runge-Kutta Method (RK4)

And compare the results with the exact solution (3).

3 Methodology

We will use $\Delta t = 3.3$, 0.1, 0.05, 0.025, 0.0125 for our analysis. For each approximate solution, we will find the error by the formula: $error = |v_n - v(t_n)|$. For each value of Δt, we will determine what the maximum error $E(\Delta t_n)$ was and then compare the maximum errors to analyze the order of accuracy by the formula:

$$\text{Order k} = \frac{log\left(\frac{E(\Delta t_n)}{E(\Delta t_{n+1})}\right)}{log\left(\frac{\Delta t_n}{\Delta t_{n+1}}\right)}$$

4 Euler Method

For the Euler method we calculate each point by starting from the previous point and moving along the trajectory of its slope for one time-step. We recall from (2) that $\frac{dv}{dt} = g - \frac{c_d}{m}v^2$, and the formula for the Euler method is therefore:

$$v_{n+1} = v_n + \Delta t\left(g - \frac{c_d}{m}v_n^2\right) \tag{4}$$

As we reduce Δt by $\frac{1}{2}$, we see that the error tends to reduce by approximately $\frac{1}{2}$. From this we can guess that the Euler method is first-order accurate and, as it turns out, it has been proven so.

Δt	Maximum Error	Order k
3.3	8.10095	-
0.1	0.181522	1.08633
0.05	0.0904899	1.00431
0.025	0.0451779	1.00214
0.0125	0.0225722	1.00107

Table 1: Convergence of the Euler method.

5 Trapezoidal Method

For the Trapezoidal Method formula we have:

$$\begin{aligned} \widetilde{v}_{n+1} &= v_n + \Delta t \left(g - \frac{c_d}{m} v_n^2 \right) \\ v_{n+1} &= v_n + \frac{\Delta t}{2} \left[\left(g - \frac{c_d}{m} v_n^2 \right) + \left(g - \frac{c_d}{m} (\widetilde{v}_{n+1})^2 \right) \right] \end{aligned} \tag{5}$$

Where we note that $\widetilde{v}_{n+1}$ is obtained by using Euler's method. This is necessary because the Trapezoidal method is implicitly defined. As we reduced Δt by $\frac{1}{2}$, the error tended to reduce by approximately $\frac{1}{4}$. This implies, though does not prove, that the Trapezoidal method is 2nd order accurate

Δt	Maximum Error	Order k
3.3	4.44347	N/A
0.1	0.00217233	2.18029
0.05	0.000538064	2.01339
0.025	0.000133895	2.00668
0.0125	3.33961e-05	2.00335

Table 2: Convergence of the trapezoidal method.

6 Classical Runge-Kutta Method (RK4)

The classical Runge-Kutta method is a one-step method that calculates the derivatives at four points and uses a weighted average of those derivatives to get a very accurate approximation of the next point. Its formula is given as:

$$\begin{aligned} f(t_n, y_n) &= g - \frac{c_d}{m} v_n^2 \\ k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{\Delta t}{2}, y_n + k_1 \frac{\Delta t}{2} \right) \\ k_3 &= f\left(t_n + \frac{\Delta t}{2}, y_n + k_2 \frac{\Delta t}{2} \right) \\ k_4 &= f\left(t_n + \Delta t, y_n + k_3 \Delta t \right) \end{aligned}$$

$$y_{n+1} = y_n + \Delta t \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \tag{6}$$

As we reduce Δt by $\frac{1}{2}$, we see that the error tends to reduce by approximately $\frac{1}{16}$, indicating that RK4 is 4th order accurate.

Δt	Maximum Error	Order k
3.3000	0.174947	N/A
0.1000	8.63907e-08	4.15303
0.0500	5.35091e-09	4.01302
0.0250	3.33038e-10	4.00602
0.0125	2.09823e-11	3.98844

Table 3: Convergence of RK4.

7 Conclusion

In general, it appears that Euler is less accurate than Trapezoidal, and both are less accurate than RK4. For small Δt, each method produced fairly accurate approximations of the solution to the ODE, but RK4 converged much more quickly and even for the largest time-step ($\Delta t = 3.3$) RK4 produced solutions very close to exact:

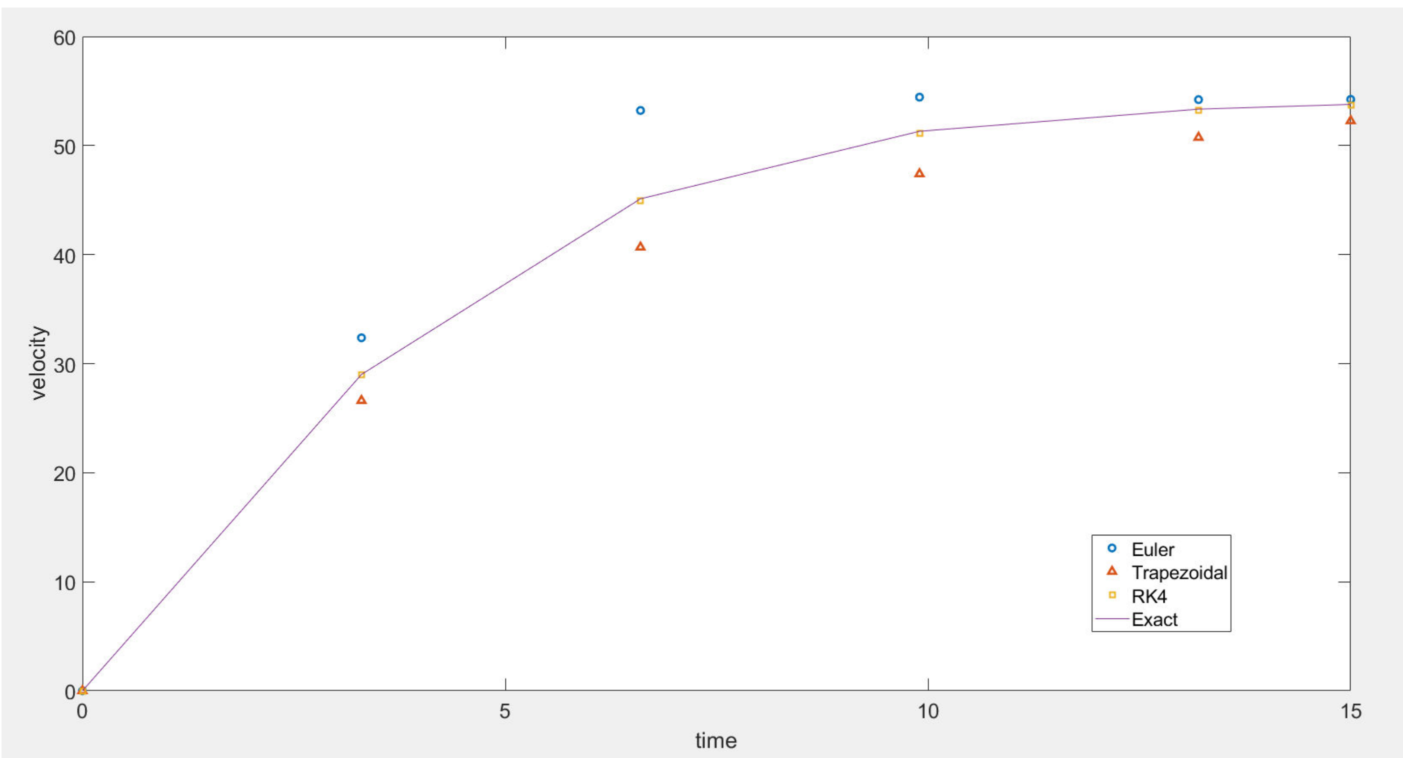

Figure 1: Comparison of approximate solutions and the exact solution. ($\Delta t = 3.3$ seconds)

We must consider which method to use depending on the problem and its parameters, but in general higher order methods will be less computationally demanding and therefore preferable.