Bouncing Ball

1 Introduction

In this paper we will approximate the solution to a system of second-order differential equations of the form:

$$\begin{cases} \frac{d^2x}{dt} = f(t, x, \frac{dx}{dt}) \\ \frac{d^2y}{dt} = f(t, y, \frac{dy}{dt}) \end{cases} \tag{1}$$

With initial conditions:

$$\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \\ \frac{dx}{dt}(t_0) = v_{x0} \\ \frac{dy}{dt}(t_0) = v_{y0} \end{cases} \tag{2}$$

We will do this by reducing this second-order system of ODE to a first-order system of ODE and applying numerical methods to approximate the solution.

2 The Problem

We are given the initial position and velocity of a ball bouncing inside of an empty container. The only force acting on the ball is the force of gravity. Thus, we have a second-order ODE:

$$\begin{cases} \frac{d^2x}{dt} = 0 \\ \frac{d^2y}{dt} = -g \\ x(0) = 0.1 \\ y(0) = 0.7 \\ \frac{dx}{dt}(0) = 3 \\ \frac{dy}{dt}(0) = 1 \\ a = 0 \\ b = 1 \\ c = 0 \\ d = 1 \end{cases} \tag{3}$$

Where $g \approx 9.81 m/s^2$ is the free-fall acceleration, $r = 0.05$m is radius of the ball, and a, b, c, and d are the position of the left, right, bottom, and top walls, respectively. We will find approximate solutions to (3) by converting this higher-order system of ODE to a first-order system of ODE, by substituting:

$$\begin{aligned} v_x &= \frac{dx}{dt} \\ \frac{dv_x}{dt} &= \frac{d^2x}{dt} \\ v_y &= \frac{dy}{dt} \\ \frac{dv_y}{dt} &= \frac{d^2y}{dt} \end{aligned} \tag{4}$$

We can now use (4) to rewrite (1) as a system of first-order ODE:

$$\begin{cases} \frac{dv_x}{dt} = f(t, x, v_x) \\ \frac{dv_y}{dt} = f(t, y, v_y) \end{cases} \tag{5}$$

Proceeding from (5) we can approximate solutions using the trapezoidal method and compare the results with the exact solution.

3 Error Analysis

We will analyze the error of our approximation at time t = 0.931 by using Δt = 0.02, 0.01, 0.005, 0.0025, 0.00125 and 0.000625 and comparing the resulting solution (for position and velocity in x and y) with the exact solution. We will then calculate the order of accuracy by using the formula:

$$\text{Order k} = \frac{log\left(\frac{E(\Delta t_n)}{E(\Delta t_{n+1})}\right)}{log\left(\frac{\Delta t_n}{\Delta t_{n+1}}\right)}$$

4 Trapezoidal Method

For the Trapezoidal Method formula we have:

$$\begin{aligned} v_x^{n+1} &= v_x^n \\ x_{n+1} &= x_n + \Delta t(v_x^n) \\ v_y^{n+1} &= v_y^n - \Delta t g \\ \widetilde{y}_{n+1} &= y_n + \Delta t \left(v_y^n\right) \\ y^{n+1} &= y_n + \frac{\Delta t}{2}\left(v_y^n + v_y^{n+1}\right) \end{aligned} \tag{6}$$

The trapezoidal method for x, and v_y simplifies to Euler's method, because the horizontal acceleration is always 0, and the force of gravity is constant (and therefore v_x is not changing and v_y is changing at a constant rate of g).

Now because the ball is inside of a box, we need to find the cases where the ball collides with the wall and bounces off it. When this occurs, we damp the normal velocity by a factor $\alpha = 0.8$ and the tangential velocity by a factor $\beta = 0.9$. The problem then is finding the exact time that these collisions occur, as we are approximating solutions with a finite time-step, and it is possible that we overshoot the walls. To solve this problem, we must calculate a smaller Δt between the last position that was within the walls and the wall itself. We call the position of the left-wall 'a', right-wall 'b', floor 'c', ceiling 'd'. For the right wall we can construct the diagram:

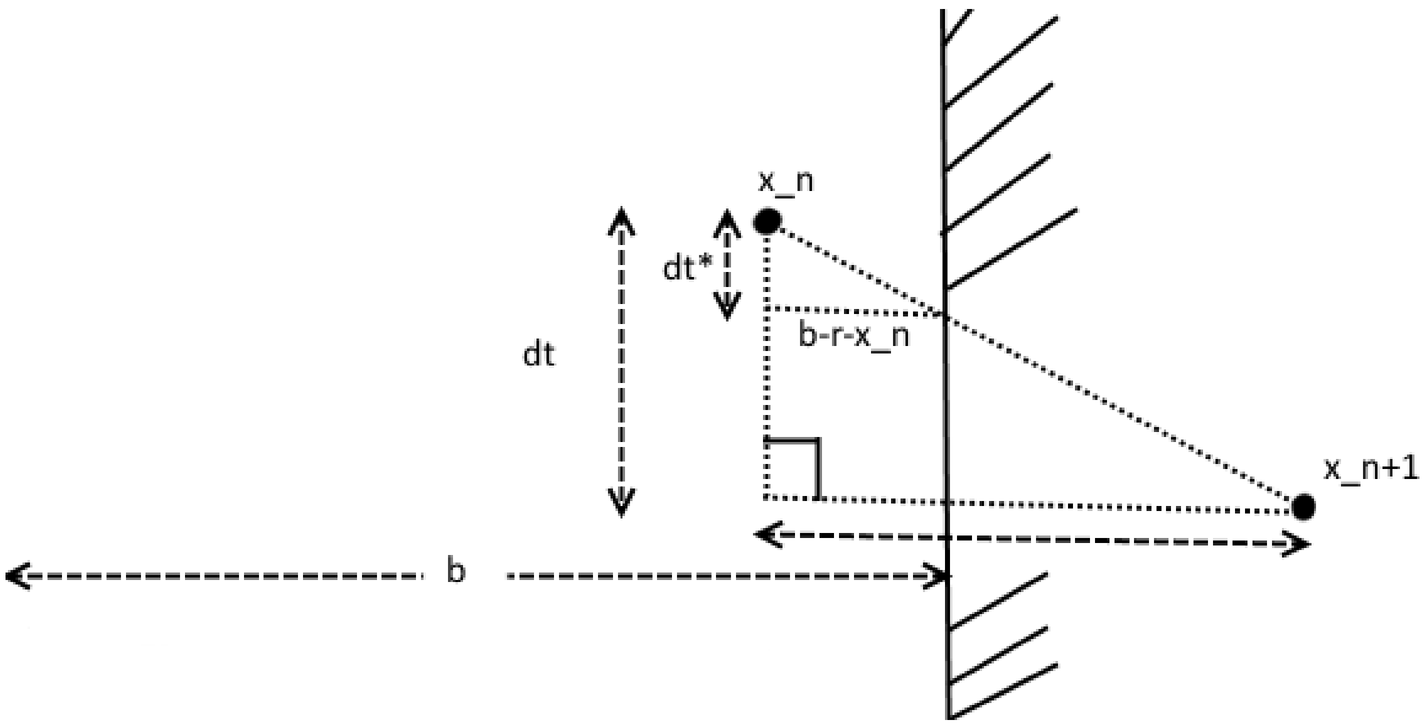

Figure 1: Example of a collision with the right-wall.

From Figure 1, we recognize by similar triangles:

$$\frac{\Delta t*}{\Delta t} = \frac{x_{n+1} - x_n}{b - r - x_n} \tag{7}$$

Rearranging (7), we can find $\Delta t*$, and use that to reapproximate a solution that will lie within the boundary of the box:

$$\Delta t* = \Delta t \frac{x_{n+1} - x_n}{b - r - x_n} \tag{8}$$

Once we have made a new approximation for x_{n+1} using $\Delta t*$, we damp the normal velocity, v_x in this case, by α and the tangential velocity (v_y) by β.

We can do the same for collisions with other walls, and the $\Delta t*$ are:

Left-wall:

$$\Delta t* = \Delta t \frac{|a + r - x_n|}{|x_{n+1} - x_n|} \tag{9}$$

Floor:

$$\Delta t* = \Delta t \frac{|c + r - y_n|}{|y_{n+1} - y_n|} \tag{10}$$

Ceiling:

$$\Delta t* = \Delta t \frac{|d - r - y_n|}{|y_{n+1} - y_n|} \tag{11}$$

For each wall we damp by α for the normal and β for the tangential velocity, and in the case of collision with the floor or ceiling, the normal velocity will be v_y and the tangential velocity will be v_x, so we damp by the appropriate factors for each.

5 Results

We now use the exact solution provided to compare with our approximate solutions for each time-step at time t = 0.931, and obtain the following results for each component of the position and velocity vectors:

Δt	x Error	Order k
0.020000	2.24134e-05	-
0.010000	5.99548e-06	1.902416
0.005000	1.31331e-06	2.190669
0.002500	3.98901e-07	1.719101
0.001250	4.63547e-08	3.105244
0.000625	1.96774e-08	1.236178

Average order: 2.030722

Table 1: Accuracy of the method for the x position

Δt	y Error	Order k
0.020000	9.08395e-05	-
0.010000	2.42514e-05	1.905251
0.005000	5.30928e-06	2.191479
0.002500	1.61245e-06	1.719259
0.001250	1.87369e-07	3.105304
0.000625	7.95371e-08	1.236182

Average order: 2.031495

Table 2: Accuracy of the method for the y position

Δt	v_x Error	Order k
0.020000	0	0.000000
0.010000	0	0.000000
0.005000	0	0.000000
0.002500	0	0.000000
0.001250	0	0.000000
0.000625	0	0.000000

Average order: 0

Table 3: Accuracy of the method for the x velocity

Δt	v_y Error	Order k
0.020000	0.00184248	0.000000
0.010000	0.000492853	1.902416
0.005000	0.000107959	2.190669
0.002500	3.27914e-05	1.719101
0.001250	3.81055e-06	3.105243
0.000625	1.61756e-06	1.236177

Average order: 2.030721

Table 4: Accuracy of the method for the y velocity

From this data, we see that the method is O(Δt^2) accurate. The reason there is no error in v_x is due to the fact that there is no force and therefore no acceleration in the x-direction. So there is nothing to approximate as v_x only changes when it is damped during a collision.

6 Conclusion

By converting the higher-order system of ODE to a first-order system of ODE and then applying the trapezoidal method we were able to very accurately approximate the position and velocity of the ball as it moves in the container, producing a realistic looking simulation when the solution is plotted in real time.

7 Appendix – Snapshots

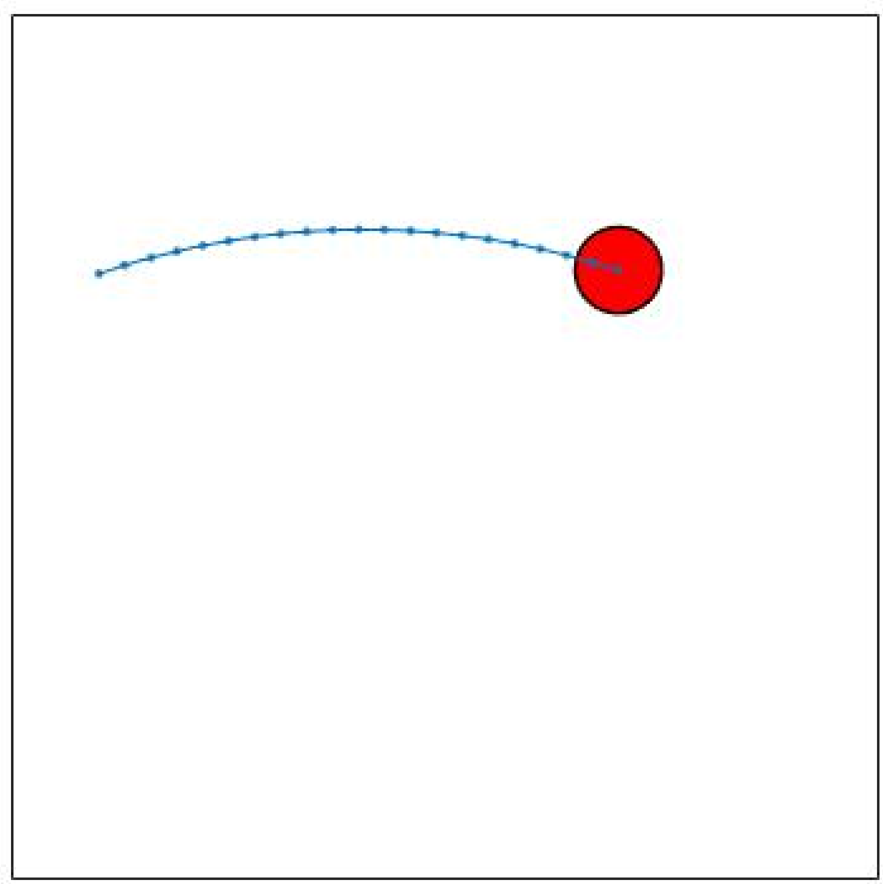

Figure 2: Position of the ball at t = 0.2 seconds

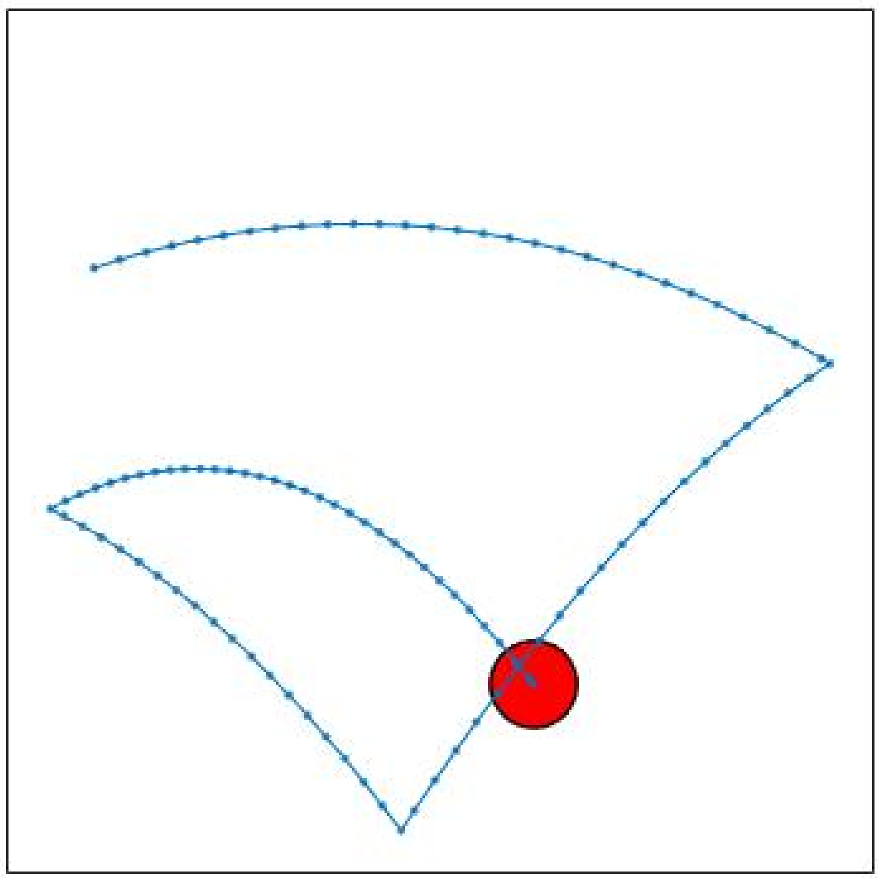

Figure 3: Position of the ball at t = 1.0 seconds

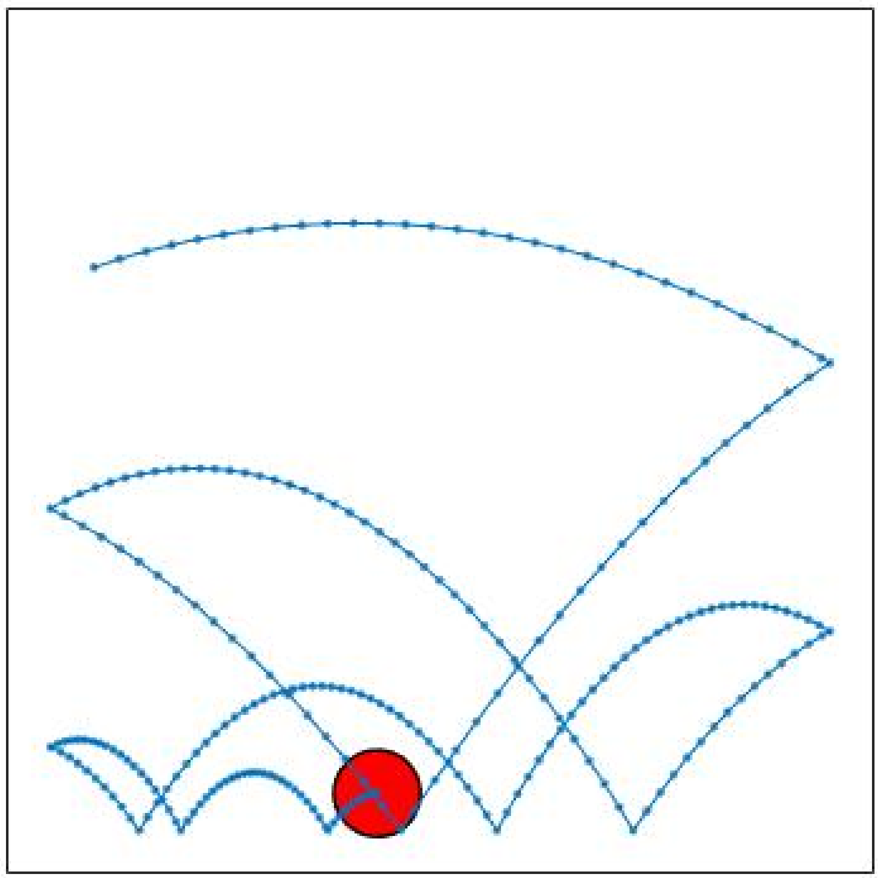

Figure 4: Position of the ball at t = 2.5 seconds