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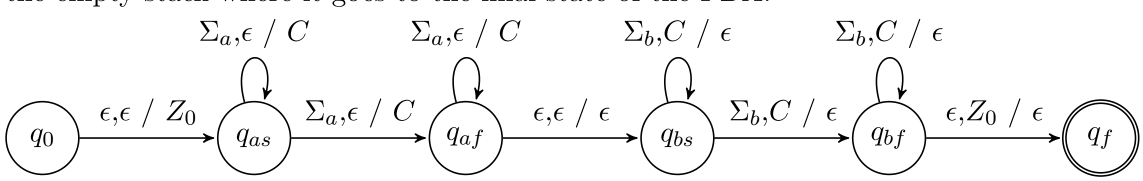
CS 510 Automata Theory
Homework 5

1. If $A$ and $B$ are languages define $A \diamond B = \{xy | x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$
Show that if $A$ and $B$ are regular languages then $A \diamond B$ is a CFL.

I will create a PDA and call it $P$, that will recognize $A \diamond B$ to show that it is a CFL, because CFL's and PDA's are equivalent. Since we are assuming $A$ and $B$ are regular they both have DFA that define all the string in $A_{DFA}$ and $B_{DFA}$ respectively. If it was just $A_{DFA}$ and $B_{DFA}$ concatenated we would know that it would be regular under the closure property, but we have to make sure that $|x| = |y|$. So we must push $x$ onto the stack, but instead of the string $x$ I'll just make it easier and push a $C$ on the stack for every character of $x$ is read. Then while reading a character of $y$ we pop, and check to see if it is a $C$ or $Z_0$. If $C$, then we continue reading $y$. If we are in the accepting state of $B_{DFA}$, and $Z_0$ is popped, then we go to the final state of $P$.

Here is a ROUGH idea how the new PDA will will work. It will start by pushing on $Z_0$. Then it will run through the DFA defined by $A_{DFA}$, when in the accepting state of $A(q_{af})$ it will nondeterministicly jump to the start $q_{bs}$, where it will run till it runs into the empty stack where it goes to the final state of the PDA.




Formally we have a DFA repersenting the regular language $A$ we will define as:
$A_{DFA} = (Q_a, \Sigma_a, \delta_a, q_{0a}, F_a)$

We also have a DFA repersenting the regular language $B$ we will define as:
$B_{DFA} = (Q_b, \Sigma_b, \delta_b, q_{0b}, F_b)$

I will define a PDA that accepts $A \diamond B$ defined as:
P $= (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

The states will be a new $q_0$ (used to put the bottom of the stack symbol) a new final state to detect the bottom of the stack. Then all the states from $A_{DFA}$ and $B_{DFA}$.
$Q = \{q_0, q_f\} \cup Q_a \cup Q_b$

The alphabet will be all the characters from language A, and B.
$\Sigma = \Sigma_a \cup \Sigma_b$

The only thing that will be on the stack is $C$ which is our count of $x$ and the bottom of the stack symbol($Z_0$).
$\Gamma = \{C, Z_0\}$

$\delta(q, a, X) =$

$\{q_{0a}, Z_0 | q = q_0, a = \epsilon, X = \epsilon\}$
move from the start state to $A_{DFA}$'s start state, while reading no input, nothing off the stack, and pushing $Z_0$

$\{\delta_a(q, a), C | q \in Q_a, a \in \Sigma_a, X = \epsilon\}$
move threw $A_{DFA}$ while reading input characters in the language of $A$, nothing off the stack, and pushing $C$

$\{q_0 b, \epsilon | q \in F_a, a = \epsilon, X = \epsilon\}$
move from a final state of $A_{DFA}$ to the start state of $B_{DFA}$, while reading no input, nothing off the stack, and pushing nothing

$\{\delta_b(q, a), C | q \in Q_b, a \in \Sigma_b, X = C\}$
move threw $B_{DFA}$ while reading input characters in the language of $B$, $C$ on the stack, and popping $C$

$\{q_f, \epsilon | q \in F_b, a = \epsilon, X = Z_0\}$
move from $B_{DFA}$'s final state to the final state, while reading no input characters, $Z_0$ on the stack, and popping $Z_0$

The new starting symbol.
$q_0 = q_0$

Bottom of stack character.
$Z_0 = Z_0$

The only accepting state is the new final state we created.
$F = \{q_f\}$

2. Lets define a $perfect\ shuffle$ of two languages $A$ and $B$ as:
$\{w|w = a_1b_1a_2b_2...a_kb_k$ where $a_1a_2...a_k \in A$ and $b_1b_2...b_k \in B\}$
Show that context free languages are not closed under perfect shuffle.

Let $A$ be CFL defined as:
$\Sigma = \{a, b\}, B_{CFL} = \{a^kb^{2k}|k \geq 1\ \}$

Let $B$ be CFL defined as:
$\Sigma = \{0, 1\}, A_{CFL} = \{0^{2k}1^k|k \geq 1\}$

(These 2 languages are CFL because they are able to push 2 characters when reading a and pop 1 character when reading b to keep track, and able to push 1 character while reading 0, and pop 2 characters while reading 1. And the only possibility for pumping to beat $A_{CFL}$ and $B_{CFL}$ are 0011 or aabb, and he can make it so you pump one a and two b's or two 0's and one 1 keep that ratio of 1:2 or 2:1 so they are CFL.)

$perfect\ shuffle$:
$\Sigma = \{a, b, 0, 1\}$
$PS = \{w|w = a_1b_1a_2b_2...a_kb_k$ where $a_1a_2...a_k \in A_{CFL}$ and $b_1b_2...b_k \in B_{CFL}\}$

When we perfect shuffle these 2 languages we are going to get the string like $a0a0b0b0b1b1$

So if we use the pumping lemma and make k $\geq$ n, the available windows that $vwx$ can be in are:
$a0a0$
$a0b0$
$b0b0$
$b0b1$
$b1b1$

If the adversary chooses $a0...a0$, $a0...a0$, or $b1...b1$ then is we unpump/pump what ever v and x he chooses it will put the 2:1 or 1:2 ratios out of whack. There is no way he can have a's and b's or 1's and 2's to keep the ratio.

If the adversary chooses $a0...b0$ or $b0...b1$ then he will be able to pick to keep the 0 and 1 ratio correct or a and b accordingly, but not both. He will be able to pick $v = a\ w = b0b$ to keep a:b ratio correct but there is a 0 stuck in there so the 0:1 ratio is off. If he picks pick $v = a\ w = \epsilon$ then the 0:1 are correct but the a:b are off. There is no way for him to win.

So the pumping lemma shows that context free languages are not closed under $perfect\ shuffle$.