Enhancing the Inverse Volatility Portfolio through Clustering

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KEY FINDINGS

- The article introduces a new portfolio construction approach called cluster-enhanced inverse volatility, designed to improve the performance of traditional inverse volatility portfolios.
- Empirical evidence, based on seven diverse datasets, supports the performance of the cluster-enhanced inverse volatility portfolio. It consistently performs at least as well as the traditional inverse volatility portfolios, showcasing its adaptability across various datasets.
- This article presents a practical way to boost portfolio performance by blending machine learning clustering techniques into the asset allocation process.

ABSTRACT

This article presents a novel approach to portfolio construction, termed *cluster-enhanced inverse volatility,* designed to enhance the effectiveness of traditional inverse volatility portfolios. The goal of the method is to cluster the data to meet the two conditions—the same Sharpe ratios across assets and equal pairwise correlations—under which the inverse volatility portfolio becomes theoretically equivalent to the mean–variance optimal portfolio. The authors show that, as the asset data increasingly meet these two conditions, the cluster-enhanced inverse volatility portfolio approaches the mean–variance optimal portfolio. Empirical evidence from various datasets indicates that the authors' cluster-enhanced inverse volatility portfolios outperform their traditional counterparts, particularly in portfolios with a large number of assets.

As discussed in de Prado (2016) and empirically shown in DeMiguel, Garlappi, and Uppal (2009), although it may be mathematically accurate, Markowitz's mean–variance optimal (MVO) portfolio produces inconsistent and poor results. The core problem lies in the use of expected returns—which are notoriously hard to forecast—and the need to invert a covariance matrix. Portfolio optimization techniques have evolved since Harry Markowitz's seminal work on the mean–variance model, offering various alternatives designed to either simplify portfolio construction or account for estimation errors in input parameters. One such alternative is the inverse volatility portfolio, a simple approach that allocates weights to assets based on the inverse of their individual volatilities. The inverse volatility portfolio is often lauded for its simplicity and lower susceptibility to estimation error compared to the mean–variance model, which make it perform well empirically (Kirby and Ostdiek 2012). In addition, the inverse volatility portfolio also possesses characteristics that make it suitable to be compared to the mean–variance optimal portfolio.[1]

However, the inverse volatility portfolio is usually not optimal from a theoretical point of view. It is equivalent to the theoretically optimal mean–variance portfolio only when certain conditions are met—namely, equal Sharpe ratios and equal pairwise correlations among assets. In this article, we develop a new portfolio construction methodology, which we name *cluster-enhanced inverse volatility*, by rearranging the data through clustering to create an environment that closely resembles the conditions under which the inverse volatility portfolio converges to the mean–variance optimal portfolio. Specifically, our methodology brings the data closer to the required assumption of equal pairwise correlations, with the use of clustering from the machine learning literature. We focus on the assumption of equal pairwise correlations because there is evidence that various asset classes have similar Sharpe ratios over a long period.[2]

We begin by laying the theoretical groundwork, elucidating the mathematical conditions under which the inverse volatility portfolio becomes equivalent to the mean–variance optimal portfolio.[3] Then, we introduce our clustering methodology, designed to group together assets with similar characteristics to approximate the ideal conditions for the optimality of the inverse volatility portfolio. Through empirical testing across diverse datasets—including multiasset portfolios and equity portfolios based on different industry classifications—we demonstrate that our cluster-enhanced inverse volatility portfolios perform at least as well as, if not better than, traditional inverse volatility portfolios.

The core idea behind our clustering approach is straightforward. Using standard machine learning clustering algorithms, we sort assets into clusters that are internally homogeneous (high correlation) and externally heterogeneous (low correlation). This approach aligns the data more closely with the assumption of equal pairwise correlations. Therefore, using the inverse volatility portfolio within these clusters leads to within-cluster portfolios that are closer to being theoretically optimal. After forming these within-cluster portfolios, we then use the inverse volatility method again between these within-cluster portfolios. Because we grouped together assets that are highly correlated, the correlation among between-clusters is expected to be low, thus again making the data closer to the optimal condition for the use of the inverse volatility portfolio.

For the case of two clusters, we show analytically that our proposed methodology converges to the mean–variance optimal portfolio when assets have the same Sharpe ratio and equal pairwise correlation. This finding offers analytical proof that the proposed clustering approach can improve risk-adjusted returns, provided the assets are suitably clustered. Furthermore, the merits of our clustering approach become especially clear when assets display a wide array of pairwise correlations that can be grouped together. On the flip side, if assets do not exhibit such variance in correlations, clustering is unlikely to offer benefits. For example, if all N assets are uncorrelated, then there are as many clusters as the number of assets and we should not expect improvements from clustering the data. Similarly, if all the N assets have a high correlation, then they would all belong to the same (unique) cluster and we, again, should not see benefits from clustering.

[1] For example, Elkamhi, Lee, and Salerno (2023) build a statistical test to check whether the inverse volatility portfolios are equivalent to the mean–variance portfolio from a statistical point of view and find that in many datasets inverse volatility and MVO portfolios are statistically equivalent, thus providing an explanation for their good empirical performance.

[2] For example, Van Binsbergen (2020) shows the same Sharpe ratio of 0.22 for both the S&P 500 and 10-year government bonds from 1970–2021.

[3] See the first section for details.

Our empirical results show that using machine learning techniques to cluster the data improves the performance of the inverse volatility portfolio in terms of Sharpe ratios. We use different clustering techniques and datasets to validate that our results are robust across various algorithms and data. We also note that the benefits are most pronounced when the asset pool is large. For example, the dataset with the 76 anomalies—which has a large number of assets—benefits from clustering the most with substantial increases in the Sharpe ratio.

In summary, this article contributes to the literature (de Prado 2016, 2020; Raffinot 2017; Lohre, Rother, and Schäfer 2020; Kaczmarek and Perez 2022) that aims to bridge the gap between traditional portfolio theory and modern computational methods, showing how machine learning techniques like clustering can be transparently applied to enhance portfolio performance. Specifically, we provide a new methodology to use clustering techniques to improve the performance of the inverse volatility portfolio, and we show the conditions under which clustering data can improve the performance of the inverse volatility portfolio. The findings have significant implications for asset management, providing a method to construct portfolios that are theoretically closer to the mean–variance frontier while being less susceptible to estimation error.

INVERSE VOLATILITY PORTFOLIO IS EQUIVALENT TO THE MEAN–VARIANCE OPTIMAL PORTFOLIO UNDER SOME CONDITIONS

We prove in this section that the inverse volatility portfolio is equivalent to the mean–variance optimal portfolio under two conditions: Assets have the same Sharpe ratios and equal pairwise correlations. To conduct our analysis, let us assume that we have N assets with a vector of excess returns and covariance matrix equal to μ and Σ, respectively. Let us define a matrix with the volatilities on the diagonal

$$D \equiv \mathrm{Diag}(\Sigma)^{1/2} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_N \end{bmatrix}$$

the weights of the inverse volatility portfolio w_{IVol} are defined as

$$w_{IVol} = \frac{D^{-1}\mathbf{1}_N}{\mathbf{1}'_N D^{-1}\mathbf{1}_N} \tag{1}$$

where $\mathbf{1}_N$ is an $N \times 1$ vector of ones. The formula implies that the weights are proportional to the reciprocal of the N asset volatilities.

The mean–variance-optimal mean–variance portfolio with weights that sum up to one is defined as

$$w_{MVO} = \frac{\Sigma^{-1}\mu}{\mathbf{1}'_N \Sigma^{-1}\mu}$$

where Σ and μ are the covariance matrix and expected excess return vector, respectively.

The following proposition formalizes that under the assumption that Sharpe ratios across assets are equal and if pairwise correlations are the same, then the inverse volatility portfolio converges to the optimal mean–variance portfolio.

Proposition 1

Let (i) $\bar{s}$ be the Sharpe ratio common to all assets such that $\mu_i/\sigma_i = \bar{s}$ for each asset $i \in \{1, 2, \ldots N\}$, and (ii) let the correlations across any two assets *i, j* be equal to the constant ρ (i.e., $\rho_{i,j} = \rho$ for each *i, j,* where $\rho_{i,j}$ is the correlation between assets *i* and *j*). Then the inverse volatility portfolio is equivalent to the optimal mean–variance portfolio (i.e., $w_{IVol} = w_{MVO}$).

In Appendix A, we provide a formal proof for Proposition 1.

HOW CAN CLUSTERING BE USEFUL IN CONJUNCTION WITH THE INVERSE VOLATILITY PORTFOLIO?

Having proven analytically that, when Sharpe ratios and pairwise correlations are equal across assets, the inverse volatility portfolio is equivalent to the optimal mean–variance portfolio in a mean–variance framework, we check whether the two assumptions hold empirically. It is known, even without examining the data, that these two assumptions come into question when considering long-term averages. Indeed, Sharpe ratios vary across different assets, as do the pairwise correlations.

However, are such differences large enough once we account for the confidence interval around them? Upon closer inspection of the data, many assets have similar Sharpe ratios that are not statistically different between each other. For example, Wright, Yam, and Pang Yung (2014) find that the hypothesis of equality for the Sharpe ratios of 18 iShares Exchange-Traded Fund (ETF) cannot be rejected at the 1% level. Ardia and Boudt (2015) find that the hypothesis of equality for the Sharpe ratios using hedge fund returns from 2008 to 2012 cannot be rejected at the 1% level for 84.22% of all pairs of two Sharpe ratios. Van Binsbergen (2020) shows the same Sharpe ratio of 0.22 for both the S&P 500 and 10-year government bond from 1970–2021. This finding implies that, particularly for major asset classes and over an extended period, Sharpe ratios may not differ significantly.

Unlike Sharpe ratios, empirical data indicate that pairwise correlations vary among assets. This sets the stage for the usefulness of our proposed methodology: We propose a method to group the assets into clusters such that they have similar correlations between each other. Within each cluster, we can then apply an inverse volatility portfolio. Because assets in each cluster exhibit high correlations with one another, they more closely align with the assumption of equal pairwise correlation. This brings the inverse volatility portfolio within each cluster closer to mean–variance optimality.

We can summarize our methodology—which we defined as *cluster-enhanced inverse volatility*—as follows: First, we group the data into clusters in which assets have nearly identical pairwise correlations. Within each of these clusters, we construct inverse volatility portfolios. Second, we build a portfolio across clusters, anticipating that correlations between clusters are lower than the correlations observed between assets that belong to the same cluster. Our goal is to cluster together assets with high internal correlations within each cluster. Having high correlations between assets within a cluster brings the data closer to the feature of equal pairwise correlations, which would make inverse volatility portfolios identical to the mean–variance portfolio from an optimality standpoint—but with the added benefit of lower estimation error.

Following de Prado (2016)—who proposed the hierarchical risk parity and provided a justification for doing so—we contribute to the literature by showing that the use of machine learning techniques like clustering can be transparently applied (i.e., do not need to be a black box) to enhance portfolio diversification and, consequently, performance.

Building Portfolios Using Clustering

While we provide the intuition behind our clustering methodology as detailed previously, we provide a formal description in this section. The machine learning literature provides a great number of clustering techniques. The common goal of the various algorithms is to generate a set of clusters, in which each cluster is distinct from the others, while the assets within each cluster are similar to each other. We consider four different methodologies: hierarchical clustering (HC) with single link, HC with complete link, HC with Ward link, and K-means.

The clustering methodology can be summarized as follows, and it is the same for all algorithms: (1) Compute the distance matrix based on the correlation matrix of the assets; (2) choose the optimal number of clusters based on the silhouette score for the chosen clustering algorithm; and (3) cluster the data using the chosen algorithm and the optimal number of clusters from (2).

First, the distance matrix is a matrix that provides information on the similarity of the various assets. We use the correlation matrix to build the distance matrix based on the Euclidean distance. Formally, we denote Q the $N \times N$ correlation matrix and DM the $N \times N$ distance matrix, where N is the number of assets. The element i, j of DM is calculated as

$$DM_{i,j} = \sqrt{\sum_{n=1}^{N} (Q_{n,i} - Q_{n,j})^2} \tag{2}$$

Equation 2 is the Euclidean distance between the correlations of asset i and asset j. We use this measure because the two assets that have a high correlation between each other and similar correlations with the other assets will have a low distance.

One could argue that we could simply use the correlation matrix as a measure of distance. However, this would imply that the distance between two assets i and j uses only the information on the correlation between asset i and asset j while the expression in Equation 2 uses also the information about the correlations of asset i and asset j with the other assets.

In the HC approach, forming clusters requires a measure of how dissimilar groups of assets are to each other. This is accomplished by using a linkage criterion, which defines the dissimilarity between asset groups based on the pairwise distances $DM_{i,j}$ among the assets within those groups. Essentially, the linkage criterion quantifies how different clusters are by computing a value derived from the pairwise distances of assets contained in those clusters. Given that the choice of linkage criterion significantly impacts the clustering results, we opt to test the robustness using three different linkage methods: single linkage, complete linkage, and Ward linkage. For two clusters A and B, their dissimilarity can be calculated using single and complete linkages according to the following expressions:

$$d_{Single}(A,B) = min(DM_{A_i,B_j}) \tag{3}$$

$$d_{Complete}(A,B) = max(DM_{A_i,B_j}) \tag{4}$$

for all assets i in cluster A (A_i represents asset i in cluster A) and for all assets j in cluster B. The dissimilarity measure using the Ward linkage ($d_{Ward}(A, B)$) can be calculated as a recursive function (i.e., the Ward's minimum variance) and we refer to Ward (1963) for details.[4]

We also use the K-means clustering algorithm, which is a widely used technique in machine learning and data science for partitioning a dataset into *K* distinct clusters. The main goal is to divide data points (assets in our case) into groups in such a way that the sum of the squared distances between the data points and the centroid of their respective clusters is minimized.[5]

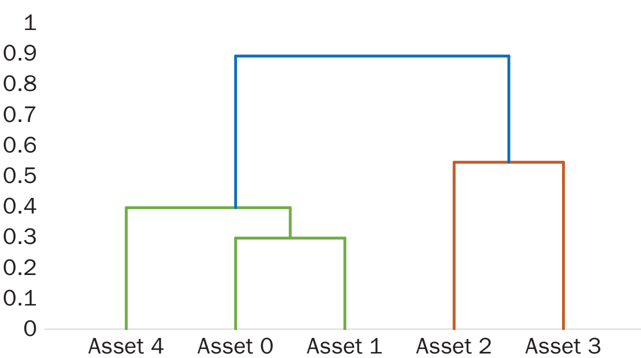
Second, it is important to have a methodology that can endogenously determine the number of clusters without requiring user input. Each algorithm possesses tuning parameters that when changed lead to a different number of clusters being detected. For example, HC requires the definition of a threshold that affects the number of clusters. We choose this tuning parameter of each algorithm by maximizing the silhouette coefficient. In the K-means algorithm, users have to specify the number of clusters they desire. We choose the number of clusters that maximizes the silhouette coefficient.

An example will help to clarify the methodology. Let us assume that we have a set of five assets and that, after grouping them using HC with single linkage, we obtain the dendrogram shown in Exhibit 1. The *y*-axis contains the distance measure between assets and clusters, whereas the *x*-axis contains the assets (five assets, numbered from 0 to 4). If we choose a value of 0.7 as the threshold to determine the number of clusters, we will have two clusters. If we choose a threshold of 0.2, we will have as many clusters as the number of assets. We develop a methodology, which is described in the following, that does not require any threshold, and it uses the full information in the dendrogram in order to cluster the data.

We choose the threshold that maximizes the average silhouette coefficient across all assets. The silhouette coefficient for asset i is calculated as $s(i) = (MNC - MIC)/\max\{MNC, MIC\}$, where MNC is the average of the nearest-cluster distances between asset i and all assets in the nearest cluster. MIC is the average of the intracluster distances between asset i and all other assets within the same cluster. The silhouette coefficient varies from –1 to 1, and it has an intuitive interpretation: When it is close to 1, it means that the asset is well matched to other assets within its own cluster and is poorly matched to assets in the nearest cluster; a value of –1 indicates that the asset is poorly matched to other assets within the same cluster and is more closely matched to assets in the nearest cluster (i.e., it is assigned to the wrong cluster). In other words, the silhouette coefficient measures how well the data are clustered; therefore, we choose this methodology instead of an arbitrarily chosen threshold. Specifically, we compute the dendrogram using the sample-based correlation matrix, and we search for the threshold value that maximizes the silhouette coefficient. Although this example is applied to HC, the same intuition can be used for the K-means.

Once the clusters have been formed, we need to define an algorithm to build a portfolio from the clustered data. We again use the dendrogram in Exhibit 1 to help us illustrate our methodology. Let us assume that, following the clustering methodology described earlier, we have two clusters: Cluster 1 contains the

EXHIBIT 1
A Dendrogram



NOTE: This exhibit shows how HC groups together various assets to form clusters.

[4] Statistical tools such as Python can calculate the Ward linkage very efficiently with built-in packages.
[5] For a detailed explanation of the K-means algorithm, we refer to Bishop and Nasrabadi (2006).

assets 0, 1, and 4 while cluster 2 contains the assets 2 and 3. We form an inverse volatility portfolio within each cluster using the sample-based estimates of the assets' volatilities within each cluster. We call these two portfolios *CP*1 (cluster portfolio 1) and *CP*2 (cluster portfolio 2). The weights of each asset *i* belonging to cluster portfolio *j* are labeled as w_i^{CPj}. We then move to calculating a portfolio between clusters. We first calculate the covariance matrix between the two cluster portfolios in our example, *CP*1 and *CP*2, by building a time series of returns for cluster portfolio *CPj* using the optimal weights (w_i^{CPj} for each asset *i* in *CPj*). This step gives us a time series of returns for each cluster portfolio that we use to calculate the covariance matrix between the two clusters. We then form an inverse volatility portfolio using the volatilities of the within-cluster portfolios to form the between-clusters overall portfolios. Each cluster portfolio is assigned a weight $\overline{w}^{CPj}$. Last, we calculate the weight of each single asset, taking into account both the within-cluster and between-clusters portfolio weights. For any asset *i* that belongs to cluster portfolio *j*, the optimal weight accounting for both within-cluster and between-clusters allocations is

$$w_i = \overline{w}^{CPj} \cdot w_i^{CPj} \tag{5}$$

We provide a simple clarifying example to visualize our methodology in Exhibit 2. In this example, we assume for simplicity that we have 14 assets divided into three well-defined groups: equities (colored in blue), bonds (colored in green), and commodities (colored in red). The inverse volatility portfolio leads to weights that are proportional to the reciprocal of their individual volatilities. In Panel A of Exhibit 2, assets are not grouped based on their characteristics (e.g., group all equities, all bonds, and all commodities). The portfolio is simply built involving all assets without clustering. In Panel B, we show a visualization of our methodology (cluster-enhanced inverse volatility). First, we cluster the assets. Then, we build the inverse volatility portfolios within the clusters (i.e., obtaining the term w_i^{CPj} in Equation 5). Then we build an inverse volatility portfolio between clusters, thus obtaining the term $\overline{w}^{CPj}$ in Equation 5. By doing so, we apply the inverse volatility portfolio on data that are closer to having equal pairwise correlations.

An Analytical Example of How Our Proposed Clustering Works

We provide an analytical example of why our clustering methodology works when assets have equal pairwise correlations and Sharpe ratios. Our goal is to compare the weights of the mean–variance portfolio with the weights implied by our clustering methodology. Let us describe the setup of our example. Let us assume that we have *N* assets that can be partitioned into two clusters, *A* and *B*. We can express their expected returns and variance-covariance matrix as follows:

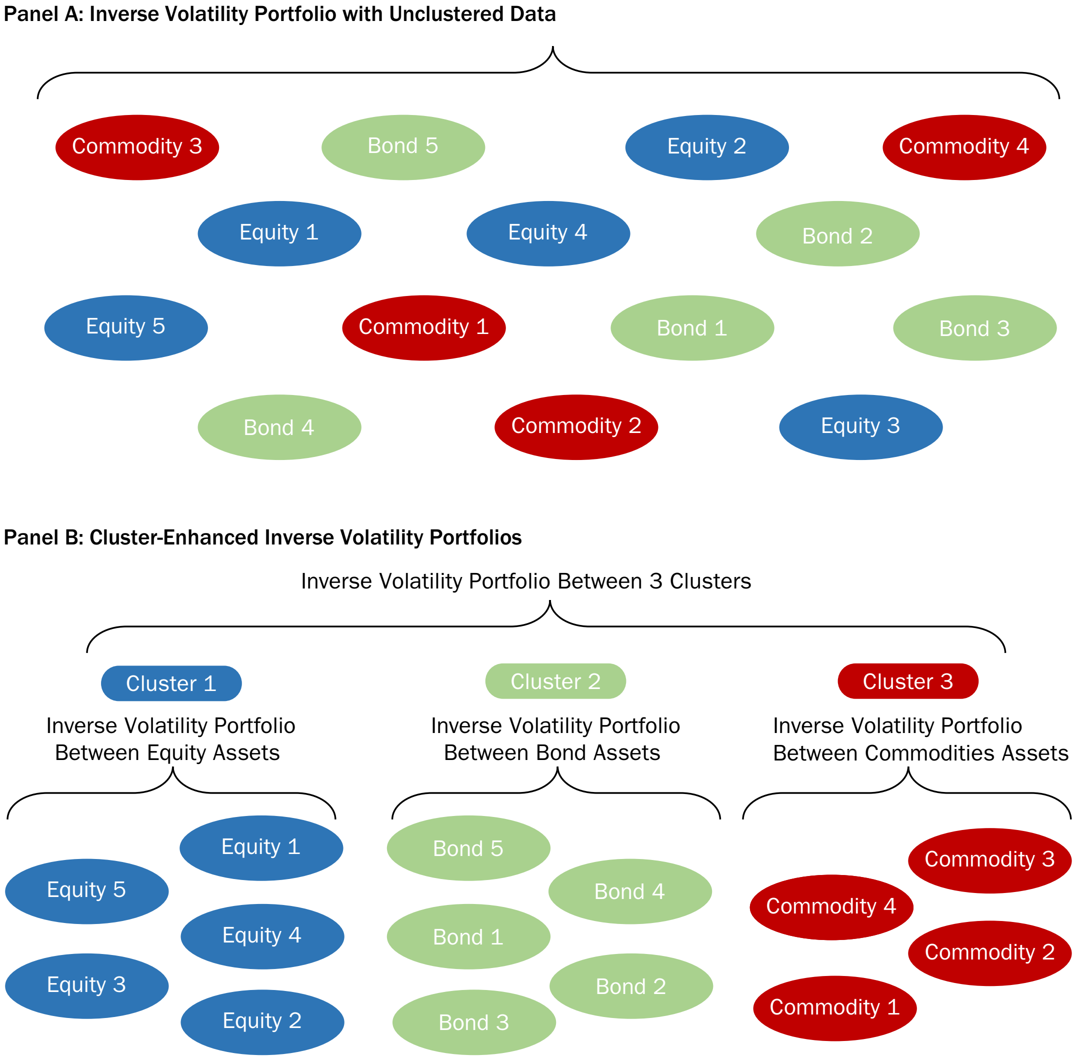
$$\mu_a = [\mu_A', \mu_B']' \tag{6}$$

$$\Sigma_a = \begin{bmatrix} \Sigma_A & 0 \\ 0 & \Sigma_B \end{bmatrix} \tag{7}$$

where μ_i is the vector of expected returns of the assets in cluster *i*, and Σ_i is the variance-covariance matrix of the assets in cluster *i*, for $i \in \{A, B\}$. The weights of the mean–variance portfolio are as easily calculated as

EXHIBIT 2

Visual Illustration of Cluster-Enhanced Inverse Volatility Portfolios



NOTES: This exhibit provides a simplified example in which we consider 14 assets categorized into three distinct groups: equities (in blue), bonds (in green), and commodities (in red). Panel A shows the traditional inverse volatility portfolio that assigns weights based on the inverse of each asset's volatility but does not group assets by their type. Panel B illustrates our method (cluster-enhanced inverse volatility) in which we first group assets into clusters based on their similarities. Within each cluster, we apply the inverse volatility portfolio strategy. Then, we apply the same strategy across these clusters.

$$w_{MVO} \propto \Sigma_a^{-1}\mu_a = \begin{bmatrix} \Sigma_A^{-1} & 0 \\ 0 & \Sigma_B^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix} = \begin{bmatrix} \Sigma_A^{-1}\mu_A \\ \Sigma_B^{-1}\mu_B \end{bmatrix} \tag{8}$$

where the first equality follows from the property of the partitioned matrixes. Under the assumption that Sharpe ratios are the same and equal to k and assets have equal pairwise correlations, we can write for each $i \in \{A, B\}$

$$\mu_i = kV_i\mathbf{1}_i \tag{9}$$

$$\Sigma_i = V_iC_iV_i \tag{10}$$

where V_i is a diagonal matrix with the volatilities of assets in cluster i on its diagonal, $\mathbf{1}_i$ is a vector of ones with the length equal to the number of assets in cluster i, and C_i is the correlation matrix of the assets in cluster i. Substituting Equations 9 and 10 in Equation 8, we can rewrite the weights of the mean–variance portfolio as

$$w_{MVO} \propto k\begin{bmatrix} V_A^{-1}C_A^{-1}\mathbf{1}_A \\ V_B^{-1}C_B^{-1}\mathbf{1}_B \end{bmatrix} \tag{11}$$

The next step is to derive an expression for the weights of the assets following our clustering methodology and then compare them to the weights described in Equation 11. According to our clustering methodology, the first step is to build within-cluster portfolios, which we label w_{IVolA} and w_{IVolB} in our example with two clusters:

$$w_{IVolA} \propto V_A^{-1}\mathbf{1}_A \Rightarrow w_{IVolA} = f_A V_A^{-1}\mathbf{1}_A \tag{12}$$

$$w_{IVolB} \propto V_B^{-1}\mathbf{1}_B \Rightarrow w_{IVolB} = f_B V_B^{-1}\mathbf{1}_B \tag{13}$$

where f_A and f_B are two constants that simply regularize the weights of the portfolios such that they sum up to one.

We now need to build the between-clusters portfolio and, to do that, we need to calculate the variance-covariance matrix between the two portfolios *IVolA* and *IVolB*, which can be expressed as

$$\begin{aligned} \Sigma_{AB} &= \begin{bmatrix} w'_{IVolA} & 0 \\ 0 & w'_{IVolB} \end{bmatrix} \cdot \Sigma_a \cdot \begin{bmatrix} w_{IVolA} & 0 \\ 0 & w_{IVolB} \end{bmatrix} \\ &= \begin{bmatrix} w'_{IVolA} & 0 \\ 0 & w'_{IVolB} \end{bmatrix} \cdot \begin{bmatrix} V_A C_A V_A & 0 \\ 0 & V_B C_B V_B \end{bmatrix} \cdot \begin{bmatrix} w_{IVolA} & 0 \\ 0 & w_{IVolB} \end{bmatrix} \\ &= \begin{bmatrix} f_A^2 \mathbf{1}'_A C_A \mathbf{1}_A & 0 \\ 0 & f_B^2 \mathbf{1}'_B C_B \mathbf{1}_B \end{bmatrix} \end{aligned} \tag{14}$$

The weights of the between-clusters portfolios are

$$w_{AB} = \begin{bmatrix} \dfrac{1}{\sqrt{f_A^2 \mathbf{1}'_A C_A \mathbf{1}_A}} \\ \dfrac{1}{\sqrt{f_B^2 \mathbf{1}'_B C_B \mathbf{1}_B}} \end{bmatrix} \tag{15}$$

In our example, w_{AB} is a 2×1 vector, but in general it will be a vector equal to the number of clusters. Combining the weights of the between-clusters and within-cluster portfolios yields the weights of each asset that, in our example, can be written as

$$w_{cluster} = g\left(\begin{bmatrix} \dfrac{1}{\sqrt{f_A^2 \mathbf{1}'_A C_A \mathbf{1}_A}} \otimes w_{IVolA} \\ \dfrac{1}{\sqrt{f_B^2 \mathbf{1}'_A C_B \mathbf{1}_B}} \otimes w_{IVolB} \end{bmatrix}\right) \tag{16}$$

where $w_{cluster}$ is a vector containing the weights of the N assets, which is formed by stacking together the weights of the assets in clusters A and B, and the function $g(\cdot)$ is a function that normalizes the weights such that they sum up to one. The intuition for the expression describing $w_{cluster}$ is simple. We multiply the weights of each within-cluster portfolio (i.e., w_{IVolA} and w_{IVolB}) by the weights assigned to them by the between-clusters portfolio.

Because we have closed-form solutions for both the mean–variance portfolio and our clustering methodology, we can now compare their weights and see when they deviate from each other. To visualize the error, we use the following example. We use five assets, three of which are in the first cluster (e.g., cluster A), and two are in the second cluster (e.g., cluster B). We assume their Sharpe ratios are equal to 0.5 and their volatilities are defined by the following vector $[0.1, 0.12, 0.15, 0.05, 0.07]$, where the first three volatilities belong to the assets in cluster A and the remaining two belong to the assets in cluster B. In the base case scenario, we assume that the correlation between clusters (ρ_{btw}) is zero (i.e., assets that belong to different clusters are uncorrelated) and the correlation within clusters is 0.9 (ρ_{win}). Using these assumptions and the closed-form solutions from Equations 11 and 16, we can calculate the weights of each asset using the mean–variance portfolio and our clustering methodology. To evaluate how much the two methodologies deviate from each other, we calculate the sum of squared deviations between the two weights: $SSD = |w_{MVO} - w_{cluster}|_2$, where $|\cdot|_2$ indicates the norm-2 operator.

In Exhibit 3, we analyze how changing the correlation between clusters ρ_{btw} and within clusters ρ_{win} affects the sum of squared deviations SSD. In Panel A, we fix all parameters to the base case scenario and we vary the correlation between clusters ρ_{btw}. As expected, when ρ_{btw} is close to zero, the deviations between our clustering methodology and the mean–variance portfolio are close to zero as well. In other words, our clustering methodology is equivalent to the mean–variance portfolio. However, as ρ_{btw} increases, then the sum of squared deviations SSD increases, showing that our clustering methodology deviates from the mean–variance portfolio. In Panel B, we repeat the exercise but this time we vary the correlation within clusters (ρ_{win}). Again, our results confirm our intuition. When ρ_{win} is close to one, the deviations between our clustering methodology and the mean–variance portfolio are close to zero, thus confirming that when the within-cluster correlation is high then our methodology converges to the mean–variance portfolio. As ρ_{win} decreases, the sum of squared deviations SSD increases, showing that our clustering methodology is not effective because it deviates from the mean–variance portfolio.

Overall, our results show that if assets have the same Sharpe ratios and we are able to cluster together assets that are highly correlated and separate those that have low correlations, then we can use the inverse volatility portfolio—which is subject to much less error than mean–variance—and still achieve a more optimal portfolio from a theoretical point of view compared to the traditional inverse volatility portfolio. In the next section, we check whether we can successfully cluster the data using our methodology and empirical data.

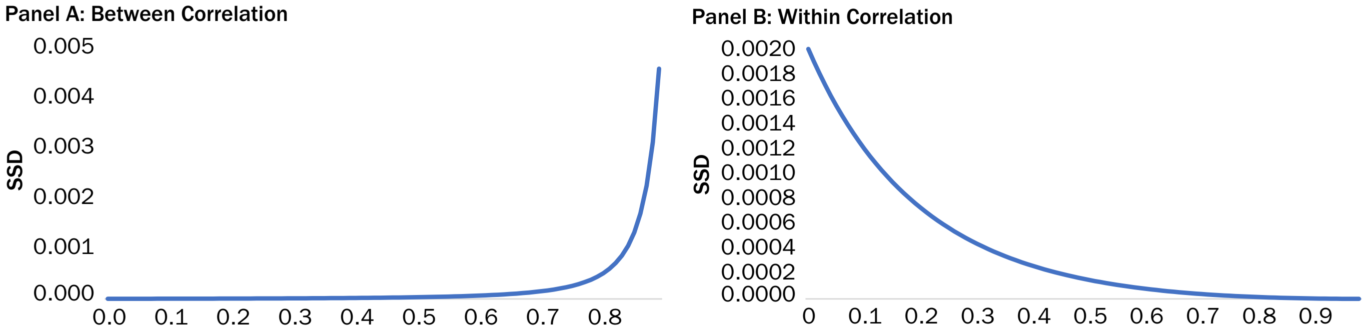
THE BENEFITS OF CLUSTERING: EMPIRICAL EVIDENCE

Description of the Datasets

We consider seven different datasets listed in Exhibit 4. We consider a large multiasset dataset using data from Global Financial Data and Bloomberg. There are a total of 31 assets in the multiasset dataset divided into equities (15 assets), government bonds (2 assets), corporate bonds (12 assets), and commodities (2 assets). In Panel B of Exhibit 4, we list all the assets used in the multiasset dataset.

EXHIBIT 3

How Different Are Cluster-Enhanced Inverse Volatility Portfolios from Mean–Variance?



NOTES: Panels A and B show how much the cluster-enhanced inverse volatility portfolios differ from the optimal tangency portfolio for various levels of correlations. We assume that there are five assets, three of which are in the first cluster (e.g., cluster *A*) and two are in the second cluster (e.g., cluster *B*). We assume their Sharpe ratios are equal to 0.5, and their volatilities are defined by the following vector [0.1, 0.12, 0.15, 0.05, 0.07], where the first three volatilities belong to the assets in cluster *A* and the remaining two belong to the assets in cluster *B*. In the base case scenario, we assume that the correlation between clusters (ρ_{btw}) is zero (i.e., assets that belong to different clusters are uncorrelated) and the correlation within clusters is 0.9 (ρ_{win}). Using these assumptions and the closed-form solutions from Equations 11 and 16, we calculate the weights of each asset using the tangency portfolio and our clustering methodology and the sum of squared deviations as $SSD = |w_{TAN} - w_{cluster}|_2$, where $|\cdot|_2$ indicates the norm-2 operator. In Panel A, we fix all parameters to the base case scenario and we vary the correlation between clusters ρ_{btw}. In Panel B, we repeat the exercise but this time we vary the correlation within clusters (ρ_{win}).

We also use five different equity datasets that are based on different definitions of industries; specifically, the 10-industry portfolios (10 industries), the 17-industry portfolios (17 industries), the 30-industry portfolios (30 industries), the 48-industry portfolios (48 industries), and the 49-industry portfolios (49 industries) from Kenneth French's website.[6] These portfolios allow us to evaluate the robustness of our results across various definitions of industries as well as a various number of assets.

Last, we consider 76 anomalies in the cross section of equity returns as datasets. The list of the anomalies can be found in Exhibit B1 of Appendix B. We compiled the set of 76 anomalies as follows. Following Hou, Xue, and Zhang (2018), we consider only anomalies that have been constructed using value-weighted returns, have NYSE break points, and generate excess returns that are statistically different from zero. If the authors of the original study publish the data, we download the anomalies from their websites. Otherwise, we build them ourselves. All the anomalies studied in this article produce excess returns that are statistically significant from zero for the period from 1981 to 2019. In Exhibit B1 of Appendix B, we list all the anomalies as well as the reference paper that we followed to replicate them or the paper of the authors from whom we obtained the data.

Performance of Portfolios That Use Clustering

In this section, we present our empirical findings on the performance of the inverse volatility portfolio when data are clustered using the methodology described earlier. Exhibit 5 reports the Sharpe ratio of the inverse volatility portfolio as well as four cluster-enhanced portfolios that use clustering to improve the performance of the inverse volatility portfolio.

[6] The data can be downloaded from Kenneth French's website at https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

EXHIBIT 4

The Datasets

Panel A: Datasets Used in This Study

Datasets	Dates	Number of Assets
Multiasset Portfolio	1990–2023	31
10 Industries	1926–2023	10
17 Industries	1926–2023	17
30 Industries	1926–2023	30
48 Industries	1969–2023	48
49 Industries	1969–2023	49
76 Anomalies	1981–2019	76

Panel B: Details on the Multiasset Dataset

Equity Indexes	Corporate Bond Indexes
S&P 500	Dow Jones Corporate Bond Return Index
Russell 1000	Bank of America Merrill Lynch US High Yield
Russell 2000	Bank of America Merrill Lynch US Corp 10–15yr
S&P 500 Finance	Bank of America Merrill Lynch US Corp 1–3yr
S&P 500 Utilities	Bank of America Merrill Lynch US Corp 15+yr
S&P 500 Energy	Bank of America Merrill Lynch US Corp 3–5yr
S&P 500 Materials	Bank of America Merrill Lynch US Corp 5–7yr
S&P 500 Industrials	Bank of America Merrill Lynch US Corp A
S&P 500 Consumer Discretionary	Bank of America Merrill Lynch US Corp AA
S&P 500 Consumer Staples	Bank of America Merrill Lynch US Corp AAA
S&P 500 Health Care	Bank of America Merrill Lynch US Corp BBB
S&P 500 Information Technology	Bank of America Merrill Lynch US Corp
S&P 500 Telecommunications	
S&P 500/Citigroup Growth	
S&P 500/Citigroup Value	
Commodity Indexes	**Government Bond Indexes**
Bloomberg Commodity Index	10-year US Treasuries
S&P GSCI Index	30-year US Treasuries

EXHIBIT 5

Enhancements in Sharpe Ratios

	Inverse Volatility	Clustering HC (complete)	Clustering HC (Ward)	Clustering HC (single)	Clustering K-Means
Multiasset Portfolio	0.227	0.235	0.241	0.228	0.228
10 Industries	0.144	0.144	0.151	0.144	0.144
17 Industries	0.135	0.141	0.138	0.135	0.142
30 Industries	0.137	0.157	0.152	0.161	0.146
48 Industries	0.132	0.141	0.150	0.153	0.154
49 Industries	0.130	0.153	0.140	0.152	0.153
76 Anomalies	0.366	0.522	0.478	0.455	0.370

NOTE: This exhibit shows the monthly Sharpe ratios for the inverse volatility portfolio—which does not use clustering—as well as four cluster-enhanced portfolios that use clustering to improve the performance of the inverse volatility portfolio.

Exhibit 5 shows that clustering works well for the multiasset portfolio. For example, using the HC with complete linkage, the cluster-enhanced portfolio achieves a Sharpe ratio of 0.235 while the inverse volatility portfolio has a Sharpe ratio of 0.227. Cluster-enhanced portfolios perform at least as well as the inverse volatility portfolio for all datasets considered here, but there is another interesting observation:

EXHIBIT 6

Clustered Data for the 30-Industry Dataset

Cluster 1
Chemicals
Construction
Steel Works
Fabricated Products and Machinery
Electrical Equipment
Aircraft, Ships, and Railroad Equipment
Business Supplies and Shipping Containers
Transportation
Financials
Cluster 2
Recreation
Printing and Publishing
Apparel
Textiles
Automobiles and Trucks
Business Equipment
Wholesale
Retail
Restaurants, Hotels, Motels
Other
Cluster 3
Utilities
Communication
Cluster 4
Food Products
Consumer Goods
Healthcare and Pharmaceuticals
Cluster 5
Beer and Liquor
Cluster 6
Mining
Coal
Oil and Natural Gas
Cluster 7
Tobacco Products
Cluster 8
Personal and Business Services

NOTE: This exhibit shows the clusters formed by the HC algorithm (with complete linkage) for the 30-industry portfolios dataset.

Clustering seems to help the most when the number of assets increases. Indeed, while for the 10-industry dataset there are small or no improvements over the inverse volatility portfolio, in datasets that have a larger number of assets (i.e., 30 or more), the improvement in Sharpe ratio is larger and consistent across the four different clustering methods considered here.

To understand how clustering helped improve performance, we present in Exhibit 6 the clusters that the HC with complete linkage created using the 30-industry dataset. The algorithm split the assets into eight clusters. Some industries have been categorized as being different enough from all other industries and therefore constitute a cluster on their own (e.g., beer and liquor, tobacco products). Other industries have been clustered together because they belong to the same line of business: For example, mining, coal, and oil and natural gas are three different industries that were grouped into a cluster. Utilities and communication have been grouped together due to their shared characteristics, even though they operate in distinct sectors of the economy. Both industries typically enjoy stable demand because consumers always need essential utilities like water and electricity, just as they need communication services, regardless of economic conditions. Additionally, both utilities and communications are subject to regulatory oversight: utilities for the fair and safe distribution of vital services like electricity and water and communications for aspects like spectrum allocation, market competition, and consumer rights.

CONCLUSION

In this article, we tackled the limitations of inverse volatility portfolios, specifically their divergence from mean–variance optimal portfolios due to real-world violations of assumptions like equal Sharpe ratios and equal pairwise correlations among assets. To bridge this gap, we introduced a novel portfolio construction methodology—cluster-enhanced inverse volatility—based on clustering techniques that takes advantage of the intrinsic properties of inverse volatility portfolios to enhance diversification benefits and, ultimately, portfolio performance.

Our approach is motivated by the mathematical equivalence between inverse volatility and mean–variance portfolios under the condition of equal Sharpe ratios and pairwise correlations. By clustering assets in a manner that approximates these ideal conditions as closely as possible, we effectively use the inverse volatility portfolio in an environment in which it is closer to its optimal mean–variance counterpart in terms of optimality. Our work thus contributes a new layer of understanding to the literature on portfolio optimization, started by de Prado (2016), highlighting the benefits of leveraging machine learning techniques like clustering to make more robust financial decisions.

APPENDIX A

PROOF THAT THE INVERSE VOLATILITY PORTFOLIO IS EQUIVALENT TO THE MEAN–VARIANCE OPTIMAL PORTFOLIO UNDER SOME CONDITIONS

In the following, we prove Proposition 1.

Proof

Using assumption (ii) from Proposition 1 that pairwise correlations are all equal, let us rewrite the correlation matrix Q as

$$Q = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & & \ddots & \vdots \\ \rho & \cdots & \cdots & 1 \end{bmatrix} = \rho \mathbf{1}_N \mathbf{1}_N' + (1-\rho)\mathbf{I}_N$$

where $\mathbf{I}_N$ is the identity matrix, and $\mathbf{1}_N$ is a column vector of ones. We can write the variance-covariance matrix as

$$\Sigma = D \cdot Q \cdot D \tag{A1}$$

By definition of the inverse volatility portfolio, we have that the weights w_{IVol} are proportional to $D^{-1} \cdot \mathbf{1}_N$. We also know that the weights of the mean–variance portfolio w_{MVO} are proportional to $\Sigma^{-1} \cdot \mu$. We can therefore write $\Sigma^{-1} \cdot \mu = D^{-1} \cdot Q^{-1} \cdot D^{-1} \cdot \mu$ using Equation A1. Formally, we can write the weights of the *IVol* and *MVO* portfolios as follows:

$$w_{IVol} \propto D^{-1} \cdot \mathbf{1}_N$$
$$w_{MVO} \propto D^{-1} \cdot Q^{-1} \cdot D^{-1} \cdot \mu$$

To prove that $w_{MVO} = w_{IVol}$ it is sufficient to show that (a) μ is proportional to $D \cdot \mathbf{1}_N$ and (b) that $Q^{-1}\mathbf{1}_N$ is proportional to $\mathbf{1}_N$.

To prove (a) (i.e., that μ is proportional to $D \cdot \mathbf{1}_N$), we use the assumption (i) that Sharpe ratios across assets are equal to s and we can write $\mu = \bar{s}D \cdot \mathbf{1}_N$. This proves that $\mu \propto D \cdot \mathbf{1}_N$ when Sharpe ratios are equal across assets.

To prove (b) (i.e., that $Q^{-1}\mathbf{1}_N$ is proportional to $\mathbf{1}_N$), we proceed in two steps.

First, we show that one of the eigenvectors of Q is $\mathbf{1}_N$. Consider the matrix $B \equiv Q + (\rho - 1)\mathbf{I}_N$ whose elements are all equal to ρ. Note that

$$B \cdot \mathbf{1}_N = \begin{bmatrix} \rho & \rho & \cdots & \rho \\ \rho & \rho & \cdots & \rho \\ \vdots & & \ddots & \vdots \\ \rho & \cdots & \cdots & \rho \end{bmatrix} \cdot \mathbf{1}_N = \begin{bmatrix} N\rho \\ N\rho \\ \vdots \\ N\rho \\ N\rho \end{bmatrix} = N\rho \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = N\rho\mathbf{1}_N$$

which proves that $\mathbf{1}_N$ is an eigenvector of B. Let $\boldsymbol{x}$ be an eigenvector of Q and λ_x be its associated eigenvalue. It follows that

$$
\begin{aligned}
&Q\boldsymbol{x} = \lambda_x \boldsymbol{x} \\
&(Q + (\rho - 1)\boldsymbol{I}_N)\boldsymbol{x} = \lambda_x \boldsymbol{x} + (\rho - 1)\boldsymbol{I}_N \boldsymbol{x} \Rightarrow \\
&(Q + (\rho - 1)\boldsymbol{I}_N)\boldsymbol{x} = (\lambda_x + \rho - 1)\boldsymbol{x} \Rightarrow \\
&B\boldsymbol{x} = (\lambda_x + \rho - 1)\boldsymbol{x}
\end{aligned} \tag{A2}
$$

The last step in Equation A2 follows from the definition of the matrix B. Equation A2 shows that the matrixes Q and B have the same eigenvectors. Therefore, because $\mathbf{1}_N$ is an eigenvector of B, it is also an eigenvector of Q.

Second, using the fact the pairwise correlations are constant, we can write $Q = \rho \mathbf{1}_N \cdot \mathbf{1}'_N + (1 - \rho)\mathbf{I}_N$. Because one of the eigenvectors of Q is $\mathbf{1}_N$, then we can write $Q \cdot \mathbf{1}_N = \lambda \mathbf{1}_N$, where λ is the associated eigenvalue. It follows that $Q^{-1} \cdot \mathbf{1}_N = \frac{1}{\lambda}\mathbf{1}_N$. This proves that, when pairwise correlations are constant across assets, $Q^{-1} \cdot \mathbf{1}_N \propto \mathbf{1}_N$.

APPENDIX B

DESCRIPTION OF THE ANOMALIES USED IN THIS ARTICLE

EXHIBIT B1

76 Anomalies Used in This Study

No.	Anomaly	Reference Paper	Name
1	ABR1	Chan, Jegadeesh, and Lakonishok (1996)	Cumulative abnormal returns around earnings announcement dates. Holding period one month.
2	ABR6	Chan, Jegadeesh, and Lakonishok (1996)	Cumulative abnormal returns around earnings announcement dates. Holding period six months.
3	ACI	Hou, Xue, and Zhang (2018)	Abnormal corporate investment
4	ADM	Chan, Lakonishok and Sourgiannis (2001)	Advertising expense-to-market
5	BAB	Frazzini and Pedersen (2014)	Betting-against-beta
6	BM	Fama and French (1993)	Sort by book-to-market equity
7	CEI	Hou, Xue, and Zhang (2018)	Composite equity issuance
8	CLA	Ball et al. (2016)	Cash-based operating profits-to-lagged assets using yearly Compustat data
9	CLAQ1	Ball et al. (2016)	Cash-based operating profits-to-lagged assets using quarterly Compustat data and holding period of one month
10	CMA	Fama and French (2015)	Conservative minus aggressive
11	COP	Ball et al. (2016)	Cash-based operating profitability
12	DA	Hou, Xue, and Zhang (2018)	Changes in short-term investments
13	DFIN	Hou, Xue, and Zhang (2018)	Changes in net financial assets
14	DLTI	Hou, Xue, and Zhang (2018)	Changes in short-term investments
15	DNCA	Hou, Xue, and Zhang (2018)	Changes in noncurrent operating assets
16	DNCO	Hou, Xue, and Zhang (2018)	Changes in net noncurrent operating assets
17	DNOA	Hou, Xue, and Zhang (2018)	Changes in net operating assets
18	DPIA	Hou, Xue, and Zhang (2018)	Changes in Property, plant, and equipment (PPE) and inventory-to-assets
19	DROE1	Hou, Xue, and Zhang (2018)	Four-quarter change in return on equity. Holding period of one month.
20	DROE12	Hou, Xue, and Zhang (2018)	Four-quarter change in return on equity. Holding period of 12 months.
21	DROE6	Hou, Xue, and Zhang (2018)	Four-quarter change in return on equity. Holding period of six months.
22	DWC	Hou, Xue, and Zhang (2018)	Changes in net noncash working capital
23	EM	Loughran and Wellman (2011)	Enterprise multiple
24	HML	Fama and French (1993)	High minus low
25	HMLD	Asness and Frazzini (2013)	The devil in HML's details
26	IA	Hou, Xue, and Zhang (2018)	Investment-to-assets
27	IG	Hou, Xue, and Zhang (2018)	Investment growth, one year

(continued)

EXHIBIT B1 *(continued)*

76 Anomalies Used in This Study

No.	Anomaly	Reference Paper	Name
28	IG2y	Hou, Xue, and Zhang (2018)	Investment growth, two years
29	IVC	Hou, Xue, and Zhang (2018)	Inventory changes
30	IVG	Hou, Xue, and Zhang (2018)	Inventory growth
31	MKT-RF		Excess market return
32	NEI1	Barth, Elliott, and Finn (1999)	The number of quarters with consecutive earnings increase
33	NOA	Hirshleifer et al. (2004)	Net operating assets
34	NOP	Boudoukh et al. (2007)	Net payout yield
35	NSI	Pontiff and Woodgate (2008)	Net stock issues
36	OA	Sloan (1996)	Operating accruals
37	OCA	Eisfeldt and Papanikolaou (2013)	Industry-adjusted organizational capital-to-assets
38	OCP	Desai, Rajgopal, and Venkatachalam (2004)	Operating cash-flow to price
39	OP	Hou, Xue, and Zhang (2018)	Payout yield
40	OPA	Ball et al. (2016)	Operating profits to assets
41	POA	Sloan (1996)	Percentage operating accruals
42	PTA	Sloan (1996)	Percentage total accruals
43	QMJ	Asness, Frazzini, and Pedersen (2019)	Quality minus junk
44	R_EG	Hou et al. (2019)	Expected growth factor
45	R_IA	Hou, Xue, and Zhang (2015)	Investment factor
46	R_ROE	Hou, Xue, and Zhang (2015)	Return on Equity (ROE) factor
47	R111	Jegadeesh and Titman (1993)	Price momentum, prior 11-month returns, holding period one month
48	R1112	Jegadeesh and Titman (1993)	Price momentum, prior 11-month returns, holding period 12 months
49	R1115A	Heston and Sadka (2008)	Years 11–15 lagged returns, annual
50	R1115n	Heston and Sadka (2008)	Years 11–15 lagged returns, nonannual
51	R116	Jegadeesh and Titman (1993)	Price momentum, prior 11-month returns, holding period six months
52	R15A	Heston and Sadka (2008)	Years 1–5 lagged returns, annual
53	R1620A	Heston and Sadka (2008)	Years 16–20 lagged returns, annual
54	R1A	Heston and Sadka (2008)	Year 1 lagged return, annual
55	R1N	Heston and Sadka (2008)	Year 1 lagged return, nonannual
56	R25A	Heston and Sadka (2008)	Years 2–5 lagged returns, annual
57	R61	Jegadeesh and Titman (1993)	Price momentum, prior six-month returns, holding period one month
58	R610A	Heston and Sadka (2008)	Years 6–10 lagged returns, annual
59	R610n	Heston and Sadka (2008)	Years 6–10 lagged returns, nonannual
60	R612	Jegadeesh and Titman (1993)	Price momentum, prior six-month returns, holding period 12 months
61	R66	Jegadeesh and Titman (1993)	Price momentum, prior six-month returns, holding period six months
62	RDM	Chan, Lakonishok and Sourgiannis (2001)	R&D expense-to-market using Compustat yearly
63	RE_1	Chan, Chan, Jegadeesh, and Lakonishok (2001)	Revisions in analysts' earnings forecasts—one-month holding period
64	RE_6	Chan, Chan, Jegadeesh, and Lakonishok (2001)	Revisions in analysts' earnings forecasts—six-month holding period
65	RER	Tuzel (2010)	Industry-adjusted real estate ratio
66	RESID11_1	Blitz, Huij, and Martens (2011)	Eleven-month residual momentum, one-month holding period
67	RESID11_12	Blitz, Huij, and Martens (2011)	Eleven-month residual momentum, 12-month holding period
68	RESID11_6	Blitz, Huij, and Martens (2011)	Eleven-month residual momentum, six-month holding period
69	RESID6_12	Blitz, Huij, and Martens (2011)	Six-month residual momentum, 12-month holding period
70	RESID6_6	Blitz, Huij, and Martens (2011)	Six-month residual momentum, six-month holding period
71	RMW	Fama and French (2015)	Robust minus weak factor
72	ROE1	Hou, Xue, and Zhang (2015)	Return on equity with holding period of one month
73	ROE6	Hou, Xue, and Zhang (2015)	Return on equity with holding period of six months
74	SP	Barbee, Mukherji, and Raines (1996)	Sales-to-price ratio
75	SUE1	Foster, Olsen, and Shevlin (1984)	Standardized unexpected earnings. Holding period of one month.
76	SUE6	Foster, Olsen, and Shevlin (1984)	Standardized unexpected earnings. Holding period of six months.

NOTES: The column "Reference Paper" refers to the paper that contains the methodology used to build the anomaly. When authors make the data available until the end of 2019, we use their data. If not, we build the anomalies ourselves.

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REFERENCES

Ardia, D., and K. Boudt. 2015. "Testing Equality of Modified Sharpe Ratios." *Finance Research Letters* 13: 97–104.

Asness, C., and A. Frazzini. 2013. "The Devil in HML's Details." *The Journal of Portfolio Management* 39 (4): 49–68.

Asness, C. S., A. Frazzini, and L. H. Pedersen. 2019. "Quality Minus Junk." *Review of Accounting Studies* 24 (1): 34–112.

Ball, R., J. Gerakos, J. T. Linnainmaa, and V. Nikolaev. 2016. "Accruals, Cash Flows, and Operating Profitability in the Cross Section of Stock Returns." *Journal of Financial Economics* 121 (1): 28–45.

Barbee, Jr, W. C., S. Mukherji, and G. A. Raines. 1996. "Do Sales–Price and Debt–Equity Explain Stock Returns Better Than Book–Market and Firm Size?" *Financial Analysts Journal* 52 (2): 56–60.

Barth, M. E., J. A. Elliott, and M. W. Finn. 1999. "Market Rewards Associated with Patterns of Increasing Earnings." *Journal of Accounting Research* 37 (2): 387–413.

Bishop, C. M., and N. M. Nasrabadi. 2006. *Pattern Recognition and Machine Learning*, Volume 4. Springer in Berlin, Germany.

Blitz, D., J. Huij, and M. Martens. 2011. "Residual Momentum." *Journal of Empirical Finance* 18 (3): 506–521.

Boudoukh, J., R. Michaely, M. Richardson, and M. R. Roberts. 2007. "On the Importance of Measuring Payout Yield: Implications for Empirical Asset Pricing." *The Journal of Finance* 62 (2): 877–915.

Chan, K., L. K. Chan, N. Jegadeesh, and J. Lakonishok. 2001. "Earnings Quality and Stock Returns." Technical report, National Bureau of Economic Research.

Chan, L. K., N. Jegadeesh, and J. Lakonishok. 1996. "Momentum Strategies." *The Journal of Finance* 51 (5): 1681–1713.

Chan, L. K., J. Lakonishok, and T. Sougiannis. 2001. "The Stock Market Valuation of Research and Development Expenditures." *The Journal of Finance* 56 (6): 2431–2456.

de Prado, M. L. 2016. "Building Diversified Portfolios That Outperform Out of Sample." *The Journal of Portfolio Management* 42 (4): 59–69.

——. 2020. *Machine Learning for Asset Managers*. Cambridge: Cambridge University Press.

DeMiguel, V., L. Garlappi, and R. Uppal. 2009. "Optimal versus Naive Diversification: How Inefficient Is the 1/*n* Portfolio Strategy?" *The Review of Financial Studies* 22 (5): 1915–1953.

Desai, H., S. Rajgopal, and M. Venkatachalam. 2004. "Value-Glamour and Accruals Mispricing: One Anomaly or Two?" *The Accounting Review* 79 (2): 355–385.

Eisfeldt, A. L., and D. Papanikolaou. 2013. "Organization Capital and the Cross-Section of Expected Returns." *The Journal of Finance* 68 (4): 1365–1406.

Elkamhi, R., J. S. H. Lee, and M. Salerno. 2023. "How Optimal Are Risk-Based Portfolios?" *The Journal of Portfolio Management* 50 (1): 19–45.

Fama, E. F., and K. R. French. 1993. "Common Risk Factors in the Returns on Stocks and Bonds." *Journal of Financial Economics* 33 (1): 3–56.

——. 2015. “A Five-Factor Asset Pricing Model.” *Journal of Financial Economics* 116 (1): 1–22.

Foster, G., C. Olsen, and T. Shevlin. 1984. “Earnings Releases, Anomalies, and the Behavior of Security Returns.” *The Accounting Review* 59 (4): 574–603.

Frazzini, A., and L. H. Pedersen. 2014. “Betting against Beta.” *Journal of Financial Economics* 111 (1): 1–25.

Heston, S. L., and R. Sadka. 2008. “Seasonality in the Cross-Section of Stock Returns.” *Journal of Financial Economics* 87 (2): 418–445.

Hirshleifer, D., K. Hou, S. H. Teoh, and Y. Zhang. 2004. “Do Investors Overvalue Firms with Bloated Balance Sheets?” *Journal of Accounting and Economics* 38: 297–331.

Hou, K., H. Mo, C. Xue, and L. Zhang. 2019. “Which Factors?” *Review of Finance* 23 (1): 1–35.

Hou, K., C. Xue, and L. Zhang. 2015. “Digesting Anomalies: An Investment Approach.” *The Review of Financial Studies* 28 (3): 650–705.

——. 2018. “Replicating Anomalies.” *The Review of Financial Studies* 33 (5): 2019–2133.

Jegadeesh, N., and S. Titman. 1993. “Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency.” *The Journal of Finance* 48 (1): 65–91.

Kaczmarek, T., and K. Perez. 2022. “Building Portfolios Based on Machine Learning Predictions.” *Economic Research-Ekonomska istraživanja* 35 (1): 19–37.

Kirby, C., and B. Ostdiek. 2012. “It’s All in the Timing: Simple Active Portfolio Strategies That Outperform Naïve Diversification.” *Journal of Financial and Quantitative Analysis* 47 (2): 437–467.

Lohre, H., C. Rother, and K. A. Schäfer. 2020. “Hierarchical Risk Parity: Accounting for Tail Dependencies in Multi-Asset Multi-Factor Allocations.” *Machine Learning for Asset Management: New Developments and Financial Applications*, 329–368. https://onlinelibrary.wiley.com/doi/10.1002/9781119751182.ch9.

Loughran, T., and J. W. Wellman. 2011. “New Evidence on the Relation between the Enterprise Multiple and Average Stock Returns.” *Journal of Financial and Quantitative Analysis* 46 (6): 1629–1650.

Pontiff, J., and A. Woodgate. 2008. “Share Issuance and Cross-Sectional Returns.” *The Journal of Finance* 63 (2): 921–945.

Raffinot, T. 2017. “Hierarchical Clustering-Based Asset Allocation.” *The Journal of Portfolio Management* 44 (2): 89–99.

Sloan, R. G. 1996. “Do Stock Prices Fully Reflect Information in Accruals and Cash Flows about Future Earnings?” *The Accounting Review* 71 (3): 289–315.

Tuzel, S. 2010. “Corporate Real Estate Holdings and the Cross-Section of Stock Returns.” *The Review of Financial Studies* 23 (6): 2268–2302.

Van Binsbergen, J. H. 2020. “Duration-Based Stock Valuation: Reassessing Stock Market Performance and Volatility.” Technical report, National Bureau of Economic Research.

Ward, Jr., J. H. 1963. “Hierarchical Grouping to Optimize an Objective Function.” *Journal of the American Statistical Association* 58 (301): 236–244.

Wright, J., S. Yam, and S. Pang Yung. 2014. “A Test for the Equality of Multiple Sharpe Ratios.” *Journal of Risk* 16 (4): 3–21.

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