Auctions and Mechanism Design

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Consider a single-item, sealed-bid auction as a game:

- Each of n bidders is a player
- Each player i has a valuation $v_i \in \mathbb{R}$ for the item being auctioned.
- If the outcome is that player i wins the item and pay a price pr, then the payoff to player i is $u_i(outcome) = v_i - pr$, and the payoff to other players are 0.
- A reasonable constraint for such an auction would be that: given the bids $b = (b_1, \ldots, b_n)$, one of the highest bidders must win and pay a price $0 \leq pr \leq \max_i b_i$. It makes sense that the bidders who bid more should win and that they should not pay a price more than their bid (nor a negative price).

We want to design an auction such that every bidder bidding their true valuations $b_i = v_i$ is a dominant strategy.

Vickrey/Second-price auction: The highest bidder, j, whose bid is $b_j = \max_i b_i$, gets the item, but pays the second highest bid price: $pr = \max_{i \neq j} b_i$. Bidding the true valuation is a weakly dominant strategy for all players (irrespective of what other players do).

To prove this, we split the scenarios into cases:

- If the player is bidding their true valuation and wins the item:
    1. if they bid higher, they still win and pay the second-price with non-negative utility
    2. if they bid a little lower (still above second-price), they still win and pay the second-price with non-negative utility
    3. if they bid sufficiently lower (below second-price) , they lose and receive 0 utility
- If the player is bidding their true valuation and loses the item:

1. if they bid a little higher (still lower than first-price), they still lose and receive 0 utility
2. if they bid sufficiently higher (higher than first-price), they win but with negative utility
3. if they bid lower, they still lose and receive 0 utility

So clearly, bidding the true valuation is a weakly dominant strategy for any player, because they can not strictly increase their payoff.

This strategy is not dominant in a first-price auction because the winning person can potentially bid less than their true valuation and still win the item.

Bayesian Games

A Bayesian game has:

- A set of $N = \{1, \ldots, n\}$ players
- A set A_i of actions for each player
- A set of possible types T_i for each player
- A payoff function for each player:

$$u_i : A_1 \times \cdots \times A_n \times T_1 \times \cdots \times T_n \mapsto \mathbb{R}$$

- A joint probability distribution over types (common prior):

$$p : T_1 \times \cdots \times T_n \mapsto [0, 1]$$

  where the probability sum to 1.

A pure strategy for a player is a function $s_i : T_i \mapsto A_i$. This means a player knows its own type t_i and chooses action $s_i(t_i)$. Player's types are chosen randomly according to p. Player i knows its type but not others' types, but everyone knows p and compute $p(t_{-i} \mid t_i)$ (likelihood of others' types given its type). The expected payoff for to player i under the pure strategy profile s is:

$$U_i(s, t_i) = \sum_{t_{-i}} p(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, s_n(t_n), t_i, t_{-i})$$

A Bayesian NE is similarly defined. Every finite Bayesian game has a Bayesian NE.

The Vickrey-Clarke-Groves (VCG) Auction Generalization of the Vickrey auction

Define a multi-item sealed-bid auction:

- Let V be a set of bidders
- Let C be a set of outcomes (an outcome is e.g. bidder 1 gets item 3 and 4, bidder 2 gets item 1, etc.)
- Each bidder $i \in V$ has a value function $v_i : C \mapsto \mathbb{R}_{\geq 0}$ which indicates the amount of money it is worth to bidder i if c is the winning outcome

Suppose we want to find an outcome c^* that maximizes the total value for all the bidders:

$$f(v_i, \ldots, v_n) = c^* \in \arg\max_c \sum_{i \in V} v_i(c)$$

But bidders can lie about their true valuation and we want to incentivise them tell the truth. In a VCG auction:

- Each bidder submits its valuation v' (v' might not be their true valuation v)
- The auction computes the optimal outcome:

$$f(v'_i, \ldots, v'_n) = c^* \in \arg\max_c \sum_{k \in V} v'_k(c)$$

- Each bidder i pays an amount $p_i(c^*)$ which is independent of v_i:

$$p_i(c^*) = \left(\max_{c' \in C} \sum_{j \in V \setminus \{i\}} v'_j(c') \right) - \sum_{j \in V \setminus \{i\}} v'_j(c^*)$$

  The first term is the sum valuation of the best outcome **without** bidder i's presence, the second term is the sum valuation of the other bidders with bidder i's presence. So bidder i pays the price it costs other bidders by joining the auction.

A VCG auction is incentive-compatible and it is a weakly dominant strategy for each bidder to submit their true valuation. Proof is mostly mathematical manipulation, omitted here.