Linear Programming and the Simplex Algorithm

Haoran Peng

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A linear programming problem instance consists of:

- A linear objective function $f : \mathbb{R}^n \mapsto \mathbb{R}$:

$$f(x_1, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n + d$$

- An optimization criterion: `Maximize/Minimize`
- A set $C(x_1, \ldots, x_n)$ of m linear constraints. Each C_i, where $i = 1, \ldots, m$, has the following form:

$$c_{i,1}x_1 + c_{i,2}x_2 + \cdots + c_{i,n}x_n \ \Delta \ b_i$$

  where $\Delta \in \{\geq, \leq, =\}$ and $a_{i,j}, b_i$ are rational numbers.

A vector v satisfy C_i if the constraint $C_i(v)$ holds true. A vector v is a solution to the system C if v satisfies every constraint in C. Let $K(C)$ denotes the set of all solutions to a system. C is feasible if $K(C) \neq \emptyset$. An optimal solution, for a maximization problem, is some $x^* \in K(C)$ such that:

$$f(x^*) = \max_{x \in K(C)} f(x)$$

There can be 3 outcomes in solving a LP problem:

- The problem is infeasible (no solution)
- The problem is feasible (solution goes to infinity)
- An optimal feasible solution exists

Geometric intuition on the simplex algorithm. An LP problem's feasible region can be seen as a convex polyhedron. The simplex algorithm pivots between different vertices of the polyhedron. Note that the problem is convex, the local minimum is the global minimum. If in every pivoting step, we can improve our solution, we just need to keep iterating until we find a vertex where no improvements can be made.

LP in Primal Form

We can convert any LP into the following form:

Maximize: $c_1x_1 + c_2x_2 + \cdots + c_nx_n + d$

Subject to:

$$\begin{gathered} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m \end{gathered}$$

$$x_1, \ldots, x_n \geq 0$$

By adding slack variables, we can turn the LP into a standard form (a dictionary):

$$\begin{gathered} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n + y_1 = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n + y_2 = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n + y_m = b_m \end{gathered}$$

$$\begin{gathered} x_1, \ldots, x_n \geq 0 \\ y_1, \ldots, y_m \geq 0 \end{gathered}$$

We also assume that $b_i \geq 0$ (if not, multiply both sides by -1). An LP in standard form has 3 properties:

- Every constraint has at least one variable with coefficient 1 that does not appear in other constraints
- Picking one such variable, y_i, from each constraint, we obtain a set of m variables $B = \{y_1, \ldots, y_m\}$ called a basis.
- The objective function f only involves non-basis variables.

Dual LP

For a primal LP:

Maximize: $c^T x$

Subject to: $Ax \leq b, x \geq 0$

Its dual LP is:

Minimize: $b^T y$

Subject to: $A^T y \geq c, y \geq 0$

Weak LP duality: If x and y are feasible solutions to the primal and dual LPs $\implies c^T x \leq g^T y$

Strong LP duality:

- If both primal and dual are feasible, and x^* and y^* are optimal solutions to the primal and dual LPs $\implies c^T x^* \leq b^T y^*$
- The primal is infeasible and the dual is unbounded
- The primal is unbounded and the dual is infeasible
- Both are infeasible