Imperial College London

Control Systems
Part III: Stability

Thomas Parisini

Department of Electrical and Electronic Engineering

Stability of Equilibrium

Imperial College London

$$\begin{cases} \dot{x} = f(x,u,t) \\ y = g(x,u,t) \end{cases}$$

$$u(t) = \bar{u}, t \geq 0$$

$$0 = f(x,\bar{u}) \longrightarrow \bar{x}$$

Consider a **perturbation of the initial state** set on the equilibrium state $\bar{x}$:

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases} \longrightarrow x(t) \neq \bar{x}, t \geq 0$$

perturbed state trajectory

Stability of Equilibrium - Definitions

- The equilibrium state $\bar{x}$ is **stable** if $\forall \varepsilon > 0\,,\ \exists \delta(\varepsilon) > 0$ such that:

$$\forall\, x(0) :\ \|\delta \bar{x}\| < \delta(\varepsilon) \implies \|x(t) - \bar{x}\| < \varepsilon,\ \forall\, t \geq 0$$

- The equilibrium state $\bar{x}$ is **asymptotically stable** if:
  - it is stable, that is, if $\forall \varepsilon > 0\,,\ \exists \delta(\varepsilon) > 0$ such that:

$$\forall\, x(0) :\ \|\delta \bar{x}\| < \delta(\varepsilon) \implies \|x(t) - \bar{x}\| < \varepsilon,\ \forall\, t \geq 0$$

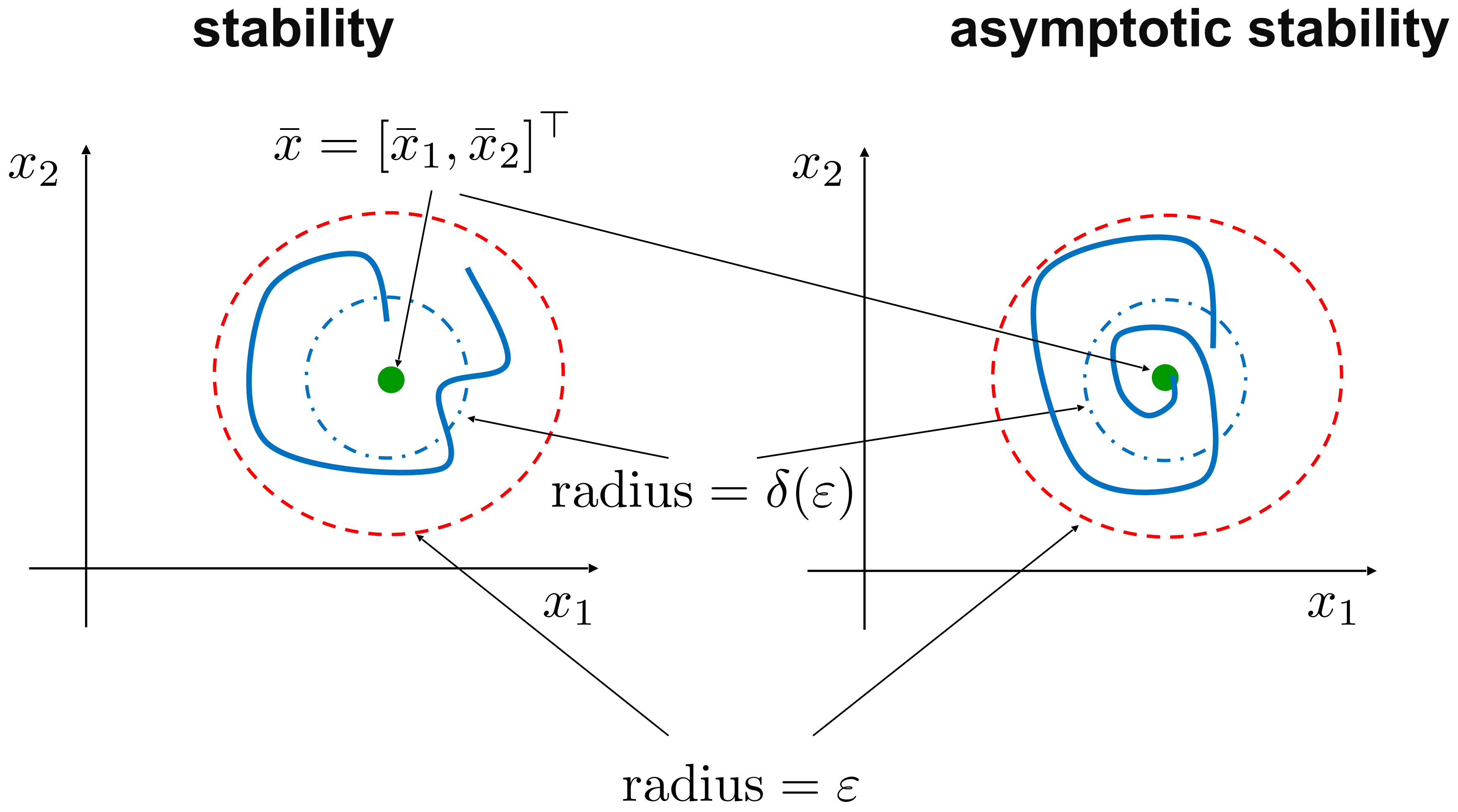
  - and

$$\lim_{t \to \infty} \|x(t) - \bar{x}\| = 0$$

- The equilibrium state $\bar{x}$ is **unstable** if it is not stable

Stability of Equilibrium - Geometric Interpretation

$$f(\bar{x}, \bar{u}) = 0$$



Stability of Linear Systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

- **a)** In equilibrium conditions:

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ \\ x(0) = \bar{x} \end{cases}$$

$$x(t) = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)} B\bar{u} d\tau = \bar{x}, \ \forall\, t \geq 0$$

Stability of Linear Systems (contd.)

- **b)** After a perturbation of the equilibrium state:

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases} \Rightarrow \quad x(t) \neq \bar{x}, t \geq 0$$

perturbed state trajectory

$$x(t) = e^{At}(\bar{x} + \delta\bar{x}) + \int_0^t e^{A(t-\tau)} B\bar{u}\, d\tau$$

$$= \bar{x} + e^{At}\delta\bar{x}$$

$$x(t) - \bar{x} = e^{At}\delta\bar{x}$$

Stability of Linear Systems (contd.)

$$\begin{cases} u(t) = \bar{u}, t \geq 0 \\ x(0) = \bar{x} + \delta\bar{x} \end{cases} \quad \Rightarrow \quad x(t) - \bar{x} = e^{At}\delta\bar{x}$$

deviation of the perturbed state trajectory from the equilibrium state

- stability properties do not depend on the specific value taken on by $\bar{x}$

  ↳ stability is not a property of the equilibrium state (as in the general case) but it is a structural property of the system as a whole

- stability properties depend on the time-behaviour of e^{At} :

  - stability $\Leftrightarrow$ e^{At} **bounded** $\forall\, t \geq 0$
  - asymptotic stability $\Leftrightarrow$ $\lim_{t \to \infty} e^{At} = 0$
  - instability $\Leftrightarrow$ e^{At} **unbounded**

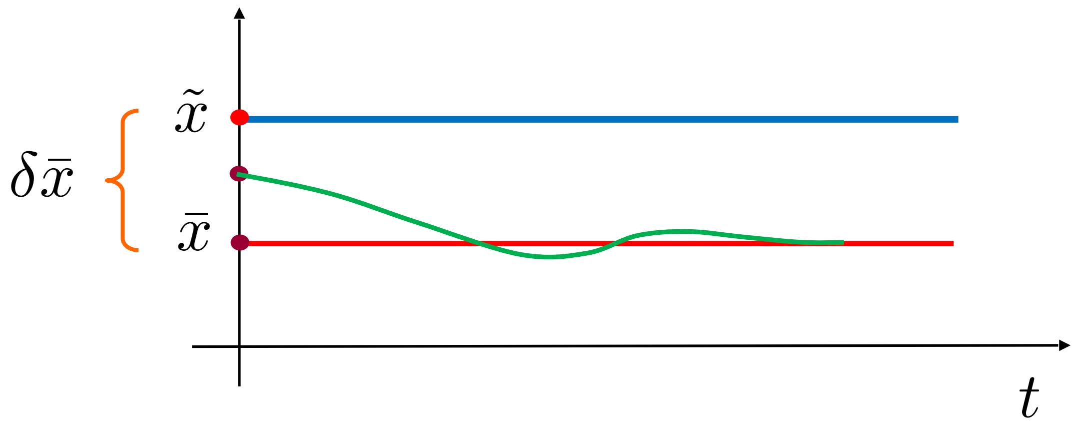
Properties of Asymptotically Stable Linear Systems

- The system state moved from equilibrium "tends" getting back to it

- Given a specific **constant input** $u(t) = \bar{u}, t \geq 0$ the corresponding equilibrium state $\bar{x}$ is **unique:**

asymptotic stability $\Rightarrow$ $\bar{x}$ $\quad \text{unique}$ $\Leftrightarrow$ $\det(A) \neq 0$

In fact, suppose that, for a given $\bar{u}$, two different equilibrium states $\bar{x}, \tilde{x}$ would exist:



contradiction with the asymptotic stability assumption!!!

Properties of Asymptotically Stable Linear Systems (contd.)

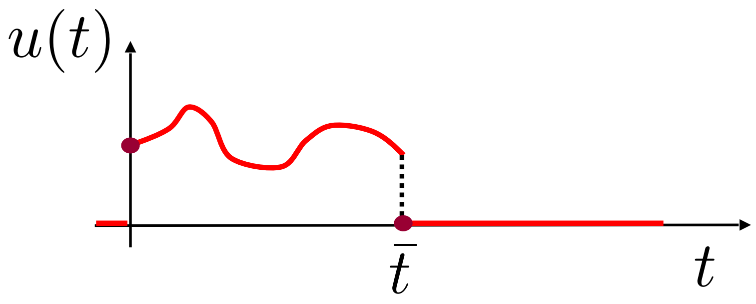
- The perturbed state trajectory asymptotically only depends on the input trajectory $u(t)$:

$$x(t) = \underbrace{x_l(t)}_{\to 0} + x_f(t)$$

For example:

- $u(t) = 0,\ t \geq 0 \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0;\ \lim_{t\to\infty} y(t) = 0,\ \forall\, x_0$

- $u(t) = \begin{cases} \text{any function} & 0 \leq t < \bar{t} \\ 0 & t \geq \bar{t} \end{cases}$

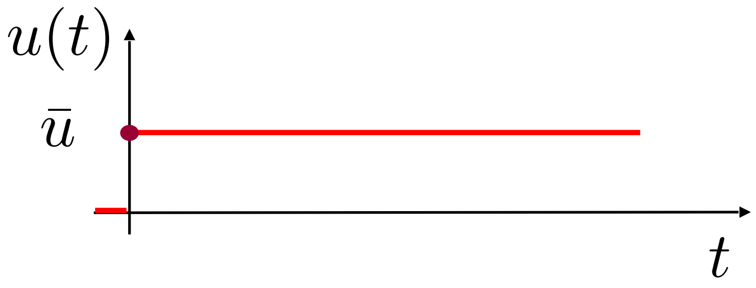


$$\lim_{t\to\infty} x(t) = 0;\ \lim_{t\to\infty} y(t) = 0,\ \forall\, x_0$$

Properties of Asymptotically Stable Linear Systems (contd.)

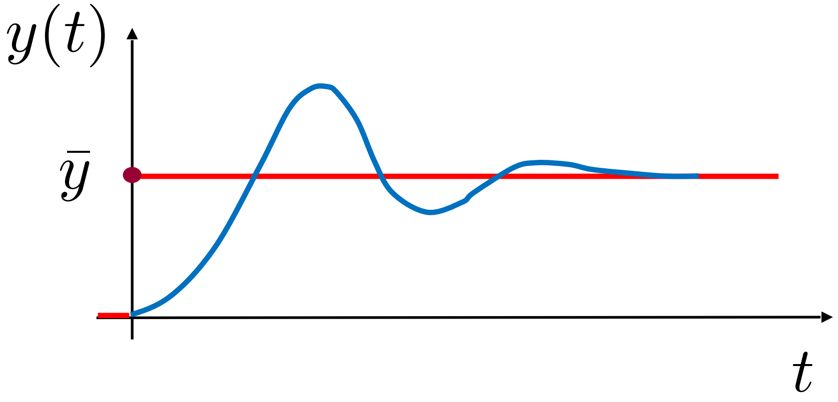
- If:

$$u(t) = \begin{cases} 0 & t < 0 \\ \bar{u} & t \geq 0 \end{cases}$$



$$\lim_{t \to \infty} y(t) = \bar{y}, \ \forall\, x_0 \qquad \text{where} \qquad \bar{y} = \mu \bar{u} = \left(-CA^{-1}B + D\right) \bar{u}$$

static gain



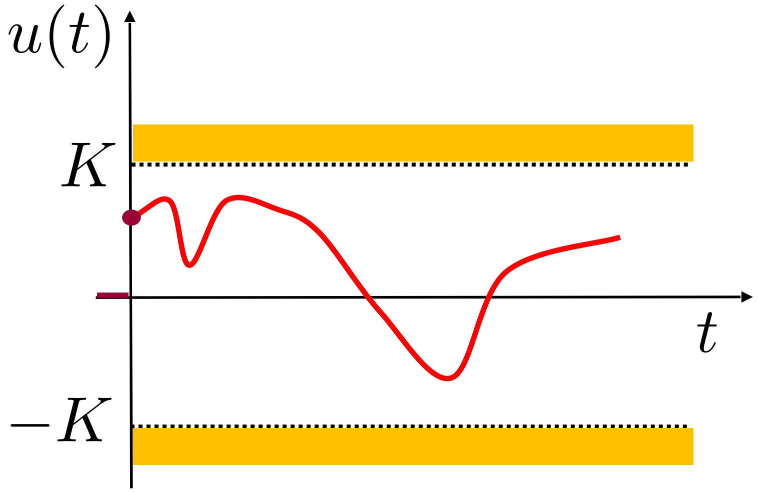
Properties of Asymptotically Stable Linear Systems (contd.)

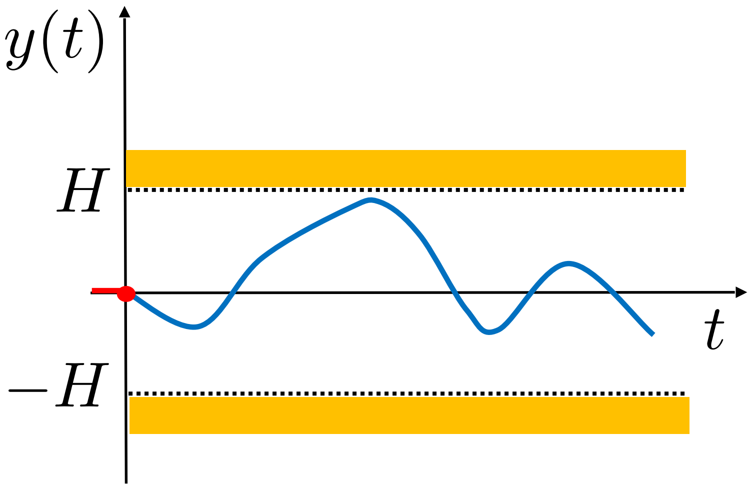
- It holds that:

$$u(t) \quad \text{bounded} \quad \Longrightarrow \quad y(t) \quad \text{bounded}$$

that is:

$$\|u(t)\| < K, \forall\, t \geq 0 \quad \Longrightarrow \quad \exists\, H : \|y(t)\| < H, \forall\, t \geq 0$$





- This property is named **Bounded Input Bounded Output** (**BIBO**) Stability
- Asymptotic Stability $\Longrightarrow$ BIBO Stability but **NOT** viceversa

Stability of Linear Systems: Summing Up

Stability properties depend on the asymptotic time-behaviour of the free state trajectories or, equivalently, of the matrix exponential:

$$x_l(t) = e^{At}x(0)$$

- stability $\Longleftrightarrow$ e^{At} **bounded** $\forall\, t \geq 0$
- asymptotic stability $\Longleftrightarrow$ $\lim\limits_{t \to \infty} e^{At} = 0$
- instability $\Longleftrightarrow$ e^{At} **unbounded**

Example: Scalar State Case

$$x_l(t) = e^{at}x(0)\,, \quad a \in \mathbb{R}$$

- stability $\Longleftrightarrow$ $a \leq 0$
- asymptotic stability $\Longleftrightarrow$ $a < 0$
- instability $\Longleftrightarrow$ $a > 0$

The stability property just depends on the **sign** of $a \in \mathbb{R}$

General Case: Analysis of e^{At}

Consider the following cases:

1. Matrix A is **diagonal**
2. Matrix A has **real and distinct eigenvalues**
3. Matrix A has **complex and distinct eigenvalues**
4. Matrix A has eigenvalues with multiplicity larger than 1
    1. Matrix A **can** be transformed into a diagonal matrix
    2. Matrix A **cannot** be transformed into a diagonal matrix

Case 1: Matrix A is Diagonal

$$A = \begin{bmatrix} s_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & s_n \end{bmatrix}$$

$s_1, s_2, \ldots, s_n \in \mathbb{R}$
eigenvalues of A

$$e^{At} = \begin{bmatrix} e^{s_1 t} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{s_n t} \end{bmatrix}$$

The matrix e^{At} is a square $n \times n$ matrix whose elements $e^{s_i t}$ are functions of time

Hence:

- stability $\Longleftrightarrow$ $s_i \leq 0\,,\ i = 1, \ldots, n$
- asymptotic stability $\Longleftrightarrow$ $s_i < 0\,,\ i = 1, \ldots, n$
- instability $\Longleftrightarrow$ $\exists\, i \in \{1, \ldots, n\}$ such that $s_i > 0$

Case 2: Matrix A has real and distinct eigenvalues

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$s_1, s_2, \ldots, s_n \in \mathbb{R},\ s_1 \neq s_2 \neq \cdots \neq s_n$
eigenvalues of A

Find matrix $T \in \mathbb{R}^{n\times n},\ \det(T) \neq 0\ :\ A = T\tilde{A}T^{-1}$, $\quad \tilde{A} = \begin{bmatrix} s_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & s_n \end{bmatrix}$

$$\begin{aligned} e^{At} &= I + At + \tfrac{1}{2}A^2t^2 + \cdots \\ &= I + T\tilde{A}T^{-1}t + \tfrac{1}{2}(T\tilde{A}T^{-1}t)(T\tilde{A}T^{-1}t) + \cdots \\ &= T\left(I + \tilde{A}t + \tfrac{1}{2}(\tilde{A}t)^2 + \cdots\right)T^{-1} \\ &= Te^{\tilde{A}t}T^{-1} \end{aligned}$$

$$\Rightarrow\quad e^{At} = T\begin{bmatrix} e^{s_1 t} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{s_n t} \end{bmatrix}T^{-1}$$

Hence:

- stability $\Longleftrightarrow$ $s_i \leq 0\,,\ i = 1, \ldots, n$
- asymptotic stability $\Longleftrightarrow$ $s_i < 0\,,\ i = 1, \ldots, n$
- instability $\Longleftrightarrow$ $\exists\, i \in \{1, \ldots, n\}$ such that $s_i > 0$

Example

$$\begin{cases} \dot{x}_1 = -2x_1 + 6x_2 \\ \dot{x}_2 = -2x_1 + 5x_2 \end{cases} \qquad A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$$

- Eigenvalues:

$$p_A(s) = \det(sI - A) = \det \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix} = (s+2)(s-5) + 12$$

$$= s^2 - 3s + 2 = (s-2)(s-1)$$

$\Rightarrow$ $s_1 = 1;\ s_2 = 2$ **Unstable** because both eigenvalues are positive

Example (contd.)

- Eigenvectors:

$$Av = s_1 v \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \overset{s_1}{1} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{cases} -2v_1 + 6v_2 = v_1 \\ -2v_1 + 5v_2 = v_2 \end{cases}$$

for example:

$$v_1 = 2v_2 \Rightarrow v^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Av = s_2 v$$

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \overset{s_2}{2} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{cases} -2v_1 + 6v_2 = 2v_1 \\ -2v_1 + 5v_2 = 2v_2 \end{cases}$$

for example:

$$v_1 = \frac{3}{2} v_2 \Rightarrow v^{(2)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Example (contd.)

- Transformation into diagonal form:

$$T = \left[v^{(1)} \middle| v^{(2)} \right] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad T^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\tilde{A} = T^{-1} A T = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example (contd.)

- Calculation of the matrix exponential e^{At} :

$$
\begin{aligned}
e^{At} &= T e^{\tilde{A}t} T^{-1} = T \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 4e^t - 3e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & -3e^t + 4e^{2t} \end{bmatrix}
\end{aligned}
$$

↳ The matrix exponential contains elements (all, in this specific example) that are **asymptotically unbounded** which is consistent with the previous statement about **instability** based on the **positive sign of the eigenvalues**

Case 3: Matrix A has complex and distinct eigenvalues

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$s_1, s_2, \ldots, s_n \in \mathbb{C}$, $s_1 \neq s_2 \neq \cdots \neq s_n$
eigenvalues of A

For simplicity, consider the case: $n = 2\,; \quad s_1 = \sigma + j\omega\,; \quad s_2 = \sigma - j\omega$

Hence, the matrix exponential e^{At} contains terms such as:

$$\gamma e^{(\sigma + j\omega)t} + \bar{\gamma} e^{(\sigma - j\omega)t} \quad \text{where} \quad \gamma = \alpha + j\beta\,; \quad \bar{\gamma} = \alpha - j\beta$$

Case 3: Matrix A has complex and distinct eigenvalues (contd.)

Then:

$$
\begin{aligned}
&\gamma e^{(\sigma+j\omega)t} + \bar{\gamma} e^{(\sigma-j\omega)t} \\
&\qquad = \gamma e^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right] + \bar{\gamma} e^{\sigma t}\left[\cos(\omega t) - j\sin(\omega t)\right] \\
&\qquad = e^{\sigma t}\left[(\gamma + \bar{\gamma})\cos(\omega t) + j(\gamma - \bar{\gamma})\sin(\omega t)\right] \\
&\qquad = e^{\sigma t}\left[2\alpha\cos(\omega t) + j(j2\beta)\sin(\omega t)\right] \\
&\qquad = 2e^{\sigma t}\underbrace{\left[\alpha\cos(\omega t) - \beta\sin(\omega t)\right]}_{\text{bounded}}
\end{aligned}
$$

this is the term responsible for boundedness/convergence/divergence over time

Case 3: Matrix A has complex and distinct eigenvalues (contd.)

Imperial College London

Hence, generalising:

- stability $\Longleftrightarrow$ $\mathrm{Re}(s_i) \leq 0\,,\; i = 1, \ldots, n$
- asymptotic stability $\Longleftrightarrow$ $\mathrm{Re}(s_i) < 0\,,\; i = 1, \ldots, n$
- instability $\Longleftrightarrow$ $\exists\, i \in \{1, \ldots, n\}$ such that $\mathrm{Re}(s_i) > 0$

Case 4: Matrix A has multiple eigenvalues

- **Example 1:**

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \qquad s_1 = s_2 = \alpha$$

$$\Rightarrow \quad e^{At} = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{bmatrix}$$

Matrix A is already in diagonal form, hence no need for resorting to an equivalent state equation. This case is equivalent to Case 1

Case 4: Matrix A has multiple eigenvalues (contd.)

Imperial College London

- **Example 2:**

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \qquad s_1 = s_2 = \alpha$$

In this case, there **does not exist** an equivalent state space transformation bringing matrix A into a diagonal form

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$$

$$A^2 = \begin{bmatrix} \alpha^2 & 2\alpha \\ 0 & \alpha^2 \end{bmatrix} \qquad A^3 = \begin{bmatrix} \alpha^3 & 3\alpha^2 \\ 0 & \alpha^3 \end{bmatrix} \quad \cdots \quad A^k = \begin{bmatrix} \alpha^k & k\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix} \quad \cdots$$

Therefore:

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \qquad s_1 = s_2 = \alpha$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} t + \begin{bmatrix} \alpha^2 & 2\alpha \\ 0 & \alpha^2 \end{bmatrix} \frac{t^2}{2!} + \cdots$$

$$+ \cdots + \begin{bmatrix} \alpha^k & k\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix} \frac{t^k}{k!} + \cdots$$

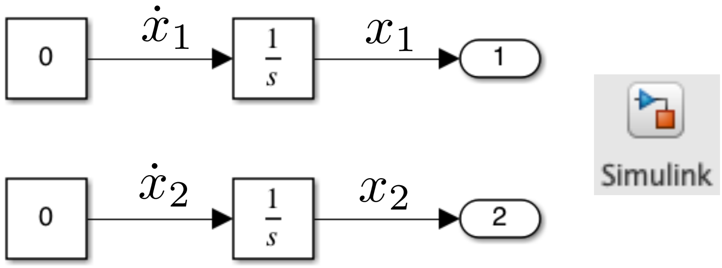
$$= \begin{bmatrix} e^{\alpha t} & t + \alpha t^2 + \cdots + \alpha^{k-1} \frac{t^k}{(k-1)!} + \cdots \\ 0 & e^{\alpha t} \end{bmatrix} = \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix}$$

Hence:

- Concerning **Example 1**:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \qquad \alpha = 0$$

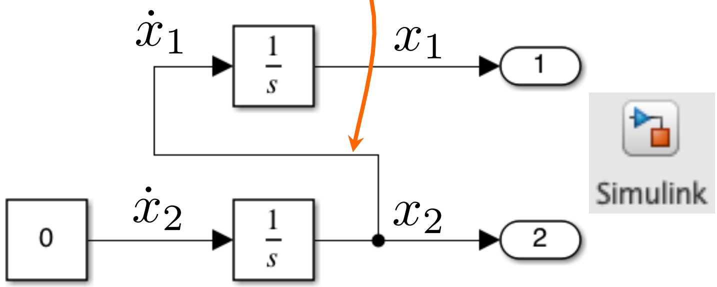
- stability $\Longleftrightarrow$ $\alpha = 0$
- asymptotic stability $\Longleftrightarrow$ $\alpha < 0$
- instability $\Longleftrightarrow$ $\alpha > 0$



- Concerning **Example 2**:

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \qquad \alpha = 0$$

- asymptotic stability $\Longleftrightarrow$ $\alpha < 0$
- instability $\Longleftrightarrow$ $\alpha \geq 0$



Case 4: Matrix A has multiple eigenvalues (contd.)

Imperial College London

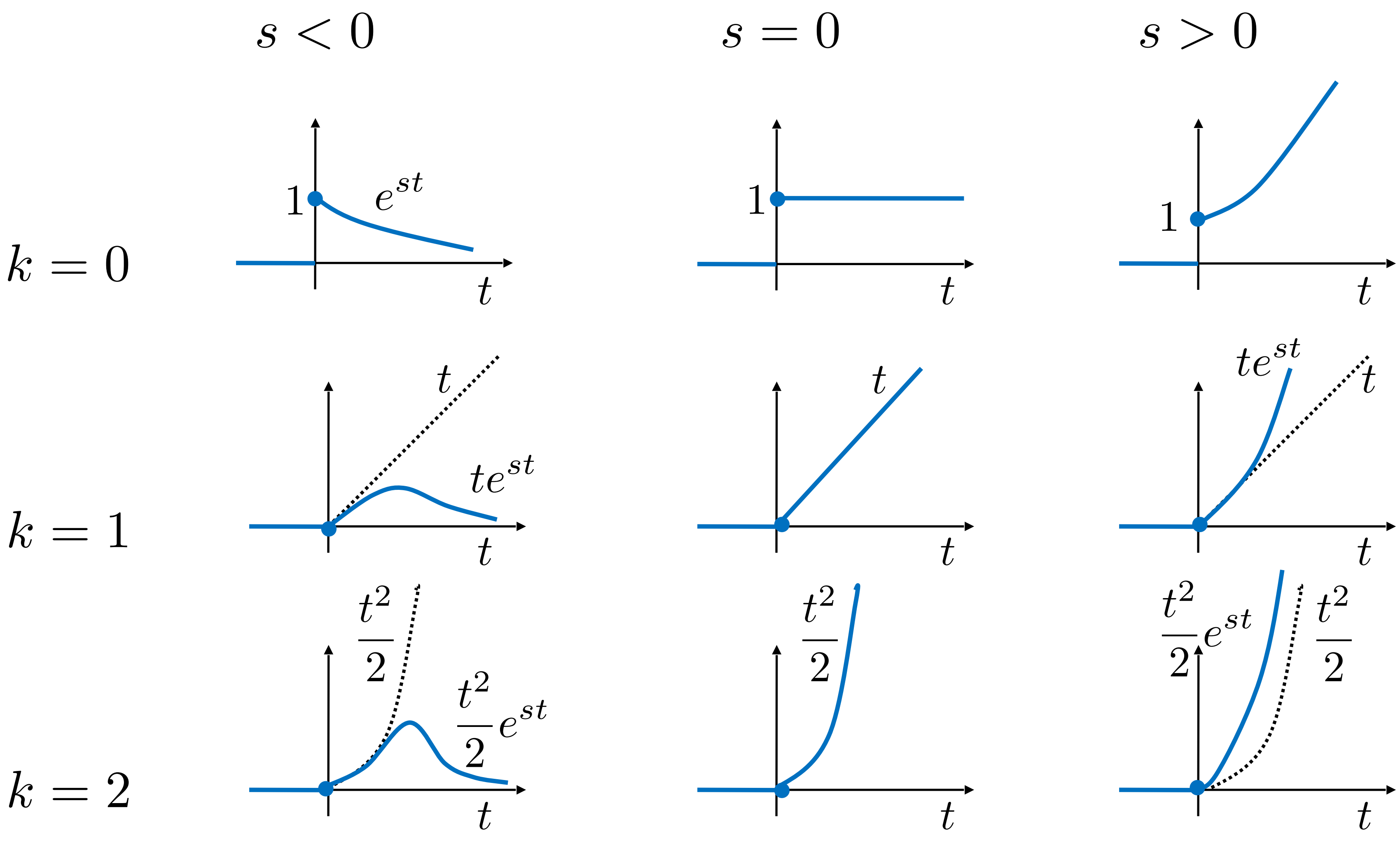
In general, consider a matrix A such that:

- A has eigenvalues with multiplicity $\nu > 1$
- matrix A **cannot** be transformed into equivalent diagonal form

$\Rightarrow$ e^{At} contains terms of the form $t^k e^{s_i t}$, $k = 1, 2, \dots, \nu - 1$

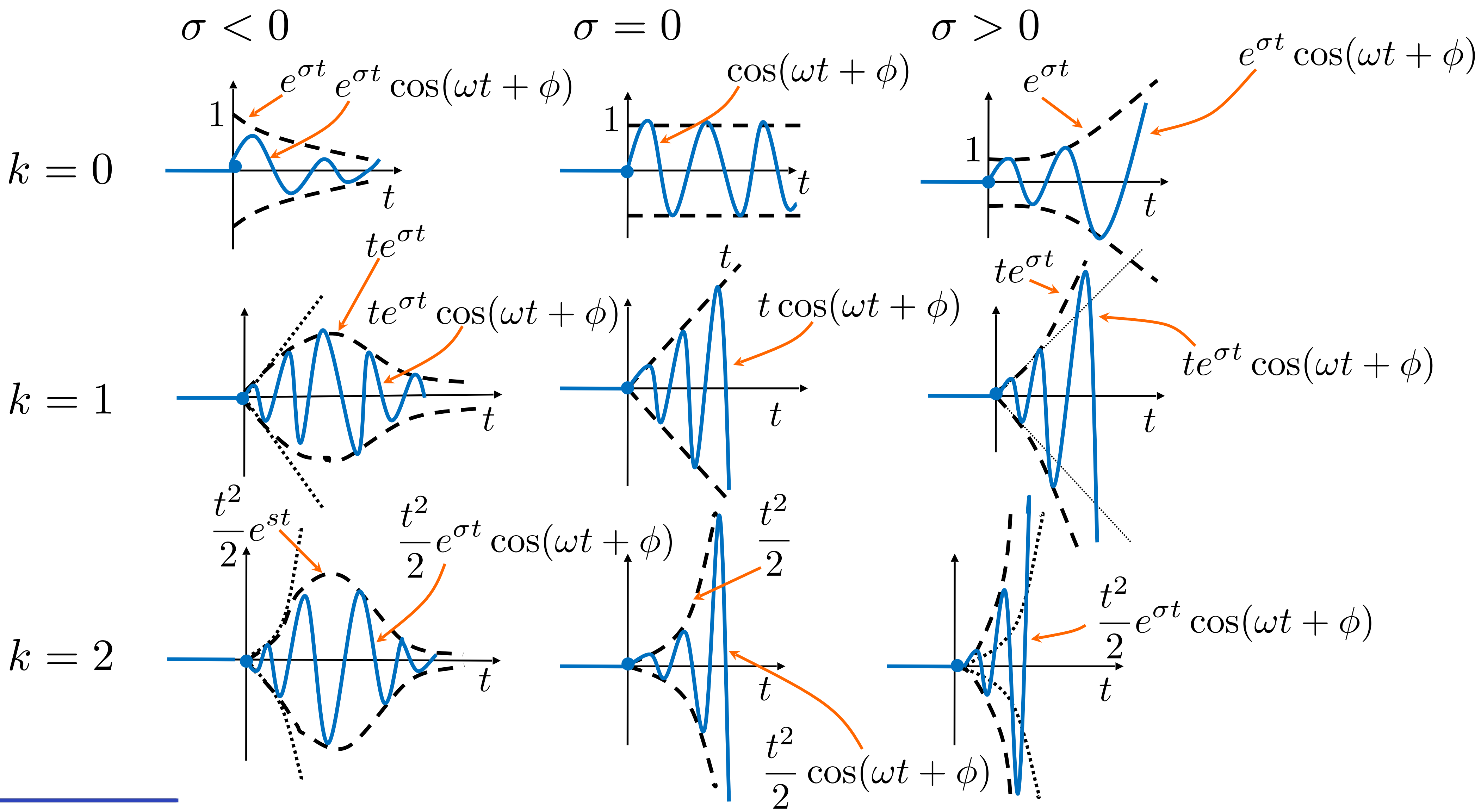
Qualitative Analysis of the generic term $t^k e^{st}$, $s \in \mathbb{C}$, $k \in \mathbb{Z}^+$

- **Case 1:** $s \in \mathbb{R}$



Qualitative Analysis of the generic term $t^k e^{st}$, $s \in \mathbb{C}$, $k \in \mathbb{Z}^+$

- **Case 2:** $s \in \mathbb{C}$, $s_1 = \sigma + j\omega$, $s_2 = \sigma - j\omega$



Stability of Linear Systems: Main Results

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$s_1, s_2, \ldots, s_n \in \mathbb{C}$
eigenvalues of A

Result 1:

$\mathrm{Re}(s_i) < 0\,,\ i = 1, \ldots, n$ $\Longleftrightarrow$ Asymptotic Stability

Result 2:

$\exists\, i \in \{1, \ldots, n\}$ such that $\mathrm{Re}(s_i) > 0$ $\Longrightarrow$ Instability

Result 3:

$$\begin{cases} \mathrm{Re}(s_i) \leq 0\,,\ i = 1, \ldots, n \\ \exists \widetilde{i} \in \{1, \ldots, n\} \text{ such that } \mathrm{Re}(s_{\tilde{i}}) = 0 \end{cases}$$

➡ No Asymptotic Stability

- If multiplicity of **all** $s_{\tilde{i}}$ such that $\mathrm{Re}(s_{\tilde{i}}) = 0$ is equal to 1

  ↳ Stability (Non-Asymptotic, also called Marginal)

- If $\exists \widetilde{i} \in \{1, \ldots, n\}$ such that $\mathrm{Re}(s_{\tilde{i}}) = 0$ with multiplicity > 1

  ↳ Instability ? or Stability (non asymptotic anyway)

Stability of Linear Systems: Other Criteria/Tools

- So far, the stability analysis has been carried out by **evaluating the eigenvalues** of matrix A and their **location** in the complex plane
- Other criteria can be devised **not requiring** the calculation of the eigenvalues of matrix A but based on the:
  - analysis of the **elements** of matrix A
  - analysis of the **characteristic polynomial** of matrix A :

$$\varphi_A(s) = \det(sI - A) = \varphi_0 s^n + \varphi_1 s^{n-1} + \cdots + \varphi_{n-1} s + \varphi_n$$

Criteria/Tools Based on Analysis Elements of Matrix A

Imperial College London

- **Criterion 1:**

  If matrix A is triangular:

  $$a_{ii} < 0,\ i = 1, \ldots, n \iff \text{Asymptotic Stability}$$

- **Criterion 2:**

  Letting $\mathrm{tr}\,(A) := \sum_{i=1}^{n} a_{ii}$:

  $$\text{Asymptotic Stability} \implies \mathrm{tr}\,(A) < 0$$

  $$\mathrm{tr}\,(A) > 0 \implies \text{Instability}$$

- **Criterion 3:**

  $$\text{Asymptotic Stability} \implies \det\,(A) \neq 0$$

Criteria/Tools Based on Characteristic Polynomial of A

Imperial College London

- **Criterion 4 (valid for second-order systems $n = 2$):**

Asymptotic Stability $\Longleftrightarrow$ $\mathrm{Re}\,(s_i) < 0,\ i = 1, 2$ $\Longleftrightarrow$ $\{\varphi_0, \varphi_1, \varphi_2\}$ $\neq 0$ same sign

- **Criterion 5:**

Asymptotic Stability $\Longleftrightarrow$ $\mathrm{Re}\,(s_i) < 0,\ i = 1, 2$ $\Longrightarrow$ $\{\varphi_0, \dots, \varphi_n\}$ $\neq 0$ same sign

Examples

$\varphi(s) = s^3 + 3s^2 + 2s$ Not Asymptotically Stable

$\varphi(s) = s^3 + 2s + 5$ Not Asymptotically Stable

$\varphi(s) = s^3 + 5s^2 - 2s + 4$ Not Asymptotically Stable

$\varphi(s) = s^3 + 5s^2 + 2s + 4$???

In the general case, a **necessary and sufficient** condition is needed

Routh Table

Imperial College London

For a given system matrix A the characteristic polynomial $\varphi_A(s)$ is:

$$\varphi_A(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \cdots + \varphi_{n-1} s + \varphi_n$$

$$\left.\begin{array}{c|cccc}
1 & \varphi_0 & \varphi_2 & \varphi_4 & \cdots \\
2 & \varphi_1 & \varphi_3 & \varphi_5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
i-2 & h_1 & h_2 & h_3 & \cdots \\
i-1 & k_1 & k_2 & k_3 & \cdots \\
i & l_1 & l_2 & l_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots
\end{array}\right\}$$

maximum $n+1$ rows

r_i

$$l_1 = -\frac{1}{k_1}\det\begin{bmatrix} h_1 & h_2 \\ k_1 & k_2 \end{bmatrix} \quad l_2 = -\frac{1}{k_1}\det\begin{bmatrix} h_1 & h_3 \\ k_1 & k_3 \end{bmatrix} \quad \ldots \quad l_j = -\frac{1}{k_1}\det\begin{bmatrix} h_1 & h_{j+1} \\ k_1 & k_{j+1} \end{bmatrix}$$

General Routh-Hurwitz Stability Criterion

Consider a given system matrix A and its characteristic polynomial $\varphi_A(s)$:

$$\varphi_A(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \cdots + \varphi_{n-1} s + \varphi_n$$

- If the Routh Table cannot be completed $\Rightarrow$ no asymptotic stability
- If the Routh Table can be completed ($n+1$ rows)
  - The number of roots of $\varphi_A(s)$ with **positive real part** is equal to the **number of sign-changes** in the first column
  - The number of roots of $\varphi_A(s)$ with **negative real part** is equal to the **number of sign-permanencies** in the first column r_i
  - **No sign-changes** in the first column r_i $\Leftrightarrow$ **asymptotic stability**

Example 1

$$\varphi_A(s) = s^3 + 5s^2 + 2s + 4$$

$$\begin{array}{c|ccc}
1 & \boxed{1} & 2 & 0 \\
2 & 5 & 4 & 0 \\
3 & \alpha & 0 & 0 \\
4 & \beta & 0 & 0
\end{array}$$

r_4

$$\alpha = -\frac{1}{5}\det\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \frac{6}{5}$$

$$\beta = -\frac{1}{\alpha}\det\begin{bmatrix} 5 & 4 \\ \alpha & 0 \end{bmatrix} = 4$$

No sign-changes in r_4

asymptotic stability

Example 2

$$\varphi_A(s) = s^4 + 2s^3 + 4s^2 + 9s + 6$$

$$\begin{array}{c|ccc}
1 & 1 & 4 & 6 \\
2 & 2 & 9 & 0 \\
3 & \alpha & 6 & 0 \\
4 & \beta & 0 & 0 \\
5 & 6 & 0 & 0
\end{array}$$

r_5

$$\alpha = -\frac{1}{2} \det \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = -\frac{1}{2}$$

$$\beta = -\frac{1}{\alpha} \det \begin{bmatrix} 2 & 9 \\ \alpha & 6 \end{bmatrix} = 33$$

Two sign-changes in r_5

instability

Example 3 (important)

Imperial College London

$$\varphi_A(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

$$\begin{array}{c|ccc}
1 & 1 & 11 & K \\
2 & 6 & 6 & 0 \\
3 & 10 & K & 0 \\
4 & \alpha & 0 & 0 \\
5 & K & 0 & 0
\end{array}$$

r_5

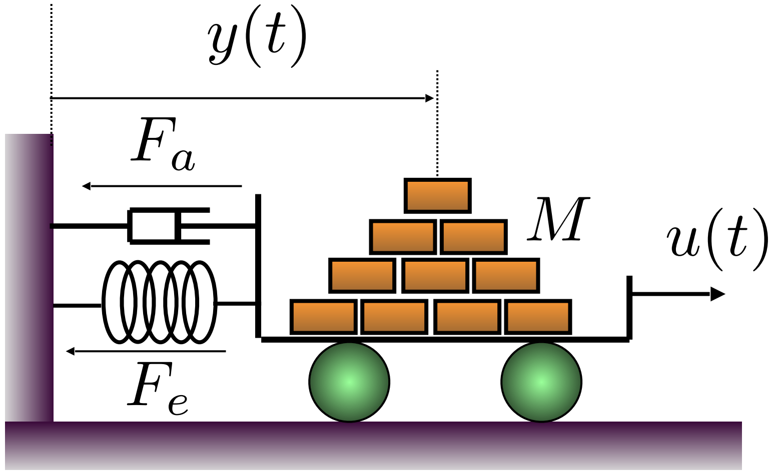
$$\alpha = -\frac{1}{10}\det\begin{bmatrix} 6 & 6 \\ 10 & K \end{bmatrix} = \frac{3}{5}(10 - K)$$

If $K > 0$ and $K < 10$

No sign-changes in r_5

asymptotic stability

Example 4



$$\begin{cases} x_1 := y \\ x_2 := \dot{y} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{M} x_1 - \dfrac{h}{M} x_2 + \dfrac{1}{M} u \\ y = x_1 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix}$$

$$\varphi_A(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ \frac{k}{M} & s + \frac{h}{M} \end{bmatrix} = s^2 + \frac{h}{M} s + \frac{k}{M}$$

where $M > 0,\ k \geq 0,\ h \geq 0$

Example 4 (contd.)

- **Case 1 (both elastic and friction forces active):** $k > 0,\ h > 0$

  ↳ $\mathrm{Re}(s_i) < 0,\ i = 1, 2$ ➡ asymptotic stability

- **Case 2 (only elastic force active):** $k > 0,\ h = 0$

  ↳ $\varphi_A(s) = s^2 + \dfrac{k}{M}$ ➡ $s_{1,2} = \pm j\sqrt{\dfrac{k}{M}}$

  ↳ $\mathrm{Re}(s_i) = 0,\ i = 1, 2$

  ↳ stability (non asymptotic)

Example 4 (contd.)

- **Case 3 (only friction force active):** $k = 0,\ h > 0$

$$\varphi_A(s) = s^2 + \frac{h}{M}s \;\Rightarrow\; s_1 = 0\,;\ s_2 = -\frac{h}{M}$$

$$\Rightarrow\; \mathrm{Re}\,(s_1) = 0,\ \mathrm{Re}\,(s_2) < 0$$

$\Rightarrow$ stability (non asymptotic)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{h}{M}\,x_2 \end{cases} \;\Rightarrow\; x_2(t) = x_{20}e^{-\frac{h}{M}t} \longrightarrow 0$$

$$\Rightarrow\; \dot{x}_1 = x_{20}e^{-\frac{h}{M}t}$$

$$\Rightarrow\; x_1(t) = x_{10} + x_{20}\int_0^t e^{-\frac{h}{M}\tau}d\tau$$

$$= x_{10} - x_{20}\frac{M}{h}\left(e^{-\frac{h}{M}t} - 1\right) \longrightarrow x_{10} + x_{20}\frac{M}{h}$$

Example 4 (contd.)

Imperial College London

- **Case 4 (no elastic nor friction force acting):** $k = 0,\ h = 0$

$$\varphi_A(s) = s^2 \quad \Rightarrow \quad s_1 = 0\,;\ \ s_2 = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{unstable}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{cases} \qquad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\Rightarrow \quad x_2(t) = x_{20}\,,\ \forall\, t$$

$$x_1(t) = x_{10} + x_{20}t \underset{t \to \infty}{\longrightarrow} \infty$$

Stability of Equilibrium of a Nonlinear System via the Linearised System

Recall from Part 2:

$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases} \implies 0 = f(x,\bar{u}) \implies \bar{x} \ \text{(equilibrium state)}$$

$$\delta\dot{x} \simeq \underbrace{f_x(\bar{x},\bar{u})}_{A}\delta x + \underbrace{f_u(\bar{x},\bar{u})}_{B}\delta u$$

$$\delta y \simeq \underbrace{g_x(\bar{x},\bar{u})}_{C}\delta x + \underbrace{g_u(\bar{x},\bar{u})}_{D}\delta u$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} ; \quad \begin{array}{l} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{array}$$

Stability of equilibrium is a **local property** (see Definitions in Part 2, slide 3).

We can take advantage of the linearised system on the specific equilibrium state to analyse its stability

Main Result:

Denoting by $s_i,\, i = 1, \ldots, n$ the eigenvalues of matrix A :

(A) $\mathrm{Re}\,(s_i) < 0,\ i = 1, \ldots, n$ ➡ $\bar{x}$ asymptotically stable equilibrium state

(B) $\exists\, i \text{ such that } \mathrm{Re}\,(s_i) > 0$ ➡ $\bar{x}$ unstable stable equilibrium state

Critical Case:

$$\mathrm{Re}\,(s_i) \leq 0,\ i = 1, \ldots, n$$

$$\exists\, i \text{ such that } \mathrm{Re}\,(s_i) = 0$$

?

$\bar{x}$ asymptotically stable equilibrium state

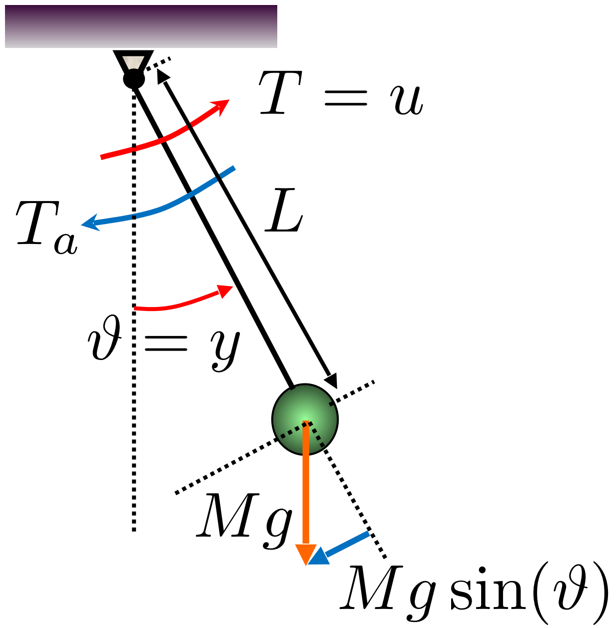
$\bar{x}$ stable equilibrium state

$\bar{x}$ unstable stable equilibrium state

In this case, **no decision** can be made on the stability of the equilibrium state based on the linearised system

Example

Recall from Part 2, slides 45, 46:



$$\begin{cases} x_1 := \vartheta \\ x_2 := \dot{\vartheta} \end{cases}, \quad x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = \dot{\vartheta} = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 + \dfrac{1}{J}u \\ y = x_1 \end{cases}$$

$$u(t) = \bar{u} = 0, \forall t \quad \Rightarrow \quad \begin{cases} 0 = x_2 \\ 0 = -\dfrac{MgL}{J}\sin(x_1) - \dfrac{h}{J}x_2 \end{cases}$$

$$\begin{cases} \bar{x}_2 = 0 \\ \sin(\bar{x}_1) = \dfrac{1}{MgL}\bar{u} \end{cases} \quad \Rightarrow \quad \begin{cases} \bar{x}_2 = 0 \\ \bar{x}_1 = k\pi\,, \ \forall k \in \mathbb{Z} \end{cases}$$

We pick the two "physical" solutions:

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Example (contd.)

The state matrices of the linearised system on the two equilibrium states are:

$$f_x(\bar{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J}\cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \bar{A}$$

$$\det(sI - \bar{A}) = s^2 + \frac{h}{J}s + \frac{MgL}{J}$$ → $\bar{x}$ asymptotically stable equilibrium state

$$f_x(\tilde{x},\bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ -\frac{MgL}{J}\cos(x_1) & -\frac{h}{J} \end{bmatrix}_{\tilde{x},\bar{u}} = \begin{bmatrix} 0 & 1 \\ +\frac{MgL}{J} & -\frac{h}{J} \end{bmatrix} = \tilde{A}$$

$$\det(sI - \bar{A}) = s^2 + \frac{h}{J}s - \frac{MgL}{J}$$ → $\tilde{x}$ unstable equilibrium state