

maximal load: $O(\log n / \log \log n)$ \rightarrow aim
 Space required: $O(\log^2 n / \log \log n)$

Construction

each function described in $O(\log n \log \log n)$
 evaluated in $O(\log n \log \log n)$

IDEA: concatenate output of $O(\log \log n)$ functions which are gradually more independent. Each function f is described using d functions h_1, \dots, h_d

$$f(x) = \underbrace{h_1(x)}_{\text{binary string}} \circ \underbrace{h_2(x) \circ \dots \circ h_d(x)}_{\text{concatenate}}$$

Gradually more independent

$h_1: O(1)$ -wise indep.
 $h_2: O(h_1)$ -wise indep.
 \dots
 $h_d: O(\log n / \log \log n)$ -wise indep.

Note: ① output length decreases at the same time

$$h_1: \mathcal{O}(\log n) \Rightarrow h_d: O(\log \log n)$$

② each of h_1, \dots, h_d could be described/computed in $O(\log n)$ bits/time

Definitions

$$[n] \Rightarrow \{1, \dots, n\}$$

$U_n \Rightarrow$ Uniform distribution over set $\{0, 1\}^n$

$x \leftarrow X \Rightarrow$ sample x from X for a r.v. X .

$x \in S \Rightarrow$ sample x uniformly from finite set S .

$SD(X, Y) \Rightarrow$ **statistical distance** between two r.v. over finite domain S
 $= \sum_{w \in S} |\Pr[X=w] - \Pr[Y=w]|$

$x \circ y \Rightarrow$ concatenate x, y bit string

unit cost RAM model

\Rightarrow elements are taken from a universe of size n and each element
can be stored using $c = O(\log n)$ bits.

k -wise δ -dependent

For a family of $f: [n] \rightarrow [v]$, it is k - δ -dependent iff

$$SD(X, Y) \leq \delta$$

where $X = \text{distribution } (f(x_1), f(x_2), \dots, f(x_k))$ for any distinct $x_1, \dots, x_k \in [n]$

$Y = \text{uniform distribution over } [v]^k$

Describe Space: $O(k \max\{\log n, \log v\})$ bits

Evaluation Time: $O(k)$

ϵ -biased Distribution [M/N 93]

r.v.s X_1, X_2, \dots, X_n over $\{0, 1\}$ is ϵ -biased if for any $S \neq \emptyset \subseteq [n]$,

$$|\Pr[\bigoplus_{i \in S} X_i = 1] - \Pr[\bigoplus_{i \in S} X_i = 0]| \leq \epsilon$$

\uparrow
XOR operation

[Achliptag '92, Sec. 5] constructs an ϵ -biased distribution over $\{0, 1\}^n$ where
each point could be specified using $O(\log(n/\epsilon))$ bits where each bit
could be calculated using $O(\log(n/\epsilon))$ times.

In RAM, t bits could be calculated in time $O(\log(n/\epsilon)t)$

may need proof

min-entropy

for a r.v. X , its min-entropy is

$$H_{\infty}(X) = -\log(\max_x \Pr[X=x])$$

negative log of most probability of $X=x$.

k -source

k -source is a r.v. X with its min-entropy $H_{\infty}(X) \geq k$.

(T, k) -block source

r.v. $X = (X_1, \dots, X_T)$, for any $i \in [T]$

$$H_{\infty}(X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \geq k$$

(T, k, ε) -block source

r.v. $X = (X_1, \dots, X_T)$, for any $i \in [T]$

$$\Pr_{(x_1, \dots, x_{i-1}) \leftarrow (X_1, \dots, X_{i-1})} [H_{\infty}(X_i | X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \geq k] \geq 1 - \varepsilon$$

Pr of X is (T, k) -block source $\geq 1 - \varepsilon$

7 Levens

Corollary 2.1

Fact

for any k an ϵ -biased distribution is also k -wise δ -dependent

$$\text{for } \delta = \epsilon 2^{\frac{k}{2}}$$

For any $u, v, v=2^i$, there exists a family of k -wise δ -dependent function $f: [u] \rightarrow [v]$ described in $O(\log u + k \log v + \log c/\delta)$ bits and calculated in $O(\log u + k \log v + \log c/\delta)$ in RAM.

check AGM92 for proof.

Corollary 2.2

Let $X_1, \dots, X_n \in \{0, 1\}$ be $2k$ -wise δ -dependent r.v. for $k \in \mathbb{N}$, $0 \leq \delta < 1$.

Let $X = \sum X_i$ and $\mu = E[X]$. Then, for any $t > 0$,

$$\Pr[|X - \mu| > t] \leq 2 \left(\frac{2^{nk}}{t^2} \right)^k + f\left(\frac{n}{t}\right)^{2k}.$$

How X diverge from μ . (proven in paper using Markov Ineq.)

Lemma 2.2 from [BR94]

proof: $\Pr(|X - \mu| > t)$

$$= \Pr(|X - \mu|^{2k} > t^{2k})$$

$$\leq \frac{E[|X - \mu|^{2k}]}{t^{2k}} \quad \leftarrow \text{markov inequality } \Pr(X \geq a) \leq \frac{E[X]}{a}.$$

$$\text{consider } E[X - \mu] = \sum_{i \in [n]} E[X_i - \mu_i]$$

↓

$$E[X - E[X]] = \sum_{i=1}^n E[X_i - E[X_i]] = \sum_{i=1}^n E[X_i - \mu_i]$$

$$\text{similarly } E[(X - \mu)^2] = E[(X - E[X])^2]$$

$$= \sum_{i=1}^n \sum_{j=1}^n E[(X_{ij} - \mu_{ij})^2]$$

$$= \dots$$

11.12.2019 ~ 11:15

\therefore we know that $E[(X-M)^{2k}]$

$$= \sum_{i_1, i_2, \dots, i_{2k} \in [n]} E\left[\prod_{j=1}^{2k} (X_{ij} - M_{ij})\right]$$

let \hat{X} be estimator value of X

$$= \sum_{i_1, i_2, \dots, i_{2k} \in [n]} E\left[\prod_{j=1}^{2k} (\hat{X}_{ij} - M_{ij})\right]$$

$\because 0 \leq \delta < 1, n, k > 0$

$$\therefore \leq \sum_{i_1, i_2, \dots, i_{2k} \in [n]} E\left[\prod_{j=1}^{2k} (\hat{X}_{ij} - M_{ij})\right] + \delta n^{2k}.$$

$$\leq \sum_{i_1, i_2, \dots, i_{2k} \in [n]} E\left[\prod_{j=1}^{2k} (\hat{X}_{ij} - M_{ij})\right] + \delta n^{2k} \quad \text{where } M_i = E[X_i]$$

$$= \frac{E[(\hat{X}-M)^{2k}]}{\epsilon^{2k}} + \delta \left(\frac{n}{\epsilon}\right)^{2k}.$$

$$\text{consider } \Pr((\hat{X}-M) \geq \epsilon) \leq \frac{E[(\hat{X}-M)^2]}{\epsilon^2}$$

$$\text{From BR94: } \Pr((\hat{X}-M) \geq A) \leq \frac{E[(\hat{X}-M)^2]}{A^2}$$

$$\leq C \epsilon \cdot \left(\frac{n}{A^2}\right)^{\frac{1}{2}}. \quad \text{where } C \leq 1.0004$$

$$\therefore \Pr((\hat{X}-M) \geq \epsilon) \leq \frac{E[(\hat{X}-M)^2]}{\epsilon^2} \leq C_{2k} \cdot \left(\frac{n \cdot 2k}{\epsilon^2}\right)^{\frac{2k}{2k}} \\ \leq 2 \cdot \left(\frac{2nk}{\epsilon^2}\right)^{k-1}$$

$$\therefore \leq 2 \left(\frac{2nk}{\epsilon^2}\right)^k + \delta \left(\frac{n}{\epsilon}\right)^{2k} \text{ as required.}$$

may not be used by our construction.

Lemma 2.4 [GW97] ← may need or proof cut some from 4.1/4.2 main results

nu. $x_1 \in [0,1]^n, x_2 \in [0,1]^{n_2}, H_\infty(x_1, x_2) \geq h_1 + h_2 - \sigma$

① $H_\infty(x_1) \geq h_1 - \sigma$

② for any $\epsilon > 0$,

$$\Pr_{x_2 \in X_2} [H_\infty(x_2 | x_1 = x_1) < h_2 - \sigma - \log(1/\epsilon)] < \epsilon$$

Corollary 2.5

nu. $x_1 \in [0,1]^n, x_2 \in [0,1]^{n_2}, H_\infty(x_1, x_2) \geq h_1 + h_2 - \sigma$

Given $n = 2^k$, there exists a $(T, n, d, \log(\gamma/\epsilon), \epsilon)$ -block source for any $\epsilon > 0$.

↑ def of (T, k, ϵ) -block source
and Lemma 2.4.

Theorem 3.1

any constant $c > 0$, integer n , $u = \text{poly}(n)$, there exists a family F of $f: [u] \rightarrow [n]$ that:

- ① describe using $O(\log n \log \log n)$ bits
- ② $f(x)$ can be computed in $O(\log n \log \log n)$ time
- ③ $\exists \gamma > 0$ such that for any $S \subseteq [n]$

$$\Pr_{f \in F} \left[\max_{v \in [n]} |f^{-1}(v) \cap S| \leq \frac{\gamma \log n}{\log \log n} \right] > 1 - \frac{1}{n^c}$$

maximum load = $O\left(\frac{\gamma \log n}{\log \log n}\right)$ with high probability.

Construction

$$\text{assume } n = 2^k$$

let $d = O(\log \log n)$, and for every $i \in [d]$, let H_i be a family of k_i -wise f -dependent functions $[u] \rightarrow \{0, 1\}^{L_i}$, where:

$$\begin{aligned} ① \quad n_0 &= n \quad n_i := \frac{n_{i-1}}{2^{L_i}} \quad \text{for every } i \in [d] \\ n_1 &= \frac{n}{2^{L_1}} \quad n_2 = \frac{n_1}{2^{L_2}} = \frac{n}{2^{L_1+L_2}} \dots n_d = \frac{n}{2^{L_1+\dots+L_d}} \end{aligned}$$

$$② \quad L_i = \left\lfloor \frac{\log n_{i-1}}{4} \right\rfloor \quad \text{for every } i \in [d-1], \quad L_d = \log n - \sum_{i=1}^{d-1} L_i$$

$$L_i = \left\lfloor \frac{\log n_{i-1}}{4} \right\rfloor = \left\lfloor \frac{\log n - \sum_{j=1}^{i-1} L_j}{4} \right\rfloor \quad \text{for } i \in [d-1]$$

$$L_d = \log n - \sum_{i=1}^{d-1} L_i$$

$$③ \quad k_i L_i = \Theta(\log n) \quad \text{for every } i \in [d-1], \quad k_d = \Theta(\log n / \log \log n)$$

$$k_i L_i = k_i \left\lfloor \frac{\log n - \sum_{j=1}^{i-1} L_j}{4} \right\rfloor$$

$$\textcircled{4} \quad f = \text{poly}(\frac{1}{n})$$

By corollary 2.1, we have a family H_i : need detail.

h_i is represented using $O(\log n + k_i l_i + \log(1/\delta)) = \log n$ space/time

We define our function family F as

$$f(x) = h_1(x) \circ h_2(x) \circ \dots \circ h_d(x)$$

for $h_i \in H_i$

We visualize the construction as a reversed tree with d layers.

has last layer \hookrightarrow Layer 0 has 1 bin with all balls.
 using h_1 \hookrightarrow Layer 1 has 2^{L_1} intermediate bins.
 has last layer \hookrightarrow Layer 2 has 2^{L_2} intermediate bins.

 using h_d \hookrightarrow Layer d has n final bins

Lemma 3.2

For any $i = \{0, \dots, d-1\}$, $\alpha = \Delta C / \log \log n$, $0 < \alpha_i < 1$ and set $S_i \subseteq [n]$ of size at most $(1+\alpha_i)n_i$,

$$\Pr_{h_{i+1} \in H_{i+1}} \left[\max_{y \in [0, 1]^{L_{i+1}}} |\tilde{h}_{i+1}^{-1}(y) \cap S_i| \leq (1+\alpha)(1+\alpha_i)n_{i+1} \right] \geq 1 - \frac{1}{n^{C+1}}$$

number of balls in any bin of layer $i+1 \leq (1 + \Delta C / \log \log n)(1 + \alpha_i)n_{i+1}$

Application

with high prob.

- ① storing elements using linear probing
- ② augmenting k -wise independence function using an construction without affect the space/time requirement

need some evidence or proof.

Proof For Lemma 3.2

- Fix $y \in \{0,1\}^{L_{i+1}}$, let $\chi = |\{j \mid c_j y_j \in S_i\}|$
- assume wlog, $|S_i| \geq \lfloor C/(d_{i+1} n_i) \rfloor$ or we could add dummy elements to enlarge S_i .
How.
- Then, χ = sum of $|S_i|$ indicator random variables that are k_{i+1} -wise f-dependent.

new $\left\{ \begin{array}{l} X_j = 1 \text{ if element } j \in S_i \text{ is hashed into } h_{i+1} \\ X = \sum_j X_j \quad E[X] = \sum_j E[X_j] = \sum_j \frac{|S_i|}{2^{L_{i+1}}} = \frac{|S_i|}{2^{L_{i+1}}} \end{array} \right.$
*if uniform, total
2^{L_{i+1}} bins*

- Since X = sum of $|S_i|$ k_{i+1} -wise f-dependent r.v., we could apply Lemma 2.2:

$$\begin{aligned} k &= \sum_{j=1}^{|S_i|}, \mu = E[X] = |S_i|/2^{L_{i+1}} \\ \Rightarrow \Pr[X > C/(d_{i+1}) \mu] &\leq 2 \left(\frac{|S_i|/k_{i+1}}{\alpha \mu^2} \right)^{k_{i+1}/2} + \delta \left(\frac{|S_i|}{\alpha \mu} \right)^{k_{i+1}/2} \\ &\stackrel{\text{replace } \mu \text{ by } |S_i|/2^{L_{i+1}}}{=} 2 \left(\frac{|S_i|/k_{i+1}}{|S_i|^2/2^{2L_{i+1}} d^2} \right)^{k_{i+1}/2} + \delta \left(\frac{|S_i|}{|S_i|/2^{L_{i+1}} d} \right)^{k_{i+1}/2} \\ &= 2 \left(\frac{2^{2L_{i+1} k_{i+1}}}{|S_i|^2 \alpha^2} \right)^{k_{i+1}/2} + \delta \left(\frac{2^{L_{i+1}}}{\alpha} \right)^{k_{i+1}/2} \end{aligned}$$

- Now, we will upper bound each section.

$$1) \text{ Notice that, in our construction, } \lceil L_{i+1} \rceil = \left\lceil \frac{\log n_i}{4} \right\rceil \leq \frac{\log(n_i)}{4}$$

$$\text{and } 2) |S_i| \geq C/(d_{i+1} n_i) - 1 \geq n_i \quad 3) \alpha = \sqrt{C/\log(n_i)}$$

$$2 \left(\frac{2^{2L_{i+1} k_{i+1}}}{\alpha^2 |S_i|^2} \right)^{k_{i+1}/2} \leq 2 \left(\frac{2^{\frac{\log(n_i)}{2} k_{i+1}}}{\alpha^2 |S_i|^2} \right)^{k_{i+1}/2} \quad \text{since ①}$$

$$= 2 \left(\frac{n_i k_{i+1}}{\alpha^2 |S_i|^2} \right)^{k_{i+1}/2}$$

$$\begin{aligned} \geq \left(\frac{k_{i+1}}{\alpha^2} \right)^{k_{i+1}/2} &\leq 2 \left(\frac{\sqrt{n_i} k_{i+1}}{\alpha^2 n_i} \right)^{k_{i+1}/2} \quad \text{since ②} \Rightarrow \frac{1}{|S_i|} < \frac{1}{n_i}. \\ &= 2 \left(\frac{k_{i+1}}{\alpha^2 \sqrt{n_i}} \right)^{k_{i+1}/2} \\ &\leq 2 \left(\frac{k_{i+1}}{\alpha^2 2^{L_{i+1}}} \right)^{k_{i+1}/2} \quad \text{since } 2^{L_{i+1}} \leq 2^{\frac{\log(n_i)}{2}} = \sqrt{n_i} \\ &= 2 \left(\frac{k_{i+1}}{\alpha^2 2^{L_{i+1}}} \right)^{k_{i+1}/2} \end{aligned}$$

$$\begin{aligned}
&= 2 \left(\frac{\alpha^{k_{i+1}}}{\alpha^{k_{i+1}} 2^{\log n_i}} \right)^{\log_2} \quad \text{by construction.} \\
&= 2 \left(\frac{\alpha^{k_{i+1}}}{\alpha^{k_{i+1}} 2^{\log n_i}} \right)^{\log_2} \quad \text{set } \log_i = \log n_i \\
&= 2 \left(\frac{\alpha^{k_{i+1}}}{\alpha^{k_{i+1}} 2^{\log n_i}} \right)^{\log_2} \cdot \frac{1}{n^c} \\
&= 2 \left(\frac{\alpha^{k_{i+1}}}{\alpha^{k_{i+1}} 2^{\log n_i}} \right)^{\log_2} \cdot \frac{1}{n^c} \\
&= 2 \left(\frac{1}{n^2} \right)^{\frac{2 \log n_i}{\log n_i}} \cdot \frac{1}{n^c} \\
&\leq 2 \frac{1}{n^2} \cdot \frac{1}{n^c} \quad \text{since } \frac{\log n_i}{\log n_i} > 1 \\
2 \left(\frac{2^{2^{L_{i+1}} k_{i+1}}}{\alpha^{2^{L_{i+1}} k_{i+1}}} \right)^{\log_2} &\leq 2 \frac{1}{n^{c+2}} \quad \Leftarrow \text{result.}
\end{aligned}$$

2) Notice that $f = \text{poly}(\frac{1}{n})$, then

$$\begin{aligned}
f \left(\frac{2^{L_{i+1}} k_{i+1}}{\alpha} \right) &= f \frac{2^{L_{i+1} k_{i+1}}}{\alpha^{k_{i+1}}} \\
&= f \frac{2^{L_{i+1} k_{i+1}}}{\alpha^{k_{i+1}}} \\
&\leq f n^c \\
&\leq \frac{1}{2^{n^{c+2}}} \cdot n^c \quad f = \frac{1}{2^{n^{c+2}}} \\
&\leq \frac{1}{2^{n^{c+2}}}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\Pr[X > c(1+\alpha)(1+\alpha_i)n_{i+1}] &= \Pr[X > c(1+\alpha)(1+\alpha_i)\frac{n_i}{2^{L_{i+1}}}] \quad \text{since } n_{i+1} = \frac{n_i}{2^{L_{i+1}}} \\
&\leq \Pr[X > c(1+\alpha)\frac{|S_i|}{2^{L_{i+1}}}] \quad \text{since } |S_i| \leq (1+\alpha_i)n_i \\
&= \Pr[X > c(1+\alpha)\mu] \quad \text{since } \mu = \frac{|S_i|}{2^{L_{i+1}}} \\
&\leq \frac{1}{2^{n^{c+2}}} + \frac{1}{2^{n^{c+2}}} \quad \text{by upper bound.} \\
&= \frac{1}{n^{c+2}}
\end{aligned}$$

We have at most n different y since $y = \{0, 1\}^{L_{i+1}} \Rightarrow 2^{L_{i+1}} \leq 2^{\log n_{i+1}} \leq 2^{\log n} = n$.

\Rightarrow union bound over y and subtract from 1

$$\frac{1}{n^{c+1}}$$

$$1 - \frac{1}{n^{c+1}}$$

□

Proof For Theorem 3.1

Description length / Evaluation time

i th layer $\Rightarrow h_i$ is k_i -wise δ -dependent function: $[u] \rightarrow \{0,1\}^{L_i}$
 $v = \sum^{L_i}, w = \text{poly}(n), f = \text{poly}(\frac{1}{w})$

By Corollary 2.1, we know there exists a family of k_i -wise δ -dependent functions for:

$$\begin{aligned}\text{space} &= O(\log u + k \log v + \log(1/\delta)) \\ &= O(\log u + k_i \log_2^{L_i} \log(\frac{1}{\text{poly}(\frac{1}{w})})) \\ &= O(\log u + k_i L_i \log(\frac{1}{\delta})) \\ &= O(\log n)\end{aligned}$$

$$\text{time} = O(\log u + k \log v + \log(1/\delta)) = O(\log n) \text{ also.}$$

Maximum Load

- Fix a set $S \subseteq [u]$ of size n . We inductively argue that:

for every level $i \in \{0, \dots, d-1\}$, with probability $\geq 1 - \frac{i}{n^{c+i}}$,

the maximum load in level i is at most $(1+\alpha)^i n$ per bin.

$$\text{for } \alpha = \sqrt{C \log(n)}$$

Base Case $i=0$

$$\text{single bin with } n_0 = n = (1+\alpha)^0 n = (1+\alpha)^0 n_0$$



Inductive Hypothesis

claim holds for level i .

Inductive Case

Lemma 3.2 with $(1+k_i) = (1+\alpha)^i$: number of balls in any bin of layer $i+1$
 $\leq (1+\alpha)^{i+1} n_{i+1}$ with $\text{prob} \geq 1 - \frac{i}{n^{c+i}}$. Union bound over all $n_i \leq n$ bins,
we could know that with $\text{prob} \geq 1 - \frac{i}{n^{c+i}} \geq 1 - \frac{i+1}{n^{c+i}}$, the maximum

Load in level $i+1$ is $(1+d)^{i+1} n_{i+1}$.

Another way:

$$\begin{aligned} & \text{Prob(number of balls in one bin of layer } i+1 > (1+d)^{i+1} n_{i+1}) \leq \frac{1}{n^{c+1}} \\ & \text{Union bound } \Rightarrow \text{Prob(there exists a bin of layer } i+1 > (1+d)^{i+1} n_{i+1}) \leq \frac{n}{n^{c+1}} \\ & \Rightarrow \text{Prob(there doesn't exist a bin of layer } i+1 \leq (1+d)^{i+1} n_{i+1}) \geq 1 - \frac{n}{n^{c+1}} \\ & \geq 1 - \frac{i+1}{n^{c+1}} \\ & \Rightarrow \text{Prob(maximum load of layer } i+1 = (1+d)^{i+1} n_{i+1}) \geq 1 - \frac{i+1}{n^{c+1}} \end{aligned}$$

Thus, claim holds in inductive case.

- Now, we want to upper bound n_{d-1} , the expected load in layer $d-1$

By the induction claim, we know, with probability at least $1 - (cd-1)/n^{c+1}$,

the maximum load in level $d-1$ is $(1+d)^{d-1} n_{d-1} \leq 2n_{d-1}$ for $d = O(\log \log n)$.

For every $i \in [d-1]$,

$$\begin{aligned} l_i &\geq c \log n_{i-1}/4 - 1 \\ \Rightarrow n_i &= n_{i-1}/l_i \leq 2 n_{i-1}^{3/4} \quad \text{since } 2^{-l_i} \leq \frac{1}{\log n_{i-1}/4} = \frac{1}{n_{i-1}^{3/4}} \\ \Rightarrow n_{i-1} &\leq 2 n_{i-2}^{3/4} \Rightarrow n_i \leq 2(2n_{i-2})^{3/4} = 2^{1+3/4} n_{i-2}^{(3/4)^2} \\ \Rightarrow n_{i-1} &\leq 2^{\sum_{j=0}^{i-1} (3/4)^j} n^{(3/4)^i} \leq 2^i n^{(3/4)^i} = 16 n^{(3/4)^i} \end{aligned}$$

Thus, for an appropriate choice of $d = O(\log \log n)$ it holds

$$n_{d-1} \leq \log n$$

For example, $d = \log_{3/4}(\frac{\log \frac{\log n}{16}}{\log n}) \in O(\log \log n)$

$$\begin{aligned} n_{d-1} &\leq 16 n^{(3/4)^d} = 16 n^{(3/4)^{\log_{3/4}(\frac{\log \frac{\log n}{16}}{\log n})}} \\ &= 16 n^{\frac{\log \frac{\log n}{16}}{\log n (\frac{\log \frac{\log n}{16}}{\log n})}} \\ &= 16 n^{\frac{\log \frac{\log n}{16}}{\log n}} \\ &= 16 \frac{\log n}{16} \\ &= \log n. \end{aligned}$$

Also, by definition of n_i , we know $n_i = \frac{n}{2^{sc_i}}$, $C_d = \log n - \sum_i^d l_i = \log n_{d-1}$

Thus, we know that $\text{since } l_d = \log n \Rightarrow 2^{l_d} = 2^{\log n} = n_{d-1}$ bins.

the maximum load on level $d-1$ vs $(1+\alpha)^{d-1} n_{d-1} \leq 2 n_{d-1} \leq 2 \log n$.

These elements are hashed into n_{d-1} bins using the function h_d which is k_d -wise f -dependent, where $k_d = \lceil 2 \log n / \log \log n \rceil$. Therefore, the probability that any $t = 2 \log n / \log \log n < k_d$ elements from level $d-1$ are hashed into only specific bin in level d is at most.

$$\begin{aligned} \binom{2n_{d-1}}{t} \left(\left(\frac{1}{n_{d-1}} \right)^t + \delta \right) &\leq \left(\frac{2n_{d-1}}{t} \right)^t \left(\left(\frac{1}{n_{d-1}} \right)^t + \delta \right) \\ &\stackrel{\substack{\uparrow \\ \text{all comb of } t \\ \text{within } 2n_{d-1}}}{=} \left(\frac{2e}{t} \right)^t + f \left(\frac{2n_{d-1}}{t} \right)^t \\ &= \left(\frac{2e \log n}{\sqrt{\log n}} \right)^{\frac{\log n}{\log \log n}} + \delta \left(\frac{2n_{d-1} \log n}{\sqrt{\log n}} \right)^{\frac{\log n}{\log \log n}} \\ &\stackrel{\text{since } n_{d-1} < \log n}{\leq} \left(\frac{2e \log n}{\sqrt{\log n}} \right)^{\frac{\log n}{\log \log n}} + f \left(\frac{2e \log n}{2} \right)^{\frac{\log n}{\log \log n}} \\ \text{detail(?)} &\rightarrow \leq \frac{1}{2n^{c+3}} + \frac{1}{2n^{c+3}} \\ &= \frac{1}{n^{c+3}} \end{aligned}$$

for $t = \frac{2 \log n}{\log \log n}$ and $\delta = \text{poly}(\frac{1}{n})$

This holds for any pair of bins in level d and $d-1$, which over them implies that:

$$\Pr(\text{a bin with more than } t \text{ elements}) \leq \frac{1}{n^{c+1}} \text{ since } 2^{l_d} < 2^{l_d} = n_{d-1} \leq n$$

This implies that

$$(\text{union bound}) \Rightarrow \Pr(\text{there exists a bin with more than } t \text{ elements}) \leq \frac{n}{n^{c+1}}$$

$$\Pr(\text{there doesn't exist a bin with more than } t \text{ elements}) \geq 1 - \frac{n}{n^{c+1}} \geq 1 - \frac{1}{n^{c+1}}$$

$$\Rightarrow \Pr(\text{maximum load is } t) \geq 1 - \frac{d}{n^{c+1}} > 1 - \frac{1}{n^c}$$

(2)