ST2131
AY21/22 SEM 2
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01. COMBINATORIAL ANALYSIS

The Basic Principle of Counting

- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, there are n_2 possible outcomes of the 2nd exp, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1!\,n_2!\dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \le r \le n$$

N5 - The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomial Coefficients

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\,n_2!\dots n_r!}$$

N6 - represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \dots, n_3$, where $n_1 + n_2 + \dots + n_r = n$

N7 - The Multinomial Theorem: $(x_1 + x_2 + \dots + x_r)^n$

$$= \sum_{(n_1,\dots,n_r):n_1+n_2+\dots+n_r=n} \frac{n!}{n_1!\,n_2!\dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \dots, r$

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- **sample space** $\rightarrow$ The *set* of all outcomes of an experiment
- **event** $\rightarrow$ Any *subset* of the sample space
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E.
- **subset** $\rightarrow E \subset F$ is all of the outcomes in E that are also in F.
  - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws: $\quad (\bigcup_{i=1}^{n} \mathbf{E_i})^\mathbf{c} = \bigcap_{i=1}^{n} \mathbf{E_i^c}$ and $(\bigcap_{i=1}^{n} \mathbf{E_i})^\mathbf{c} = \bigcup_{i=1}^{n} \mathbf{E_i^c}$

Axioms of Probability

definition 1: relative frequency

$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$. problems: (1) $\frac{n(E)}{n}$ may not converge when $n \to \infty$.
(2) $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated.

Axioms (definition 2)

Consider an experiment with sample space S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

1. $0 \le P(E) \le 1$ 2. $P(S) = 1$
3. For mutually exclusive events, $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. same for finite case

mutually exclusive $\rightarrow$ events for which $E_i E_j = \emptyset$ when $i \ne j$

Simple Propositions

N1 - $P(\emptyset) = 0$

N6 - **probability function** $\iff$ it satisfies the 3 axioms.

N8 - if $E \subset F$, then $P(E) \le P(F)$

N10 - Inclusion-Exclusion identity where $n = 3$

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

N11 - **Inclusion-Exclusion identity** -

$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} E_{i_2}) + \dots$
$+ (-1)^{r+1} \sum_{i_1<i_2<\dots<i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) + \dots + (-1)^{n+1} P(E_1 E_2 \dots E_n)$

(i) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i)$ (based on Inclusion-Exclusion identity)

(ii) $P(\bigcup_{i=1}^n E_i) \ge \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j)$

(iii) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j) + \sum_{k<j<i} P(E_i E_j E_k)$

(iv) and so on.

Sample Space having Equally Likely Outcomes

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E, $P(E) = \frac{\text{\# of outcomes in } E}{\text{\# of outcomes in } S} = \frac{\text{\# of outcomes in } E}{n}$

increasing sequence of events $\{E_n, n \ge 1\} \rightarrow E_1 \subset E_2 \subset \dots \subset E_n \subset \dots$

decreasing sequence of events $\{E_n, n \ge 1\} \rightarrow E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$

increasing: $\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$ decreasing: $\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$

N2 - for both *increasing* and *decreasing* sequence, $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

03. CONDITIONAL PROBABILITY AND INDEPENDENCE

Conditional Probability

if $P(F) > 0$, then $P(E|F) = \frac{P(E \cap F)}{P(F)}$

multiplication rule:

$$P(E_1 \dots E_n) = P(E_1) P(E_2|E_1) P(E_3|E_1 E_2) \dots P(E_n|E_1 E_2 \dots E_{n-1})$$

N3 - **axioms of probability** apply to conditional probability

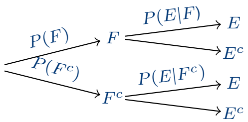
1. $0 \le P(E|F) \le 1$ 2. $P(S|F) = 1$ where S is the sample space
3. If E_i $(i \in \mathbb{Z}_{\ge 1})$ are mutually exclusive, then $P(\bigcup_1^{\infty} E_i|F) = \sum_1^{\infty} P(E_i|F)$

N4 - If we define $Q(E) = P(E|F)$, then all previously proven results apply.

- $P(E_1 \cup E_2|F) = P(E_1|F) + P(E_2|F) - P(E_1 E_2|F)$

Total Probability & Bayes' Theorem

conditioning formula - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$



$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(F) \cdot P(E|F)}{P(E)}$$

$$P(F^c|E) = \frac{P(EF^c)}{P(E)} = \frac{P(F^c) \cdot P(E|F^c)}{P(E)}$$

Total Probability

theorem of total probability - Suppose $F_1, F_2, \dots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$, then $P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(F_i) P(E|F_i)$

Bayes Theorem

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(F_j) P(E|F_j)}{\sum_{i=1}^{n} P(F_i) P(E|F_i)}$$

application of bayes' theorem

$$P(B_1 \mid A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)}$$

Let A be the event that the person test positive for a disease.
B_1: the person has the disease. B_2: the person does not have the disease.

true positives: $P(B_1 \mid A)$	false negatives: $P(\bar{A} \mid B_1)$
false positives: $P(A \mid B_2)$	true negatives: $P(\bar{A} \mid B_2)$

Independent Events

N1 - E and F are independent $\iff P(EF) = P(E) \cdot P(F)$

N2 - E and F are independent $\iff P(E|F) = P(E)$

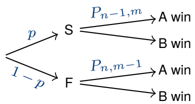
N3 - E and F are independent $\iff E$ and F^c are independent.

N4 - if E, F, G are independent, then E will be independent of any event formed from F and G. (e.g. $F \cup G$)

N6 - (E and F are indep) $\wedge$ (E and G are indep) $\not\Rightarrow$ E and FG are independent

N7 - For independent trials with probability p of success, probability of m successes before n failures, for $m, n \ge 1$,

method 1



method 2

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$
$$= P(\ge n \text{ successes in } m+n-1 \text{ trials})$$

04. RANDOM VARIABLES

- **random variable** $\rightarrow$ a real-valued function defined on the sample space

Types of Random Variables

r.v.	-	$E(X)$
binomial	X = # of successes in n trials w/ replacement	np
negative binomial	X = # of trials until k successes	k/p
geometric	X = # of trials until a success	$1/p$
hypergeometric	X = # of successes in n trials, no replacement	rn/N

- X is a **Bernoulli r.v.** with parameter p if $\rightarrow$
$$p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1-p, & x = 0 \quad \text{('failure')} \end{cases}$$
- Y is a **Binomial r.v.** with parameters n and $p \rightarrow \quad Y = X_1 + X_2 + \dots + X_n$ where $X_1, X_2, \dots, X_n$ are independent Bernoulli r.v.'s with parameter p.
  - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $P(k$ successes from n independent trials each with probability p of success$)$
  $$E(Y) = np, \quad Var(Y) = np(1-p)$$
- **Negative Binomial** $\rightarrow X$ = number of trials until k successes are obtained
  - e.g. number of balls drawn (with replacement) until k red balls are obtained
- **Geometric** $\rightarrow X$ = number of trials until a success is obtained
  - $P(X = k) = (1-p)^{k-1} \cdot p$ where k is the number of trials needed
  - e.g. number of balls drawn (with replacement) until 1 red ball is obtained
- **Hypergeometric** $\rightarrow X$ = number of trials until success, *without replacement*
  - $P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, k = 0, 1, \dots, n$ (for m red balls of N balls)
  - e.g. number of red balls out of n balls drawn without replacement

Properties

N1 - if $X \sim$ Binomial(n, p), and $Y \sim$ Binomial$(n-1, p)$, then

$$E(X^k) = np \cdot E[(Y+1)^{k-1}]$$

N2 - if $X \sim$ Binomial(n, p), then for $k \in \mathbb{Z}^+$,

$$P(X = k) = \frac{(n-k+1)p}{k(1-p)} \cdot P(X = k-1)$$

Coupon Collector Problem

Q. Suppose there are N distinct types of coupons. If T denotes the number of coupons needed to be collected for a complete set, what is $P(T = n)$?

A. $P(T > n-1) = P(T \geq n) = P(T = n) + P(T > n)$

$\Rightarrow P(T = n) = P(T > n-1) - P(T > n)$

Let $A_j = \{$no type j coupon is contained among the first $n\}$

$P(T > n) = P(\bigcup_{j=1}^{N} A_j)$

$P(T > n) = \sum_j P(A_j)$ - coupon j is not among the first n collected

$-\sum\sum_{j_1 < j_2} P(A_{j_1} A_{j_2})$ - coupon j_1 and j_2 are not the first n

$+ \cdots + (-1)^{N+1} P(A_1 A_2 \cdots A_N)$ by *inclusion-exclusion identity*

$P(A_{j_1} A_{j_2} \cdots A_{j_k}) = (\frac{N-k}{N})^n$

Hence $P(T > n) = \sum_{i=1}^{N-1} \binom{N}{i} (\frac{N-i}{N})^n (-1)^{i+1}$

Probability Mass Function

probability mass function, pmf of $X \to$ (*discrete*) $\quad p(a) = P(X = a)$

- if X assumes one of the values $x_1, x_2, \ldots$, then $\sum_{i=1}^{\infty} p(x_i) = 1$
- cdf, $F(a) = \sum p(x)$ for all $x \leq a$
- the pmf $p(a)$ is positive for at most a countable number of values of a

Cumulative Distribution Function

- **cumulative distribution function (cdf)** of a r.v. $X \to$ the function F defined by

  $F(x) = P(X \leq x), \quad -\infty < x < \infty$

  - $F(x)$ is defined on the entire real line. (aka *distribution function*)

pmf, $\begin{array}{c|ccc} a & 1 & 2 & 4 \\ \hline p(a) & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}$ **cdf**, $F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \leq a < 2 \\ \frac{3}{4}, & 2 \leq a < 4 \\ 1, & a \geq 4 \end{cases}$

Expected Value

- aka population mean/sample mean, μ
- if X is a discrete random variable having pmf $p(x)$, the **expectation** or the **expected value** of X is defined as $E(X) = \sum_x x \cdot p(x)$

N1 - if a and b are constants, then $E(aX + b) = aE(X) + b$

N2 - the n^{th} moment of of X is given as $E(X^n) = \sum_x x^n \cdot p(x)$

- I is an indicator variable for event A if $I = \begin{cases} 1, \text{ if } A \text{ occurs} \\ 0, \text{ if } A^c \text{ occurs} \end{cases}$. then $E(I) = P(A)$.

finding expectation of f(x)

- method 1, using pmf of Y: let $Y = f(X)$. Find corresponding X for each Y.
- method 2, using pmf of X: $E[f(x)] = \sum_x f(x)p(x)$

Variance

If X is a r.v. with mean $\mu = E[X]$, then the **variance** of X is defined by

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E(x^2) - [E(x)]^2 \\ &= \sum_{x_i} (x_i - \mu)^2 \cdot p(x_i) \quad \text{(deviation} \cdot \text{weight)} \end{aligned}$$

- $Var(aX + b) = a^2 Var(x)$

Poisson Random Variable

a r.v. X is said to be a **Poisson r.v.** with parameter λ if for some $\lambda > 0$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$
$$E(X) = \lambda, \quad Var(X) = \lambda$$

- $\sum_{i=0}^{\infty} P(X = i) = 1$
- **Poisson Approximation of Binomial** - if $X \sim$ Binomial(n, p), where n is large and p is small, then $X \dot\sim$Poisson(λ) where $\lambda = np$.
  - ✓ *weak* dependence is ok
- **2 ways** to look at the Poisson distribution
  1. an approximation to the binomial distribution with large n and small p
  2. counting the number of events that occur at *random* at certain points in time

Poisson distribution as random events

Let $N(t)$ be the number of events that occur in time interval $[0, t]$.

N1 - If the 3 assumptions are true, then $N(t) \sim$ Poisson(λt).

N2 - If λ is the *rate of occurrences* of events per unit time, then the number of occurrences in an interval of length t has a Poisson distribution with mean λt.

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \text{ for } k \in \mathbb{Z}_{\geq 0}$$

o(h) notation

$o(h)$ stands for any function $f(h)$ such that $\lim_{h \to 0} \frac{f(h)}{h} = 0$

- $o(h) + o(h) = o(h)$
- $\frac{\lambda t}{n} + o(\frac{t}{n}) \doteq \frac{\lambda t}{n}$ for large n

Expected Value of sum of r.v.

For a r.v. X, let $X(s)$ denote the value of X when $s \in \mathcal{S}$

N1 - $E(x) = \sum_i x_i P(X = x_i) = \sum_{s \in \mathcal{S}} X(s)p(s)$ where $\mathcal{S}_i = \{s : X(s) = x_i\}$

N2 - $E(\sum_{i=1}^{n}) = \sum_{i=1}^{n} E(X_i)$ for r.v. $X_1, X_2, \ldots, X_n$

e.g. distribution of time to next event

Q. suppose an accident happens at a rate of 5 per day. Find the distribution of time, starting from now, until the next accident.

A. Let $X =$ time (in days) until the next accident.

Let $V =$ be the number of accidents during time period $[0, t]$.

$V \sim$ Poisson$(5t) \quad \Rightarrow P(V = k) = \frac{e^{-5t} \cdot (5t)^k}{k!}$

$P(X > t) = P($no accidents happen during $[0, t]) = P(V = 0) = e^{-5t}$

$P(X \leq t) - 1 - e^{-5t}$

05. CONTINUOUS RANDOM VARIABLES

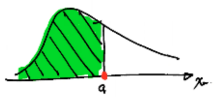
X is a **continuous r.v.** $\to$ if there exists a nonnegative function f defined for all real $x \in (-\infty, \infty)$, such that $P(X \in B) = \int_B f(x)\,dx$

N1 - $P(X \in (-\infty, \infty)) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

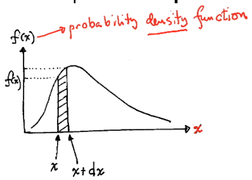
N2 - $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

N3 - $P(X = a) = \int_a^a f(x)\,dx = 0$

N4 - $P(X < a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$



N5 - interpretation of **probability density function**



$$P(x < X < x + dx) = \int_x^{x+dx} f(y)\,dy$$
$$\approx f(x) \cdot dx$$

pdf at x, $f(x) \approx \frac{P(x < X < x+dx)}{dx}$

N6 - if X is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$, then $f(x) = \frac{d}{dx}F(x)$. (*Fundamental Theorem of Calculus*)

N7 - median of X, x occurs where $F(x) = \frac{1}{2}$

Generating a Uniform r.v.

if X is a continuous r.v. with cdf $F(x)$, then

- **N8** - $F(X) = U \sim uniform(0, 1)$.

  Proof. let $Y = F(X)$. then cdf of Y, $F_Y(y) =$
  $P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.

- **N9** - $X = F^{-1}(U) \sim$ cdf $F(x)$.
  - generating a r.v. from a uniform(0, 1) r.v. and a r.v. with cdf $F(x)$.

Expectation & Variance

N1 - **expectation of** X, $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

N2 - if X is a continuous r.v. with pdf $f(x)$, then for any real-valued function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

N2a $E[aX + b] = \int_{-\infty}^{\infty} (aX + b) \cdot f(x)\,dx = a \cdot E(X) + b$

N3 - for a non-negative r.v. Y, $E(Y) = \int_0^{\infty} P(Y > y)\,dy$

Proof. $\int_0^{\infty} P(Y > y)\,dy = \int_0^{\infty} \int_y^{\infty} f_Y(x)\,dx\,dy$ (because $f(x) = \frac{d}{dx}F(x)$)

$= \int_0^{\infty} x f_Y(x)\,dx$

$= E(Y)$

N4 - **variance** of X, $Var(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2$

example

Q - Find the pdf of $(b-a)X + a$ where a, b are constants, $b > a$. The pdf of X is given by $f(x) = \begin{cases} 1, & 0 \leq X \leq 1 \\ 0, & \text{otherwise} \end{cases}$.

A. Let $Y = (b-a)X + a$.

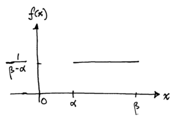
cdf, $F_Y(y) = P(Y \leq y) = P((b-a)X + a \leq y) = P(X \leq \frac{y-a}{b-a})$

$F_Y(y) = \int_0^{\frac{y-a}{b-a}} 1\,dx = \frac{y-a}{b-a}, \quad a < y < b$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{b-a}, & a < y < b \\ 0, & \text{otherwise} \end{cases}$

Uniform Random Variable

X is a **uniform r.v.** on the interval (α, β), $X \sim Uniform(\alpha, \beta)$ if its pdf is given by



$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

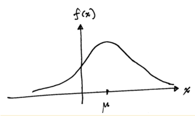
$$E(X) = \frac{\alpha + \beta}{2}, \quad Var(X) = \frac{(\beta - \alpha)^2}{12}$$

if $X \sim Uniform(\alpha, \beta)$, then $\frac{x - \alpha}{\beta - \alpha} \sim Uniform(0, 1)$

Normal Random Variable

X is a **normal r.v.** with parameters μ and σ^2, $X \sim N(\mu, \sigma^2)$ if the pdf of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x}{\mu}\sigma)^2}, \quad -\infty < x < \infty$$
$$E(X) = \mu, \quad Var(X) = \sigma^2$$



if $X \sim N(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim N(0, 1)$

if $Y \sim N(\mu, \sigma^2)$ and a is a constant, $F_y(a) = \Phi(\frac{a - \mu}{\sigma})$

standard normal distribution $\to X \sim N(0, 1)$

- $F(x) = P(X \leq x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy = \Phi(x)$

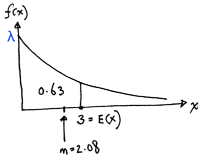
Normal Approximation to the Binomial Distribution

if $S_n \sim Binomial(n, p)$, then $\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0, 1)$ for large n.

$$\mu = np, \quad \sigma^2 = np(1-p)$$

Exponential Random Variable

a *continuous* r.v. X is a **exponential r.v.**, $X \sim Exponential(\lambda)$ or $Exp(\lambda)$
if for some $\lambda > 0$, its pdf is given by



$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

$$P(X < a) = \int_0^a \lambda e^{-\lambda x}\, dx$$

- an exponential r.v. is *memoryless*.
  - a non-negative r.v. is **memoryless** $\rightarrow$ if
    $P(X > s + t \mid X > t) = P(X > s)$ for all $s, t > 0$.

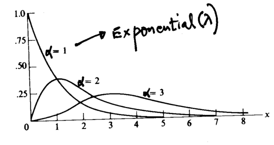
Gamma Distribution

a r.v. X has a **gamma distribution**, $X \sim Gamma(\alpha, \lambda)$
with parameters (α, λ), $\lambda > 0$ and $\alpha > 0$ if its pdf is given by

$$f(x) \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E(X) = \frac{\alpha}{\lambda} \quad Var(X) = \frac{\alpha}{\lambda^2}$$

where the gamma function $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$.



Gamma densities (λ = 1).

N1 - $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

Proof. using integration by parts of LHS to RHS

N2 - if α is an integer n, then $\Gamma(n) = (n-1)!$

N3 - if $X \sim Gamma(\alpha, \lambda)$ and $\alpha = 1$, then $X \sim Exp(\lambda)$.

N4 - for events occurring randomly in time following the 3 assumptions of poisson distribution, the **amount of time elapsed** until a total of n events has occurred is a gamma r.v. with parameters (n, λ).

- time at which the n-th event occurs, $T_n \sim Gamma(n, \lambda)$
- number of events in time period $[0, t]$, $N(t) \sim Poisson(\lambda t)$

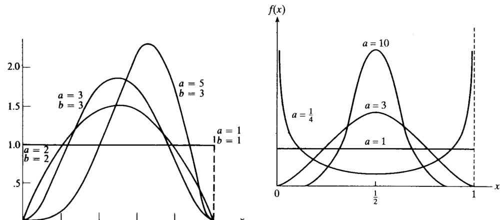
N5 - $Gamma(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}) = \chi^2_n$ (chi-square distribution to n degrees of freedom)

Beta Distribution

a r.v. X is said to have a **beta distribution**, $X \sim Beta(a, b)$

$$f(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a}{a+b} \quad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$$



Beta densities with parameters (a, b) when $a = b$.

N1 - $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$

N2 - $\beta(a = 1, b = 1) = Uniform(0, 1)$

N3 - $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Cauchy Distribution

a r.v. X has a cauchy distribution, $X \sim Cauchy(\theta)$
with parameter θ, $\infty < \theta < \infty$ if its density is given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x-\theta)^2}, \; -\infty < x < \infty$$

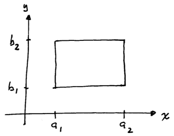
06. JOINTLY DISTRIBUTED RANDOM VARIABLES

Joint Distribution Function

the **joint cumulative distribution function** of the pair of r.v. X and Y is $\rightarrow$
$F(x, y) = P(X \leq x, Y \leq y), -\infty < x < \infty, \; -\infty < y < \infty$

N1 - **marginal cdf of** X, $F_X(x) = \lim_{y \to \infty} F(x, y)$.

N2 - **marginal cdf of** Y, $F_Y(y) = \lim_{x \to \infty} F(x, y)$.



N3 - $P(X > a, Y > b) = 1 - F_X(a) - F_Y(b) + F(a, b)$

N4 - $P(a_1 < X \leq a_2, b_1 < Y \leq b_2)$
$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)$

Joint Probability Mass Function

if X and Y are both discrete r.v., then their **joint pmf** is defined by
$p(i, j) = P(X = i, Y = j)$

N1 - **marginal pmf of** X, $P(X = i) = \sum_j P(X = i, Y = j)$

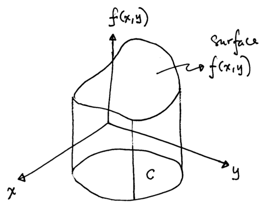
N2 - **marginal pmf of** Y, $P(Y = i) = \sum_i P(X = i, Y = j)$

Joint Probability Density Function

the r.v. X and Y are said to be *jointly continuous* if there is a function $f(x, y)$ called the **joint pdf**, such that for any two-dimensional set C,

$$P[(X, Y) \in C] = \iint_C f(x, y)\, dx\, dy$$

$=$ volume under the surface over the region C.

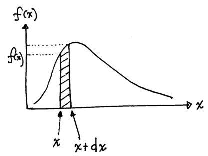


N1 - if $C = \{(x, y) : x \in A, y \in B\}$, then
$P(X \in A, Y \in B) = \int_B \int_A f(x, y)\, dx\, dy$

N2 - $F(a, b)$
$= P\Big(X \in (-\infty, a], Y \in (-\infty, b]\Big)$
$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$

N3 - $f(a, b) = \frac{\delta^2}{\delta a \delta b} F(a, b)$

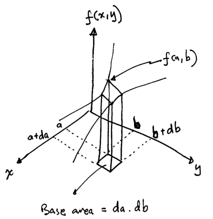
interpretation of pdf



$$P(x < X < x + dx) = \int_x^{x+dx} f(y)\, dy$$
$$\approx f(x)\, dx$$

pdf at x, $f(x) \approx \frac{P(x < X < x + dx)}{dx}$

interpretation of joint pdf



$P(a < X < a + da, b < Y < b + db)$
$= \int_b^{b+db} \int_a^{a+da} f(x, y)\, dx\, dy$
$\approx f(a, b)\, da\, db$ (density of probability)

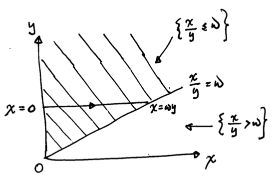
marginal pdf of X, $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

marginal pdf of Y, $f_Y(x) = \int_{-\infty}^{\infty} f(x, y)\, dx$

how to do a double integral

example - given the joint pdf of X and Y, find the pdf of r.v. X/Y.

ans. set dummy variable $W = X/Y$, then
$F_W(w) = P(W \leq w) = P(\frac{X}{Y} \leq w)$
$P(\frac{X}{Y} \leq w) = \int_0^\infty \int_0^{wy} e^{-x-y}\, dx\, dy$



Independent Random Variables

N1 - X, Y are **independent** $\rightarrow P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

N2 - X and Y are **independent** $\rightarrow P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$
or $F(a, b) = F_X(a) \cdot F_Y(b)$ $\Rightarrow$ joint cdf is the product of the marginal cdfs

N3 - *discrete case*: discrete r.v. X and Y are **independent** $\iff$
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all x, y.

N4 - *continuous case*: jointly continuous r.v. X and Y are **independent** $\iff$
$f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y.
$-\infty < x < \infty, \quad -\infty < y < \infty$

N5 - independence is a **symmetric** relation $\rightarrow X$ indep of $Y \iff Y$ indep of X

if X and Y are independent, then for any functions h and g,
$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

Sum of Independent Random Variables

N1 - for independent, continuous r.v. X and Y having pdf f_X and f_Y,
$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\, dy$
$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a - y) f_Y(y)\, dy$

Distribution of Sums of Independent r.v.

for $i = 1, 2, \ldots, n$,

1. $X_i \sim Gamma(t_i, \lambda) \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim Gamma(\sum_{i=1}^{n} t_i, \lambda)$
2. $X_i \sim Exp(\lambda) \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$
3. $Z_i \sim N(0, 1) \quad \Rightarrow \quad \sum_{i=1}^{n} z_i^2 \sim \chi^2_n = Gamma(\frac{n}{2}, \frac{1}{2})$
4. $X_i \sim N(\mu_i, \sigma_i^2) \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$
5. $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2) \Rightarrow X + Y \sim Poisson(\lambda_1 + \lambda_2)$
6. $X \sim Binom(n, p), Y \sim Binom(m, p) \Rightarrow X + Y \sim Binom(n + m, p)$

Conditional Distribution (discrete)

for discrete r.v. X and Y, the **conditional pmf** of X given that $Y = y$ is
$P_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)}$

for discrete r.v. X and Y, the **conditional pdf** of X given that $Y = y$ is
$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{a \leq x} \frac{P(X=a, Y=y)}{P(Y=y)} = \sum_{a \leq x} P_{X|Y}(a|y)$

N0 - equivalent notation:

- $P_{X|Y}(x|y) = P(X = x|Y = y)$
- $P_X(x) = P(X = x)$

N1 - if X is independent of Y, then $P_{X|Y}(x|y) = P_X(x)$

Conditional Distribution (continuous)

for X and Y with joint pdf $f(x, y)$, the **conditional pdf** of X given that $Y = y$ is
$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ for all y s.t. $f_Y(y) > 0$
$f_{X|Y}(a|y) = P(X \leq a|Y = y) = \int_{-\infty}^{a} f_{X|Y}(x|y)\, dx$

N1 - for any set A, $P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)\, dy$

N2 - if X is independent of Y, then $f_{X|Y}(x|y) = f_X(x)$.

! "find the marginal/conditional pdf of Y" $\Rightarrow$ must include the **range** too!!

Joint Probability Distribution of Functions of r.v.

Let X_1 and X_2 be jointly continuous r.v. with joint pdf $f_{x_1,x_2}(x_1,x_2)$. Suppose $Y_1 = g_1(X_1,X_2)$ and $Y_2 = g_2(X_1,X_2)$ satisfy

1. the equations $y_1 = g_1(X_1,X_2)$ and $y_2 = g_2(X_1,X_2)$ can be *uniquely* solved for x_1, x_2 in terms of y_1 and y_2
2. $g_1(x_1,x_2)$ and $g_2(x_1,x_2)$ have continuous partial derivatives at all points (x_1,x_2) such that $J(x_1,x_2) = \begin{vmatrix} \frac{\delta g_1}{\delta x_1} & \frac{\delta g_1}{\delta x_2} \\ \frac{\delta g_2}{\delta x_1} & \frac{\delta g_2}{\delta x_2} \end{vmatrix} = \frac{\delta g_1}{\delta x_1}\cdot\frac{\delta g_2}{\delta x_2} - \frac{\delta g_2}{\delta x_1}\cdot\frac{\delta g_1}{\delta x_2} \neq 0$

then

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\frac{1}{|J(x_1,x_2)|}$$

where $x_1 = h_1(y_1,y_2), x_2 = h_2(y_1,y_2)$

07. PROPERTIES OF EXPECTATION

recap:

- for a **discrete** r.v. X, $E(X) = \sum_x x\cdot p(x) = \sum_x \cdot P(X=x)$
- for a **continuous** r.v. X, $E(X) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx$
- for a **non-negative integer-valued** r.v. Y, $E(Y) = \sum_{i=1}^{\infty} P(Y \geq i)$
- for a **non-negative** r.v. Y, $E(Y) = \int_{-\infty}^{\infty} P(Y>y)\,dy$

Expectations of Sums of Random Variables

for X and Y with joint pmf $p(x,y)$ and joint pdf $f(x,y)$,

$$E[g(x,y)] = \sum_y\sum_x g(x,y)p(x,y)$$

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy$$

N2 - if $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$
N4 - for r.v.s X and Y, if $X \geq Y$, then $E(X) \geq E(Y)$
N5 - let $X_1,\ldots,X_n$ be independent and identically distributed r.v.s having distribution $P(X_i \leq x) = F(x)$ and expected value $E(X_i) = \mu$.

if $\bar{X} = \sum\limits_{i=1}^{n}\frac{X_i}{n}$, then $E(\bar{X}) = \mu$

N6 - $\bar{X}$ is the **sample mean**. $\Rightarrow$ sample mean = population mean
! trick: express a r.v. as a sum of r.v. with easier to find expectation

examples

- hypergeometric with r red balls out of N balls with n trials
  - indicator r.v. $= 1$ if the ith ball selected is red
  - $P(Y_i = 1) = \frac{r}{N} \Rightarrow E(Y_i) = \frac{r}{N} \Rightarrow E(X) = \sum_{i=1}^{n} Y_i = n\frac{r}{N}$
- coupon collector problem:
  - let X = number of coupons collected for a complete set
  - let X_i = additional number to be collected to obtain distinct type after i distinct types have been collected. $X_i \sim Geometric(p = \frac{N-i}{N})$
  - $E(X) = \sum_{i=1}^{N-1} E(X_i) = 1 + \frac{1}{\frac{N-1}{N}} + \frac{1}{\frac{N-2}{N}} + \cdots + \frac{1}{\frac{1}{N}}$
    $= N(\frac{1}{N} + \frac{1}{N-1} + \cdots + 1)$

Covariance, Variance of Sums and Correlations

covariance $\rightarrow$ measure of *linear relationship*

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

N1 - X and Y are independent $\Rightarrow Cov(X,Y) = 0$
N2 - $Cov(X,Y) = 0 \not\Rightarrow X$ and Y are independent. *Proof.* let $E(X) = 0$, $E(XY) = 0 \Rightarrow Cov(X,Y) = 0$, but not independent e.g. non-linear relationship

Covariance properties

1. $Cov(X,Y) = Cov(Y,X)$
2. $Cov(X,X) = Var(X)$
3. $Cov(aX,Y) = aCov(X,Y)$
4. $Cov(\sum\limits_{i=1}^{n} X_i, \sum\limits_{j=1}^{m} Y_j) = \sum\limits_{i=1}^{n}\sum\limits_{j-1}^{m} Cov(X_i,Y_j)$

for variance:

N1 - $Var(\sum\limits_{i=1}^{n} X_i) = \sum\limits_{i=1}^{n} Var(X_i) + 2\sum\sum\limits_{i<j} Cov(X_i,X_j)$

N2 - if $X_1,\ldots,X_n$ are *pairwise independent*, $Var(\sum\limits_{i=1}^{n} X_i) = \sum\limits_{i=1}^{n} Var(X_i)$

N3 - for n independent and identically distributed r.v. with variance σ^2,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$Var(\bar{X}) = \frac{\sigma^2}{n} \qquad E(S^2) = \sigma^2$$

$\Rightarrow S^2$ is an *unbiased estimator* for σ^2.

Correlation

correlation of two r.v. X and Y, $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$

N1 - $-1 \leq \rho(X,Y) \leq 1$ where -1 and 1 denote a perfect negative and positive linear relationship respectively.
N2 - $\rho(X,Y) = 0 \Rightarrow$ no *linear* relationship - uncorrelated
N3 - $\rho(X,Y) = 1 \Rightarrow Y = aX + b, a = \frac{\delta y}{\delta x} > 0$
N4 - for independent events A, B with indicator r.v. I_A, I_B: $Cov(I_A, I_B) = 0$.
N5 - deviation is not correlated with the sample mean. For independent & identically distributed r.v. $X_1, X_2, \ldots, X_n$ with variance σ^2, then $Cov(X_i - \bar{X}, \bar{X}) = 0$.

Conditional Expectation

the **conditional expectation** of X given that $Y = y$, $\forall y$ s.t. $P_Y(y) > 0$, is:

$$E[X|Y=y] = \sum_x x\cdot P(X=x|Y=y) = \sum_x x\cdot p_{X|Y}(x|y)$$

$$E(X|Y=y) = \int_{-\infty}^{\infty} x\cdot f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x\cdot\frac{f(x,y)}{f_Y(y)}\,dx$$

! note the range for $f_{X|Y}(x|y)$

N1 - If $X, Y \sim Geometric(p)$,
then $P(X = i|X+Y=n) = \frac{1}{n-1}$, a uniform distribution.
N2 - $E(X|X+Y=n) = \sum_{i=1}^{n-1} i\cdot P(X=i|X+Y=n) = \frac{n}{2}$

discrete case: $E[g(x)|Y=y] = \sum\limits_x g(x)P_{X|Y}(x|y)$

continuous case: $E[g(x)|Y=y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)$

then $E(X) = E_{\text{w.r.t. } y}(E_{\text{w.r.t. } X|Y=y}(X|Y))$

Deriving Expectation

$E(X) = E_Y(E_X(X|Y))$

discrete case: $E(X) = \sum\limits_y E(X|Y=y)P(Y=y)$

continuous case: $E(X) = \int_{-\infty}^{\infty} E(X|Y=y)f_Y(y)\,dy$

N3 - 3 methods for finding $E(X)$ given $f(x,y)$

1. using $E(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy$ $\Rightarrow$ let $g(x,y) = x$
2. using $E(X) = \int_{-\infty}^{\infty} xf_X(x)\,dx$
3. using $E(X) = \int_{-\infty}^{\infty} E(X|Y=y)f_Y(y)\,dy$

N4 - $E(\sum\limits_{i=1}^{N} X_i) = E_N(E(\sum\limits_{i=1}^{N} X_i|N)) = \sum\limits_{n=0}^{\infty} E(\sum\limits_{i=1}^{N} X_i|N=n)\cdot P(N=n)$

Computing Probabilities by Conditioning

discrete: $P(E) = \sum\limits_y P(E|Y=y)P(Y=y)$

continuous: $P(E) = \int\limits_{-\infty}^{\infty} P(E|Y=y)f_Y(y)\,dy$

Proof. X is an indicator r.v.; $E(X|Y=y) = P(X=1|Y=y) = P(E|Y=y)$

N5 - $P(X<Y) = \int P(X<Y|Y=y)\cdot f_Y(y)$

Conditional Variance

$$\begin{aligned} Var(X|Y) &= E[(X - E(X|Y))^2 \mid Y] \\ &= E(X^2|Y) - [E(X|Y)]^2 \end{aligned}$$

N6 - $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$
N7 - $E(f(Y)) = E(f(Y)|Y=t) = E(f(t)|Y=t)$
$= E(f(t))$ if $N(t)$ and Y are independent

Moment Generating Functions

moment generating function $M(t)$ of the r.v. $X \rightarrow$

$M(t) = E(e^{tX})$ for all real values of t

- if X is *discrete* with pmf $p(x)$, $M(t) = \sum_x e^{tx}\cdot p(x)$
- if X is *continuous* with pdf $f(x)$, $M(t) = \int_{-\infty}^{\infty} e^{tx}f(x)\,dx$

$M(t)$ is called the **mgf** because *all moments of* X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.]

- $M'(0) = E(X)$, $M''(0) = E(X^2)$, $M^n(0) = E(X^n)$, $n \geq 1$
- $M'(t) = E(X^n e^{tX})$, $n \geq 1$

if X and Y are independent and have mgf's $M_X(t)$ and $M_Y(t)$ respectively,
N10 - the mgf of $X+Y$ is $M_{X+Y}(t) = M_X(t)\cdot M_Y(t)$
N11 - if $M_X(t)$ exists and is finite in some region about $t = 0$, then the distribution of X is **uniquely** determined. $M_X(t) = M_Y(t) \iff X = Y$

Common mgf's

- $X \sim Normal(0,1)$, $M(t) = e^{e^2/2}$
- $X \sim Binomial(n,p)$, $M(t) = (pe^t + (1-p))^n$
- $X \sim Poisson(\lambda)$, $M(t)\exp[\lambda(e^t - 1)]$
- $X \sim Exp(\lambda)$, $M(t) = \frac{\lambda}{\lambda - t}$

08. LIMIT THEOREMS

Markov's Inequality $\rightarrow$ if X is a non-negative r.v., $\forall a > 0$, $P(X \geq a) \leq \frac{E(x)}{a}$.
Chebyshev's inequality $\rightarrow$ if X is an r.v. with finite mean μ and variance σ^2, then for any value of $k > 0$, $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$.
N1 - if $Var(X) = 0$, then $P(X = E[X]) = 1$
weak law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s, each with finite mean $E[X_i] = \mu$. Then, for any $\epsilon > 0$, $P\{|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$
central limit theorem $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s each having mean μ and variance σ^2. Then the distribution of $\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \rightarrow \infty$.

- aka: $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \rightarrow z \sim N(0,1)$
- for $-\infty < a < \infty$, as $n \rightarrow \infty$,
  $P(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a) \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx = F(a)$ - cdf of $N(0,1)$

N2 - Let $Z_1, Z_2, \ldots$ be a sequence of r.v.s with distribution functions F_{Z_n} and moment generating functions $M_{Z_n}, n \geq 1$. Let Z be a r.v. with distribution function F_Z and mgf M_Z.
If $M_{Z_n}(t) \rightarrow M_Z(t)$ for all t, then $F_{Z_n}(t) \rightarrow F_Z(t)$ for all t at which $F_Z(t)$ is continuous.
strong law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distribution r.v.s, each having finite mean $\mu = E[X_i]$.
Then, with probability 1, $\frac{X_1 + \cdots + X_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$

$$\boxed{\text{approximations - } \lim_{n\to\infty}(1 - \frac{\lambda}{n})^n = e^{-\lambda}}$$