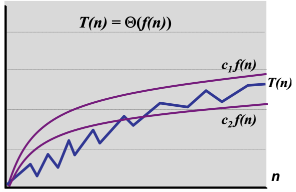
CS2040S
AY20/21 sem 2
github.com/jovyntls

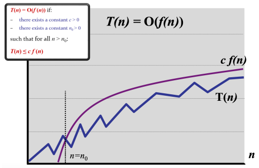
ORDERS OF GROWTH

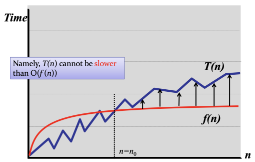
definitions

$$T(n) = \Theta(f(n))$$
$$\iff T(n) = O(f(n)) \text{ and } T(n) = \Omega(f(n))$$



$$T(n) = O(f(n))$$
if $\exists c, n_0 > 0$ such that for all $n > n_0, T(n) \leq cf(n)$
$$T(n) = \Omega(f(n))$$
if $\exists c, n_0 > 0$ such that for all $n > n_0, T(n) \geq cf(n)$





properties

Let $T(n) = O(f(n))$ and $S(n) = O(g(n))$

- addition: $T(n) + S(n) = O(f(n) + g(n))$
- multiplication: $T(n) * S(n) = O(f(n) * g(n))$
- composition: $f_1 \circ f_2 = O(g_1 \circ g_2)$
  - only if both functions are increasing
- if/else statements: $\text{cost} = \max(c1, c2) \leq c1 + c2$
- max: $\max(f(n), g(n)) \leq f(n) + g(n)$

notable

- $\sqrt{n} \log n$ is $O(n)$
- $O(2^{2n}) \neq O(2^n)$
- $O(\log(n!)) = O(n \log n) \rightarrow$ sterling's approximation
- $T(n-1) + T(n-2) + \cdots + T(1) = 2T(n-1)$

master theorem

$$T(n) = aT(\tfrac{n}{b}) + f(n) \quad a \geq 0, b > 1$$
$$= \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) < n^{\log_b a} \text{ polynomially} \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) = n^{\log_b a} \\ \Theta(f(n)) & \text{if } f(n) > n^{\log_b a} \text{ polynomially} \end{cases}$$

space complexity

- $\Theta(f(n))$ time complexity $\Rightarrow O(f(n))$ space complexity
- the maximum space incurred **at any time at any point**
- NOT the maximum space incurred altogether!
- assumption: once we exit the function, we release all memory that was used

SORTING

overview

- **BubbleSort** - compare adjacent items and swap
- **SelectionSort** - takes the smallest element, swaps into place
- **InsertionSort** - from left to right: swap element leftwards until it's smaller than the next element. repeat for next element
  - tends to be faster than the other $O(n^2)$ algorithms
- **MergeSort** - mergeSort 1st half; mergeSort 2nd half; merge
- **QuickSort**
  - partition algorithm: $O(n)$
  - stable quicksort: $O(\log n)$ space
    - first element as partition. 2 pointers from left to right
      - left pointer moves until element > pivot
      - right pointer moves until element < pivot
      - swap elements until left = right.
    - then swap partition and left=right index.

optimisations of QuickSort

- array of duplicates: $O(n^2)$ without 3-way partitioning
- stable if the partitioning algo is stable.
- extra memory allows quickSort to be stable.

choice of pivot

- worst case $O(n^2)$: first/last/middle element
- worst case $O(n \log n)$: median/random element
  - split by fractions: $O(n \log n)$
- choose at random: runtime is a random variable

quickSelect

- $O(n)$ - to find the k^{th} smallest element
- after partitioning, the partition is always in the correct position

TREES

binary search trees (BST)

- a BST is either empty, or a node pointing to 2 BSTs.
- tree balance depends on order of insertion
- balanced tree: $O(h) = O(\log n)$
- for a full-binary tree of size $n, \exists k \in \mathbb{Z}^+$ s.t. $n = 2^k - 1$

BST operations

- `height, h(v) = max(h(v.left), h(v.right))`
  - leaf nodes: `h(v)` = 0
- modifying operations
  - `search`, `insert` - $O(h)$
  - `delete` - $O(h)$
    - case 1: no children - remove the node
    - case 2: 1 child - remove the node, connect parent to child
    - case 3: 2 children - delete the successor; replace node with successor
- query operations
  - `searchMin` - $O(h)$ - recurse into left subtree
  - `searchMax` - $O(h)$ - recurse into right subtree
  - `successor` - $O(h)$
    - if node has a right subtree: `searchMin(v.right)`
    - else: traverse upwards and return the first parent that contains the key in its left subtree

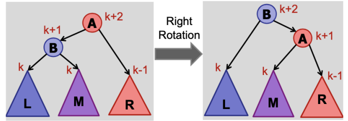
AVL Trees

- **height-balanced** (maintained with rotations)
  - $\iff$ `|v.left.height - v.right.height|` ≤ 1
- each node is augmented with its height - `v.height = h(v)`
- space complexity: $O(LN)$ for N strings of length L

rebalancing

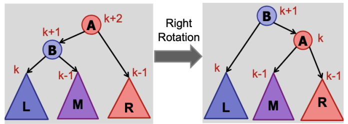
[case 1] B is **balanced: right-rotate**

$$h(L) = h(M), \quad h(R) = h(M) - 1$$



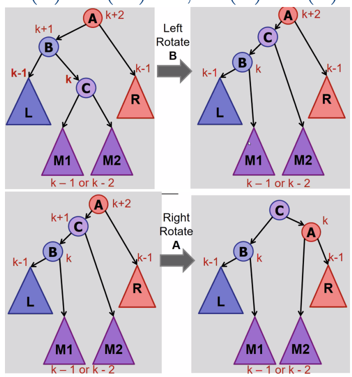
[case 2] B is **left-heavy: right-rotate**

$$h(L) = h(M) + 1, \quad h(R) = h(M)$$



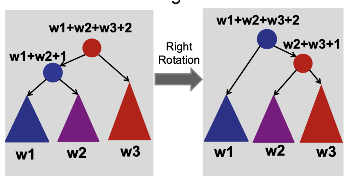
[case 3] B is **right-heavy: left-rotate(v.left), right-rotate(v)**

$$h(L) = h(M) - 1, \quad h(R) = h(L)$$

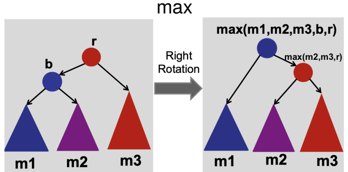


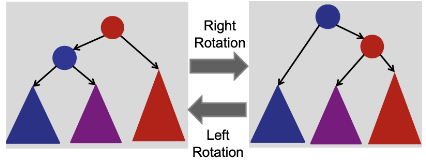
updating nodes after rotation

weights



max





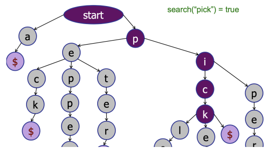
- insertion: max. 2 rotations
- deletion: recurse all the way up
- rotations can create every possible tree shape.

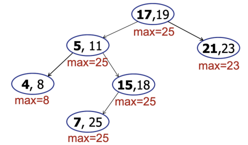
Trie

- `search, insert` - $O(L)$ (for string of length L)
- space: $O(\text{size of text} \cdot \text{overhead})$

interval trees

- `search(key)` $\Rightarrow O(\log n)$
  - if value is in root interval, return
  - if value > max(left subtree), recurse right
  - else recurse left (go left only when can't go right)
- all-overlaps $\Rightarrow O(k \log n)$ for k overlapping intervals

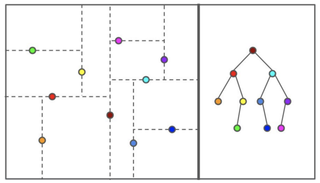




orthogonal range searching

- binary tree; leaves store points, internal nodes store max value in left subtree
- `buildTree(points[])` $\Rightarrow O(n \log n)$ (space is $O(n)$)
- `query(low, hight)` $\Rightarrow O(k + \log n)$ for k points
  - `v=findSplit()` $\Rightarrow O(\log n)$ - find node b/w low & high
  - `leftTraversal(v)` $\Rightarrow O(k)$ - either output all the right subtree and recurse left, or recurse right
  - `rightTraversal(v)` - symmetric
- `insert(key), insert(key)` $\Rightarrow O(\log n)$
- `2D_query()` $\Rightarrow O(\log^2 n + k)$ (space is $O(n \log n)$)
  - build x-tree from x-coordinates; for each node, build a y-tree from y-coordinates of subtree
- `2D_buildTree(points[])` $\Rightarrow O(n \log n)$

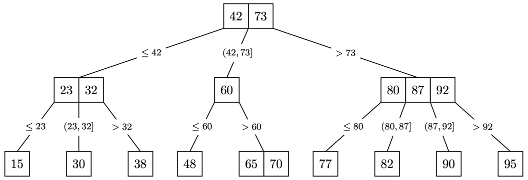
kd-Tree



- stores geometric data (points in an (x, y) plane)
- alternates splitting (partitioning) via x and y coordinates
- `construct(points[])` $\Rightarrow O(n \log n)$
- `search(point)` $\Rightarrow O(h)$
- `searchMin()` $\Rightarrow T(n) = 2T(\frac{n}{4}) + O(1) \Rightarrow O(\sqrt{n})$

(a, b)-trees

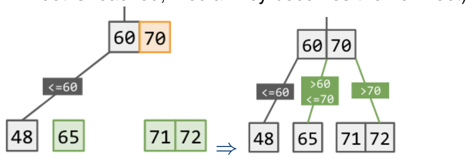
e.g. a (2, 4)-tree storing 18 keys



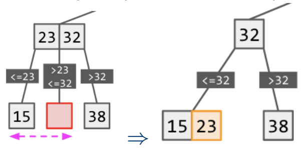
- rules
  1. (a, b)-child policy where $2 \leq a \leq (b+1)/2$

	# keys		# children	
node type	min	max	min	max
root	1	$b-1$	2	b
internal	$a-1$	$b-1$	a	b
leaf	$a-1$	$b-1$	0	0

  2. an internal node has 1 more child than its number of keys
  3. all leaf nodes must be at the **same depth** from the root
- terminology (for a node z)
  - key range - range of keys covered in subtree rooted at z
  - keylist - list of keys within z
  - treelist - list of z's children
- max height $= O(\log_a n) + 1$
- min height $= O(\log_b n)$
- `search(key)` $\Rightarrow O(\log n)$
  - $= O(\log_2 b \cdot \log_a n)$ for binary search at each node
- `insert(key)` $\Rightarrow O(\log n)$
- `split()` a node with too many children
  1. use median to split the keylist into 2 halves
  2. move median key to parent; re-connect remaining nodes
  3. (if the parent is now unbalanced, recurse upwards; if the root is reached, median key becomes the new root)



- `delete(key)` $\Rightarrow O(\log n)$
  - if the node becomes empty, `merge(y, z)` - join it with its left sibling & replace it with their parent



  - if the combined nodes exceed max size: `share(y, z)` = `merge(y, z)` then `split()`

B-Tree

- $(B, 2B)$-trees $\Rightarrow$ (a, b)-tree where $a = B, b = 2B$
- possible augmentation: use a linkedList to connect between each level

Merkle Trees

- binary tree - nodes augmented with a hash of their children
- same root value = identical tree

HASH TABLES

- disadvantage: no successor/predecessor operation

hashing

Let the m be the table size; let n be the number of items; let $cost(h)$ be the cost of the hash function

- $load$(hash table), $\alpha = \frac{n}{m}$
  - = average number of items per bucket
  - = expected number of items per bucket

hashing assumptions

- **simple uniform hashing assumption**
  - every key has an equal probability of being mapped to every bucket
  - keys are mapped independently
- **uniform hashing assumption**
  - every key is equally likely to be mapped to every permutation, independent of every other key.
  - NOT fulfilled by linear probing

properties of a good hash function

1. able to enumerate all possible buckets - $h : U \to \{1..m\}$
   - for every bucket j, $\exists i$ such that $h(key, i) = j$
2. simple uniform hashing assumption

hashCode

rules for the `hashCode()` method

1. always returns the same value, if the object hasn't changed
2. if two objects are equal, they return the same hashCode

rules for the `equals` method

- reflexive - `x.equals(x) => true`
- symmetric - `x.equals(y)` $\Rightarrow$ `y.equals(x)`
- transitive - `x.equals(y), y.equals(z)` $\Rightarrow$ `x.equals(z)`
- consistent - always returns the same answer
- null is null - `x.equals(null) => false`

chaining

- time complexity
  - `insert(key, value)` - $O(1 + cost(h)) \Rightarrow O(1)$
    - for n items: expected maximum cost
      - $= O(\log n)$
      - $= \Theta(\frac{\log n}{\log(\log(n))})$
  - `search(key)`
    - worst case: $O(n + cost(h)) \Rightarrow O(n)$
    - expected case: $O(\frac{n}{m} + cost(h)) \Rightarrow O(1)$
- total space: $O(m + n)$

open addressing - linear probing

- redefined hash function: $h(k, i) = h(k, 1) + i \bmod m$
- `delete(key)`
  - use a **tombstone value** - DON'T set to `null`
- **performance**
  - if the table is $\frac{1}{4}$ full, there will be clusters of size $\Theta(\log n)$
  - expected cost of an operation, $E[\#probes] \leq \frac{1}{1-\alpha}$ (assume $\alpha < 1$ and uniform hashing)
- **advantages**
  - saves space (use empty slots vs linked list)
  - better cache performance (table is one place in memory)
  - rarely allocate memory (no new list-node allocation)
- **disadvantages**
  - more sensitive to choice of hash function (clustering)
  - more sensitive to load (as $\alpha \to 1$, performance degrades)

double hashing

for 2 functions f, g, define

$$h(k, i) = f(k) + i \cdot g(k) \bmod m$$

- if $g(k)$ is relatively prime to m, then $h(k, i)$ hits all buckets
  - e.g. for $g(k) = n^k$, n and m should be coprime.

table size

assume chaining & simple uniform hashing
let m_1 = size of the old hash table; m_2 = size of the new hash table; n = number of elements in the hash table

- growing the table: $O(m_1 + m_2 + n)$
- rate of growth

table growth	resize	insert n items
increment by 1	$O(n)$	$O(n^2)$
double	$O(n)$	$O(n)$, average $O(1)$
square	$O(n^2)$	$O(n)$

PROBABILITY THEORY

- if an event occurs with probability p, the expected number of iterations needed for this event to occur is $\frac{1}{p}$.
- for **random variables**: expectation is always equal to the probability
- **linearity of expectation**: $E[A + B] = E[A] + E[B]$

UNIFORMLY RANDOM PERMUTATION

- for an array of n items, every of the $n!$ possible permutations are producible with probability of exactly $\frac{1}{n!}$
  - the number of outcomes should distribute over each permutation uniformly. (i.e. $\frac{\text{\# of outcomes}}{\text{\# of permutations}} \in \mathbb{N}$)
- probability of an item remaining in its initial position $= \frac{1}{n}$
- **KnuthShuffle** $\Rightarrow O(n)$ - for every element in array A, swap it with a random index in array A.

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sort	best	average	worst	stable?	memory
bubble	$\Omega(n)$	$O(n^2)$	$O(n^2)$	✓	$O(1)$
selection	$\Omega(n^2)$	$O(n^2)$	$O(n^2)$	×	$O(1)$
insertion	$\Omega(n)$	$O(n^2)$	$O(n^2)$	✓	$O(1)$
merge	$\Omega(n \log n)$	$O(n \log n)$	$O(n \log n)$	✓	$O(n)$
quick	$\Omega(n \log n)$	$O(n \log n)$	$O(n^2)$	×	$O(1)$

sorting invariants

sort	**invariant** (after k iterations)
bubble	largest k elements are sorted
selection	smallest k elements are sorted
insertion	first k slots are sorted
merge	given subarray is sorted
quick	partition is in the right position

searching

search	average
linear	$O(n)$
binary	$O(\log n)$
quickSelect	$O(n)$
interval	$O(\log n)$
all-overlaps	$O(k \log n)$
1D range	$O(k + \log n)$
2D range	$O(k + \log^2 n)$

data structures assuming $O(1)$ comparison cost

data structure	search	insert
sorted array	$O(\log n)$	$O(n)$
unsorted array	$O(n)$	$O(1)$
linked list	$O(n)$	$O(1)$
tree (kd/(a, b)/binary)	$O(\log n)$ or $O(h)$	$O(\log n)$ or $O(h)$
trie	$O(L)$	$O(L)$
dictionary	$O(\log n)$	$O(\log n)$
symbol table	$O(1)$	$O(1)$
chaining	$O(n)$	$O(1)$
open addressing	$\frac{1}{1-\alpha} = O(1)$	$O(1)$

orders of growth

$$1 < \log n < \sqrt{n} < n < n \log n < n^2 < n^3 < 2^n < 2^{2n}$$
$$\log_a n < n^a < a^n < n! < n^n$$

orders of growth

$$T(n) = 2T(\frac{n}{2}) + O(n) \quad \Rightarrow O(n \log n)$$
$$T(n) = T(\frac{n}{2}) + O(n) \quad \Rightarrow O(n)$$
$$T(n) = 2T(\frac{n}{2}) + O(1) \quad \Rightarrow O(n)$$
$$T(n) = T(\frac{n}{2}) + O(1) \quad \Rightarrow O(\log n)$$
$$T(n) = 2T(n-1) + O(1) \quad \Rightarrow O(2^n)$$
$$T(n) = 2T(\frac{n}{2}) + O(n \log n) \quad \Rightarrow O(n(\log n)^2)$$
$$T(n) = 2T(\frac{n}{4}) + O(1) \quad \Rightarrow O(\sqrt{n})$$
$$T(n) = T(n-c) + O(n) \quad \Rightarrow O(n^2)$$