CS3230
AY21/22 SEM 2
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01. COMPUTATIONAL MODELS

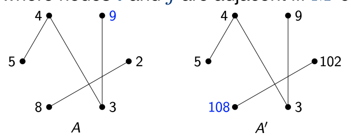
- **algorithm** $\rightarrow$ a well-defined procedure for finding the correct solution to the input
- **correctness**
  - **worst-case correctness** $\rightarrow$ correct on *every valid input*
  - other types of correctness: correct on random input/with high probability/approximately correct
- **efficiency** / **running time** $\rightarrow$ measures the number of steps executed by an algorithm as a function of the *input size* (depends on computational model used)
  - number input: typically the length of binary representation
  - **worst-case** running time $\rightarrow$ *max* number of steps executed when run on an input of size n

Comparison Model

- algorithm can **compare** any two elements in one time unit ($x > y, x < y, x = y$)
- running time = number of comparisons made
- array can be manipulated at no cost

Maximum Problem

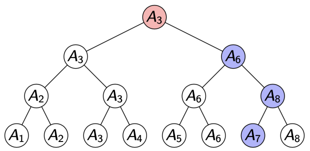
- *problem*: find the largest element in an array A of n distinct elements
- *proof that $n-1$ comparisons are needed*:
  - fix an algorithm M that solves the Maximum problem on all inputs using $< n-1$ comparisons. construct graph G where nodes i and j are adjacent iff M compares i and j.



  - M cannot differentiate A and A'.
- **adversary argument** $\rightarrow$ inputs are decided such that they have different solutions

Second Largest Problem

- *problem*: find the second largest element in $< 2n-3$ comparisons (2x Maximum $\Rightarrow (n-1)+((n-1)-1)=2n-3$)
- *solution*: **knockout tournament** $\Rightarrow n + \lceil \lg n \rceil - 2$



1. bracket system: $n-1$ matches
   - every non-winner has lost exactly once
2. then compare the elements that have lost to the largest
   - the second-largest element must have lost to the winner
   - compares $\lceil \lg n \rceil$ elements that have lost to the winner using $\lceil \lg n \rceil - 1$ comparisons

Sorting

- there is a sorting algorithm that requires $\leq n \lg n - n + 1$ comparisons.
- *proof:* every sorting algorithm must make $\geq \lg(n!)$ comparisons.
  1. let set $\mathcal{U}$ be the set of all permutations of the set $\{1, \dots, n\}$ that the adversary could choose as array A. $|\mathcal{U}| = n!$
  2. for each query "is $A_i > A_j$?", if $\mathcal{U}_{yes} = \{A \in \mathcal{U} : A_i > A_j\}$ is of size $\geq |\mathcal{U}|/2$, set $\mathcal{U} := \mathcal{U}_{yes}$. else: $\mathcal{U} := \mathcal{U} \backslash \mathcal{U}_{yes}$
  3. the size of $\mathcal{U}$ decreases by at most half with each comparison
  4. for $> \lg(n!)$ comparisons, $\mathcal{U}$ will still contain at least 2 permutations

$$n! \geq (\tfrac{n}{e})^n$$
$$\Rightarrow \lg(n!) \geq n \lg(\tfrac{n}{e}) = n \lg n - n \lg e$$
$$\approx n \lg n - 1.44n$$

$\Rightarrow$ roughly $n \lg n$ comparisons are **required** and **sufficient** for sorting n numbers

String Model

- input: string of n bits
- each query: find out **one bit** of the string
- n queries are **necessary** and **sufficient** to check if the input string is all 0s.

Graph Model

- input: (symmetric) adjacency matrix of an n-node undirected graph
- each query: find out if an edge is present between two chosen nodes
- *proof*: $\binom{n}{2}$ queries are necessary to decide whether the graph is connected or not
  1. suppose M is an algorithm making $\leq \binom{n}{2}$ queries.
  2. whenever M makes a query, the algorithm tries not adding this edge, but adding all remaining unqueried edges.
     2.1. if the resulting graph is connected, M replies 0 (i.e. edge does not exist)
     2.2. else: replies 1 (edge exists)
  3. after $< \binom{n}{2}$ queries, at least one entry of the adjacency matrix is unqueried.

02. ASYMPTOTIC ANALYSIS

- **algorithm** $\rightarrow$ a *finite* sequence of well-defined instructions to solve a given computational problem
- **runtime** $\rightarrow$ measured in number of instructions taken in **word-RAM** model
  - operators, comparisons, if, return, etc

Asymptotic Notations

upper bound ($\leq$): $f(n) = O(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq f(n) \leq cg(n)$

lower bound ($\geq$): $f(n) = \Omega(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq cg(n) \leq f(n)$

tight bound: $f(n) = \Theta(g(n))$
if $\exists c_1 > 0, c_2 > 0, n_0 > 0$ such that
$\forall n \geq n_0, \quad 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

o **notation** ($<$): $f(n) = o(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$0 \leq f(n) < cg(n)$

ω**-notation** ($>$): $f(n) = \omega(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$0 \leq cg(n) < f(n)$

Set definitions

- *upper*: $O(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 \mid \forall n \geq n_0, 0 \leq f(n) \leq cg(n)\}$
- *lower*: $\Omega(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 \mid \forall n \geq n_0, 0 \leq cg(n) \leq f(n)\}$

Proof. that $2n^2 = O(n^3)$
let $f(n) = 2n^2$. then $f(n) = 2n^2 \leq n^3$ when $n \geq 2$.
set $c = 1$ and $n_0 = 2$.
we have $f(n) = 2n^2 \leq c \cdot n^3$ for $n \geq n_0$. □

Limits

Assume $f(n), g(n) > 0$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) = o(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) = O(g(n))$$
$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) = \Theta(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) = \Omega(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) = \omega(g(n))$$

Proof. using delta epsilon definition

Properties of Big O

$$\boxed{\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))}$$

- **transitivity** - applies for $O, \Theta, \Omega, o, \omega$
  $f(n) = O(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
- **reflexivity** - for $O, \Omega, \Theta, \quad f(n) = O(f(n))$
- **symmetry** - $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$
- **complementarity** -
  - $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
  - $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

insertion sort: $O(n^2)$ with worst case $\Theta(n^2)$

$$\boxed{\log \log n < \log n < (\log n)^k < n^k < k^n}$$

03. ITERATION, RECURSION, DIVIDE-AND-CONQUER

Iterative Algorithms

loop invariant implies correctness if
- *initialisation* - true before the first iteration
- *maintenance* - if true before an iteration, remains true at the beginning of the next iteration
- *termination* - true at the end

Divide-and-Conquer

powering a number

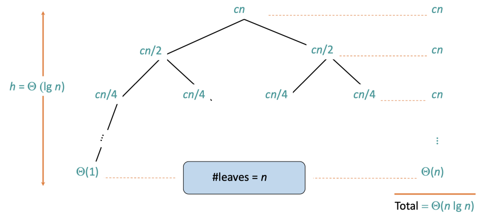
- problem: compute $f(n, m) = a^n \pmod m$ for all integer n, m
- observation: $f(x+y, m) = f(x, m) * f(y, m) \pmod m$
- **naive solution**: recursively compute and combine $f(n-1, m) * f(1, m) \pmod m$
  - $T(n) = T(n-1) + T(1) + \Theta(1) \Rightarrow T(n) = \Theta(n)$
- **better solution**: divide and conquer
  - divide: trivial
  - conquer: recursively compute $f(\lfloor n/2 \rfloor, m)$
  - combine:
    - $f(n, m) = f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if n is even
    - $f(n, m) = f(1, m) * f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if odd
  - $T(n) = T(n/2) + \Theta(1) \Rightarrow \Theta(\log n)$

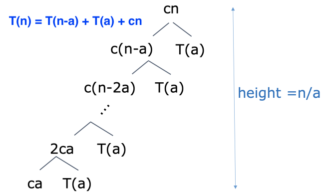
Solving Recurrences

for a sub-problems of size $\frac{n}{b}$ where $f(n)$ is the time to divide and combine, $T(n) = aT(\frac{n}{b}) + f(n)$

Recursion tree

total = height $\times$ number of leaves





Master method

$T(n) = aT(\frac{n}{b}) + f(n)$
$a \geq 0, b > 1, f$ is asymptotically positive

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) < n^{\log_b a} \text{ polynomially} \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) = n^{\log_b a} \\ \Theta(f(n)) & \text{if } f(n) > n^{\log_b a} \text{ polynomially} \end{cases}$$

harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k} \approx \ln k = \Theta(\lg n)$

Substitution method

1. guess that $T(n) = O(f(n))$.
   i.e. $\exists c$ such that $T(n) \leq c \cdot f(n)$, for $n \geq n_0$.
2. verify by induction:
   2.1. set $c = \max\{2, q\}$ and $n_0 = 1$
   2.2. base case ($n = n_0 = 1$)
   2.3. recursive case ($n > 1$):
   - by strong induction, assume $T(k) = c \cdot f(k)$ for $n > k > 1$
   - T(n) = <recurrence> ... $\leq c \cdot f(n)$

   2.4. hence $T(n) \leq c \cdot f(n)$ for $n \geq 1$.

! may not be a tight bound!

example

$T(n) = 4T(n/2) + n^2/\lg n \Rightarrow \Theta(n^2 \lg \lg n)$

Proof. $T(n) = 4T(n/2) + \frac{n^2}{\lg n}$
$= 4(4T(n/4) + \frac{(n/2)^2}{\lg n - lg 2}) + \frac{n^2}{\lg n}$
$= 16T(n/4) + \frac{n^2}{\lg n - \lg 2} + \frac{n^2}{\lg n}$
$= \sum_{k=1}^{\lg n} \frac{n^2}{\lg n - k}$
$= n^2 \lg \lg n$ by approx. of harmonic series ($\sum \frac{1}{k}$)

04. AVERAGE-CASE ANALYSIS & RANDOMISED ALGORITHMS

Quicksort Analysis

- divide & conquer, linear-time $\Theta(n)$ partitioning subroutine
- assume we select the first array element as pivot
- if the pivot produces subarrays of size j and $(n - j - 1)$, then $T(n) = T(j) + T(n - j - 1) + \Theta(n)$

time analysis

- **worst-case**: $T(n) = T(0) + T(n-1) + \Theta(n) \Rightarrow \Theta(n^2)$
- **average case** $\rightarrow$[$A(n)$] expected running time when the input is chosen uniformly at random from the set of all $n!$ permutations
- **average case**, $A(n) = \frac{1}{n!} \sum_\pi Q(\pi)$ where $Q(\pi)$ is the time complexity when the input is permutation π.

Proof. for quicksort, $A(n) = O(n \log n)$

let $P(i)$ be the set of all those permutations of elements $\{e_1, e_2, \ldots, e_n\}$ that begins with e_i.
Let $G(n, i)$ be the average running time of quicksort over $P(i)$. Then
$G(n) = A(i-1) + A(n-i) + (n-1)$.
$A(n) = \frac{1}{n} \sum_{i=1}^n G(n, i)$
$= \frac{1}{n} \sum_{i=1}^n (A(i-1) + A(n-i) + (n-1))$
$= \frac{2}{n} \sum_{i=1}^n A(i-1) + n - 1$
$= O(n \log n)$ by taking it as area under integration

quicksort vs mergesort

	average	best	worst
quicksort	$1.39n \lg n$	$n \lg n$	$n(n-1)$
mergesort	$n \lg n$	$n \lg n$	$n \lg n$

- disadvantages of mergesort:
  - overhead of temporary storage
  - cache misses
- advantages of quicksort
  - in place
  - reliable (as $n \uparrow$, chances of deviation from avg case $\downarrow$)
- issues with quicksort
  - **distribution-sensitive** $\rightarrow$ time taken depends on the initial (input) permutation

Randomised Algorithms

- **randomised algorithms** $\rightarrow$ output and running time are **functions** of the **input** and **random bits chosen**
  - vs non-randomised: output & running time are functions of the *input only*
- **randomised quicksort**: choose pivot at random
  - probability that the runtime of *randomised* quicksort exceeds average by $x\% = n^{-\frac{x}{100} \ln \ln n}$
  - P(time takes at least double of the average) = 10^{-15}
  - distribution insensitive

Randomised Quicksort Analysis

$T(n) = n - 1 + T(q-1) + T(n-q)$
Let $A(n) = \mathbb{E}[T(n)]$ where the expectation is over the randomness in expectation.
Taking expectations and applying linearity of expectation:
$A(n) = n - 1 + \frac{1}{n} \sum_{q=1}^n (A(q-1) + A(n-q))$
$= n - 1 + \frac{2}{n} \sum_{q=1}^{n-1} A(q)$
$A(n) = n \log n \quad \Rightarrow$ same as average case quicksort

Randomised Quickselect

- $O(n)$ to find the k^{th} smallest element
- randomisation: unlikely to keep getting a bad split

Types of Randomised Algorithms

- randomised **Las Vegas** algorithms
  - output is always correct
  - runtime is a *random variable*
  - e.g. randomised quicksort
- randomised **Monte Carlo** algorithms
  - output may be incorrect with some small probability
  - runtime is *deterministic*

examples

- *smallest enclosing circle:* given n points in a plane, compute the smallest radius circle that encloses all n points
  - best **deterministic** algorithm: $O(n)$, but complex
  - las vegas: average $O(n)$, simple solution
- *minimum cut:* given a connected graph G with n vertices and m edges, compute the smallest set of edges whose removal would disconnect G.
  - best **deterministic** algorithm: $O(mn)$
  - **monte carlo**: $O(m \log n)$, error probability n^{-c} for any c
- *primality testing:* determine if an n bit integer is prime
  - best **deterministic** algorithm: $O(n^6)$
  - **monte carlo**: $O(kn^2)$, error probability 2^{-k} for any k

Geometric Distribution

Let X be the number of trials repeated until success.
X is a random variable and follows a geometric distribution with probability p.
Expected number of trials, $E[X] = \frac{1}{p}$