CS3230
AY21/22 SEM 2
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01. COMPUTATIONAL MODELS

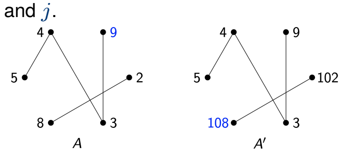
- **algorithm** $\rightarrow$ a well-defined procedure for finding the correct solution to the input
- **correctness**
  - **worst-case correctness** $\rightarrow$ correct on *every valid input*
  - other types of correctness: correct on random input/with high probability/approximately correct
- **efficiency** / **running time** $\rightarrow$ measures the number of steps executed by an algorithm as a function of the *input size* (depends on computational model used)
  - number input: typically the length of binary representation
  - **worst-case** running time $\rightarrow$ *max* number of steps executed when run on an input of size n

Comparison Model

- algorithm can **compare** any two elements in one time unit ($x > y, x < y, x = y$)
- running time = number of comparisons made
- array can be manipulated at no cost

Maximum Problem

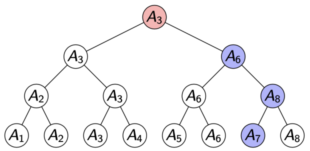
- *problem*: find the largest element in an array A of n distinct elements
- *proof that $n-1$ comparisons are needed*:
  - fix an algorithm M that solves the Maximum problem on all inputs using $< n-1$ comparisons. construct graph G where nodes i and j are adjacent iff M compares i and j.



  - M cannot differentiate A and A'.
- **adversary argument** $\rightarrow$ inputs are decided such that they have different solutions

Second Largest Problem

- *problem*: find the second largest element in $< 2n-3$ comparisons (2x Maximum $\Rightarrow$ $(n-1)+((n-1)-1)=2n-3$)
- *solution*: **knockout tournament** $\Rightarrow n + \lceil \lg n \rceil - 2$



1. bracket system: $n-1$ matches
   - every non-winner has lost exactly once
2. then compare the elements that have lost to the largest
   - the second-largest element must have lost to the winner
   - compares $\lceil \lg n \rceil$ elements that have lost to the winner using $\lceil \lg n \rceil - 1$ comparisons

Sorting

- there is a sorting algorithm that requires $\leq n \lg n - n + 1$ comparisons.
- *proof:* every sorting algorithm must make $\geq \lg(n!)$ comparisons.
  1. let set $\mathcal{U}$ be the set of all permutations of the set $\{1, \dots, n\}$ that the adversary could choose as array A. $|\mathcal{U}| = n!$
  2. for each query "is $A_i > A_j$?", if $\mathcal{U}_{yes} = \{A \in \mathcal{U} : A_i > A_j\}$ is of size $\geq |\mathcal{U}|/2$, set $\mathcal{U} := \mathcal{U}_{yes}$. else: $\mathcal{U} := \mathcal{U} \backslash \mathcal{U}_{yes}$
  3. the size of $\mathcal{U}$ decreases by at most half with each comparison
  4. for $> \lg(n!)$ comparisons, $\mathcal{U}$ will still contain at least 2 permutations

$$\begin{aligned} n! &\geq (\tfrac{n}{e})^n \\ \Rightarrow \lg(n!) &\geq n \lg(\tfrac{n}{e}) = n \lg n - n \lg e \\ &\approx n \lg n - 1.44n \end{aligned}$$

$\Rightarrow$ roughly $n \lg n$ comparisons are **required** and **sufficient** for sorting n numbers

String Model

- input: string of n bits
- each query: find out **one bit** of the string
- n queries are **necessary** and **sufficient** to check if the input string is all 0s.

Graph Model

- input: (symmetric) adjacency matrix of an n-node undirected graph
- each query: find out if an edge is present between two chosen nodes
- *proof*: $\binom{n}{2}$ queries are necessary to decide whether the graph is connected or not
  1. suppose M is an algorithm making $\leq \binom{n}{2}$ queries.
  2. whenever M makes a query, the algorithm tries not adding this edge, but adding all remaining unqueried edges.
     2.1. if the resulting graph is connected, M replies 0 (i.e. edge does not exist)
     2.2. else: replies 1 (edge exists)
  3. after $< \binom{n}{2}$ queries, at least one entry of the adjacency matrix is unqueried.

02. ASYMPTOTIC ANALYSIS

- **algorithm** $\rightarrow$ a *finite* sequence of well-defined instructions to solve a given computational problem
- **runtime** $\rightarrow$ measured in number of instructions taken in **word-RAM** model
  - operators, comparisons, if, return, etc

Asymptotic Notations

upper bound ($\leq$): $f(n) = O(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq f(n) \leq cg(n)$

lower bound ($\geq$): $f(n) = \Omega(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq cg(n) \leq f(n)$

tight bound: $f(n) = \Theta(g(n))$
if $\exists c_1 > 0, c_2 > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

o **notation** ($<$): $f(n) = o(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq f(n) < cg(n)$

ω**-notation** ($>$): $f(n) = \omega(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0, \quad 0 \leq cg(n) < f(n)$

Set definitions

- *upper*: $O(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 \mid \forall n \geq n_0, 0 \leq f(n) \leq cg(n)\}$
- *lower*: $\Omega(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 \mid \forall n \geq n_0, 0 \leq cg(n) \leq f(n)\}$

Proof. that $2n^2 = O(n^3)$
let $f(n) = 2n^2$. then $f(n) = 2n^2 \leq n^3$ when $n \geq 2$.
set $c = 1$ and $n_0 = 2$.
we have $f(n) = 2n^2 \leq c \cdot n^3$ for $n \geq n_0$. $\square$

Limits

Assume $f(n), g(n) > 0$.

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 &\Rightarrow f(n) = o(g(n)) \\ \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty &\Rightarrow f(n) = O(g(n)) \\ 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty &\Rightarrow f(n) = \Theta(g(n)) \\ \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 &\Rightarrow f(n) = \Omega(g(n)) \\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty &\Rightarrow f(n) = \omega(g(n)) \end{aligned}$$

Proof. using delta epsilon definition

Properties of Big O

$$\boxed{\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))}$$

- **transitivity** - applies for $O, \Theta, \Omega, o, \omega$
  $f(n) = O(g(n)) \land g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
- **reflexivity** - for $O, \Omega, \Theta, \quad f(n) = O(f(n))$
- **symmetry** - $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$
- **complementarity** -
  - $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
  - $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

insertion sort: $O(n^2)$ with worst case $\Theta(n^2)$

$$\boxed{\log \log n < \log n < (\log n)^k < n^k < k^n}$$