MA1521

AY20/21 Sem 1

by jovyntls

01. FUNCTIONS & LIMITS

Rules of Limits

1. $\lim_{x\to a}(f\pm g)(x) = L \pm L'$
2. $\lim_{x\to a}(fg)(x) = LL'$
3. $\lim_{x\to a}\frac{f}{g}(x) = \frac{L}{L'}$, provided $L' \neq 0$
4. $\lim_{x\to a} kf(x) = kL$ for any real number k.

02. DIFFERENTIATION

extreme values:

- $f'(x) = 0$
- $f'(x)$ does not exist
- end points of the domain of f

parametric differentiaton: $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

Differentiation Techniques

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x)a^{f(x)}$
$\log_a f(x)$	$\log_a e \cdot \frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\ \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\ \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

L'Hopital's Rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

- for indeterminate forms ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), cannot directly substitute $x = a$.
- for other forms: convert to ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) then apply L'Hopital's rule
- for exponents: use $\ln$, then sub into $e^{f(x)}$

03. INTEGRATION

Integration Techniques

$f(x)$	$\int f(x)$
$\tan x$	$\ln(\sec x),\ \lvert x\rvert < \frac{\pi}{2}$
$\cot x$	$\ln(\sin x),\ 0 < x < \pi$
$\csc x$	$-\ln(\csc x + \cot x),\ 0 < x < \pi$
$\sec x$	$\ln(\sec x + \tan x),\ \lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}(\frac{x}{a})$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(\frac{x}{a}),\ \lvert x\rvert < a$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln(\frac{x-a}{x+a}),\ x > a$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln(\frac{x+a}{x-a}),\ x < a$
a^x	$\frac{a^x}{\ln a}$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

- **indefinite integral** — the integral of the function without any limits
- **antiderivative** — any function whose derivative will be the same as the original function

substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

by parts: $\int uv'\,dx = uv - \int u'v\,dx$

Volume of Revolution

about x-axis:

- (with hollow area) $V = \pi\int_a^b [f(x)]^2 - [g(x)]^2 dx$
- (about line $y = k$) $V = \pi\int_a^b [f(x) - k]^2 dx$

04. SERIES

Geometric Series

sum (**divergent**) $\frac{a(1-r^n)}{1-r}$

sum (**convergent**) $\frac{a}{1-r}$

Power Series

power series about $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \ldots$$

power series about $x = a$ (a is the centre of the power series)

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

Taylor series

$$\sum_{k=0}^{\infty}\frac{f^k(a)}{k!}(x-a)^k$$

MacLaurin series:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$

Taylor polynomial of f at a:

$$P_n(x) = \sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k$$

Radius of Convergence

power series converges where $\lim_{n\to\infty}\lvert\frac{u_{n+1}}{u_n}\rvert < 1$

converge at	R	$\lim_{n\to\infty}\lvert\frac{u_{n+1}}{u_n}\rvert$
$x = a$	0	∞
$(x-h, x+h)$	$h, \frac{1}{N}$	$N\cdot\lvert x-a\rvert$
all x	∞	0

MacLaurin Series

For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2}$$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

Differentiation/Integration

For $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $a - h < x < a + h$,

differentiation of power series:

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

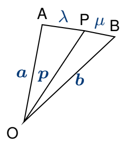
integration of power series:

$$\int f(x)dx = \sum_{0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1} + c$$

if $R = \infty$, $f(x)$ can be integrated to $\int_0^1 f(x)dx$

05. VECTORS

unit vector, $\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$



ratio theorem

$\boldsymbol{p} = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$

midpoint theorem

$\boldsymbol{p} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2}$

Dot product

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$\boldsymbol{a}\perp\boldsymbol{b} \Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\boldsymbol{a}\parallel\boldsymbol{b} \Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

$\boldsymbol{a}\cdot\boldsymbol{b} > 0$: $\boldsymbol{a}$ is acute

$\boldsymbol{a}\cdot\boldsymbol{b} < 0$: $\boldsymbol{a}$ is obtuse

Cross product

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - 1_3b_2\\-(a_1b_3 - a_3b_1)\\a_1b_2 - a_2b_1\end{pmatrix}$$

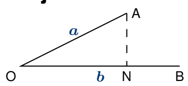
$\boldsymbol{a}\perp\boldsymbol{b} \Rightarrow \boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

$\boldsymbol{a}\parallel\boldsymbol{b} \Rightarrow \boldsymbol{a}\times\boldsymbol{b} = 0$

$\boldsymbol{a}\times\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{a})$

$\lambda\boldsymbol{a}\times\mu\boldsymbol{b} = \lambda\mu(\boldsymbol{a}\times\boldsymbol{b})$

Projection



- $|\overrightarrow{ON}| = |\boldsymbol{a}\cdot\hat{\boldsymbol{b}}| = \frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{|\boldsymbol{b}|}$
- $\overrightarrow{ON} = (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{|\boldsymbol{b}|^2}\boldsymbol{b}$

Planes

Equation of a Plane

$\boldsymbol{n}$ is a perpendicular to the plane; A is a point on the plane.

- parametric: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
- scalar product: $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
- standard form: $\boldsymbol{r}\cdot\hat{\boldsymbol{n}} = d$
- cartesian: $\boldsymbol{a}x + \boldsymbol{b}y + \boldsymbol{c}z = p$

Length of projection of $\boldsymbol{a}$ on $\boldsymbol{n}$ $= |\boldsymbol{a}\cdot\hat{\boldsymbol{n}}| = \perp$ from O to π

Distance from a point to a plane

Shortest distance from a point $S(x_0, y_0, z_0)$ to a plane $\Pi : ax + by + c = d$ is given by:

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

06. PARTIAL DIFFERENTIATION

Partial Derivatives

For $f(x, y)$,

first-order partial derivatives:

$f_x = \frac{d}{dx}f(x, y)$

$f_y = \frac{d}{dy}f(x, y)$

second-order partial derivatives:

$f_{xx} = (f_x)_x = \frac{d}{dx}f_x$

$f_{yy} = (f_y)_y = \frac{d}{dy}f_y$

$f_{xy} = (f_x)_y = \frac{d}{dy}f_x$

$f_{yx} = (f_y)_x = \frac{d}{dx}f_y$

Chain Rule

For $z(t) = f(x(t), y(t))$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

For $z(s, t) = f(x(s, t), y(s, t))$,

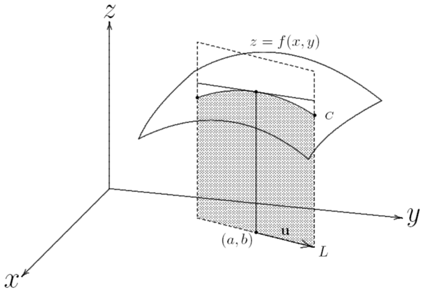
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Directional Derivatives

The directional derivative of f at (a, b) in the direction of unit vector $\hat{\boldsymbol{u}} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is

$$D_u f(a,b) = f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$



- **geometric meaning**: $D_u f(a,b)$ is the gradient of the tangent at (a,b) to curve C on a surface $z = f(x,y)$
  - rate of change of $f(x,y)$ at (a,b) in the direction of $\boldsymbol{u}$

Gradient Vector

The **gradient** at $f(x,y)$ is the vector

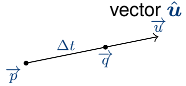
$$\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j}$$

$$D_u f(a,b) = \nabla f(a,b)\cdot\hat{\boldsymbol{u}}$$
$$= |\nabla f(a,b)|\cos\theta$$

- f increases most rapidly in the direction $\nabla f(a,b)$
- f decreases most rapidly in the direction $-\nabla f(a,b)$
- largest possible value of $D_u f(a,b) = |\nabla f(a,b)|$
  - occurs in the same direction as $f_x(a,b)\boldsymbol{i} + f_y(a,b)\boldsymbol{j}$

Physical Meaning

Suppose a point p moves a small distance Δt along a unit vector $\hat{\boldsymbol{u}}$ to a new point $\boldsymbol{q}$.



increment in f,

$$\Delta f \approx D_u f(\boldsymbol{p})(\Delta t)$$

Maximum & Minimum Values

$f(x,y)$ has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) near (a,b).

$f(x,y)$ has a **local minimum** at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) near (a,b).

Critical Points

- $f_x(a,b)$ or $f_y(a,b)$ does not exist; OR
- $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Saddle Points

- $f_x(a,b) = 0, f_y(a,b) = 0$
- neither a local minimum nor a local maximum

Second Derivative Test

Let $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

D	$f_{xx}(a,b)$	**local**
+	+	min
+	-	max
-	any	saddle point
0	any	no conclusion

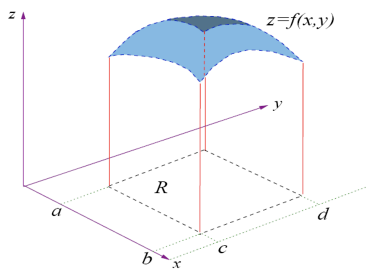
07. DOUBLE INTEGRALS

Let ΔA_i be the area of R_i and (x_i, y_i) be a point on R_i. Let $f(x,y)$ be a function of two variables. The **double integral** of f over R is

$$\iint_R f(x,y)dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\Delta A_i$$

Geometric Meaning

$\iint_R f(x,y)dA$ is the volume under hte surface $z = f(x,y)$ and above the xy-plane over the region R.



Properties of Double Integrals

1. $\iint_R \big(f(x,y) + g(x,y)\big)dA$
   $= \iint_R f(x,y)dA + \iint_R g(x,y)dA$
2. $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$, where c is a constant
3. If $f(x,y) \ge g(x,y)$ for all $(x,y)\in\mathbb{R}$, then
   $\iint_R f(x,y)dA \ge \iint_R g(x,y)dA$
4. If $R = R1 \cup R2$, $R1$ and $R2$ do not overlap, then
   $\iint_R f(x,y)dA = \iint_{R1} f(x,y)dA + \iint_{R2} f(x,y)dA$
5. The area of R,
   $A(R) = \iint_R dA = \iint_R 1dA$
6. If $m \le f(x,y) \le M$ for all $(x,y)\in R$, then
   $mA(R) \le \iint_R f(x,y)dA \le MA(R)$

Rectangular Regions

For a rectangular region R in the xy-plane,

$$a \le x \le b, \quad c \le y \le d$$

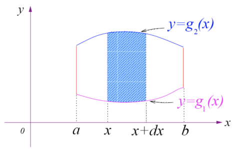
$$\iint_R f(x,y)dA = \int_c^d\left[\int_a^b f(x,y)dx\right]dy$$
$$= \int_a^b\left[\int_c^d f(x,y)dy\right]dx$$

If $f(x,y) = g(x)h(y)$, then

$$\iint_R g(x)h(y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

General Regions

Type A



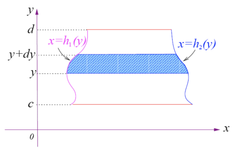
lower/upper bounds:
$g_1(x) \le y \le g_2(x)$

left/right bounds:
$a \le x \le b$

The region R is given by

$$\iint_R f(x,y)dA = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right]dx$$

Type B



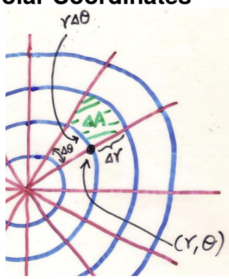
lower/upper bounds:
$c \le y \le d$

left/right bounds:
$h_1(y) \le x \le h_2(y)$

The region R is given by

$$\iint_R f(x,y)dA = \int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right]dy$$

Polar Coordinates



$x = r\cos\theta$
$y = r\sin\theta$
$dxdy \Rightarrow rdrd\theta$

$\Delta A \approx (r\Delta\theta)(\Delta r)$
$= r\Delta r\Delta\theta$

$\boxed{dA = rdrd\theta}$

The region R is given by

$$R: a \le r \le b,\ \alpha \le \theta \le \beta$$

$$\iint_R f(x,y)dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta$$

Applications

Volume

Suppose D is a solid under the surface of $z = f(x,y)$ over a plane region R

$$\text{Volume of } D = \iint_R f(x,y)dA$$

Surface Area

For area S of that portion of the surface $z = f(x,y)$ that projects onto R,

$$S = \iint_R\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1dA}$$

08. ORDINARY DIFFERENTIAL EQUATIONS

- **general solution**: solution containing arbitrary constants
- **particular solution**: gives specific values to arbitrary constants
- the general solution of the n-th order DE will have n arbitrary constants

Separable Equations

A first-order DE is **separable** if it can be written in the form

$M(x) - N(y)y' = 0$ or $M(x)dx = N(y)dy$

Reductions to Separable Form

form	change of variable
$y' = g(\frac{y}{x})$	set $v = \frac{y}{x}$
$y' = f(ax+by+c)$ $\Rightarrow y' = \frac{ax+by+c}{\alpha x+\beta y+\gamma}$	set $v = ax + by$
$y' + P(x)y = Q(x)$	$R = e^{\int Pdx}$ $\Rightarrow y = \frac{1}{R}\int RQdx$
$y' + P(x)y = Q(x)y^n$	set $z = y^{1-n}$ $\Rightarrow y' = \frac{y^n}{1-n}z'$ $R = e^{\int Pdx}$ $\Rightarrow y = \frac{1}{R}\int RQdx$

Logistic Models

$$N = \frac{N_{t=\infty}}{1+(\frac{N_{t=\infty}}{N_{t=0}}-1)e^{-Bt}}$$

- N - number
- B - birth rate
- t - time