

MA1102R

AY20/21 sem 2

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00. FUNCTIONS & SETS

sets

$$A = \{x \mid \text{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

operations on sets

- union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$

notations of sets

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$
- $\mathbb{N} = \mathbb{Z}^+$
- \emptyset : empty set

notations of intervals

- closed interval (inclusive):
 $[a, b] = \{x \mid a \leq x \leq b\}$
- open interval (exclusive):
 $(a, b) = \{x \mid a < x < b\}$
- $(a, \infty) = \{x \mid a < x\}$

functions

- **existence**: $\forall a \in A, f(a) \in B$
- **uniqueness**: $\forall a \in A$ has only one image in B .
- for $f : A \rightarrow B$
 - domain: A
 - codomain: B
 - range: $\{f(x) \mid x \in A\}$
- for this mod:
 - $A, B \subseteq \mathbb{R}$
 - if A is not stated, the domain of f is the largest possible set for which f is defined
 - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set
 $G(f) := \{(x, f(x)) \mid x \in A\}$

- if $A, B \subseteq \mathbb{R}$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane \mathbb{R}^2

algebra of functions

function	domain
$(f + g)(x) := f(x) + g(x)$	$A \cap B$
$(f - g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

types of functions

- **rational function**: $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
 - every polynomial is a rational function ($Q(x) = 1$)
- **algebraic function**: constructed from polynomials using algebraic operations

- a function f is **increasing** on a set I if
 $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if
 $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.
- even/odd:
 - **even function**: $\forall x, f(-x) = f(x)$
 - symmetric about the y -axis
 - **odd function**: $\forall x, f(-x) = -f(x)$
 - symmetric about the origin O
- any function defined on \mathbb{R} can be decomposed *uniquely* into the sum of an even function and an odd function
- **power function**: x^n
 - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

01. LIMITS

definition

if $f(x)$ is arbitrarily close to L by taking x to be sufficiently close (but not equal to) a , then we write

$$\lim_{x \rightarrow a} f(x) = L$$

or $x \rightarrow a \Rightarrow f(x) \rightarrow L$

- the limit $\lim_{x \rightarrow a} f(x)$
 - depends only on the values of $f(x)$ for x near a
 - is independent to the value of $f(x)$ at a .

limit laws

- Let $c \in \mathbb{R}$. $\lim_{x \rightarrow a} c = c$
- $\lim_{x \rightarrow a} x = a$

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant.

- $\lim_{x \rightarrow a} (cf(x)) = cL = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

inequalities on limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

lemma

if $f(x) \leq g(x)$ for all x near a (except possibly at a), then $L \leq M$.

lemma

If $f(x) \geq 0$ for all x , then $L \geq 0$.

direct substitution property

Let f be a polynomial or rational function.

If a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

If $f(x) = g(x)$ for all x near a except possibly at a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

applications

- if a is not in the domain (e.g. 0 denominator), don't apply directly
- convert to an equivalent function and then sub in

one-sided limits

- limit laws also hold for one-sided limits

If as x is close to a from the right, $f(x)$ is close to L , the right-hand limit of f as x approaches a equals L .

$$(x \rightarrow a^+ \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^+} f(x) = L$$

If as x is close to a from the left, $f(x)$ is close to L , the left-hand limit of f as x approaches a equals L .

$$(x \rightarrow a^- \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^-} f(x) = L$$

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

$$f(x) \rightarrow L \Leftrightarrow x \rightarrow a \Leftrightarrow \begin{cases} x \rightarrow a^+ \Rightarrow f(x) \rightarrow L \\ x \rightarrow a^- \Rightarrow f(x) \rightarrow L \end{cases}$$

infinite limits

Suppose f is defined on both sides of a (except possibly at a).

If $f(x)$ is arbitrarily large by taking x sufficiently close to a ,

$$\lim_{x \rightarrow a} f(x) = \infty$$

If $f(x)$ is arbitrarily negatively large \dots ,

$$\lim_{x \rightarrow a} f(x) = -\infty$$

Suppose f is defined on $[M, \infty)$ for some real number M .

If $f(x)$ is arbitrarily close to L by taking x sufficiently large,

$$\lim_{x \rightarrow \infty} f(x) = L$$

squeeze theorem

- Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where
 - $g(x) \leq f(x) \leq h(x)$ for all x near a (except at a),
 - and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$.

$$\text{Then } \lim_{x \rightarrow a} f(x) = L$$

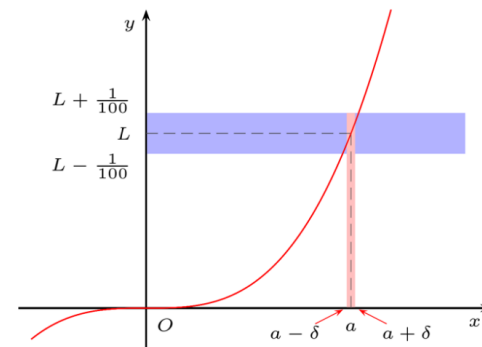
definition of limits

Let f be a function defined on an open interval containing a , except possibly at a .

The limit of $f(x)$ as x approaches a , equals L if. for every

$\epsilon > 0$ there is $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$



informally,

- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a .
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L .

definition of one-sided limits

$$\text{LH Limit: } \lim_{x \rightarrow a^-} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that
 $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$

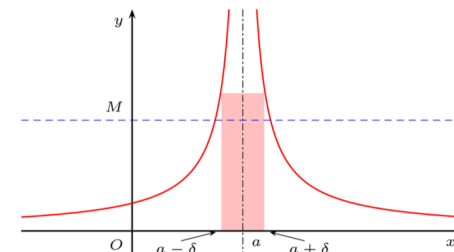
$$\text{RH Limit: } \lim_{x \rightarrow a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that
 $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$

definition of infinite limit

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for every $M > 0$ there exists $\delta > 0$ such that
 $0 < |x - a| < \delta \Rightarrow f(x) > M$



negative infinite limit:

$$0 < |x - a| < \delta \Rightarrow f(x) < M$$

02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at a** \Leftrightarrow

f is continuous from the left and from the right at a

$$\lim_{x \rightarrow a} f(x) = f(a)$$

a function f is *continuous at an interval* if it is continuous at every number in the interval.

f is continuous on **open interval** (a, b)
 $\Leftrightarrow f$ is continuous at every $x \in (a, b)$
 f is continuous on **closed interval** $[a, b]$
 $\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$

continuity test

- f is continuous at $a \Leftrightarrow$
- f is defined at a (a is in the domain of f)
 - $\lim_{x \rightarrow a} f(x)$ exists
 - $\lim_{x \rightarrow a} f(x) = f(a)$

precise definition of continuity

a function f is continuous at a number a if $\forall \epsilon > 0, \exists \delta > 0$
such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

examples of discontinuity

- removable discontinuity
- infinite discontinuity
- jump discontinuity

properties of continuous functions

let f and g be functions continuous at a . let c be a constant.

- cf is continuous at a
- $f + g$ is continuous at a
- $f - g$ is continuous at a
- fg is continuous at a
- $\frac{f}{g}$ is continuous at a , provided $g(a) \neq 0$

other properties

- a polynomial is continuous everywhere;
- a rational function is continuous on its domain
- let c be a real number. $f(x) = c$ is continuous on \mathbb{R} .
- $f(x) = x$ is continuous on \mathbb{R} .

trigonometric functions

- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous everywhere
- $\tan x, \sec x$ are continuous whenever $\cos x \neq 0$
- $\cot x, \csc x$ are continuous whenever $\sin x \neq 0$
 - domain: $\mathbb{R} \setminus \{0, \pm\pi, \pm2\pi, \dots\}$

triangle inequality

$$|a - b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R}$$