MA1521 Cheatsheet

AY20/21 Sem 1 | Chapter 1-3

01. FUNCTIONS & LIMITS

Rules of Limits

1. $\lim\limits_{x\to a}(f\pm g)(x) = L \pm L'$
2. $\lim\limits_{x\to a}(fg)(x) = LL'$
3. $\lim\limits_{x\to a}\frac{f}{g}(x) = \frac{L}{L'}$, provided $L' \neq 0$
4. $\lim\limits_{x\to a} kf(x) = kL$ for any real number k.

Estimation

first order estimate: $y' \approx y + \Delta x \times \frac{dy}{dx}\Big|_{x=2}$

second order estimate:

$y' \approx$ 1st estimate $+ (\frac{(\triangle x)^2}{2} \times \frac{d^2y}{dx^2}\Big|_{x=2})$

Stats

pop. variance: $\sigma^2 = \frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n}$

pop. covariance: cov$(x,y) = \frac{\sum xy^2 - \frac{\sum x \sum y}{n}}{n}$

pop. correlation: $\frac{\text{cov}(x,y)}{\sigma_x \times \sigma_y}$

02. DIFFERENTIATION

extreme values:
- $f'(x) = 0$
- $f'(x)$ does not exist
- end points of the domain of f

parametric differentiaton: $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

Differentiation Techniques

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x)a^{f(x)}$
$\log_a f(x)$	$\log_a e \cdot \frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\ \ \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}},\ \ \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

L'Hopital's Rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

- for indeterminate forms ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), cannot directly substitute $x = a$.
- for other forms: convert to ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) then apply L'Hopital's rule
- for exponents: use $\ln$, then sub into $e^{f(x)}$

03. INTEGRATION

Integration Techniques

$f(x)$	$\int f(x)$
$\tan x$	$\ln(\sec x)$, $\lvert x\rvert < \frac{\pi}{2}$
$\cot x$	$\ln(\sin x)$, $0 < x < \pi$
$\csc x$	$-\ln(\csc x + \cot x)$, $0 < x < \pi$
$\sec x$	$\ln(\sec x + \tan x)$, $\lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}(\frac{x}{a})$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(\frac{x}{a})$, $\lvert x\rvert < a$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln(\frac{x-a}{x+a})$, $x > a$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln(\frac{x+a}{x-a})$, $x < a$
a^x	$\frac{a^x}{\ln a}$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

- **indefinite integral** — the integral of the function without any limits
- **antiderivative** — any function whose derivative will be the same as the original function

substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

by parts: $\int uv'\,dx = uv - \int u'v\,dx$

Volume of Revolution

about x-axis:

- (with hollow area) $V = \pi\int_a^b [f(x)]^2 - [g(x)]^2 dx$
- (about line $y = k$) $V = \pi\int_a^b [f(x) - k]^2 dx$

Logistic Models

$$N = \frac{N_{t=\infty}}{1 + (\frac{N_{t=\infty}}{N_{t=0}} - 1)e^{-Bt}}$$

- N - number
- B - birth rate
- t - time

04. SERIES

Geometric Series

sum (**divergent**)	sum (**convergent**)
$\frac{a(1-r^n)}{1-r}$	$\frac{a}{1-r}$

MacLaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} x^2 n$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n_1)!}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

Power Series

power series about $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

power series about $x = a$ (a is the centre of the power series)

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

Radius of Convergence

power series converges where $\lim\limits_{n\to\infty}\lvert\frac{u_{n+1}}{u_n}\rvert < 1$

converge at	R	$\lim\limits_{n\to\infty}\lvert\frac{u_{n+1}}{u_n}\rvert$
$x = a$	0	∞
$(x-h, x+h)$	h, $\frac{1}{N}$	$N \cdot \lvert x-a\rvert$
all x	∞	0

Differentiation/Integration

For $f(x) = \sum\limits_{n=0}^{\infty} c_n(x-a)^n$ and $a - h < x < a + h$

differentiation of power series:

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

integration of power series:

$$\int f(x)dx = \sum_0^{\infty} c_n\frac{(x-1)^{n+1}}{n+1} + c$$

if $R = \infty$, $f(x)$ can be integrated to $\int_0^1 f(x)dx$

Taylor series

$$\sum_{k=0}^{\infty}\frac{f^k(a)}{k!}(x-a)^k$$

MacLaurin series:

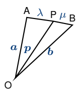
$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$

Taylor polynomial of f at a:

$$P_n(x) = \sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k$$

05. VECTORS

unit vector, $\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{\lvert\boldsymbol{p}\rvert}$



- **ratio theorem**: $\boldsymbol{p} = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$
- **midpoint theorem**: $\boldsymbol{p} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2}$

Dot product

$$\boldsymbol{a}\cdot\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos\theta$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$\boldsymbol{a}\perp\boldsymbol{b} \Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$	$\boldsymbol{a}\cdot\boldsymbol{b} > 0$: $\boldsymbol{a}$ is acute
$\boldsymbol{a}\parallel\boldsymbol{b} \Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert$	$\boldsymbol{a}\cdot\boldsymbol{b} < 0$: $\boldsymbol{a}$ is obtuse

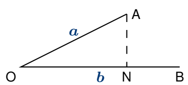
Cross product

$$\boldsymbol{a}\times\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta\hat{\boldsymbol{n}}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - 1_3b_2\\-(a_1b_3 - a_3b_1)\\a_1b_2 - a_2b_1\end{pmatrix}$$

$\boldsymbol{a}\perp\boldsymbol{b} \Rightarrow \boldsymbol{a}\times\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert$	$\boldsymbol{a}\times\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{a})$
$\boldsymbol{a}\parallel\boldsymbol{b} \Rightarrow \boldsymbol{a}\times\boldsymbol{b} = 0$	$\lambda\boldsymbol{a}\times\mu\boldsymbol{b} = \lambda\mu(\boldsymbol{a}\times\boldsymbol{b})$

Projection



- $\lvert\overrightarrow{ON}\rvert = \lvert\boldsymbol{a}\cdot\hat{\boldsymbol{b}}\rvert = \frac{\lvert\boldsymbol{a}\cdot\boldsymbol{b}\rvert}{\lvert\boldsymbol{b}\rvert}$
- $\overrightarrow{ON} = (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \frac{\lvert\boldsymbol{a}\cdot\boldsymbol{b}\rvert}{\lvert\boldsymbol{b}\rvert^2}\boldsymbol{b}$

Planes

Equation of a Plane

$\boldsymbol{n}$ is a perpendicular to the plane; A is a point on the plane.

- parametric: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
- scalar product: $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
- standard form: $\boldsymbol{r}\cdot\hat{\boldsymbol{n}} = d$
- cartesian: $\boldsymbol{a}x + \boldsymbol{b}y + \boldsymbol{c}z = p$

Length of projection of $\boldsymbol{a}$ on $\boldsymbol{n}$ = $\lvert\boldsymbol{a}\cdot\hat{\boldsymbol{n}}\rvert = \perp$ from O to π

Distance from a point to a plane

Shortest distance from a point $S(x_0, y_0, z_0)$ to a plane $\Pi : ax + by + c = d$ is given by:

$$\frac{\lvert ax_0 + by_0 + cz_0 + d\rvert}{\sqrt{a^2+b^2+c^2}}$$