ST2132
AY23/24 SEM 1
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01. PROBABILITY

- **probability** of an event $\rightarrow$ the limiting relative frequency of its occurrence as the experiment is repeated many times
- the **realisation** x is a constant, and X is a generator
  - running r experiments gives us r realisations $x_1, \dots, x_r$

expectation

expectation of X

discrete: mass function	**continuous**: density function
$E(X) := \sum_{i=1}^{n} x_i p_i$	$E(X) := \int_{-\infty}^{\infty} x f(x)\, dx$

expectation of a function $h(X)$

$$E\{h(X)\} = \begin{cases} \sum_{i=1}^{n} h(x_i) p_i & X \text{ is discrete} \\ \int_{-\infty}^{\infty} h(x) f(x)\, dx & X \text{ is continuous} \end{cases}$$

variance

variance, $\mathrm{var}(X) := E\{(X-\mu)^2\}$
standard deviation, $SD(X) := \sqrt{\mathrm{var}(X)}$

law of large numbers

LLN: for a function h, as number of realisations $r \to \infty$,
$\bar{x} \to E(X)$, $v \to \mathrm{var}(X)$
$\frac{1}{r}\sum_{i=1}^{r} h(x_i) \to E\{h(X)\}$

mean of realisations, $\bar{x} := \frac{1}{r}\sum_{i=1}^{r} x_i$

variance of realisations, $v := \frac{1}{r}\sum_{i=1}^{r}(x_i - \bar{x})^2$

Monte Carlo approximation

$$E\{h(X)\} \approx \frac{1}{r}\sum_{i=1}^{r} h(x_i)$$

by LLN, as $r \to \infty$, the approximation becomes exact

joint distribution

- **discrete**: mass function $\Pr(X = x_i, Y = y_j) = p_{ij}$ where $x_1, \dots, x_i$ and $y_1, \dots, y_j$ are all possible values of X and Y
- **continuous**: density function $f : \mathbb{R}^2 \to [0, \infty)$, $\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

for $h : \mathbb{R}^2 \to \mathbb{R}$,

$$E\{h(X,Y)\} = \begin{cases} \sum_{i=1}^{I}\sum_{j=1}^{J} h(x_i, y_j) p_{ij} & X \text{ is discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy & Y \text{ is continuous} \end{cases}$$

algebra of RV's

let X, Y be RVs and a, b, c be constants
- $Z = aX + bY + c$ is also an RV
  - $z = ax + by + c$ is a realisation of Z
- linearity of expectation - $E(Z) = aE(X) + bE(Y) + c$

covariance

let $\mu_X = E(X), \mu_Y = E(Y)$.

covariance, $\mathrm{cov}(X,Y) = E\{(X - \mu_X)(Y - \mu_Y)\}$

- $\mathrm{cov}(X,Y) = E(XY) - \mu_X \mu_Y$
- $\mathrm{cov}(X,Y) = \mathrm{cov}(Y,X)$
- $\mathrm{cov}(X,X) = \mathrm{var}(X)$
- $\mathrm{cov}(W, aX + bY + c) = a\,\mathrm{cov}(W,X) + b\,\mathrm{cov}(W,Y)$
- $\mathrm{var}(aX + bY + c) = a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y) + 2ab\,\mathrm{cov}(X,Y)$

joint, marginal & conditional distributions

let $f(x,y)$ be the **joint** density and $f_X(x), f_Y(y)$ be the **marginal** densities. then

$$f(x,y) = f_X(x) f_Y(y|x) = f_Y(y) f_X(x|y), \quad x, y \in \mathbb{R}$$

$f_Y(\cdot|x)$ is the **conditional** density of Y given $X = x$
$f_X(\cdot|y)$ is the **conditional** density of X given $Y = y$

independence

X, Y are independent $\iff \forall x, y \in \mathbb{R}$,
1. $f(x,y) = f_X(x) f_Y(y)$
2. $f_Y(y|x) = f_Y(y)$
3. $f_X(x|y) = f_Y(x)$

X, Y are independent $\Rightarrow$
- $E(XY) = E(X)E(Y)$
- $\mathrm{cov}(X,Y) = 0$

(the converse does not hold)

Distributions

if X is iid, then $\mathrm{var}(\sum_{i=-1}^{n} x_i) = \sum_{i=1}^{n} \mathrm{var}(x_i)$

bernoulli

- $X \sim Bernoulli(p)$ $\Rightarrow$ coin flip with probability p

binomial

- $X \sim Bin(n,p)$ $\Rightarrow n$ coin flips with probability p
- $X_i \overset{i.i.d.}{\sim} Bernoulli(p)$

$$E(X) = np, \quad \mathrm{var}(X) = np(1-p)$$

multinomial

- $X \sim Multinomial(n, \mathbf{p})$ $\Rightarrow n$ runs of an experiment with k outcomes with probability vector $\mathbf{p}$
  - An experiment with k outcomes $E_1, \dots, E_k$, $Pr(E_i) = p_i$. For some $1 \le i \le k$, let X_i be the number of times E_i occurs in n runs.

$(X_1, \dots, X_k)$ has the multinomial distribution:

$$Pr(X_1 = x_1, \dots, X_k = x_k) = \binom{n}{x_1 \dots x_k} \Pi_{i=1}^{k} p_i^{x_i}$$

- combinatorially, $\binom{n}{x_1 \dots x_k} = \frac{n!}{x_1! x_2! \dots x_k!}$

$$E(X) = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix}, \quad \mathrm{var}(X_i) = np_i(1-p_i)$$

$\mathrm{var}(X) =$ *covariance matrix* M with

$$m_{ij} = \begin{cases} \mathrm{var}(X_i) & \text{if } i = j \\ \mathrm{cov}(X_i, X_j) & \text{if } i \neq j \end{cases}$$

- $\mathrm{cov}(X_i, X_j) < 0$
- $X_i \sim Bin(n, p_i)$
  - $E(X_i) = np_i, \quad \mathrm{var}(X_i) = np_i(1-p_i)$
- $X_i + X_j \sim Bin(n, p_i + p_j)$
  - $\mathrm{var}(X_i + X_j) = n(p_i + p_j)(1 - p_i - p_j)$

Conditional expectation

discrete case

for r.v.s (X, Y), let $f_Y(\cdot|x_i)$ be the conditional mass function of Y given $X = x_i$.

$$E[Y|x_i] := \sum_{j=1}^{J} y_j f_Y(y_j|x_i)$$

$$\mathrm{var}[Y|x_i] := \sum_{j=1}^{J} (y_j - E[Y|x_i])^2 f_Y(y_j|x_i)$$

$E[Y|x_i]$ is like $E(Y)$, with conditional distribution replacing marginal distribution $f_Y(\cdot)$. likewise $\mathrm{var}[Y|x_i]$ is like $\mathrm{var}(Y)$

continuous case

$$E[Y|x] := \int_{-\infty}^{\infty} y f_Y(y|x)\, dy$$

$$\mathrm{var}[Y|x] := \int_{-\infty}^{\infty} (y - E[Y|x])^2 f_Y(y|x)\, dy$$

02. PROBABILITY (2)

mean square error (MSE)

mean square error, $MSE = E\{(Y-c)^2\}$