ST2131
AY21/22 SEM 2
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01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- **combinatorial analysis** $\rightarrow$ the mathematical theory of counting
- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \cdots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1!\,n_2!\ldots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \le r \le n$

Proof. If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.
If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - **The Binomial Theorem** - $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Proof. by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\,n_2!\ldots n_r!}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_3$, where $n_1 + n_2 + \cdots + n_r = n$

Proof. using basic counting principle,

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_{r-1}}{n_r}$$
$$= \frac{n!}{(n-n_1)!\,n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{r-1})}{0!\,n_r!}$$
$$= \frac{n!}{n_1!\,n_2!\ldots n_r!}$$

N7 - **The Multinomial Theorem**: $(x_1 + x_2 + \cdots + x_r)^n$
$$= \sum_{(n_1,\ldots,n_r):n_1+n_2+\cdots+n_r=n} \frac{n!}{n_1!\,n_2!\ldots n_r!} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying $x_1 + x_2 + \cdots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \ldots, r$

! cannot be directly applied to *N8* as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying $x_1 + x_2 + \cdots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \cdots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- **sample space** $\rightarrow$ The *set* of all outcomes of an experiment (where outcomes are not predictable with certainty)
- **event** $\rightarrow$ Any *subset* of the sample space
- **union** of events E and $F \rightarrow E \cup F$ is the event that contains all outcomes that are either in E or F (or both).
- **intersection** of events E and $F \rightarrow E \cap F$ or EF is the event that contains all outcomes that are both in E and in F.
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E.
- **subset** $\rightarrow E \subset F$ is all of the outcomes in E that are also in F.
  - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws

$$\left(\bigcup_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcap_{i=1}^{n} \mathbf{E_i^c}$$

Proof. to show LHS $\subset$ RHS: let $x \in (\bigcup_{i=1}^n E_i)^c$
$\Rightarrow x \notin \bigcup_{i=1}^n E_i \Rightarrow x \notin E_1$ and $x \notin E_2 \ldots$ and $x \notin E_n$
$\Rightarrow x \in E_1^c$ and $x \in E_2^c \ldots$ and $x \in E_n^c$
$\Rightarrow x \in \bigcap_{i=1}^n E_i^c$
to show RHS $\subset$ LHS: let $x \in \bigcap_{i=1}^n E_i^c$

$$\left(\bigcap_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcup_{i=1}^{n} \mathbf{E_i^c}$$

Proof. using the first law of DeMorgan, negate LHS to get RHS

Axioms of Probability

definition 1: relative frequency

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

problems with this definition:

1. $\frac{n(E)}{n}$ may not converge when $n \to \infty$
2. $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated

definition 2: Axioms

Consider an experiment with sample space S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (i.e., events for which $E_i E_j = \emptyset$ when $i \ne j$),
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ is the probability of event E.

Simple Propositions

N1 - $P(\emptyset) = 0$

N2 - $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ (aka axiom 3 for a finite n)

N3 - **strong law of large numbers** - if an experiment is repeated over and over again, then with probability 1, the proportion of time during which any specific event E occurs will be equal to $P(E)$.

N6 - the definitions of probability are mathematical definitions. They tell us which set functions can be called **probability functions**. They do not tell us what value a probability function $P(\cdot)$ assigns to a given event E.

probability function $\iff$ it satisfies the 3 axioms.

N7 - $P(E_c) = 1 - P(E)$

N8 - if $E \subset F$, then $P(E) \le P(F)$

N9 - $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

N10 - Inclusion-Exclusion identity where $n = 3$
$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$

N11 - **Inclusion-Exclusion identity** -
$$\begin{aligned} P(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} E_{i_2}) + \ldots \\ &\quad + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r}) + \ldots \\ &\quad + (-1)^{n+1} P(E_1 E_2 \ldots E_n) \end{aligned}$$

Proof. Suppose an outcome with probability ω is in exactly m of the events E_i, where $m > 0$. Then
LHS: the outcome is in $E_1 \cup E_2 \cup \cdots \cup E_n$ and ω will be counted once in $P(E_1 \cup E_2 \cup \cdots \cup E_n)$
RHS:

- the outcome is in exactly m of the events E_i and ω will be counted exactly $\binom{m}{1}$ times in $\sum_{i=1}^{n} P(E_i)$
- the outcome is contained in $\binom{m}{2}$ subsets of the type $E_{i_1} E_{i_2}$ and ω will be counted $\binom{m}{2}$ times in $\sum_{i_1<i_2} P(E_{i_1} E_{i_2})$
- $\ldots$ and so on

hence RHS = $\binom{m}{1}\omega - \binom{m}{2}\omega + \binom{m}{3}\omega - \cdots \pm \binom{m}{m}\omega$
$= \omega \sum_{i=0}^{m} \binom{m}{i}(-1)^i =$ binomial theorem where $x = -1, y = 1$
$= 0 =$ LHS

e.g. For an outcome with probability ω and $n = 3$

- *Case 1.* $w = P(E_1 E_2)$
  LHS = ω
  RHS = (ω + ω + 0) - (ω + 0 + 0) + 0 = ω
- *Case 2.* $\omega = P(E_1 \cap E_2 \cap E_3)$
  LHS = ω
  RHS = (ω + ω + ω) - (ω + ω + ω) + ω = ω

N12 -

(i) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i)$

(ii) $P(\bigcup_{i=1}^n E_i) \ge \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j)$

(iii) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_i E_j) + \sum_{k<j<i} P(E_i E_j E_k)$

(iv) and so on.

Proof. $\bigcup_{i=1}^{n} E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \cdots \cup E_1^c E_2^c \ldots E_{n-1}^c E_n$
$P(\bigcup_{i=1}^{n} E_i) = P(E_1) + P(E_1^c E_2) + P(E_1^c E_2^c E_3) + \cdots + P(E_1^c E_2^c \ldots E_{n-1}^c E_n)$

Sample Space having Equally Likely Outcomes

tricky - 14, 15, 16, 18, 19, 20

Consider an experiment with sample space $S = \{e_1, e_2, \ldots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \cdots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E, $P(E) = \frac{\#\text{ of outcomes in }E}{\#\text{ of outcomes in }S} = \frac{\#\text{ of outcomes in }E}{n}$

increasing sequence of events $\{E_n, n \ge 1\} \rightarrow$ $E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \ldots$

$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$

decreasing sequence of events $\{E_n, n \geq 1\} \to$
$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \ldots$

$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$

03. CONDITIONAL PROBABILITY AND INDEPENDENCE

tricky - E6, urns (p.37)

Conditional Probability

N1 - if $P(F) > 0$. then $P(E|F) = \frac{P(E \cap F)}{P(F)}$

N2 - **multiplication rule** - $P(E_1 E_2 \ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1E_2\ldots E_{n-1})$

N3 - **axioms of probability** apply to conditional probability

1. $0 \leq P(E|F) \leq 1$
2. $P(S|F) = 1$ where S is the sample space
3. If E_i $(i \in \mathbb{Z}_{\geq 1})$ are mutually exclusive events, then
$P(\bigcup_1^{\infty} E_i|F) = \sum_1^{\infty} P(E_i|F)$

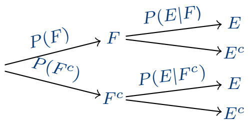
N4 - If we define $Q(E) = P(E|F)$, then $Q(E)$ can be regarded as a probability function on the events of S, hence all results previously proved for probabilities apply.

- $Q(E_1 \cup E_2) = Q(E_1) + Q(E_2) - Q(E_1E_2)$
- $P(E_1 \cup E_2|F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$

Total Probability & Bayes' Theorem

conditioning formula - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$

tree diagram -



$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(F)\cdot P(E|F)}{P(E)}$

$P(F^c|E) = \frac{P(EF^c)}{P(E)} = \frac{P(F^c)\cdot P(E|F^c)}{P(E)}$

Total Probabililty

theorem of total probability - Suppose $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$, then $P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(F_i)P(E|F_i)$

Bayes Theorem

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(F_j)P(E|F_j)}{\sum_{i=1}^{n} P(F_i)P(E|F_i)}$$

application of bayes' theorem

$$P(B_1 \mid A) = \frac{P(A|B_1)\cdot P(B_1)}{P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2)}$$

Let A be the event that the person test positive for a disease.
B_1: the person has the disease. B_2: the person does not have the disease.

true positives: $P(B_1 \mid A)$	false negatives: $P(\bar{A} \mid B_1)$
false positives: $P(A \mid B_2)$	true negatives: $P(\bar{A} \mid B_2)$

Independent Events

N1 - E and F are independent $\iff$ $P(EF) = P(E)\cdot P(F)$

N2 - E and F are independent $\iff$ $P(E|F) = P(E)$

N3 - if E and F are independent, then E and F^c are independent.

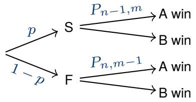
N4 - if E, F, G are independent, then E will be independent of any event formed from F and G. (e.g. $F \cup G$)

N5 - if E, F, G are independent, then $P(EFG) = P(E)P(F)P(G)$

N6 - if E and F are independent and E and G are independent,
$\not\Rightarrow$ E and FG are independent

N7 - For independent trials with probability p of success, probability of m successes before n failures, for $m, n \geq 1$,

method 1



method 2

$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k(1-p)^{m+n-1-k}$
$= P(\text{exactly } k \text{ successes in } m+n-1 \text{ trials})$

recursive approach to solving probabilities: see page 85 alternative approach

04. RANDOM VARIABLES

- **random variable** $\to$ a real-valued function defined on the sample space

Types of Random Variables

- X is a **Bernoulli r.v.** with parameter p if $\to$
$p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1-p, & x = 0 \quad \text{('failure')} \end{cases}$
- Y is a **Binomial r.v.** with parameters n and $p \to$ $Y = X_1 + X_2 + \cdots + X_n$ where $X_1, X_2, \ldots, X_n$ are independent Bernoulli r.v.'s with parameter p.
  - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
  - $P(k$ successes from n independent trials each with probability p of success$)$
  - e.g. number of red balls out of n balls drawn with replacement
  - $E(Y) = np, \quad Var(Y) = np(1-p)$
- **Negative Binomial** $\to X =$ number of trials until k successes are obtained
  - e.g. number of balls drawn (with replacement) until k red balls are obtained
- **Geometric** $\to X =$ number of trials until a success is obtained
  - $P(X = k) = (1-p)^{k-1}\cdot p$ where k is the number of trials needed
  - e.g. number of balls drawn (with replacement) until 1 red ball is obtained
- **Hypergeometric** $\to X =$ number of trials until success, *without replacement*
  - e.g. number of red balls out of n balls drawn without replacement

Summary

binomial	X = # of successes in n trials w/ replacement	np
negative binomial	X = # of trials until k successes	k/p
geometric	X = # of trials until a success	$1/p$
hypergeometric	X = # of successes in n trials, no replacement	rn/N

Properties

N1 - if $X \sim$ Binomial(n, p), and $Y \sim$ Binomial$(n-1, p)$,
then $E(X^k) = np \cdot E[(Y+1)^{k-1}]$

N2 - if $X \sim$ Binomial(n, p), then for $k \in \mathbb{Z}^+$,
$P(X = k) = \frac{(n-k+1)p}{k(1-p)} \cdot P(X = k-1)$

Coupon Collector Problem

Q. Suppose there are N distinct types of coupons. If T denotes the number of coupons needed to be collected for a complete set, what is $P(T = n)$?

A. $P(T > n-1) = P(T \geq n) = P(T = n) + P(T > n)$
$\Rightarrow P(T = n) = P(T > n-1) - P(T > n)$ Let
$A_j = \{\text{no type } j \text{ coupon is contained among the first } n\}$
$P(T > n) = P(\bigcup_{j=1}^{N} A_j)$
Using the inclusion-exclusion identity,
$P(T > n) = \sum_j P(A_j)$ - coupon j is not among the first n collected
$-\sum_{j_1}\sum_{j_2} P(A_{j_1}A_{j_2})$ - coupon j_1 and j_2 are not the first n
$+ \cdots + (-1)^{k+1} \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k} P(A_{j_1}A_{j_2}\cdots A_{j_n}) + \ldots$
$+(-1)^{N+1} P(A_1 A_2 \cdots A_N)$

$P(A_{j_1}A_{j_2}\cdots A_{j_n}) = (\frac{N-k}{N})^n$

Hence $P(T > n) = \sum_{i=1}^{N-1} \binom{N}{i} \left(\frac{N-i}{N}\right)^n (-1)^{i+1}$

Probability Mass Function

- for a *discrete* r.v., we define the **probability mass function** (pmf) of X by $p(a) = P(X = a)$
  - cdf, $F(a) = \sum p(x)$ for all $x \leq a$
  - if X assumes one of the values $x_1, x_2, \ldots$, then $\sum_{i=1}^{\infty} p(x_i) = 1$
  - the pmf $p(a)$ is positive for at most a countable number of values of a
- e.g. $\begin{array}{c|ccc} a & 1 & 2 & 4 \\ \hline p(a) & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}$
- **discrete** variable $\to$ a random variable that can take on at most a countable number of possible values

Cumulative Distribution Function

- for a r.v. X, the function F defined by $F(x) = P(X \leq x), \quad -\infty < x < \infty$, is called the **cumulative distribution function (cdf)** of X.
  - aka *distribution function*
  - $F(x)$ is defined on the entire real line
- e.g. $F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \leq a < 2 \\ \frac{3}{4}, & 2 \leq a < 4 \\ 1, & a \geq 4 \end{cases}$

Expected Value

- aka population mean/sample mean, μ
- if X is a discrete random variable having pmf $p(x)$, the **expectation** or the **expected value** of X is defined as $E(X) = \sum_x x \cdot p(x)$

N1 - if a and b are constants, then $E(aX + b) = aE(X) + b$

N2 - the n^{th} moment of of X is given as $E(X^n) = \sum_x x^n \cdot p(x)$

- I is an indicator variable for event A if $I = \begin{cases} 1, \text{ if } A \text{ occurs} \\ 0, \text{ if } A^c \text{ occurs} \end{cases}$. then $E(I) = P(A)$.

Proof of N1. $E(aX + b) = \sum_x (aX + b)p(x)$
$= a \cdot \sum_x xp(x) + b \cdot \sum_x p(x) = a \cdot E(X) + b$

finding expectation of f(x)

- method 1, using pmf of Y: let $Y = f(X)$. Find corresponding X for each Y.
- method 2, using pmf of X: $E[g(x)] = \sum_i g(x_i)p(x_i)$
  - where X is a discrete r.v. that takes on one of the values of x_i with the respective probabilities of $p(x_i)$, and g is any real-valued function g

Variance

If X is a r.v. with mean $\mu = E[X]$, then the variance of X is defined by
$Var(X) = E[(X - \mu)^2]$
$= \sum x_i(x_i - \mu)^2 \cdot p(x_i)$ (deviation $\cdot$ weight)
$= E(x^2) - [E(x)]^2$

- $Var(aX + b) = a^2 Var(x)$

Poisson Random Variable

a r.v. X is said to be a **Poisson r.v.** with parameter λ if for some $\lambda > 0$,

$$P(X=i)=e^{-\lambda}\cdot\frac{\lambda^i}{i!}$$

- notation: $X\sim$ Poisson(λ)
- $\sum_{i=0}^{\infty} P(X=i)=1$
- **Poisson Approximation of Binomial** - if $X\sim$ Binomial(n,p), n is large and p is small, then $X\dot{\sim}$Poisson(λ) where $\lambda=np$.
  - For n independent trials with probability p of success, the number of successes is approximately a *Poisson r.v.* with parameter $\lambda=np$ if n is large & p is small.
  - Poisson approximation remains even when the trials are not independent, provided that their *dependence is weak*.
- **2 ways** to look at the Poisson distribution
  1. an approximation to the binomial distribution with large n and small p
  2. counting the number of events that occur at *random* at certain points in time

Mean and Variance

if $X\sim$ Poisson(λ), then $\quad E(X)=\lambda,\quad Var(X)=\lambda$

Poisson distribution as random events

Let $N(t)$ be the number of events that occur in time interval $[0,t]$.
N1 - If the 3 assumptions are true, then $N(t)\sim$ Poisson(λt).
N2 - If λ is the rate of occurrences of events per unit time, then the number of occurrences in an interval of length t has a Poisson distribution with mean λt.

$$P(N(t)=k)=\frac{e^{-\lambda t}(\lambda t)^k}{k!},\ \text{for } k\in\mathbb{Z}_{\geq 0}$$

o(h) notation

$o(h)$ stands for any function $f(h)$ such that $\lim_{h\to 0}\frac{f(h)}{h}=0$

- a function of h that is *small* compared to h when h is small
- $o(h)+o(h)=o(h)$
- $\frac{\lambda t}{n}+o(\frac{t}{n})\doteq\frac{\lambda t}{n}$ for large n

Expected Value of sum of r.v.

For a r.v. X, let $X(s)$ denote the value of X when $s\in\mathcal{S}$
N1 - $E(x)=\sum_i x_iP(X=x_i)=\sum_{s\in\mathcal{S}}X(s)p(s)$ where $\mathcal{S}_i=\{s:X(s)=x_i\}$
N2 - $E(\sum_{i-1}^{n})=\sum_{i=1}^{n}E(X_i)\quad$ for r.v. $X_1,X_2,\ldots,X_n$

examples

Selecting hats problem

Let n be the number of men who select their own hats. Let I_E be an indicator r.v. for E. E_i is the event that the i-th man selects his own hat. Let X be the number of men that select their own hats.

- $X=I_{E_1}+I_{E_2}+\cdots+I_{E_n}$
- $P(E_i)=\frac{1}{n}$
- $P(E_i|E_j)=\frac{1}{n-1}\neq P(E_j)$ for $j<i$ (hence E_i and E_j are not independent)
  - but dependence is weak for large n
- X satisfies the other conditions for binomial r.v., besides independence (n trials with equal probability of success)
- Poisson approximation of X : $X\sim$ Poisson(λ)
  - $\lambda=n\cdot P(E_i)=n\cdot\frac{1}{n}=1$
  - $P(X=i)=\frac{e^{-1}1^i}{i!}=\frac{e^{-1}}{i!}$
  - $P(X=0)=e^{-1}\approx 0.37$

No 2 people have the same birthday

For $\binom{n}{2}$ pairs of individuals i and j, $i\neq j$, let E_{ij} be the event where they have the same birthday. Let X be the number of pairs with the same birthday.

- $X=I_{E_1}+I_{E_2}+\cdots+I_{E_n}$
- Each E_{ij} is only *pairwise independent*. $P(E_{ij})=\frac{1}{365}$
  - i.e. E_{ij} and E_{mn} are independent
  - but E_{12} and $(E_{13}\cap E_{23})$ are not independent $\quad\Rightarrow P(E_{12}|E_{13}\cap E_{23})=1$
- $X\dot{\sim}$Poisson(λ), $\lambda=\frac{\binom{n}{2}}{365}=\frac{n(n-1)}{730}\qquad\Rightarrow P(X=0)=e^{-\frac{n(n-1)}{730}}$
  - for $P(X=0)\leq\frac{1}{2}$, $n\geq 23$

distribution of time to next event
Q. suppose an accident happens at a rate of 5 per day. Find the distribution of time, starting from now, until the next accident.

A. Let $X=$ time (in days) until the next accident.
Let $V=$ be the number of accidents during time period $[0,t]$.

$V\sim$ Poisson$(5t)\qquad\Rightarrow P(V=k)=\frac{e^{-5t}\cdot(5t)^k}{k!}$

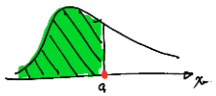
$P(X>t)=P($no accidents happen during $[0,t])=P(V=0)=e^{-5t}$

$P(X\leq t)-1-e^{-5t}$

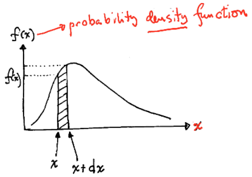
05. CONTINUOUS RANDOM VARIABLES

X is a **continuous r.v.** $\rightarrow$ if there exists a nonnegative function f defined for all real $x\in(-\infty,\infty)$, such that $P(X\in B)=\int_B f(x)\,dx$

N1 - $P(X\in(-\infty,\infty))=\int_{-\infty}^{\infty}f(x)\,dx=1$
N2 - $P(a\leq X\leq b)=\int_a^b f(x)\,dx$
N3 - $P(X=a)=\int_a^a f(x)\,dx=0$
N4 - $P(X<a)=P(X\leq a)=\int_{-\infty}^{a}f(x)\,dx$



N5 - interpretation of **probability density function**



$$P(x<X<x+dx)=\int_x^{x+dx}f(y)\,dy$$
$$\approx f(x)\cdot dx$$

pdf at x, $f(x)\approx\frac{P(x<X<x+dx)}{dx}$

N6 - if X is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$, then $f(x)=\frac{d}{dx}F(x)$.
(*Fundamental Theorem of Calculus*)
N7 - median of X, x occurs where $F(x)=\frac{1}{2}$

Generating a Uniform r.v.

if X is a continuous r.v. with cdf $F(x)$, then

- **N8** - $F(X)=U\sim uniform(0,1)$.

Proof. let $Y=F(X)$. then cdf of Y, $F_Y(y)$ =
$P(Y\leq y)=P(F(X)\leq y)=P(X\leq F^{-1}(y))=F(F^{-1}(y))=y$.
hence Y is a uniform r.v.

- **N9** - $X=F^{-1}(U)\sim$ cdf $F(x)$.
  - generating a r.v. from a uniform(0, 1) r.v. and a r.v. with cdf $F(x)$.

Expectation & Variance

expectation

N1 - **expectation of X**, $E(X)=\int_{-\infty}^{\infty}x\cdot f(x)\,dx$
N2 - if X is a continuous r.v. with pdf $f(x)$, then for any real-valued function g,
$$E[g(x)]=\int_{-\infty}^{\infty}g(x)f(x)\,dx$$
N2a $E[aX+b]=\int_{-\infty}^{\infty}(aX+b)\cdot f(x)\,dx=a\cdot E(X)+b$
N3 - for a non-negative r.v. Y, $E(Y)=\int_0^{\infty}P(Y>y)\,dy$

Proof. $\int_0^{\infty}P(Y>y)\,dy=\int_0^{\infty}\int_y^{\infty}f_Y(x)\,dx\,dy$ (because $f(x)=\frac{d}{dx}F(x)$)
$=\int_0^{\infty}\int_0^{x}f_Y(x)\,dy\,dx$ (draw diagram to convert integration)
$=\int_0^{\infty}f_Y(x)\int_0^{x}dy\,dx$
= $\int_0^{\infty}xf_Y(x)\,dx$ (because $\int_0^{x}dy=x$)
$=E(Y)$

variance

N1 - variance of X, $Var(X)=E[(X-\mu)^2]=E(X^2)-[E(X)]^2$

example

Q - Find the pdf of $(b-a)X+a$ where a,b are constants, $b>a$. The pdf of X is given by $f(x)=\begin{cases}1, & 0\leq X\leq 1\\ 0, & \text{otherwise}\end{cases}$.

A. Let $Y=(b-a)X+a$.
cdf, $F_Y(y)=P(Y\leq y)=P((b-a)X+a\leq y)=P(X\leq\frac{y-a}{b-a})$

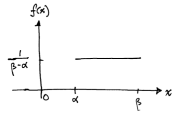
$F_Y(y)=\int_0^{\frac{y-a}{b-a}}1\,dx=\frac{y-a}{b-a},\quad a<y<b$

$f_Y(y)=\frac{d}{dy}F_Y(y)=\begin{cases}\frac{1}{b-a}, & a<y<b\\ 0, & \text{otherwise}\end{cases}$

Uniform Random Variable

X is a **uniform r.v.** on the interval (α,β), $X\sim Uniform(\alpha,\beta)$
if its pdf is given by

$$f(x)=\begin{cases}\frac{1}{\beta-\alpha}, & \alpha<x<\beta\\ 0, & \text{otherwise}\end{cases}$$
$$E(X)=\frac{\alpha+\beta}{2},\quad Var(X)=\frac{(\beta-\alpha)^2}{12}$$

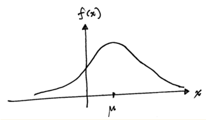


if $X\sim Uniform(\alpha,\beta)$, then $\frac{x-\alpha}{\beta-\alpha}\sim Uniform(0,1)$

Normal Random Variable

X is a **normal r.v.** with parameters μ and σ^2, $X\sim N(\mu,\sigma^2)$
if the pdf of X is given by

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x}{\mu}\sigma)^2},\quad -\infty<x<\infty$$
$$E(x)=\mu,\quad Var(X)=\sigma^2$$



if $X\sim N(\mu,\sigma^2)$, then $\frac{X-\mu}{\sigma}\sim N(0,1)$
if $Y\sim N(\mu,\sigma^2)$ and a is a constant, $F_y(a)=\Phi(\frac{a-\mu}{\sigma})$

standard normal distribution $\rightarrow X\sim N(0,1)$

- $F(x)=P(X\leq x)=\frac{1}{\sqrt{\tau\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}y^2}\,dy=\Phi(x)$

Normal Approximation to the Binomial Distribution

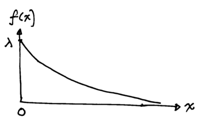
if $S_n\sim Binomial(n,p)$, then $\frac{S_n-np}{\sqrt{np(1-p)}}\sim N(0,1)$ for large n.

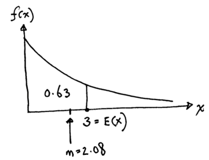
$$\mu=np,\quad \sigma^2=np(1-p)$$

Exponential Random Variable

a *continuous* r.v. X is a **exponential r.v.**, $\quad X\sim Exponential(\lambda)$ or $Exp(\lambda)$
if for some $\lambda>0$, its pdf is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & \text{otherwise}\end{cases}$$
$$E(X)=\frac{1}{\lambda},\quad Var(X)=\frac{1}{\lambda^2}$$





$$P(X<a)=\int_0^a\lambda e^{-\lambda x}\,dx$$

- an exponential r.v. is *memoryless*.
  - a non-negative r.v. is **memoryless** $\rightarrow$ if $P(X > s+t \mid X > t) = P(X > s)$ for all $s, t > 0$.

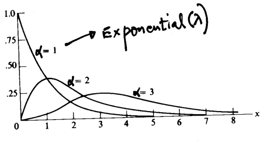
Gamma Distribution

a r.v. X has a **gamma distribution**, $X \sim Gamma(\alpha, \lambda)$ with parameters (α, λ), $\lambda > 0$ and $\alpha > 0$ if its pdf is given by

$$f(x) \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E(X) = \frac{\alpha}{\lambda} \quad Var(X) = \frac{\alpha}{\lambda^2}$$

where the gamma function $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$.



Gamma densities (λ = 1).

N1 - $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

Proof. using integration by parts of LHS to RHS

N2 - if α is an integer n, then $\Gamma(n) = (n-1)!$

N3 - if $X \sim Gamma(\alpha, \lambda)$ and $\alpha = 1$, then $X \sim Exp(\lambda)$.

N4 - for events occurring randomly in time following the 3 assumptions of poisson distribution, the amount of time elapsed until a total of n events has occurred is a gamma r.v. with parameters (n, λ).

- time at which event n occurs, $T_n \sim Gamma(n, \lambda)$
- number of events in time period $[0, t]$, $N(t) \sim Poisson(\lambda t)$

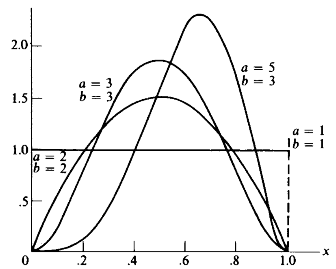
N5 - $Gamma(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}) = \chi_n^2$ (chi-square distribution to n degrees of freedom)

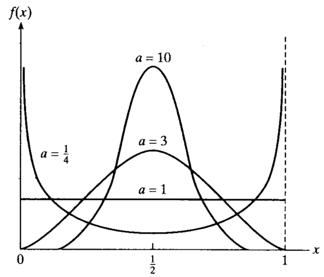
Beta Distribution

a r.v. X is said to have a **beta distribution**, $X \sim Beta(a, b)$ if its density is given by

$$f(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a}{a+b} \quad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$$





Beta densities with parameters (a, b) when $a = b$.

N1 - $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$

N2 - $\beta(a=1, b=1) = Uniform(0, 1)$

N3 - $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Cauchy Distribution

a r.v. X has a cauchy distribution, $X \sim Cauchy(\theta)$ with parameter θ, $\infty < \theta < \infty$ if its density is given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, -\infty < x < \infty$$

Proof. $E(X^n)$ does not exist for $n \in \mathbb{Z}^+$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \infty - \infty \text{ (undefined)}$$

06. JOINTLY DISTRIBUTED RANDOM VARIABLES

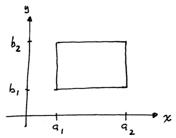
Joint Distribution Function

the **joint cumulative distribution function** of the pair of r.v. X and Y is $\rightarrow$

$$F(x, y) = P(X \leq x, Y \leq y), -\infty < x < \infty, \ -\infty < y < \infty$$

N1 - **marginal cdf of** X, $F_X(x) = \lim_{y \to \infty} F(x, y)$.

N2 - **marginal cdf of** Y, $F_Y(y) = \lim_{x \to \infty} F(x, y)$.



N3 - $P(X > a, Y > b) = 1 - F_X(a) - F_Y(b) + F(a, b)$

N4 - $P(a_1 < X \leq a_2, b_1 < Y \leq b_2)$
$= F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)$

Joint Probability Mass Function

if X and Y are both discrete r.v., then their **joint pmf** is defined by

$$p(i, j) = P(X = i, Y = j)$$

N1 - **marginal pmf of** X, $P(X = i) = \sum_j P(X = i, Y = j)$

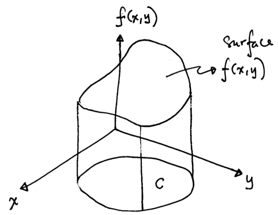
N2 - **marginal pmf of** Y, $P(Y = i) = \sum_i P(X = i, Y = j)$

Joint Probability Density Function

the r.v. X and Y are said to be *jointly continuous* if there is a function $f(x, y)$ called the **joint pdf**, such that for any two-dimensional set C,

$$P[(X, Y) \in C] = \iint_C f(x, y)\, dx\, dy$$

$=$ volume under the surface over the region C.



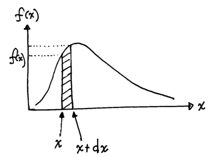
N1 - if $C = \{(x, y) : x \in A, y \in B\}$, then $P(X \in A, Y \in B) = \int_B \int_A f(x, y)\, dx\, dy$

N2 - $F(a, b) = P\Big(X \in (-\infty, a], Y \in (-\infty, b]\Big) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$

for double integral: when integrating dx, take y as a constant

N3 - $f(a, b) = \frac{\delta^2}{\delta a \delta b} F(a, b)$

interpretation of pdf



$$P(x < X < x + dx) = \int_x^{x+dx} f(y)\, dy$$

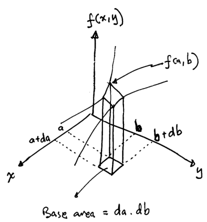
$$\approx f(x)\, dx$$

pdf at x, $f(x) \approx \frac{P(x < X < x+dx)}{dx}$

N4 - pdf of X, $f_X(x) = \int_0^\infty f(x, y)\, dy$

N5 - pdf of Y, $f_Y(y) = \int_0^\infty f(x, y)\, dx$

interpretation of joint pdf



$P(a < X < a + da, b < Y < b + db)$
$= \int_b^{b+db} \int_a^{a+da} f(x, y)\, dx\, dy$
$\approx f(a, b)\, da\, db$ (density of probability)

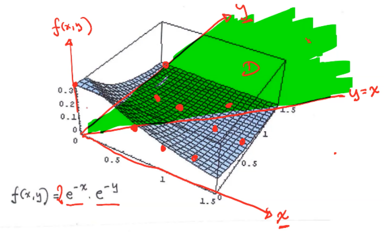
marginal pdf of X, $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

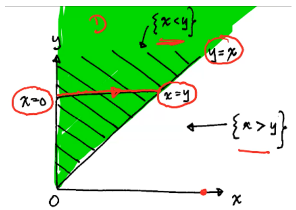
marginal pdf of Y, $f_Y(x) = \int_{-\infty}^{\infty} f(x, y)\, dx$

how to do a double integral

e.g. find $P(X < Y)$ where the joint pdf of X and Y are given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

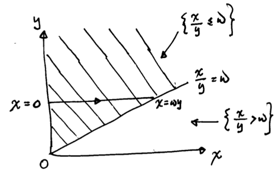




1. to get the bounds for dx and dy, plot $X < Y$
   1.1. draw horizontal lines to determine the bounds for x, from $x = a$ to $x = b$
   1.2. draw vertical lines to determine the bounds for y, from $y = c$ to $y = d$
2. integrate $\int_c^d \int_a^b f(x)\, dx\, dy$

example - given the joint pdf of X and Y, find the pdf of r.v. X/Y.

ans. set dummy variable $W = X/Y$, then
$F_W(w) = P(W \leq w) = P(\frac{X}{Y} \leq w)$
$P(\frac{X}{Y} \leq w) = \int_0^\infty \int_0^{wy} e^{-x-y}\, dx\, dy$



Independent Random Variables

N1 - X and Y are **independent** $\rightarrow$
$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

N2 - X and Y are **independent** $\rightarrow \forall a, b$,
$P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$
or $F(a, b) = F_X(a) \cdot F_Y(b)$ $\Rightarrow$ joint cdf is the product of the marginal cdfs

N3 - *discrete case*: discrete r.v. X and Y are **independent** $\iff$
$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all x, y.

N4 - *continuous case*: jointly continuous r.v. X and Y are **independent** $\iff$
$f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y.

N5 - independence is a **symmetric** relation $\rightarrow X$ is independent of $Y \iff Y$ is independent of X

Sum of Independent Random Variables

N1 - for independent, continuous r.v. X and Y having pdf f_X and f_Y,

$$F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\, dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

impt example - E52 (pdf of $X + Y$)

Distribution of Sums of Independent r.v.

for $i = 1, 2, \ldots, n$,

1. $X_i \sim Gamma(t_i, \lambda) \Rightarrow \sum_{i=1}^n X_i \sim Gamma(\sum_{i=1}^n t_i, \lambda)$
2. $X_i \sim Exp(\lambda) \Rightarrow \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$
3. $Z_i \sim N(0, 1) \Rightarrow \sum_{i=1}^n z_i^2 \sim \chi_n^2 = Gamma(\frac{n}{2}, \frac{1}{2})$
4. $X_i \sim N(\mu_i, \sigma_i^2) \Rightarrow \sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
5. $X \sim Poisson(\lambda_1), Y \sim Poisson(\lambda_2) \Rightarrow X + Y \sim Poisson(\lambda_1 + \lambda_2)$
6. $X \sim Binom(n, p), Y \sim Binom(m, p) \Rightarrow X + Y \sim Binom(n + m, p)$

Conditional Distribution (discrete)

for discrete r.v. X and Y, the **conditional pmf** of X given that $Y = y$ is

$$P_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

for discrete r.v. X and Y, the **conditional pdf** of X given that $Y = y$ is

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{a \leq x} \frac{P(X=a,Y=y)}{P(Y=y)} = \sum_{a \leq x} P_{X|Y}(a|y)$$

N0 - equivalent notation:
- $P_{X|Y}(x|y) = P(X = x|Y = y)$
- $P_X(x) = P(X = x)$

N1 - if X is independent of Y, then $P_{X|Y}(x|y) = P_X(x)$

Conditional Distribution (continuous)

for X and Y with joint pdf $f(x, y)$, the **conditional pdf** of X given that $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{for all } y \text{ s.t. } f_Y(y) > 0$$

$$f_{X|Y}(a|y) = P(X \leq a|Y = y) = \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx$$

N1 - for any set A, $P(X \in A|Y = y) = \int_A f_{X|Y}(x|y)\,dy$

N2 - if X is independent of Y, then $f_{X|Y}(x|y) = f_X(x)$.

! "find the marginal/conditional pdf of Y" $\Rightarrow$ must include the **range** too!!
(see Ex. 69(b, c))

Joint Probability Distribution of Functions of r.v.

Let X_1 and X_2 be jointly continuous r.v. with joint pdf $f_{x_1,x_2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ satisfy

1. the equations $y_1 = g_1(X_1, X_2)$ and $y_2 = g_2(X_1, X_2)$ can be *uniquely* solved for x_1, x_2 in terms of y_1 and y_2
2. $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ have continuous partial derivatives at all points (x_1, x_2) such that $J(x_1, x_2) = \begin{vmatrix} \frac{\delta g_1}{\delta x_1} & \frac{\delta g_1}{\delta x_2} \\ \frac{\delta g_2}{\delta x_1} & \frac{\delta g_2}{\delta x_2} \end{vmatrix} = \frac{\delta g_1}{\delta x_1} \cdot \frac{\delta g_2}{\delta x_2} - \frac{\delta g_2}{\delta x_1} \cdot \frac{\delta g_1}{\delta x_2} \neq 0$

then

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)\frac{1}{|J(x_1,x_2)|}$$

where $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$

07. PROPERTIES OF EXPECTATION

recap:
- for a **discrete** r.v. X, $E(X) = \sum_x x \cdot p(x) = \sum_x \cdot P(X = x)$
- for a **continuous** r.v. X, $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$
- for a **non-negative integer-valued** r.v. Y, $E(Y) = \sum_{i=1}^{\infty} P(Y \geq i)$
- for a **non-negative** r.v. Y, $E(Y) = \int_{-\infty}^{\infty} P(Y > y)\,dy$

Expectations of Sums of Random Variables

for X and Y with joint pmf $p(x, y)$ and joint pdf $f(x, y)$,

$$E[g(x, y)] = \sum_y \sum_x g(x, y)p(x, y)$$

$$E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)\,dx\,dy$$

N2 - if $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$

N3 - if $E(X)$ and $E(Y)$ are finite, $E(X + Y) = E(X) + E(Y)$

Proof. using N1, integrate $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)f(x, y)\,dx\,dy$
$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx + \int_{-\infty}^{\infty} yf_Y(y)\,dy = E(X) + E(Y)$

N4 - if, for r.v.s X and Y, if $X \geq Y$, then $E(X) \geq E(Y)$

N5 - let $X_1, \ldots, X_n$ be independent and identically distributed r.v.s having distribution $P(X_i \leq x) = F(x)$ and expected value $E(X_i) = \mu$.

if $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$, then $E(\bar{X}) = \mu$

Proof. $E(\bar{X}) = E(\sum_{i=1}^{n} \frac{X_i}{n}) = \frac{1}{n}(\sum_{i=1}^{n} E(X_i)) = \frac{1}{n} \cdot n\mu = \mu$

$\Rightarrow$ sample mean = population mean

N6 - $\bar{X}$ is the **sample mean**.

N7 - if $X \sim Binom(n, p)$, then $E(X) = np$.

Proof. express X as a sum of Bernoulli r.v. $\Rightarrow$ sum of indicator r.v. = np.

examples

! trick: express a r.v. as a sum of r.v. with easier to find expectation
- negative binomial = sum of geometric $= k/p$
- hypergeometric with r red balls out of N balls with n trials
  - indicator r.v. $= 1$ if the ith ball selected is red
  - $P(Y_i = 1) = \frac{r}{N} \Rightarrow E(Y_i) = \frac{r}{N} \Rightarrow E(X) = \sum_{i=1}^{n} Y_i = n\frac{r}{N}$
- hat throwing problem: expected number of people that select their own hat
  - P(select your own hat back) = $\frac{1}{N} \Rightarrow E(X) = N \cdot \frac{1}{N} = 1$
- coupon collector problem:
  - let X = number of coupons collected for a complete set
  - let X_i = number of *additional* coupons that need to be collected to obtain another distinct type after i distinct types have been collected
    - $X_i \sim Geometric(p = \frac{N-i}{N})$
  - $E(X) = \sum_{i=1}^{N-1} E(X_i) = 1 + \frac{1}{\frac{N-1}{N}} + \frac{1}{\frac{N-2}{N}} + \cdots + \frac{1}{\frac{1}{N}}$
    $= N(\frac{1}{N} + \frac{1}{N-1} + \cdots + 1)$

Covariance, Variance of Sums and Correlations

if X and Y are independent, then for any functions h and g,
$$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$$

covariance $\rightarrow$ measure of *linear relationship*

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

N1 - X and Y are independent $\Rightarrow Cov(X, Y) = 0$

N2 - $Cov(X, Y) = 0 \not\Rightarrow X$ and Y are independent

Proof. let $E(X) = 0, E(XY) = 0 \Rightarrow Cov(X, Y) = 0$, but not independent

e.g. non-linear relationship

Covariance properties

1. $Cov(X, Y) = Cov(Y, X)$
2. $Cov(X, X) = Var(X)$
3. $Cov(aX, Y) = aCov(X, Y)$
4. $Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n} \sum_{j-1}^{m} Cov(X_i, Y_j)$

for variance:

N1 - $Var(\sum_{i=1}^{n} X_i) - \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j)$

N2 - if $X_1, \ldots, X_n$ are *pairwise independent* (X_i, X_j are independent $\forall i \neq j$), then $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

N3 - for n independent and identically distributed r.v. with expected value μ and variance σ^2,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad S^2 \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$
$$Var(\bar{X}) = \frac{\sigma^2}{n} \qquad E(S^2) = \sigma^2$$

$\Rightarrow S^2$ is an *unbiased estimator* for σ^2.

Correlation

correlation of two r.v. X and Y, $\rho(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$

N1 - $-1 \leq \rho(X, Y) \leq 1$ where -1 and 1 denote a perfect negative and positive linear relationship respectively.

N2 - $\rho(X, Y) = 0 \Rightarrow$ no *linear* relationship - uncorrelated

N3 - $\rho(X, Y) = 1 \Rightarrow Y = aX + b, a = \frac{\delta y}{\delta x} > 0$

N4 for events A and B with indicator r.v. I_A and I_B, then $Cov(I_A, I_B) = 0$ when they are independent events.

N5 - deviation is not correlated with the sample mean. For independent & identically distributed r.v. $X_1, X_2, \ldots, X_n$ with variance σ^2, then $Cov(X_i - \bar{X}, \bar{X}) = 0$.

Proof. $Cov(X_i - \bar{X}, \bar{X}) = Cov(X_i, \bar{X}) - Cov(\bar{X}, \bar{X})$
$= Cov(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j) - Var(\bar{X})$
$= \frac{1}{n}\sum_{j=1}^{n} Cov(X_i, X_j) - Var(\bar{X})$
$= \frac{1}{n}Cov(X_i, X_i) - \frac{\sigma^2}{n}$ since $\forall i \neq j, Cov(x_i, x_j) = 0$
$= \frac{1}{n}Var(x_i) - \frac{\sigma^2}{n} = 0$

Conditional Expectation

the **conditional expectation** of X,
given that $Y = y$, for all values of y such that $P_Y(y) > 0$ is defined by

$$E[X|Y = y] = \sum_x x \cdot P(X = x|Y = y) = \sum_x x \cdot p_{X|Y}(x|y)$$

$$E(X|Y = y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)}\,dx$$

! note the range for $f_{X|Y}(x|y)$

N1 - If $X, Y \sim Geometric(p)$,
then $P(X = i|X + Y = n) = \frac{1}{n-1}$, a uniform distribution.

N2 - $E(X|X + Y = n) = \sum_{i=1}^{n-1} i \cdot P(X = i|X + Y = n) = \frac{n}{2}$

Conditional expectations also satisfy properties of ordinary expectations.
$\Rightarrow$ an ordinary expectation on a *reduced sample space* consisting only of outcomes for which $Y = y$

discrete case: $E[g(x)|Y = y] = \sum_x g(x)P_{X|Y}(x|y)$

continuous case: $E[g(x)|Y = y] = \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)$

then $E(X) = E_{\text{w.r.t. } y}(E_{\text{w.r.t. } X|Y=y}(X|Y))$

Deriving Expectation

$E(X) = E_Y(E_X(X|Y))$

discrete case: $E(X) = \sum_y E(X|Y = y)P(Y = y)$

continuous case: $E(X) = \int_{-\infty}^{\infty} E(X|Y = y)f_Y(y)\,dy$

N3 - 3 methods for finding $E(X)$ given $f(x, y)$
1. using $E(g(x, y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)\,dx\,dy \quad \Rightarrow$ let $g(x, y) = x$
2. using $E(X) = \int_{-\infty}^{\infty} xf_X(x)\,dx$
3. using $E(X) = \int_{-\infty}^{\infty} E(X|Y = y)f_Y(y)\,dy$

N4 - $E(\sum_{i=1}^{N} X_i) = E_N(E(\sum_{i=1}^{N} X_i|N)) = \sum_{n=0}^{\infty} E(\sum_{i=1}^{N} X_i|N = n) \cdot P(N = n)$

Computing Probabilities by Conditioning

$$P(E) = \sum_y P(E|Y = y)P(Y = y) \text{ if } Y \text{ is discrete}$$

$$P(E) = \int_{-\infty}^{\infty} P(E|Y = y)f_Y(y)\,dy \text{ if } Y \text{ is continuous}$$

Proof. let X be an indicator r.v. for E. $\quad \Rightarrow E(X) = P(E)$
$E(X|Y = y) = P(X = 1|Y = y) = P(E|Y = y)$

N5 - find $P((X, Y) \in C)$ given $f(x, y)$: see p.57
also: $P(X < Y) = \int P(X < Y|Y = y) \cdot f_Y(y)$

Conditional Variance

$$Var(X|Y) = E[(X - E(X|Y))^2 \mid Y]$$
$$Var(X|Y) = E(X^2|Y) - [E(X|Y)]^2$$

N6 - $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

N7 - $E(f(Y)) = E(f(Y)|Y = t) = E(f(y)|Y = t)$
$= E(f(t))$ if $N(t)$ and Y are independent

Moment Generating Functions

moment generating function $M(t)$ of the r.v. $X \rightarrow$
$M(t) = E(e^{tX})$ for all real values of t

- if X is *discrete* with pmf $p(x)$, $M(t) = \sum_x e^{tx} \cdot p(x)$
- if X is *continuous* with pdf $f(x)$, $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$M(t)$ is called the **mgf** because *all moments of* X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.
($M'(0) = E(X), M''(0) = E(X^2)$, etc)
in general,

- $M'(t) = E(X^n e^{tX})$, $n \geq 1$
- $M^n(0) = E(X^n)$, $n \geq 1$

N8 - binomial expansion: $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$
(see other series for useful expansions on other distributions)
N9 - integrating over a pdf from ∞ to $-\infty$ always gives 1

if X and Y are independent and have mgf's $M_X(t)$ and $M_Y(t)$ respectively,
N10 - the mgf of $X + Y$ is $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

Proof. $M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} \cdot e^{tY}] = E(e^{tX})E(e^{tY})$
$= M_X(t) \cdot M_Y(t)$

N11 - if $M_X(t)$ exists and is finite in some region about $t = 0$, then the distribution of X is **uniquely** determined. $M_X(t) = M_Y(t) \iff X = Y$

Common mgf's

- $X \sim Normal(0, 1)$, $M(t) = e^{e^2/2}$
- $X \sim Binomial(n, p)$, $M(t) = (pe^t + (1 - p))^n$
- $X \sim Poisson(\lambda)$, $M(t) \exp[\lambda(e^t - 1)]$
- $X \sim Exp(\lambda)$, $M(t) = \frac{\lambda}{\lambda - t}$

08. LIMIT THEOREMS

Markov's Inequality $\rightarrow$ if X is a non-negative r.v., for any $a > 0$,
$P(X \geq a) \leq \frac{E(x)}{a}$.

Proof. let I be an indicator r.v. $= 1$ when $X \geq a$.
Then $I \leq \frac{X}{a}$, and $E(I) \leq \frac{E(X)}{a}$, and $P(X \geq a) \leq \frac{E(X)}{a}$.

Chebyshev's inequality $\rightarrow$ if X is an r.v. with finite mean μ and variance σ^2, then for any value of $k > 0$, $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$.

Proof. $P[(X - \mu)^2 \geq k^2] \leq \frac{E[(X-\mu)^2]}{k^2}$ by Markov's inequality
Since $(X - \mu)^2 \geq k^2 \iff |X - \mu| \geq k$, then $P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$

N1 - if $Var(X) = 0$, then $P(X = E[X]) = 1$

Proof. let $\mu = E[X]$. by Chebyshev's inequality, for any $n \geq 1$,
$P(|X - \mu| > \frac{1}{n}) \leq \frac{Var(X)}{(\frac{1}{n})^2} = 0$
then $P(X \neq \mu) = 0 \Rightarrow P(X = \mu) = 1$

weak law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s, each with finite mean $E[X_i] = \mu$. Then, for any $\epsilon > 0$,
$P\{|\frac{X_1+\cdots+X_n}{n} - \mu| \geq \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$

central limit theorem $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed r.v.s each having mean μ and variance σ^2. Then the distribution of $\frac{X_1+\cdots+X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \rightarrow \infty$.

- aka: $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \rightarrow z \sim N(0, 1)$
- for $-\infty < a < \infty$,
$P(\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \leq a) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx = F(a)$ (cdf of standard normal) as $n \rightarrow \infty$

N2 - Let $Z_1, Z_2, \ldots$ be a sequence of r.v.s with distribution functions F_{Z_n} and moment generating functions $M_{Z_n}, n \geq 1$. Let Z be a r.v. with distribution function F_Z and mgf M_Z.
If $M_{Z_n}(t) \rightarrow M_Z(t)$ for all t, then $F_{Z_n}(t) \rightarrow F_Z(t)$ for all t at which $F_Z(t)$ is continuous.

strong law of large numbers $\rightarrow$ let $X_1, X_2, \ldots$ be a sequence of independent and identically distribution r.v.s, each having finite mean $\mu = E[X_i]$.
Then, with probability 1, $\frac{X_1+\cdots+X_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$

Chernoff bounds $\rightarrow$

- $P(X \geq a) \leq e^{-ta} M(t)$ for all $t > 0$
- $P(X \leq a) \leq e^{-ta} M(t)$ for all $t < 0$

commutative	$E \cup F = F \cup E$	$E \cap F = F \cap E$
associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(E \cap F) \cap G = E \cap (F \cap G)$
distributive	$(E \cup F) \cap G = (E \cap F) \cup (F \cap G)$	$(E \cap F) \cup G = (E \cup F) \cap (F \cup G)$
DeMorgan's	$(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i^c$	$(\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i^c$