MA1102R
AY20/21 sem 2
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00. FUNCTIONS & SETS

sets

$$A = \{x \mid \textit{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

operations on sets

- union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- difference: $A \backslash B = \{x \mid x \in A \wedge x \notin B\}$

notations of sets

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$
- $\mathbb{N} = \mathbb{Z}^+$
- $\emptyset$: empty set

notations of intervals

- closed interval (inclusive): $[a, b] = \{x \mid a \le x \le b\}$
- open interval (exclusive): $(a, b) = \{x \mid a < x < b\}$
  - $(a, \infty) = \{x \mid a < x\}$

functions

- **existence**: $\forall a \in A, f(a) \in B$
- **uniqueness**: $\forall a \in A$ has only one image in B.
- for $f : A \to B$
  - domain: A
  - codomain: B
  - range: $\{f(x) \mid x \in A\}$
- for this mod:
  - $A, B \subseteq \mathbb{R}$
  - if A is not stated, the domain of f is the largest possible set for which f is defined
  - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set
$$G(f) := \{(x, f(x)) \mid x \in A\}$$

- if $A, B \subseteq R$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane $\mathbb{R}^2$

algebra of functions

function	domain
$(f+g)(x) := f(x) + g(x)$	$A \cap B$
$(f-g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

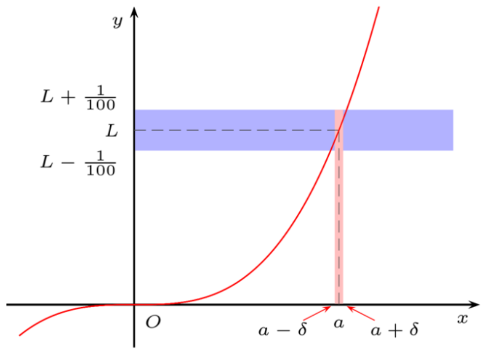
types of functions

- **rational function**: $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
  - every polynomial is a rational function ($Q(x) = 1$)
- **algebraic function**: constructed from polynomials using algebraic operations
- a function f is **increasing** on a set I if $x_q < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if $x_q < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.
- even/odd:
  - **even function**: $\forall x, f(-x) = f(x)$
    - $\star$ symmetric about the y-axis
  - **odd function**: $\forall x, f(-x) = -f(x)$
    - $\star$ symmetric about the origin O
  - any function defined on $\mathbb{R}$ can be decomposed *uniquely* into the sum of an even function and an odd function
- **power function**: x^n
  - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

01. LIMITS

precise definition of limits

Let f be a function defined on an open interval containing a, except possibly at a.
The limit of $f(x)$ as x approaches a, equals L if, for every $\epsilon > 0$ there is $\delta > 0$ such that
$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$



informally,
- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a.
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L.

limit laws

- Let $c \in \mathbb{R}$. $\lim\limits_{x \to a} c = c$
- $\lim\limits_{x \to a} x = a$

Suppose $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant.

- $\lim\limits_{x \to a} (cf(x)) = cL = c \lim\limits_{x \to a} f(x)$
- $\lim\limits_{x \to a} (f(x) + g(x)) = L + M = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} (f(x)g(x)) \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $\lim\limits_{x \to a} g(x) \neq 0$
- $\lim\limits_{x \to a} (f(x))^n = \left(\lim\limits_{x \to a} f(x)\right)^n$
- $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

$$\boxed{\text{if } \lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to a} g(x) = 0, \text{ then } \lim_{x \to a} f(x) = 0}$$

inequalities on limits

Suppose $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$.

lemma
if $f(x) \le g(x)$ for all x near a (except possibly at a), then $L \le M$.

lemma
If $f(x) \ge 0$ for all x, then $L \ge 0$.

direct substitution property

Let f be a polynomial or rational function.
If a is in the domain of f, then
$$\lim_{x \to a} f(x) = f(a)$$
If $f(x) = g(x)$ for all x near a except possibly at a, then
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

applications
- if a is not in the domain (e.g. 0 denominator), don't apply directly
- convert to an equivalent function and then sub in

one-sided limits

- limit laws also hold for one-sided limits

If as x is close to a from the right, $f(x)$ is close to L, the right-hand limit of f as x approaches a equals L.
$$(x \to a^+ \Rightarrow f(x) \to L) \Rightarrow \lim_{x \to a^+} f(x) = L$$
If as x is close to a from the left, $f(x)$ is close to L, the left-hand limit of f as x approaches a equals L.
$$(x \to a^- \Rightarrow f(x) \to L) \Rightarrow \lim_{x \to a^-} f(x) = L$$

$$\boxed{\lim_{x \to a} f(x) = L \leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L}$$

$$f(x) \to L \Leftarrow x \to a \Leftrightarrow \begin{cases} x \to a^+ \Rightarrow f(x) \to L \\ x \to a^- \Rightarrow f(x) \to L \end{cases}$$

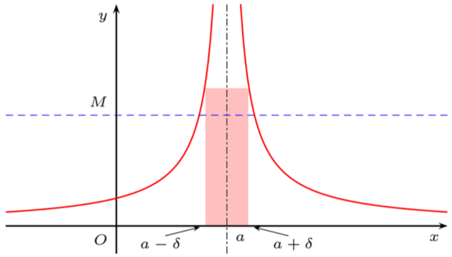
definition of one-sided limits

LH Limit: $\lim\limits_{x \to a^-} f(x) = L$
if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$$

RH Limit: $\lim\limits_{x \to a^+} f(x) = L$
if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$$

definition of infinite limits

$$\lim_{x \to a} f(x) = \infty$$
if for every $M > 0$ there exists $\delta > 0$ such that
$$0 < |x - a| < \delta \Rightarrow f(x) > M$$



negative infinite limit:
$$0 < |x - a| < \delta \Rightarrow f(x) < M$$

limits to infinity

$\lim\limits_{x \to \infty} f(x) = L$:
For every $\epsilon > 0$, there exists N such that
$$x > N \Rightarrow |f(x) - L| < \epsilon$$

$\lim\limits_{x \to \infty} f(x) = \infty$:
For every $M > 0$, there exists N such that
$$x > N \Rightarrow f(x) > M$$

squeeze theorem

- Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where
  - $g(x) \le f(x) \le h(x)$ for all x near a (except at a),
  - and $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$.

Then $\lim\limits_{x \to a} f(x) = L$

02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at** $a \Leftrightarrow$
f is continuous from the left and from the right at a.
$$\lim_{x \to a} f(x) = f(a)$$

a function f is *continuous at an interval* if it is continuous at every number in the interval.

f is continuous on **open interval** (a, b)
$\Leftrightarrow f$ is continuous at every $x \in (a, b)$
f is continuous on **closed interval** [a, b]
$$\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$$

continuity test

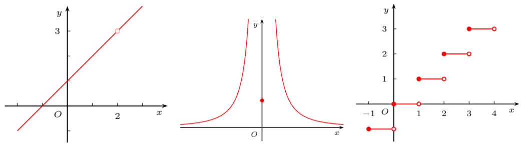
f is continuous at $a \Leftrightarrow$
1. f is defined at a (a is in the domain of f)
2. $\lim\limits_{x \to a} f(x)$ exists
3. $\lim\limits_{x \to a} f(x) = f(a)$

precise definition of continuity

a function f is continuous at a number a if $\forall \epsilon > 0, \exists \delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \epsilon$

examples of discontinuity

- removable discontinuity
- infinite discontinuity
- jump discontinuity



properties of continuous functions

let f and g be functions continuous at a. let c be a constant.

1. cf is continuous at a
2. $f+g$ is continuous at a
3. $f-g$ is continuous at a
4. fg is continuous at a
5. $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$

other properties

- a polynomial is continuous everywhere;
- a rational function is continuous on its domain
- let c be a real number. $f(x) = c$ is continuous on $\mathbb{R}$.
- $f(x) = x$ is continuous on $\mathbb{R}$.

trigonometric functions

- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous everywhere
- $\tan x, \sec x$ are continuous whenever $\cos x \neq 0$
- $\cot x, \csc x$ are continuous whenever $\sin x \neq 0$
  - domain: $\mathbb{R} \backslash \{0, \pm\pi, \pm 2\pi, \cdots\}$

composite of continuous functions

if f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

$$\lim_{x\to a} (f\circ g)(x) = (f\circ g)(a)$$

substitution theorem

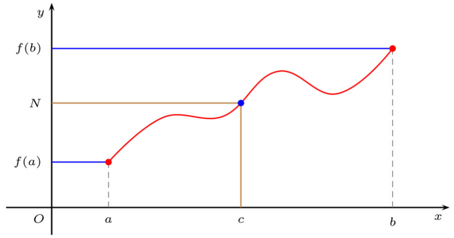
Suppose $y = f(x)$ such that $\lim_{x\to a} f(x) = b$. If

1. g is continuous at b, OR
2. $\forall x$ near a, except at a, $f(x) \neq b$ and $\lim_{y\to b} g(y)$ exists

Then $\lim_{x\to a} g(f(x)) = \lim_{y\to b} g(y)$

intermediate value theorem

Let f be a function continuous on $[a,b]$ with $f(a) \neq f(b)$. Let N be a number between $f(a)$ and $f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = N$.



03. DERIVATIVES

tangent line

the **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:

$$y = f'(a)(x-a) + f(a)$$

definition of derivatives

- f is differentiable at a if $f'(a)$ exists
- $f'(a)$ is the slope of $y = f(x)$ at $x = a$

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

- $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x) = \cdots$
- $\frac{dy}{dx} := \lim_{x\to 0} \frac{\Delta y}{\Delta x}$ (derivative of y with respect to x)
- $f'(a) = \frac{dy}{dx}|_{x=a}$

differentiable functions

- f is differentiable at a if $f'(a) := \lim_{x\to 0} \frac{f(a+h)-f(a)}{h}$ exists.
- f is differentiable on (a,b) if f is differentiable at every $c \in (a,b)$

differentiability & continuity

- if f is differentiable at a, then f is continuous at a.
  - differentiability $\Rightarrow$ continuity
- continuity $\not\Rightarrow$ differentiability

differentiation

- every polynomial and rational function is differentiable on its domain
  - the domain of f' may be smaller than the domain of f.
- trigonometric functions are differentiable on the domain

chain rule

If g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and

$$F'(a) = (f\circ g)'(a) = f'(b)g'(a) = f'(g(a))g'(a)$$

If $z = f(y)$ and $y = g(x)$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$\frac{dz}{dx}|_{x=a} = \frac{dz}{dy}|_{y=b}\frac{dy}{dx}|_{x=a}$$

generalised chain rule

h is differentiable at a; g is differentiable at $B = h(a)$; f is differentiable at $c = g(b)$.

$$(f\circ(g\circ h))' = f' \circ (g\circ h)\cdot(g\circ h)'$$
$$= f'(c)g'(b)h'(a)$$

Leibniz notation:
If $y = h(x), z = g(y), w = f(z)$,
$\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx}$

implicit differentiation

- assumes that $\frac{dy}{dx}$ exists

second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$
$$f' = D(f) \Rightarrow f'' := D^2(f)$$

higher derivatives

$$f^{(0)} := f$$

For any positive integer n, $f^{(n)} := (f^{(n-1)})'$

if $y = f(x)$, then $f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x)$

04. APPLICATIONS OF DIFFERENTIATION

extreme values of functions

Let f be a function with domain D.

global (absolute) max/min

- aka absolute max/min
- extreme values = absolute maximum and absolute minimum

f has a global **maximum** at $c \in D$
$\Leftrightarrow f(c) \geq f(x)$ for all $x \in D$
f has a global **minimum** at $c \in D$
$\Leftrightarrow f(c) \leq f(x)$ for all $x \in D$

local max/min

- aka relative max/min aka "turning points"

f has a local **maximum** at $c \in D$
$\Leftrightarrow f(c) \geq f(x)$ for all x near c
f has a local **minimum** at $c \in D$
$\Leftrightarrow f(c) \leq f(x)$ for all x near c

extreme value theorem

existence

if f is *continuous* on a *finite closed* interval $[a,b]$, then f attains extreme values on $[a,b]$.

value

the extreme value occurs at either *critical numbers* or the *endpoints* ($x = a, x = b$).

critical numbers

Then $c \in D$ is a *critical number* of f if $f'(c) = 0$, or $f'(c)$ does not exist.

fermat's theorem

Suppose f has a local maximum or minimum at c. If $f'(c)$ exists, then $f'(c) = 0$.

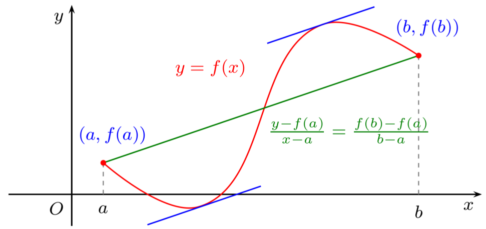
Rolle's Theorem

Let f be a function such that f is *continuous* on $[a,b]$, f is *differentiable* on (a,b), and $f(a) = f(b)$. Then there is a number $c \in (a,b)$ such that $f'(c) = 0$.

mean value theorem

Let f be a function such that f is *continuous* on $[a,b]$ and f is *differentiable* on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



- generalisation of Rolle's theorem when $f(a) = f(b)$.

ordinary differential equations

Let f and g be continuous on $[a,b]$. If $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x) = g(x) + C$ on $[a,b]$ for a constant C.

increasing/decreasing test

Let f be continuous on $[a,b]$ and differentiable on (a,b).

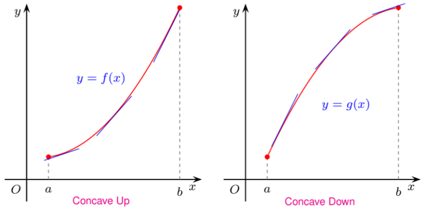
- $f'(x) > 0$ for any $x \in (a,b) \Rightarrow f$ is increasing.
  - f is increasing $\Rightarrow f(x) \geq 0$
- $f'(x) < 0$ for any $x \in (a,b) \Rightarrow f$ is decreasing.
  - f is decreasing $\Rightarrow f(x) \leq 0$
- $f'(x) = 0 \Rightarrow f$ could be increasing OR decreasing.

first derivative test

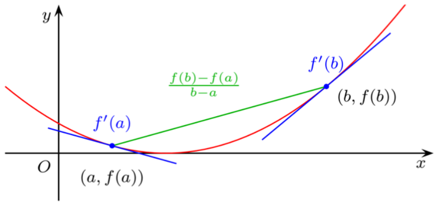
Let f be continuous and c be a critical number of f. Suppose f is differentiable near c (except possibly at c). At c, if f' changes from:

- (+) to (-) $\to f$ has a local **maximum** at c
- (-) to (+) $\to f$ has a local **minimum** at c
- no change in sign $\to f$ has neither local max/min at c.

concavity



f is **concave up** on an open interval I
if $f(x) > f'(y)(x-y) + f(y)$ for any $x \neq y \in I$
for $a < b \in I$, $f'(a) < f'(b)$
concave up $\Leftrightarrow$ f' is increasing
f is **concave down** on an open interval I
if $f(x) < f'(y)(x-y) + f(y)$ for any $x \neq y \in I$
for $a < b \in I$, $f'(a) > f'(b)$
concave down $\Leftrightarrow$ f' is decreasing



concavity test

- $f'' > 0$ on $I \Rightarrow f$ is concave up on I
- $f'' < 0$ on $I \Rightarrow f$ is concave down on I

second derivative test

If $f'(c) = 0$ and $f''(c)$ exists,
- $f''(c) > 0 \Rightarrow f$ has a **local maximum** at c.
- $f''(c) < 0 \Rightarrow f$ has a **local minimum** at c.
- $f''(c) = 0 \Rightarrow$ inconclusive

inflection point

- A point P on the curve $y = f(x)$ is an inflection point if
  - f is continuous at P, and
  - the concavity of the curve changes at P.
- if c is an inflection point and f is twice differentiable at c, then $f''(c) = 0$.

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n,$$

where $R_n = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)}$ for c between x and a

Taylor Series

As $R - n \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

L'Hopital's Rule $\left(\frac{0}{0}\right)$

Let f and g be functions such that
- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
- f and g are differentiable near a (except at a).

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$,
provided that the RHS limit exists or is $\pm\infty$

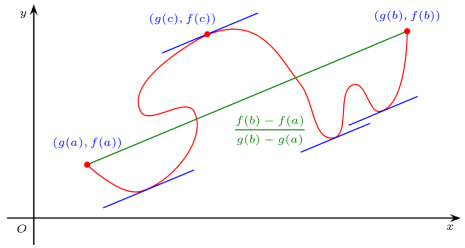
L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$

Suppose that
- $\lim_{x\to a}|f(x)| = \lim_{x\to a}|g(x)| = \infty$,
- f and g are differentiable near a (except at a),
- $g'(x) \neq 0$ near a (except at a)

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
provided that the RHS limit exists or is $\pm\infty$

Cauchy's Mean Value Theorem

Let f, g be continuous on $[a,b]$, differentiable on (a,b),
and $g'(x) \neq 0$ for any $x \in (a,b)$.
Then there exists $c \in (a,b)$ such that
$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$



misc

triangle inequality

$$|a = b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R}$$

binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$
$$= a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

where the binomial coefficient is given by
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

factorisation

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

misc

- $\forall x \in (0, \frac{\pi}{2}), \sin x < x < \tan x$