

01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- combinatorial analysis** → the mathematical theory of counting
- basic principle of counting** → Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- generalized basic principle of counting** → If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1! n_2! \dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)! r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, $1 \leq r \leq n$

Proof. If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.

If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - The Binomial Theorem - $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Proof. by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes n_1, n_2, \dots, n_r , where $n_1 + n_2 + \dots + n_r = n$

Proof. using basic counting principle,

$$\begin{aligned} &= \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots \binom{n-n_1-n_2-\dots-n_{r-1}}{n_r} \\ &= \frac{n!}{(n-n_1)! n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)! n_2!} \times \dots \times \frac{(n-n_1-n_2-\dots-n_{r-1})!}{0! n_r!} \\ &= \frac{n!}{n_1! n_2! \dots n_r!} \end{aligned}$$

N7 - The Multinomial Theorem: $(x_1 + x_2 + \dots + x_r)^n = \sum_{(n_1, \dots, n_r): n_1 + n_2 + \dots + n_r = n} \frac{n!}{n_1! n_2! \dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors (x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$, $x_i > 0$, $i = 1, 2, \dots, r$

! cannot be directly applied to N8 as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors

(x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- sample space** → The *set* of all outcomes of an experiment (where outcomes are not predictable with certainty)
- event** → Any *subset* of the sample space
- union** of events E and $F \rightarrow E \cup F$ is the event that contains all outcomes that are either in E or F (or both).
- intersection** of events E and $F \rightarrow E \cap F$ or EF is the event that contains all outcomes that are both in E and in F .
- complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E .
- subset** → $E \subset F$ if all of the outcomes in E that are also in F .
 - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws

$$\left(\bigcup_{i=1}^n E_i \right)^c = \bigcap_{i=1}^n E_i^c$$

Proof. to show $LHS \subset RHS$: let $x \in \left(\bigcup_{i=1}^n E_i \right)^c$
 $\Rightarrow x \notin \bigcup_{i=1}^n E_i \Rightarrow x \notin E_1$ and $x \notin E_2 \dots$ and $x \notin E_n$
 $\Rightarrow x \in E_1^c$ and $x \in E_2^c \dots$ and $x \in E_n^c$
 $\Rightarrow x \in \bigcap_{i=1}^n E_i^c$
 to show $RHS \subset LHS$: let $x \in \bigcap_{i=1}^n E_i^c$

$$\left(\bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c$$

Proof. using the first law of DeMorgan, negate LHS to get RHS

Axioms of Probability

definition 1: relative frequency

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

problems with this definition:

- $\frac{n(E)}{n}$ may not converge when $n \rightarrow \infty$
- $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated

definition 2: Axioms

Consider an experiment with sample space S . For each event E of the sample space S , we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- For any sequence of mutually exclusive events E_1, E_2, \dots (i.e., events for which $E_i E_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ is the probability of event E .

Simple Propositions

N1 - $P(\emptyset) = 0$

N2 - $P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$ (aka axiom 3 for a finite n)

N3 - strong law of large numbers - if an experiment is repeated over and over again, then with probability 1, the proportion of time during which any specific event E occurs will be equal to $P(E)$.

N6 - the definitions of probability are mathematical definitions. They tell us which set functions can be called **probability functions**. They do not tell us what value a probability function $P(\cdot)$ assigns to a given event E .

probability function \iff it satisfies the 3 axioms.

N7 - $P(E^c) = 1 - P(E)$

N8 - if $E \subset F$, then $P(E) \leq P(F)$

N9 - $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

N10 - Inclusion-Exclusion identity where $n = 3$

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$

N11 - Inclusion-Exclusion identity -

$$\begin{aligned} P(E_1 \cup E_2 \cup \dots \cup E_n) &= \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots \\ &\quad + (-1)^{r+1} \sum_{i_1 < i_2 < \dots < i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) + \dots \\ &\quad + (-1)^{n+1} P(E_1 E_2 \dots E_n) \end{aligned}$$

Proof. Suppose an outcome with probability ω is in exactly m of the events E_i , where $m > 0$. Then

LHS: the outcome is in $E_1 \cup E_2 \cup \dots \cup E_n$ and ω will be counted once in $P(E_1 \cup E_2 \cup \dots \cup E_n)$

RHS:

- the outcome is in exactly m of the events E_i and ω will be counted exactly $\binom{m}{1}$ times in $\sum_{i=1}^n P(E_i)$

- the outcome is contained in $\binom{m}{2}$ subsets of the type $E_{i_1} E_{i_2}$ and ω will be counted $\binom{m}{2}$ times in $\sum_{i_1 < i_2} P(E_{i_1} E_{i_2})$

- ... and so on

hence $RHS = \binom{m}{1} \omega - \binom{m}{2} \omega + \binom{m}{3} \omega - \dots \pm \binom{m}{m} \omega$

$$\begin{aligned} &= \omega \sum_{i=0}^m \binom{m}{i} (-1)^i = \text{binomial theorem where } x = -1, y = 1 \\ &= 0 = LHS \end{aligned}$$

e.g. For an outcome with probability ω and $n = 3$

- Case 1.** $w = P(E_1 E_2)$
 LHS = ω
 RHS = $(\omega + \omega + 0) - (\omega + 0 + 0) + 0 = \omega$
- Case 2.** $\omega = P(E_1 \cap E_2 \cap E_3)$
 LHS = ω
 RHS = $(\omega + \omega + \omega) - (\omega + \omega + \omega) + \omega = \omega$

N12 -

- $P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i)$
- $P\left(\bigcup_{i=1}^n E_i\right) \geq \sum_{i=1}^n P(E_i) - \sum_{j < i} P(E_i E_j)$
- $P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i) - \sum_{j < i} P(E_i E_j) + \sum_{k < j < i} P(E_i E_j E_k)$
- and so on.

Proof. $\bigcup_{i=1}^n E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \dots \cup E_1^c E_2^c \dots E_{n-1}^c E_n$

$$P\left(\bigcup_{i=1}^n E_i\right) = P(E_1) + P(E_1^c E_2) + P(E_1^c E_2^c E_3) + \dots + P(E_1^c E_2^c \dots E_{n-1}^c E_n)$$

Sample Space having Equally Likely Outcomes

tricky - 14, 15, 16, 18, 19, 20

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E , $P(E) = \frac{\# \text{ of outcomes in } E}{\# \text{ of outcomes in } S} = \frac{\# \text{ of outcomes in } E}{n}$

increasing sequence of events $\{E_n, n \geq 1\} \rightarrow$

$E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$

$\lim_{n \rightarrow \infty} E_n = \bigcup_{i=1}^{\infty} E_i$

decreasing sequence of events $\{E_n, n \geq 1\} \rightarrow$

$E_1 \supset E_2 \supset \dots \supset E_n \supset E_{n+1} \supset \dots$

$\lim_{n \rightarrow \infty} E_n = \bigcap_{i=1}^{\infty} E_i$

03. CONDITIONAL PROBABILITY AND INDEPENDENCE

tricky - E6, urns (p.37)

Conditional Probability

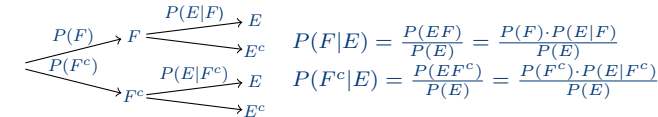
N1 - if $P(F) > 0$. then $P(E|F) = \frac{P(E \cap F)}{P(F)}$

N2 - multiplication rule - $P(E_1 E_2 \dots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \dots P(E_n|E_1 E_2 \dots E_{n-1})$

Total Probability & Bayes' Theorem

conditioning formula - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$

tree diagram -



Total Probability

theorem of total probability - Suppose F_1, F_2, \dots, F_n are mutually exclusive

events such that $\bigcup_{i=1}^n F_i = S$, then $P(E) = \sum_{i=1}^n P(EF_i) = \sum_{i=1}^n P(F_i)P(E|F_i)$

Bayes Theorem

$$P(F_j|E) = \frac{P(EF_j)}{P(E)}$$
$$= \frac{P(F_j)P(E|F_j)}{\sum_{i=1}^n P(F_i)P(E|F_i)}$$

application of bayes' theorem

$$P(B_1 | A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)}$$

Let A be the event that the person test positive for a disease.

B_1 : the person actually has the disease.

B_2 : the person does not have the disease.

true positives: $P(B_1 A)$	false negatives: $P(\bar{A} B_1)$
false positives: $P(A B_2)$	true negatives: $P(\bar{A} B_2)$

Independent Events

N1 - E and F are independent \iff

$P(EF) = P(E) \cdot P(F)$ or $P(E|F) = P(E)$

N2 - if E and F are independent, then E and F^c are independent.

N3 - if E, F, G are independent, then E will be independent of any event formed from F and G . (e.g. $F \cup G$)

04. RANDOM VARIABLES

Random Variables

- random variable \rightarrow a real-valued function defined on the sample space
- X is a Bernoulli r.v. with parameter p if \rightarrow

$$p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1 - p, & x = 0 \text{ ('failure')} \end{cases}$$

commutative	$E \cup F = F \cup E$	$E \cap F = F \cap E$
associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(E \cap F) \cap G = E \cap (F \cap G)$
distributive	$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$	$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$
DeMorgan's	$(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i^c$	$(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$