ST2131
AY21/22 SEM 2
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01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- **combinatorial analysis** $\rightarrow$ the mathematical theory of counting
- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1!\,n_2!\dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \le r \le n$

Proof. If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.
If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - **The Binomial Theorem** - $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Proof. by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\,n_2!\dots n_r!}$ represents the number of possible divisions of n distrinct objects into r distinct groups of respective sizes $n_1, n_2, \dots, n_3$, where $n_1 + n_2 + \dots + n_r = n$

Proof. using basic counting principle,

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-n_{r-1}}{n_r}$$
$$= \frac{n!}{(n-n_1)!\,n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!} \times \dots \times \frac{(n-n_1-n_2-\dots-n_{r-1})}{0!\,n_r!}$$
$$= \frac{n!}{n_1!\,n_2!\dots n_r!}$$

N7 - **The Multinomial Theorem**: $(x_1 + x_2 + \dots + x_r)^n$
$$= \sum_{(n_1,\dots,n_r):n_1+n_2+\dots+n_r=n} \frac{n!}{n_1!\,n_2!\dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \dots, r$

! cannot be directly applied to *N8* as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- **sample space** $\rightarrow$ The *set* of all outcomes of an experiment (where outcomes are not predictable with certainty)
- **event** $\rightarrow$ Any *subset* of the sample space
- **union** of events E and $F \rightarrow E \cup F$ is the event that contains all outcomes that are either in E or F (or both).
- **intersection** of events E and $F \rightarrow E \cap F$ or EF is the event that contains all outcomes that are both in E and in F.
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E.
- **subset** $\rightarrow E \subset F$ is all of the outcomes in E that are also in F.
  - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws

$$\left(\bigcup_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcap_{i=1}^{n} \mathbf{E_i^c}$$

Proof. to show LHS $\subset$ RHS: let $x \in (\bigcup_{i=1}^n E_i)^c$
$\Rightarrow x \notin \bigcup_{i=1}^n E_i \Rightarrow x \notin E_1$ and $x \notin E_2 \dots$ and $x \notin E_n$
$\Rightarrow x \in E_1^c$ and $x \in E_2^c \dots$ and $x \in E_n^c$
$\Rightarrow x \in \bigcap_{i=1}^n E_i^c$
to show RHS $\subset$ LHS: let $x \in \bigcap_{i=1}^n E_i^c$

$$\left(\bigcap_{i=1}^{n} \mathbf{E_i}\right)^{\mathbf{c}} = \bigcup_{i=1}^{n} \mathbf{E_i^c}$$

Proof. using the first law of DeMorgan, negate LHS to get RHS

Axioms of Probability

definition 1: relative frequency

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

problems with this definition:

1. $\frac{n(E)}{n}$ may not converge when $n \to \infty$
2. $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated

definition 2: Axioms

Consider an experiment with sample space S. For each event E of the sample space S, we assume that a number $P(E)$ is definned and satisfies the following 3 axioms:

1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. For any sequence of mutually exclusive events $E_1, E_2, \dots$ (i.e., events for which $E_iE_j = \emptyset$ when $i \neq j$),
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ is the probability of event E.

Simple Propositions

N1 - $P(\emptyset) = 0$

N2 - $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ (aka axiom 3 for a finite n)

N3 - **strong law of large numbers** - if an experiment is repeated over and over again, then with probability 1, the proportion of time during which any specific event E occurs will be equal to $P(E)$.

N6 - the definitions of probability are mathematical definitions. They tell us which set functions can be called **probability functions**. They do not tell us what value a probability function $P(\cdot)$ assigns to a given event E.

probability function $\iff$ it satisfies the 3 axioms.

N7 - $P(E_c) = 1 - P(E)$

N8 - if $E \subset F$, then $P(E) \le P(F)$

N9 - $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

N10 - Inclusion-Exclusion identity where $n = 3$

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$

N11 - **Inclusion-Exclusion identity** -

$$\begin{aligned} P(E_1 \cup E_2 \cup \dots \cup E_n) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \dots \\ &\quad + (-1)^{r+1} \sum_{i_1<i_2<\dots<i_r} P(E_{i_1}E_{i_2}\dots E_{i_r}) + \dots \\ &\quad + (-1)^{n+1} P(E_1E_2\dots E_n) \end{aligned}$$

Proof. Suppose an outcome with probability ω is in exactly m of the events E_i, where $m > 0$. Then
LHS: the outcome is in $E_1 \cup E_2 \cup \dots \cup E_n$ and ω will be counted once in $P(E_1 \cup E_2 \cup \dots \cup E_n)$
RHS:

- the outcome is in exactly m of the events E_i and ω will be counted exactly $\binom{m}{1}$ times in $\sum_{i=1}^{n} P(E_i)$
- the outcome is contained in $\binom{m}{2}$ subsets of the type $E_{i_1}E_{i_2}$ and ω will be counted $\binom{m}{2}$ times in $\sum_{i_1<i_2} P(E_{i_1}E_{i_2})$
- $\dots$ and so on

hence RHS = $\binom{m}{1}\omega - \binom{m}{2}\omega + \binom{m}{3}\omega - \dots \pm \binom{m}{m}\omega$
$= \omega \sum_{i=0}^{m} \binom{m}{i}(-1)^i =$ binomial theorem where $x = -1, y = 1$
$= 0 =$ LHS

e.g. For an outcome with probability ω and $n = 3$

- *Case 1.* $w = P(E_1E_2)$
  LHS = ω
  RHS = (ω + ω + 0) - (ω + 0 + 0) + 0 = ω
- *Case 2.* $\omega = P(E_1 \cap E_2 \cap E_3)$
  LHS = ω
  RHS = (ω + ω + ω) - (ω + ω + ω) + ω = ω

N12 -

(i) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i)$

(ii) $P(\bigcup_{i=1}^n E_i) \ge \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_iE_j)$

(iii) $P(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^{n} P(E_i) - \sum_{j<i} P(E_iE_j) + \sum_{k<j<i} P(E_iE_jE_k)$

(iv) and so on.

Proof. $\bigcup_{i=1}^{n} E_i = E_1 \cup E_1^cE_2 \cup E_1^cE_2^cE_3 \cup \dots \cup E_1^cE_2^c \dots E_{n-1}^cE_n$
$P(\bigcup_{i=1}^{n} E_i) = P(E_1) + P(E_1^cE_2) + P(E_1^cE_2^cE_3) + \dots + P(E_1^cE_2^c \dots E_{n-1}^cE_n)$

Sample Space having Equally Likely Outcomes

tricky - 14, 15, 16, 18, 19, 20

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E, $P(E) = \frac{\text{\# of outcomes in } E}{\text{\# of outcomes in } S} = \frac{\text{\# of outcomes in } E}{n}$

increasing sequence of events $\{E_n, n \ge 1\} \rightarrow E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$$

decreasing sequence of events $\{E_n, n \geq 1\} \rightarrow$
$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \ldots$

$$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$$

commutative	$E \cup F = F \cup E$	$E \cap F = F \cap E$
associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(E \cap F) \cap G = E \cap (F \cap G)$
distributive	$(E \cup F) \cap G = (E \cap F) \cup (F \cap G)$	$(E \cap F) \cup G = (E \cup F) \cap (F \cup G)$
DeMorgan's	$(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i^c$	$(\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i^c$