

MA1102R

AY20/21 sem 2

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00. FUNCTIONS & SETS

sets

$$A = \{x \mid \text{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \iff A \subseteq B \wedge B \subseteq A$
- operations on sets**
 - union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
 - intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
 - difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$
- common notations on sets:**
 - $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ where $\mathbb{N} = \mathbb{Z}^+$
 - \emptyset : empty set

closed interval (inclusive):

$$[a, b] = \{x \mid a \leq x \leq b\}$$

open interval (exclusive):

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, \infty) = \{x \mid a < x\}$$

functions

- existence:** $\forall a \in A, f(a) \in B$
- uniqueness:** $\forall a \in A$ has only one image in B .
- for $f: A \rightarrow B$
 - domain: A , codomain: B
 - range: $\{f(x) \mid x \in A\}$
- for this mod:
 - $A, B \subseteq \mathbb{R}$
 - if A is not stated, the domain of f is the largest possible set for which f is defined
 - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set
 $G(f) := \{(x, f(x)) \mid x \in A\}$

- if $A, B \subseteq \mathbb{R}$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane \mathbb{R}^2

algebra of functions

function	domain
$(f+g)(x) := f(x) + g(x)$	$A \cap B$
$(f-g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

types of functions

- rational function:** $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
 - every polynomial is a rational function ($Q(x) = 1$)
- algebraic function:** constructed from polynomials using algebraic operations
- a function f is **increasing** on a set I if
 $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if
 $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.

• even/odd:

- even function:** $\forall x, f(-x) = f(x)$
 - symmetric about the y -axis
- odd function:** $\forall x, f(-x) = -f(x)$
 - symmetric about the origin O
- any function defined on \mathbb{R} can be decomposed *uniquely* into the sum of an even function and an odd function
- power function:** x^n
 - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

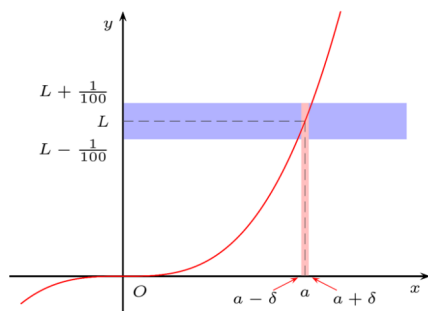
01. LIMITS

precise definition of limits

Let f be a function defined on an open interval containing a , except possibly at a .

The limit of $f(x)$ (as x approaches a) equals L if,

$$\text{for every } \epsilon > 0 \text{ there is } \delta > 0 \text{ such that} \\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$



informally,

- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a .
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L .

limit laws

you cannot apply any laws on limits UNLESS you have shown that the limit exists!!

- Let $c \in \mathbb{R}$. $\lim_{x \rightarrow a} c = c$

- $\lim_{x \rightarrow a} x = a$

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant.

- $\lim_{x \rightarrow a} (cf(x)) = cL = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

$$\text{if } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \rightarrow a} g(x) = 0, \text{ then } \lim_{x \rightarrow a} f(x) = 0$$

inequalities on limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

lemma

if $f(x) \leq g(x)$ for all x near a (except possibly at a), then $L \leq M$.

lemma

If $f(x) \geq 0$ for all x , then $L \geq 0$.

direct substitution property

Let f be a polynomial or rational function.

If a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

If $f(x) = g(x)$ for all x near a except possibly at a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

If a is not in the domain (e.g. 0 denominator), don't apply directly - convert to an equivalent function and then sub in

one-sided limits

- limit laws also hold for one-sided limits

If as x is close to a from the right, $f(x)$ is close to L , the right-hand limit of f as x approaches a equals L .
 $(x \rightarrow a^+ \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^+} f(x) = L$

If as x is close to a from the left, $f(x)$ is close to L , the left-hand limit of f as x approaches a equals L .
 $(x \rightarrow a^- \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^-} f(x) = L$

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

$$f(x) \rightarrow L \Leftarrow x \rightarrow a \Leftrightarrow \begin{cases} x \rightarrow a^+ \Rightarrow f(x) \rightarrow L \\ x \rightarrow a^- \Rightarrow f(x) \rightarrow L \end{cases}$$

definition of one-sided limits

$$\text{LH Limit: } \lim_{x \rightarrow a^-} f(x) = L$$

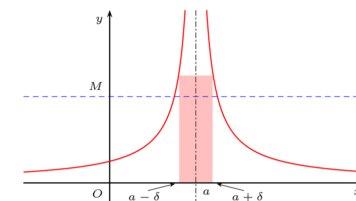
if for every $\epsilon > 0$ there exists $\delta > 0$ such that
 $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$

$$\text{RH Limit: } \lim_{x \rightarrow a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that
 $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$

definition of infinite limits

$\lim_{x \rightarrow a} f(x) = \infty$
if for every $M > 0$ there exists $\delta > 0$ such that
 $0 < |x - a| < \delta \Rightarrow f(x) > M$



negative infinite limit:

$$0 < |x - a| < \delta \Rightarrow f(x) < M$$

- ∞ is NOT a number \Rightarrow an infinite limit does NOT exist

limits to infinity

Suppose f is defined on $[M, \infty)$ for some $M \in \mathbb{R}$:

$$\lim_{x \rightarrow \infty} f(x) = L:$$

For every $\epsilon > 0$, there exists N such that
 $x > N \Rightarrow |f(x) - L| < \epsilon$

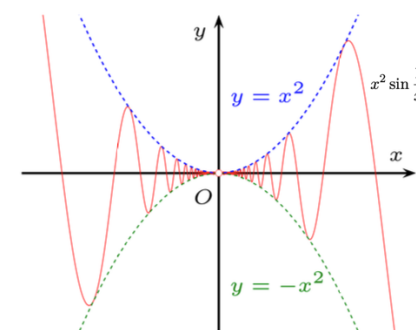
$$\lim_{x \rightarrow \infty} f(x) = \infty:$$

For every $M > 0$, there exists N such that
 $x > N \Rightarrow f(x) > M$

squeeze theorem

- Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where
- $g(x) \leq f(x) \leq h(x)$ for all x near a (except at a), and
 - $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$.

Then $\lim_{x \rightarrow a} f(x) = L$.



02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at a** \iff
 f is continuous from the left and from the right at a .
$$\lim_{x \rightarrow a} f(x) = f(a)$$

- f is continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = a$
- f is continuous from the left at a if $\lim_{x \rightarrow a^-} f(x) = a$

a function f is **continuous at an interval** if it is continuous at every number in the interval.

$$\begin{aligned} &f \text{ is continuous on } \textbf{open interval } (a, b) \\ &\Leftrightarrow f \text{ is continuous at every } x \in (a, b) \\ &f \text{ is continuous on } \textbf{closed interval } [a, b] \\ &\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases} \end{aligned}$$

precise definition of continuity

a function f is **continuous** at a number a if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

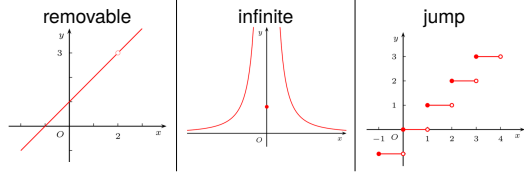
- aka $\lim_{x \rightarrow a} f(x) = f(a)$

continuity test

f is continuous at $a \Leftrightarrow$

- f is defined at a (a is in the domain of f)
- $\lim_{x \rightarrow a} f(x)$ exists
- $\lim_{x \rightarrow a} f(x) = f(a)$

examples of discontinuity



properties of continuous functions

let f and g be functions continuous at a . let c be a constant.

- cf is continuous at a
- $f + g$ is continuous at a
- $f - g$ is continuous at a
- fg is continuous at a
- f/g is continuous at a , provided $g(a) \neq 0$

other properties

- a polynomial is continuous everywhere
- a rational function is continuous on its domain
 - if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ is continuous whenever $Q(x) \neq 0$.
- $f(x) = c$ is continuous on \mathbb{R} for all $c \in \mathbb{R}$.
- $f(x) = x$ is continuous on \mathbb{R} .

trigonometric functions

- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous everywhere
- $\tan x, \sec x$ are continuous whenever $\cos x \neq 0$
 - domain: $\mathbb{R} \setminus \{\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots\}$
- $\cot x, \csc x$ are continuous whenever $\sin x \neq 0$
 - domain: $\mathbb{R} \setminus \{0, \pm \pi, \pm 2\pi, \dots\}$

composite of continuous functions

if f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b)$$

if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .

$$\lim_{x \rightarrow a} (f \circ g)(x) = (f \circ g)(a)$$

substitution theorem

Suppose $y = f(x)$ such that $\lim_{x \rightarrow a} f(x) = b$. If

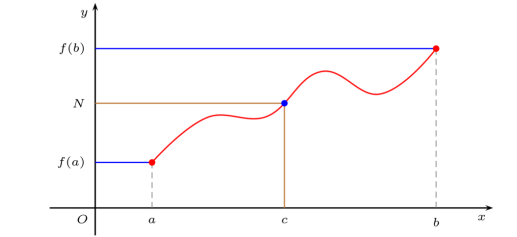
- g is continuous at b , OR
- $\forall x$ near a , except at a , $f(x) \neq b$ and $\lim_{y \rightarrow b} g(y)$ exists

- aka, $\lim_{y \rightarrow b} g(y)$ exists and f is one-to-one.

Then $\lim_{x \rightarrow a} g(f(x)) = \lim_{y \rightarrow b} g(y)$

intermediate value theorem

Let f be a function continuous on $[a, b]$ with $f(a) \neq f(b)$. Let N be a number between $f(a)$ and $f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = N$.



03. DERIVATIVES

tangent line

the **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:

$$y = f'(a)(x - a) + f(a)$$

definition of derivatives

- f is differentiable at a if $f'(a)$ exists
- $f'(a)$ is the slope of $y = f(x)$ at $x = a$
 - $f'(a) = \frac{dy}{dx} \Big|_{x=a}$
 - $\frac{dy}{dx} := \lim_{x \rightarrow 0} \frac{\Delta y}{\Delta x}$ (derivative of y with respect to x)
- $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x) = \dots$

$$\begin{aligned} &\text{the derivative of a function } f \\ &f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &\text{the derivative of a function } f \text{ at a number } a \text{ is} \\ &f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

differentiable functions

- f is differentiable at a if
 - $f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.
- f is differentiable on (a, b) if
 - f is differentiable at every $c \in (a, b)$

differentiability & continuity

- differentiability \Rightarrow continuity
 - if f is differentiable at a , then f is continuous at a .
- continuity \nRightarrow differentiability

differentiation

- every polynomial and rational function is differentiable on its domain
 - the domain of f' may be smaller than the domain of f .
- trigonometric functions are differentiable on the domain

differentiation of trigonometric functions

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \Bigg| \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

chain rule

If g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and

$$F'(a) = (f \circ g)'(a) = f'(b)g'(a) = f'(g(a))g'(a)$$

$$\begin{aligned} &\text{If } z = f(y) \text{ and } y = g(x), \text{ then} \\ &\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} \\ &\frac{dz}{dx} \Big|_{x=a} = \frac{dz}{dy} \Big|_{y=b} \frac{dy}{dx} \Big|_{x=a} \end{aligned}$$

generalised chain rule

h is differentiable at a ; g is differentiable at $B = h(a)$; f is differentiable at $c = g(b)$.

$$\begin{aligned} (f \circ (g \circ h))' &= f' \circ (g \circ h) \cdot (g \circ h)' \\ &= f'(c)g'(b)h'(a) \end{aligned}$$

$$\begin{aligned} &\text{Leibniz notation:} \\ &\text{If } y = h(x), z = g(y), w = f(z), \\ &\frac{dw}{dx} = \frac{dw}{dz} \frac{dz}{dy} \frac{dy}{dx} \end{aligned}$$

implicit differentiation

- assumes that $\frac{dy}{dx}$ exists

second derivative

$$\begin{aligned} f''(x) &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} \\ f' &= D(f) \Rightarrow f'' := D^2(f) \end{aligned}$$

higher derivatives

$$\begin{aligned} f^{(0)} &:= f \\ &\text{For any positive integer } n, f^{(n)} := (f^{(n-1)})' \\ &\text{if } y = f(x), \text{ then } f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x) \end{aligned}$$

- for $f(x) = \frac{1}{x}$, $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$
- for $f(x) = x^m$, $f^{(n)}(x) = \begin{cases} \frac{m! x^{m-n}}{(m-n)!} & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$

04. APPLICATIONS OF DIFFERENTIATION

extreme values of functions

Let f be a function with domain D .

global (absolute) max/min

- aka absolute max/min
- extreme values = absolute maximum and absolute minimum

$$\begin{aligned} &f \text{ has a global } \textbf{maximum} \text{ at } c \in D \\ &\Leftrightarrow f(c) \geq f(x) \text{ for all } x \in D \\ &f \text{ has a global } \textbf{minimum} \text{ at } c \in D \\ &\Leftrightarrow f(c) \leq f(x) \text{ for all } x \in D \end{aligned}$$

local max/min

- aka relative max/min aka "turning points"
- "all x near c " = for all x in an open interval containing c

$$\begin{aligned} &f \text{ has a local } \textbf{maximum} \text{ at } c \in D \\ &\Leftrightarrow f(c) \geq f(x) \text{ for all } x \text{ near } c \\ &f \text{ has a local } \textbf{minimum} \text{ at } c \in D \\ &\Leftrightarrow f(c) \leq f(x) \text{ for all } x \text{ near } c \end{aligned}$$

- local max/min \nRightarrow global max/min
- global max/min \nRightarrow local max/min

extreme value theorem

existence

if f is continuous on a finite closed interval $[a, b]$, then f attains extreme values on $[a, b]$.

value

the extreme value occurs at either *critical numbers* or the *endpoints* ($x = a, x = b$).

critical numbers

$c \in D$ is a *critical number* of f if $f'(c) = 0$, or $f'(c)$ does not exist.

fermat's theorem

If f has a local maximum or minimum at c , then c is a critical number.

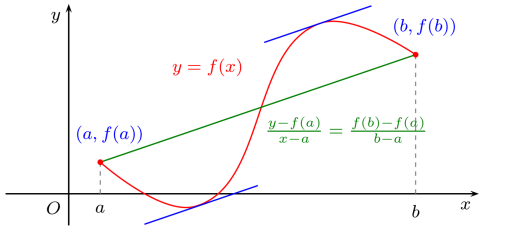
If $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function such that f is continuous on $[a, b]$, f is differentiable on (a, b) , and $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

mean value theorem

Let f be a function such that f is continuous on $[a, b]$ and f is differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$


- generalisation of Rolle's theorem when $f(a) \neq f(b)$.

ordinary differential equations

Let f and g be continuous on $[a, b]$.
If $f'(x) = g'(x)$ for all $x \in (a, b)$,
then $f(x) = g(x) + C$ on $[a, b]$ for a constant C .

increasing/decreasing test

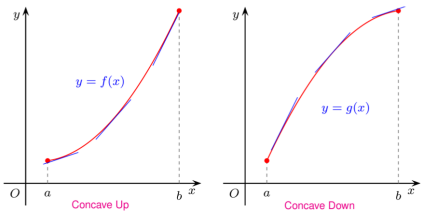
- Let f be continuous on $[a, b]$ and differentiable on (a, b) .
- $f'(x) > 0$ for any $x \in (a, b) \Rightarrow f$ is increasing.
 - f is increasing $\Rightarrow f(x) \geq 0$
 - $f'(x) < 0$ for any $x \in (a, b) \Rightarrow f$ is decreasing.
 - f is decreasing $\Rightarrow f(x) \leq 0$
 - $f'(x) = 0 \Rightarrow f$ could be increasing OR decreasing.

first derivative test

Let f be continuous and c be a critical number of f . Suppose f is differentiable near c (except possibly at c). At c , if f' changes from:

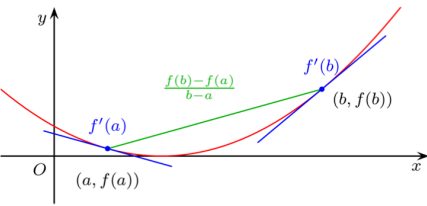
- (+) to (-) $\Rightarrow f$ has a local **maximum** at c
- (-) to (+) $\Rightarrow f$ has a local **minimum** at c
- no change in sign $\Rightarrow f$ has neither local max/min at c .

concavity



f is **concave up** on an open interval I
if $f(x) > f'(y)(x - y) + f(y)$ for any $x \neq y \in I$
for $a < b \in I, f'(a) < f'(b)$
concave up $\Leftrightarrow f'$ is increasing

f is **concave down** on an open interval I
if $f(x) < f'(y)(x - y) + f(y)$ for any $x \neq y \in I$
for $a < b \in I, f'(a) > f'(b)$
concave down $\Leftrightarrow f'$ is decreasing



concavity test

- $f'' > 0$ on $I \Rightarrow f$ is concave up on I
- $f'' < 0$ on $I \Rightarrow f$ is concave down on I

second derivative test

If $f'(c) = 0$ and $f''(c)$ exists,

- $f''(c) < 0 \Rightarrow f$ has a **local maximum** at c .
- $f''(c) > 0 \Rightarrow f$ has a **local minimum** at c .
- $f''(c) = 0 \Rightarrow$ inconclusive

inflection point

- A point P on the curve $y = f(x)$ is an inflection point if
 - f is continuous at P , and
 - the concavity of the curve changes at P .
- if c is an inflection point and f is twice differentiable at c , then $f''(c) = 0$.

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n,$$

where $R_n = \frac{f^{(n+1)}(a)}{(n+1)!}(x - a)^{(n+1)}$ for c between x and a

Taylor Series

As $R - n \rightarrow 0$ as $n \rightarrow \infty$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

L'Hopital's Rule ($\frac{0}{0}$)

Let f and g be functions such that

- $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$
- f and g are differentiable near a (except at a).

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$,
provided that the RHS limit exists or is $\pm\infty$

L'Hopital's Rule ($\frac{\infty}{\infty}$)

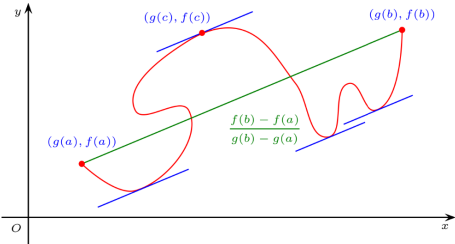
Suppose that

- $\lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$,
- f and g are differentiable near a (except at a),
- $g'(x) \neq 0$ near a (except at a)

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$
provided that the RHS limit exists or is $\pm\infty$

Cauchy's Mean Value Theorem

Let f, g be continuous on $[a, b]$, differentiable on (a, b) ,
and $g'(x) \neq 0$ for any $x \in (a, b)$.
Then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$


05. INTEGRALS

definite integral

Let f be a continuous function on $[a, b]$ divided into n intervals.

Riemann sum

$$[f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)] \Delta x = \sum_{i=1}^n f(x_i^*) \Delta x$$

- the lengths of subintervals are not necessarily equal
 - $\max\{|x_i - x_{i-1}| : i = 1, \dots, n\} \rightarrow 0$

definite integral of f from a to b :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

where $\Delta x = \frac{b-a}{n}$

- f is **integrable** from a to b if $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists.
- continuous functions are integrable.
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- $\int_a^a f(x) dx = 0$

properties

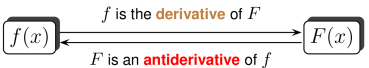
let f and g be continuous functions.

- $\int_a^b c dx = (b - a)c$
- $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^c f(x) dx = \int_b^c f(x) dx \pm \int_a^b f(x) dx$
- suppose $f(x) \geq 0$ on $[a, b]$. Then $\int_a^b f(x) dx \geq 0$.
- suppose $f(x) \geq g(x)$ on $[a, b]$.
 - Then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.
- suppose $m \leq f(x) \leq M$ on $[a, b]$.
 - Then $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$.

fundamental theorem of calculus

for $g(x) = \int_a^x f(t) dt$ ($a \leq x \leq b$),

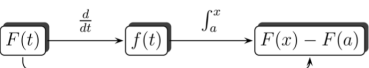
- g is continuous on $[a, b]$
- g is differentiable on (a, b)
- $g'(x) = f(x)$ on (a, b) or $\frac{d}{dx} \int_a^x f(t) dt = f(x)$



if F is continuous on $[a, b]$, and $F' = f$ on (a, b) ,

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_{x=a}^{x=b}$$

$$\int_a^x \frac{d}{dx} F(t) dt = F(x) - F(a)$$



indefinite integral

- indefinite integral** of f , $\int f(x) dx = F(x) + c$
- antiderivative** (of a continuous function f): a continuous function F such that $F' = f$.
 - antiderivatives of f are functions of form $F + c$
 - indefinite integral is a family of antiderivatives
- properties of indefinite integral
 - $\int (af(x) \pm bg(x)) dx = a \int f(x) dx \pm b \int g(x) dx$

integration by parts

$$u dv = uv - \int v du$$

substitution rule (I)

let $u = g(x)$ be a differentiable function.

indefinite integral

if f and g' are continuous,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

definite integral

if g' are continuous on $[a, b]$,
and f is continuous on the range of $u = g(x)$,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

substitution rule (II)

let f and g' be continuous functions, and
 $x = g(t)$ is a one-to-one differentiable function.

$$\int f(x) dx = \int f(g(t))g'(t) dt$$

improper integral

for discontinuous integrands

if f is continuous on $[a, b)$ and discontinuous at b ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if f is continuous on $(a, b]$ and discontinuous at a ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

- $\int_a^b f(x) dx$ is the limit of integrals.
 - converges** if the limit exists
 - diverges** if the limit does not exist

discontinuity in the interior of the interval

suppose f has discontinuity at $c \in (a, b)$. then

$$\int_a^b f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

over infinite intervals

∫_{-∞}^∞ f(x) dx = ∫_{-∞}^a f(x) dx + ∫_a^∞ f(x) dx

if ∫_a^t f(x) dx exists for every t ≥ a, then the **improper integral** of f from a to ∞ is

∫_a^∞ f(x) dx = lim_{t→∞} ∫_a^t f(x) dx

if ∫_t^b f(x) dx exists for every t ≤ b, then the **improper integral** of f from -∞ to b is

∫_{-∞}^b f(x) dx = lim_{t→-∞} ∫_t^b f(x) dx

• NOTE: ∫_{-∞}^∞ f(x) dx ≠ lim_{a→∞} ∫_{-a}^a f(x) dx

06. INVERSE FUNCTIONS & INTEGRATION

one to one functions

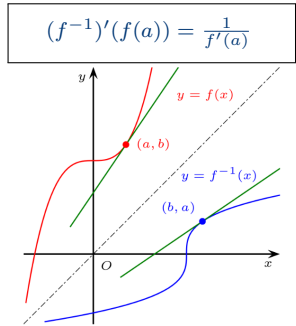
let f be a function with domain D.
f is **one-to-one** if, for any a, b ∈ D,
a ≠ b ⇒ f(a) ≠ f(b)
OR f(a) = f(b) ⇒ a = b

inverse function

let f be a one-to-one function with domain A and range B.
• its **inverse function** f⁻¹ is the function with
• domain B and range A, and
• f⁻¹(y) = x ⇔ y = f(x) for any x ∈ A, y ∈ B
• f⁻¹ ∘ f = id_A and f ∘ f⁻¹ = id_B
• (f⁻¹)⁻¹ = f
• NOTE: (f(x))⁻¹ is the reciprocal of the value of f(x)

properties

let f be a *one-to-one continuous* function on an open interval I.
• the inverse function f⁻¹ is also continuous.
• if f is differentiable at a ∈ I, and f'(a) ≠ 0, then
• f⁻¹ is differentiable at b = f(a)
• (f⁻¹)'(b) = 1 / f'(a)



techniques of integration

common trigonometric substitutions

- √(a² - x²), x = a sin t, t ∈ [-π/2, π/2]
- √(x² - a²), x = a sec t, t ∈ [0, π/2) ∪ (π, 3π/2]
- a² + x², x = a tan t, t ∈ (-π/2, π/2)

integration of rational functions

for f = A(x)/B(x),
• manipulate such that deg A(x) < deg B(x), then decompose into partial fractions
• common rational functions:

• ∫ 1/(x+a)^k dx = { ln|x+a| + K, if k = 1; (x+a)^(1-k)/(1-k) + K, if k ≥ 2 }

• ∫ u/(u² + d²)^r du = { 1/2 ln(u² + d²), if r = 1; (u² + d²)^(1-r)/2(1-r), if r ≥ 2 }

• ∫ 1/(u² + d²)^r du = 1/d^(2r-1) ∫ 1/(t² + 1)^r dt

universal trigonometric substitution

any rational expression in sin x and cos x can be integrated using the substitution t = tan x/2, x ∈ (-π, π).
sin x = 2t/(1+t²), cos x = (1-t²)/(1+t²), dx/dt = 2/(1+t²)

derivatives of trigonometric functions

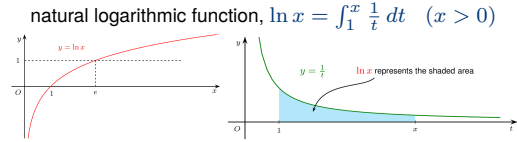
function	derivative	function	derivative
sin ⁻¹ x	1/√(1-x²)	csc ⁻¹ x	-1/(x√(x²-1))
cos ⁻¹ x	-1/√(1-x²)	sec ⁻¹ x	1/(x√(x²-1))
tan ⁻¹ x	1/(1+x²)	cot ⁻¹ x	-1/(1+x²)

trigonometric identities

• tan⁻¹ x + cot⁻¹ x = π/2

• sec⁻¹ x + csc⁻¹ x = { π/2, if x ≥ 1; 5π/2, if x ≤ -1 }

natural logarithmic function



- ln x < 0 for 0 < x < 1; ln x > 0 for x > 1; ln 1 = 0
- ln x is increasing on ℝ⁺ (d/dx ln x > 0)

logarithmic differentiation I

aka take ln on both sides and implicitly differentiate

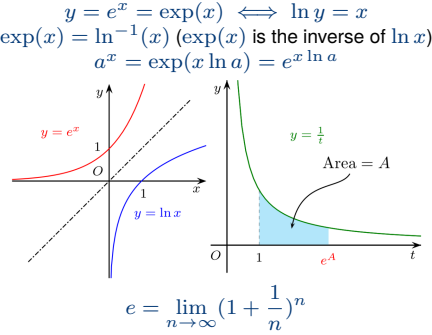
for y = f₁(x)f₂(x)⋯f_n(x) (product of nonzero functions),
ln|y| = ln|f₁(x)| + ln|f₂(x)| + ⋯ + ln|f_n(x)|
dy/dx = [f'₁(x)/f₁(x) + f'₂(x)/f₂(x) + ⋯ + f'_n(x)/f_n(x)] y
= [f'₁(x)/f₁(x) + f'₂(x)/f₂(x) + ⋯ + f'_n(x)/f_n(x)] f₁(x)f₂(x)⋯f_n(x)

logarithmic differentiation II

for y = f(x)^{g(x)} (f(x) > 0),
ln y = g(x) ln f(x) ⇒ dy/dx = y * d/dx [g(x) ln f(x)]

lim_{x→a} (f(x)^{g(x)}) = lim_{x→a} exp(g(x) ln f(x))
= exp(lim_{x→a} g(x) ln f(x))

exponential function



- ln(e^x) = x for x ∈ ℝ and e^{ln y} = y for y ∈ ℝ⁺
- common equations
 - lim_{x→∞} e^x = ∞, lim_{x→-∞} e^x = 0
 - lim_{x→∞} x^n / e^x = 0 for n ∈ ℤ⁺
 - e^x = ∑_{n=0}^∞ x^n / n! = 1 + x + x²/2! + x³/3! + ...

properties

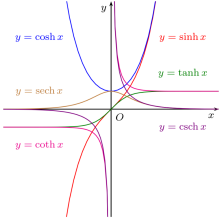
- a^ua^v = a^{u+v}
- a^{-u} = 1/a^u
- (a^u)^v = a^{uv}
- (a^x)' = a^x ln a
- d/dx x^r = r x^{r-1}
- lim_{x→∞} e^x = ∞, lim_{x→-∞} e^x = 0
- lim_{x→∞} x^n / e^x = 0 for n ∈ ℤ⁺
- e^x = ∑_{n=0}^∞ x^n / n! = 1 + x + x²/2! + ...
- ∫ x^r dx = { x^(r+1)/(r+1) + C if r ≠ -1, ln x + C if r = -1, if r is irrational, then x^r is only defined for x ≥ 0.

hyperbolic trigonometric functions

sinh x = (e^x - e^-x)/2, (sinh x)' = cosh x
cosh x = (e^x + e^-x)/2, (cosh x)' = sinh x

properties

- cosh² x - sinh² x = 1
- parametrization represents a **hyperbola** -
let { x = cosh t, y = sinh t. Then x² - y² = 1



tanh x = sinh x / cosh x
coth x = cosh x / sinh x
sech x = 1 / cosh x
csch x = 1 / sinh x

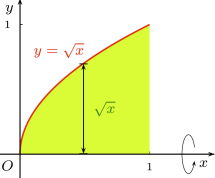
inverse hyperbolic functions:
sinh⁻¹ x = y ⇔ x = sinh y
cosh⁻¹ x = y ⇔ x = cosh y

- properties
 - d/dx sinh⁻¹ x = 1/√(x²+1)
 - d/dx cosh⁻¹ x = 1/√(x²-1)
 - sinh⁻¹ x = ln(x + √(x²+1)), x ∈ ℝ
 - cosh⁻¹ x = ln(x + √(x²-1)), x ≥ 1
 - tanh⁻¹ x = 1/2 ln((1+x)/(1-x)), -1 < x < 1

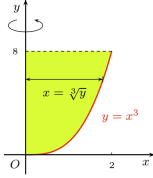
07. APPLICATIONS OF INTEGRALS

volume

disk/washer method

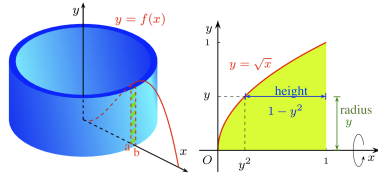


rotate about the **x-axis**:
V = π ∫_a^b [f(x)]² dx



rotate about the **y-axis**:
V = π ∫_c^d [f(y)]² dy

method of cylindrical shells



rotation about **x-axis** from y = a to y = b:
V = 2π ∫_a^b y f(y) dy = 2π ∫ (radius · height) dy

rotation about **y-axis** from x = a to x = b:
V = 2π ∫_a^b x f(x) dx = 2π ∫ (radius · height) dx

arc length

- a function f is **smooth** if f' is continuous.
- arc length,
L = ∫_a^b √(1 + (f'(x))²) dx

surface area of revolution

Let f be a smooth function such that f(x) ≥ 0 on [a, b].
Then the area of the surface obtained by rotating the curve y = f(x), a ≤ x ≤ b about the x-axis is

A = ∫_a^b 2π f(x) √(1 + (f'(x))²) dx

08. ORDINARY DIFFERENTIAL EQUATIONS

dy/dx = f(x) => y = integral f(x) dx
dy/dx = f(y) => x = integral 1/f(y) dy

separation of variables

dy/dx = f(x)g(y) => 1/g(y) dy = f(x) dx
=> integral 1/g(y) dy = integral f(x) dx

singular solution

- if y = C is a solution to g(y) = 0, then it is a singular solution to dy/dx = f(x)g(x).
- singular solution disappears if the equation is 1/g(x) dy/dx = f(x)
- (can ignore singular solutions in this course)

homogenous equations

Suppose dy/dx = F(x, y) is not separable.

- suppose F(x, y) is homogenous of degree zero
 - i.e. F(x, y) = F(tx, ty) for all t in R \ {0}
- let z = y/x. Then
 - y = xz and dy/dx = x dz/dx + z
 - F(x, y) = F(y/x, y/x) = F(1, z)
- x dz/dx + z = F(1, z) => separable!

first order linear differential equations

general equation: dy/dx + p(x)y = q(x)

- find P(x) = integral p(x) dx
- multiply both sides by integrating factor v(x) = e^P(x):
 - e^P(x) dy/dx + e^P(x)p(x)y = e^P(x)q(x)
 - d/dx (e^P(x)y) = e^P(x)q(x)
- integrate with respect to x
 - e^P(x) = integral e^P(x)q(x) dx

y = 1/e^P(x) integral e^P(x)q(x) dx

note: if the equation is not linear in y but is linear in x, can take the reciprocal and use dx/dy instead.

Bernoulli's equation

dy/dx + p(x)y = q(x)y^n

- if n = 0 or n = 1:
 - the system is linear
- if n != 0, 1:
 - let z = y^(1-n) => dz/dx = (1-n)y^(-n) dy/dx
 - multiply both sides of the equation by (1-n)y^(-n)
 - equation is reduced to a linear equation
 - dz/dx + (1-n)p(x)z = (1-n)q(x)

applications

- compound interest - let r be the interest rate, A be the money
 - ODE: dA/dt = rA; A(0) = C
 - solve for A(t) = Ce^(rt)
- radiocarbon dating - let lambda be the half life, C be Carbon left
 - ODE: dC/dt = kC; C(0) = 1; k = -ln 2 / lambda
 - solve C(t) = e^(kt)
- population growth - let M be max. population (carrying capacity), r be the rate of change of population
 - ODE: dP/dt = r(M - P)P
 - solve P(t) = M / (1 + (M/P(0) - 1)e^(-rt))

misc

triangle inequality

|a + b| <= |a| + |b| for all a, b in R

binomial theorem

(a + b)^n = sum_{k=0}^n (n choose k) a^(n-k) b^k
= a^n + (n choose 1) a^(n-1) b + ... + (n choose n-1) a b^(n-1) + b^n

where the binomial coefficient is given by

(n choose k) = n! / (k!(n-k)!)

factorisation

a^n - b^n = (a - b)(a^(n-1) + a^(n-2)b + ... + ab^(n-2) + b^(n-1))
a^3 - b^3 = (a - b)(a^2 + ab + b^2)
a^3 + b^3 = (a + b)(a^2 - ab + b^2)

misc

- forall x in (0, pi/2), sin x < x < tan x