ST2131
AY21/22 SEM 2
github/jovyntls

01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- **combinatorial analysis** $\rightarrow$ the mathematical theory of counting
- **basic principle of counting** $\rightarrow$ Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** $\rightarrow$ If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1!\,n_2!\dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n$

Proof: If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.
If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - **The Binomial Theorem** - $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Proof: by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\,n_2!\dots n_r!}$ represents the number of possible divisions of n distrinct objects into r distinct groups of respective sizes $n_1, n_2, \dots, n_3$, where $n_1 + n_2 + \dots + n_r = n$

Proof: using basic counting principle,

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-n_{r-1}}{n_r}$$
$$= \frac{n!}{(n-n_1)!\,n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!} \times \dots \times \frac{(n-n_1-n_2-\dots-n_{r-1})}{0!\,n_r!}$$
$$= \frac{n!}{n_1!\,n_2!\dots n_r!}$$

N7 - **The Multinomial Theorem**: $(x_1 + x_2 + \dots + x_r)^n$
$$= \sum_{(n_1,\dots,n_r):n_1+n_2+\dots+n_r=n} \frac{n!}{n_1!\,n_2!\dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n, \quad x_i > 0, \quad i = 1, 2, \dots, r$

! cannot be directly applied to *N8* as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors $(x_1, x_2, \dots, x_r)$ satisfying $x_1 + x_2 + \dots + x_r = n$

Proof: let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$