MA1102R
AY20/21 sem 2
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00. FUNCTIONS & SETS

sets

$$A = \{x \mid \textit{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \iff A \subseteq B \land B \subseteq A$
- **operations on sets**
  - union: $A \cup B = \{x \mid x \in A \lor x \in B\}$
  - intersection: $A \cap B = \{x \mid x \in A \land x \in B\}$
  - difference: $A \backslash B = \{x \mid x \in A \land x \notin B\}$
- **common notations on sets**:
  - $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ where $\mathbb{N} = \mathbb{Z}^+$
  - $\emptyset$: empty set

closed interval (inclusive): $[a, b] = \{x \mid a \le x \le b\}$

open interval (exclusive): $(a, b) = \{x \mid a < x < b\}$, $(a, \infty) = \{x \mid a < x\}$

functions

- **existence**: $\forall a \in A, f(a) \in B$
- **uniqueness**: $\forall a \in A$ has only one image in B.
- for $f : A \to B$
  - domain: A, codomain: B
  - range: $\{f(x) \mid x \in A\}$
- for this mod:
  - $A, B \subseteq \mathbb{R}$
  - if A is not stated, the domain of f is the largest possible set for which f is defined
  - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set
$$G(f) := \{(x, f(x)) \mid x \in A\}$$

- if $A, B \subseteq R$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane $\mathbb{R}^2$

algebra of functions

function	domain
$(f+g)(x) := f(x) + g(x)$	$A \cap B$
$(f-g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

types of functions

- **rational function**: $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
  - every polynomial is a rational function ($Q(x) = 1$)
- **algebraic function**: constructed from polynomials using algebraic operations
- a function f is **increasing** on a set I if $x_q < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if $x_q < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.
- even/odd:
  - **even function**: $\forall x, f(-x) = f(x)$
    - symmetric about the y-axis
  - **odd function**: $\forall x, f(-x) = -f(x)$
    - symmetric about the origin O
  - any function defined on $\mathbb{R}$ can be decomposed *uniquely* into the sum of an even function and an odd function
- **power function**: x^n
  - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

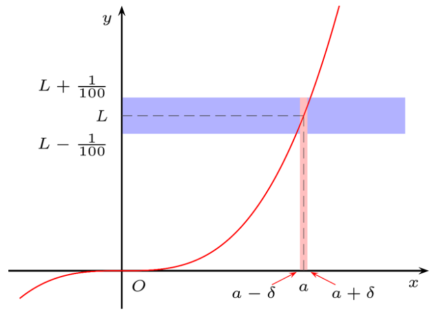
01. LIMITS

precise definition of limits

Let f be a function defined on an open interval containing a, except possibly at a.

The limit of $f(x)$ (as x approaches a) equals L if,

for every $\epsilon > 0$ there is $\delta > 0$ such that
$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$



informally,

- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a.
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L.

limit laws

you cannot apply any laws on limits UNLESS you have shown that the limit exists!

- Let $c \in \mathbb{R}$. $\lim_{x \to a} c = c$
- $\lim_{x \to a} x = a$

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant.

- $\lim_{x \to a} (cf(x)) = cL = c \lim_{x \to a} f(x)$
- $\lim_{x \to a} (f(x) + g(x)) = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- $\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $\lim_{x \to a} g(x) \neq 0$
- $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$
- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

if $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$

direct substitution property

Let f be a polynomial or rational function.

If a is in the domain of f, then
$$\lim_{x \to a} f(x) = f(a)$$

If $f(x) = g(x)$ for all x near a except possibly at a, then
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

If a is not in the domain (e.g. 0 denominator), don't apply directly - convert to an equivalent function and then sub in

inequalities on limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

lemma
if $f(x) \le g(x)$ for all x near a (except possibly at a), then $L \le M$.

lemma
If $f(x) \ge 0$ for all x, then $L \ge 0$.

one-sided limits

- limit laws also hold for one-sided limits

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

$$f(x) \to L \Leftarrow x \to a \Leftrightarrow \begin{cases} x \to a^+ \Rightarrow f(x) \to L \\ x \to a^- \Rightarrow f(x) \to L \end{cases}$$

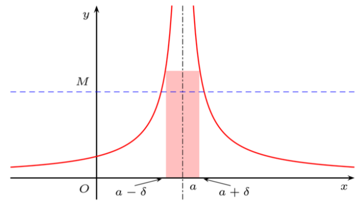
definition of one-sided limits ($\lim f(x) = \infty$)

LH Limit: $\lim_{x \to a^-} f(x) = L$
if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$

RH Limit: $\lim_{x \to a^+} f(x) = L$
if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$

definition of infinite limits

$\lim_{x \to a} f(x) = \infty$
if for every $M > 0$ there exists $\delta > 0$ such that
$0 < |x - a| < \delta \Rightarrow f(x) > M$



negative infinite limit:
$0 < |x - a| < \delta \Rightarrow f(x) < M$

- ∞ is NOT a number $\Rightarrow$ an infinite limit does NOT exist

limits to infinity ($\lim_{x \to \infty}$)

Suppose f is defined on $[M, \infty)$ for some $M \in \mathbb{R}$:

$\lim_{x \to \infty} f(x) = L$:
For every $\epsilon > 0$, there exists N such that
$x > N \Rightarrow |f(x) - L| < \epsilon$

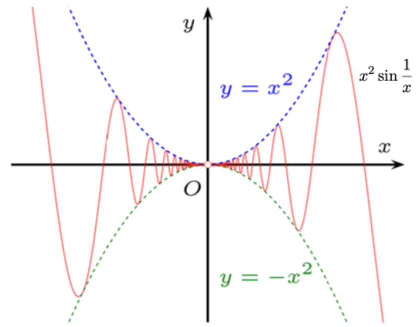
$\lim_{x \to \infty} f(x) = \infty$:
For every $M > 0$, there exists N such that
$x > N \Rightarrow f(x) > M$

squeeze theorem

Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where

- $g(x) \le f(x) \le h(x)$ for all x near a (except at a), and
- $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} f(x) = L$.



02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at** $a \iff$
f is continuous from the left and from the right at a.
$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

a function f is **continuous at an interval** if it is continuous at every number in the interval.

f is continuous on **open interval** (a, b)
$\Leftrightarrow f$ is continuous at every $x \in (a, b)$
f is continuous on **closed interval** $[a, b]$
$\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$

precise definition of continuity

a function f is **continuous** at a number a if
for all $\epsilon > 0$, there exists $\delta > 0$ such that
$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

- aka $\lim_{x \to a} f(x) = f(a)$

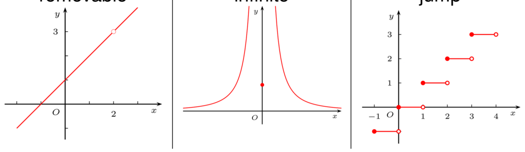
continuity test

f is continuous at a $\Leftrightarrow$

1. f is defined at a (a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

examples of discontinuity

removable infinite jump



properties of continuous functions

let f and g be functions continuous at a. let c be a constant.

1. cf is continuous at a
2. $f+g$ is continuous at a
3. $f-g$ is continuous at a
4. fg is continuous at a
5. f/g is continuous at a, provided $g(a) \neq 0$

other properties

- a polynomial is continuous everywhere
- a rational function is continuous on its domain
  - if $P(x)$ and $Q(x)$ are polynomials, $\frac{P(x)}{Q(x)}$ is continuous whenever $Q(x) \neq 0$.
- $f(x) = c$ is continuous on $\mathbb{R}$ for all $c \in \mathbb{R}$.
- $f(x) = x$ is continuous on $\mathbb{R}$.

trigonometric functions

- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous everywhere
- $\tan x, \sec x$ are continuous whenever $\cos x \neq 0$
  - domain: $\mathbb{R}\backslash\{\pm\frac{pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\}$
- $\cot x, \csc x$ are continuous whenever $\sin x \neq 0$
  - domain: $\mathbb{R}\backslash\{0, \pm\pi, \pm 2\pi, \cdots\}$

composite of continuous functions

if f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\boxed{\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)}$$

if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

$$\boxed{\lim_{x\to a} (f \circ g)(x) = (f \circ g)(a)}$$

substitution theorem

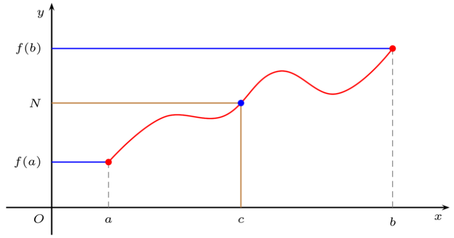
Suppose $y = f(x)$ such that $\lim_{x\to a} f(x) = b$. If

1. g is continuous at b, OR
2. $\lim_{y\to b} g(y)$ exists and f is one-to-one.
   - $\forall x$ near a, except at a, $f(x) \neq b$ and $\lim_{y\to b} g(y)$ exists

Then $\lim_{x\to a} g(f(x)) = \lim_{y\to b} g(y)$

intermediate value theorem

Let f be a function continuous on $[a,b]$ with $f(a) \neq f(b)$.
Let N be a number between $f(a)$ and $f(b)$.
Then there exists $c \in (a,b)$ such that $f(c) = N$.



03. DERIVATIVES

definition of derivatives

- f is differentiable at a if $f'(a)$ exists
- $f'(a)$ is the slope of $y = f(x)$ at $x = a$
  - $f'(a) = \frac{dy}{dx}|_{x=a}$
  - $\frac{dy}{dx} := \lim_{x\to 0} \frac{\Delta y}{\Delta x}$ (derivative of y with respect to x)
- $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x) = \cdots$

the **derivative** of a function f

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

the **derivative** of a function f at a number a is

$$f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

tangent line

the **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:

$$\boxed{y = f'(a)(x-a) + f(a)}$$

differentiable functions

- f is differentiable at a if
  - $f'(a) := \lim_{x\to 0} \frac{f(a+h)-f(a)}{h}$ exists.
- f is differentiable on (a,b) if
  - f is differentiable at every $c \in (a,b)$

differentiability & continuity

- differentiability $\Rightarrow$ continuity
  - if f is differentiable at a, then f is continuous at a.
- continuity $\not\Rightarrow$ differentiability

differentiability

- every polynomial and rational function is differentiable on its domain
  - the domain of f' may be smaller than the domain of f.
- trigonometric functions are differentiable on the domain

differentiation

differentiation of trigonometric functions

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$$

chain rule

If g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and

$$F'(a) = (f \circ g)'(a) = f'(b)g'(a) = f'(g(a))g'(a)$$

If $z = f(y)$ and $y = g(x)$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$\frac{dz}{dx}|_{x=a} = \frac{dz}{dy}|_{y=b}\frac{dy}{dx}|_{x=a}$$

generalised chain rule

h is differentiable at a; g is differentiable at $B = h(a)$; f is differentiable at $c = g(b)$.

$$(f \circ (g \circ h))' = f' \circ (g \circ h) \cdot (g \circ h)'$$
$$= f'(c)g'(b)h'(a)$$

Leibniz notation:
If $y = h(x), z = g(y), w = f(z)$,

$$\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx}$$

implicit differentiation

- assumes that $\frac{dy}{dx}$ exists

second derivative

$$f''(x) = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$$
$$f' = D(f) \Rightarrow f'' := D^2(f)$$

higher derivatives

$$f^{(0)} := f$$

For any positive integer n, $f^{(n)} := (f^{(n-1)})'$

if $y = f(x)$, then $f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x)$

- for $f(x) = \frac{1}{x}$, $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$
- for $f(x) = x^m$, $f^{(n)}(x) = \begin{cases} \frac{m! x^{m-n}}{(m-n)!} & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$

04. APPLICATIONS OF DIFFERENTIATION

extreme values of functions

Let f be a function with domain D.

- local max/min $\not\Rightarrow$ global max/min
- global max/min $\not\Rightarrow$ local max/min

global (absolute) max/min

- aka extreme values

f has a global **maximum** at $c \in D$
$\Leftrightarrow f(c) \geq f(x)$ for all $x \in D$
f has a global **minimum** at $c \in D$
$\Leftrightarrow f(c) \leq f(x)$ for all $x \in D$

local (relative) max/min

- aka "turning points"
- "all x near c" = for all x in an open interval containing c

f has a local **maximum** at $c \in D$
$\Leftrightarrow f(c) \geq f(x)$ for all x near c
f has a local **minimum** at $c \in D$
$\Leftrightarrow f(c) \leq f(x)$ for all x near c

extreme value theorem

existence

if f is *continuous* on a *finite closed* interval $[a,b]$, then f attains extreme values on $[a,b]$.

value

the extreme value occurs at either critical numbers or the endpoints ($x = a, x = b$).

critical numbers

$c \in D$ is a **critical number** of f if $f'(c) = 0$, or $f'(c)$ does not exist.

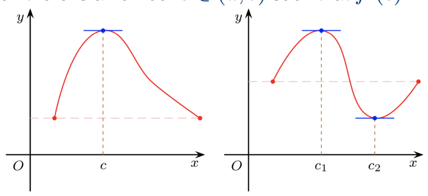
fermat's theorem

If f has a local maximum or minimum at c, then

1. c is a critical number.
2. If $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

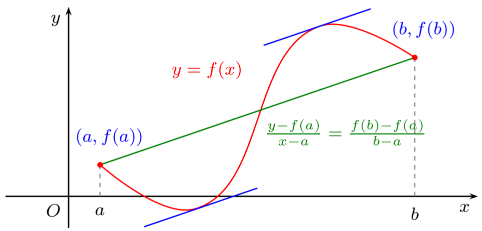
Let f be a function such that f is *continuous* on $[a,b]$, f is *differentiable* on (a,b), and $f(a) = f(b)$.
Then there is a number $c \in (a,b)$ such that $f'(c) = 0$.



mean value theorem

Let f be a function such that f is *continuous* on $[a,b]$ and f is *differentiable* on (a,b).
Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



- generalisation of Rolle's theorem when $f(a) = f(b)$.

ordinary differential equations

Let f and g be continuous on $[a,b]$.
If $f'(x) = g'(x)$ for all $x \in (a,b)$,
then $f(x) = g(x) + C$ on $[a,b]$ for a constant C.

increasing/decreasing test

Let f be continuous on $[a,b]$ and differentiable on (a,b).

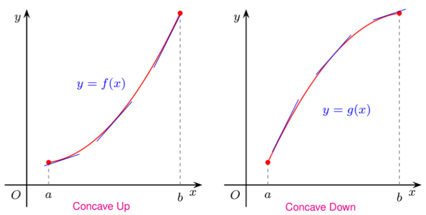
- $f'(x) > 0$ for any $x \in (a,b) \Rightarrow f$ is increasing.
  - f is increasing $\Rightarrow f'(x) \geq 0$ on (a,b)
- $f'(x) < 0$ for any $x \in (a,b) \Rightarrow f$ is decreasing.
  - f is decreasing $\Rightarrow f'(x) \leq 0$ on (a,b)
- $f'(x) = 0 \Rightarrow f$ could be increasing OR decreasing.

first derivative test

Let f be continuous and c be a critical number of f.
Suppose f is differentiable near c (except possibly at c).
At c, if f' changes from:

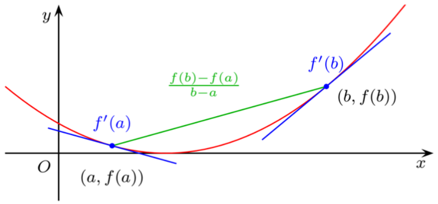
- (+) to (-) $\Rightarrow$ f has a local **maximum** at c
- (-) to (+) $\Rightarrow$ f has a local **minimum** at c
- no change in sign $\Rightarrow$ f has neither local max/min at c.

concavity



f is **concave up** on an open interval I $\Leftrightarrow$ f' is increasing
$\Leftrightarrow$ for $a < b \in I$, $f'(a) < f'(b)$
$\Leftrightarrow$ $f(x) > f'(y)(x-y) + f(y)$ for any $x \neq y \in I$

f is **concave down** on an open interval I $\Leftrightarrow$ f' is decreasing
$\Leftrightarrow$ for $a < b \in I$, $f'(a) > f'(b)$
$\Leftrightarrow$ $f(x) < f'(y)(x-y) + f(y)$ for any $x \neq y \in I$



concavity test

- $f'' > 0$ on I $\Rightarrow$ f is concave up on I
- $f'' < 0$ on I $\Rightarrow$ f is concave down on I

second derivative test

If $f'(c) = 0$ and $f''(c)$ exists,

- $f''(c) < 0 \Rightarrow f$ has a **local maximum** at c.
- $f''(c) > 0 \Rightarrow f$ has a **local minimum** at c.
- $f''(c) = 0 \Rightarrow$ inconclusive

inflection point

- A point P on the curve $y = f(x)$ is an inflection point if
  - f is continuous at P, and
  - the concavity of the curve changes at P.
- if c is an inflection point and f is twice differentiable at c, then $f''(c) = 0$.

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n,$$

where $R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$ for c between x and a

Taylor Series

As $R-n \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

L'Hopital's Rule

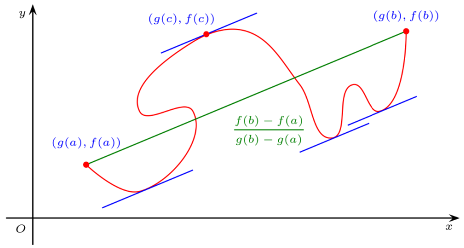
Let f and g be functions such that

- $(\frac{0}{0})$ $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, OR
  $(\frac{\infty}{\infty})$ $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$,
- f and g are differentiable near a (except at a),
- $g'(x) \neq 0$ near a (except at a).

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
provided that the RHS limit exists or is $\pm\infty$

Cauchy's Mean Value Theorem

Let f, g be continuous on $[a,b]$, differentiable on (a,b), and $g'(x) \neq 0$ for any $x \in (a,b)$. Consider a curve defined by $t \mapsto (g(t), f(t))$.
Then there exists $c \in (a,b)$ such that
$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$



05. INTEGRALS

definite integral

Let f be a continuous function on $[a,b]$ divided into n intervals.

Riemann sum

$$[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]\,\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- the lengths of subintervals are not necessarily equal
  - $\max\{|x_i - x_{i-1}| : i = 1, \cdots, n\} \to 0$

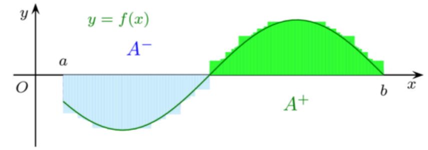
definite integral of f from a to b:

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

where $\Delta x = \frac{b-a}{n}$

- f is **integrable** from a to b if $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists.
- continuous functions are integrable.
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$

geometric meaning



$$\int_a^b f(x)\,dx = A^+ - A^- \text{ (net area)}$$

properties

let f and g be continuous functions.

- $\int_a^b c\,dx = (b-a)c$
- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx = \int_b^c f(x)\,dx \pm \int_a^b f(x)\,dx$
- suppose $f(x) \geq 0$ on $[a,b]$. Then $\int_a^b f(x)\,dx \geq 0$.
- suppose $f(x) \geq g(x)$ on $[a,b]$.
- • Then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- suppose $m \leq f(x) \leq M$ on $[a,b]$.
  - Then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

fundamental theorem of calculus

for $g(x) = \int_a^x f(t)\,dt \quad (a \leq x \leq b)$,

- g is continuous on $[a,b]$
- g is differentiable on (a,b)
- $g'(x) = f(x)$ on (a,b) or $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$\begin{tikzcd} \boxed{f(x)} \arrow[r, shift left, "f \text{ is the derivative of } F"] & \boxed{F(x)} \arrow[l, shift left, "F \text{ is an antiderivative of } f"] \end{tikzcd}$$

if F is continuous on $[a,b]$, and $F' = f$ on (a,b),

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$$

$$\int_a^x \frac{d}{dx}F(t)\,dt = F(x) - F(a)$$

$$\begin{tikzcd} \boxed{f(t)} \arrow[r, "\int_a^x"] \arrow[rr, bend right] & \boxed{F(x)} \arrow[r, "\frac{d}{dx}"] & \boxed{f(x)} \end{tikzcd}$$

$$\begin{tikzcd} \boxed{F(t)} \arrow[r, "\frac{d}{dt}"] \arrow[rr, bend right] & \boxed{f(t)} \arrow[r, "\int_a^x"] & \boxed{F(x)-F(a)} \end{tikzcd}$$

indefinite integral

- **indefinite integral** of f, $\int f(x)\,dx = F(x) + c$
- **antiderivative** (of a continuous function f): a continuous function F such that $F' = f$.
  - antiderivatives of f are functions of form $F + c$
  - indefinite integral is a family of antiderivatives
- properties of indefinite integral
  - $\int (af(x) \pm bg(x))\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx$

integration by parts

$$u\,dv = uv - \int v\,du$$

substitution rule (I)

let $u = g(x)$ be a differentiable function.

indefinite integral

if f and g' are continuous,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

definite integral

if g' are continuous on $[a,b]$,
and f is continuous on the range of $u = g(x)$,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

substitution rule (II)

let f and g' be continuous functions, and
$x = g(t)$ is a one-to-one differentiable function.

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

improper integral

for discontinuous integrands

if f is continuous on $[a,b)$ and discontinuous at b,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if f is continuous on $(a,b]$ and discontinuous at a,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

- $\int_a^b f(x)\,dx$ is the limit of integrals.
  - **converges** if the limit exists
  - **diverges** if the limit does not exist

discontinuity in the interior of the interval

suppose f has discontinuity at $c \in (a,b)$. then

$$\int_a^b f(x)\,dx = \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

over infinite intervals

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

if $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then
the **improper integral** of f from a to ∞ is

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

if $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then
the **improper integral** of f from $-\infty$ to b is

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

- NOTE: $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx$

06. INVERSE FUNCTIONS & INTEGRATION

one to one functions

let f be a function with domain D.
f is **one-to-one** if, for any $a, b \in D$,
$a \neq b \Rightarrow f(a) \neq f(b)$
OR $f(a) = f(b) \Rightarrow a = b$

inverse function

let f be a one-to-one function with domain A and range B.

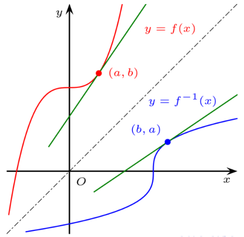
- its **inverse function** f^{-1} is the function with
  - domain B and range A, and
  - $f^{-1}(y) = x \iff y = f(x)$ for any $x \in A, y \in B$
- $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$
- $(f^{-1})^{-1} = f$
- NOTE: $(f(x))^{-1}$ is the reciprocal of the value of $f(x)$

properties

let f be a *one-to-one continuous* function on an open interval I.

- the inverse function f^{-1} is also continuous.
- if f is differentiable at $a \in I$, and $f'(a) \neq 0$, then
  - f^{-1} is differentiable at $b = f(a)$
  - $(f^{-1})'(b) = \frac{1}{f'(a)}$

$$\boxed{(f^{-1})'(f(a)) = \frac{1}{f'(a)}}$$



techniques of integration

integration of rational functions

for $f = \frac{A(x)}{B(x)}$,

- manipulate such that $\deg A(x) < \deg B(x)$, then decompose into partial fractions
- common rational functions:
  - $\displaystyle\int \frac{1}{(x+a)^k}\,dx = \begin{cases} \ln|x+a| + K, & \text{if } k = 1 \\ \frac{(x+a)^{1-k}}{1-k} + K, & \text{if } k \geq 1 \end{cases}$
  - $\displaystyle\int \frac{u}{(u^2+d^2)^r}\,du = \begin{cases} \frac{1}{2}\ln(u^2+d^2), & \text{if } r = 1 \\ \frac{(u^2+d^2)^{1-r}}{2(1-r)}, & \text{if } r \geq 2 \end{cases}$
  - $\displaystyle\int \frac{1}{(u^2+d^2)^r}\,du = \frac{1}{d^{2r-1}} \int \frac{1}{(t^2+1)^r}\,dt$
  -

partial fractions

- for each linear factor $(x+a)^k$:
  - $\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_k}{(x+a)^k}$
- for each quadratic factor $(x^2+bx+c)^r$:
  - $\frac{B_1x+C_1}{x^2+bx+c} + \cdots + \frac{B_rx+C_r}{(x^2+bx+c)^r}$

common trigonometric substitutions

- $\sqrt{a^2-x^2}$, $\quad x = a\sin t$, $\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sqrt{x^2-a^2}$, $\quad x = a\sec t$, $\quad t \in [0, -\frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}]$
- a^2+x^2, $\quad x = a\tan t$, $\quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

universal trigonometric substitution

any rational expression in $\sin x$ and $\cos x$ can be integrated using the substitution $t = \tan\frac{x}{2}$, $\quad x \in (-\pi, \pi)$.

$\sin x = \frac{2t}{1+t^2}$, $\quad \cos x = \frac{1-t^2}{1+t^2}$, $\quad \frac{dx}{dt} = \frac{2}{1+t^2}$

derivatives of trigonometric functions

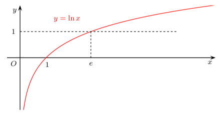
function	derivative	function	derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$\frac{-1}{1+x^2}$

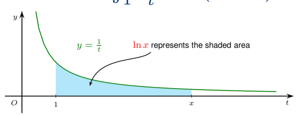
trigonometric identities

- $\tan^{-1} x + \cot^{-1} x - \frac{\pi}{2}$
- $\sec^{-1} x + \csc^{-1} x = \begin{cases} \frac{\pi}{2}, & \text{if } x \geq 1 \\ \frac{5\pi}{2}, & \text{if } x \leq -1 \end{cases}$

natural logarithmic function

natural logarithmic function, $\ln x = \int_1^x \frac{1}{t}\,dt \quad (x > 0)$





- $\ln x < 0$ for $0 < x < 1$; $\quad \ln x > 0$ for > 1; $\quad \ln 1 = 0$
- $\ln x$ is increasing on $\mathbb{R}^n$ ($\frac{d}{dx}\ln x > 0$)

logarithmic differentiation I

aka take $\ln$ on both sides and implicitly differentiate

for $y = f_1(x)f_2(x)\cdots f_n(x)$ (product of nonzero functions),

$$\ln|y| = \ln|f_1(x)| + \ln|f_2(x)| + \cdots + \ln|f_n(x)|$$
$$\frac{dy}{dx} = \left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right] y$$
$$= \left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right] f_1(x)f_2(x)\cdots f_n(x)$$

logarithmic differentiation II

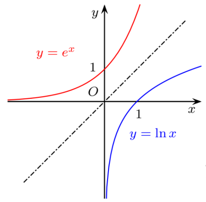
for $y = f(x)^{g(x)} (f(x) > 0)$,

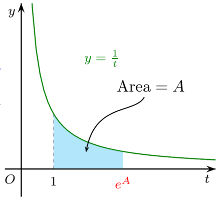
$$\ln y = g(x)\ln f(x) \Rightarrow \frac{dy}{dx} = y\frac{d}{dx}[g(x)\ln f(x)]$$

$$\lim_{x\to a}(f(x)^{g(x)}) = \lim_{x\to a}\exp(g(x)\ln f(x))$$
$$= \exp\left(\lim_{x\to a} g(x)\ln f(x)\right)$$

exponential function

$y = e^x = \exp(x) \iff \ln y = x$
$\exp(x) = \ln^{-1}(x)$ ($\exp(x)$ is the inverse of $\ln x$)
$a^x = \exp(x\ln a) = e^{x\ln a}$





$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

- $\ln(e^x) = x$ for $x \in \mathbb{R}$ and $e^{\ln y} = y$ for $y \in \mathbb{R}^+$
- common equations
  - $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$
  - $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}^+$
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

properties

- $a^u a^v = a^{u+v}$
- $a^{-u} = \frac{1}{a^u}$
- $(a^u)^v = a^{uv}$
- $(a^x)' = a^x \ln a$
- $\frac{d}{dx}x^r = rx^{r-1}$

- $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}^+$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$

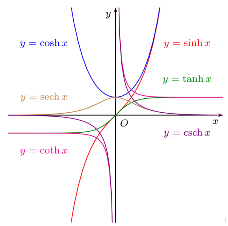
- $\int x^r\,dx = \begin{cases} \frac{x^{r+1}}{r+1} + C & \text{if } r \neq -1, \\ \ln x + C & \text{if } r = -1, \end{cases}$
  - if r is irrational, then x^r is only defined for $x \geq 0$.

hyperbolic trigonometric functions

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\quad (\sinh x)' = \cosh x$
$\cosh x = \frac{e^x + e^{-x}}{2}$, $\quad (\cosh x)' = \sinh x$

- $\cosh^2 x - \sinh^2 x = 1$
- parametrization represents a **hyperbola** -

let $\begin{cases} x = \cosh t, \\ y = \sinh t. \end{cases}$ Then $x^2 - y^2 = 1$



$\tanh x = \frac{\sinh x}{\cosh x}$
$\coth x = \frac{\cosh x}{\sinh x}$
$sech\, x = \frac{1}{\cosh x}$
$csch\, x = \frac{1}{\sinh x}$

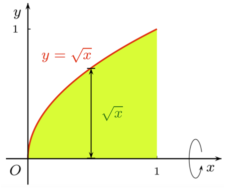
inverse hyperbolic functions:
$\sinh^{-1} x = y \Leftrightarrow x = \sinh y$
$\cosh^{-1} x = y \Leftrightarrow x = \cosh y$

- properties
  - $\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), x \in \mathbb{R}$
  - $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), x \geq 1$
  - $\tanh^{-1} x = \frac{1}{2}\ln(\frac{1+x}{1-x}), \quad -1 < x < 1$
  - $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$
  - $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
  - $\frac{d}{dx}\tanh^{-1} x = sech\, x$

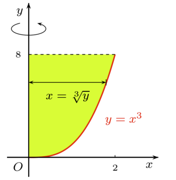
07. APPLICATIONS OF INTEGRALS

volume

disk/washer method

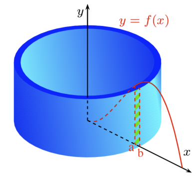


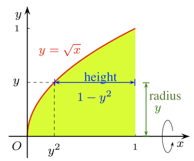
rotate about the **x-axis**:
$$V = \pi\int_a^b [f(x)]^2\,dx$$



rotate about the **y-axis**:
$$V = \pi\int_c^d [f(y)]^2\,dy$$

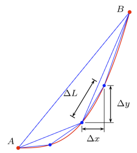
method of cylindrical shells





rotation about **x-axis** from $y = a$ to $y = b$:
$$V = 2\pi\int_a^b yf(y)\,dy = 2\pi\int (radius \cdot height)\,dy$$
rotation about **y-axis** from $x = a$ to $x = b$:
$$V = 2\pi\int_a^b xf(x)\,dx = 2\pi\int (radius \cdot height)\,dx$$

arc length



- a function f is **smooth** if f' is continuous.
- arc length,

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx}$$

parametric curves:
arc length = $\displaystyle\int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$

surface area of revolution

Let f be a smooth function such that $f(x) \geq 0$ on $[a, b]$. Then the area of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$ about the x-axis is

$$\boxed{A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx}$$

08. ORDINARY DIFFERENTIAL EQUATIONS

$\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\,dx$
$\frac{dy}{dx} = f(y) \Rightarrow x = \int \frac{1}{f(y)}\,dy$

separation of variables

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)}\,dy = f(x)\,dx$$
$$\Rightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

singular solution

- if $y = C$ is a solution to $g(y) = 0$, then it is a **singular solution** to $\frac{dy}{dx} = f(x)g(x)$.
- singular solution disappears if the equation is $\frac{1}{g(x)}\frac{dy}{dx} = f(x)$
- (can ignore singular solutions in this course)

homogenous equations

Suppose $\frac{dy}{dx} = F(x, y)$ is not separable.

- suppose $F(x, y)$ is **homogenous of degree zero**
  - i.e. $F(x, y) = F(tx, ty)$ for all $t \in \mathbb{R}\backslash\{0\}$
- let $z = \frac{y}{x}$. Then
  - $y = xz$ and $\frac{dy}{dx} = x\frac{dz}{dx} + z$
  - $F(x, y) = F(\frac{x}{x}, \frac{y}{x}) = F(1, z)$
- $x\frac{dz}{dx} + z = F(1, z) \Rightarrow$ separable!

first order linear differential equations

general equation: $\frac{dy}{dx} + p(x)y = q(x)$

1. find $P(x) = \int p(x)\,dx$
2. multiply both sides by integrating factor $v(x) = e^{P(x)}$:
   - $e^{P(x)}\frac{dy}{dx} + e^{P(x)}p(x)y = e^{P(x)}q(x)$
   - $\frac{d}{dx}(e^{P(x)}y) = e^{P(x)}q(x)$
3. integrate with respect to x
   - $e^{P(x)} = \int e^{P(x)}q(x)\,dx$

$$\boxed{y = \frac{1}{e^{P(x)}}\int e^{P(x)}q(x)\,dx}$$

note: if the equation is not linear in y but is linear in x, can take the reciprocal and use $\frac{dx}{dy}$ instead.

Bernoulli's equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

- if $n = 0$ or $n = 1$:
  - the system is linear
- if $n \neq 0, 1$:
  - let $z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$
  - multiply both sides of the equation by $(1-n)y^{-n}$
  - equation is reduced to a linear equation
    - $\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$

applications

- compound interest
  - let r be the interest rate (%), A be the money
  - ODE: $\frac{dA}{dt} = rA; \quad A(0) = C$
  - solve for $A(t) = Ce^{rt}$
- radiocarbon dating
  - let λ be the half life, C be % of Carbon left
  - ODE: $\frac{dC}{dt} = kC; \quad C(0) = 1; \quad k = -\frac{\ln 2}{\lambda}$
  - solve $C(t) = e^{kt}$
- population growth - let M be max. population (carrying capacity), r be the rate of change of population
  - ODE: $\frac{dP}{dt} = rP(M - P)$
  - solve $P(t) = \frac{M}{1 + (\frac{M}{P(0)} - 1)e^{-rt}}$
- newton's law of cooling
  - let T_S be the surrounding temperature, $r > 0$ be the rate of heat loss
  - ODE: $\frac{dT}{dt} = -r \cdot (T - T_S)$
  - $\ln|T - T_S| = -rt + C$
- draining tank problem (torricelli's law)
  - the rate at which water flows out is proportional to the square root of the water's depth
  - let A be the base area of the tank, R be the rate of flow
  - ODE: $A\frac{dh}{dt} = -R$

misc

triangle inequality

$$|a + b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R}$$

binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$
$$= a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

where the binomial coefficient is given by
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

factorisation

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

misc

- $\forall x \in (0, \frac{\pi}{2}), \sin x < x < \tan x$
- $\sin\theta = \frac{\tan\theta}{\sqrt{\tan^2\theta + 1}}$

differentiation

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$

integration

$f(x)$	$\int f(x)$
$\tan x$	$\ln(\sec x), \lvert x\rvert < \frac{\pi}{2}$
$\cot x$	$\ln(\sin x), 0 < x < \pi$
$\csc x$	$-\ln(\csc x + \cot x), 0 < x < \pi$
$\sec x$	$\ln(\sec x + \tan x), \lvert x\rvert < \frac{\pi}{2}$