

01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- **combinatorial analysis** → the mathematical theory of counting
- **basic principle of counting** → Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- **generalized basic principle of counting** → If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1! n_2! \dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)! r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, $1 \leq r \leq n$

Proof: If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.

If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - The Binomial Theorem - $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Proof: by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes n_1, n_2, \dots, n_r , where $n_1 + n_2 + \dots + n_r = n$

Proof: using basic counting principle,

$$\begin{aligned} &= \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots \binom{n-n_1-n_2-\dots-n_{r-1}}{n_r} \\ &= \frac{n!}{(n-n_1)! n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)! n_2!} \times \dots \times \frac{(n-n_1-n_2-\dots-n_{r-1})}{0! n_r!} \\ &= \frac{n!}{n_1! n_2! \dots n_r!} \end{aligned}$$

N7 - The Multinomial Theorem: $(x_1 + x_2 + \dots + x_r)^n$
 $= \sum_{(n_1, \dots, n_r): n_1 + n_2 + \dots + n_r = n} \frac{n!}{n_1! n_2! \dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors (x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$, $x_i > 0$, $i = 1, 2, \dots, r$

! cannot be directly applied to N8 as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors

(x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$

Proof: let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- **sample space** → The *set* of all outcomes of an experiment (where outcomes are not predictable with certainty)
- **event** → Any *subset* of the sample space
- **union** of events E and $F \rightarrow E \cup F$ is the event that contains all outcomes that are either in E or F (or both).
- **intersection** of events E and $F \rightarrow E \cap F$ or EF is the event that contains all outcomes that are both in E and in F .
- **complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E .
- **subset** → $E \subset F$ is all of the outcomes in E that are also in F .
 - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws

$$\left(\bigcup_{i=1}^n E_i \right)^c = \bigcap_{i=1}^n E_i^c$$

Proof: to show LHS \subset RHS: let $x \in \left(\bigcup_{i=1}^n E_i \right)^c$
 $\Rightarrow x \notin \bigcup_{i=1}^n E_i \Rightarrow x \notin E_1$ and $x \notin E_2 \dots$ and $x \notin E_n$
 $\Rightarrow x \in E_1^c$ and $x \in E_2^c \dots$ and $x \in E_n^c$
 $\Rightarrow x \in \bigcap_{i=1}^n E_i^c$
to show RHS \subset LHS: let $x \in \bigcap_{i=1}^n E_i^c$

$$\left(\bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c$$

Proof: using the first law of DeMorgan, negate LHS to get RHS

Axioms of Probability

Defining Probability of an Event

definition 1: relative frequency

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

problems with this definition:

1. $\frac{n(E)}{n}$ may not converge when $n \rightarrow \infty$
2. $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated

definition 2

Consider an experiment with sample space S . For each event E of the sample space S , we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. For any sequence of mutually exclusive events E_1, E_2, \dots (i.e., events for which $E_i E_j = \emptyset$ when $i \neq j$),
 $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

$P(E)$ is the probability of event E .

commutative	$E \cup F = F \cup E$	$E \cap F = F \cap E$
associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(E \cap F) \cap G = E \cap (F \cap G)$
distributive	$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$	$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$
DeMorgan's	$(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i^c$	$(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$