CS3230
AY21/22 SEM 2
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01. COMPUTATIONAL MODELS

- **algorithm** $\rightarrow$ a well-defined procedure for finding the correct solution to the input
- **correctness**
  - **worst-case correctness** $\rightarrow$ correct on *every valid input*
  - other types of correctness: correct on random input/with high probability/approximately correct
- **efficiency** / **running time** $\rightarrow$ measures the number of steps executed by an algorithm as a function of the *input size* (depends on computational model used)
  - number input: typically the length of binary representation
  - **worst-case** running time $\rightarrow$ *max* number of steps executed when run on an input of size n

adversary argument $\rightarrow$
inputs are decided such that they have different solutions

Comparison Model

- algorithm can **compare** any two elements in one time unit ($x > y, x < y, x = y$)
- running time = number of pairwise comparisons made
- array can be manipulated at no cost

Decision Tree

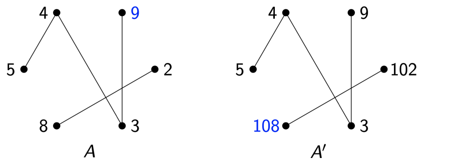
- each node is a comparison
- each branch is an outcome of the comparison
- each leaf is a class label (decision after *all* comparisons)
- **worst-case** runtime = height of tree
- # of leaves = # of permutations $\Rightarrow \lg(n!) = \Theta(n \lg n)$

Max Problem

problem: find largest element in array A of n distinct elements

Proof. $n-1$ comparisons are needed

fix an algorithm M that solves the Max problem on all inputs using $< n-1$ comparisons. construct graph G where nodes i and j are adjacent iff M compares i & j.

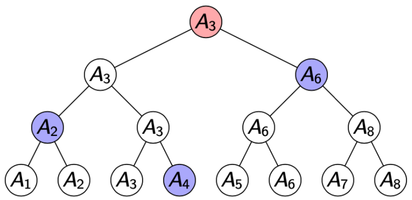


M cannot differentiate A and A'.

Second Largest Problem

problem: find the second largest element in $< 2n-3$ comparisons (2x Maximum $\Rightarrow (n-1)+((n-1)-1)=2n-3$)

- *solution*: **knockout tournament** $\Rightarrow n + \lceil \lg n \rceil - 2$



1. bracket system: $n-1$ matches
   - every non-winner has lost exactly once
2. then compare the elements that have lost to the largest
   - the 2nd largest element must have lost to the winner
   - compares $\lceil \lg n \rceil$ elements that have lost to the winner using $\lceil \lg n \rceil - 1$ comparisons

Sorting

Claim. there is a sorting algorithm that requires $\leq n \lg n - n + 1$ comparisons.

Proof. every sorting algorithm must make $\geq \lg(n!)$ comparisons.

1. let set $\mathcal{U}$ be the set of all permutations of the set $\{1, \ldots, n\}$ that the adversary could choose as array A. $|\mathcal{U}| = n!$
2. for each query "is $A_i > A_j$?", if $\mathcal{U}_{yes} = \{A \in \mathcal{U} : A_i > A_j\}$ is of size $\geq |\mathcal{U}|/2$, set $\mathcal{U} := \mathcal{U}_{yes}$. else: $\mathcal{U} := \mathcal{U} \backslash \mathcal{U}_{yes}$
3. the size of $\mathcal{U}$ decreases by at most half with each comparison
4. with $< \lg(n!)$ comparisons, $\mathcal{U}$ will still contain at least 2 permutations

$$n! \geq (\tfrac{n}{e})^n$$
$$\Rightarrow \lg(n!) \geq n \lg(\tfrac{n}{e}) = n \lg n - n \lg e$$
$$\approx n \lg n - 1.44n$$

$\Rightarrow$ roughly $n \lg n$ comparisons are **required** and **sufficient** for sorting n numbers

String Model

input	string of n bits
each query	find out **one bit** of the string

- n queries are **necessary** and **sufficient** to check if the input string is all 0s.
- **query complexity** $\rightarrow$ number of bits of the input string queried by the algorithm
- **evasive** $\rightarrow$ a problem requiring n query complexity

Graph Model

input	(symmetric) adjacency matrix of an n-node undirected graph
each query	find out if an edge is present between two chosen nodes (one entry of G)

- **evasive** $\rightarrow$ requires $\binom{n}{2}$ queries
- *Proof.* determining whether the graph is connected is evasive (requires $\binom{n}{2}$ queries)
  1. suppose M is an algorithm making $\leq \binom{n}{2}$ queries.
  2. whenever M makes a query, the algorithm tries not adding this edge, but adding all remaining unqueried edges.
     2.1. if the resulting graph is connected, M replies 0 (i.e. edge does not exist)
     2.2. else: replies 1 (edge exists)
  3. after $< \binom{n}{2}$ queries, at least one entry of the adjacency matrix is unqueried.

$\square$

02. ASYMPTOTIC ANALYSIS

- **algorithm** $\rightarrow$ a *finite* sequence of well-defined instructions to solve a given computational problem
- **word-RAM model** $\rightarrow$ runtime is the total number of instructions executed
  - operators, comparisons, if, return, etc
  - each instruction operates on a *word* of data (limited size) $\Rightarrow$ fixed constant amount of time

Asymptotic Notations

upper bound ($\leq$): $f(n) = O(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq f(n) \leq cg(n)}$

lower bound ($\geq$): $f(n) = \Omega(g(n))$
if $\exists c > 0, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq cg(n) \leq f(n)}$

tight bound: $f(n) = \Theta(g(n))$
if $\exists c_1, c_2, n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)}$

o**-notation** ($<$): $f(n) = o(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq f(n) < cg(n)}$

ω**-notation** ($>$): $f(n) = \omega(g(n))$
if $\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
$\boxed{0 \leq cg(n) < f(n)}$

Limits

Assume $f(n), g(n) > 0$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \quad \Rightarrow f(n) = o(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \quad \Rightarrow f(n) = O(g(n))$$
$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \quad \Rightarrow f(n) = \Theta(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \quad \Rightarrow f(n) = \Omega(g(n))$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \quad \Rightarrow f(n) = \omega(g(n))$$

Proof. using delta epsilon definition

Properties of Big O

$\boxed{\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))}$

- **transitivity** - applies for $O, \Theta, \Omega, o, \omega$
  $f(n) = O(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
- **reflexivity** - for $O, \Omega, \Theta, \quad f(n) = O(f(n))$
- **symmetry** - $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$
- **complementarity** -
  - $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
  - $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$
- **misc**
  - if $f(n) = \omega(g(n))$, then $f(n) = \Omega(g(n))$
  - if $f(n) = o(g(n))$, then $f(n) = O(g(n))$

$\boxed{\log \log n < \log n < (\log n)^k < n^k < k^n}$

insertion sort: $O(n^2)$ with worst case $\Theta(n^2)$

03. ITERATION, RECURSION, DIVIDE-AND-CONQUER

Iterative Algorithms

- **iterative** $\rightarrow$ loop(s), sequentially processing input elements
- **loop invariant** implies correctness if
  - *initialisation* - true before the first iteration of the loop
  - *maintenance* - if true before an iteration, it remains true at the beginning of the next iteration
  - *termination* - true when the algorithm terminates

examples

- **insertionSort**: with loop variable as j, $A[1..J-1]$ is sorted.
- **selectionSort**: with loop variable as j, the array $A[1..j-1]$ is sorted and contains the $j-1$ smallest elements of A.
- **Misra-Gries algorithm** (determines which bit occurs more in an n-bit array A):
  - if there is an equal number of 0's and 1's, then $id = \bot$ and $count = 0$
  - if $z \in \{0, 1\}$ is the majority element, then $id = z$ and $count$ equals the difference between the count of the bits.

Divide-and-Conquer

powering a number

problem: compute $f(n, m) = a^n \pmod m$ for all $n, m \in \mathbb{Z}$

- observation: $f(x + y, m) = f(x, m) * f(y, m) \pmod m$
- **naive solution**: recursively compute and combine $f(n-1, m) * f(1, m) \pmod m$
  - $T(n) = T(n-1) + T(1) + \Theta(1) \Rightarrow T(n) = \Theta(n)$
- **better solution**: divide and conquer
  - divide: trivial
  - conquer: recursively compute $f(\lfloor n/2 \rfloor, m)$
  - combine:
    - $f(n, m) = f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if n is even
    - $f(n, m) = f(1, m) * f(\lfloor n/2 \rfloor, m)^2 \pmod m$ if odd
  - $T(n) = T(n/2) + \Theta(1) \Rightarrow \Theta(\log n)$

Solving Recurrences

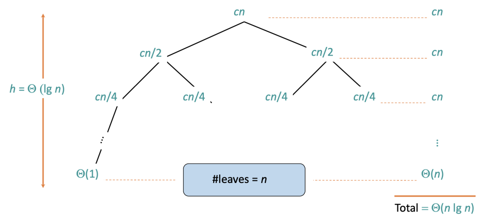
for a sub-problems of size $\frac{n}{b}$ where $f(n)$ is the time to divide and combine,

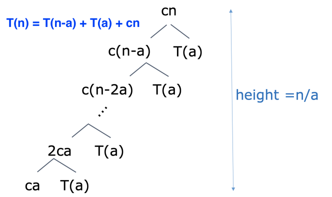
$$T(n) = aT(\tfrac{n}{b}) + f(n)$$

Recursion tree

total = height $\times$ number of leaves

- each node represents the cost of a single subproblem
- height of the tree = longest path from root to leaf





Master method

$a \geq 1, b > 1,$ and f is asymptotically positive

$T(n) = aT(\frac{n}{b}) + f(n) =$

$$\begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) < n^{\log_b a} \text{ polynomially} \\ \Theta(n^{\log_b a}\log n) & \text{if } f(n) = n^{\log_b a} \\ \Theta(f(n)) & \text{if } f(n) > n^{\log_b a} \text{ polynomially} \end{cases}$$

three common cases

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$,
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ by n^ϵ factor.
   - then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a}\log^k n)$ for some $k \geq 0$,
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.
   - then $T(n) = \Theta(n^{\log_b a}\log n)$
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$,
   - and $f(n)$ satisfies the **regularity condition**
     - $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n,
     - this guarantees that the sum of subproblems is smaller than $f(n)$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ by n^ϵ factor
   - then $T(n) = \Theta(f(n))$.

Substitution method

1. guess that $T(n) = O(f(n))$.
2. verify by induction:
   2.1. to show that for $n \geq n_0, T(n) \leq c \cdot f(n)$
   2.2. set $c = \max\{2, q\}$ and $n_0 = 1$
   2.3. verify base case(s): $T(n_0) = q$
   2.4. recursive case ($n > n_0$):
   - by strong induction, assume $T(k) \leq c \cdot f(k)$ for $n > k \geq n_0$
   - T(n) = <recurrence> ... $\leq c \cdot f(n)$

   2.5. hence $T(n) = O(f(n))$.

! may not be a tight bound!

example

Proof. $T(n) = 4T(n/2) + n^2/\lg n \Rightarrow \Theta(n^2 \lg\lg n)$

$$T(n) = 4T(n/2) + \frac{n^2}{\lg n}$$
$$= 4(4T(n/4) + \frac{(n/2)^2}{\lg n - lg 2}) + \frac{n^2}{\lg n}$$
$$= 16T(n/4) + \frac{n^2}{\lg n - \lg 2} + \frac{n^2}{\lg n}$$
$$= \sum_{k=1}^{\lg n} \frac{n^2}{\lg n - k}$$

$= n^2 \lg\lg n$ by approx. of harmonic series ($\sum \frac{1}{k}$)

Proof. $T(n) = 4T(n/2) + n \Rightarrow O(n^2)$

To show that for all $n \geq n_0, T(n) \leq c_1 n^2 - c_2 n$

1. Set $c_1 = q + 1, c_2 = 1, n_0 = 1$.
2. Base case ($n = 1$): subbing into $c_1 n^2 - c_2 n$, $T(1) = q \leq (q+1)(1)^2 - (1)(1)$
3. Recursive case ($n > 1$):
   - by strong induction, assume $T(k) \leq c_1 \cdot k^2 - c_2 \cdot k$ for all $n > k \geq 1$
   - $T(n) = 4T(n/2) + n$

$$= 4(c_1(n/2)^2 - c_2(n/2)) + n$$
$$= c_1 n^2 - 2c_2 n + n$$
$$= c_1 n^2 - c_2 n + (1 - c_2)n$$
$$= c_1 n^2 - c_2 n \quad \text{since } c_2 = 1 \Rightarrow 1 - c_2 = 0$$

□

04. AVERAGE-CASE ANALYSIS & RANDOMISED ALGORITHMS

- **average case** $A(n) \rightarrow$ expected running time when the input is chosen uniformly at random from the set of all $n!$ permutations
  - $A(n) = \frac{1}{n!}\sum_\pi Q(\pi)$ where $Q(\pi)$ is the time complexity when the input is permutation π.
  - $A(n) = \mathop{\mathbb{E}}_{x\sim\mathcal{D}_n}$ [Runtime of Alg on x]
    - $\mathbb{E}_{x\sim\mathcal{D}_n}$ is a probability distribution on U restricted to inputs of size n.

Quicksort Analysis

- divide & conquer, linear-time $\Theta(n)$ partitioning subroutine
- assume we select the first array element as pivot
- $T(n) = T(j) + T(n - j - 1) + \Theta(n)$
  - if the pivot produces subarrays of size j and $(n - j - 1)$
- **worst-case**: $T(n) = T(0) + T(n-1) + \Theta(n) \Rightarrow \Theta(n^2)$

Proof. for quicksort, $A(n) = O(n \log n)$

let $P(i)$ be the set of all those permutations of elements $\{e_1, e_2, \dots, e_n\}$ that begins with e_i.

Let $G(n, i)$ be the average running time of quicksort over $P(i)$. Then

$G(n) = A(i-1) + A(n-i) + (n-1)$.

$$A(n) = \frac{1}{n}\sum_{i=1}^n G(n,i)$$
$$= \frac{1}{n}\sum_{i=1}^n (A(i-1) + A(n-i) + (n-1))$$
$$= \frac{2}{n}\sum_{i=1}^n A(i-1) + n - 1$$

$= O(n\log n)$ by taking it as area under integration

quicksort vs mergesort

	average	best	worst
quicksort	$1.39n\lg n$	$n\lg n$	$n(n-1)$
mergesort	$n\lg n$	$n\lg n$	$n\lg n$

- disadvantages of mergesort:
  - overhead of temporary storage
  - cache misses
- advantages of quicksort
  - in place
  - reliable (as $n \uparrow$, chances of deviation from avg case $\downarrow$)
- issues with quicksort
  - **distribution-sensitive** $\rightarrow$ time taken depends on the initial (input) permutation

Randomised Algorithms

- **randomised algorithms** $\rightarrow$ output and running time are **functions** of the **input** and **random bits chosen**
  - vs non-randomised: output & running time are functions of the *input only*
- expected running time = worst-case running time = $E(n) = \max_{\text{input } x \text{ of size } n} \mathbb{E}$[Runtime of RandAlg on x]
- **randomised quicksort**: choose pivot at random
  - probability that the runtime of *randomised* quicksort exceeds average by $x\% = n^{-\frac{x}{100}\ln\ln n}$
  - P(time takes at least double of the average) = 10^{-15}
  - distribution insensitive

Randomised Quicksort Analysis

$T(n) = n - 1 + T(q-1) + T(n-q)$

Let $A(n) = \mathbb{E}[T(n)]$ where the expectation is over the randomness in expectation.

Taking expectations and applying linearity of expectation:

$$A(n) = n - 1 + \frac{1}{n}\sum_{q=1}^n (A(q-1) + A(n-q))$$
$$= n - 1 + \frac{2}{n}\sum_{q=1}^{n-1} A(q)$$

$A(n) = n\log n \quad \Rightarrow$ same as average case quicksort

Randomised Quickselect

- $O(n)$ to find the k^{th} smallest element
- randomisation: unlikely to keep getting a bad split

Types of Randomised Algorithms

- randomised **Las Vegas** algorithms
  - output is always correct
  - runtime is a *random variable*
  - e.g. randomised quicksort, randomised quickselect
- randomised **Monte Carlo** algorithms
  - output may be incorrect with some small probability
  - runtime is *deterministic*

examples

- *smallest enclosing circle:* given n points in a plane, compute the smallest radius circle that encloses all n points
  - best **deterministic** algorithm: $O(n)$, but complex
  - las vegas: average $O(n)$, simple solution
- *minimum cut:* given a connected graph G with n vertices and m edges, compute the smallest set of edges whose removal would disconnect G.
  - best **deterministic** algorithm: $O(mn)$
  - **monte carlo**: $O(m\log n)$, error probability n^{-c} for any c
- *primality testing:* determine if an n bit integer is prime
  - best **deterministic** algorithm: $O(n^6)$
  - **monte carlo**: $O(kn^2)$, error probability 2^{-k} for any k

Geometric Distribution

Let X be the number of trials repeated until success.

X is a random variable and follows a geometric distribution with probability p.

Expected number of trials, $E[X] = \frac{1}{p}$

$$Pr[X = k] = q^{k-1}p$$

Linearity of Expectation

For any two events X, Y and a constant a,

$$E[X + Y] = E[X] + E[Y]$$
$$E[aX] = aE[X]$$

Coupon Collector Problem

n types of coupon are put into a box and randomly drawn with replacement. What is the expected number of draws needed to collect at least one of each type of coupon?

- let T_i be the time to collect the i-th coupon after the $i - 1$ coupon has been collected.
  - Probability of collecting a new coupon, $p_i = \frac{(n-(i-1))}{n}$
  - T_i has a **geometric distribution**
  - $E[T_i] = 1/p_i$
- total number of draws, $T = \sum_{i=1}^n T_i$
- $E[T] = E[\sum_{i=1}^n T_i] = \sum_{i=1}^n E[T_i]$ by linearity of expectation

$$= \sum_{i=1}^n \frac{n}{n-(i-1)} = n \cdot \sum_{i=1}^n \frac{1}{i} = \Theta(n\lg n)$$

05. HASHING

Dictionary ADT

- different types:
  - **static** - fixed set of inserted items; only care about queries
  - **insertion-only** - only insertions and queries
  - **dynamic** - insertions, deletions, queries
- implementations
  - sorted list (static) - $O(\log N)$ query
  - balanced search tree (dynamic) - $O(\log N)$ all operations
  - direct access table
    - × needs items to be represented as non-negative integers (**prehashing**)
    - × huge space requirement
- using $\mathcal{H}$ for dictionaries: need to store both the hash table and the matrix A.
  - additional storage overhead $= \Theta(\log N \cdot \log |U|)$, if $M = \Theta(N)$
  - other universal hashing constructions may have more efficient hash function evaluation
- *associative array* - has both key and value (dictionary in this context has only key)

Hashing

- **hash function**, $h : U \rightarrow \{1, \dots, M\}$ gives the location of where to store in the hash table
  - notation: $[M] = \{1, \dots, M\}[M] = \{1, \dots, M\}$
  - storing N items in hash table of size M
- **collision** $\rightarrow$ for two different keys x and y, $h(x) = h(y)$
  - resolve by **chaining**, **open addressing**, etc
- desired properties
  - ✓ minimise collisions - `query(x)` and `delete(x)` take time $\Theta(|h(x)|)$
  - ✓ minimise storage space - aim to have $M = O(N)$
  - ✓ function h is easy to compute (assume constant time)
- if $|U| \geq (N-1)M + 1$, for any $h : U \rightarrow [M]$, there is a set of N elements having the same hash value.
  - *Proof*: pigeonhole principle
- use **randomisation** to overcome the adversary

- e.g. randomly choose between two *deterministic* hash functions h_1 and h_2
  $\Rightarrow$ for any pair of keys, with probability $\geq \frac{1}{2}$, there will be no collision

Universal Hashing

Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$.

$$\mathcal{H} \text{ is } \textbf{universal} \text{ if } \forall\, x \neq y,\ \frac{|h \in \mathcal{H} : h(x) = h(y)|}{|H|} \leq \frac{1}{M}$$
$$\text{or } \Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$$

- aka: for any $x \neq y$, if h is chosen uniformly at random from a universal $\mathcal{H}$, then there is at most $\frac{1}{M}$ probability that $h(x) = h(y)$
- probability where h is sampled uniformly from $\mathcal{H}$
- aka: for any $x \neq y$, the fraction of hash functions with collisions is at most $\frac{1}{M}$.

Properties of universal hashing

Collision Analysis

- for any N elements $x_1, \ldots, x_N \in \mathcal{U}$, the **expected number of collisions** between x_N and other elements is $< N/M$.
  - it follows that for K operations, the expected cost of the last operation is $< K/M = O(1)$ if $M > K$.

Proof. by definition of Universal Hashing, each element $x_1, \ldots, x_{N-1} \in \mathcal{U}$ has at most $\frac{1}{M}$ probability of collision with x_N (over random choice of h).

by indicator r.v., $E[A_i] = P(A_i = 1) \leq \frac{1}{M}$. expected number of collisions = $(N-1) \cdot \frac{1}{M} < \frac{N}{M}$.

- if $x_1, \ldots, x_N$ are added to the hash table, and $M > N$, the expected **number of pairs** (i, j) with collisions is $< 2N$.

Proof. let A_{ij} be an indicator r.v. for collision.

$$\mathbb{E}[\sum_{1 \leq i,j \leq N} A_{ij}] = \sum_{i=1}^{N} \mathbb{E}[A_{ii}] + \sum_{i \neq j} \mathbb{E}[A_{ij}]$$
$$\leq N \cdot 1 + N(N-1) \cdot \frac{1}{M} < 2N$$

Expected Cost

- for any sequence of N operations, if $M > N$, then the **expected total cost** for executing the sequence is $O(N)$.

Proof. linearity of expectation; sum up expected costs

Construction of Universal Family

Obtain a universal family of hash functions with $M = O(N)$.

- Suppose U is indexed by u-bit strings and $M = 2^m$.
- For any $m \times u$ binary matrix A, $h_A(x) = Ax \pmod 2$
  - each element `x => x % 2`
  - x is a $u \times 1$ matrix $\Rightarrow Ax$ is $m \times 1$
- *Claim*: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal
- e.g. $U = \{00, 01, 10, 11\}, M = 2$
  - h_{ab} means $A = [a\ b]$

	00	01	10	11
h_{00}	0	0	0	0
h_{01}	0	1	0	1
h_{10}	0	0	1	1
h_{11}	0	1	1	0

Proof. Let $x \neq y$. Let $z = x - y$. We know $z \neq 0$.

Collision: $P(Ax = Ay) = P[A(x-y) = 0] = P(Az = 0)$.

To show $P(Az = 0) \leq \frac{1}{M}$.

Special case - Suppose z is 1 at the i-th coordinate but 0 everywhere else. Then Az is the i-th column of A. Since the i-th column is uniformly random, $P(Az = 0) = \frac{1}{2^m} = \frac{1}{M}$.

General case - Suppose z is 1 at the i-th coordinate. Let $z = [z_1\ z_2\ \ldots\ z_u]^T$. $A = [A_1\ A_2\ \ldots\ A_u]$ hence A_k is the k-th column of A.

Then $Az = z_1 A_1 + z_2 A_2 + \cdots + z_u A_u$.

$Az = 0 \Rightarrow z_1 A_1 = -(z_2 A_2 + \cdots + z_u A_u) \quad (*)$

We fix $z_1 A_1$ to be an arbitrary $m \times 1$ matrix of 1s and 0s. The probability that $(*)$ holds is $\frac{1}{2^m}$.

Perfect Hashing

static case - N fixed items in the dictionary $x_1, x_2, \ldots, x_N$

To perform `Query` *in $O(1)$ worst-case time.*

Quadratic Space: $M = N^2$

if $\mathcal{H}$ is universal and $M = N^2$, and h is sampled uniformly from $\mathcal{H}$, then the expected number of collisions is < 1.

Proof. for $i \neq j$, let indicator r.v. A_{ij} be equal to 1 if $h(x_i) = h(x_j)$, or 0 otherwise.

By universality, $E[A_{ij}] = P(A_{ij} = 1) \leq 1/N^2$

$$E[\text{\# collisions}] = \sum_{i<j} E[A_{ij}] \leq \binom{N}{2} \frac{1}{N^2} < 1$$

It follows that there exists $h \in \mathcal{H}$ causing no collisions (because if not, $\mathbb{E}$[#collisions] would be ≥ 1).

2-Level Scheme: $M = N$

- No collision and less space needed

Construction

Choose $h : U \to [N]$ from a universal hash family.

- Let L_k be the number of x_i's for which $h(x_i) = k$.
- Choose $h_1, \ldots, h_N$ **second-level** hash functions $h_k : [N] \to [(L_k)^2]$ s.t. there are no collisions among the L_k elements mapped to k by h.
  - quadratic second-level table $\rightarrow$ ensures no collisions using quadratic space

Analysis

if $\mathcal{H}$ is universal and h is sampled uniformly from $\mathcal{H}$, then

$$E\left[\sum_k L_k^2\right] < 2N$$

Proof. For $i, j \in [1, N]$, define indicator r.v. $A_{ij} = 1$ if $h(x_i) = h(x_j)$, or 0 otherwise.

A_{ij} = # possible collisions = # pairs * 2 $= L_k^2$

Hence $\sum_k L_k^2 = \sum_{i,j} A_{ij}$

$$E[\sum_{i,j} A_{ij}] = \sum_i E[A_{ii}] + \sum_{i \neq j} E[A_{ij}]$$
$$\leq N \cdot 1 + N(N-1) \cdot \frac{1}{N}$$
$$< 2N$$

Hash Table Resizing

- when number of inserted items, N is not known
  - **reshashing** - choose a new hash function of a larger size and re-hash all elements
  - costly but infrequent $\Rightarrow$ amortize

06. FINGERPRINTING & STREAMING

String Pattern Matching

problem: does the pattern string P occur as a substring of the text string T?

m = length of P, n = length of T, ℓ = size of alphabet

- assumption: operations on strings of length $O(\log n)$ can be executed in $O(1)$ time. (word-RAM model)
- naive solution: $\Theta(n^2)$

Fingerprinting approach (Karp-Rabin)

- faster string equality check:
  - for substring X, check $h(X) == h(P)$ for a hash function $h \Rightarrow \Theta(1)$ + cost of hashing instead of $\Theta(|X|)$
- **Rolling Hash**: $O(m + n)$
  - update the hash from what we already have from the previous hash - $O(1)$
  - compute $n - m + 1$ hashes in $O(n)$ time
  - *Monte Carlo* algorithm

Division Hash

Choose a random **prime** number p in the range $\{1, \ldots, K\}$. For integer x, $h_p(x) = x \pmod p$

- if p is small and x is b-bits long in binary, hashing $\Rightarrow O(b)$
- hash family $\{h_p\}$ is approximately universal
- if $0 \leq x < y < 2^b$, then $\Pr_h[h_p(x) = h_p(y)] < \frac{b \ln K}{K}$

Proof. $h_p(x) = h_p(y)$ when $y - x = 0 \pmod p$.

Let $z = y - x$.

Since $z < 2^b$, then z can have at most b distinct prime factors.

p divides z if p is one of these $\leq b$ prime factors.

number of primes in range $\{1, \ldots, K\}$ is $> \frac{K}{\ln K}$,

hence the probability is $b / \frac{K}{\ln K} = \frac{b \ln K}{K}$

values of K

- higher K = lower probability of false positive
  - for $\delta = \frac{1}{100n}$, P(false positive) < 1%.

$\forall \delta > 0$, if $X \neq Y$ and $K = \frac{2m}{\delta} \cdot \lg \ell \cdot \lg(\frac{2m}{\delta} \lg \ell)$, then $Pr[h(X) = h(Y)] < \delta$

Streaming

problem: Consider a sequence of insertions or deletions of items from a large universe $\mathcal{U}$. At the end of the stream, the *frequency* f_i of item i is its net count.

Let M be the sum of all frequencies at the end of stream.

naive solutions

- direct access table - $\Omega(U)$ space
- sorted list - $\Omega(M)$ space, no $O(1)$ update
- binary search tree - $O(M)$ space

Frequency Estimation

an approximation $\hat{f}_i$ is **ϵ-approximate** if

$$f_i - \epsilon M \leq \hat{f}_i \leq f_i + \epsilon M$$

Using Hash Table

$$f_i \leq \mathbb{E}[\hat{f}_i] \leq f_i + M/k$$

- increment/decrement $A[h(j)]$ on an empty table A of size k
- collision $\Rightarrow$ false positives $\Rightarrow$ may give overestimate of f_i
  - $A[h(i)] = \sum_{j : h(j) = h(i)} f_j \geq f_i$
- if h is drawn from a universal family, overestimate, $\mathbb{E}[A[h(i)] - f_i] \leq M/k$
- space: $O(\frac{1}{\epsilon} \cdot \lg M + \lg U \cdot \lg M)$
  let $k = \frac{1}{\epsilon}$ for some $\epsilon > 0$.
  - number of rows $= O(\frac{1}{\epsilon})$
  - size of each row $= O(\lg M)$
  - size of hash function (using universal hash family from ch.05) $= O(\lg U \cdot \lg M)$
- **Count-Min Sketch** $\rightarrow$ gives a bound on the probability that $\hat{f}_i$ deviates from f_i instead of a bound on the expectation of the gap

07. AMORTIZED ANALYSIS

- **amortized analysis** $\rightarrow$ guarantees the *average* performance of each operation in the *worst case*.
- For a sequence of n operations $o_1, o_2, \ldots, o_n$,
  - let $t(i)$ be the time complexity of the i-th operation o_i
  - let $f(n)$ be the *worst-case* time complexity for *any* of the n operations
  - let $T(n)$ be the time complexity of all n operations

$$T(n) = \sum_{i=1}^{n} t(i) = n f(n)$$

Types of Amortized Analysis

Aggregate method

- look at the whole sequence, sum up the cost of operations and take the average - simpler but less precise
- e.g. binary counter - amortized $O(1)$
- e.g. queues (with `INSERT` and `EMPTY`) - amortized $O(1)$

Accounting method

- charge the i-th operation a fictitious amortized cost $c(i)$
  - **amortized cost** $c(i)$ is a fixed cost for each operation
  - **true cost** $t(i)$ depends on when the operation is called
- amortized cost $c(i)$ must satisfy:

$$\sum_{i=1}^{n} t(i) \leq \sum_{i=1}^{n} c(i) \text{ for all } n$$

- take the extra amount for cheap operations early on as "credit" paid in advance for expensive operations
  - **invariant**: bank balance never drops below 0
- the total amortized cost provides an **upper bound** on the total true cost

Potential method

- ϕ : potential function associated with the algo/DS
- $\phi(i)$: potential at the end of the i-th operation
- c_i : amortized cost of the i-th operation
- t_i : true cost of the i-th operation

$$c_i = t_i + \phi(i) - \phi(i-1)$$
$$\sum_{i=1}^{n} c_i = \phi(n) - \phi(0) + \sum_{i=1}^{n} t_i$$

- hence as long as $\phi(n) \geq 0$, then amortized cost is an upper bound of the true cost.

$$\sum_{i=1}^{n} c_i \geq \sum_{i=1}^{n} t_i$$

- usually take $\phi(0) = 0$

- **e.g.** for queue:
  - let $\phi(i)$ = # of elements in queue after the i-th operation
  - amortized cost for insert:
    $c_i = t_i + \phi(i) - \phi(i-1) = 1 + 1 = 2$
  - amortized cost for empty (for k elements):
    $c_i = t_i + \phi(i) - \phi(i-1) = k + 0 - k = 0$

e.g. Dynamic Table (insertion only)

Aggregate method

cost of n insertions =

$$\sum_{i=1}^{n} t(i) \le n + \sum_{j=1}^{\lfloor \log(n-1) \rfloor} 2^j \le 3n$$

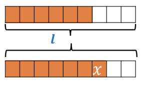
i	1	2	3	4	5	6	7	8	9	10	
$size_i$	1	2	4	4	8	8	8	8	16	16	
t(i)	1	1	1	1	1	1	1	1	1	1	Cost for *i* th insert
		1	2		4				8		Cost for copying due to overflow

Accounting method

- charge \$3 per insertion
  - \$1 for insertion itself
  - \$1 for moving itself when the table expands
  - \$1 for moving one of the existing items when the table expands

Potential method

let $\phi(i) = 2i - size(T)$



2(i-1) - (i-1) =
$\phi(i\text{-}1) = 2(i-1) - l$ Before **Insert**(x)

2i - 2(i-1) =
$\phi(i) = 2i - l$ After **Insert**(x)

Operation **Insert**(x)	Actual Cost	$\Delta\phi_i$	Amortized Cost
Case 1: when table is not full	1	**2**	**3**
Case 2: when table is already full	i	$3 - i$	**3**

Amortized cost of n insertions = $3n = O(n)$

Actual cost of n insertions = $O(n)$

helpful approximations

stirling's approximation: $T(n) = \sum_{i=0}^{n} \log(n-i) = \log \prod_{i=0}^{n}(n-i) = \Theta(n \log n)$

harmonic number, $H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\lg n)$

basel problem: $\sum_{n=1}^{N} \frac{1}{n^2} \leq 2 - \frac{1}{N} \xrightarrow{N \to \infty} 2$

because $\sum_{n=1}^{N} \frac{1}{N^2} \leq 1 + \sum_{x=2}^{\log_3 n} \frac{1}{(x-1)x} = 1 + \sum_{n=2}^{N}(\frac{1}{n-1} - \frac{1}{n}) = 1 + 1 - \frac{1}{N} = 2 - \frac{1}{N}$

number of primes in range $\{1, \ldots, K\}$ is $> \frac{K}{\ln K}$

asymptotic bounds

$1 < \log n < \sqrt{n} < n < n \log n < n^2 < n^3 < 2^n < 2^{2n}$
$\log_a n < n^a < a^n < n! < n^n$
for any $a, b > 0, \quad \log_a n < n^b$

multiple parameters

for two functions $f(m,n)$ and $g(m,n)$, we say that $f(m,n) = O(g(m,n))$ if there exists constants c, m_0, n_0 such that $0 \leq f(m,n) \leq c \cdot g(m,n)$ for all $m \geq m_0$ or $n \geq n_0$.

set notation

- $O(g(n)) = \{f(n) : \exists c, n_0 > 0 \mid \forall n \geq n_0,\ 0 \leq f(n) \leq cg(n)\}$
- $\Omega(g(n)) = \{f(n) : \exists c, n_0 > 0 \mid \forall n \geq n_0,\ 0 \leq cg(n) \leq f(n)\}$
- $\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\} = O(g(n)) \cap \Omega(g(n))$
- $o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq f(n) < cg(n)\}$
- $\omega(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \mid \forall n \geq n_0, \quad 0 \leq cg(n) < f(n)\}$

example proofs

Proof. that $2n^2 = O(n^3)$
let $f(n) = 2n^2$. then $f(n) = 2n^2 \leq n^3$ when $n \geq 2$.
set $c = 1$ and $n_0 = 2$.
we have $f(n) = 2n^2 \leq c \cdot n^3$ for $n \geq n_0$. □

Proof. $n = o(n^2)$
For any $c > 0$, use $n_0 = 2/c$.

Proof. $n^2 - n = \omega(n)$
For any $c > 0$, use $n_0 = 2(c+1)$.

Example. let $f(n) = n$ and $g(n) = n^{1+\sin(n)}$.
Because of the oscillating behaviour of the sine function, there is no n_0 for which f dominates g or vice versa.
Hence, we cannot compare f and g using asymptotic notation.

Example. let $f(n) = n$ and $g(n) = n(2 + \sin(n))$.
Since $\frac{1}{3}g(n) \leq f(n) \leq g(n)$ for all $n \geq 0$, then $f(n) = \Theta(g(n))$. (note that limit rules will not work here)

mentioned algorithms

- ch.3 - **Misra Gries** - space-efficient computation of the majority bit in array A
- ch.3 - **Euclidean** - efficient computation of GCD of two integers
- ch.3 - **Tower of Hanoi** - $T(n) = 2^n - 1$
  1. move the top $n-1$ discs from the first to the second peg using the third as temporary storage.
  2. move the biggest disc directly to the empty third peg.
  3. move the $n-1$ discs from the second peg to the third using the first peg for temporary storage.
- ch.3 - **MergeSort** - $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n)$
- ch.3 - **Karatsuba Multiplication** - multiply two n-digit numbers x and y in $O(n^{\log_2 3})$
  - worst-case runtime: $T(n) = 3T(\lceil n/2 \rceil) + \Theta(n)$

uncommon notations

- $\perp$ - false