

MA1102R

AY20/21 sem 2

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00. FUNCTIONS & SETS

sets

$$A = \{x \mid \text{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

operations on sets

- union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$

notations of sets

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$
- $\mathbb{N} = \mathbb{Z}^+$
- \emptyset : empty set

notations of intervals

- closed interval (inclusive): $[a, b] = \{x \mid a \leq x \leq b\}$
- open interval (exclusive): $(a, b) = \{x \mid a < x < b\}$
- $(a, \infty) = \{x \mid a < x\}$

functions

- existence:** $\forall a \in A, f(a) \in B$
- uniqueness:** $\forall a \in A$ has only one image in B .
- for $f : A \rightarrow B$
 - domain: A
 - codomain: B
 - range: $\{f(x) \mid x \in A\}$
- for this mod:
 - $A, B \subseteq \mathbb{R}$
 - if A is not stated, the domain of f is the largest possible set for which f is defined
 - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set $G(f) := \{(x, f(x)) \mid x \in A\}$

- if $A, B \subseteq \mathbb{R}$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane \mathbb{R}^2

algebra of functions

function	domain
$(f + g)(x) := f(x) + g(x)$	$A \cap B$
$(f - g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

types of functions

- rational function:** $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
 - every polynomial is a rational function ($Q(x) = 1$)
- algebraic function:** constructed from polynomials using algebraic operations

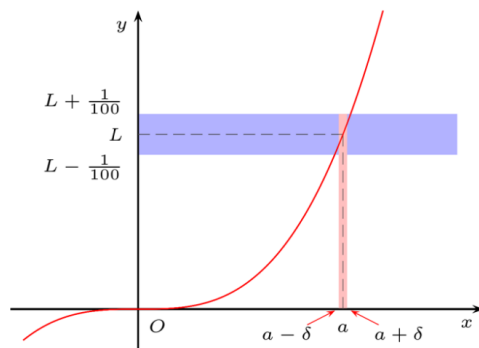
- a function f is **increasing** on a set I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.
- even/odd:
 - even function:** $\forall x, f(-x) = f(x)$
 - * symmetric about the y -axis
 - odd function:** $\forall x, f(-x) = -f(x)$
 - * symmetric about the origin O
- any function defined on \mathbb{R} can be decomposed *uniquely* into the sum of an even function and an odd function
- power function:** x^n
 - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

01. LIMITS

precise definition of limits

Let f be a function defined on an open interval containing a , except possibly at a .

The limit of $f(x)$ as x approaches a , equals L if, for every $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$



informally,

- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a .
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L .

limit laws

- Let $c \in \mathbb{R}$. $\lim_{x \rightarrow a} c = c$
- $\lim_{x \rightarrow a} x = a$
- Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant.
 - $\lim_{x \rightarrow a} (cf(x)) = cL = c \lim_{x \rightarrow a} f(x)$
 - $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
 - $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
 - $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
 - $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$
 - $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
 - $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

$$\text{if } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \rightarrow a} g(x) = 0, \text{ then } \lim_{x \rightarrow a} f(x) = 0$$

inequalities on limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

lemma

if $f(x) \leq g(x)$ for all x near a (except possibly at a), then $L \leq M$.

lemma

If $f(x) \geq 0$ for all x , then $L \geq 0$.

direct substitution property

Let f be a polynomial or rational function.

If a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

If $f(x) = g(x)$ for all x near a except possibly at a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

applications

- if a is not in the domain (e.g. 0 denominator), don't apply directly
- convert to an equivalent function and then sub in

one-sided limits

- limit laws also hold for one-sided limits

If as x is close to a from the right, $f(x)$ is close to L , the right-hand limit of f as x approaches a equals L .
 $(x \rightarrow a^+ \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^+} f(x) = L$

If as x is close to a from the left, $f(x)$ is close to L , the left-hand limit of f as x approaches a equals L .
 $(x \rightarrow a^- \Rightarrow f(x) \rightarrow L) \Rightarrow \lim_{x \rightarrow a^-} f(x) = L$

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

$$f(x) \rightarrow L \Leftrightarrow x \rightarrow a \Leftrightarrow \begin{cases} x \rightarrow a^+ \Rightarrow f(x) \rightarrow L \\ x \rightarrow a^- \Rightarrow f(x) \rightarrow L \end{cases}$$

definition of one-sided limits

$$\text{LH Limit: } \lim_{x \rightarrow a^-} f(x) = L$$

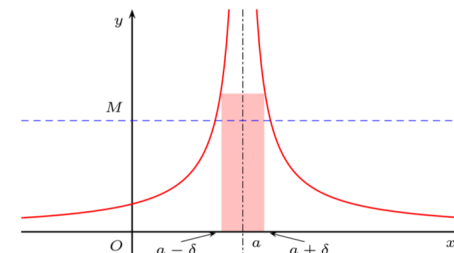
if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$

$$\text{RH Limit: } \lim_{x \rightarrow a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$

definition of infinite limits

$\lim_{x \rightarrow a} f(x) = \infty$
 if for every $M > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > M$



negative infinite limit:

$0 < |x - a| < \delta \Rightarrow f(x) < M$

limits to infinity

$$\lim_{x \rightarrow \infty} f(x) = L:$$

For every $\epsilon > 0$, there exists N such that $x > N \Rightarrow |f(x) - L| < \epsilon$

$$\lim_{x \rightarrow \infty} f(x) = \infty:$$

For every $M > 0$, there exists N such that $x > N \Rightarrow f(x) > M$

squeeze theorem

- Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where
 - $g(x) \leq f(x) \leq h(x)$ for all x near a (except at a),
 - and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$.
 Then $\lim_{x \rightarrow a} f(x) = L$

02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at a** \Leftrightarrow
 f is continuous from the left and from the right at a .
 $\lim_{x \rightarrow a} f(x) = f(a)$

a function f is **continuous at an interval** if it is continuous at every number in the interval.

f is continuous on **open interval** (a, b)
 $\Leftrightarrow f$ is continuous at every $x \in (a, b)$
 f is continuous on **closed interval** $[a, b]$
 $\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$

continuity test

f is continuous at $a \Leftrightarrow$

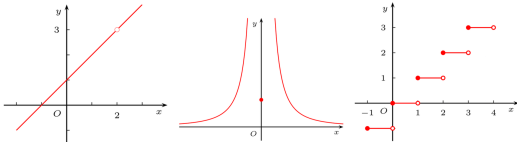
- f is defined at a (a is in the domain of f)
- $\lim_{x \rightarrow a} f(x)$ exists
- $\lim_{x \rightarrow a} f(x) = f(a)$

precise definition of continuity

a function f is continuous at a number a if $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

examples of discontinuity

- removable discontinuity
- infinite discontinuity
- jump discontinuity



properties of continuous functions

let f and g be functions continuous at a . let c be a constant.

- cf is continuous at a
- $f + g$ is continuous at a
- $f - g$ is continuous at a
- fg is continuous at a
- $\frac{f}{g}$ is continuous at a , provided $g(a) \neq 0$

other properties

- a polynomial is continuous everywhere;
- a rational function is continuous on its domain
- let c be a real number. $f(x) = c$ is continuous on \mathbb{R} .
- $f(x) = x$ is continuous on \mathbb{R} .

trigonometric functions

- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous everywhere
- $\tan x, \sec x$ are continuous whenever $\cos x \neq 0$
- $\cot x, \csc x$ are continuous whenever $\sin x \neq 0$
 - domain: $\mathbb{R} \setminus \{0, \pm\pi, \pm2\pi, \dots\}$

composite of continuous functions

if f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$

if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .
 $\lim_{x \rightarrow a} (f \circ g)(x) = (f \circ g)(a)$

substitution theorem

Suppose $y = f(x)$ such that $\lim_{x \rightarrow a} f(x) = b$. If

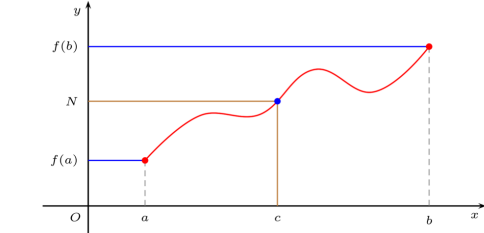
- g is continuous at b , OR
- $\forall x$ near a , except at a , $f(x) \neq b$ and $\lim_{y \rightarrow b} g(y)$ exists

Then $\lim_{x \rightarrow a} g(f(x)) = \lim_{y \rightarrow b} g(y)$

intermediate value theorem

Let f be a function continuous on $[a, b]$ with $f(a) \neq f(b)$.

Let N be a number between $f(a)$ and $f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = N$.



03. DERIVATIVES

tangent line

the **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:
 $y = f'(a)(x - a) + f(a)$

definition of derivatives

- f is differentiable at a if $f'(a)$ exists
- $f'(a)$ is the slope of $y = f(x)$ at $x = a$

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

- $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x) = \dots$
- $\frac{dy}{dx} := \lim_{x \rightarrow 0} \frac{\Delta y}{\Delta x}$ (derivative of y with respect to x)
- $f'(a) = \frac{dy}{dx} |_{x=a}$

differentiable functions

- f is differentiable at a if $f'(a) := \lim_{x \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists.
- f is differentiable on (a, b) if f is differentiable at every $c \in (a, b)$

differentiability & continuity

- if f is differentiable at a , then f is continuous at a .
 - differentiability \Rightarrow continuity
- continuity \nRightarrow differentiability

differentiation

- every polynomial and rational function is differentiable on its domain
 - the domain of f' may be smaller than the domain of f .
- trigonometric functions are differentiable on the domain

chain rule

If g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and $F'(a) = (f \circ g)'(a) = f'(b)g'(a) = f'(g(a))g'(a)$

If $z = f(y)$ and $y = g(x)$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$
$$\frac{dz}{dx} |_{x=a} = \frac{dz}{dy} |_{y=b} \frac{dy}{dx} |_{x=a}$$

generalised chain rule

h is differentiable at a ; g is differentiable at $B = h(a)$; f is differentiable at $c = g(b)$.

$$(f \circ (g \circ h))' = f' \circ (g \circ h) \cdot (g \circ h)' = f'(c)g'(b)h'(a)$$

Leibniz notation:

If $y = h(x), z = g(y), w = f(z)$,

$$\frac{dw}{dx} = \frac{dw}{dz} \frac{dz}{dy} \frac{dy}{dx}$$

implicit differentiation

- assumes that $\frac{dy}{dx}$ exists

second derivative

$$f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}$$
$$f' = D(f) \Rightarrow f'' := D^2(f)$$

higher derivatives

$$f^{(0)} := f$$

For any positive integer $n, f^{(n)} := (f^{(n-1)})'$

if $y = f(x)$, then $f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x)$

04. APPLICATIONS OF DIFFERENTIATION

extreme values of functions

Let f be a function with domain D .

global (absolute) max/min

- aka absolute max/min
- extreme values = absolute maximum and absolute minimum

f has a global **maximum** at $c \in D$
 $\Leftrightarrow f(c) \geq f(x)$ for all $x \in D$

f has a global **minimum** at $c \in D$
 $\Leftrightarrow f(c) \leq f(x)$ for all $x \in D$

local max/min

- aka relative max/min aka "turning points"

f has a local **maximum** at $c \in D$
 $\Leftrightarrow f(c) \geq f(x)$ for all x near c

f has a local **minimum** at $c \in D$
 $\Leftrightarrow f(c) \leq f(x)$ for all x near c

extreme value theorem

- existence**
if f is *continuous* on a *finite closed* interval $[a, b]$, then f attains extreme values on $[a, b]$.
- value**
the extreme value occurs at either *critical numbers* or the *endpoints* ($x = a, x = b$).

critical numbers

Then $c \in D$ is a *critical number* of f if $f'(c) = 0$, or $f'(c)$ does not exist.

fermat's theorem

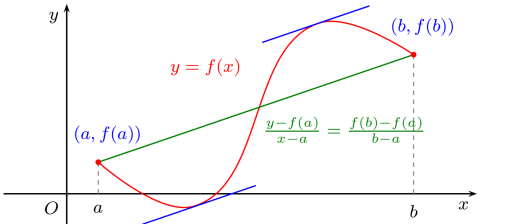
Suppose f has a local maximum or minimum at c . If $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function such that f is *continuous* on $[a, b]$, f is *differentiable* on (a, b) , and $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

mean value theorem

Let f be a function such that f is *continuous* on $[a, b]$ and f is *differentiable* on (a, b) . Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



- generalisation of Rolle's theorem when $f(a) = f(b)$.

ordinary differential equations

Let f and g be continuous on $[a, b]$. If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ on $[a, b]$ for a constant C .

misc

triangle inequality

$$|a + b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R}$$

binomial theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$
$$= a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$

where the binomial coefficient is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

factorisation

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + \dots + a b^{n-2} + b^{n-1})$$