

01. COMBINATORIAL ANALYSIS

tricky - E18, E20-22, E23, E26

The Basic Principle of Counting

- combinatorial analysis** → the mathematical theory of counting
- basic principle of counting** → Suppose that two experiments are performed. If exp1 can result in any one of m possible outcomes and if, for each outcome of exp1, there are n possible outcomes of exp2, then together there are mn possible outcomes of the two experiments.
- generalized basic principle of counting** → If r experiments are performed such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes, and if ..., then there is a total of $n_1 \cdot n_2 \cdot \dots \cdot n_r$ possible outcomes of r experiments.

Permutations

factorials - $1! = 0! = 1$

N1 - if we know how to count the number of different ways that an event can occur, we will know the probability of the event.

N2 - there are $n!$ different arrangements for n objects.

N3 - there are $\frac{n!}{n_1! n_2! \dots n_r!}$ different arrangements of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike.

Combinations

N4 - $\binom{n}{r} = \frac{n!}{(n-r)! r!}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

N4b - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, $1 \leq r \leq n$

Proof. If object 1 is chosen $\Rightarrow \binom{n-1}{r-1}$ ways of choosing the remaining objects.

If object 1 is not chosen $\Rightarrow \binom{n-1}{r}$ ways of choosing the remaining objects.

N5 - The Binomial Theorem - $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Proof. by mathematical induction: $n = 1$ is true; expand; sub dummy variable; combine using N4b; combine back to final term

Multinomial Coefficients

N6 - $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes n_1, n_2, \dots, n_r , where $n_1 + n_2 + \dots + n_r = n$

Proof. using basic counting principle,

$$\begin{aligned} &= \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots \binom{n-n_1-n_2-\dots-n_{r-1}}{n_r} \\ &= \frac{n!}{(n-n_1)! n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)! n_2!} \times \dots \times \frac{(n-n_1-n_2-\dots-n_{r-1})!}{0! n_r!} \\ &= \frac{n!}{n_1! n_2! \dots n_r!} \end{aligned}$$

N7 - The Multinomial Theorem: $(x_1 + x_2 + \dots + x_r)^n = \sum_{(n_1, \dots, n_r): n_1 + n_2 + \dots + n_r = n} \frac{n!}{n_1! n_2! \dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$

Number of Integer Solutions of Equations

N8 - there are $\binom{n-1}{r-1}$ distinct *positive* integer-valued vectors (x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$, $x_i > 0$, $i = 1, 2, \dots, r$

! cannot be directly applied to N8 as 0 value is not included

N9 - there are $\binom{n+r-1}{r-1}$ distinct *non-negative* integer-valued vectors

(x_1, x_2, \dots, x_r) satisfying $x_1 + x_2 + \dots + x_r = n$

Proof. let $y_k = x_k + 1 \Rightarrow y_1 + y_2 + \dots + y_r = n + r$

02. AXIOMS OF PROBABILITY

Sample Space and Events

- sample space** → The *set* of all outcomes of an experiment (where outcomes are not predictable with certainty)
- event** → Any *subset* of the sample space
- union** of events E and $F \rightarrow E \cup F$ is the event that contains all outcomes that are either in E or F (or both).
- intersection** of events E and $F \rightarrow E \cap F$ or EF is the event that contains all outcomes that are both in E and in F .
- complement** of $E \rightarrow E^c$ is the event that contains all outcomes that are *not* in E .
- subset** → $E \subset F$ if all of the outcomes in E that are also in F .
 - $E \subset F \wedge F \subset E \Rightarrow E = F$

DeMorgan's Laws

$$\left(\bigcup_{i=1}^n E_i \right)^c = \bigcap_{i=1}^n E_i^c$$

Proof. to show $LHS \subset RHS$: let $x \in \left(\bigcup_{i=1}^n E_i \right)^c$
 $\Rightarrow x \notin \bigcup_{i=1}^n E_i \Rightarrow x \notin E_1$ and $x \notin E_2 \dots$ and $x \notin E_n$
 $\Rightarrow x \in E_1^c$ and $x \in E_2^c \dots$ and $x \in E_n^c$
 $\Rightarrow x \in \bigcap_{i=1}^n E_i^c$
 to show $RHS \subset LHS$: let $x \in \bigcap_{i=1}^n E_i^c$

$$\left(\bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c$$

Proof. using the first law of DeMorgan, negate LHS to get RHS

Axioms of Probability

definition 1: relative frequency

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

problems with this definition:

- $\frac{n(E)}{n}$ may not converge when $n \rightarrow \infty$
- $\frac{n(E)}{n}$ may not converge to the same value if the experiment is repeated

definition 2: Axioms

Consider an experiment with sample space S . For each event E of the sample space S , we assume that a number $P(E)$ is defined and satisfies the following 3 axioms:

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- For any sequence of mutually exclusive events E_1, E_2, \dots (i.e., events for which $E_i E_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E)$ is the probability of event E .

Simple Propositions

N1 - $P(\emptyset) = 0$

N2 - $P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$ (aka axiom 3 for a finite n)

N3 - strong law of large numbers - if an experiment is repeated over and over again, then with probability 1, the proportion of time during which any specific event E occurs will be equal to $P(E)$.

N6 - the definitions of probability are mathematical definitions. They tell us which set functions can be called **probability functions**. They do not tell us what value a probability function $P(\cdot)$ assigns to a given event E .

probability function \iff it satisfies the 3 axioms.

N7 - $P(E^c) = 1 - P(E)$

N8 - if $E \subset F$, then $P(E) \leq P(F)$

N9 - $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

N10 - Inclusion-Exclusion identity where $n = 3$

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$

N11 - Inclusion-Exclusion identity -

$$\begin{aligned} P(E_1 \cup E_2 \cup \dots \cup E_n) &= \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \dots \\ &\quad + (-1)^{r+1} \sum_{i_1 < i_2 < \dots < i_r} P(E_{i_1} E_{i_2} \dots E_{i_r}) + \dots \\ &\quad + (-1)^{n+1} P(E_1 E_2 \dots E_n) \end{aligned}$$

Proof. Suppose an outcome with probability ω is in exactly m of the events E_i , where $m > 0$. Then

LHS: the outcome is in $E_1 \cup E_2 \cup \dots \cup E_n$ and ω will be counted once in $P(E_1 \cup E_2 \cup \dots \cup E_n)$

RHS:

- the outcome is in exactly m of the events E_i and ω will be counted exactly $\binom{m}{1}$ times in $\sum_{i=1}^n P(E_i)$

- the outcome is contained in $\binom{m}{2}$ subsets of the type $E_{i_1} E_{i_2}$ and ω will be counted $\binom{m}{2}$ times in $\sum_{i_1 < i_2} P(E_{i_1} E_{i_2})$

- ... and so on

hence $RHS = \binom{m}{1} \omega - \binom{m}{2} \omega + \binom{m}{3} \omega - \dots \pm \binom{m}{m} \omega$

$$\begin{aligned} &= \omega \sum_{i=0}^m \binom{m}{i} (-1)^i = \text{binomial theorem where } x = -1, y = 1 \\ &= 0 = LHS \end{aligned}$$

e.g. For an outcome with probability ω and $n = 3$

- Case 1.** $w = P(E_1 E_2)$
 LHS = ω
 RHS = $(\omega + \omega + 0) - (\omega + 0 + 0) + 0 = \omega$
- Case 2.** $\omega = P(E_1 \cap E_2 \cap E_3)$
 LHS = ω
 RHS = $(\omega + \omega + \omega) - (\omega + \omega + \omega) + \omega = \omega$

N12 -

- $P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i)$
- $P\left(\bigcup_{i=1}^n E_i\right) \geq \sum_{i=1}^n P(E_i) - \sum_{j < i} P(E_i E_j)$
- $P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i) - \sum_{j < i} P(E_i E_j) + \sum_{k < j < i} P(E_i E_j E_k)$
- and so on.

Proof. $\bigcup_{i=1}^n E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \dots \cup E_1^c E_2^c \dots E_{n-1}^c E_n$

$$P\left(\bigcup_{i=1}^n E_i\right) = P(E_1) + P(E_1^c E_2) + P(E_1^c E_2^c E_3) + \dots + P(E_1^c E_2^c \dots E_{n-1}^c E_n)$$

Sample Space having Equally Likely Outcomes

tricky - 14, 15, 16, 18, 19, 20

Consider an experiment with sample space $S = \{e_1, e_2, \dots, e_n\}$. Then $P(\{e_1\}) = P(\{e_2\}) = \dots = P(\{e_n\}) = \frac{1}{n}$ or $P(\{e_i\}) = \frac{1}{n}$.

N1 - for any event E , $P(E) = \frac{\# \text{ of outcomes in } E}{\# \text{ of outcomes in } S} = \frac{\# \text{ of outcomes in } E}{n}$

increasing sequence of events $\{E_n, n \geq 1\} \rightarrow$

$E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$

lim_{n -> infinity} E_n = union_{i=1}^infinity E_i

decreasing sequence of events {E_n, n ≥ 1} -> E_1 supset E_2 supset ... supset E_n supset E_{n+1} supset ...

lim_{n -> infinity} E_n = intersection_{i=1}^infinity E_i

03. CONDITIONAL PROBABILITY AND INDEPENDENCE

tricky - E6, urns (p.37)

Conditional Probability

- N1 - if P(F) > 0. then P(E|F) = (P(E ∩ F) / P(F))
- N2 - multiplication rule - P(E_1 E_2 ... E_n) = P(E_1) P(E_2 | E_1) P(E_3 | E_1 E_2) ... P(E_n | E_1 E_2 ... E_{n-1})
- N3 - axioms of probability apply to conditional probability

- 1. 0 ≤ P(E|F) ≤ 1
- 2. P(S|F) = 1 where S is the sample space
- 3. If E_i (i ∈ Z_{≥1}) are mutually exclusive events, then

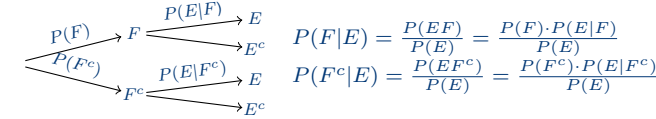
P((union_{i=1}^infinity E_i) | F) = sum_{i=1}^infinity P(E_i | F)

- N4 - If we define Q(E) = P(E|F), then Q(E) can be regarded as a probability function on the events of S, hence all results previously proved for probabilities apply.
- Q(E_1 ∪ E_2) = Q(E_1) + Q(E_2) - Q(E_1 E_2)
- P(E_1 ∪ E_2 | F) = P(E_1 | F) + P(E_2 | F) - P(E_1 E_2 | F)

Total Probability & Bayes' Theorem

conditioning formula - P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)

tree diagram -



Total Probability

theorem of total probability - Suppose F_1, F_2, ..., F_n are mutually exclusive events such that union_{i=1}^n F_i = S, then P(E) = sum_{i=1}^n P(EF_i) = sum_{i=1}^n P(F_i)P(E|F_i)

Bayes Theorem

P(F_j | E) = (P(EF_j) / P(E)) = (P(F_j)P(E|F_j) / sum_{i=1}^n P(F_i)P(E|F_i))

application of bayes' theorem

P(B_1 | A) = (P(A|B_1) * P(B_1)) / (P(A|B_1) * P(B_1) + P(A|B_2) * P(B_2))

Let A be the event that the person test positive for a disease.
B_1: the person has the disease. B_2: the person does not have the disease.

true positives: P(B_1 A)	false negatives: P(A-bar B_1)
false positives: P(A B_2)	true negatives: P(A-bar B_2)

Independent Events

- N1 - E and F are independent <=> P(EF) = P(E) * P(F)
- N2 - E and F are independent <=> P(E|F) = P(E)
- N3 - if E and F are independent, then E and F^c are independent.
- N4 - if E, F, G are independent, then E will be independent of any event formed from F and G. (e.g. F ∪ G)
- N5 - if E, F, G are independent, then P(EFG) = P(E)P(F)P(G)

- N6 - if E and F are independent and E and G are independent, $\nRightarrow E$ and FG are independent
- N7 - For independent trials with probability p of success, probability of m successes before n failures, for $m, n \geq 1$, $method\ 1$ $method\ 2$ $P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$ = P(exactly k successes in $m + n - 1$ trials)

recursive approach to solving probabilities: see page 85 alternative approach

04. RANDOM VARIABLES

- random variable -> a real-valued function defined on the sample space

Types of Random Variables

- X is a Bernoulli r.v. with parameter p if -> $p(x) = \begin{cases} p, & x = 1, \text{ ('success')} \\ 1-p, & x = 0 \text{ ('failure')} \end{cases}$
- Y is a Binomial r.v. with parameters n and p -> $Y = X_1 + X_2 + \dots + X_n$ where X_1, X_2, \dots, X_n are independent Bernoulli r.v.'s with parameter p.
 - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
 - P(k successes from n independent trials each with probability p of success)
 - e.g. number of red balls out of n balls drawn with replacement
 - $E(Y) = np, \text{ Var}(Y) = np(1-p)$
- Negative Binomial -> X = number of trials until k successes are obtained
 - e.g. number of balls drawn (with replacement) until k red balls are obtained
- Geometric -> X = number of trials until a success is obtained
 - $P(X = k) = (1-p)^{k-1} \cdot p$ where k is the number of trials needed
 - e.g. number of balls drawn (with replacement) until 1 red ball is obtained
- Hypergeometric -> X = number of trials until success, without replacement
 - e.g. number of red balls out of n balls drawn without replacement

Summary

binomial	X = # of successes in n trials w/ replacement	np
negative binomial	X = # of trials until k successes	k/p
geometric	X = # of trials until a success	1/p
hypergeometric	X = # of successes in n trials, no replacement	rn/N

Properties

- N1 - if $X \sim \text{Binomial}(n, p)$, and $Y \sim \text{Binomial}(n - 1, p)$, then $E(X^k) = np \cdot E[(Y + 1)^{k-1}]$
- N2 - if $X \sim \text{Binomial}(n, p)$, then for $k \in \mathbb{Z}^+$, $P(X = k) = \frac{(n-k+1)p}{k(1-p)} \cdot P(X = k - 1)$

Coupon Collector Problem

Q. Suppose there are N distinct types of coupons. If T denotes the number of coupons needed to be collected for a complete set, what is P(T = n)?

- A. $P(T > n - 1) = P(T \geq n) = P(T = n) + P(T > n)$
=> $P(T = n) = P(T > n - 1) - P(T > n)$ Let
$A_j = \{\text{no type } j \text{ coupon is contained among the first } n\}$
$P(T > n) = P(\cup_{j=1}^N A_j)$

Using the inclusion-exclusion identity,
$P(T > n) = \sum P(A_j) - \sum_{j_1, j_2} P(A_{j_1} A_{j_2}) + \dots + (-1)^{k+1} \sum_{j_1, j_2} \dots \sum_{j_k} P(A_{j_1} A_{j_2} \dots A_{j_n}) + \dots + (-1)^{N+1} P(A_1 A_2 \dots A_N)$

P(A_{j_1} A_{j_2} ... A_{j_n}) = ((N-k)/N)^n

Hence P(T > n) = sum_{i=1}^{N-1} (N choose i) ((N-1)/N)^n (-1)^{i+1}

Probability Mass Function

- for a discrete r.v., we define the probability mass function (pmf) of X by $p(a) = P(X = a)$
 - cdf, $F(a) = \sum p(x)$ for all $x \leq a$
 - if X assumes one of the values x_1, x_2, \dots, then $\sum_{i=1}^{\infty} p(x_i) = 1$
 - the pmf $p(a)$ is positive for at most a countable number of values of a
- e.g. $\frac{a}{p(a)} \mid \frac{1}{2}, \frac{2}{4}, \frac{4}{4}$
- discrete variable -> a random variable that can take on at most a countable number of possible values

Cumulative Distribution Function

- for a r.v. X, the function F defined by $F(x) = P(X \leq x)$, <math>-\infty < x < \infty</math>, is called the cumulative distribution function (cdf) of X.
 - aka distribution function
 - F(x) is defined on the entire real line
- e.g. <math>F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \leq a < 2 \\ \frac{3}{4}, & 2 \leq a < 4 \\ 1, & a \geq 4 \end{cases}</math>

Expected Value

- aka population mean/sample mean, μ
- if X is a discrete random variable having pmf p(x), the expectation or the expected value of X is defined as $E(X) = \sum_x x \cdot p(x)$

N1 - if a and b are constants, then $E(aX + b) = aE(X) + b$

N2 - the nth moment of of X is given as $E(X^n) = \sum_x x^n \cdot p(x)$

- I is an indicator variable for event A if $I = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } A^c \text{ occurs} \end{cases}$. then $E(I) = P(A)$.

Proof of N1. $E(aX + b) = \sum_x (aX + b)p(x) = a \cdot \sum_x xp(x) + b \cdot \sum_x p(x) = a \cdot E(X) + b$

finding expectation of f(x)

- method 1, using pmf of Y: let $Y = f(X)$. Find corresponding X for each Y.
- method 2, using pmf of X: $E[g(x)] = \sum_i g(x_i) p(x_i)$
 - where X is a discrete r.v. that takes on one of the values of x_i with the respective probabilities of p(x_i), and g is any real-valued function g

Variance

- If X is a r.v. with mean $\mu = E[X]$, then the variance of X is defined by $Var(X) = E[(X - \mu)^2]$
=$\sum x_i (x_i - \mu)^2 \cdot p(x_i)$ (deviation · weight)
=$E(x^2) - [E(x)]^2$
- $Var(aX + b) = a^2 Var(x)$

Poisson Random Variable

a r.v. X is said to be a **Poisson r.v.** with parameter λ if for some $\lambda > 0$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

- notation: $X \sim \text{Poisson}(\lambda)$
- $\sum_{i=0}^\infty P(X = i) = 1$
- Poisson Approximation of Binomial** - if $X \sim \text{Binomial}(n, p)$, n is large and p is small, then $X \sim \text{Poisson}(\lambda)$ where $\lambda = np$.
 - For n independent trials with probability p of success, the number of successes is approximately a *Poisson r.v.* with parameter $\lambda = np$ if n is large & p is small.
 - Poisson approximation remains even when the trials are not independent, provided that their *dependence is weak*.
- 2 ways** to look at the Poisson distribution
 - an approximation to the binomial distribution with large n and small p
 - counting the number of events that occur at *random* at certain points in time

Mean and Variance

if $X \sim \text{Poisson}(\lambda)$, then $E(X) = \lambda, \text{Var}(X) = \lambda$

Poisson distribution as random events

Let $N(t)$ be the number of events that occur in time interval $[0, t]$.

N1 - If the 3 assumptions are true, then $N(t) \sim \text{Poisson}(\lambda t)$.

N2 - If λ is the rate of occurrences of events per unit time, then the number of occurrences in an interval of length t has a Poisson distribution with mean λt .

$$P(N(t) = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \text{ for } k \in \mathbb{Z}_{\geq 0}$$

o(h) notation

$o(h)$ stands for any function $f(h)$ such that $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$

- a function of h that is *small* compared to h when h is small
- $o(h) + o(h) = o(h)$
- $\frac{\lambda t}{n} + o(\frac{t}{n}) \approx \frac{\lambda t}{n}$ for large n

Expected Value of sum of r.v.

For a r.v. X , let $X(s)$ denote the value of X when $s \in S$

N1 - $E(x) = \sum_i x_i P(X = x_i) = \sum_{s \in S} X(s) p(s)$ where $S_i = \{s : X(s) = x_i\}$

N2 - $E(\sum_{i=1}^n) = \sum_{i=1}^n E(X_i)$ for r.v. X_1, X_2, \dots, X_n

examples

Selecting hats problem

Let n be the number of men who select their own hats. Let I_E be an indicator r.v. for E . E_i is the event that the i -th man selects his own hat. Let X be the number of men that select their own hats.

- $X = I_{E_1} + I_{E_2} + \dots + I_{E_n}$
- $P(E_i) = \frac{1}{n}$
- $P(E_i | E_j) = \frac{1}{n-1} \neq P(E_j)$ for $j < i$ (hence E_i and E_j are not independent)
 - but dependence is weak for large n
- X satisfies the other conditions for binomial r.v., besides independence (n trials with equal probability of success)
- Poisson approximation of X : $X \sim \text{Poisson}(\lambda)$
 - $\lambda = n \cdot P(E_i) = n \cdot \frac{1}{n} = 1$
 - $P(X = i) = \frac{e^{-1} 1^i}{i!} = \frac{e^{-1}}{i!}$
 - $P(X = 0) = e^{-1} \approx 0.37$

No 2 people have the same birthday

For $\binom{n}{2}$ pairs of individuals i and $j, i \neq j$, let E_{ij} be the event where they have the same birthday. Let X be the number of pairs with the same birthday.

- $X = I_{E_{12}} + I_{E_{13}} + \dots + I_{E_{nn}}$
- Each E_{ij} is only *pairwise independent*. $P(E_{ij}) = \frac{1}{365}$

- i.e. E_{ij} and E_{mn} are independent
- but E_{12} and $(E_{13} \cap E_{23})$ are not independent $\Rightarrow P(E_{12} | E_{13} \cap E_{23}) = 1$
- $X \sim \text{Poisson}(\lambda), \lambda = \frac{\binom{n}{2}}{365} = \frac{n(n-1)}{730} \Rightarrow P(X = 0) = e^{-\frac{n(n-1)}{730}}$
 - for $P(X = 0) \leq \frac{1}{2}, n \geq 23$

distribution of time to next event

Q. suppose an accident happens at a rate of 5 per day. Find the distribution of time, starting from now, until the next accident.

- A. Let X = time (in days) until the next accident.
Let V = be the number of accidents during time period $[0, t]$.

$V \sim \text{Poisson}(5t) \Rightarrow P(V = k) = \frac{e^{-5t} \cdot (5t)^k}{k!}$

$P(X > t) = P(\text{no accidents happen during } [0, t]) = P(V = 0) = e^{-5t}$

$P(X \leq t) = 1 - e^{-5t}$

05. CONTINUOUS RANDOM VARIABLES

X is a **continuous r.v.** \rightarrow if there exists a nonnegative function f defined for all real $x \in (-\infty, \infty)$, such that $P(X \in B) = \int_B f(x) dx$

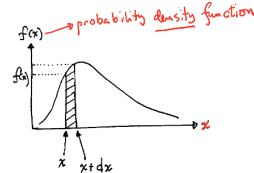
N1 - $P(X \in (-\infty, \infty)) = \int_{-\infty}^\infty f(x) dx = 1$

N2 - $P(a \leq X \leq b) = \int_a^b f(x) dx$

N3 - $P(X = a) = \int_a^a f(x) dx = 0$

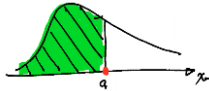
N4 - $P(X < a) = P(X \leq a) = \int_{-\infty}^a f(x) dx$

N5 - interpretation of **probability density function**



$$P(x < X < x + dx) = \int_x^{x+dx} f(y) dy \approx f(x) \cdot dx$$

pdf at $x, f(x) \approx \frac{P(x < X < x + dx)}{dx}$



N6 - if X is a continuous r.v. with pdf $f(x)$ and cdf $F(x)$, then $f(x) = \frac{d}{dx} F(x)$. (Fundamental Theorem of Calculus)

N7 - median of X, x occurs where $F(x) = \frac{1}{2}$

Generating a Uniform r.v.

if X is a continuous r.v. with cdf $F(x)$, then

• **N8** - $F(X) = U \sim \text{uniform}(0, 1)$.

Proof. let $Y = F(X)$. then cdf of $Y, F_Y(y) = P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.
hence Y is a uniform r.v.

- N9** - $X = F^{-1}(U) \sim \text{cdf } F(x)$.
 - generating a r.v. from a uniform(0, 1) r.v. and a r.v. with cdf $F(x)$.

Expectation & Variance

expectation

N1 - **expectation of X** , $E(X) = \int_{-\infty}^\infty x \cdot f(x) dx$

N2 - if X is a continuous r.v. with pdf $f(x)$, then for any real-valued function g ,
 $E[g(x)] = \int_{-\infty}^\infty g(x) f(x) dx$

N2a $E[aX + b] = \int_{-\infty}^\infty (aX + b) \cdot f(x) dx = a \cdot E(X) + b$

N3 - for a non-negative r.v. $Y, E(Y) = \int_0^\infty P(Y > y) dy$

Proof. $\int_0^\infty P(Y > y) dy = \int_0^\infty \int_y^\infty f_Y(x) dx dy$ (because $f(x) = \frac{d}{dx} F(x)$)
 $= \int_0^\infty \int_0^x f_Y(x) dy dx$ (draw diagram to convert integration)
 $= \int_0^\infty f_Y(x) \int_0^x dy dx$
 $= \int_0^\infty x f_Y(x) dx$ (because $\int_0^x dy = x$)
 $= E(Y)$

variance

N1 - variance of $X, \text{Var}(X) = E[(X - \mu)^2] = E(X^2) - [E(X)]^2$

example

Q - Find the pdf of $(b - a)X + a$ where a, b are constants, $b > a$. The pdf of X is

given by $f(x) = \begin{cases} 1, & 0 \leq X \leq 1 \\ 0, & \text{otherwise} \end{cases}$.

A. Let $Y = (b - a)X + a$.

cdf, $F_Y(y) = P(Y \leq y) = P((b - a)X + a \leq y) = P(X \leq \frac{y-a}{b-a})$

$F_Y(y) = \int_0^{\frac{y-a}{b-a}} 1 dx = \frac{y-a}{b-a}, \quad a < y < b$

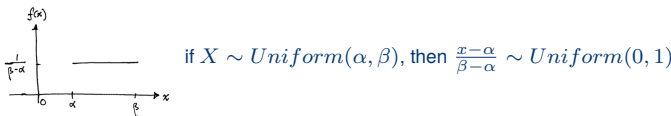
$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{b-a}, & a < y < b \\ 0, & \text{otherwise} \end{cases}$

Uniform Random Variable

X is a **uniform r.v.** on the interval $(\alpha, \beta), X \sim \text{Uniform}(\alpha, \beta)$

if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$
$$E(X) = \frac{\alpha+\beta}{2}, \quad \text{Var}(X) = \frac{(\beta-\alpha)^2}{12}$$

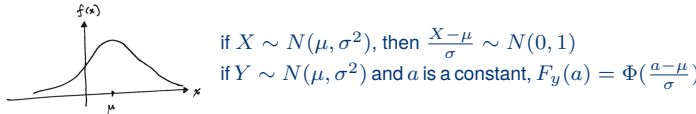


Normal Random Variable

X is a **normal r.v.** with parameters μ and $\sigma^2, X \sim N(\mu, \sigma^2)$

if the pdf of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \quad -\infty < x < \infty$$
$$E(x) = \mu, \quad \text{Var}(X) = \sigma^2$$



standard normal distribution $\rightarrow X \sim N(0, 1)$

• $F(x) = P(X \leq x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy = \Phi(x)$

Normal Approximation to the Binomial Distribution

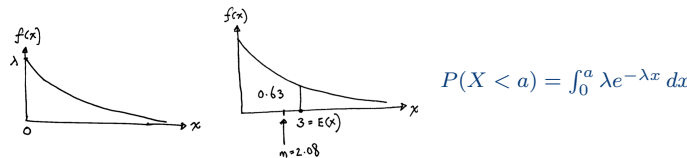
if $S_n \sim \text{Binomial}(n, p)$, then $\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0, 1)$ for large n .
 $\mu = np, \quad \sigma^2 = np(1 - p)$

Exponential Random Variable

a *continuous* r.v. X is a **exponential r.v.**, $X \sim \text{Exponential}(\lambda)$ or $\text{Exp}(\lambda)$

if for some $\lambda > 0$, its pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$
$$E(X) = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$



- an exponential r.v. is *memoryless*.
 - a non-negative r.v. is **memoryless** \rightarrow if $P(X > s + t | X > t) = P(X > s)$ for all $s, t > 0$.

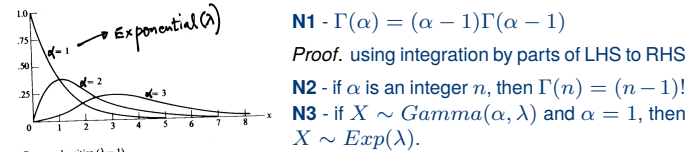
Gamma Distribution

a r.v. X has a **gamma distribution**, $X \sim \text{Gamma}(\alpha, \lambda)$ with parameters (α, λ) , $\lambda > 0$ and $\alpha > 0$ if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E(X) = \frac{\alpha}{\lambda} \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}$$

where the gamma function $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$.



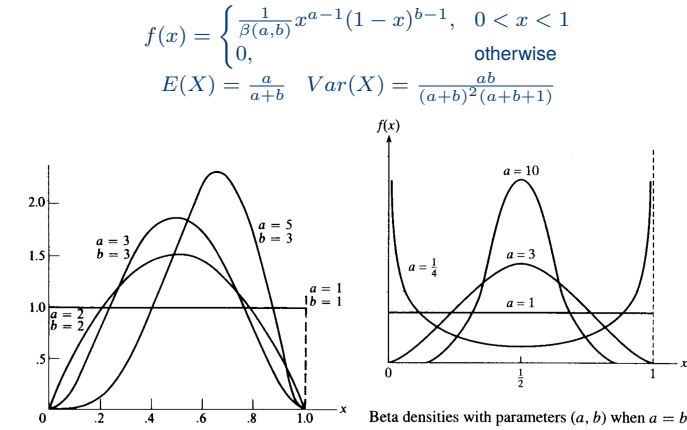
N4 - for events occurring randomly in time following the 3 assumptions of poisson distribution, the amount of time elapsed until a total of n events has occurred is a gamma r.v. with parameters (n, λ) .

- time at which event n occurs, $T_n \sim \text{Gamma}(n, \lambda)$
- number of events in time period $[0, t]$, $N(t) \sim \text{Poisson}(\lambda t)$

N5 - $\text{Gamma}(\alpha = \frac{n}{2}, \lambda = \frac{1}{2}) = \chi_n^2$ (chi-square distribution to n degrees of freedom)

Beta Distribution

a r.v. X is said to have a **beta distribution**, $X \sim \text{Beta}(a, b)$ if its density is given by



- N1** - $\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$
N2 - $\beta(a = 1, b = 1) = \text{Uniform}(0, 1)$
N3 - $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Cauchy Distribution

a r.v. X has a cauchy distribution, $X \sim \text{Cauchy}(\theta)$ with parameter θ , $-\infty < \theta < \infty$ if its density is given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, \quad -\infty < x < \infty$$

Proof. $E(X^n)$ does not exist for $n \in \mathbb{Z}^+$
 $E(X) = \int_{-\infty}^\infty x \cdot f(x) dx = \infty - \infty$ (undefined)

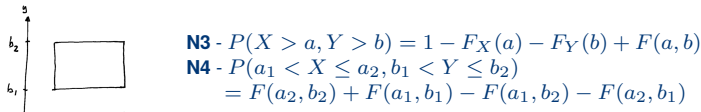
06. JOINTLY DISTRIBUTED RANDOM VARIABLES

Joint Distribution Function

the **joint cumulative distribution function** of the pair of r.v. X and Y is \rightarrow
 $F(x, y) = P(X \leq x, Y \leq y), \quad -\infty < x < \infty, \quad -\infty < y < \infty$

N1 - **marginal cdf of X** , $F_X(x) = \lim_{y \rightarrow \infty} F(x, y)$.

N2 - **marginal cdf of Y** , $F_Y(y) = \lim_{x \rightarrow \infty} F(x, y)$.



Joint Probability Mass Function

if X and Y are both discrete r.v.s, then their **joint pmf** is defined by
 $p(i, j) = P(X = i, Y = j)$

N1 - **marginal pmf of X** , $P(X = i) = \sum_j P(X = i, Y = j)$

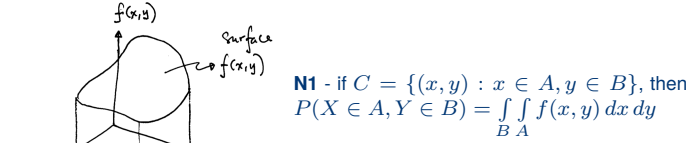
N2 - **marginal pmf of Y** , $P(Y = i) = \sum_j P(X = i, Y = j)$

Joint Probability Density Function

the r.v. X and Y are said to be *jointly continuous* if there is a function $f(x, y)$ called the **joint pdf**, such that for any two-dimensional set C ,

$$P[(X, Y) \in C] = \iint_C f(x, y) dx dy$$

= volume under the surface over the region C .

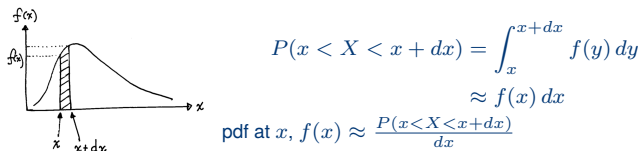


N2 - $F(a, b) = P(X \in (-\infty, a], Y \in (-\infty, b]) = \int_{-\infty}^b \int_{-\infty}^a f(x, y) dx dy$

for double integral: when integrating dx , take y as a constant

N3 - $f(a, b) = \frac{\delta^2}{\delta a \delta b} F(a, b)$

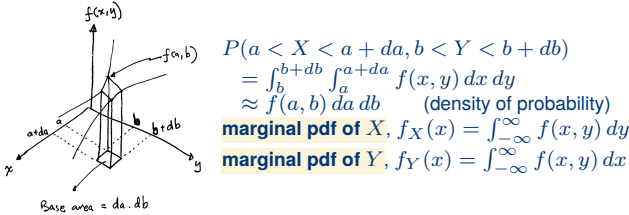
interpretation of pdf



N4 - pdf of X , $f_X(x) = \int_0^\infty f(x, y) dy$

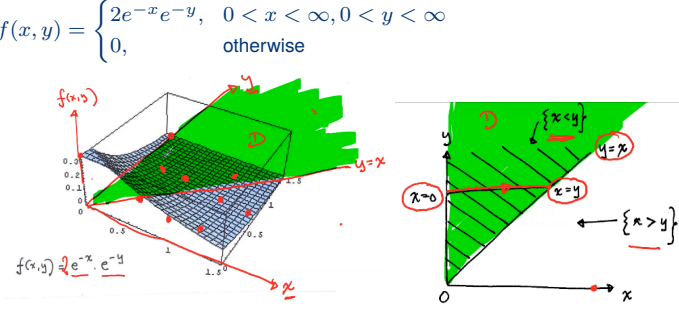
N5 - pdf of Y , $f_Y(y) = \int_0^\infty f(x, y) dx$

interpretation of joint pdf



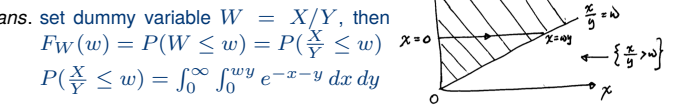
how to do a double integral

e.g. find $P(X < Y)$ where the joint pdf of X and Y are given by



- to get the bounds for dx and dy , plot $X < Y$
 - draw horizontal lines to determine the bounds for x , from $x = a$ to $x = b$
 - draw vertical lines to determine the bounds for y , from $y = c$ to $y = d$
- integrate $\int_c^d \int_a^b f(x) dx dy$

example - given the joint pdf of X and Y , find the pdf of r.v. X/Y .



Independent Random Variables

N1 - X and Y are **independent** \rightarrow
 $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

N2 - X and Y are **independent** $\rightarrow \forall a, b$,
 $P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$
 or $F(a, b) = F_X(a) \cdot F_Y(b) \rightarrow$ joint cdf is the product of the marginal cdfs

N3 - *discrete case*: discrete r.v. X and Y are **independent** \iff
 $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all x, y .

N4 - *continuous case*: jointly continuous r.v. X and Y are **independent** \iff
 $f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y .

N5 - independence is a **symmetric** relation $\rightarrow X$ is independent of $Y \iff Y$ is independent of X

Sum of Independent Random Variables

N1 - for independent, continuous r.v. X and Y having pdf f_X and f_Y ,
 $f_{X+Y}(a) = \int_{-\infty}^\infty f_X(a-y) f_Y(y) dy$
 $f_{X+Y}(a) = \int_{-\infty}^\infty f_X(a-y) f_Y(y) dy$

impt example - E52 (pdf of $X + Y$)

Distribution of Sums of Independent r.v.

for $i = 1, 2, \dots, n$,

- $X_i \sim \text{Gamma}(t_i, \lambda) \Rightarrow \sum_{i=1}^n X_i \sim \text{Gamma}(\sum_{i=1}^n t_i, \lambda)$
- $X_i \sim \text{Exp}(\lambda) \Rightarrow \sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda)$
- $Z_i \sim N(0, 1) \Rightarrow \sum_{i=1}^n z_i^2 \sim \chi_n^2 = \text{Gamma}(\frac{n}{2}, \frac{1}{2})$
- $X_i \sim N(\mu_i, \sigma_i^2) \Rightarrow \sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
- $X \sim \text{Poisson}(\lambda_1), Y \sim \text{Poisson}(\lambda_2) \Rightarrow X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$
- $X \sim \text{Binom}(n, p), Y \sim \text{Binom}(m, p) \Rightarrow X + Y \sim \text{Binom}(n + m, p)$

Conditional Distribution (discrete)

for discrete r.v. X and Y , the **conditional pmf** of X given that $Y = y$ is

$$P_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

for discrete r.v. X and Y , the **conditional pdf** of X given that $Y = y$ is

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{a \leq x} \frac{P(X=a,Y=y)}{P(Y=y)} = \sum_{a \leq x} P_{X|Y}(a|y)$$

N0 - equivalent notation:

- $P_{X|Y}(x|y) = P(X = x|Y = y)$
- $P_X(x) = P(X = x)$

N1 - if X is independent of Y , then $P_{X|Y}(x|y) = P_X(x)$

Conditional Distribution (continuous)

for X and Y with joint pdf $f(x, y)$, the **conditional pdf** of X given that $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{for all } y \text{ s.t. } f_Y(y) > 0$$

$$f_{X|Y}(a|y) = P(X \leq a|Y = y) = \int_{-\infty}^a f_{X|Y}(x|y) \, dx$$

N1 - for any set A , $P(X \in A|Y = y) = \int_A f_{X|Y}(x|y) \, dy$

N2 - if X is independent of Y , then $f_{X|Y}(x|y) = f_X(x)$.

! "find the marginal/conditional pdf of Y " \Rightarrow must include the **range** too!!
(see Ex. 69(b, c))

Joint Probability Distribution of Functions of r.v.

Let X_1 and X_2 be jointly continuous r.v. with joint pdf $f_{x_1,x_2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ satisfy

1. the equations $y_1 = g_1(X_1, X_2)$ and $y_2 = g_2(X_1, X_2)$ can be *uniquely* solved for x_1, x_2 in terms of y_1 and y_2
2. $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ have continuous partial derivatives at all points

$$(x_1, x_2) \text{ such that } J(x_1, x_2) = \begin{vmatrix} \frac{\delta g_1}{\delta x_1} & \frac{\delta g_1}{\delta x_2} \\ \frac{\delta g_2}{\delta x_1} & \frac{\delta g_2}{\delta x_2} \end{vmatrix} = \frac{\delta g_1}{\delta x_1} \cdot \frac{\delta g_2}{\delta x_2} - \frac{\delta g_2}{\delta x_1} \cdot \frac{\delta g_1}{\delta x_2} \neq 0$$

then

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) \frac{1}{|J(x_1, x_2)|}$$

where $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$

07. PROPERTIES OF EXPECTATION

recap:

- for a **discrete** r.v. X , $E(X) = \sum_x x \cdot p(x) = \sum_x \cdot P(X = x)$
- for a **continuous** r.v. X , $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$
- for a **non-negative integer-valued** r.v. Y , $E(Y) = \sum_{i=1}^{\infty} P(Y \geq i)$
- for a **non-negative** r.v. Y , $E(Y) = \int_{-\infty}^{\infty} P(Y > y) \, dy$

Expectations of Sums of Random Variables

for X and Y with joint pmf $p(x, y)$ and joint pdf $f(x, y)$,

$$E[g(x, y)] = \sum_y \sum_x g(x, y) p(x, y)$$

$$E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, dx \, dy$$

N2 - if $P(a \leq X \leq b) = 1$, then $a \leq E(X) \leq b$

N3 - if $E(X)$ and $E(Y)$ are finite, $E(X + Y) = E(X) + E(Y)$

Proof. using N1, integrate $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f(x, y) \, dx \, dy$
 $= \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx + \int_{-\infty}^{\infty} y f_Y(y) \, dy = E(X) + E(Y)$

N4 - if, for r.v.s X and Y , if $X \geq Y$, then $E(X) \geq E(Y)$

N5 - let X_1, \dots, X_n be independent and identically distributed r.v.s having distribution $P(X_i \leq x) = F(x)$ and expected value $E(X_i) = \mu$.

if $\bar{X} = \sum_{i=1}^n \frac{X_i}{n}$, then $E(\bar{X}) = \mu$

Proof. $E(\bar{X}) = E(\sum_{i=1}^n \frac{X_i}{n}) = \frac{1}{n} (\sum_{i=1}^n E(X_i)) = \frac{1}{n} \cdot n\mu = \mu$

\Rightarrow sample mean = population mean

N6 - \bar{X} is the **sample mean**.

N7 - if $X \sim \text{Binom}(n, p)$, then $E(X) = np$.

Proof. express X as a sum of Bernoulli r.v. \Rightarrow sum of indicator r.v. = np .

examples

! trick: express a r.v. as a sum of r.v. with easier to find expectation

- negative binomial = sum of geometric = k/p
- hypergeometric with r red balls out of N balls with n trials
 - indicator r.v. = 1 if the i th ball selected is red
 - $P(Y_i = 1) = \frac{r}{N} \Rightarrow E(Y_i) = \frac{r}{N} \Rightarrow E(X) = \sum_{i=1}^n Y_i = n \frac{r}{N}$
- hat throwing problem: expected number of people that select their own hat
 - P(select your own hat back) = $\frac{1}{N} \Rightarrow E(X) = N \cdot \frac{1}{N} = 1$
- coupon collector problem:
 - let X = number of coupons collected for a complete set
 - let X_i = number of *additional* coupons that need to be collected to obtain another distinct type after i distinct types have been collected
 - $X_i \sim \text{Geometric}(p = \frac{N-i}{N})$
 - $E(X) = \sum_{i=1}^{N-1} E(X_i) = 1 + \frac{1}{\frac{N-1}{N}} + \frac{1}{\frac{N-2}{N}} + \dots + \frac{1}{\frac{1}{N}}$
 $= N(\frac{1}{N} + \frac{1}{N-1} + \dots + 1)$

Covariance, Variance of Sums and Correlations

if X and Y are independent, then for any functions h and g ,

$$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$$

covariance \rightarrow measure of linear relationship

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

N1 - X and Y are independent $\Rightarrow Cov(X, Y) = 0$

N2 - $Cov(X, Y) = 0 \nRightarrow X$ and Y are independent

Proof. let $E(X) = 0, E(XY) = 0 \Rightarrow Cov(X, Y) = 0$, but not independent
e.g. non-linear relationship

Covariance properties

1. $Cov(X, Y) = Cov(Y, X)$
2. $Cov(X, X) = Var(X)$
3. $Cov(aX, Y) = aCov(X, Y)$
4. $Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)$

for variance:

N1 - $Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i) + 2 \sum_{i < j} Cov(X_i, X_j)$

N2 - if X_1, \dots, X_n are *pairwise independent* (X_i, X_j are independent $\forall i \neq j$),

then $Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i)$

commutative	$E \cup F = F \cup E$	$E \cap F = F \cap E$
associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(E \cap F) \cap G = E \cap (F \cap G)$
distributive	$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$	$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$
DeMorgan's	$(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i^c$	$(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$