STATS 260 Class 19

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1. 22 and 23 Estimation

2. Point estimate
Single number that serves as an estimate for the true value of the parameter.

The estimate comes from a statistic called the **estimator**

It is calculated from the sample data

Example

We might not know the population mean μ (also called the **true mean**) of some population. We take a sample of n observations, and then find the value of the sample mean $\bar{X}$, which is our estimator. The value we find, $\bar{x}$, is our point estimate for the true mean μ.

Here we use $\bar{X}$ to estimate μ

3. Properties of a good estimator

- The estimator should be **unbiased**: this means that the estimator does not tend to overestimate, and does not tend to underestimate.

  If $\hat{\theta}$ is as unbiased estimator for some parameter θ then:

  $$E(\hat{\theta}) = \theta$$

  In other words: the long- run average value of the estimate will be parameter which we are estimating.

- The estimator should be **consistent**; this means the value of the estimate will approach the true value of the parameter as $n \to \infty$ where n is the sample size.

  $$V(\hat{\theta}) \to 0$$

  $$n \to \infty$$

Example

Suppose we have a population with an unknown true mean μ. We select n members of the population as our sample, and wish to use $\bar{X}$, the sample mean, as the estimator which will give us the point estimate of our true mean,

is $\bar{X}$ unbiased for μ? Is $\bar{X}$ as consistent estimator?

$$E(\bar{X}) = \mu \quad \textit{unbiased}$$

$$V(\bar{X}) = \frac{\sigma^2}{n}, \quad n \to \infty \quad V(\bar{X}) \to 0$$

Example

Suppose a population has unknown mean μ and variance σ^2. We take n observations, $X_1, \ldots, X_n$ from this population.

We can show that $S^2 = \dfrac{\sum(X_1 - \bar{X})^2}{n-1}$ the sample variance, is an unbiased estimator for σ^2 (i.e. we can show $E(S^2) = \sigma^2$).

S^2 is unbiased for σ^2

$E(S) \neq \sigma$ Therefore S is biased estimation for σ

If we have a choice between estimators, we would prefer the estimator with the grater **efficiency**. The less variability an estimator has in estimating the parameter, the more efficient it is.

If two estimators are both unbiased, then the estimator with the smaller variance is the more efficient estimator.

features of a good estimator

Unbiased	Does not over estimate aka $E(\hat{\theta}) = \theta$
Consistent	$\lim\limits_{n\to\infty} V(\hat{\theta}) = 0$ as $n \to \infty$
Efficient	Smaller variance or smaller standard deviation

now use

$\bar{X}$ to estimate μ

S to estimate σ

S^2 to estimate σ^2

3.1. interval estimate

An estimate between and interval e.g. Between 10 and 15 or (10, 15)

A **pivotal quantity** is a function of observations with a distribution that does not depend on the value of any unknown parameters.

Example

Suppose we have a random sample of n, either from population with mean μ and standard deviation σ. Also, suppose the population is normal, or that n is large (or both).

Then $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a pivotal quantity

does not depend on value of μ

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

NOTE: N(0,1) does not depend on μ even though $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ does depend on μ

By the Central Limit Theorem, we know that $\bar{X}$ is normal with mean μ and sdandart deviation $\sigma/\sqrt{n}$. so the expression above with standard normal distribution, regardless of what the value of μ, σ are.

(i) $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ if X_i is normal (population) n doesn't matter

(i) $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \approx N(0,1)$ if $n \geq 30$ (rule of thumb for CLT)

Computation is the same if it is approximate.

What happens to $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ if X_i is not normal and $n < 30$

↑ Trick question we ignore this in this class

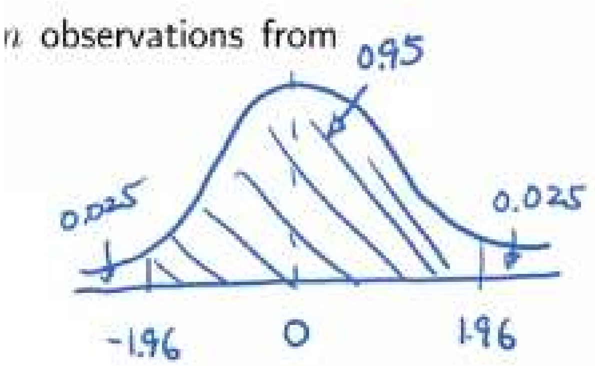
4. Random intervals

A random interval is an interval for real numbers whose endpoints are random variables.

We can use **pivotal quantities** to construct a **random interval** for one of the parameters.

Example

Suppose we have a random sample of n observations from some normal population. We can find that



$$P\left(-1.96 \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le 1.96\right) = 0.95$$

This can be re written as:

$$P\left(\bar{X} - 1.96\,\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\,\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

Written as a random interval this is $\left(\bar{X} - 1.96\,\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\,\frac{\sigma}{\sqrt{n}}\right) = 0.95$

4.1. Example from video

the average test scores in a physics class is normally distributed with a standard deviation of 5.4. 50 scores with a sample mean of 79 where selected at random.

(A) Find a 95% confidence interval for the population mean test score.

Given

$$\sigma 5.4$$

$$n = 50$$

$$\bar{X} = 79$$

$$CL = 0.95\%$$

$$CL\ is\ for\ confidence\ level$$

The equation to determine the confidence interval is

$$\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + x_\alpha \frac{\sigma}{\sqrt{n}}$$

and the actual interval is

$$CI \rightarrow (\bar{X} - error\ bound\ \text{for}\ the\ mean, \bar{X} + error\ bound\ \text{for}\ the\ mean)$$

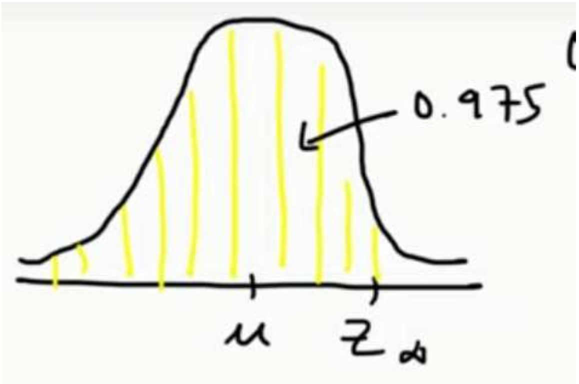
aka

$$CI \rightarrow \bar{X} \pm z_\alpha \frac{\sigma}{\sqrt{n}}$$

How do I find z_α

z_α is some point right of μ.

as seen here



Here is how you calculate the area to the left

$$A_L = \frac{CL+1}{2}$$

$$A_L = \frac{0.95+1}{2} = 0.975$$

Now simply find where in the positive z score table the probability = 0.975

In this case it is at 1.96

$\therefore$ the z score is 1.96 and $z_\alpha = 1.96$

Next we need to calculate $\sigma_{\bar{X}}$

This is just $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{5.4}{\sqrt{50}}$

0.76368

$$95\%CI \rightarrow 79 \pm 1.96(0.76368)$$
$$\rightarrow 79 \pm 1.4968$$

$\boxed{\therefore \text{ the confidence interval is } (77.5032, 80.4968)}$

This means that we are 95% confident that the mean for the population falls somewhere between 77.5032 and

(B) What is the value of the margin of error?

We need to find **error bound for the mean** aka EBM

This is just $EBM = Z_{\alpha} \frac{\sigma}{\sqrt{n}}$

$$= 1.96 \frac{(5.4)}{\sqrt{50}} = 1.4968$$

4.2. Another way to approach this

For a large-sample size scenario (n ≥ 40), the following is an approximate 100(1 − α)% confidence interval:

$$\left(\bar{x} - z_{a/2}, \bar{x} + z_{a/2} \frac{s}{\sqrt{n}}\right)$$

$Z_{\frac{\alpha}{2}}$ is known as the critical value for the confidence interval.

Here is how to find $Z_{\frac{\alpha}{2}}$

If the confidence level is 95 then we know that $95 = 100(1 - \alpha)\%$

$$\alpha = 0.05$$

$$\frac{\alpha}{2} = 0.025$$

$Z_{\frac{\alpha}{2}}$ is the 97.5th percentile

We get this using the formula

$$1 - \frac{\alpha}{2}$$

We then just need to find what Z value on the table has the probability of 0.975

$$\boxed{Z_{0.025} = 1.96}$$

In summary

$$\textit{estimate} \pm (\textit{criticalvalue})(\textit{standarderror})$$

$$\bar{X} \quad \pm \quad Z_{\frac{\alpha}{2}} \quad \frac{\sigma}{\sqrt{n}}$$

Common Critical Values:

% Confidence	Critical Value
90	$z_{\alpha/2} = z_{0.05} = 1.645$
95	$z_{\alpha/2} = z_{0.025} = 1.96$
99	$z_{\alpha/2} = z_{0.005} = 2.575$

Note: While these confidence levels are common, you should also be able to find the critical value for *any* confidence level.

Interpretation

We should not interpret this as meaning that there is a 95% chance that the true mean height is between 1.66864 and 1.73136 meters.

4.3. Margin of error (d)

d = (critical value) (standard error)

$$d = z_{a/2} \frac{\sigma}{\sqrt{n}}$$

rearranging this we get

$$n = \left(\frac{Z_{\frac{\alpha}{2}} \cdot \sigma}{d} \right)^2$$

NOTE: Width of confidence interval, is the distance between the upper and lower confidence limits. This is $2D$ is the width of a CI.

4.4. Scaling of confidence level

CL $\uparrow$	$Z_{\frac{\alpha}{2}} \uparrow$	$d \uparrow$
	$\sigma \uparrow$	$d \uparrow$
	n $\uparrow$	$d \uparrow$

NOTE: we want a smaller *d* since this means we have more confidence

Example

Example: Suppose the lifetime of certain type of lightbulb is normally distributed and has a standard deviation of σ = 200 hours. How many samples do we need to be create a 95% confidence interval for μ, the mean lifespan, with a <u>margin of error of 10 hours?</u>

$$d = 10$$

$$Z_{\frac{\alpha}{2}} = 1.96$$

$$\sigma = 200$$

$$n = \left(\frac{1.96 \cdot 200}{10}\right)^2$$

$$= 1536.64 \Rightarrow 1.537$$

NOTE: Since *n* aka sample size must be a whole number we always round up.

Example

How many observations do we need to create a 95% confidence interval for μ with a width of 40 hours?

$$d = 20$$

$$n = \left(\frac{1.96 \cdot 200}{20}\right)^2 = 384.16$$

remember to round up

$$= 385$$

wider confidence interval requires smaller n

5. Estimated Standard Error

For a large-sample size scenario (n ≥ 40), the following is an approximate 100(1 − α)% confidence interval:

We can replace σ with s to estimate

$$\left(\bar{x} - Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \bar{x} + Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right)$$

We call $s/\sqrt{n}$ the **estimated standard error**

Example

At a particular location, fifty daily measurements of wind speed (in m/s) are made. It is found that $x = 15.9$ m/s and $s = 7.7$ m/s. Find a 98% confidence interval for μ, the average daily wind speed. Assume that the measurements constitute a random sample from the population of all wind speed measurements.

$$\alpha = 1 - 98 = 0.02$$

$$\frac{\alpha}{2} = 0.01$$

$$\text{reverse lookup for probability} = 1 - \frac{\alpha}{2} = 0.99$$

$$Z_{0.01} = 2.33$$

$$\bar{x} \pm Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

$$15.9 \pm 2.33 \frac{7.7}{\sqrt{50}}$$

$$15.9 \pm 2.5$$

$$\boxed{(1.3.363, 18.437)}$$

6. t-Distribution

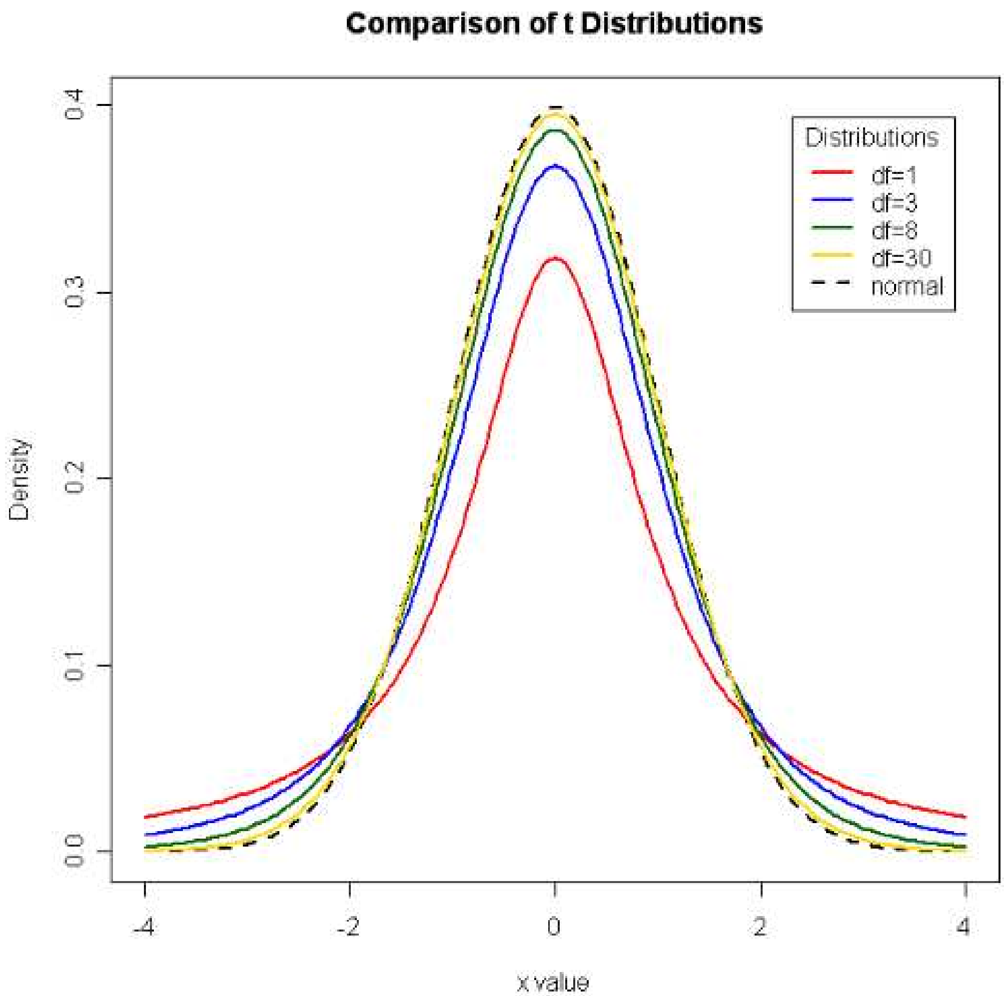
If the sample is $n < 40$ then we use a t-distribution, with $n - 1$ degrees of freedom.

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

6.1. Properties of the t distribution:

1. The t distribution is continuous, and defined on $(-\infty, \infty)$.
2. The t distribution is **symmetric**, **bell-shaped,** and **centered** at zero.
3. The number of degrees of freedom affect the shape of the distribution; as the number of degrees of freedom increases, the distribution becomes more peaked, and the tails become thinner.
4. When the number of degrees of freedom is large (30 or more), the t-distribution is approximately a standard normal distribution.

NOTE:To denote a t-distribution with k degrees of freedom, we write



6.2. Confidence Interval for Population Mean

for sample sizes < 40 we use

$$\bar{x} \pm t_{n-1,\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

$t_{n-1,\alpha/2}$ acts as the critical value for a t-distrobution with n- 1 degrees

NOTE:sample sizes < 40 we and sigma must not be known

Example

The following data is collected on the mass (in grams) of adult white mice.

$$14.6, 13.2, 19.5, 10.1, 8.8, 15.5, 16.1$$

Assuming that the weights of mice are normally distributed, find a 95% confidence interval for μ, the mean weight of adult white mice.

$$n = 7 < 40,\ \alpha = 1 - 0.95 = 0.05,\ \alpha/2 = 0.025,\ s = 3.655003$$

$$\mu = 13.97143$$

$$t_{7-1,0.025} = t_{6,0.025}$$

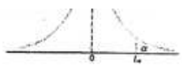
Using the table in Appendix D we use v = degrees of freedon and α as α

degree of freedom

t_1.

$t_{6,0.025}$

α



v	0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306

$$t_{6,0.025} = 2.447$$

$$13.971 \pm 2.447 \cdot \frac{3.655}{\sqrt{7}}$$

$$\boxed{(10.591, 17.351)}$$

in R

```
> tmp=c(14.6, 13.2, 19.5, 10.1, 8.8, 15.5, 16.1)
> t.test(tmp, conf.level = .95)

        One Sample t-test

data:  tmp
t = 10.114, df = 6, p-value = 5.431e-05
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 10.59111 17.35174
sample estimates:
mean of x
 13.97143
```