STATS 260 Class 11

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1. Set 12

2. Poisson Experiment

An experiment having the following properties.

1. The number of successes that occur in any interval is independent of the number of successes occurring in any other interval. *non-overlapping interval*

2. The probability of success in an interval is proportional to the size of the interval. *Larger the interval larger the probability*

3. If two intervals have the same size, then the probability of a success is the same for both intervals.

3. Poisson Random Variable

If in a Poisson experiment, X counts the number of successes that occur in one interval of time/space, then X is a Poisson random variable. We write $X \sim Poisson(\lambda)$.

Where λ is the average number of successes per region/interval.

NOTE: Some books will use μ rather than λ for the parameter of the Poisson random variable.

Example

At a bank, customers use the bank machine at an average rate of 40 customers per hour. Let X count the number of customers that use the machine in a 30-minute interval.

40 customers per hour

$$\lambda = 40 \text{ per hour}$$

we use 20 for a 30 minute interval

$$X \sim Poisson(\lambda = 20)$$

$$\boxed{\text{NO n}}$$

Example

At a busy intersection, it is noted that on average 5 cars pass through the intersection per minute. Let X count the number of cars which pass through the intersection in an hour.

$$X \sim Poisson(\lambda = 300)$$

Example

Suppose that a typist makes on average 10 errors while typing 300 pages of text. Let X count the number of errors on one page of text.

Errors per page

$$X \sim Poisson(\lambda = \frac{10}{300})$$

Example

We examine ten pages of text. Let Y count the number of pages with at least one error. The random variable Y is **not** Poisson. Why?

Assume pages are independent

$$n = 10,\ p = P(\text{ at least one error per page })$$

Binary

$$y \sim Bin(10, p)$$

Poisson

$$X \sim Poisson(\lambda = \frac{1}{30})$$

The difference is that y counts the # of pages out of the 10 pages

4. Poisson Probability Distribution

$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Remember

Binomial has a set endpoint eg 1, 2 ,..., n

Poisson has no fixed end eg 1, 2 , ...

Example

Suppose a machine makes defective items at an average rate of 5 defective items per hour. What is the probability that the machine will make exactly 4 defective items in an hour?

X = # of defective items per hour

$$X \sim Poisson(\lambda = 3)$$

$$P(X = 4) = \frac{e^{-5} \cdot 5^{4}}{4!}$$

$$= 0.1755$$

4.1. Expected Value and Variance

if $X \sim Poisson(\lambda)$

$$E(X) = \lambda \text{ and } V(X) = \lambda$$

Example

What is the expected number of defective items made by the machine in an hour? What is the variance?

$$\lambda = \mu = E(X) = 5 \text{ defective items (per hour)}$$

$$\sigma^2 = V(X) = 5 \text{ item}^2$$

$$\sigma = \sqrt{5} \text{ defective items}$$

4.2. Cumulative Distribution Tables

These tables give $P(X \leq x)$ for "nice" values of λ

Example

Suppose the machine is watched for three hours. What is the probability that it will make no more than 12 defective items?

$$\lambda = 5 \text{ per hour}$$

$$X \sim Poisson(\lambda = 15)$$

(Recall that the machine makes on average 5 defective items per hour)

From table

$$P(X \leq 12) = 0.2676$$

Example

What is the probability that at least 6 defective items will be made?

$$P(X \geq 6) = 1 - P(X \leq 5)$$

$$= 1 - 0.0028$$

$$= 0.9972$$

Example

What is the probability that exactly 13 defective items will be made?

$$P(X = 13) = \frac{e^{-15} \cdot 15^{13}}{13!}$$

$$= P(X \leq 13) - P(X \leq 12)$$

$$= 0.3632 - 0.2676$$

$$= 0.0956$$

Example

Suppose that a typist makes on average of 2 errors per page. [Poisson] Suppose the typist is creating a ten-page document. What is the probability that exactly three of the pages do not contain any errors?

let X be the number of errors per page

$$X \sim Poisson(\lambda = 2 perpage)$$

let y be a number of pages that contain no errors (success)

Assuming they are independent

$$y \sim Bin(n = 10, p = P(X = 0)) = 0.1353$$

$$P(y = 3) = \binom{10}{3} 0.1353^3 (1 - 0.1353)^7$$

$$= 0.1074$$

5. Poisson approximation to Binomial

If X is a binomial random variable where n is very large and p is very small then X can be approximated with a Poisson distribution with λ = np.

NOTE: Provided n $\geq$ 100 and np $\leq$ 10, the approximation will be quite good. It will still be reasonably good when n $\geq$ 20, as long as p $\leq$ 0.05.

Example

Brugada syndrome is a rare disease which afflicts 0.02% of the population. Suppose 10,000 people are selected at random and tested for Brugada syndrome. What is the probability that no more than 3 of the tested people will have Brugada syndrome?

$$X \sim Bin(n = 10000, p = 0.0002)$$

No table to look up

$$P(X \leq 3)$$

$$= P(X \leq 3)$$

$$X \sim Poisson(\lambda = 10000 \times 0.0002 = 2)$$

$$= 0.8571$$

6. Sets 13 and 14

7. Continuous Random Variable

A random variable which can assume an uncountable number of values (i.e. some interval of real numbers).

For a random variable, the **probability distribution** or **probability density function** (pdf) is a function f (x) satisfying

NOTE: Discrete random variable support is countable
a.k.a finite number of outcomes or countably infinite [Poisson]

$$P(a \leq X \leq b) \int_a^b f(x)dx$$

For any two numbers a and b with $a \leq b$

Some immediate consequences

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$