STATS 260 Class 12

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1. Sets 13 and 14

2. Continuous Random Variable

A random variable which can assume an uncountable number of values (i.e. some interval of real numbers).

For a random variable, the **probability distribution** or **probability density function** (pdf) is a function f (x) satisfying

NOTE: Discrete random variable support is countable
a.k.a finite number of outcomes or countably infinite [Poisson]

$$P(a \leq X \leq b) \int_a^b f(x)dx$$

For any two numbers a and b with $a \leq b$

Some immediate consequences

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Note: Since a valid pdf must never be below the x axis, we can interpret P ($a \le X \le b$) as the area under f (x) on the interval [a, b].

Some further consequences for any valid pdf:

1. P (X = a) = 0 for any a.

$$P(X = a) = P(a \le X \le a) = \int_a^a f(x)dx = 0$$

Discrete = P(X = a) > 0 a in support of X.

2. P ($X \ge a$) = P (X > a) and P ($X \le a$) = P(X < a)

$$= P(X < a) + P(X = a)$$

where P(X = a) = 0

3. P ($X \ge a$) = 1 - P($X \le a$)

if all Random Variables

$$= 1 - P(X < a)$$

Continuous

$$= 1 - P(X \le a)$$

4. P ($a \le X \le b$) = P($X \le b$) - P(X < a) (provieded $a \le b$)

$$= P(X \le b) - P(X < a)$$

Example of a Continuous Random variable

3. Uniform Probability Distribution

For a uniform probability distribution, the pdf is:

$$f(x;a,b) = \frac{1}{b-a} \text{ where } a \le x \le b$$

NOTE: f(x) ≠ P(X=x) in Continuous Random Variable

The graph of f (x) is a horizontal line segment from a to b with height 1/(b - a).

$$P(x_1 \le X \le x_2) = (height) \times (width) = \left(\frac{1}{b-a}\right)(x_1 - x_2)$$

eg

$$X \sim Uniform(1,3)$$

$$f(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x \le 3 \\ 0 & x > 3 \end{cases}$$

Example

Suppose that the continuous rv X has the following pdf:

$$X \sim Uniform(1, 3)$$

$$f(x) = \begin{cases} \frac{4}{609}x^3 & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(3 \le X \le 4)$.

$$= \int_3^4 \frac{4}{609}x^3 dx$$

$$= \left.\frac{x^4}{609}\right|_3^4 = \frac{4^4}{609} - \frac{3^4}{609}$$

$$\frac{25}{87}$$

Check that

1. $f(x) \ge 0$

1. $\int_{-\infty}^{\infty} f(x) = \int_2^5 \frac{4x^3}{609} = 1$

Example

Find an expression for P (X ≤ b), where b is some number in [2, 5].

$$F(b) = P(X \le b)$$

$$= \int_2^b \frac{4}{609}x^3 dx$$

$$= \left.\frac{x^4}{609}\right|_2^b$$

$$= \frac{b^4}{609} - \frac{16}{609}$$

When b < 2 F(b) = 0

When b > 5

Put it together to get

$$f(x) = \begin{cases} 0 & x < 2 \\ \dfrac{x^4}{609} - \dfrac{16}{609} & 2 \le x \le 5 \\ 0 & x > 5 \end{cases}$$

NOTE: The fundamental theorem of calculus tells us that for every x at which $F'(x)$ exists, that $F'(x) = f(x)$.

Example

Suppose the random variable X has the following cdf:

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{x+1} & x \ge 0 \end{cases}$$

Find the pdf for the random variable X

$$f(x) - F'(x) = \left(\frac{x}{x+1}\right)^1$$

$$= \frac{1(x+1) - x \cdot 1}{x+1^2}$$

$$= \frac{1}{x+1^2} \ge 0$$

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{x+1^2} & x \ge 0 \end{cases}$$