Vector Spaces of finite dimension

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Generated Vector Subspace (review)

Theorem

Let $\{v_1, \ldots, v_n\}$ a finite set of vectors of a Vector Space E over $\mathbb{K}$. Then :

- *The set of linear combinations of the vectors $\{v_1, \ldots, v_n\}$ is a Vector Subspace of E, and*
- *It is the smallest Vector Subspace of E (in the sense of inclusion) containing $v_1, \ldots, v_n$.*

This Vector Subspace is called ***generated Subspace by*** $\mathbf{v}_1, \ldots, \mathbf{v_n}$ *and is denoted $Vect(v_1, \ldots, v_n)$. We thus have*

$$\boxed{u \in Vect(v_1, \ldots, v_n) \Longleftrightarrow \exists(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n,\ u = \lambda_1 v_1 + \ldots + \lambda_n v_n}$$

I) Independent and dependent sets

Definition

The generated vector spaces by a finite number of vectors (called a **set of vectors**) are said to be **vector spaces of finite dimension**.

Reminder

Let $n \in \mathbb{N}, n \geq 1$ and $v_1, v_2, \ldots, v_n$ n vectors in a Vector Space E over $\mathbb{K}$. Then every vector $u \in E$ of the form

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n,$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ scalars in $\mathbb{K}$, is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$, and the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called **coefficients** of the linear combination.

Definitions

- A set of vectors $\{v_1, v_2, \ldots, v_n\}$ of a vector space E over $\mathbb{K}$ is said to be **linearly independent** if and only if

$$\boxed{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0_E} \quad \Rightarrow \quad \boxed{\lambda_1 = 0,\ \lambda_2 = 0,\ \ldots,\ \text{and } \lambda_n = 0}$$

- By contra-position, if
$\exists i \in \{1, ..., n\}$ such that $\lambda_i \neq 0$ and $\lambda_1 v_1 + \ldots + \lambda_i v_i + \ldots + \lambda_n v_n = 0_E$
then we say that the set $\{v_1, v_2, \ldots, v_n\}$ is **linearly dependent**.
- If a set of vectors is linearly dependent, we call **dependence relation** the expression of one vector as function of the others.
- In order to determine if a set of vectors $\{v_1, v_2, \ldots, v_n\}$ in the vector space $\mathbb{R}^n$ is linearly dependent or independent, we need to solve a linear system.

Examples :

1) Determine whether the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent or independent in $\mathbb{R}^3$:

a) $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$,

b) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

2) Determine whether the set of polynomials $\{P_1, P_2, P_3\}$ is linearly dependent or independent in $\mathbb{R}_2[x]$, with $P_1(x) = 2 - x$, $P_2(x) = 1 - 2x + x^2$ and $P_3(x) = 3 + 2x - x^2$.

Proposition

Let E be a vector space over $\mathbb{K}$, then

- a set of one vector $v \in E$ is linearly independent if $v \neq 0_E$,
- a set $\{v_1, v_2\}$ is linearly dependent if and only if v_1 is a multiple of v_2 or v_2 is a multiple of v_1.

Theorem

Let E be a vector space over $\mathbb{K}$. A set $\mathcal{S} = \{v_1, v_2, \ldots, v_n\}$ $(n \geq 2)$ is linearly dependent if and only if

$$\exists i \in \{1, \ldots, n\},\ v_i = \sum_{j=1, i \neq j}^{n} \lambda_j v_j,$$

i.e., at least one vector of $\mathcal{S}$ is a linear combination of the others.

Interpretations :

- In $\mathbb{R}^2$, two vectors are linearly dependent if they are colinear and form a vectorial line,
- In $\mathbb{R}^3$, three vectors are linearly dependent if they are coplanar and form a vectorial plane.

Proposition

Let $\mathcal{S} = \{v_1, v_2, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$. If $p > n$, then the set $\mathcal{S}$ is linearly dependent.

Exercise :

For which values of $t \in \mathbb{R}$ the set $\mathcal{S}$ is linearly independent ?

a) $\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ -t \end{pmatrix} \right\}$ in $\mathbb{R}^2$?

b) $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \begin{pmatrix} t^2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$?

II) Generating set

Definition

Let E be a Vector Space over $\mathbb{K}$ and $v_1, v_2, \ldots, v_n$ vectors in E. We say that $\mathcal{S} = \{v_1, v_2, \ldots, v_n\}$ is a **generating set** (**spanning set**) of the Vector Space E if

$$\forall v \in E, \exists \lambda_1, \ldots, \lambda_n \in \mathbb{K}, \ v = \lambda_1 v_1 + \ldots + \lambda_n v_n$$

- We say that the set $\mathcal{S}$ **generates** (**spans**) the Vector Space E.
- Remark :
  if a set $\mathcal{S} = \{v_1, v_2, \ldots, v_p\}$ generates a Vector Space E, then we get back the previous concept of a generated Vector Space by the vectors $v_1, v_2, \ldots, v_p$:

$$E = Vect(v_1, v_2, \ldots, v_p)$$

Examples :

1) Which Vector Space generates the set $\mathcal{S}$?

a) $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

b) $\mathcal{S} = \{1, X, X^2, \dots, X^n\}$ the set of polynomials of degree $n \geq 1$.

c) $\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

2) Is $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ a generating set of $\mathbb{R}^2$?

Proposition

Let $\mathcal{S} = \{v_1, \ldots, v_n\}$ a generating set of E. Then $\mathcal{S}' = \{v_1', \ldots, v_n'\}$ is also a generating set of E if and only if every vector in $\mathcal{S}'$ is a linear combination of the vectors of $\mathcal{S}$ and vice-versa.

Exercise :

For which values of $t \in \mathbb{R}$ the set $\mathcal{S} = \left\{ \begin{pmatrix} 0 \\ t-1 \end{pmatrix}, \begin{pmatrix} t \\ -t \end{pmatrix} \right\}$ is a generating set of $\mathbb{R}^2$?

III) Basis of a Vector Space

Definition

Let E be a Vector Space over $\mathbb{K}$. A set $\mathcal{F} = \{v_1, v_2, \ldots, v_n\}$ of vectors in E is said to be a **basis** of E if it is :

- a generating set of E, and
- linearly independent.

Theorem

Let $\mathcal{F} = \{v_1, v_2, \ldots, v_n\}$ be a basis of a Vector Space E. Then, every vector $v \in E$ is expressed in a unique way as a linear combination of elements of $\mathcal{F}$. I.e.,

$$\forall v \in E, \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}, \; v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n.$$

$\Rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *are called the **coordinates** of the vector v in the basis $\mathcal{F}$.*

Examples :

- Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then (e_1, e_2) is the so-called **canonical basis** of $\mathbb{R}^2$,
- Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then (e_1, e_2, e_3) is the so-called **canonical basis** of $\mathbb{R}^3$.
- Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$. Then $(e_1, e_2, \ldots, e_n)$ is the so-called **canonical basis** of $\mathbb{R}^n$.
- The canonical basis of $\mathbb{R}_n[X]$ is the set $\mathcal{F} = \{1, X, X^2, \ldots, X^n\}$.
- The canonical basis of $M_2(\mathbb{R})$ is the set $\mathcal{F} = \{A, B, C, D\}$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Exercise :

Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$. Show that the set $\mathcal{F} = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

Remarks :

- To show that a set of n vectors $\mathcal{F} = \{v_1, v_2, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, we simply need to determine whether the matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$ is invertible or not.
- The basis of a Vector Space is not unique.

Theorem 1 : Existence of a basis

Every Vector Space with a generating set has a basis.

Theorem 2

Let E be a Vector Space over $\mathbb{K}$ with a finite generating set.

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Let E be a Vector Space over $\mathbb{K}$ with a finite generating set.

- **Incomplete basis theorem** :
  Every linearly independent set $\mathcal{I}$ in E can be completed to a basis. I.e., there exists a set of elements $\mathcal{S}$ in E such that $\mathcal{I} \cup \mathcal{S}$ is a generating and linearly independent set.
- **Extracted basis theorem** :
  From every generating set $\mathcal{G}$ of E we can extract a basis of E. I.e., there exists a set of elements $\mathcal{B} \subset \mathcal{G}$ such that $\mathcal{B}$ is a generating and linearly independent set of E.

Theorem 3

Let $\mathcal{G}$ a finite generating set and $\mathcal{I}$ a linearly independent set of E. Then, there exists a set $\mathcal{S} \subset \mathcal{G}$ such that $\mathcal{I} \cup \mathcal{S}$ is a basis of E.

Exercise :

1) Let E be the Vector Subspace of the $\mathbb{R}$-Vector Space $\mathbb{R}[X]$ generated by the set $\mathcal{G} = \{P_1, P_2, P_3, P_4, P_5\}$ defined as :

$$P_1(X) = 1,\ P_2(X) = X,\ P_3(X) = X + 1,\ P_4(X) = 1 + X^3,\ P_5(X) = X - X^3$$

Find a basis $\mathcal{B}$ of E.
2) Show that the set $\mathcal{S} = \{v_1, v_2, v_3\}$, with $v_1 = (1, 0, 2, 3)$, $v_2 = (0, 1, 2, 3)$ and $v_3 = (1, 2, 0, 3)$, can be completed to a basis.

IV) Dimension of a Vector Space

Definition

A Vector Space E over $\mathbb{K}$ with a basis of finite elements is said to be of **finite dimension**.

Theorem

All the bases of a Vector Space E of finite dimension have the same number of elements.

Definition

The **dimension** of a Vector Space of finite dimension, denoted $\dim(E)$, corresponds to the number of elements of a basis of E.

Remarks :

1) In order to determine the dimension of a Vector Space of finite dimension,

- Find a basis of E (generating and linearly independent set),
- determine the cardinal (number of elements) of this basis.

2) The dimension of the Vector Space $\{0_E\}$ is 0.

Examples :

1) Determine the dimension of $\mathbb{R}^2$, $\mathbb{R}^n$ and $\mathbb{R}_n[X]$,
2) The Vector Spaces $\mathbb{R}[X]$ and $\mathcal{F}(\mathbb{R}, \mathbb{R})$ are not of finite dimension.

Exercise :

Let (S) be the following linear system :

$$\begin{cases} 2x_1 & +2x_2 & -x_3 & & +x_5 & =0 \\ -\ x_1 & -x_2 & +2x_3 & -3x_4 & +x_5 & =0 \\ x_1 & +x_2 & -2x_3 & & -x_5 & =0 \\ & & x_3 & +x_4 & +x_5 & =0 \end{cases}$$

1) Is the solution set of S a Vector Space ?

2) Determine the solution set of S,

3) Determine the dimension of this Vector Space.

Proposition 1

Let E be a Vector Space, $\mathcal{I}$ a linearly independent set and $\mathcal{G}$ a generating set of E. Then $\text{card}(\mathcal{I}) \leq \text{card}(\mathcal{G})$.

$\rightarrow$ We admit this result.

Proposition 2

Let E be a Vector Space with a basis of n elements. Then,

- Every linearly independent set of E has at most n elements,
- Every generating set of E has at least n elements.

Proposition 3

If E is a Vector Space with a basis of n elements, then every basis of E is composed of n elements.

Theorem

Let E be a Vector Space over $\mathbb{K}$ of dimension n, and $\mathcal{S} = (v_1, \ldots, v_n)$ a set of n elements of E. Then, the following statements :

1. *$\mathcal{S}$ is a basis of E,*
2. *$\mathcal{S}$ is a linearly independent set of E,*
3. *$\mathcal{S}$ is a generating set of E,*

are equivalent. I.e.,

$$\boxed{(1) \Leftrightarrow (2) \Leftrightarrow (3)}$$

Exercise :

For which values of $t \in \mathbb{R}$ the set $\mathcal{S} = (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$?

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 3 \\ t \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$$

V) Dimension of Vector Subspaces

Theorem

Let E be a $\mathbb{K}$-Vector Space of finite dimension. Then

- *Every Vector Subspace F of E is of finite dimension,*
- $dim(F) \leq dim(E)$,
- $F = E \Leftrightarrow dim(F) = dim(E)$

Example :
Find the Vector Subspaces of the $\mathbb{K}$-Vector Space E of dimension 2, and determine their dimensions.

Definition

Let E be a $\mathbb{K}$-Vector Space of dimension n. We call a **hyperplane** every Vector Subspace of E of dimension $n-1$.

Proposition

Let E be a $\mathbb{K}$-Vector Space of finite dimension and F, G Vector Subspaces of E. If $G \subset F$, then

$$F = G \Leftrightarrow \dim(F) = \dim(G)$$

Example :

Show that the following Vector Subspace of $\mathbb{R}^3$:

$$F = \{(x, y, z) \in \mathbb{R}^3 | 2x - 3y + z = 0\}$$

$$G = \text{Vect}(u, v), \text{ where } u = (1, 1, 1) \text{ and } v = (2, 1, -1)$$

are equal.

Theorem

Let E be a Vector Space of finite dimension and F, G Vector Subspaces of E. Then

$$dim(F+G) = dim(F) + dim(G) - dim(F \cap G)$$

Proposition

If $E = F \oplus G$, then $\dim(E) = \dim(F) + \dim(G)$.

Proposition

Every Vector Subspace of a Vector Space E of finite dimension has a supplementary in E.

Exercise : Let $v_1 = (1, t, -1)$, $v_2 = (t, 1, 1)$ and $v_3 = (1, 1, 1)$, with $t \in \mathbb{R}$.

Consider the following Vector Subspaces of $\mathbb{R}^3$:

$$F = Vect(v_1, v_2) \text{ and } G = Vect(v_3)$$

Determine the dimensions of $F, G, F \cap G$ and $F + G$ as function of t.

END

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