Homework 2
Vector Spaces of finite dimension

Exercise 1

1) Determine whether the following vectors form a linearly independent set of the given Vector Space. If not, give the dependence relation between the vectors.
   a) $v_1 = (1, 0, 1)$, $v_2 = (0, 0, 2)$ and $v_3 = (3, 7, 1)$ in $\mathbb{R}^3$.
   b) $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 1)$ and $v_3 = (1, 1, 1)$ in $\mathbb{R}^3$.
   c) $v_1 = (1, 2, 1, 2, 1)$, $v_2 = (2, 1, 2, 1, 2)$, $v_3 = (1, 0, 1, 1, 0)$ and $v_4 = (0, 1, 0, 0, 1)$ in $\mathbb{R}^5$.
   d) $v_1 = (2, 4, 3, -1, -2, 1)$, $v_2 = (1, 1, 2, 1, 3, 1)$, $v_3 = (0, -1, 0, 3, 6, 2)$ in $\mathbb{R}^6$.
2) Show that $(1, \sqrt{2}, \sqrt{3})$ is a linearly independent set of the $\mathbb{Q}$-Vector Space.

Exercise 2

1) Determine whether the following vectors form a generating set of the given Vector Space :
   a) $v_1 = (0, 1, 1)$, $v_2 = (1, -1, 0)$, $v_3 = (1, 0, 2)$ and $v_4 = (1, -1, 2)$ in $\mathbb{R}^3$.
   b) $v_1 = (1, 2, -2, 0)$, $v_2 = (0, 4, 1, -2)$, $v_3 = (2, 0, -5, 2)$ and $v_4 = (1, -6, -4, 4)$ in $\mathbb{R}^4$.
2) Show that the following functions in the $\mathbb{R}$-Vector Space $E = \mathcal{F}([0, 2\pi], \mathbb{R})$ defined as :
$$\forall x \in \mathbb{R},\ f_1(x) = \cos(x),\ f_2(x) = x\cos(x),\ f_3(x) = \sin(x),\ f_4(x) = x\sin(x)$$
   form a linearly independent set.
3) Let E a $\mathbb{K}$-Vector Space and e_1, e_2, e_3 vectors in E such that the set (e_1, e_2, e_3) is linearly independent. Show that the set (f_1, f_2, f_3), with $f_1 = e_1 - e_2$, $f_2 = e_2 + e_3$ and $f_3 = e_1 - e_3$, is linearly independent.
4) We define for every $a \in \mathbb{R}$ the application f_a as :
$$\forall x \in \mathbb{R},\ f_a(x) = |x - a|$$
   Show that the set (f_0, f_1, f_2) a linearly independent set of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Exercise 3

1) Let $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$ and $v_3 = (0, 0, 1)$. Is $\mathcal{B} = (v_1, v_2, v_3)$ a basis of $\mathbb{R}^3$?
2) Let E a $\mathbb{K}$-Vector Space and $\mathcal{B} = (e_1, e_2, e_3)$ a basis of E. Show that the set $\mathcal{B}' = (f_1, f_2, f_3)$, with $f_1 = e_2 + 2e_3$, $f_2 = e_3 - e_1$ and $f_3 = e_1 + 2e_2$, is a basis of E.
3) Let $F = \{(x, y, z, t) \in \mathbb{R}^4 | 2x - y + z - t = 0\}$.
   a) Show that F is a Vector Subspace of $\mathbb{R}^4$.
   b) Find a generating set of F and deduce a basis of F.
4) Show that the set $\mathcal{S} = \{v_1, v_2\}$, with $v_1 = (1, -1, 1, -1)$ and $v_2 = (1, 1, 1, 1)$, can be completed to a basis.
5) Let G the generated Vector Subspace by the set $\mathcal{S} = \{v_1, v_2, v_3, v_4\}$, with $v_1 = (1, 2, 1, 0)$, $v_2 = (-1, 1, 1, 1)$, $v_3 = (2, -1, 0, 1)$ and $v_4 = (2, 2, 2, 2)$. Determine a basis of G.

Exercise 4

Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and F the set defined as

$$\{A \in M_2(\mathbb{R}) | A \times N = N \times A\}$$

1) Show that F is a Vector Subspace of $M_2(\mathbb{R})$.
2) What are the conditions on a, b, c, d in order to have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F$?
3) Determine a basis of F.

Exercise 5

Let $f, g, h, k \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined as

$$\forall x \in \mathbb{R},\ f(x) = -1,\ g(x) = (x-1)^2,\ h(x) = x - 3,\ k(x) = x^2 + 1.$$

Let P be the Vector Subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ generated by (f, g, h, k).

1) Find a basis of P. What is the dimension of P ?

2) Let $p, q, r \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined as

$$\forall x \in \mathbb{R},\ p(x) = 1,\ q(x) = x,\ r(x) = x^2.$$

Show that the set (p, q, r) is also a basis of P.

3) Let G be the set defined as

$$G = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) |\ f \text{ 3-times differentiable and } \forall x \in \mathbb{R}, f^{(3)}(x) = 0\}$$

Show that G is a Vector Subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ and $G = P$.

Exercise 6

1) Let $F = \text{Vect}(v_1, v_2, v_3)$ and $G = \text{Vect}(v_4, v_5)$ be two Vector Subspaces of $\mathbb{R}^4$ with

$$v_1 = (1, 2, 3, 4),\ v_2 = (1, 1, 1, 3),\ v_3 = (2, 1, 1, 1),\ v_4 = (-1, 0, -1, 2),\ v_5 = (2, 3, 0, 1).$$

Determine the dimensions of F, G, $F \cap G$ and $F + G$.

2) Consider in $\mathbb{R}^4$ the following Vector Subspaces

$$F = \text{Vect}\big((1, 2, 1, 3), (2, 0, 0, 1)\big) \text{ and } G = \{(x, y, z, t) \in \mathbb{R}^4 | 2x + y + z = 0 \text{ and } x = y\}.$$

a) Determine the dimensions of F and G,
b) Show that $F \cap G = \{0_{\mathbb{R}^4}\}$,
c) Deduce that $F \oplus G = \mathbb{R}^4$.

Exercise 7

Let E a $\mathbb{K}$-Vector Space of finite dimension $n \geq 2$.

1) Let H_1 and H_2 be two distinct hyperplanes of E. Determine the dimension of $H_1 \cap H_2$.

2) Let H be a hyperplane of E and F a Vector Subspace of E not included in H.

Show that $\dim(F \cap H) = \dim(F) - 1$.

Hw2 Corrections

1. a) $v_1=(1,0,1)$, $v_2=(0,0,2)$ $v_3=(3,7,1)$

$$\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=0_{\mathbb{R}^3}$$

$$\lambda_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda_2\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda_3\begin{pmatrix}3\\7\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}\lambda_1+3\lambda_3=0\\ 7\lambda_3=0 \Rightarrow \lambda_3=0\\ \lambda_1+2\lambda_2+\lambda_3=0\end{cases} \Rightarrow \begin{array}{l}\lambda_1+3\cancel{\lambda_3}^{\,0}=0 \Rightarrow \lambda_1=0\\ \lambda_1+2\lambda_2=0 \Rightarrow \lambda_2=0\end{array}$$

$\Rightarrow \lambda_1=0,\ \lambda_2=0,\ \lambda_3=0$

$\Rightarrow$ S is L.I. in $\mathbb{R}^3$

b) $v_1=(1,0,0)$, $v_2=(0,1,1)$, $v_3=(1,1,1)$

$$\lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda_2\begin{pmatrix}0\\1\\1\end{pmatrix}+\lambda_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}\lambda_1+\lambda_3=0\\ \lambda_2+\lambda_3=0\\ \lambda_2+\lambda_3=0\end{cases} \Rightarrow \begin{array}{l}\lambda_1=-\lambda_3\\ \lambda_2=-\lambda_3\end{array} \Rightarrow \text{infinitely many solutions.}$$

$\Rightarrow S=\{v_1,v_2,v_3\}$ is Linearly dependent in $\mathbb{R}^3$ and the dependence relation is:

$$\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=0$$
$$-\lambda_3 v_1-\lambda_3 v_2+\lambda_3 v_3=0$$
$$\Rightarrow v_3=v_1+v_2$$

c) $v_1=(1,2,1,2,1)$, $v_2=(2,1,2,1,2)$, $v_3=(1,0,1,1,0)$, $v_4=(0,1,0,0,1)$

$$\lambda_1\begin{pmatrix}1\\2\\1\\2\\1\end{pmatrix}+\lambda_2\begin{pmatrix}2\\1\\2\\1\\2\end{pmatrix}+\lambda_3\begin{pmatrix}1\\0\\1\\1\\0\end{pmatrix}+\lambda_4\begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&2&1&0\\2&1&0&1\\1&2&1&0\\2&1&1&0\\1&2&0&1\end{pmatrix}\xrightarrow[\substack{L_2\leftarrow L_2-2L_1\\ L_3\leftarrow L_3-L_1\\ L_4\leftarrow L_4-2L_1\\ L_5\leftarrow L_5-L_1}]{}\begin{pmatrix}1&2&1&0\\0&-3&-2&1\\0&0&0&0\\0&-3&-1&0\\0&0&-1&1\end{pmatrix}\xrightarrow{L_4\leftarrow L_4-L_2}\begin{pmatrix}1&2&1&0\\0&-3&-2&1\\0&0&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}$$

$\rightarrow \lambda_1+2\lambda_2+\lambda_3=0$

$\rightarrow -3\lambda_2-2\lambda_3+\lambda_4=0$

$\rightarrow \lambda_3=\lambda_4$

$\Rightarrow -3\lambda_2 - 2\lambda_3 + \lambda_3 = 0 \Rightarrow -3\lambda_2 = \lambda_3$

$\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_1 + 2\lambda_2 - 3\lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$

$\Rightarrow S = \{v_1, v_2, v_3, v_4\}$ is Linearly dependent in $\mathbb{R}^5$ and dependence relation is:

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$$
$$\lambda_2 v_1 + \lambda_2 v_2 - 3\lambda_2 v_3 - 3\lambda_2 v_4 = 0$$
$$\Rightarrow 3v_4 = 3v_3 - v_1 - v_2$$
$$v_4 = v_3 - \frac{1}{3}v_1 - \frac{1}{3}v_2$$

d) $v_1 = (2, 4, 3, -1, -2, 1)$, $v_2 = (1, 1, 2, 1, 3, 1)$, $v_3 = (0, -1, 0, 3, 6, 2)$

$$\lambda_1 \begin{pmatrix} 2 \\ 4 \\ 3 \\ -1 \\ -2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 3 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$2\lambda_1 + \lambda_2 = 0 \longrightarrow 2\lambda_1 = \lambda_2$

$4\lambda_1 + \lambda_2 - \lambda_3 = 0 \qquad \Rightarrow \lambda_1 = \lambda_2 = 0$

$3\lambda_1 + 2\lambda_2 = 0 \longrightarrow 3\lambda_1 = -2\lambda_2$

$-\lambda_1 + \lambda_2 + 3\lambda_3 = 0$

$-2\lambda_1 + 3\lambda_2 + 6\lambda_3 = 0 \qquad \Rightarrow \lambda_3 = 0$

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$

$\Rightarrow S = \{v_1, v_2, v_3\}$ is L. I. in $\mathbb{R}^6$.

2) Let $\{a, b, c\} \in \mathbb{Q}$

$$a + \sqrt{2}b + \sqrt{3}c = 0$$

$a + \sqrt{2}b = -\sqrt{3}c$

$(a + \sqrt{2}b)^2 = 3c^2$

$a^2 + 2\sqrt{2}b + 2b^2 = 3c^2$

$\underbrace{2\sqrt{2}ab}_{\notin \mathbb{Q}} = \underbrace{3c^2}_{\in \mathbb{Q}} - \underbrace{a^2}_{\in \mathbb{Q}} - \underbrace{2b^2}_{\in \mathbb{Q}}$

since $\sqrt{2} \notin \mathbb{Q} \Rightarrow ab = 0$

If $b=0$: $a+\sqrt{3}c=0$

$$\underbrace{a}_{\in\mathbb{Q}} = \underbrace{-\sqrt{3}c}_{\notin\mathbb{Q}} \Rightarrow c=0$$

$\Rightarrow a=0,\ b=0,\ c=0$

If $a=0$: $\sqrt{2}b=-\sqrt{3}c$

$2b^2=3c^2$

if $c\neq 0$, $\left(\frac{b}{c}\right)^2=\frac{3}{2}$

$$\underbrace{\frac{b}{c}}_{\in\mathbb{Q}} = \underbrace{\pm\sqrt{\frac{3}{2}}}_{\notin\mathbb{Q}} \Rightarrow b=0$$

$\Rightarrow c=0$

$\Rightarrow a=0,\ b=0,\ c=0$

$\Rightarrow \forall a,b,c\in\mathbb{Q}$, $a+\sqrt{2}b+\sqrt{3}c=0$

$\Rightarrow a=b=c=0$

2 1) a) <u>Method 1</u>

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & x \\ 1 & -1 & 0 & -1 & y \\ 1 & 0 & 2 & 2 & z \end{array}\right) \xrightarrow[L_3\leftarrow L_3-L_2]{} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & x \\ 1 & -1 & 0 & -1 & y \\ 0 & 1 & 2 & 3 & z-y \end{array}\right)$$

$$\xrightarrow[L_3\leftarrow L_3-L_1]{} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & x \\ 1 & -1 & 0 & -1 & y \\ 0 & 0 & 1 & 2 & z-y-x \end{array}\right) \xrightarrow[L_2\leftarrow L_2+L_1]{} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & x \\ 1 & 0 & 1 & 0 & y+x \\ 0 & 0 & 1 & 2 & z-y-x \end{array}\right)$$

$$\xrightarrow[L_1\leftarrow L_1-L_3]{} \left(\begin{array}{cccc|c} 0 & 1 & 0 & -1 & x-(z-y-x) \\ 1 & 0 & 1 & 0 & y+x \\ 0 & 0 & 1 & 2 & z-y-x \end{array}\right)$$

$$\Rightarrow \begin{cases} \lambda_3+2\lambda_4=z-y-x \\ \lambda_1+\lambda_3=y+x \\ \lambda_2-\lambda_4=2x-z+y \end{cases} \Rightarrow \begin{aligned} \lambda_3 &= z-y-x-2\lambda_4 \\ \lambda_1 &= y+x-\lambda_3 = y+x-(z-y-x-2\lambda_4) \\ &= 2y+2x-z+2\lambda_4 \\ \lambda_2 &= 2x-z+y+\lambda_4 \end{aligned}$$

where $\lambda_4\in\mathbb{R}$

Since we can represent (uniquely) $\lambda_1,\lambda_2,\lambda_3$ in terms of x,y,z for any $\lambda_4\in\mathbb{R}$, $\{v_1,v_2,v_3,v_4\}$ is a G. S. of $\mathbb{R}^3$.
(you may plug in $\lambda_1,\lambda_2,\lambda_3$ to the equation to check your answer)

Method 2 Take any of the 3 vectors out of 4 and find the determinant. If the determinant is not 0, then the 3 vectors is the generating set of $\mathbb{R}^3$ and thus adding the 4^{th} vector will not change the conclusion since the 4^{th} vector will be linearly dependent of the other vectors.
(Remember 4 vectors in $\mathbb{R}^3$ are L. D.)

lets take V_1, V_2, V_3

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0 \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -2 + 1 = -1 \neq 0$$

$\Rightarrow \{V_1, V_2, V_3\}$ is a G. S. of $\mathbb{R}^3$.

$\Rightarrow \{V_1, V_2, V_3\} \cup \{V_4\}$ is also a G. S. of $\mathbb{R}^3$

B) let $S = \{V_1, V_2, V_3, V_4\}$ in $\mathbb{R}^4$

Check if the determinant of the matrix formed by the 4 vectors in $\mathbb{R}^4$ is 0 or not.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & 4 & 0 & -6 \\ -2 & 1 & -5 & -4 \\ 0 & -2 & 2 & 4 \end{vmatrix} \xrightarrow[L_3 \leftarrow L_3 + 2L_1]{L_2 \leftarrow L_2 - 2L_1} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -4 & -8 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 2 & 4 \end{vmatrix} \xrightarrow[L_4 \leftarrow L_4 + 2L_3]{} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -4 & -8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ the matrix is not invertible.

$\Rightarrow$ S can't be a G. S. of $\mathbb{R}^4$ because if the matrix is not invertible that means λ's can't be represented uniquely ~~as~~ by x, y, z, w is we write $\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

2) $E = \mathcal{F}([0,2\pi], \mathbb{R})$

$\forall x \in \mathbb{R}$, $f_1(x) = \cos(x)$

$f_2(x) = x\cos(x)$

$f_3(x) = \sin(x)$

$f_4(x) = x\sin(x)$

let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ such that

$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \alpha_4 f_4(x) = 0 \quad \forall x \in [0,2\pi]$

lets evaluate the functions at 4 different values of x. in the interval $[0,2\pi]$

For simplicity, lets choose $x = 0, \pi/2, \pi, 3\pi/2$

If $x = 0$, $\alpha_1 = 0$

If $x = \frac{\pi}{2}$, $\alpha_3 + \frac{\pi}{2}\alpha_4 = 0 \Rightarrow \alpha_3 = -\frac{\pi}{2}\alpha_4$

If $x = \pi$, $\alpha_1 + \pi\alpha_2 = 0 \Rightarrow \alpha_2 = 0$

If $x = \frac{3\pi}{2}$, $\alpha_3 + \frac{3\pi}{2}\alpha_4 = 0 \Rightarrow \alpha_3 = \frac{3\pi}{2}\alpha_4$

$\alpha_3 = \alpha_4 = 0$

$\Rightarrow \{f_1, f_2, f_3, f_4\}$ is L.I.

3) (e_1, e_2, e_3) is a L.I. set

let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$,

$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$

$\lambda_1(e_1 - e_2) + \lambda_2(e_2 + e_3) + \lambda_3(e_1 - e_3) = 0$

$\underline{\lambda_1 e_1} - \underline{\lambda_1 e_2} + \underline{\lambda_2 e_2} + \lambda_2 e_3 + \underline{\lambda_3 e_1} - \lambda_3 e_3 = 0$

$(\lambda_1 + \lambda_3)e_1 + (\lambda_2 - \lambda_1)e_2 + (\lambda_2 - \lambda_3)e_3 = 0$

since we know (e_1, e_2, e_3) is L.I., coefficients of e_1, e_2, e_3 must be 0.

$\Rightarrow \lambda_1 + \lambda_3 = 0$

$\lambda_2 - \lambda_1 = 0 \rightarrow \lambda_1 = \lambda_2 \Rightarrow \begin{matrix}\lambda_2 + \lambda_3 = 0 \\ \lambda_2 - \lambda_3 = 0\end{matrix} \Rightarrow \lambda_2 = \lambda_3 = 0$

$\lambda_2 - \lambda_3 = 0$

$\Rightarrow \lambda_1 = 0$

Thus, $\{f_1, f_2, f_3\}$ is L.I.

4) $\forall x \in \mathbb{R}, \quad f_a(x) = |x-a|$

$f_0(x) = |x-0| = |x|$

$f_1(x) = |x-1|$

$f_2|x| = |x-2|$

let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$

Lets evaluate the functions at 3 different values of X and for simplicity, lets choose $X = 0, 1, 2$

If $X=0$, $\quad \alpha_2 + 2\alpha_3 = 0$

$X=1$, $\quad \alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3 \Rightarrow$

$X=2$, $\quad 2\alpha_1 + \alpha_2 = 0$

$\alpha_2 + 2\alpha_3 = 0$ ①

$-2\alpha_3 + \alpha_2 = 0$ ②

$\Rightarrow \alpha_2 = 2\alpha_3$

$2\alpha_3 + 2\alpha_3 = 0$

$\alpha_3 = 0$

$\Rightarrow \alpha_2 = 0$

$\Rightarrow \alpha_1 = 0$

Thus, (f_0, f_1, f_2) is a L.I. set of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

③ 1) $v_1 = (1,1,1), \quad v_2 = (1,1,0), \quad v_3 = (0,0,1)$

Is $B = (v_1, v_2, v_3)$ a basis of $\mathbb{R}^3$?

Method 1 $\rightarrow$ Is B L.I.?

$$\lambda_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\lambda_1 + \lambda_2 = 0$

$\lambda_1 + \lambda_2 = 0 \quad \Rightarrow \lambda_1 = -\lambda_2 \quad \Rightarrow \lambda_2 = \lambda_3$

$\lambda_1 + \lambda_3 = 0 \quad \quad \lambda_1 = -\lambda_3$

since B is Linearly dependent, it can't be a basis.

Method 2 Simply find the determinant $\begin{vmatrix}1 & 1 & 0\\1 & 1 & 0\\1 & 0 & 1\end{vmatrix} = 0$

Since the determinant is 0, B can't be a G.S. and L.I. set of $\mathbb{R}^3$. Thus, not a basis of $\mathbb{R}^3$

2) $B = (e_1, e_2, e_3)$ is a basis of E.

$B' = (f_1, f_2, f_3)$. Is B' also a basis of E?

$f_1 = e_2 + 2e_3$
$f_2 = e_3 - e_1$
$f_3 = e_1 + 2e_2$

- Is B' L.I.?

$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

$\lambda_1(e_2 + 2e_3) + \lambda_2(e_3 - e_1) + \lambda_3(e_1 + 2e_2) = 0$

$\lambda_1 e_2 + 2\lambda_1 e_3 + \lambda_2 e_3 - \lambda_2 e_1 + \lambda_3 e_1 + 2\lambda_3 e_2 = 0$

$(\lambda_3 - \lambda_2)e_1 + (\lambda_1 + 2\lambda_3)e_2 + (2\lambda_1 + \lambda_2)e_3 = 0$

since (e_1, e_2, e_3) is L.I, $\lambda_3 - \lambda_2 = 0 \rightarrow \lambda_2 = \lambda_3$
$\lambda_1 + 2\lambda_3 = 0$ $\quad\quad \lambda_1 + 2\lambda_3 = 0$ ①
$2\lambda_1 + \lambda_2 = 0$ $\quad\quad 2\lambda_1 + \lambda_3 = 0$ ②

② − ① $\Rightarrow \lambda_1 = \lambda_3$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

$\Rightarrow B'$ is L.I.

- Is B' G.S. of E?

let $u \in E$, $u = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$

Knowing that (e_1, e_2, e_3) is a G.S. of E
and $u = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$u = \lambda_1(e_2 + 2e_3) + \lambda_2(e_3 - e_1) + \lambda_3(e_1 + 2e_2)$
$= (\lambda_3 - \lambda_2)e_1 + (\lambda_1 + 2\lambda_3)e_2 + (2\lambda_1 + \lambda_2)e_3$

$\Rightarrow \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = (\lambda_3 - \lambda_1)e_1 + (\lambda_1 + 2\lambda_3)e_2 + (2\lambda_1 + \lambda_2)e_3$

$\Rightarrow \alpha_1 = \lambda_3 - \lambda_2$
$\alpha_2 = \lambda_1 + 2\lambda_3$
$\alpha_3 = 2\lambda_1 + \lambda_2$

can we represent λ's in terms of α's? If yes, then we can conclude B' is G.S. of E

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & \alpha_1 \\ 1 & 0 & 2 & \alpha_2 \\ 2 & 1 & 0 & \alpha_3 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - 2L_2} \left(\begin{array}{ccc|c} 0 & -1 & 1 & \alpha_1 \\ 1 & 0 & 2 & \alpha_2 \\ 0 & 1 & -4 & \alpha_3 - 2\alpha_2 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 + L_1} \left(\begin{array}{ccc|c} 0 & -1 & 1 & \alpha_1 \\ 1 & 0 & 2 & \alpha_2 \\ 0 & 0 & -3 & \alpha_3 - 2\alpha_2 - \alpha_1 \end{array}\right)$$

$\Rightarrow \lambda_3 = -\frac{1}{3}(\alpha_3 - 2\alpha_2 - \alpha_1)$

$\lambda_1 = \alpha_2 - 2\lambda_3 = \alpha_2 + \frac{2}{3}(\alpha_3 - 2\alpha_2 - \alpha_1)$

$\lambda_2 = \lambda_3 + \alpha_1 = -\frac{1}{3}(\alpha_3 - 2\alpha_2 - \alpha_1) + \alpha_1$

$\Rightarrow$ B' is a G. S. of E

And hence, B' is also a basis of E

3) $F = \{ (x, y, z, t) \in \mathbb{R}^4 \mid 2x - y + z - t = 0 \}$

a) $0_{\mathbb{R}^4} \in F$? Yes, since $2 \cdot 0 - 0 + 0 - 0 = 0$

let $u = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ and $u' = \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \in F$

Then $2x - y + z - t = 0$

$2x' - y' + z' - t' = 0$

$2(x + x') - (y + y') + (z + z') - (t + t') =$

$= \underbrace{2x - y + z - t}_{0} + \underbrace{2x' - y' + z' - t'}_{0} = 0$

let $u \in F$, $\lambda \in \mathbb{R}$, is $\lambda u \in F$?

$2\lambda x - \lambda y + \lambda z - \lambda t = \lambda(2x - y + z - t) = 0$

$\Rightarrow \lambda u \in F$

thus, F is a vector subspace of $\mathbb{R}^4$.

b) let $u = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in F$ $\Rightarrow$ $2x - y + z - t = 0$

$t = 2x - y + z$

$$u = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 2x - y + z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow u \in \mathrm{Vect} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Thus, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a G.S. of F

Verify that these vectors are L.I. $\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$ $\Rightarrow$ L. I.

Thus, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis of F.

4) $S = \{v_1, v_2\}$

$v_1 = (1, -1, 1, -1)$

$v_2 = (1, 1, 1, 1)$

S is L.I. since v_1 and v_2 are non-collinear

Using the incomplete basis theorem, we can complete S to a basis using the vectors from the canonical basis of $\mathbb{R}^4$.

Canonical basis of $\mathbb{R}^4$ is: (e_1, e_2, e_3, e_4)

$e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$

First, add vector e_1. $S_1 = S \cup \{e_1\} = \{v_1, v_2, e_1\}$

Is S_1 L.I.?

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 + L_1}]{L_2 \leftarrow L_1 + L_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{L_4 \leftarrow L_4 - L_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow -\lambda_3 = 0$

$\lambda_3 = 0$

$\Rightarrow 2\lambda_2 + \lambda_3 = 0$

$\Rightarrow \lambda_2 = 0$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$

$\Rightarrow \lambda_1 = 0$

$\Rightarrow S_1$ is L.I.

Next, add e_2. $S_2 = S_1 \cup \{e_2\} = \{v_1, v_2, e_1, e_2\}$

Is S_2 L.I.?

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 e_1 + \lambda_4 e_2 = 0$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 + L_1}]{L_2 \leftarrow L_2 + L_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \xrightarrow{L_4 \leftarrow L_4 - L_2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\Rightarrow \lambda_4 = 0$

$\lambda_3 = 0$

$\lambda_2 = 0$

$\lambda_1 = 0$

$\Rightarrow S_2$ is L.I. $\Rightarrow \mathrm{card}(S_2) = 4 = \dim(\mathbb{R}^4)$

$\Rightarrow S_2$ is the basis of $\mathbb{R}^4$.

5) $S = \{v_1, v_2, v_3, v_4\}$

$v_1 = (1,2,1,0)$, $v_2 = (-1,1,1,1)$ $v_3 = (2,-1,0,1)$, $v_4 = (2,2,2,2)$

S is the generating set of G.

To extract a basis from a G.S., we will use extracted basis theorem.

Step 0 $\quad B_0 = \{\emptyset\}$ is not a G.S. of G

Step 1 $\quad B_1 = \{v_1\}$ $\quad v_1 \neq 0_{\mathbb{R}^4}$, $\{v_1\}$ is L.I.

Step 2 $\quad B_2 = B_1 \cup \{v_2\} = \{v_1, v_2\}$

Is $\{v_1, v_2\}$ L.I.?

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1 - \lambda_2 = 0$
$2\lambda_1 + \lambda_2 = 0$ $\quad \Rightarrow \lambda_1 = 0.$ $\quad B_2$ is L.I.
$\lambda_1 + \lambda_2 = 0$
$\lambda_2 = 0$

Step 3 $\quad B_3 = B_2 \cup \{v_3\} = \{v_1, v_2, v_3\}$. Is B_3 L.I.?

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1 - \lambda_2 + 2\lambda_3 = 0 \Rightarrow \lambda_1 + \lambda_1 + 2\lambda_1 = 0 \Rightarrow \lambda_1 = 0 \quad \Rightarrow \lambda_2 = \lambda_3 = 0$
$2\lambda_1 + \lambda_2 - \lambda_3 = 0 \Rightarrow 2\lambda_1 - \lambda_3 - \lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_3$
$\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -\lambda_1$ $\quad B_3$ is L.I.
$\lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -\lambda_3$

Step 4 $\quad B_4 = B_3 \cup \{v_4\} = \{v_1, v_2, v_3, v_4\}$ $\quad$ Is B_4 L.I.?

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & -5 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & -5 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & -5 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad 2\lambda_2 - 2\lambda_3 = 0 \qquad \lambda_2 = \lambda_3$

$3\lambda_2 - 5\lambda_3 - 2\lambda_4 = 0 \Rightarrow 3\lambda_3 - 5\lambda_3 = 2\lambda_4$

$\Rightarrow \lambda_4 = -\lambda_3$

$\Rightarrow \quad \lambda_1 - \lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$

$\lambda_1 - \lambda_3 + 2\lambda_3 - 2\lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_3$

$\Rightarrow$ infinitely many solutions

$\Rightarrow$ B_4 is not L. I.

$\Rightarrow$ B_3 is L. I. and G. S. of G.

$\Rightarrow$ B_3 is the basis of G.

4. Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$F = \{ A \in M_2(\mathbb{R}) \mid A \times N = N \times A \}$

1) • Is $O_{M_2(\mathbb{R})} \in F$?

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times N = N \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow O_{M_2(\mathbb{R})} \in F$

• let $A, B \in F \Rightarrow A \times N = N \times A$

$B \times N = N \times B$

$(A+B) \times N = A \times N + B \times N = N \times A + N \times B = N(A+B)$

• let $\lambda \in \mathbb{R}$, $A \in F$

$(\lambda A) \times N = N \times (\lambda A)$

$\Rightarrow \lambda A \in F$

2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \Rightarrow \boxed{c = 0,\ a = d}$

3) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix} = d \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I_2} + b \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N} = d I_2 + bN$

F is generated by $\{I_2, N\}$ and $\{I_2, N\}$ is L. I.

$\Rightarrow \{I_2, N\}$ is the basis of F

5. $f, g, h, K \in \mathcal{F}(\mathbb{R}, \mathbb{R})$

$\forall x \in \mathbb{R}$, $f(x) = -1$, $g(x) = (x-1)^2$, $h(x) = x-3$, $K(x) = x^2+1$

$\{f, g, h, K\}$ is the G.S. of P

a) we use the extracted basis theorem.

Step 0 Consider $B_0 = \{\phi\}$ is not a G.S. of P

step 1 Consider $B_1 = B_0 \cup \{f\} = \{f\}$, $\{f\}$ is L.I.

step 2 Consider $B_2 = B_1 \cup \{g\} = \{f, g\}$

Is B_2 L.I. ?

$\lambda_1(-1) + \lambda_2(x-1)^2 = 0$

$-\lambda_1 + \lambda_2 x^2 - 2\lambda_2 x + 2\lambda_2 = 0$

$\Rightarrow \lambda_2 = 0, \lambda_1 = 0$

step 3 Consider $B_3 = B_2 \cup \{h\} = \{f, g, h)$

Is B_3 L.I. ?

$\lambda_1(-1) + \lambda_2(x-1)^2 + \lambda_3(x-3) = 0$

$-\lambda_1 + \lambda_2 x^2 - 2\lambda_2 x + \lambda_2 + \lambda_3 x - 3\lambda_3 = 0$

$\lambda_2 x^2 + (\lambda_3 - 2\lambda_2)x - \lambda_1 + \lambda_2 - 3\lambda_3 = 0$

$\Rightarrow \lambda_2 = 0$, $\lambda_3 - 2\lambda_2 = 0 \Rightarrow \lambda_3 = 0$

$\Rightarrow \lambda_1 = 0$

$\Rightarrow B_3$ is L.I.

step 4 Consider $B_4 = B_3 \cup \{K\} = \{f, g, h, K\}$, Is B_4 L.I. ?

$\lambda_1(-1) + \lambda_2(x-1)^2 + \lambda_3(x-3) + \lambda_4(x^2+1) = 0$

$-\lambda_1 + \lambda_2 x^2 - 2\lambda_2 x + \lambda_2 + \lambda_3 x - 3\lambda_3 + \lambda_4 x^2 + \lambda_4 = 0$

$x^2(\lambda_2 + \lambda_4) + (\lambda_3 - 2\lambda_2)x - \lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 = 0$

$\Rightarrow \lambda_2 + \lambda_4 = 0 \Rightarrow \lambda_2 = -\lambda_4$

$\lambda_3 - 2\lambda_2 = 0 \qquad \lambda_3 = 2\lambda_2$

$-\lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 = -\lambda_1 + \lambda_2 - 6\lambda_2 + (-\lambda_4) = 0$

$\Rightarrow -\lambda_1 - 5\lambda_2 - \lambda_4 = 0$

$\Rightarrow$ infinity many solutions $\lambda_1 = -5\lambda_2 - \lambda_4$

$\Rightarrow B_4$ is not L.I.

$\Rightarrow$ B_3 is L.I. and G.S. of P.

$\Rightarrow$ $B_3 = \{f, g, h\}$ is the basis of P.

$\dim(P) = \mathrm{card}(B_3) = 3$

2) $p, q, r \in \mathcal{F}(\mathbb{R}, \mathbb{R})$

$\forall x \in \mathbb{R},\ p(x) = 1,\ q(x) = x,\ r(x) = x^2$

let $S = \{p, q, r\}$

Check L.I.

$\lambda_1 p + \lambda_2 q + \lambda_3 r = 0$

$\lambda_1 (1) + \lambda_2 x + \lambda_3 x^2 = 0$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$

Since S is L.I. and cardinality of S is 3, which is the same as the dimension of P, we can say that S is also a basis of P. (no need to check generating set here since we know the dimension of P).

3) $G = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f$ is 3 times differentiable and $\forall x \in \mathbb{R},\ f^{(3)}(x) = 0 \}$

Showing that G is a vector subspace:

- $0_{\mathcal{F}(\mathbb{R},\mathbb{R})} \in G$ since $0^{(3)}_{\mathcal{F}(\mathbb{R},\mathbb{R})} = 0$
- let $f, g \in G$, then $f^{(3)}(x) = 0$

  $g^{(3)}(x) = 0$

  $(f+g)^{(3)}(x) = f^{(3)}(x) + g^{(3)}(x) = 0 + 0 = 0$

  $\Rightarrow f + g \in G$
- let $f \in G$, $\lambda \in \mathbb{R}$, $(\lambda f)^{(3)}(x) = \lambda f^{(3)}(x) = 0$

  $\lambda f \in G$

$\Rightarrow$ G is a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$

Showing $G = P$:

let $f \in G$ then $f^{(3)}(x) = 0$

$$\Rightarrow f(x) = \underbrace{\alpha_1 + \alpha_2 x + \alpha_3 x^2}_{\text{f must be in this form for } f'''(x)=0}$$

So, the elements of G are polynomials generated by $\{1, x, x^2\} = \{p, q, r\}$

$\Rightarrow f \in \mathrm{Vect}\{p, q, r\}$

$\Rightarrow G = \mathrm{Vect}(p, q, r)$

Then; from part 2) we deduce that $G = P$, since P is also generated by the set $\{p, q, r\}$ and since $\{p, q, r\}$ is L.I., it is the basis of both vector subspaces.

Also, recall the theorem, since $\dim(P) = \dim(G) = 3$, and $G \subset P$, $G = P$.

6. 1) $F = \mathrm{Vect}(v_1, v_2, v_3)$

$G = \mathrm{Vect}(v_4, v_5)$

$v_1 = (1, 2, 3, 4)$, $v_2 = (1, 1, 1, 3)$ $v_3 = (2, 1, 1, 1)$ $v_4 = (-1, 0, -1, 2)$

$v_5 = (2, 3, 0, 1)$

$\dim(F) = ?$ Since $\{v_1, v_2, v_3\}$ is G.S. of F, we only check L.I.

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow[\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}]{} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & -2 & -5 \\ 0 & -1 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & -2 & -5 \\ 0 & 0 & -4 \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_3 = 0 \\ \lambda_2 = 0 \\ \lambda_1 = 0 \end{array}$$

$\Rightarrow \{v_1, v_2, v_3\}$ is the basis of F and the card. of the basis is 3, $\dim(F) = 3$

$\dim(G) = ?$

$G = \text{Vect}\{v_4, v_5\} \Rightarrow \{v_4, v_5\}$ is the G.S. of G
and since (v_4, v_5) are non-collinear, they are L.I.
$\Rightarrow \{v_4, v_5\}$ is the basis of G, card. of the basis is 2. Thus, $\dim(G) = 2$

$\dim(F \cap G) = ?$

$$\begin{cases} F \cap G \subset G \\ F \cap G \subset F \end{cases} \Rightarrow \dim(F \cap G) \leq 2.$$

Also, since $\underbrace{\dim(F+G)}_{\leq 4} = \underbrace{\dim(F)}_{3} + \underbrace{\dim(G)}_{2} - \dim(F \cap G)$

since $F + G \subset \mathbb{R}^4$

It follows that $\dim(F \cap G) \geq 1$

$$\Rightarrow 1 \leq \dim(F \cap G) \leq 2$$

So, $\dim(F \cap G)$ is either 1 or 2. But we must determine which one it is exactly.

Lets assume $\dim(F \cap G) = 2$
since $\dim(G)$ is also 2, and $F \cap G \subset G$,
the it means $F \cap G = G$
and since $F \cap G \subset F$, it means $G \subset F$

If $G \subset F$, it means we can take any vector from G and write it as a linear combination of vectors in F. lets take v_4 and see if $v_4 = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & -1 \\ 4 & 3 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -3 & 2 \\ 0 & -2 & -5 & 2 \\ 0 & -1 & -7 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -3 & 2 \\ 0 & -2 & -5 & 2 \\ 0 & 0 & -4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -3 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -4 & 4 \end{array}\right)$$

$\Rightarrow \begin{array}{l} \lambda_3 = -2 \\ \lambda_3 = -1 \end{array} \Rightarrow$ impossible

Thus, $G \not\subset F$

Then, $\dim(F \cap G) \neq 2$

$\Rightarrow \dim(F \cap G) = 1$

Another long method would be to determine the basis of $F \cap G$ and deduce its dimension

$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G) = 3 + 2 - 1 = 4$

$\Rightarrow \dim(F+G) = 4$ $\Rightarrow$ $F + G = \mathbb{R}^4$.
and since $F + G \subset \mathbb{R}^4$

2) $F = \text{Vect}((1,2,1,3),(2,0,0,1))$
$G = \{(x,y,z,t) \in \mathbb{R}^4 \mid 2x + y + z = 0 \text{ and } x = y\}$

a) $\{(1,2,1,3),(2,0,0,1)\}$ is the G.S. of F
and ~~[illegible]~~ is L.I.
Thus, $\{(1,2,1,3),(2,0,0,1)\}$ is the basis of F
and $\dim(F) = 2$

For $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in G$, $x = y$ and $z = -2x - y$

Then, $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} y \\ y \\ -2y - y \\ t \end{pmatrix} = y \underbrace{\begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix}}_{v_1} + t \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$

$\{v_1, v_2\}$ is the G.S. of G and since they are non-collinear, $\{v_1, v_2\}$ is L.I. and thus, basis of G

$\Rightarrow \dim(G) = 2$

b) Let $u \in F \cap G$

$\Rightarrow u \in F$ and $u \in G$

$$u \in F \Rightarrow u = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2\lambda_1 \\ \lambda_1 \\ 3\lambda_1 + \lambda_2 \end{pmatrix}$$

$$u \in G \Rightarrow \begin{cases} 2x + y + z = 0 \\ x = y \end{cases}$$

$$\Rightarrow 2\lambda_1 = \lambda_1 + 2\lambda_2 \Rightarrow \lambda_1 = 2\lambda_2$$

$$2(\lambda_1 + 2\lambda_2) + 2\lambda_1 + \lambda_1 = 0$$

$$2\lambda_1 + 4\lambda_2 + 3\lambda_1 = 0$$

$$5\lambda_1 + 4\lambda_2 = 0$$

$$10\lambda_2 + 4\lambda_2 = 0 \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_1 = 0$$

$$\Rightarrow u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow F \cap G = \{0_{\mathbb{R}^4}\}$$

c) Since $F \cap G = \{0_{\mathbb{R}^4}\}$, $\dim(F \cap G) = 0$

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G)$$

$$= 2 + 2 = 4$$

$\dim(F + G) = 4$ and since $\dim(\mathbb{R}^4) = 4$

$F + G \subset \mathbb{R}^4$

$$\Rightarrow F + G = \mathbb{R}^4$$

$$\Rightarrow F \oplus G = \mathbb{R}^4$$

7. E is a vector space

$\dim(E) = n \geq 2$

a) H_1 and H_2 are hyperplanes of E

$\Rightarrow \dim(H_1) = n-1$

$\dim(H_2) = n-1$

$$\underbrace{\dim(H_1 + H_2)}_{\leq n} = \underbrace{\dim(H_1)}_{n-1} + \underbrace{\dim(H_2)}_{n-1} - \dim(H_1 \cap H_2)$$

since $H_1 + H_2 \subset E$

$\Rightarrow \dim(H_1 \cap H_2) \geq n-2$

and since $H_1 \cap H_2 \subset H_1$, $H_1 \cap H_2 \subset H_2$ $\Rightarrow \dim(H_1 \cap H_2) \leq n-1$

$$n-2 \leq \dim(H_1 \cap H_2) \leq n-1$$

Assume $\dim(H_1 \cap H_2) = n-1$

then since $\dim(H_1) = \dim(H_2) = n-1$,

and $H_1 \cap H_2 \subset H_1$ (H_2) $\Rightarrow$ $H_1 \cap H_2 = H_1$, $H_1 \cap H_2 = H_2$

$\Rightarrow H_1 = H_2$

but this is impossible since H_1 and H_2 are distinct.

$\Rightarrow \dim(H_1 \cap H_2) = n-2$

2) H is a hyperplane of E

F is a vect. subspace of E not included in H

$\dim(F + H) = \dim(F) + \dim(H) - \dim(F \cap H)$

$n-1 \leq \dim(F + H) \leq n$

If $\dim(F + H) = n-1$, then since $\dim(H) = n-1$ and $H \subset F + H$, $H = F + H$

but this is impossible since $F \not\subset H$

$\Rightarrow \dim(F + H) = n$

Then, $n = \dim(F) + (n-1) - \dim(F \cap H)$

$$\Rightarrow \dim(F \cap H) = \dim(F) + (n-1) - n$$
$$= \dim(F) - 1$$