Algorithms and Complexity (Adv)

The Stable Matching Problem

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A bit of history

Residence program established in the early 20th century in US

Imbalances in the labor market causes problems:

- In 1945, hospitals hired students after their 1st year in med school
- In 1946-8, AAMC issued guidelines forbidding universities to give out student transcripts before their last year in school
- Hospitals countered by shortening waiting period from offer to acceptance: In 1945, students had 10 make up their mind; in1950, only12 hours.

Basic problem:

- Student accepts an offer and then receives an offer from a better hospital
- Hospital is turned down late in the season y cannot find good students

Abstract model

Let $\{m_1, m_2, \ldots, m_n\}$ be a set of n men, and

$\{w_1, w_2, \ldots, w_n\}$ be a set of n women

Each man has an individual ranking of the women and vice-versa

m_1	w_1	w_3	w_2
m_2	w_1	w_2	w_3
m_3	w_3	w_2	w_1

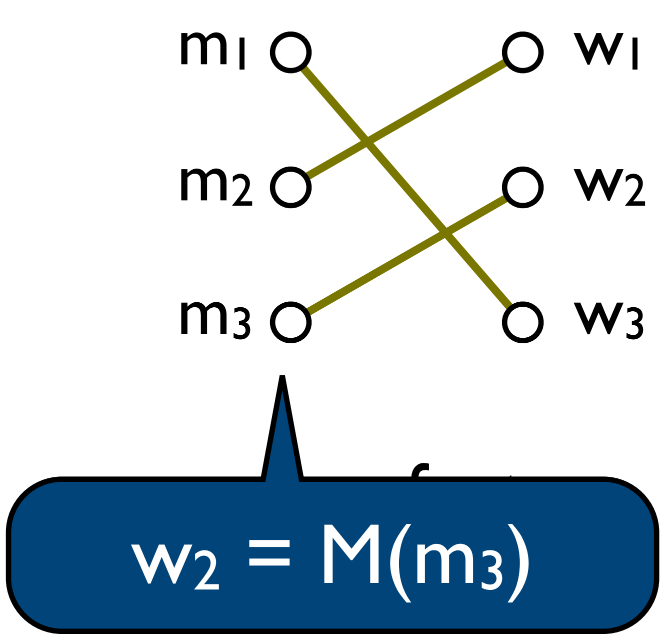
w_1	m_3	m_2	m_1
w_2	m_1	m_2	m_3
w_3	m_3	m_2	m_1

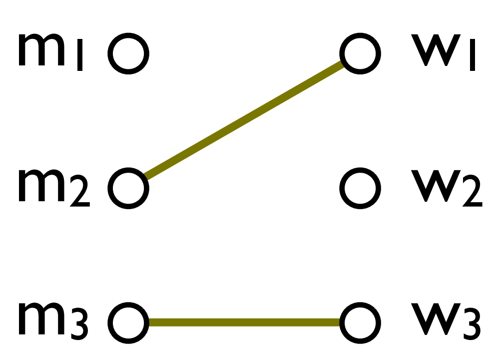
$w_3 >_{m3} w_2$

Matchings

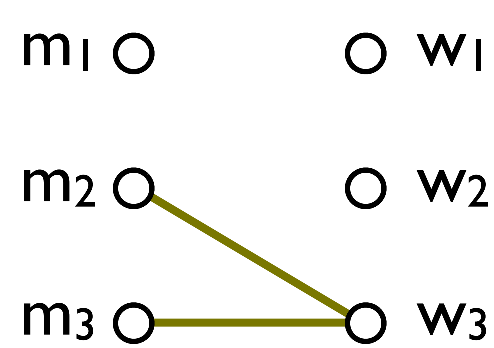
Def.: A matching M is a one-to-one correspondence between a subset of the men and the women

Def.: A matching M is perfect if nobody is unassigned





matching



not a matching

Stable matching

Def.: A matching M is stable if there is no (m,w) such that

- m is free in M or is matched to $w' = M(m)$ and $w >_m w'$
- w is free in M or is matched to $m' = M(w)$ and $m >_w m'$

If it existed, we say that (m, w) blocks M

Def.: Given a set of men and women with their individual rankings, the stable matching (SM) problem is to find a stable matching, if one exists

Back to hospitals and residents

Assuming each hospital has one position, this (almost) captures the student-hospital problem from before

In 1952, a centralized system (NRMP) was established

- Hospitals and students submitted their preferences to NRMP
- NRMP suggested a global matching
- 95% participation in its first year even though participation was voluntary and there was mechanism in place to enforce NRMP's matching

It turns out that NRMP produced stable matchings! However, the algorithm was not published until 1962, when it was re-discovered by Gale and Shapley.

Gale-Shapley algorithm

<u>Teo.</u>: Every SM instance admits at least one stable matching

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Gale-Shapley(P)

M ← empty matching
while there is a free man in M
  m ← some free man
  w ← most desired woman that m has not proposed yet
  if w is free
    add (m,w) to M
  if w is not free, but prefers m to m'=M(w)
    remove (m',w) from M
    add (m,w) to M
return M
```

We must show that such woman always exists

are engaged

We say m' and w break their engagement

Example

Matching:

{ (m_2, w_1), (m_3, w_3), (m_1, w_2) }

Free men:

{ }

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Gale-Shapley(P)

M ← empty matching
while there is a free man in M
  m ← some free man
  w ← most desired woman by m not yet proposed
  if w is free
    add (m,w) to M
  if w is not free, but prefers m to m'=M(w)
    remove (m',w) from M
    add (m,w) to M
return M
```

m_1	~~w_1~~	~~w_3~~	w_2
m_2	w_1	w_2	w_3
m_3	~~w_1~~	w_3	w_2

w_1	m_2	m_3	m_1
w_2	m_2	m_1	m_3
w_3	m_3	m_2	m_1

Analysis of GS algorithm

<u>Prop.</u>: Once a woman becomes engaged, she is never free again, and the quality of her partner can only improve with time

$w \,|\, \cdots\ m\ \cdots\ m'\ \cdots$

```
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```

<u>Prop.</u>: GS algo always terminates and returns a perfect matching

Analysis of GS algorithm

<u>Prop.</u>: The GS algorithm returns a stable matching

Suppose GS returns a perfect matching M and that (m, w) blocks M :

- m prefers w to w' = M(m)
- w prefers m to w' = M(w)

m | ··· ~~w~~ ··· w' ···

w | ··· m ··· m' ···

Contradiction!

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Gale-Shapley(P)

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```

Analysis of GS algorithm

Prop.: The GS algorithm runs for at most n^2 iterations

Each time a man proposes to a woman we can think of him crossing out that entry from his preference list

Each man has n women in his list and there are n men, so there are n^2 entries in total. Thus at most n^2 iterations

Properties of GS

Teo.: The GS algorithm produces a stable matching where each man gets his best "stable partner" and each woman gets her worst "stable partner"

Let M be a stable matching and m a man such that GS gives m a worse woman than w = M(m). Then w must reject m for another man m'. Let this be the first such rejection.

$$w \,|\, \cdots\ m'\ \cdots\ m\ \cdots \quad \text{where } m = M(w)$$

$$m' \,|\, \cdots\ w\ \cdots\ w'\ \cdots \quad \text{where } w' = M(m')$$

$\Rightarrow$ (m', w) blocks M. Contradiction!

Beyond the basic setup

Preference are typically not complete

Hospital usually have several positions

Preferences are usually not strict

Applicants are not necessarily independent

Can people cheat by misrepresenting their true preferences?

Shapley won the 2011 Nobel prize in Economics for his work on Stable Matchings