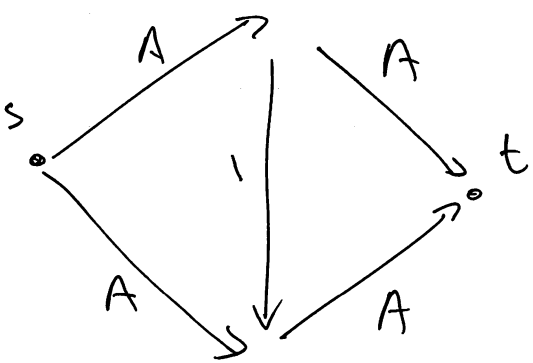
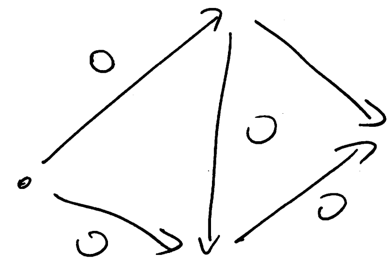
Ford Fulkerson Algorithm

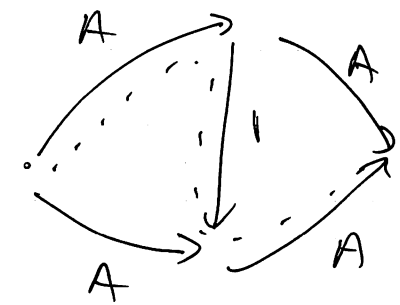
In each iteration finds an s-t path in G^f

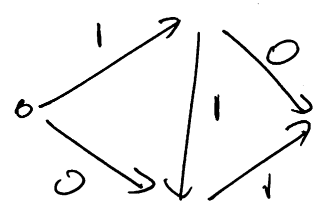
and pushes flow along the path

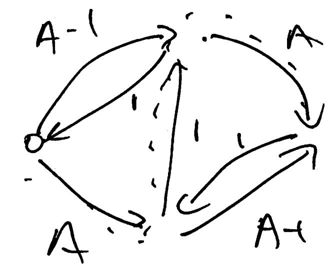
iterations $\leq C = c^{out}(s)$

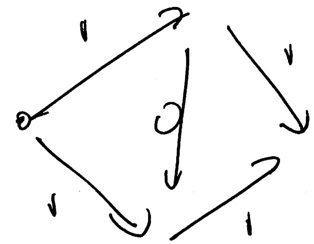












repeat another

$2(A-1)$ times

Better rules for selecting aug path p in G^f

(i) maximize $\Delta = \min_{e \in p} \{ \text{residual cap. of } e \}$

(ii) minimize $|p| = \#$ edges in p

What do we know?

(i) $\#$ iterations $= O(\log C \cdot m)$

(ii) $\#$ iterations $= O(nm)$

weakly polynomial running time

strongly polynomial running time

\# iterations for rule (i) $= O(\log C \; m)$

Let $\Delta_1, \Delta_2, \Delta_3 \ldots\ldots$ be amount pushed in each iteration

and let k be such that $2^{k-1} < \Delta_1 \leq 2^k$

<u>Def</u>: Let i_j be the first index such that $\Delta_{i_j} \leq 2^{k-j}$

Phase j consists of iterations $[\Delta_{i_j}, \Delta_{i_{j+1}})$

<u>Obs</u>: Iteration phase j pushes $\geq 2^{k-j-1}$ flow

<u>Obs</u>: \# phases $= O(\log C)$

$$\#\text{phases} \leq \log_2 \Delta_1 + 1 \leq \log_2 \max_{e \in \delta(s)} c(e) + 1 \leq \log_2 C + 1$$

<u>Obs</u>: If we can show that each phase runs for $O(m)$ iteration we are done!

Claim: Phase j runs for $O(m)$ iterations

Let f be flow at beg. of phase i

we will find s-t cut (A, B) such that $c(A,B) \leq v(f) + m\,2^{k-j}$

Let $A = \left\{ u \in V : \text{can reach } u \text{ from } s \text{ using edges with res. cap} > 2^{k-j} \right\}$

Note that $t \notin A$, otherwise we wouldn't have started phase j

$$c(A,B) = \sum_{\substack{(u,v)\in E \\ u\in A,\, v\in B}} c(u,v) = \sum_{\substack{(u,v)\in E \\ u\in A,\, v\in B}} f(u,v) + \sum_{\substack{(u,v) \text{ forward in } G^f \\ u\in A,\, v\in B}} \text{res-cap}(u,v) - f^{in}(A) + f^{in}(A)$$

$$= f^{out}(A) + f^{in}(A) + \sum_{\substack{(u,v) \text{ forward in } G^f \\ u\in A,\, v\in B}} \text{res-cap}(u,v) + \sum_{\substack{(u,v) \text{ backward in } G^f \\ v\in B,\, u\in A}} \text{res-cap}(u,v)$$

$$\leq v(f) + m\,2^{k-j}$$

Each iteration in phase j pushes $\geq 2^{k-j-1}$ units of flow

Let f' be flow at end of phase j then

$$v(f') \geq v(f) + 2^{k-j-1} \times (\#\text{iteration in phase } j)$$

but $v(f') \leq c(A,B)$ so

$$v(f) + 2^{k-j-1} \times \left(\begin{array}{c}\#\text{iterations} \\ \text{in phase } j\end{array}\right) \leq v(f') \leq c(A,B) \leq v(f) + m\,2^{k-j}$$

$$\Rightarrow \quad \begin{array}{c}\#\text{iterations} \\ \text{in phase } j\end{array} \leq 2m$$