

Lecture 4:

Divide & Conquer



General techniques in this course

- Greedy algorithms [Lecture 3]
- Divide & Conquer algorithms [today]
- Sweepline algorithms [28 Aug]
- Dynamic programming algorithms [4 and 11 Sep]
- Network flow algorithms [18 Sep and 9 Oct]

Divide-and-Conquer

- **Divide-and-conquer.**
 - Break up problem into several parts.
 - Solve each part recursively.
 - Combine solutions to sub-problems into overall solution.
- **Most common usage.**
 - Break up problem of size n into **two** equal parts of size $\frac{1}{2}n$.
 - Solve two parts recursively.
 - Combine two solutions into overall solution in **linear time**.

Searching

Input: A sorted sequence S of n numbers a_1, a_2, \dots, a_n , stored in an array $A[1..n]$.

Question: Given a number x , is x in S ?

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	---	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	---	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	---	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	---	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	---	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	--------------	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	---	---	---	--------------	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
---	---	--------------	---	---	--------------	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
--------------	---	--------------	---	---	--------------	----	----	----	----	----	----

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
--------------	---	--------------	---	---	--------------	----	----	----	----	----	----

Analysis: $T(n) = 1 + T(n/2)$

Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

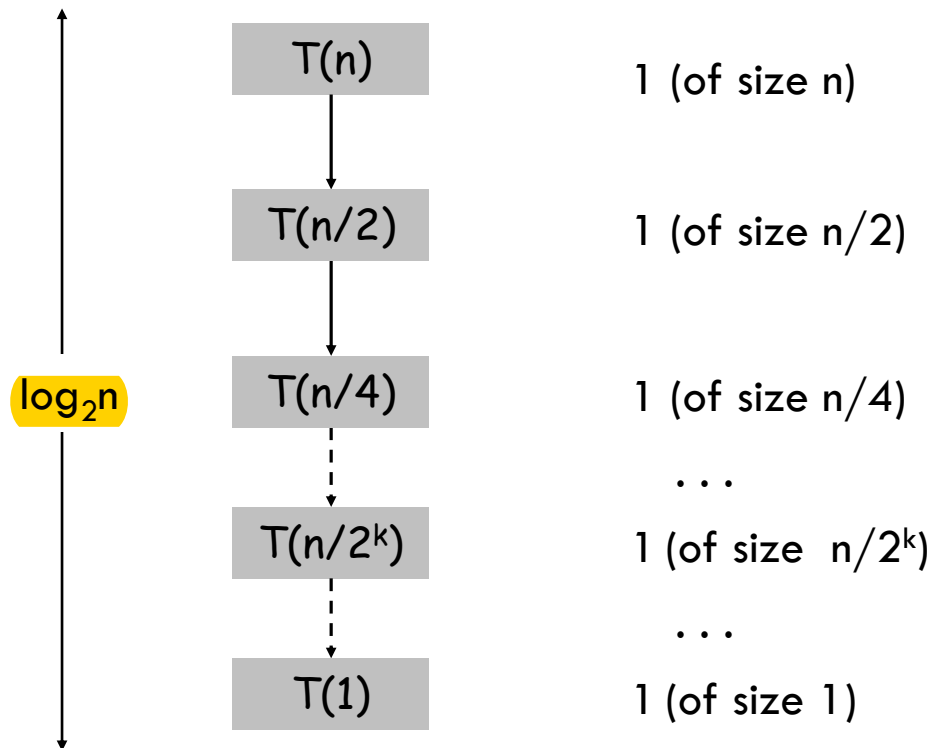
Example: $x=1$ (non-integers are rounded up)

0	1	3	4	5	7	10	13	15	18	19	23
--------------	---	--------------	---	---	--------------	----	----	----	----	----	----

Analysis: $T(n) = 1 + T(n/2)$

Analyze recursion

$$T(n) = T(n/2) + O(1)$$



Binary Search

- Compare x to the middle element of the array ($A[n/2]$).
- If $A[n/2] = x$ then “Yes”
- Otherwise, if $A[n/2] < x$ then recursively Search $A[1 \dots n/2-1]$.
- Otherwise, if $A[n/2] > x$ then recursively Search $A[n/2+1 \dots n]$

Example: $x=1$ (non-integers are rounded up)



Analysis: $T(n) = 1 + T(n/2) = O(\log n)$

Sorting

- **Sorting.** Given n elements, rearrange in ascending order.

Obvious sorting applications.

- List files in a directory.

- Organize an MP3 library.

- List names in a phone book.

- Display Google PageRank results.

Problems become easier once sorted.

- Find the median.

- Find the closest pair.

- Binary search in a database.

- Identify statistical outliers.

- Find duplicates in a mailing list.

Non-obvious sorting applications.

- Data compression.

- Computer graphics.

- Interval scheduling.

- Computational biology.

- Minimum spanning tree.

- Supply chain management.

- Simulate a system of particles.

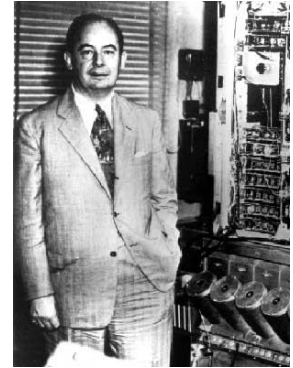
- Book recommendations on Amazon.

- Load balancing.

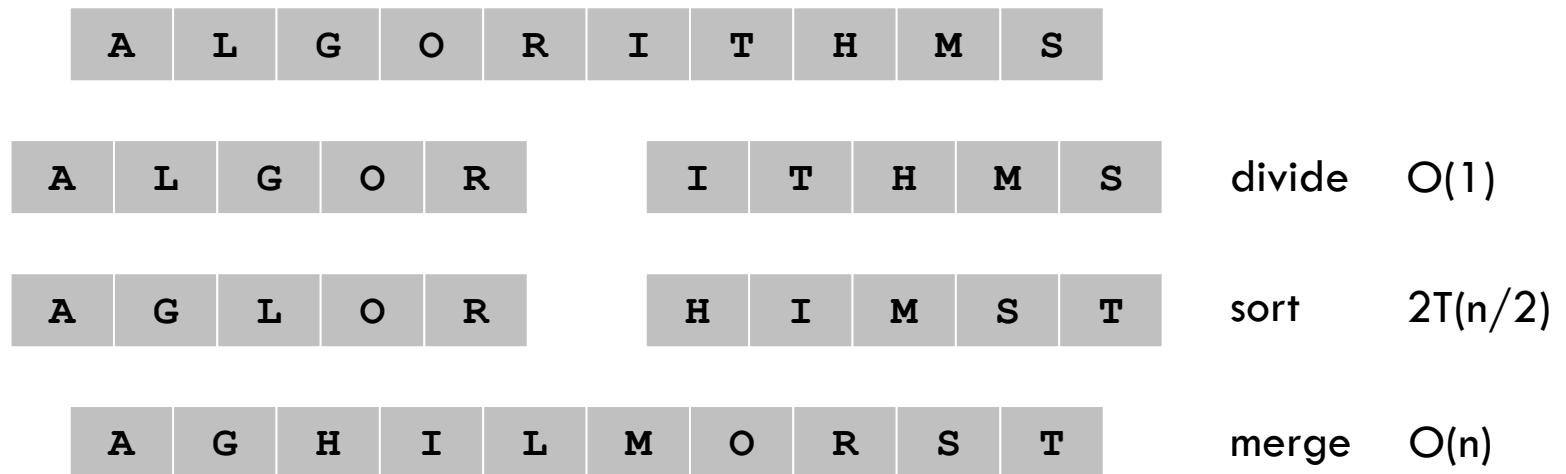
- ...

Mergesort

- Mergesort.
 - Divide array into two halves.
 - Recursively sort each half.
 - Merge two halves to make sorted whole.



Jon von Neumann (1945)



Merging

- **Merging.** Combine two pre-sorted lists into a sorted whole.
- How to merge efficiently?
 - **Linear number of comparisons.**
 - Use temporary array.



Merging

- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.

smallest



A	G	L	O	R
---	---	---	---	---

smallest



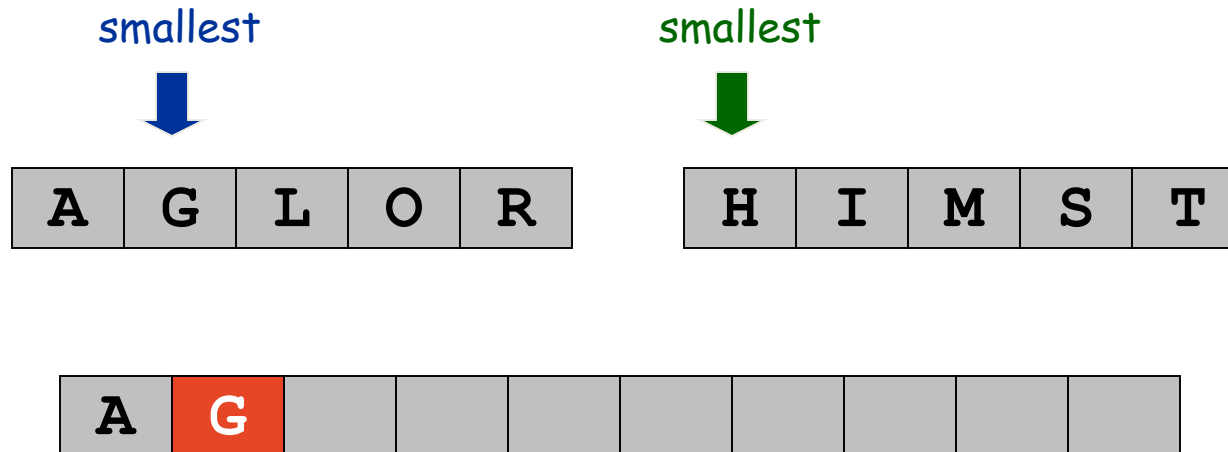
H	I	M	S	T
---	---	---	---	---

A									
---	--	--	--	--	--	--	--	--	--

auxiliary array

Merging

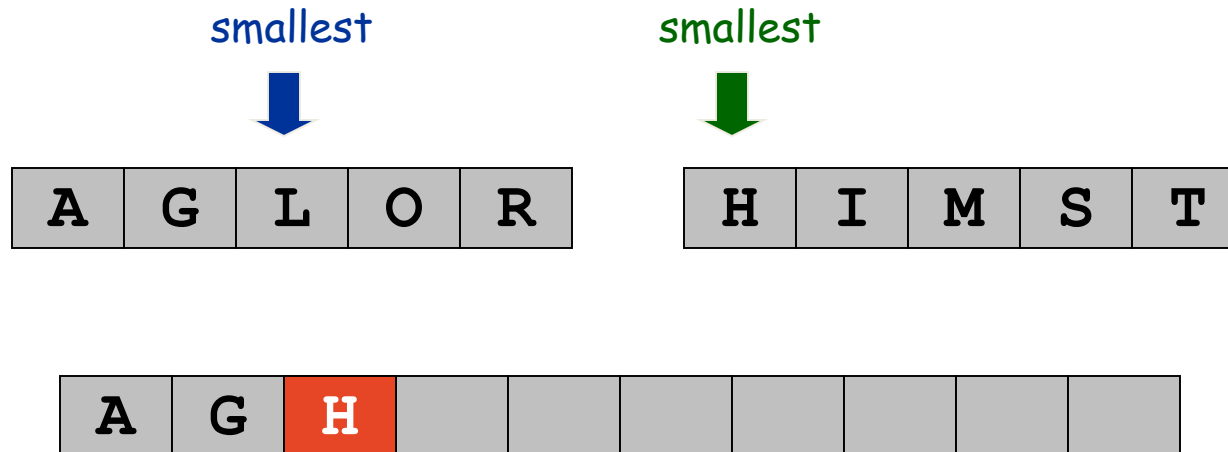
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

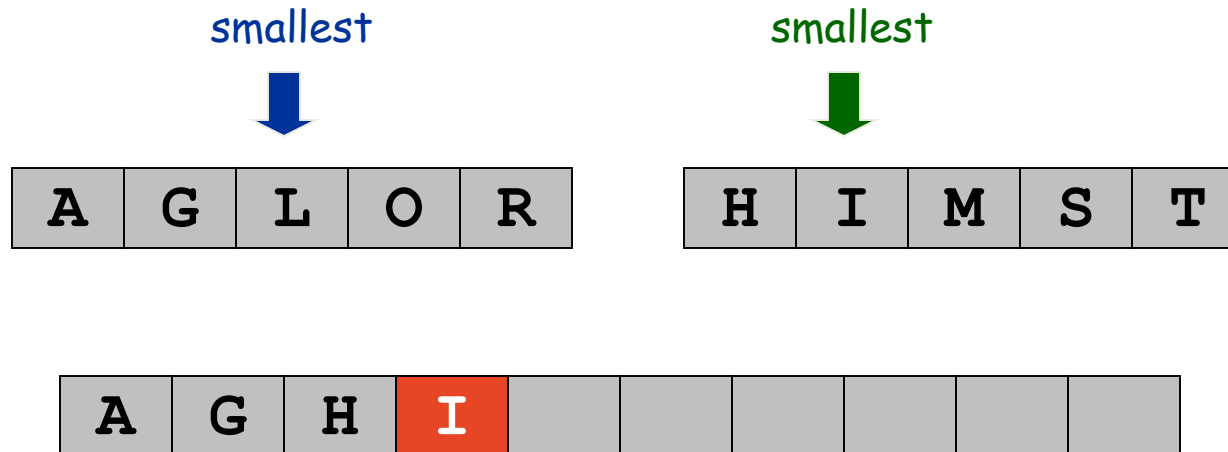
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

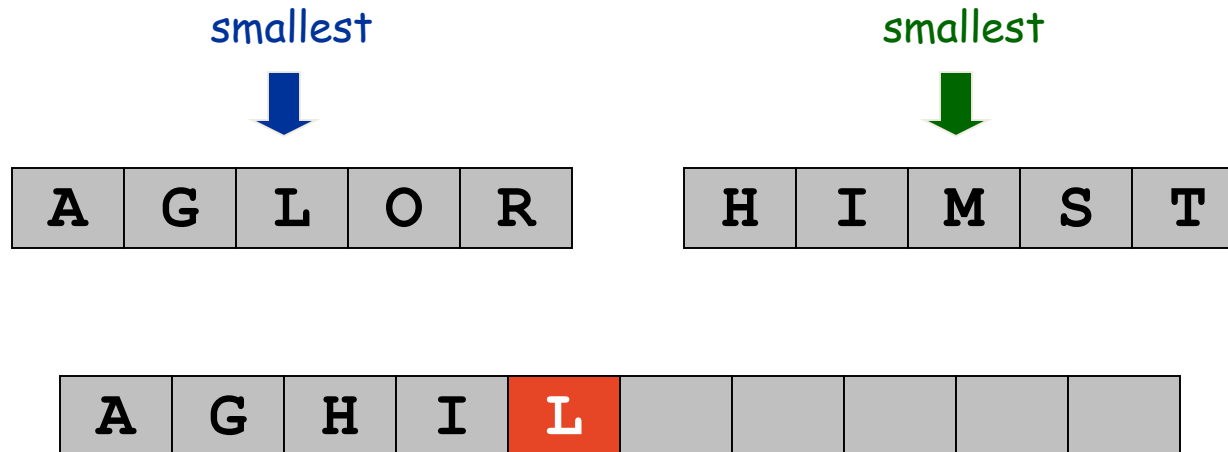
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

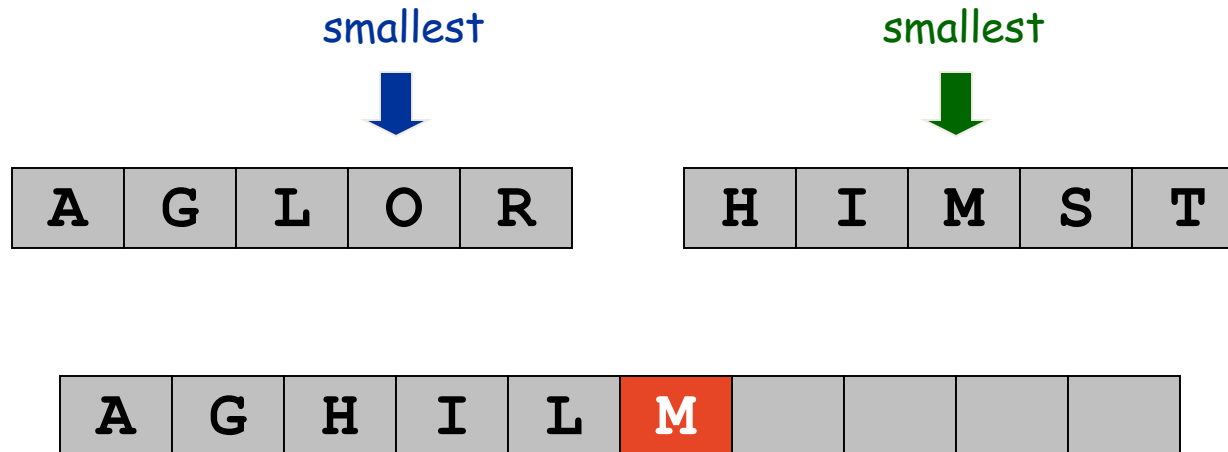
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

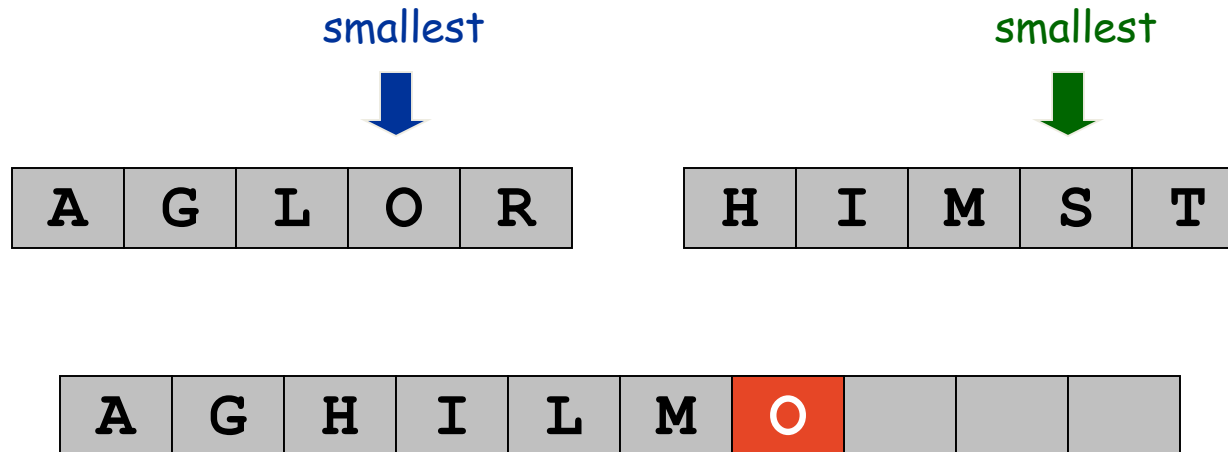
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

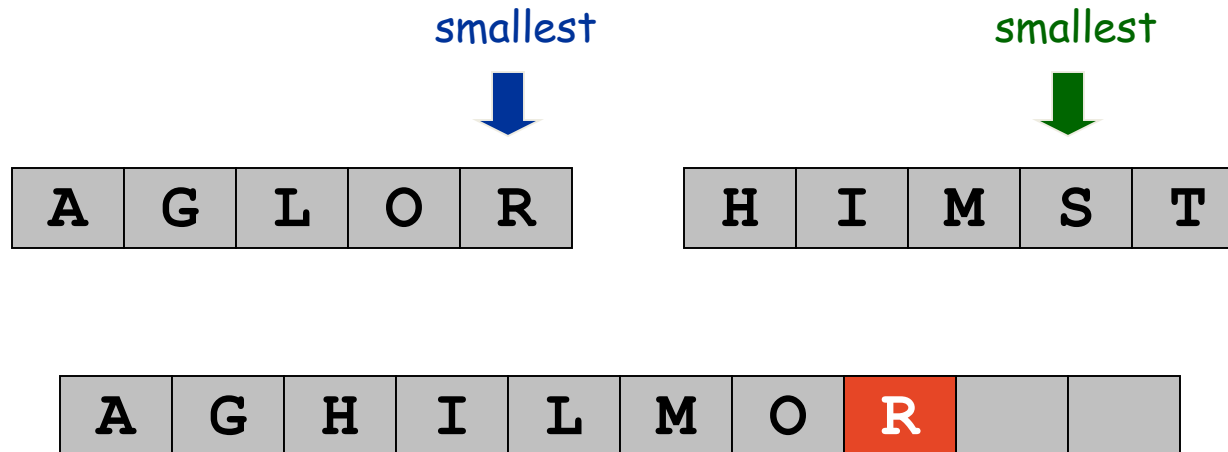
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

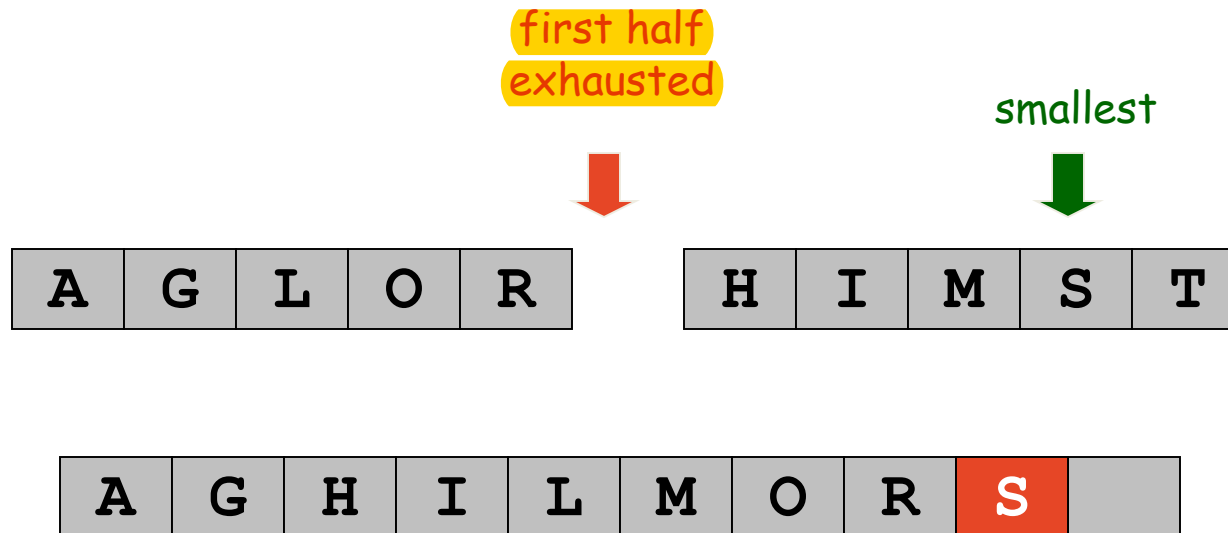
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

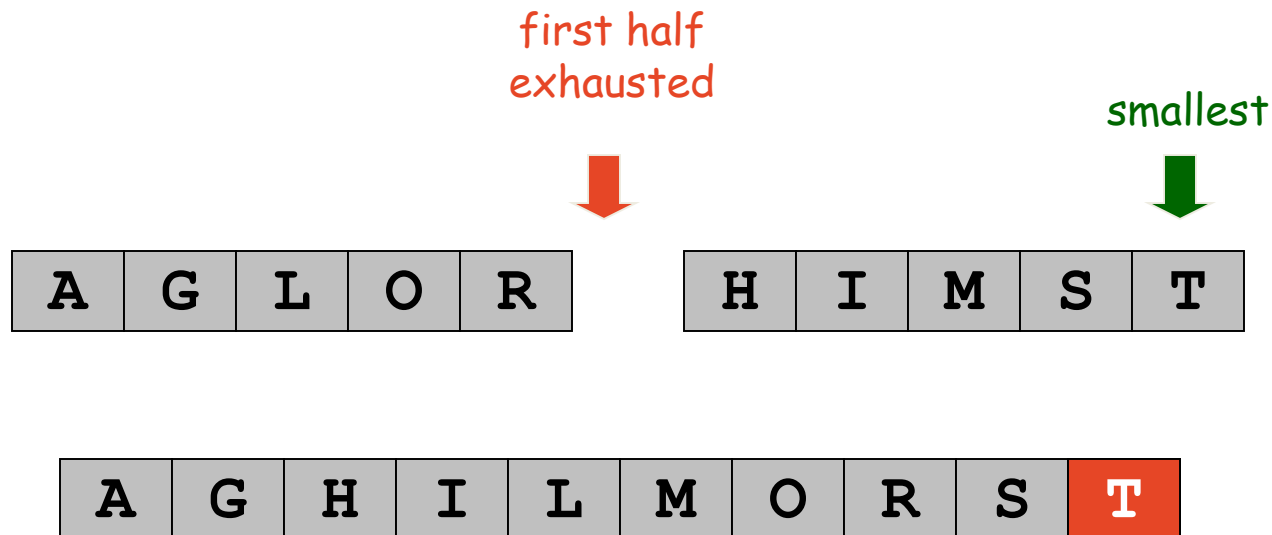
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

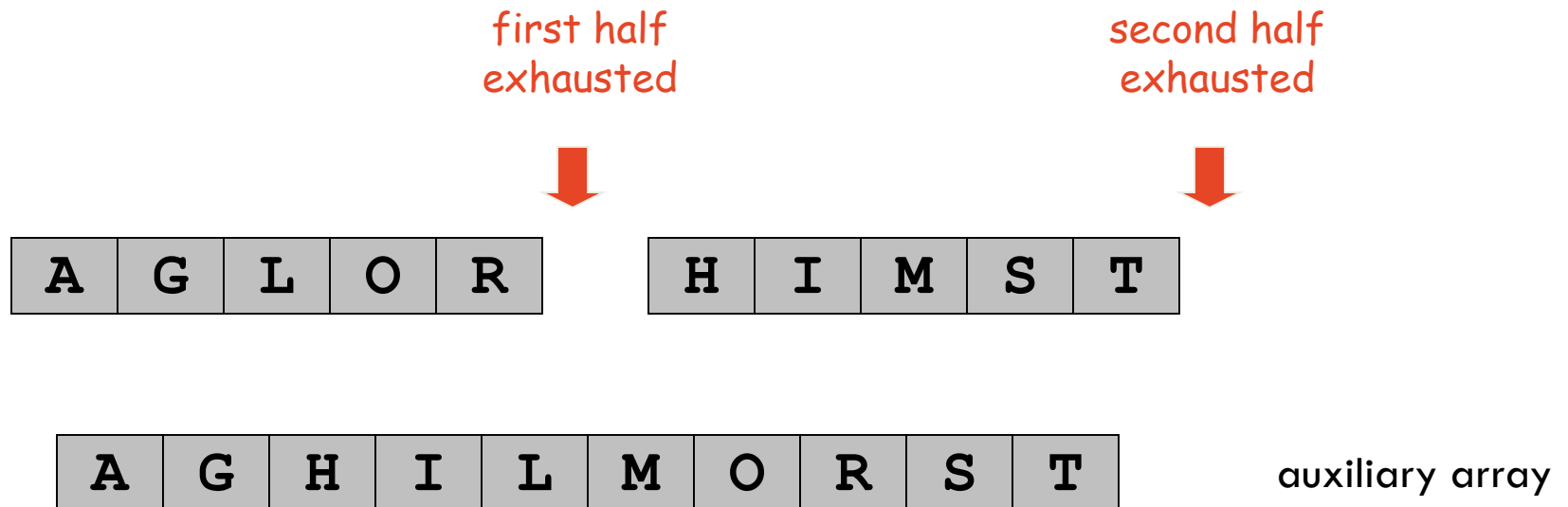
- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



auxiliary array

Merging

- Merge.
 - Keep track of smallest element in each sorted half.
 - Insert smallest of two elements into auxiliary array.
 - Repeat until done.



A Useful Recurrence Relation

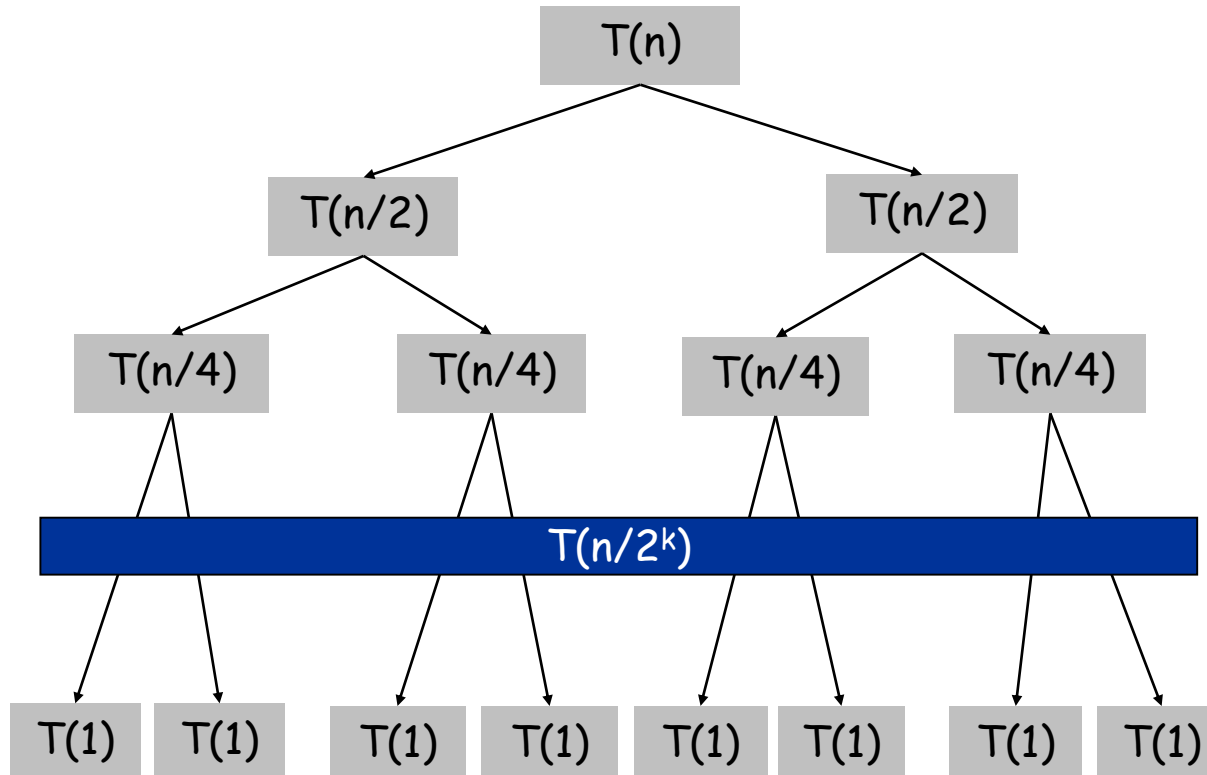
- **Definition:** $T(n)$ = number of comparisons to mergesort an input of size n .
- **Mergesort recurrence.**

$$T(n) \leq \begin{cases} 0 & \text{if } n=1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

- **Solution:** $T(n) = ?$

Proof by unrolling

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$



1 (of size n)

2 (of size $n/2$)

4 (of size $n/4$)

...

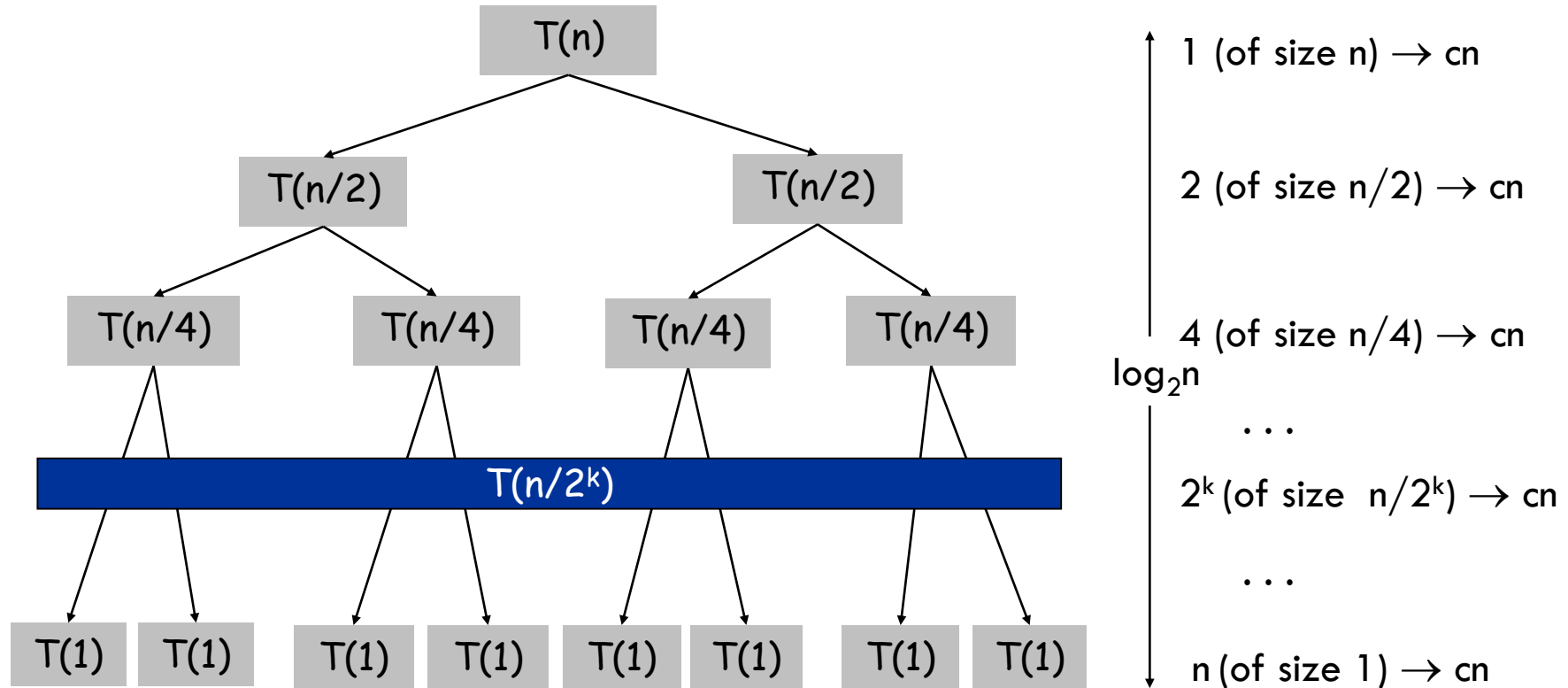
2^k (of size $n/2^k$)

...

n (of size 1)

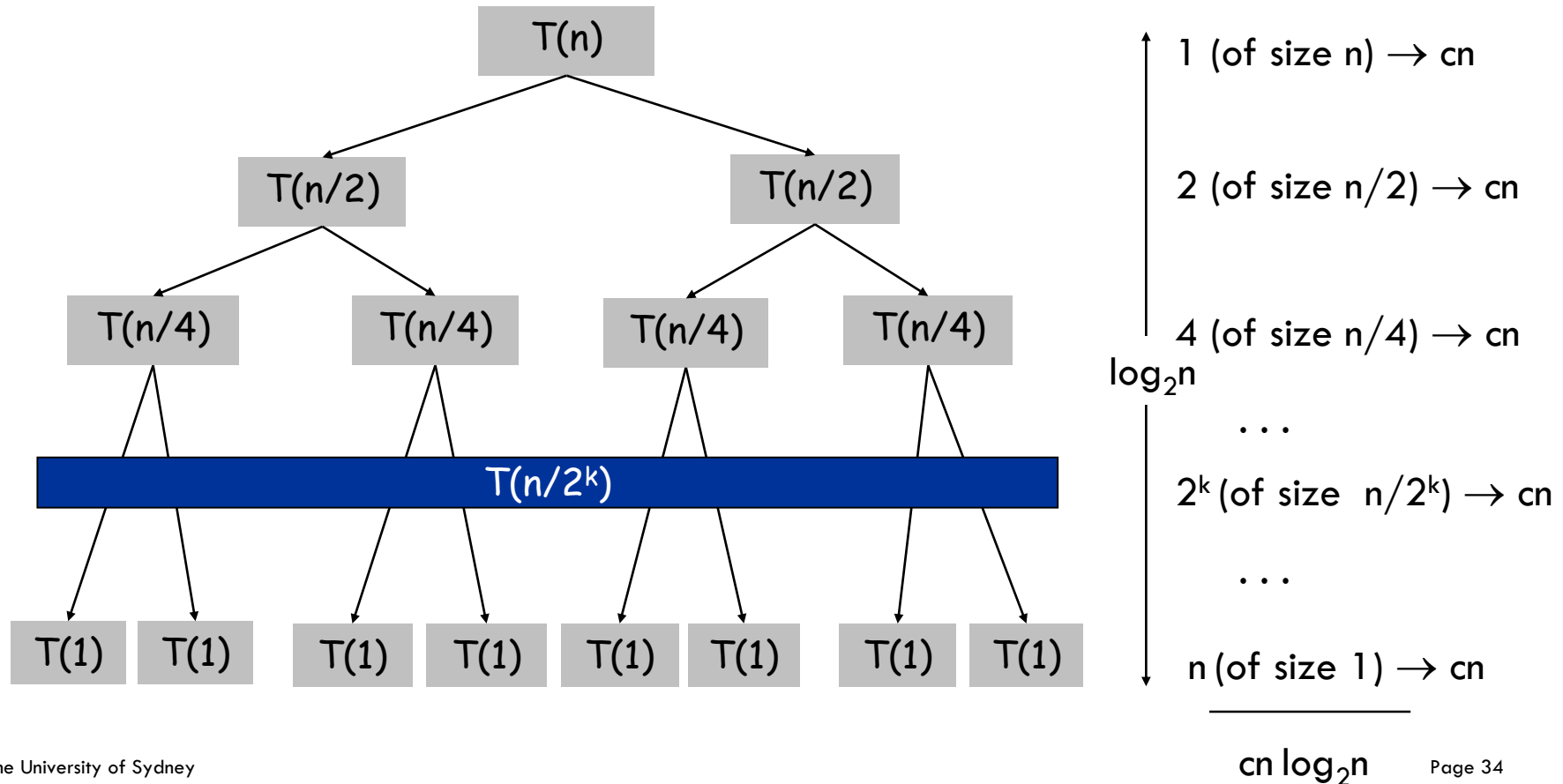
Proof by unrolling

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by unrolling

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$



A Useful Recurrence Relation

- **Definition:** $T(n)$ = number of comparisons to mergesort an input of size n .
- **Mergesort recurrence.**

$$T(n) \leq \begin{cases} 0 & \text{if } n=1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

- **Solution:** $T(n) = O(n \log_2 n)$.

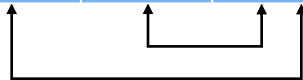
Counting Inversions

Counting Inversions

- Music site tries to match your song preferences with others.
 - You rank n songs.
 - Music site consults database to find people with **similar** tastes.
- Similarity metric: number of inversions between two rankings.
 - My rank: $1, 2, \dots, n$.
 - Your rank: a_1, a_2, \dots, a_n .
 - Songs i and k **inverted** if $i < k$, but $a_i > a_k$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5



Inversions

3-2, 4-2

- Brute force: check all $\Theta(n^2)$ pairs i and k .

Applications

- Applications.
 - Voting theory.
 - Collaborative filtering.
 - Measuring the "sortedness" of an array.
 - Sensitivity analysis of Google's ranking function.
 - Rank aggregation for meta-searching on the Web.
 - Nonparametric statistics (e.g., Kendall's Tau distance).

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
 - **Divide**: separate list into two pieces.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Divide: $O(1)$.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
 - Divide: separate list into two pieces.
 - **Conquer**: recursively count inversions in each half.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Divide: $O(1)$.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Conquer: $2T(n/2)$

5 blue-blue inversions

8 green-green inversions

5-4, 5-2, 4-2, 8-2, 10-2

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

– Divide-and-conquer.

- **Divide:** separate list into two pieces.
- **Conquer:** recursively count inversions in each half.
- **Combine:** count inversions where a_i and a_j are in different halves, and return sum of three quantities.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Divide: $O(1)$.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

5 blue-blue inversions

8 green-green inversions

Conquer: $2T(n/2)$

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = $5 + 8 + 9 = 22$.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is **sorted**.
- Count inversions where a_i and a_j are in different halves.
- **Merge** two sorted halves into sorted whole.

3	7	10	14	18	19
---	---	----	----	----	----

5 blue-blue inversions

2	11	16	17	23	25
---	----	----	----	----	----

6 3 2 2 0 0

8 green-green inversions

How many blue-green inversions?

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

$i = 6$



3	7	10	14	18	19
---	---	----	----	----	----



2	11	16	17	23	25
---	----	----	----	----	----

two sorted halves

--	--	--	--	--	--	--	--	--	--	--	--

auxiliary array

Total:

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

$i = 6$



two sorted halves

6



auxiliary array

Total: 6

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

$i = 6$



two sorted halves

6



auxiliary array

Total: 6

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

$i = 6$



two sorted halves

6

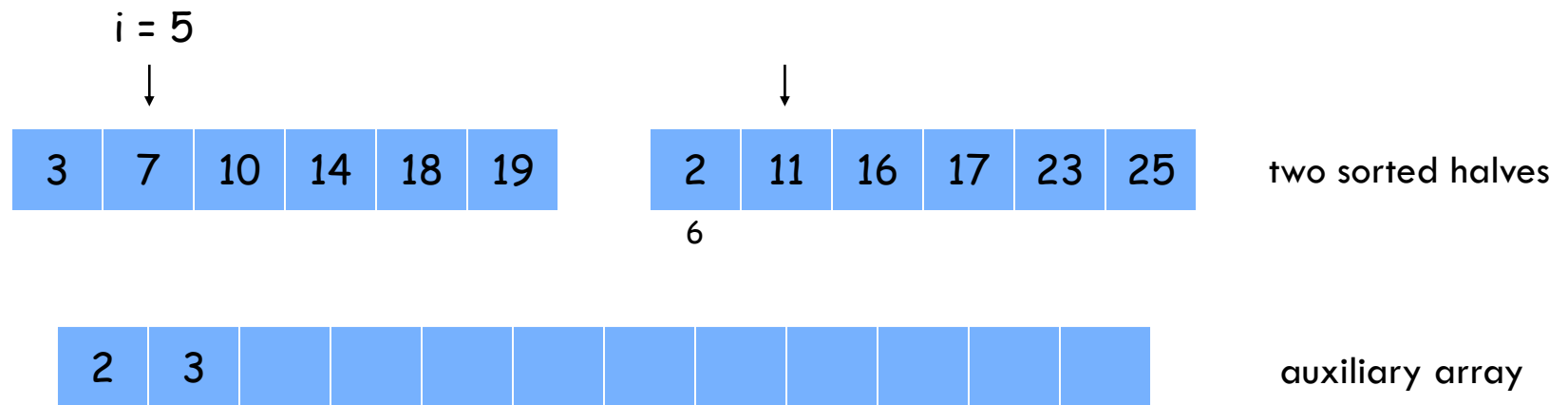


auxiliary array

Total: 6

Merge and Count

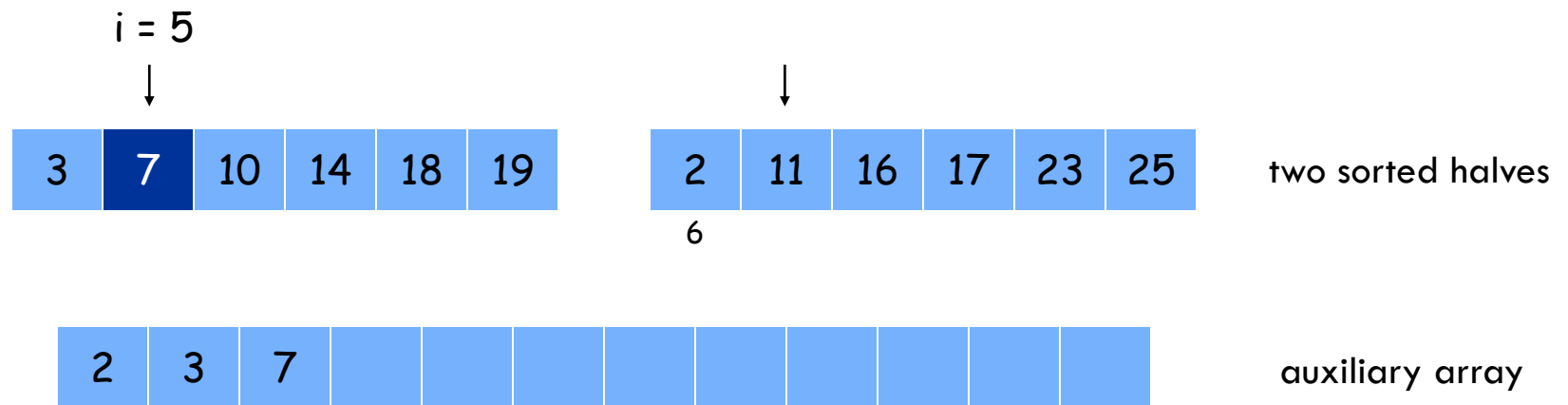
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

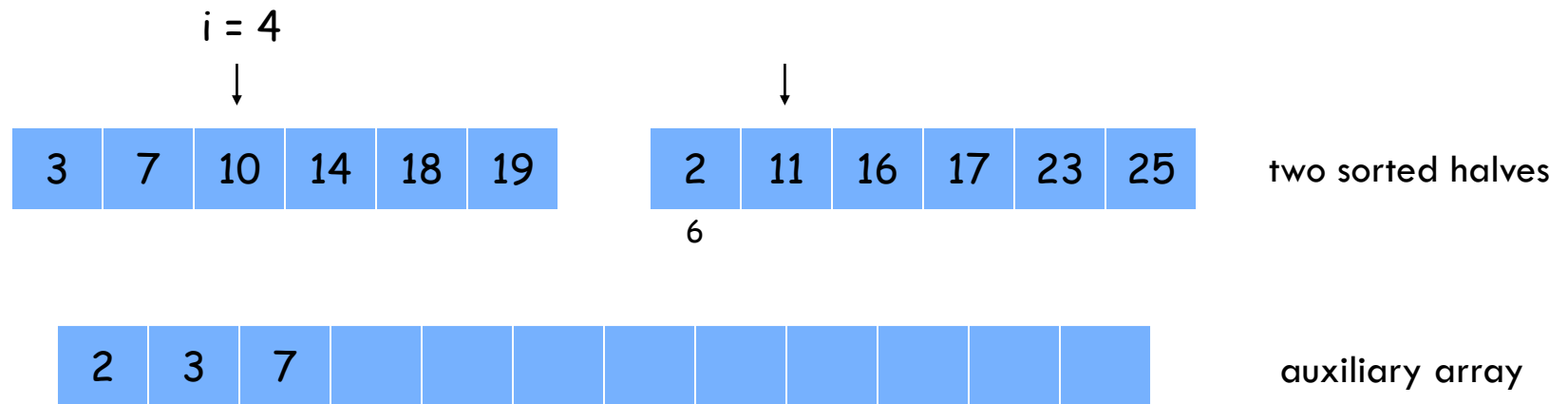
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

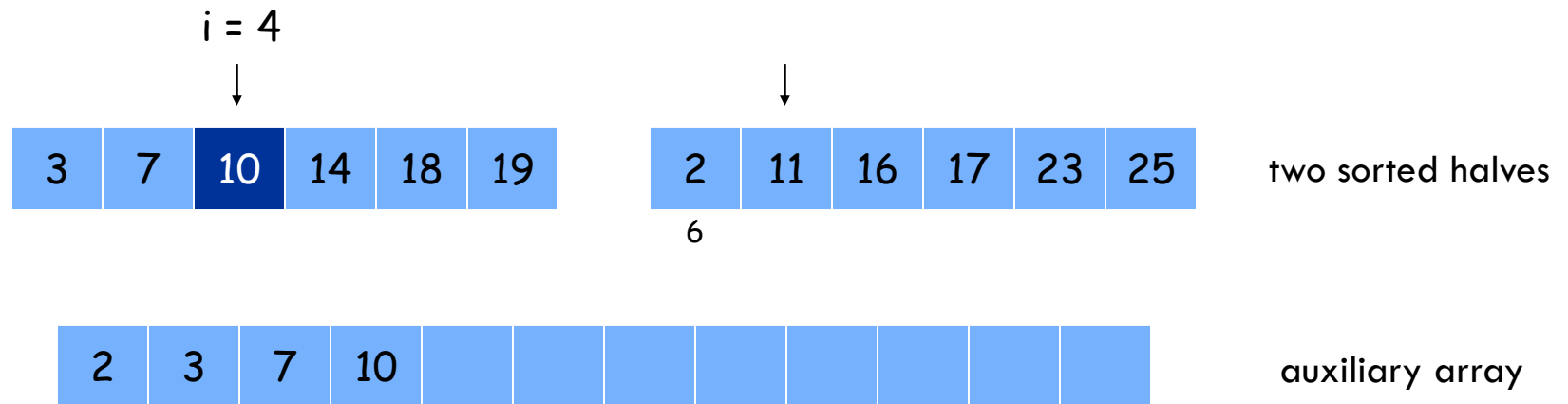
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

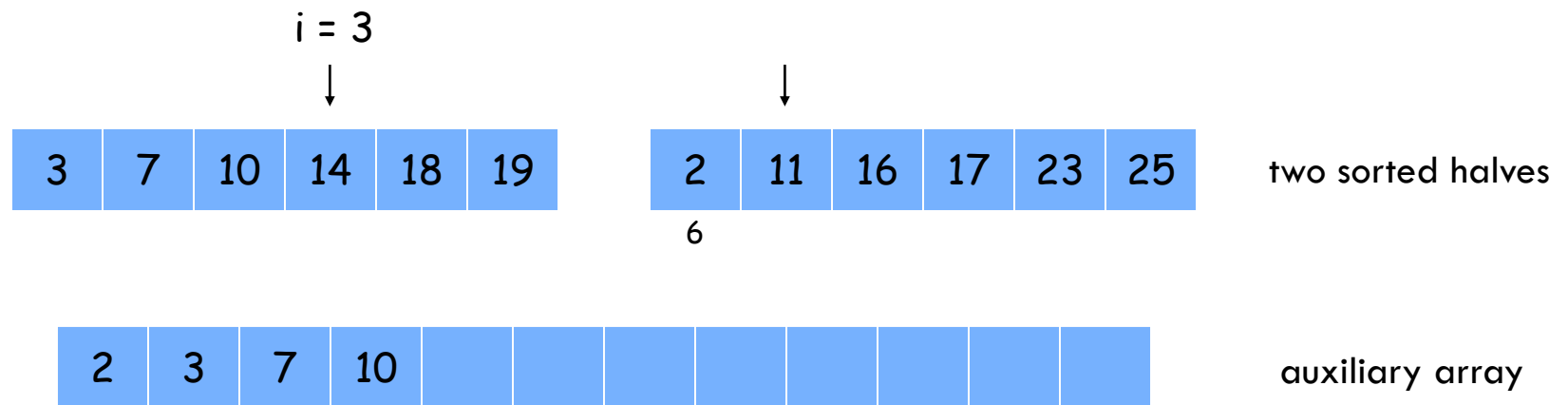
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

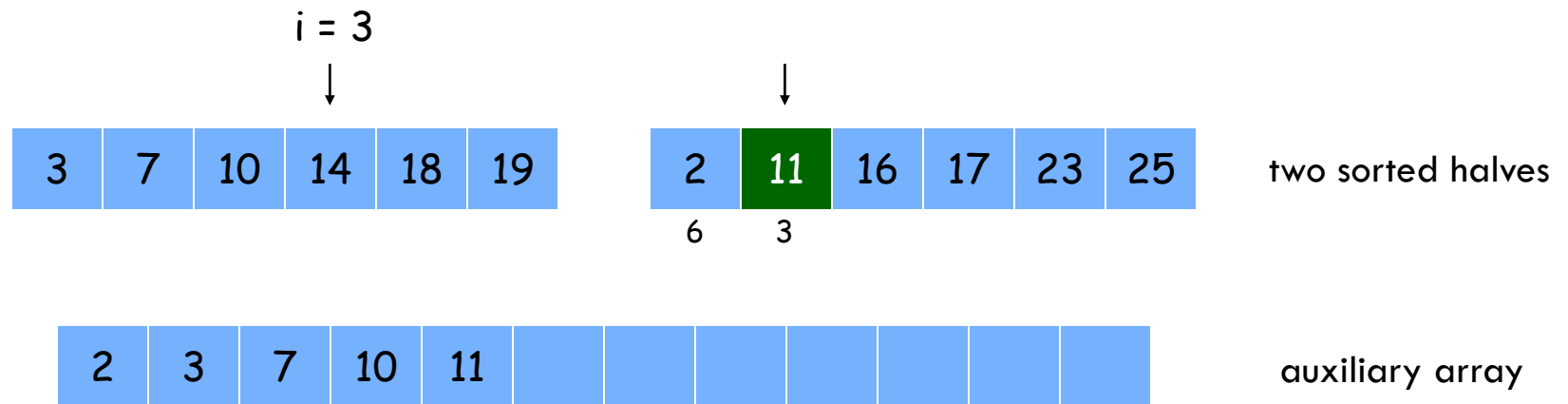
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

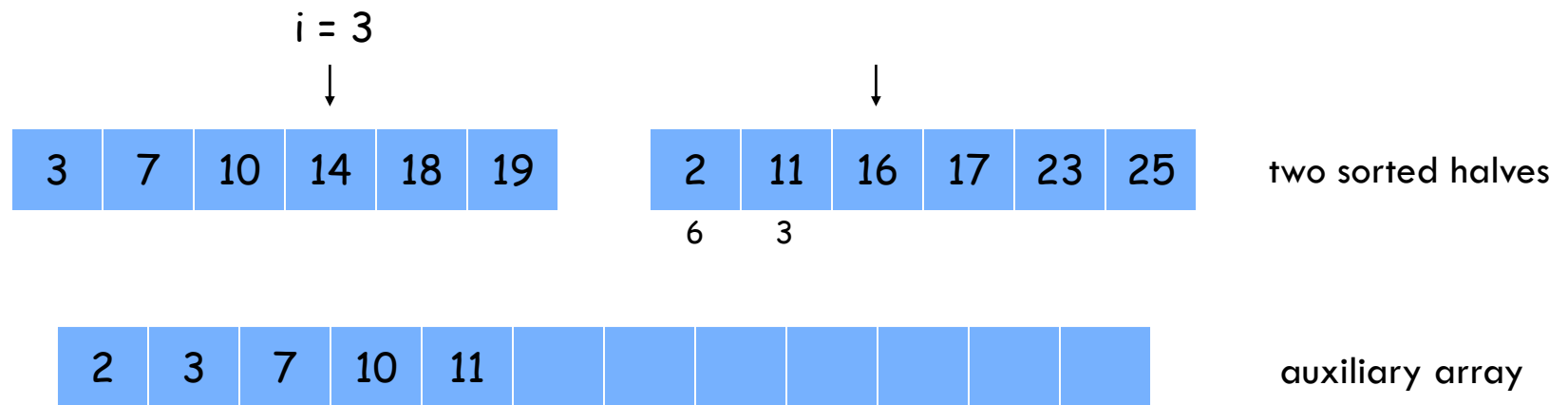
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6 + 3

Merge and Count

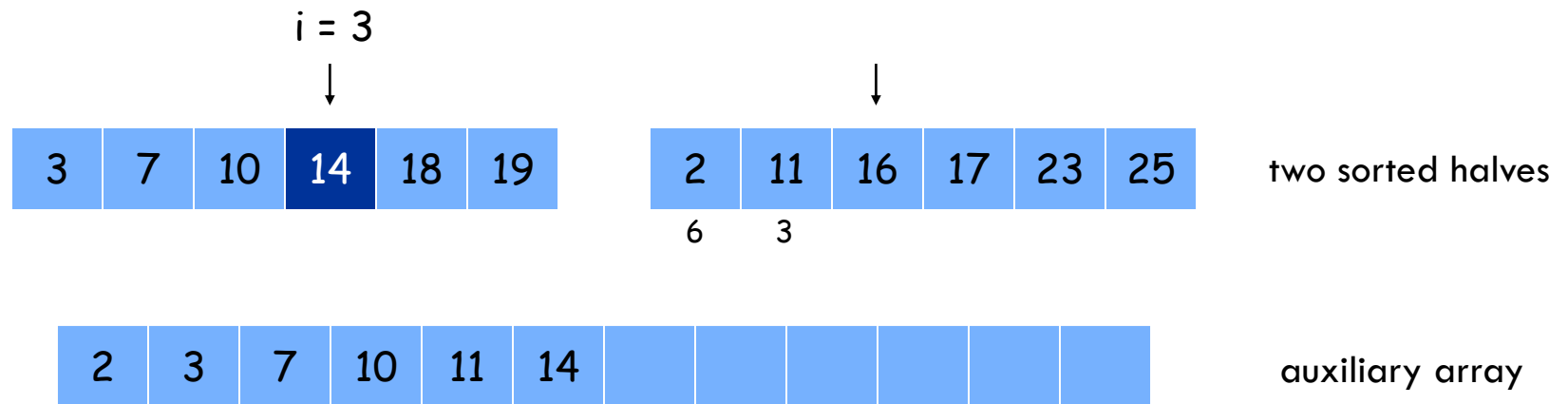
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6 + 3

Merge and Count

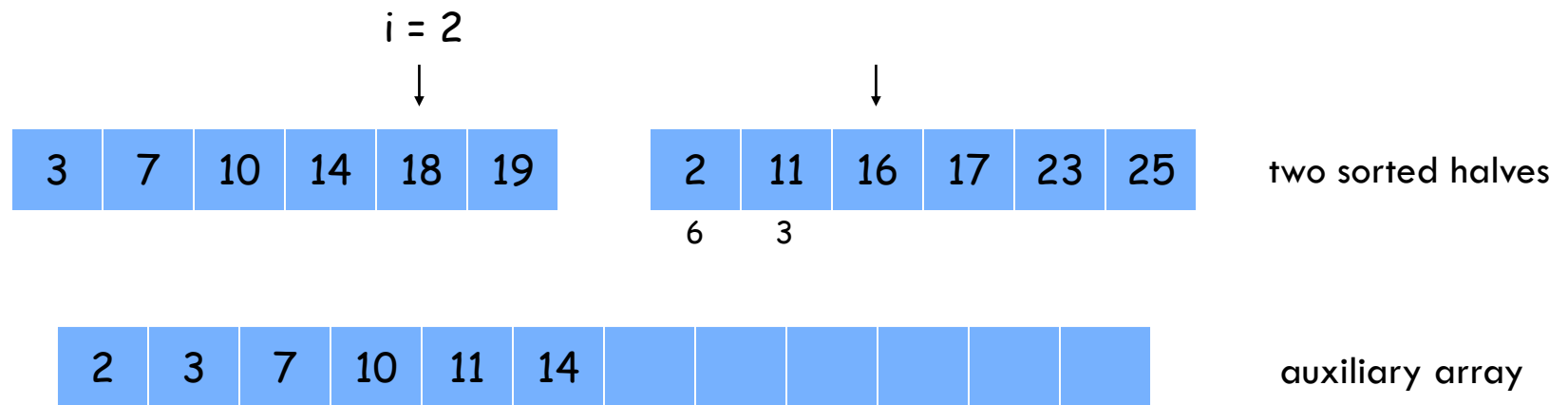
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6 + 3

Merge and Count

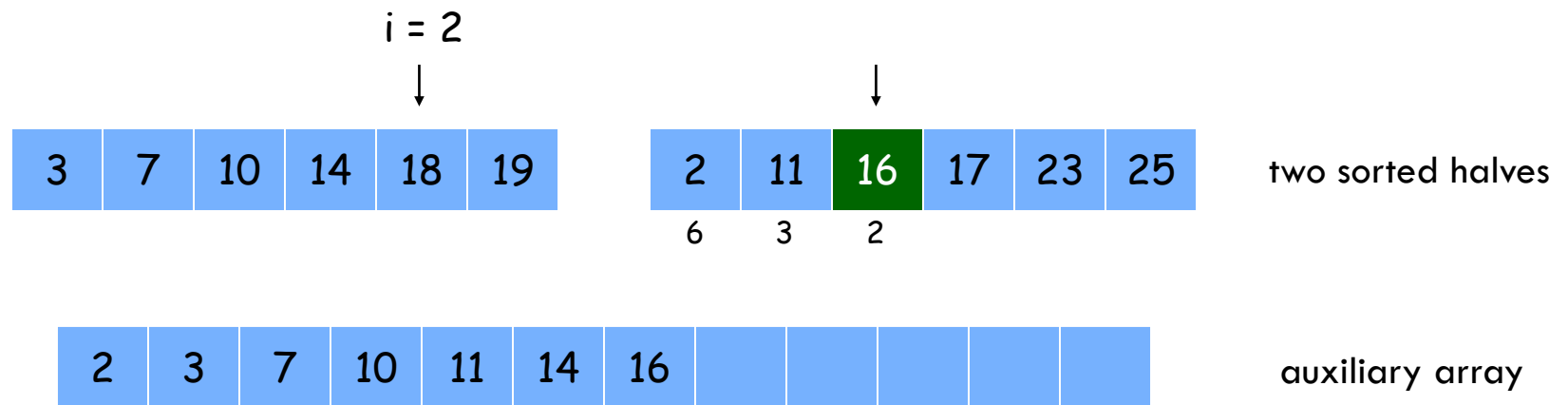
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: 6 + 3

Merge and Count

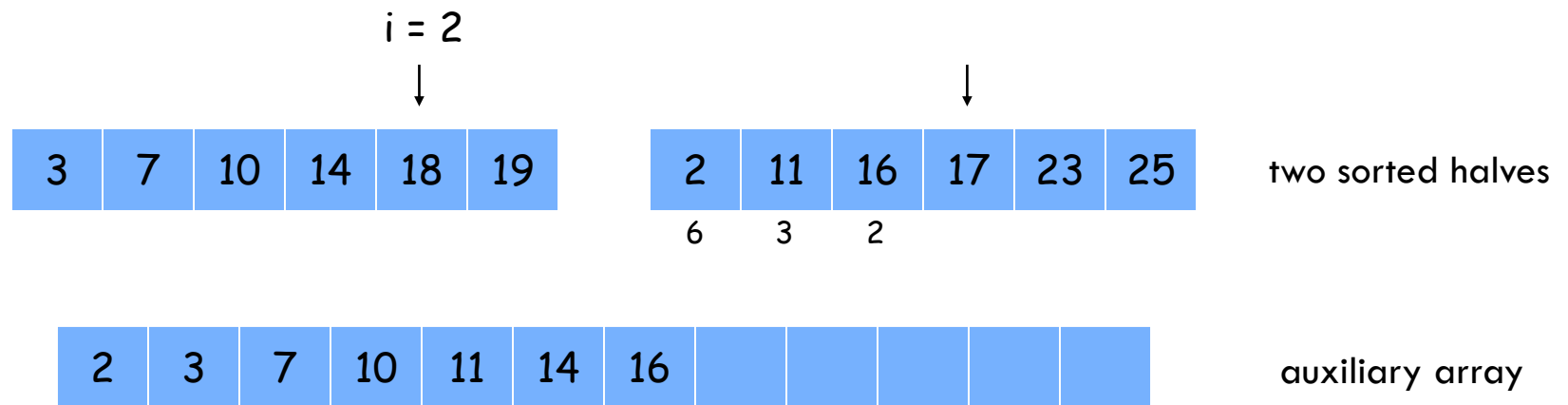
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2$

Merge and Count

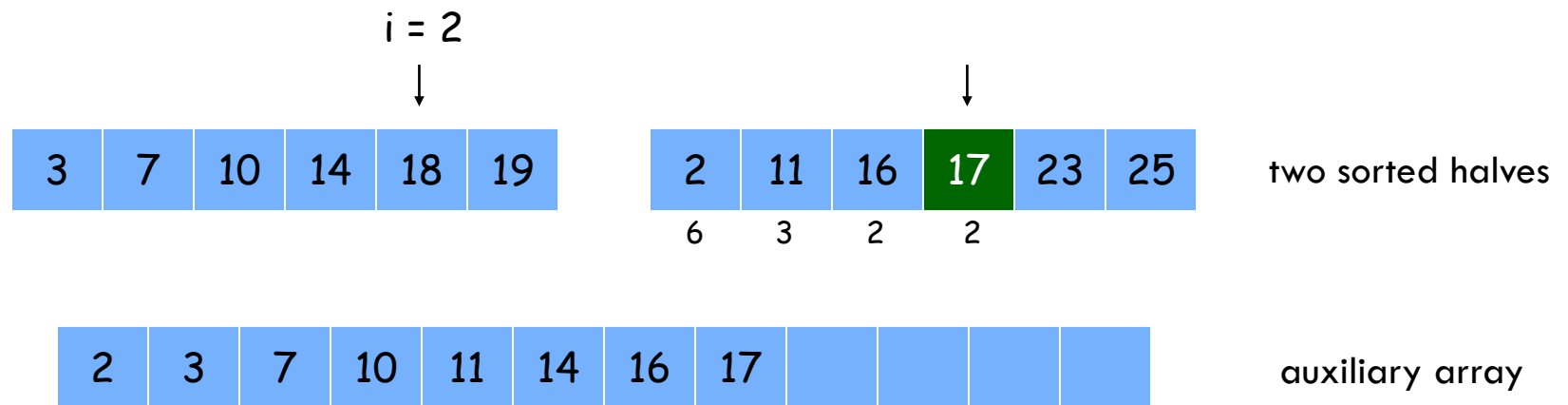
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2$

Merge and Count

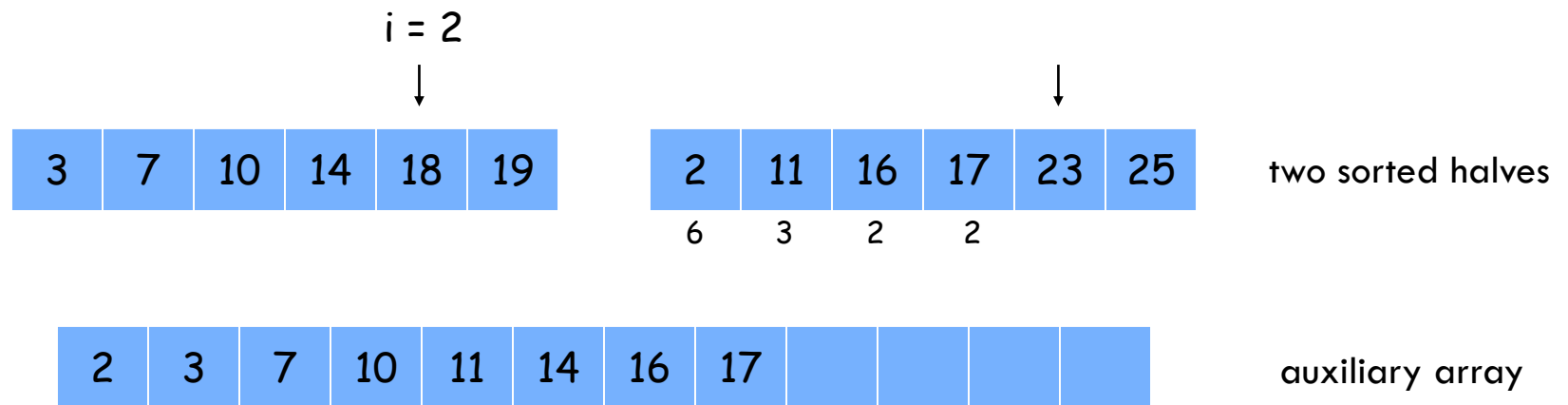
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

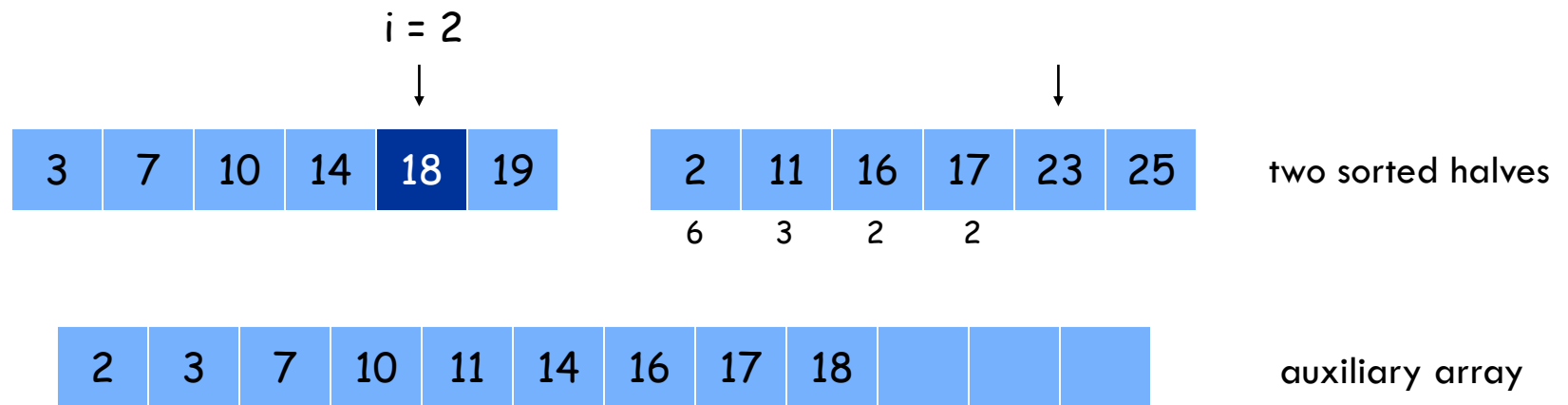
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

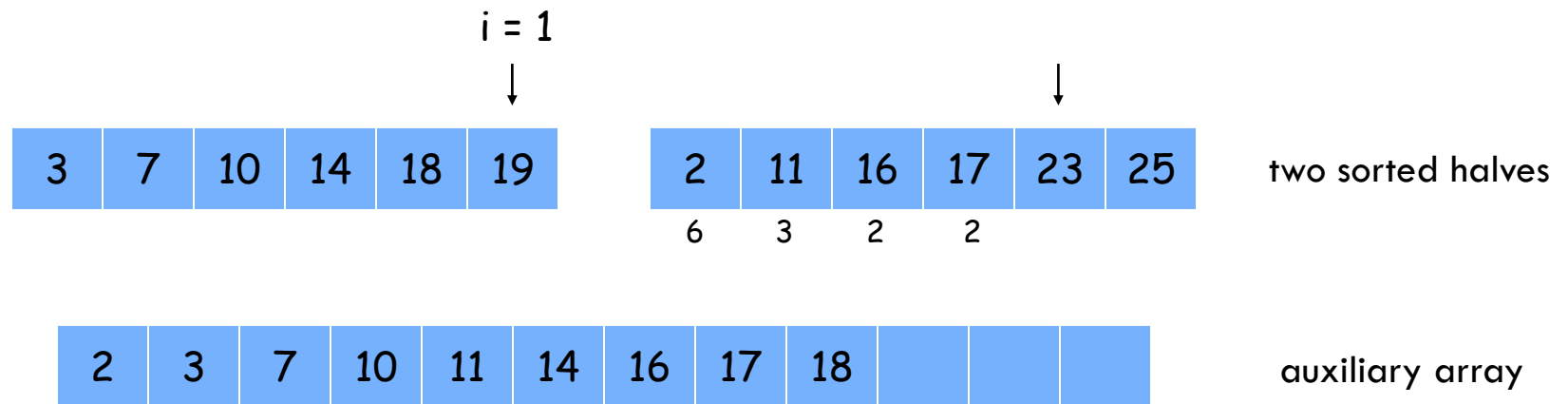
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

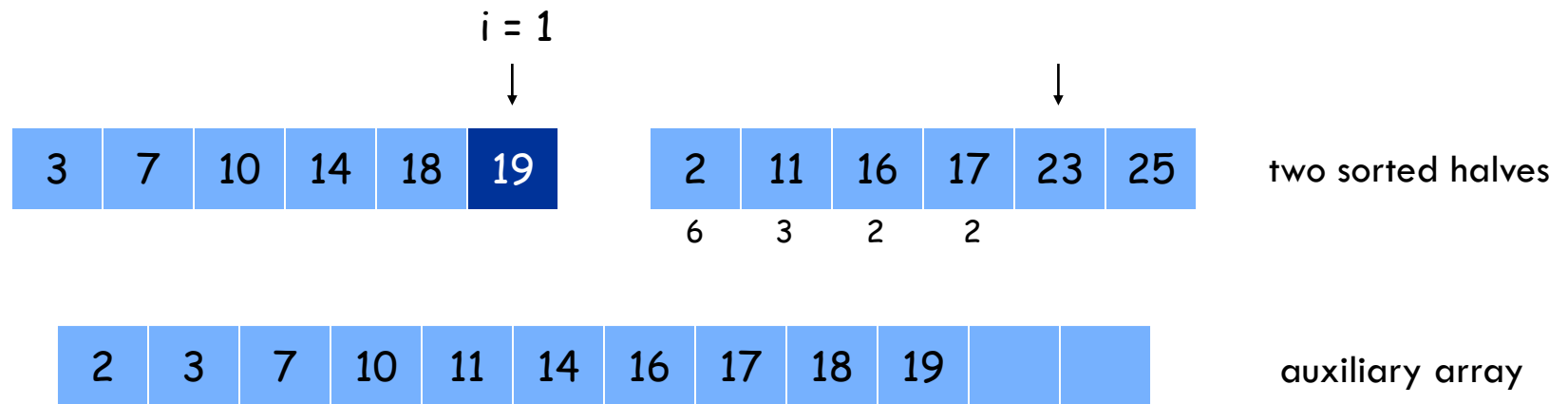
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

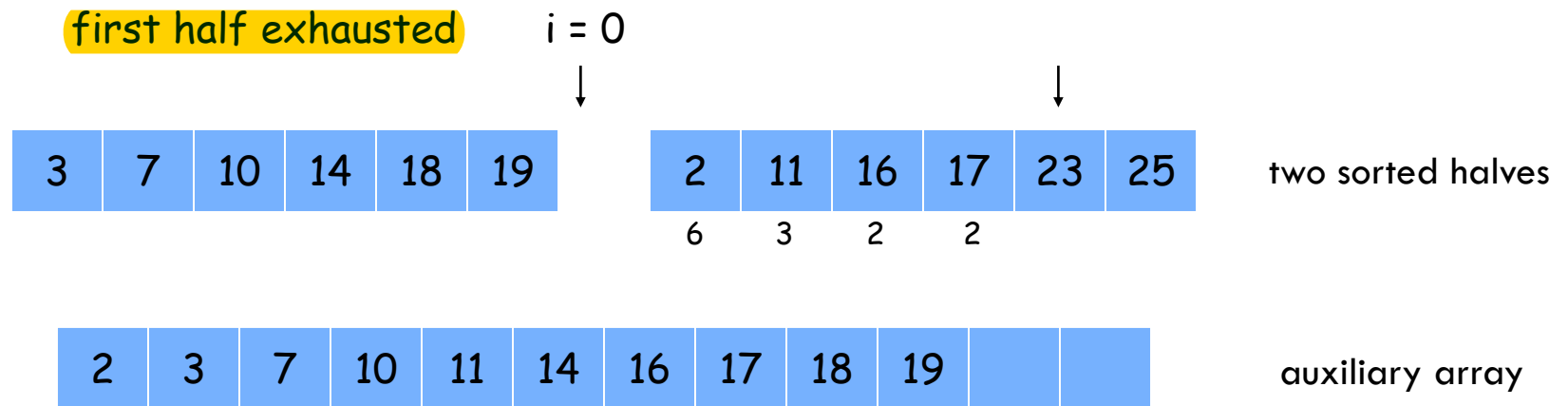
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

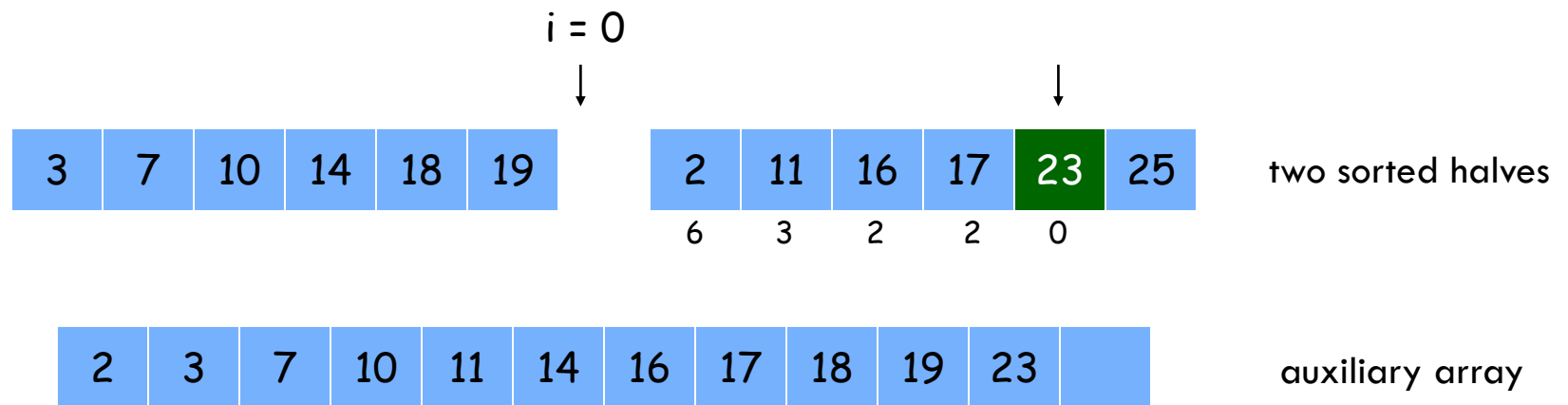
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2$

Merge and Count

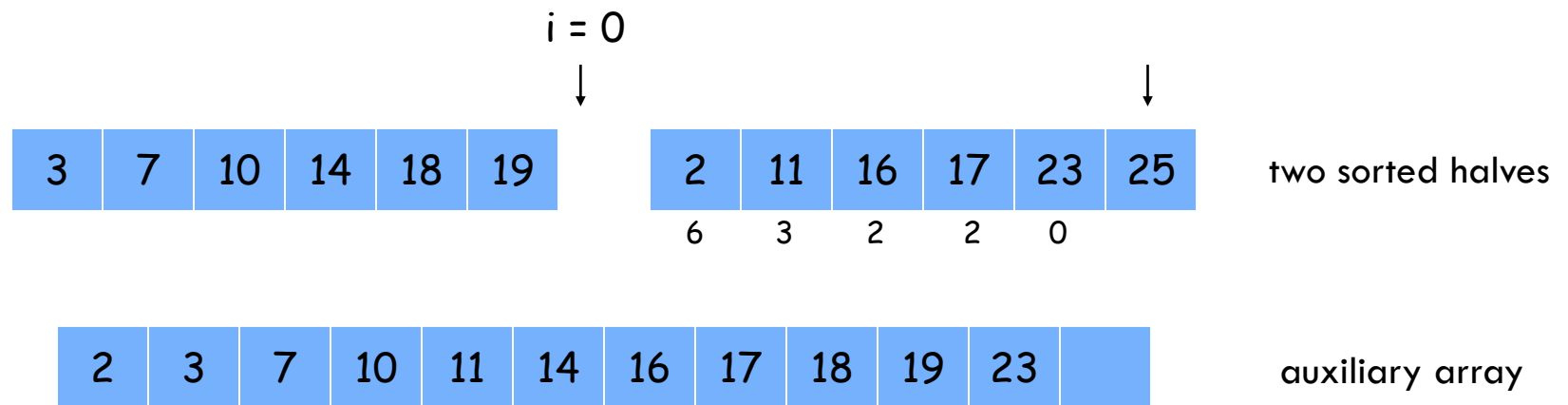
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2 + 0$

Merge and Count

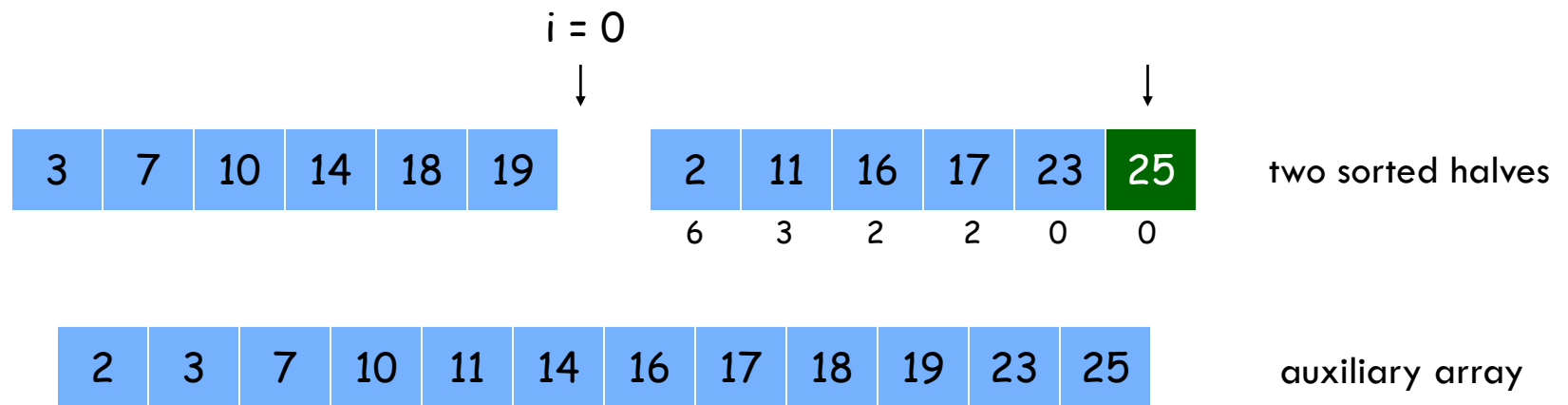
- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2 + 0$

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

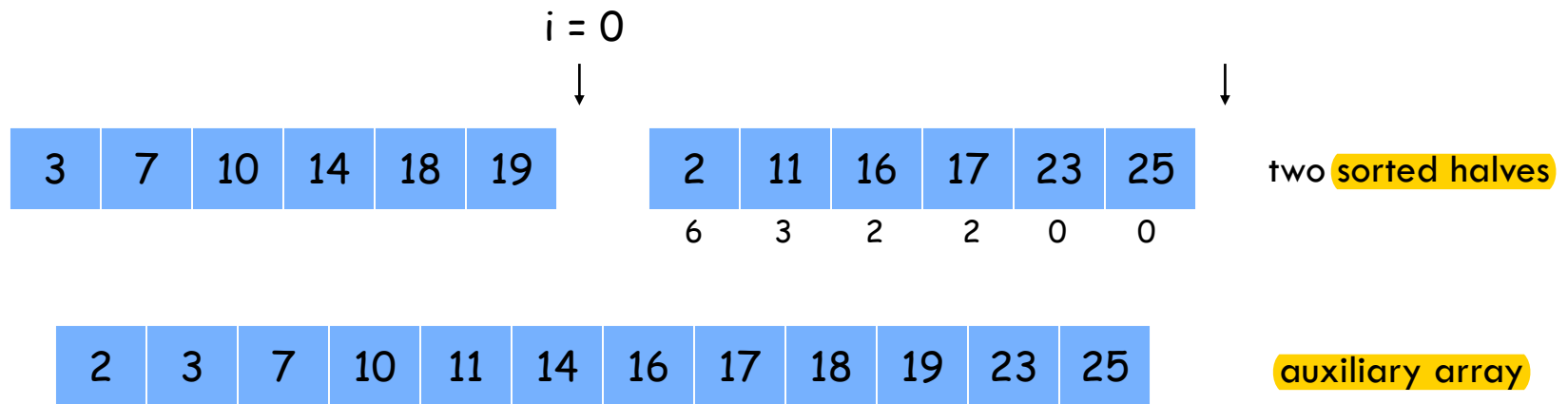


Total: $6 + 3 + 2 + 2 + 0 + 0$

Merge and Count

- Merge and count step.
 - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
 - Combine two sorted halves into sorted whole.

Assume each half are sorted.



Total: $6 + 3 + 2 + 2 + 0 + 0 = 13$

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

3	7	10	14	18	19
---	---	----	----	----	----

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: $6 + 3 + 2 + 2 + 0 + 0$

Count: $O(n)$

2	3	7	10	11	14	16	17	18	19	23	25
---	---	---	----	----	----	----	----	----	----	----	----

Merge: $O(n)$

Time: $T(n) = 2T(n/2) + O(n) = O(n \log n)$

Counting Inversions: Implementation

- Pre-condition. **[Merge-and-Count]** A and B are sorted.
- Post-condition. **[Sort-and-Count]** L is sorted.

```
Sort-and-Count(L) {  
    if list L has one element  
        return 0 and the list L  
  
    Divide the list into two halves A and B  
    ( $r_A$ , A)  $\leftarrow$  Sort-and-Count(A)  
    ( $r_B$ , B)  $\leftarrow$  Sort-and-Count(B)  
    ( $r_B$ , L)  $\leftarrow$  Merge-and-Count(A, B)  
  
    return  $r = r_A + r_B + r$  and the sorted list L  
}
```

Closest Pair of Points

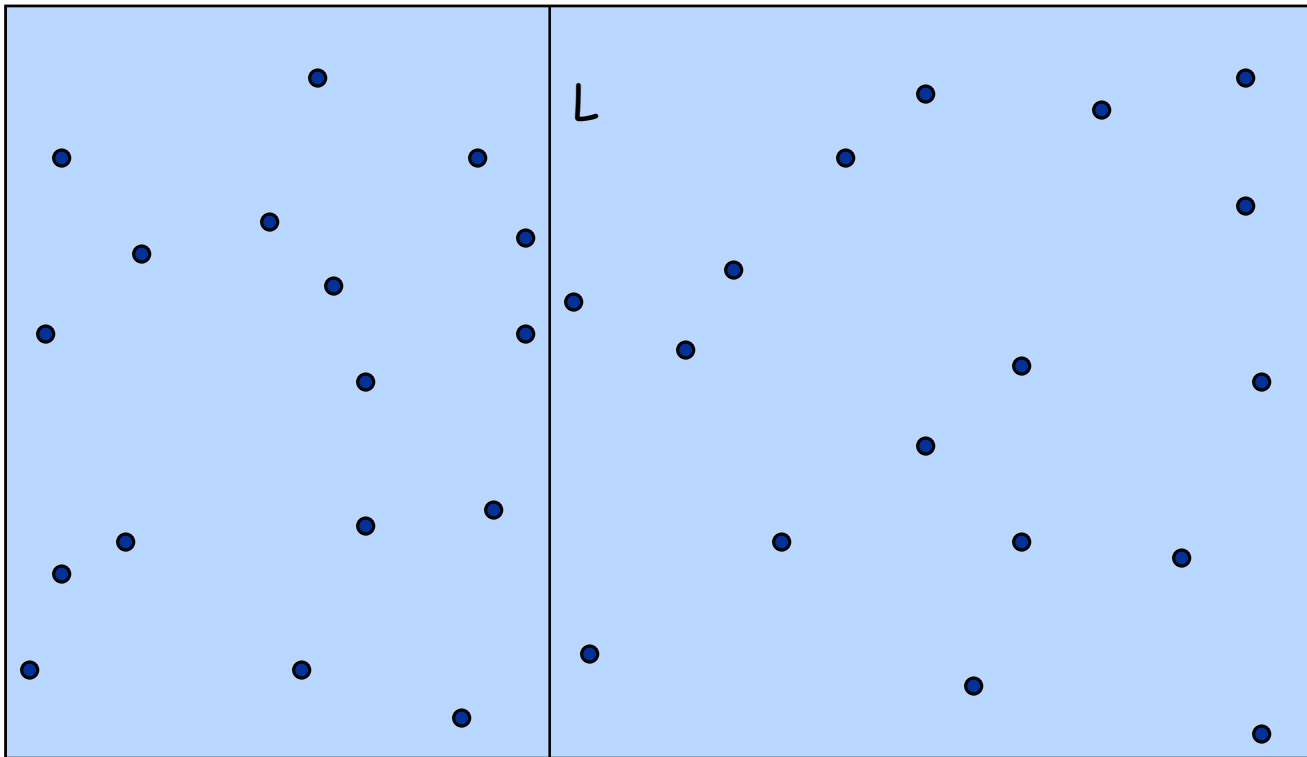
Closest Pair of Points

- **Closest pair.** Given n points in the plane, find a pair with smallest Euclidean distance between them.
- **Fundamental geometric primitive.**
 - Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
 - Special case of nearest neighbor, Euclidean MST, Voronoi.
- **Brute force.** Check all pairs of points p and q with $\Theta(n^2)$ comparisons.
- **1-D version.** $O(n \log n)$ easy if points are on a line.
- **Assumption.** No two points have same x coordinate.

Closest Pair of Points

- **Algorithm.**

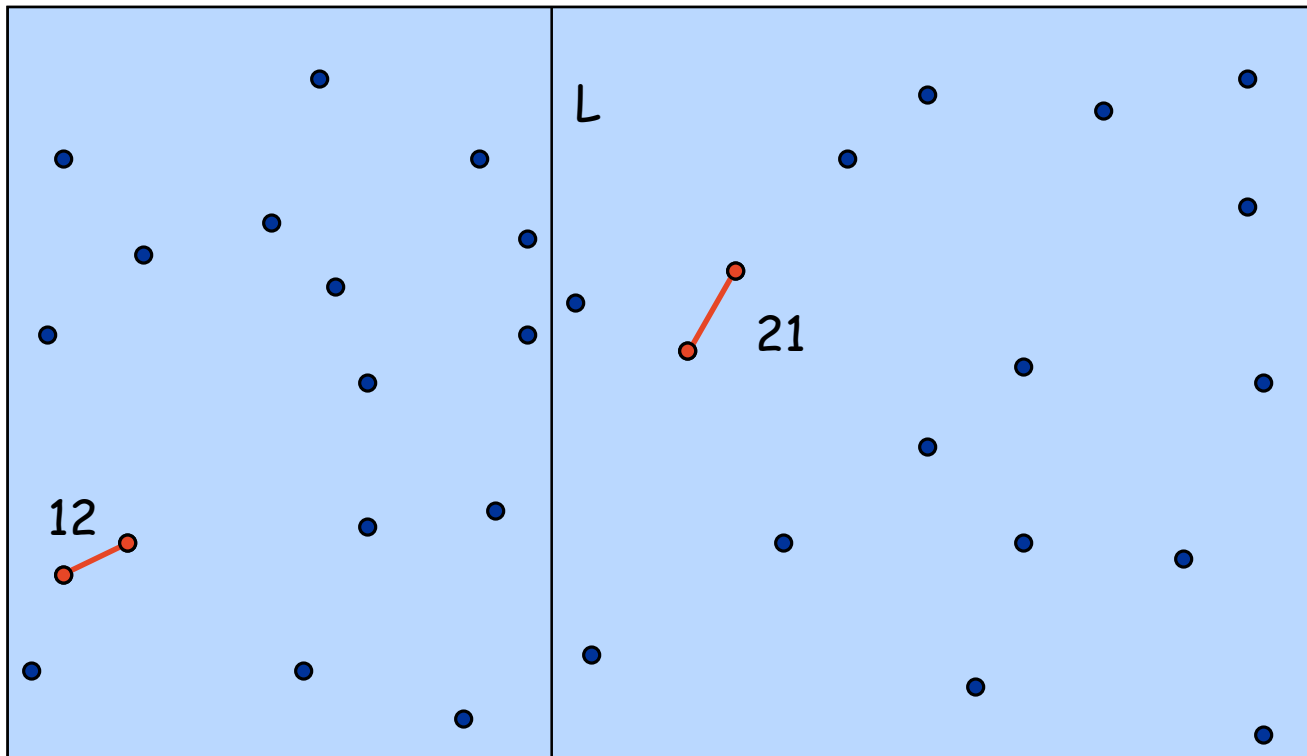
- **Divide:** draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

– Algorithm.

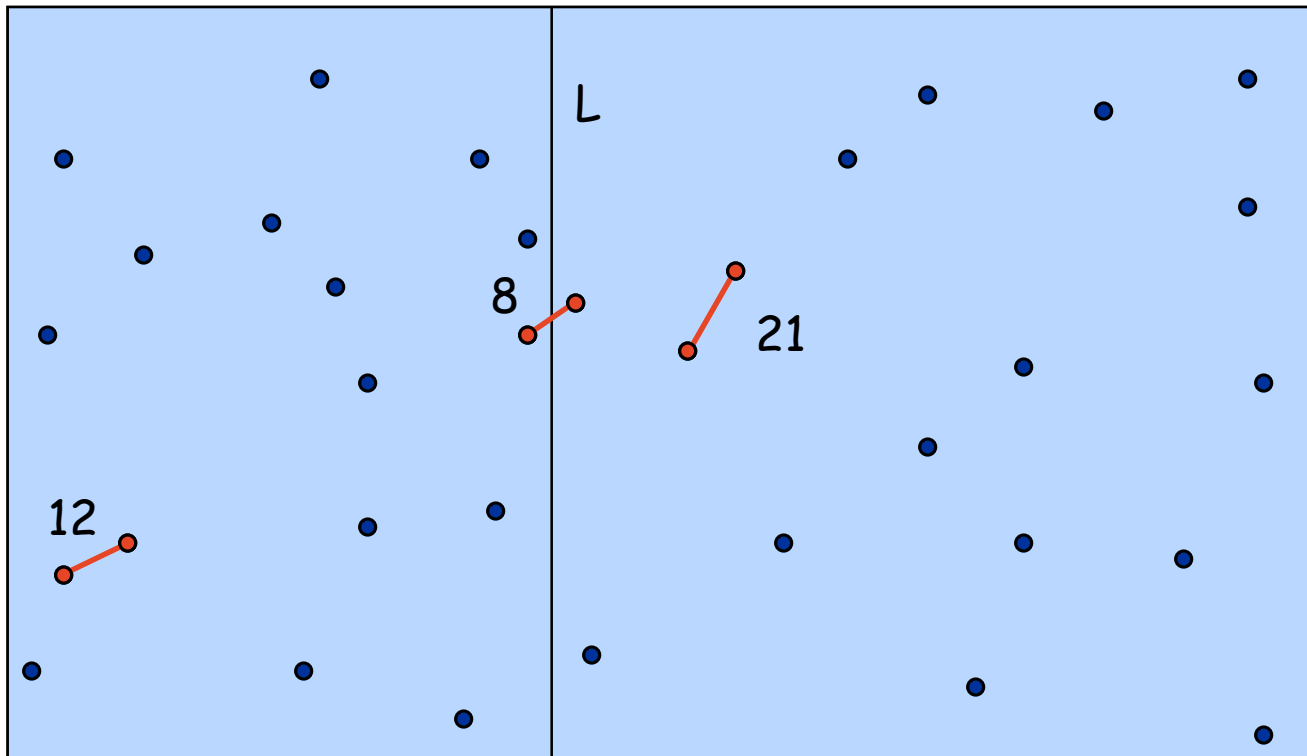
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

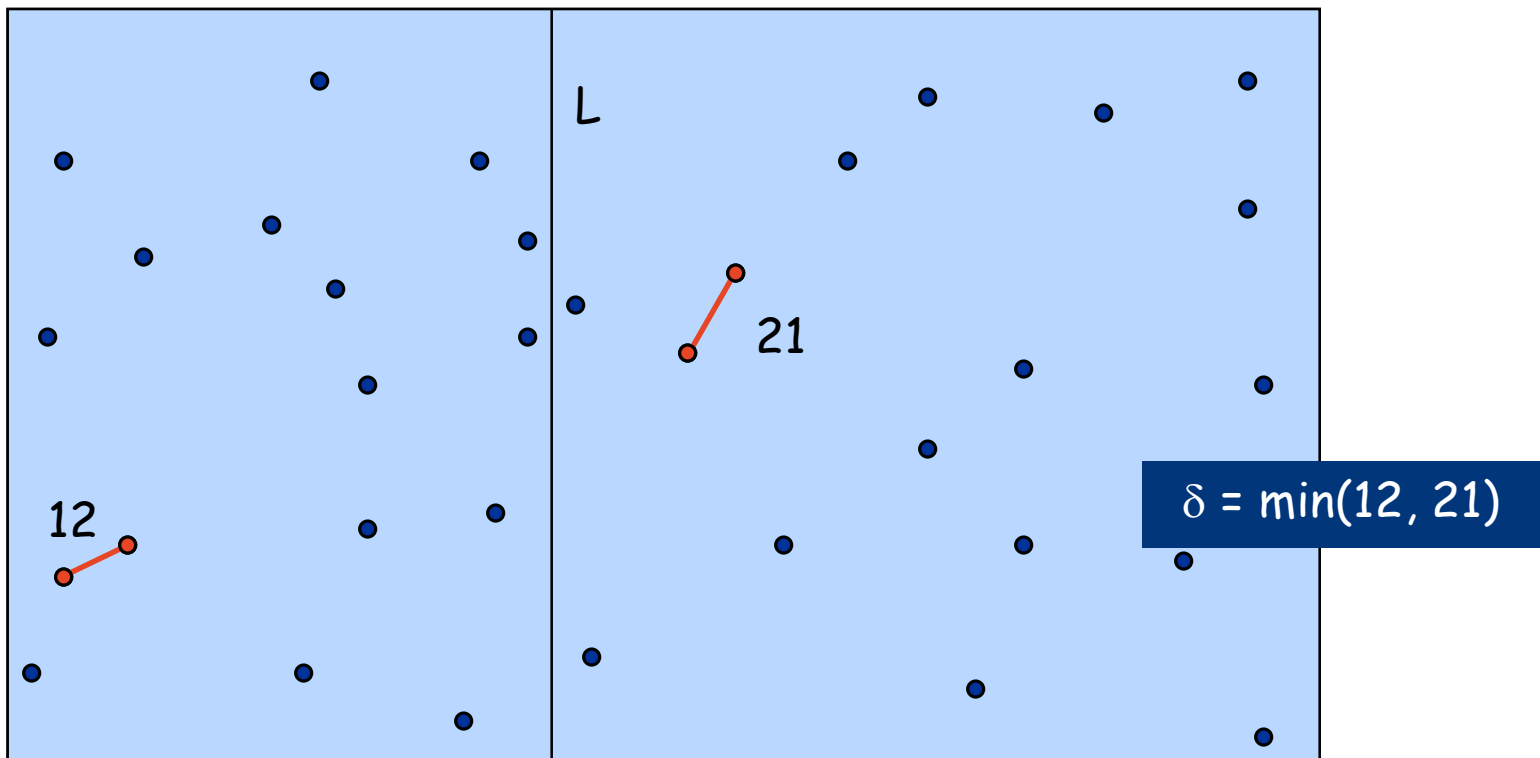
– Algorithm.

- **Divide:** draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- **Conquer:** find closest pair in each side recursively.
- **Combine:** find closest pair with one point in each side.
- **Return best of 3 solutions.**



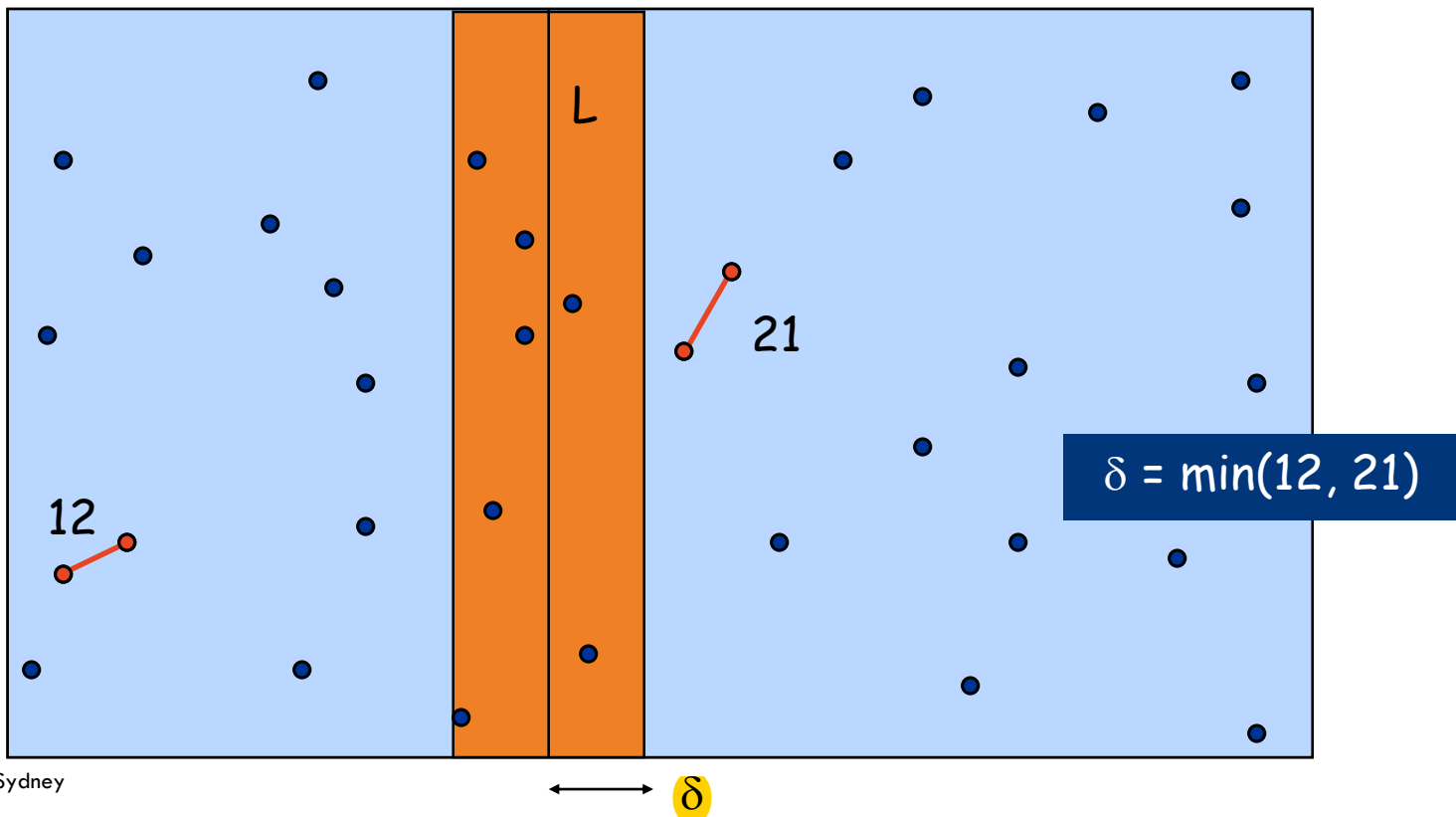
Closest Pair of Points

- Find closest pair with one point in each side, **assuming that distance $< \delta$.**



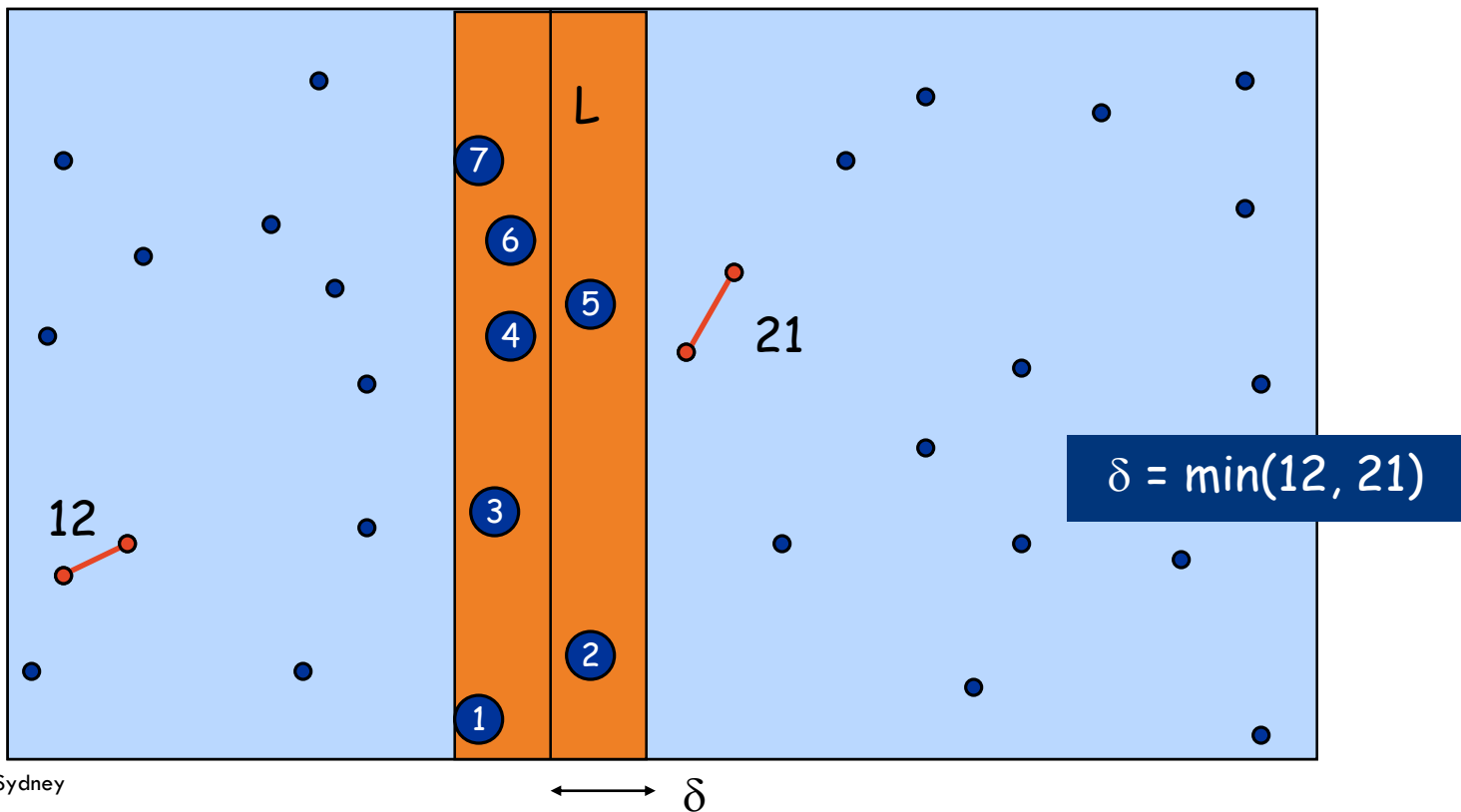
Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance $< \delta$.
 - **Observation:** only need to consider points within δ of line L .



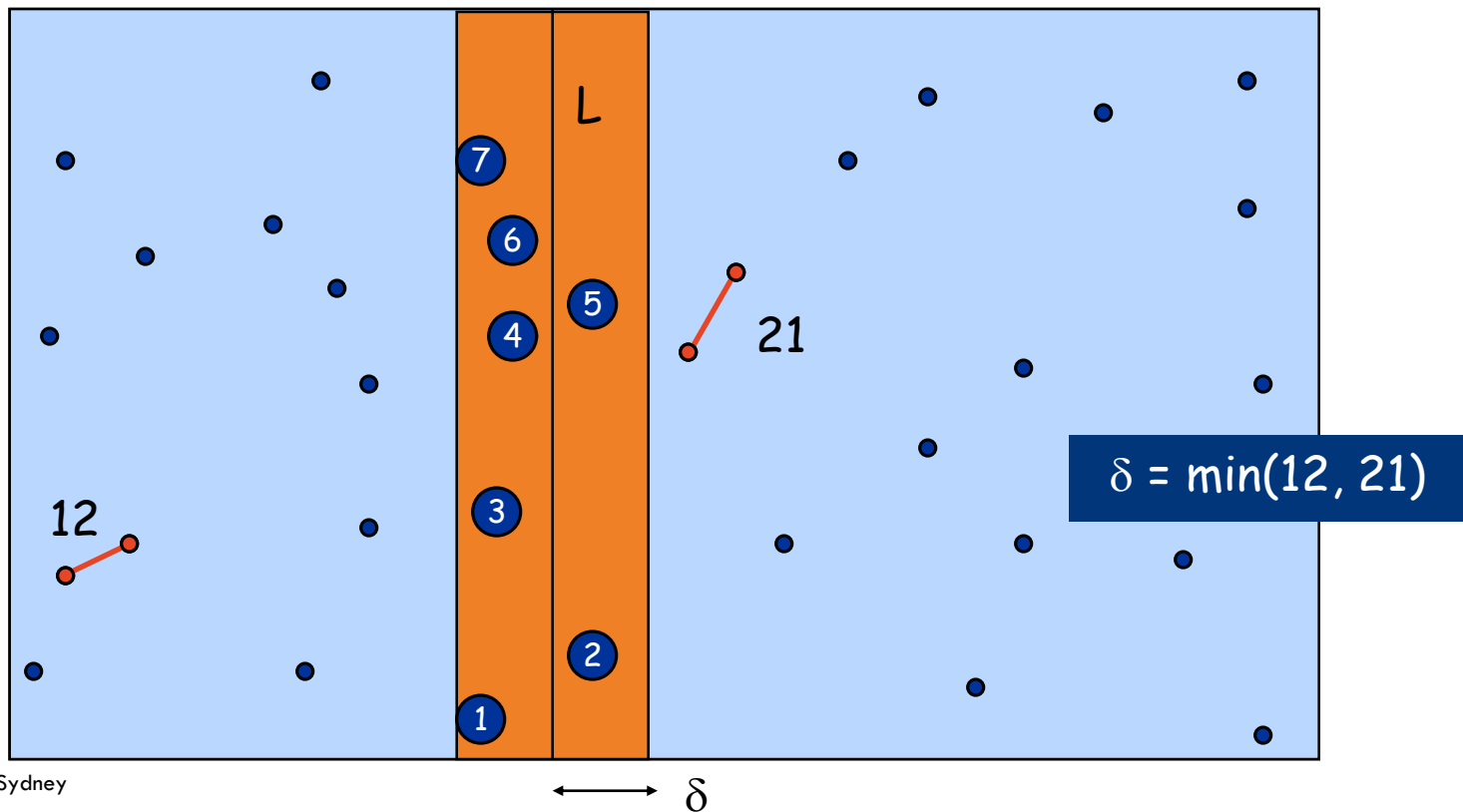
Closest Pair of Points

- Find closest pair with one point in each side, **assuming that distance $< \delta$.**
 - **Observation:** only need to consider points within δ of line L .
 - **Sort points in 2δ -strip by their y coordinate.**



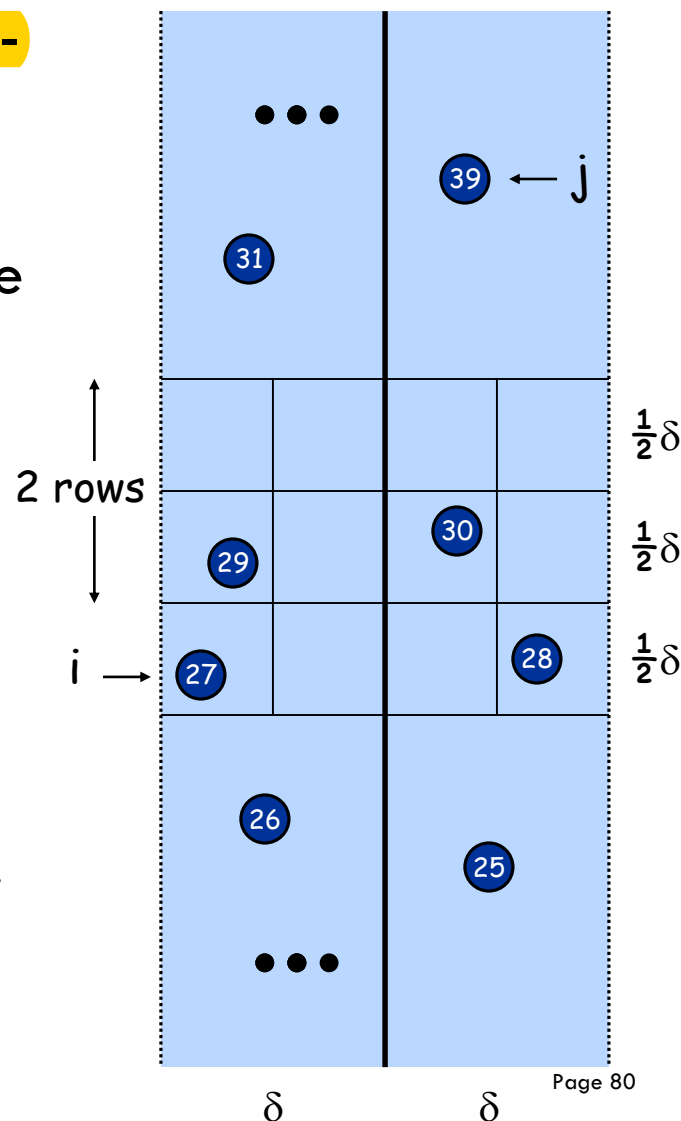
Closest Pair of Points

- Find closest pair with one point in each side, **assuming that distance $< \delta$** .
 - **Observation:** only need to consider points within δ of line L .
 - Sort points in 2δ -strip by their y coordinate.
 - Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

- **Definition:** Let s_i be the point in the 2δ -strip, with the i^{th} smallest y -coordinate.
- **Claim:** If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ .
- **Proof:**
 - No two points lie in same $\frac{1}{2}\delta$ -by- $\frac{1}{2}\delta$ box.
 - Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▀
- **Fact:** Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair( $p_1, \dots, p_n$ ) {  
    Compute separation line  $L$  such that half the points  
    are on one side and half on the other side.  $O(n \log n)$   
  
     $\delta_1 = \text{Closest-Pair}(\text{left half})$   
     $\delta_2 = \text{Closest-Pair}(\text{right half})$   $2T(n / 2)$   
     $\delta = \min(\delta_1, \delta_2)$   
  
    Delete all points further than  $\delta$  from separation line  $L$   $O(n)$   
  
    Sort remaining points by  $y$ -coordinate.  $O(n \log n)$   
  
    Scan points in  $y$ -order and compare distance between  
    each point and next 11 neighbors. If any of these  
    distances is less than  $\delta$ , update  $\delta$ .  $O(n)$   
  
    return  $\delta$ .  
}
```

Closest Pair of Points: Analysis

- **Running time**

$$T(n) \leq 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

- **Question:** Can we achieve $O(n \log n)$?

□

- **Answer:** Yes. **Don't sort points in strip from scratch each time.**

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by **merging** two pre-sorted lists.

$$T(n) \leq 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

Solving recursions

Unrolling and the Master method

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

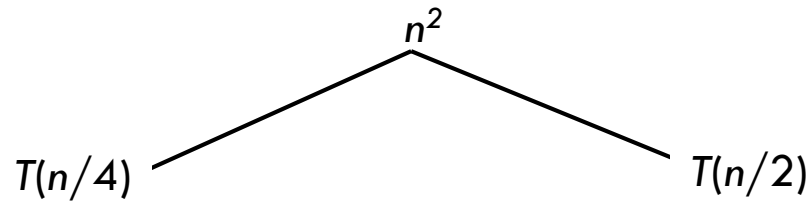
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$T(n)$

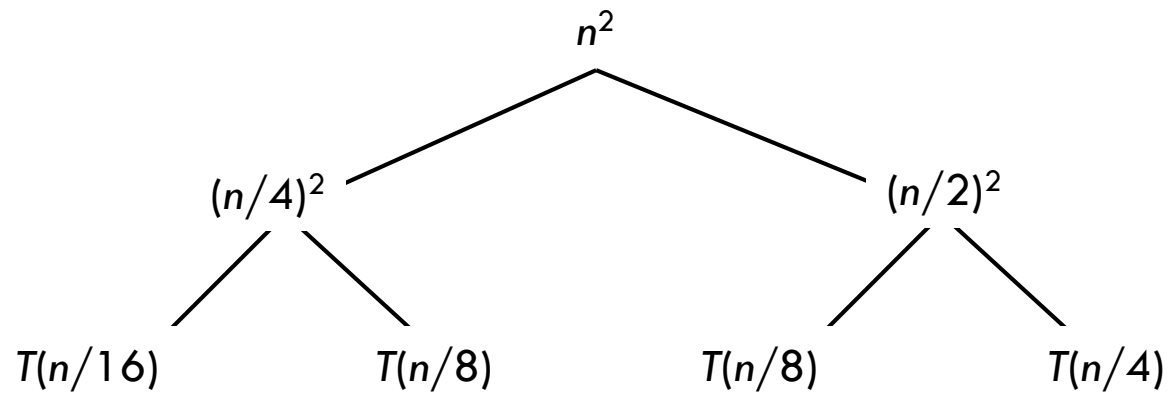
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



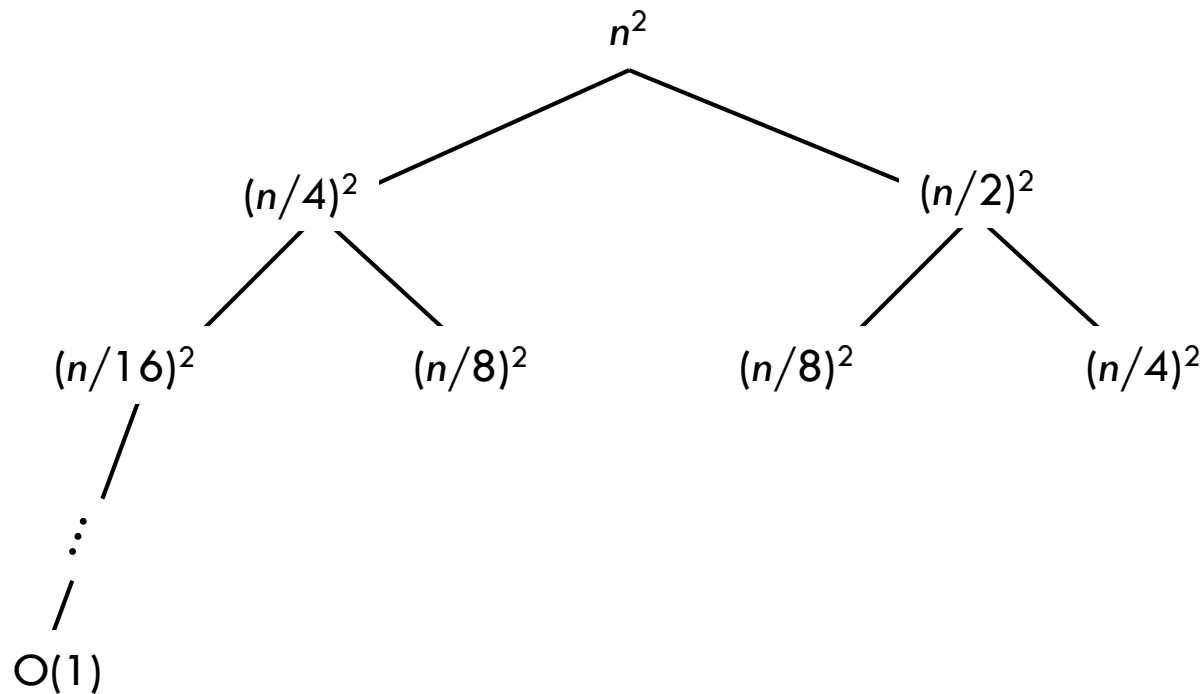
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



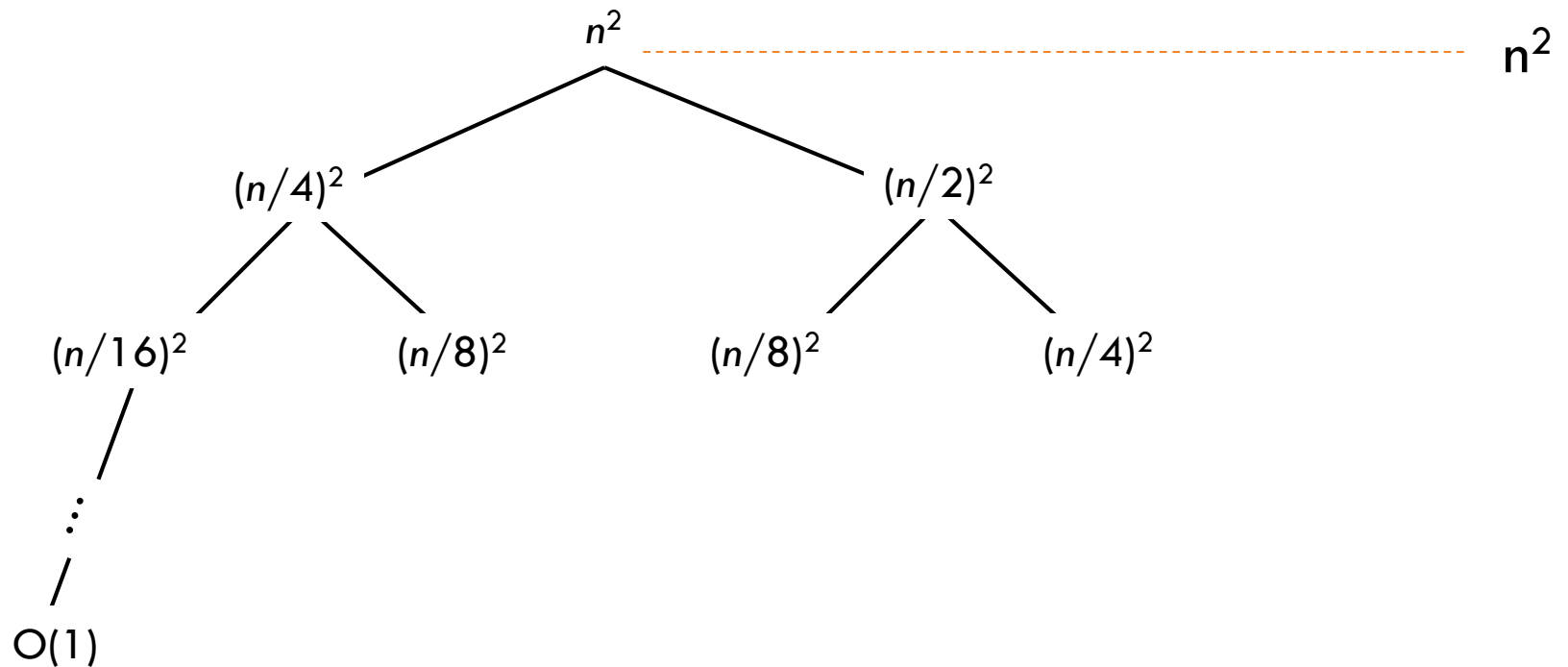
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



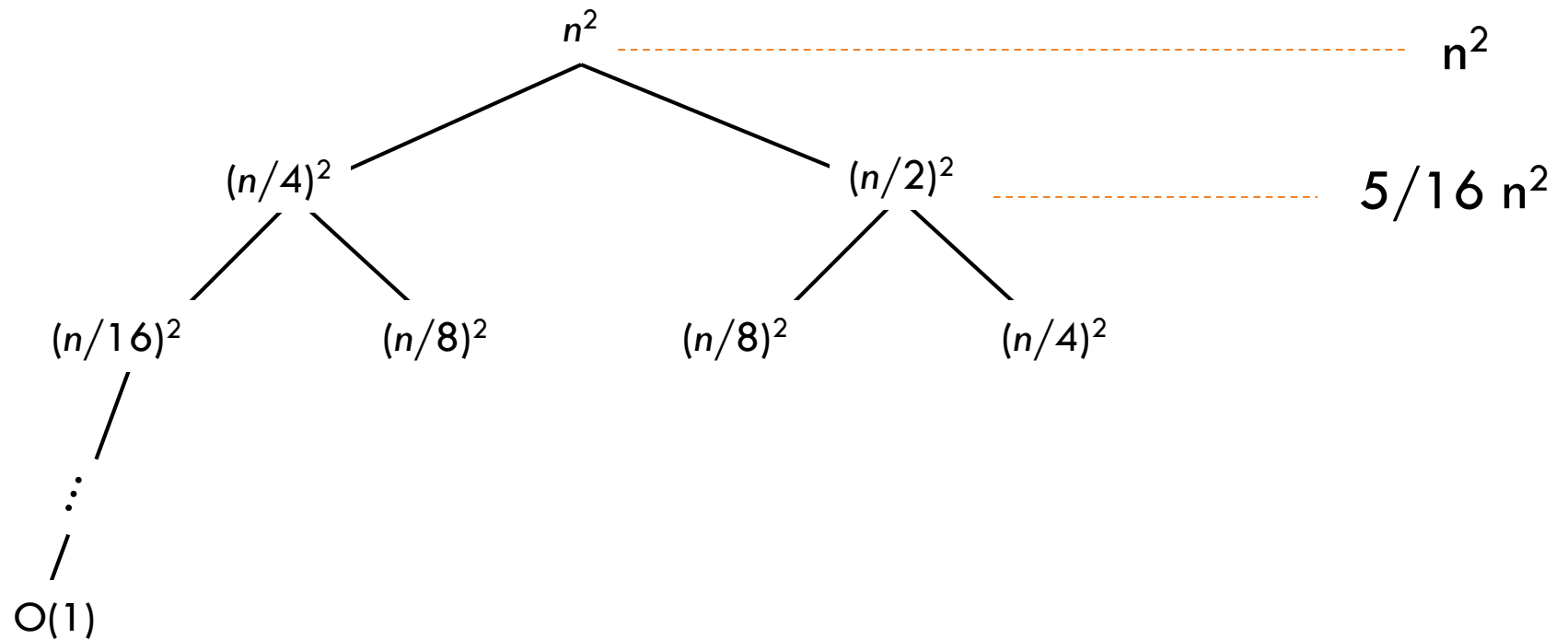
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



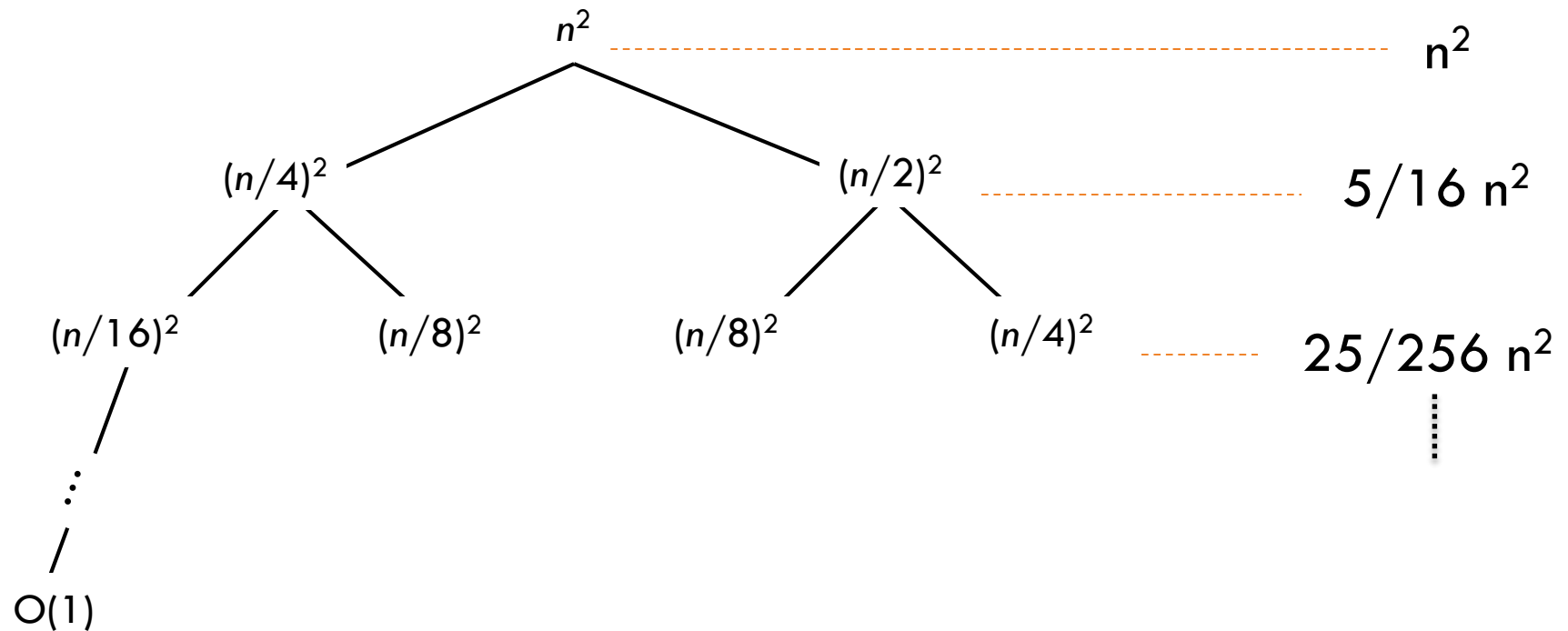
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



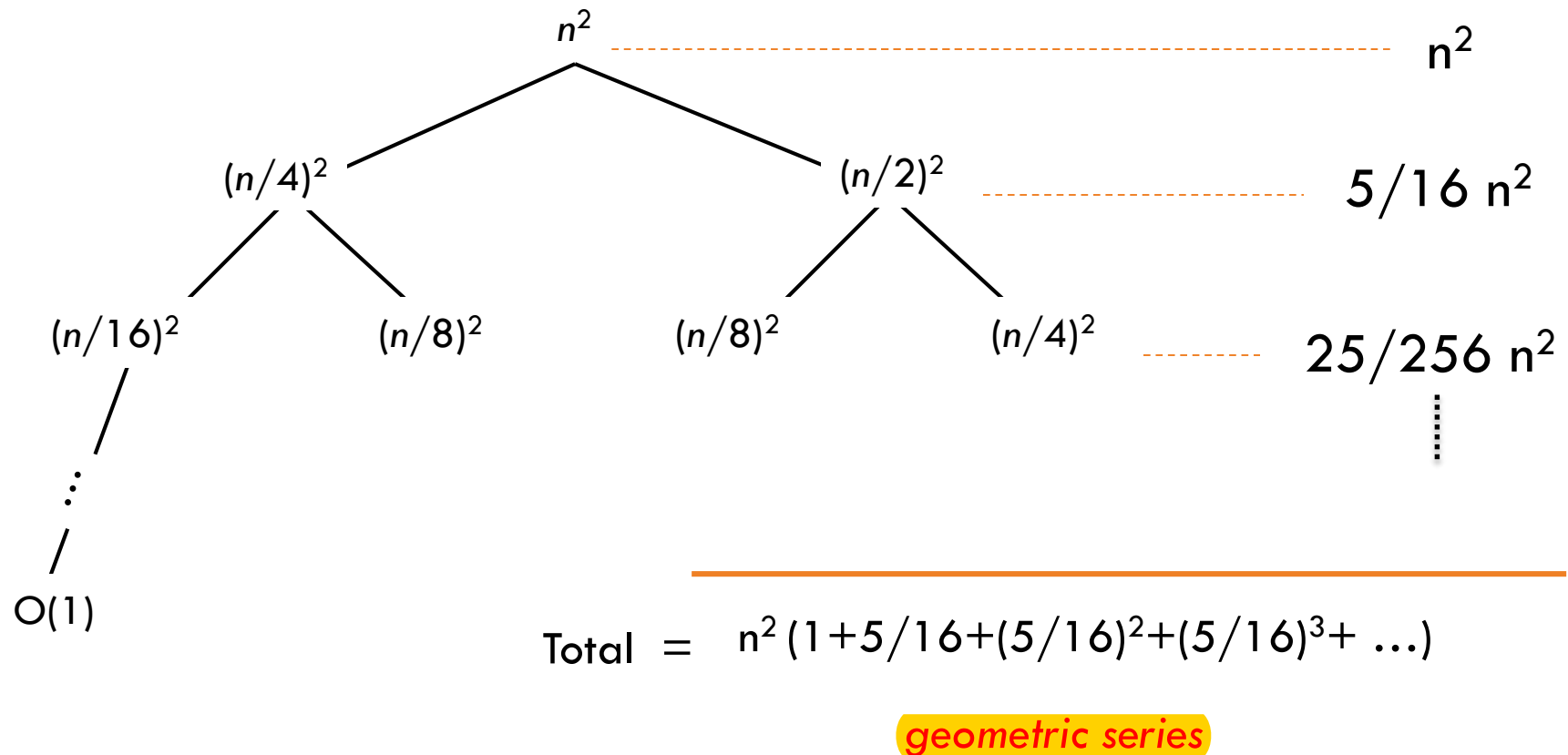
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



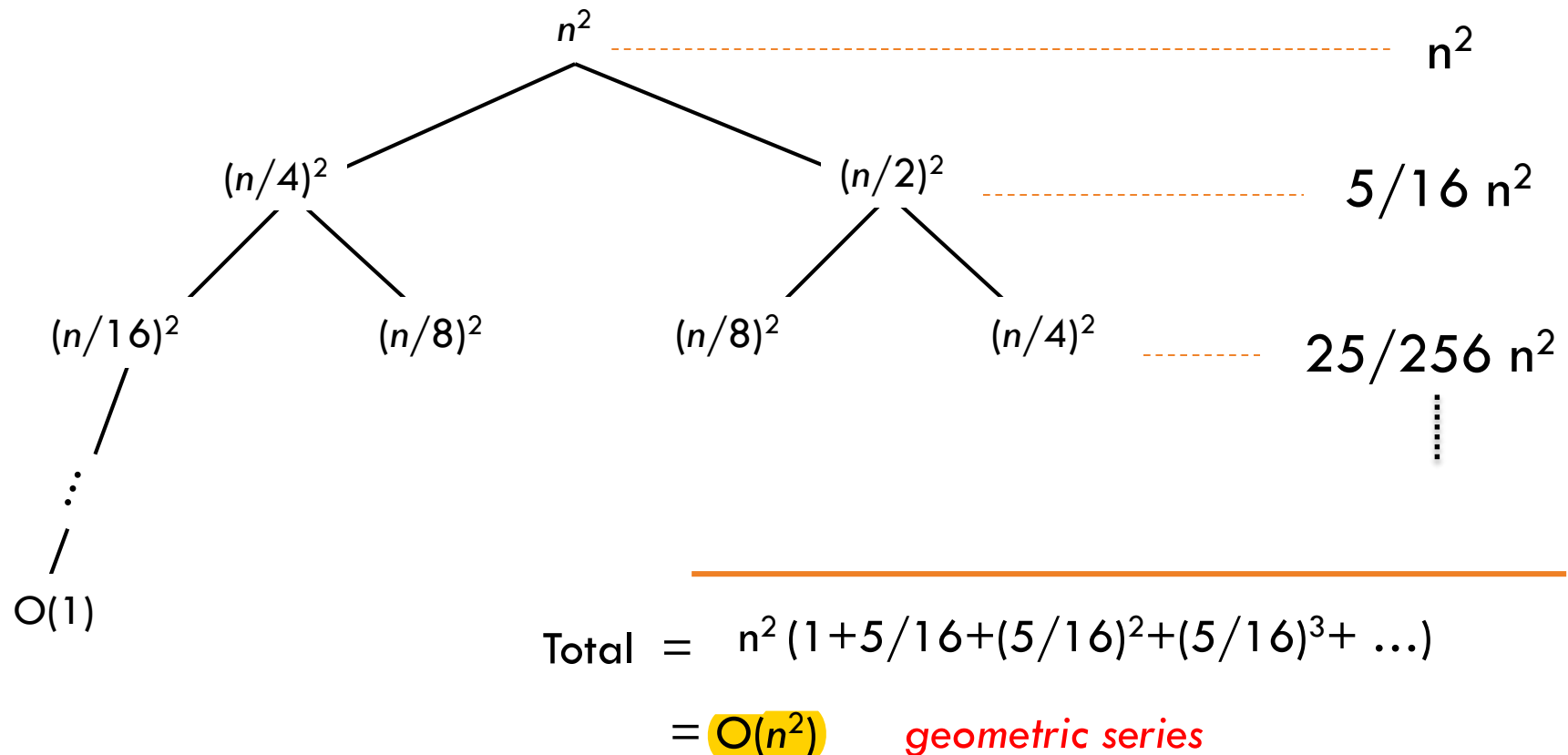
Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



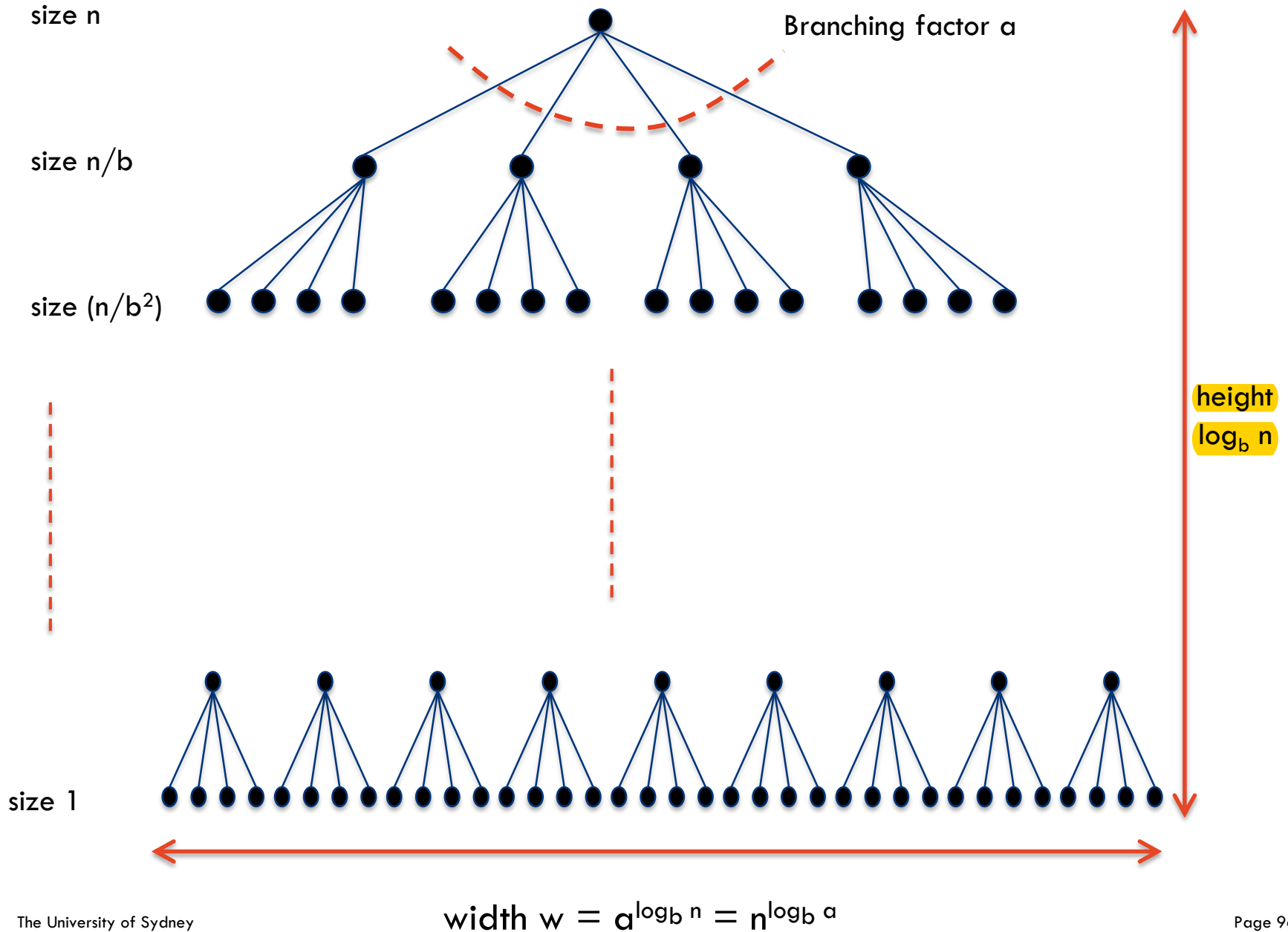
The master method

The master method applies to recurrences of the form

$$T(n) = a \cdot T(n/b) + f(n) ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

$$T(n) = a \cdot T(n/b) + f(n)$$



$$T(n) = a \cdot T(n/b) + f(n)$$

size n
cost $f(n)$

Branching factor a

size n/b
cost $a \cdot f(n/b)$

size (n/b^2)
cost $a^2 \cdot f(n/b^2)$

height
 $\log_b n$

size 1
cost $w \cdot T(1)$

$$\text{width } w = a^{\log_b n} = n^{\log_b a}$$

Three common cases

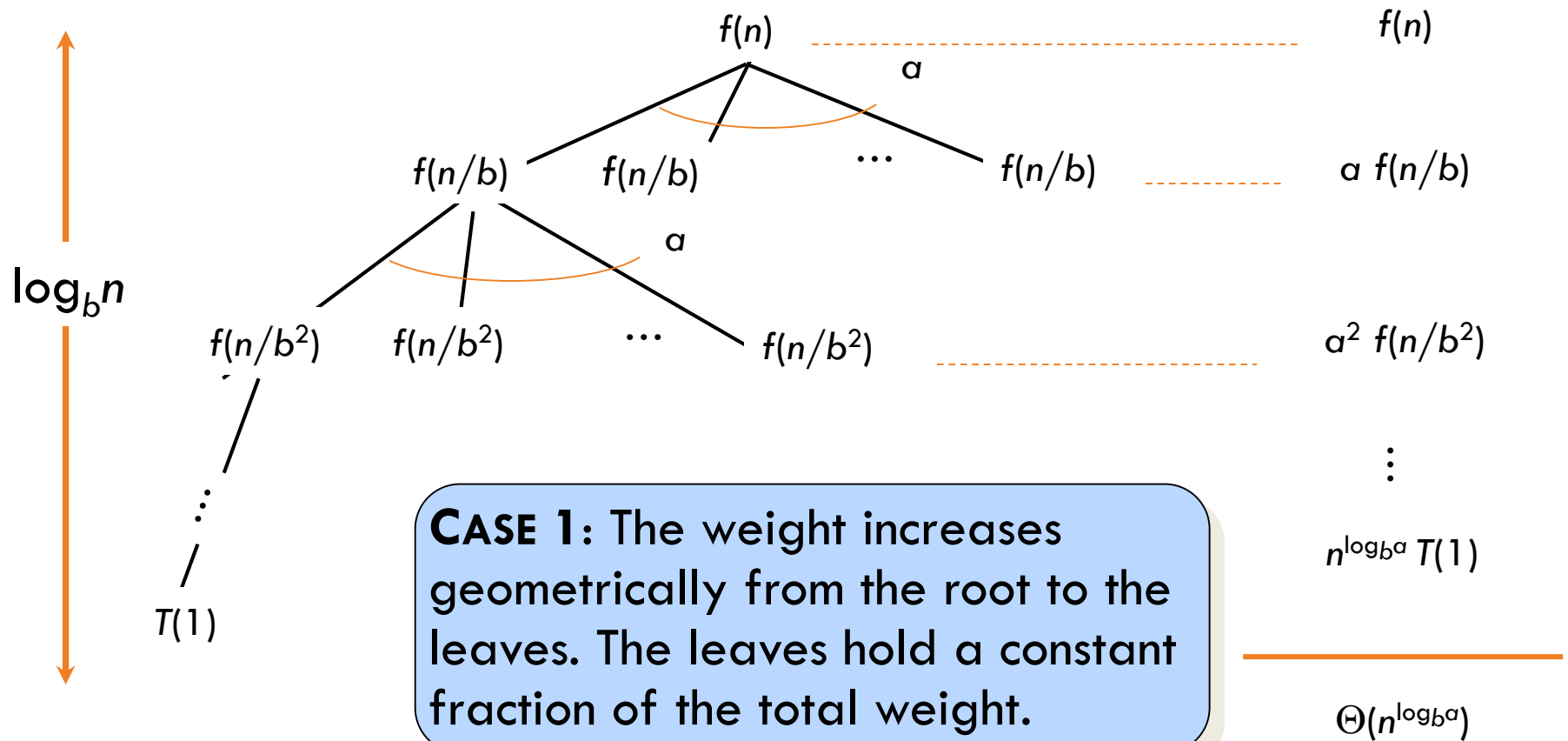
Compare $f(n)$ with $n^{\log_b a}$:

Case 1:

If $f(n) = O(n^{\log_b a - \varepsilon})$ for any constant $\varepsilon > 0$ then $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^ε factor).

Solution: $T(n) = \Theta(n^{\log_b a})$.

Idea of master theorem: Case 1



Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

Case 1: Example

$$T(n) = 8T(n/2) + 10n^2$$

$$\Rightarrow a = 8, b = 2 \text{ and } f(n) = 10n^2$$

$$f(n) = 10n^2 = O(n^{\log_b a - \varepsilon}) = O(n^{\log_2 8 - \varepsilon}) = O(n^{3 - \varepsilon}) \text{ for } \varepsilon = 1 > 0.$$

\Rightarrow Case 1 holds.

$$\textbf{Solution: } T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$$

Three common cases (cont'd)

Compare $f(n)$ with $n^{\log_b a}$:

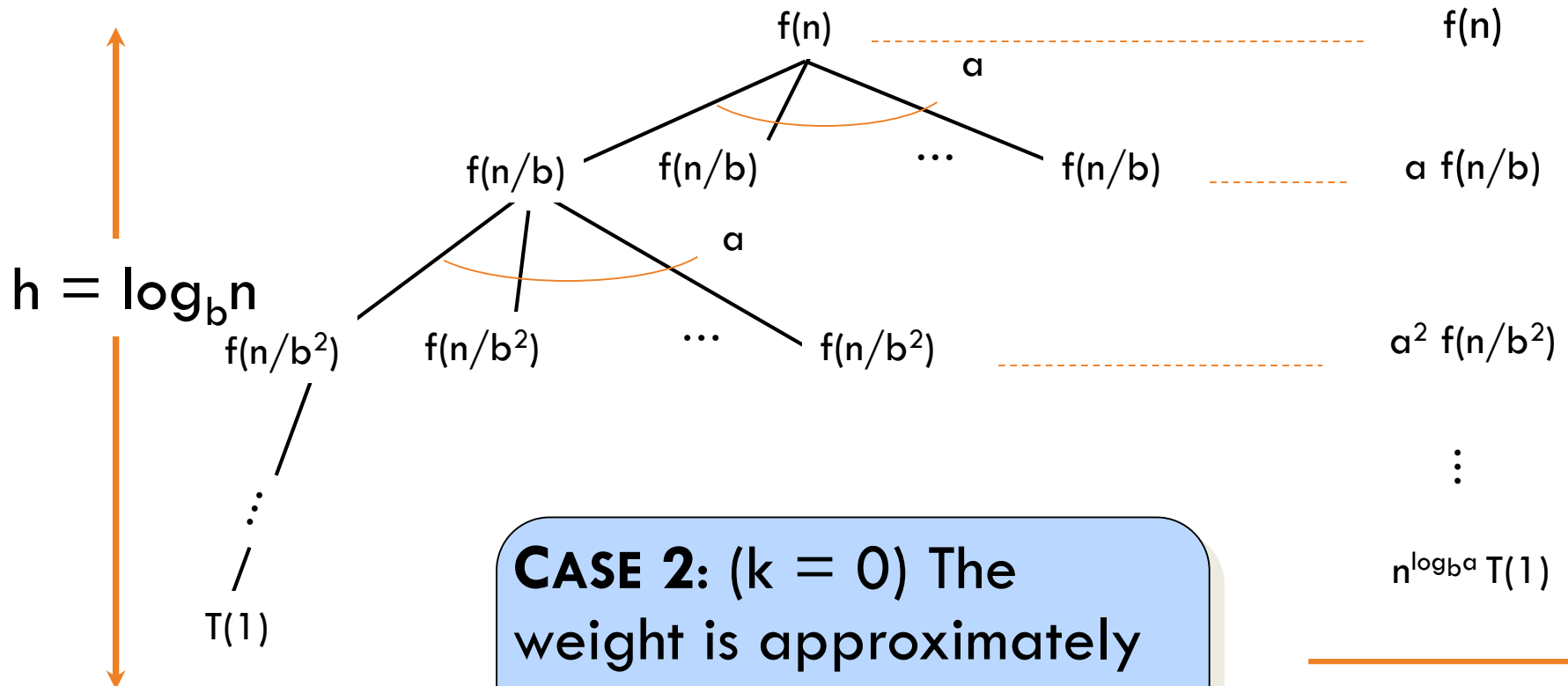
Case 2:

If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$ then $f(n)$ and $n^{\log_b a}$ grow at similar rates.

Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Idea of master theorem

Recursion tree:



CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$$\Theta(n^{\log_b a} h)$$

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

Case 2: Example

$$T(n) = 2T(n/2) + n \log n$$

$$\Rightarrow a = 2, b = 2 \text{ and } f(n) = n \log n$$

$$f(n) = n \log n = \Theta(n^{\log_b a} \log^k n) = \Theta(n \log n) \text{ for } k=1$$

\Rightarrow Case 2 holds.

$$\textbf{Solution: } T(n) = \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n \log^2 n)$$

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

Case 3:

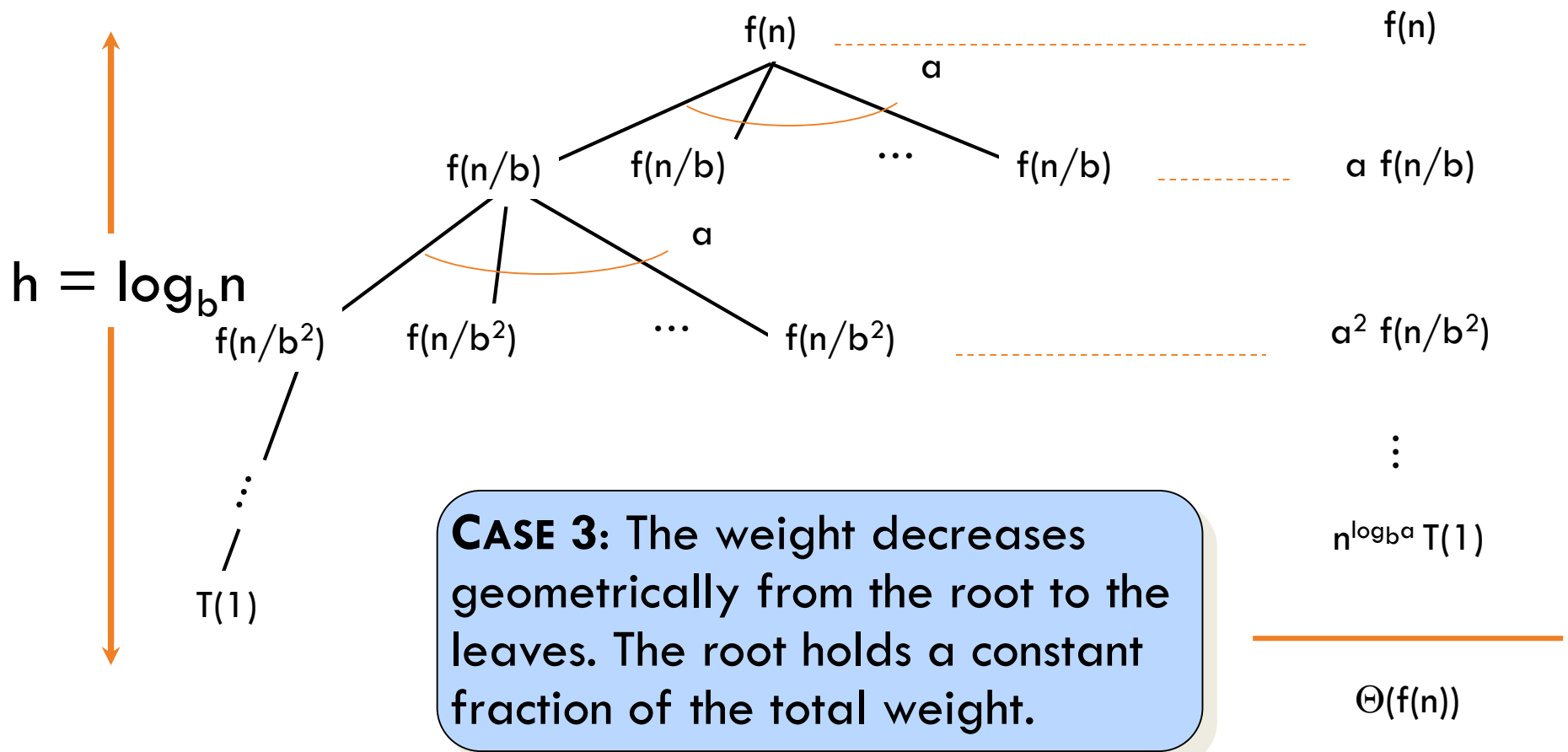
If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.

- $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^ε factor), **and**
- $f(n)$ satisfies the **regularity condition** that $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.

Idea of master theorem

Recursion tree:



Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

$$T(n) = 4T(n/2) + n^3$$

$$a = 4, b = 2 \quad \Rightarrow \quad n^{\log_b a} = n^2 \text{ and } f(n) = n^3.$$

CASE 3: $f(n) = \Omega(n^{2 + \varepsilon})$ for $\varepsilon = 1$

and

$$4(cn/2)^3 \leq cn^3 \text{ (reg. cond.) for } c = 1/2.$$

Solution: $T(n) = \Theta(n^3)$.

Integer Multiplication

Integer Arithmetic

- **Add.** Given two n -digit integers a and b , compute $a + b$.
 - $O(n)$ bit operations.
- **Multiply.** Given two n -digit integers a and b , compute $a \times b$.
 - Brute force solution: $\Theta(n^2)$ bit operations.

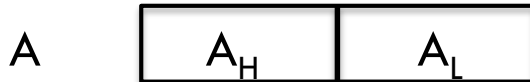
1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

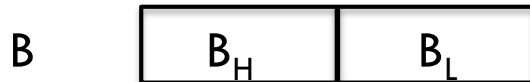
[illegible]

Divide-and-Conquer Multiplication: Warmup

- To multiply two n -digit integers:
 - Given two numbers A and B with n bits each.
 - Partition the n bits into the $n/2$ “high” bits and the $n/2$ “low” bits.



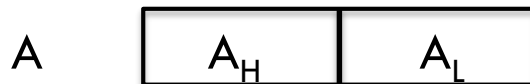
$$A = A_H \cdot 2^{n/2} + A_L$$



$$B = B_H \cdot 2^{n/2} + B_L$$

Divide-and-Conquer Multiplication: Warmup

- To multiply two n -digit integers:
 - Given two numbers A and B with n bits each.
 - Partition the n bits into the $n/2$ “high” bits and the $n/2$ “low” bits.



$$A = A_H \cdot 2^{n/2} + A_L$$



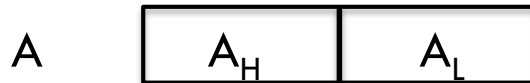
$$B = B_H \cdot 2^{n/2} + B_L$$

$$\begin{aligned} \rightarrow A \cdot B &= (A_H \cdot 2^{n/2} + A_L) \cdot (B_H \cdot 2^{n/2} + B_L) \\ &= A_H B_H \cdot 2^n + A_H B_L \cdot 2^{n/2} + A_L B_H \cdot 2^{n/2} + A_L B_L \end{aligned}$$

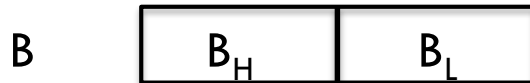
- 4 multiplications of $n/2$ -bit numbers: $A_H B_H$, $A_H B_L$, $A_L B_H$, $A_L B_L$, additions and shifts. Multiplications by powers of 2 are just shifts.

Divide-and-Conquer Multiplication: Warmup

- To multiply two n -digit integers:
 - Given two numbers A and B with n bits each.
 - Partition the n bits into the $n/2$ “high” bits and the $n/2$ “low” bits.



$$A = A_H \cdot 2^{n/2} + A_L$$



$$B = B_H \cdot 2^{n/2} + B_L$$

$$\begin{aligned} \rightarrow A \cdot B &= (A_H \cdot 2^{n/2} + A_L) \cdot (B_H \cdot 2^{n/2} + B_L) \\ &= A_H B_H \cdot 2^n + A_H B_L \cdot 2^{n/2} + A_L B_H \cdot 2^{n/2} + A_L B_L \end{aligned}$$

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

Karatsuba Multiplication [1960]

– Multiply two n-digit integers



$$A = A_H \cdot 2^{n/2} + A_L$$

$$B = B_H \cdot 2^{n/2} + B_L$$

Observation:

$$A \cdot B = A_H B_H \cdot 2^n + [(A_H + A_L) \cdot (B_H + B_L) - A_H B_H - A_L B_L] \cdot 2^{n/2} + A_L B_L$$

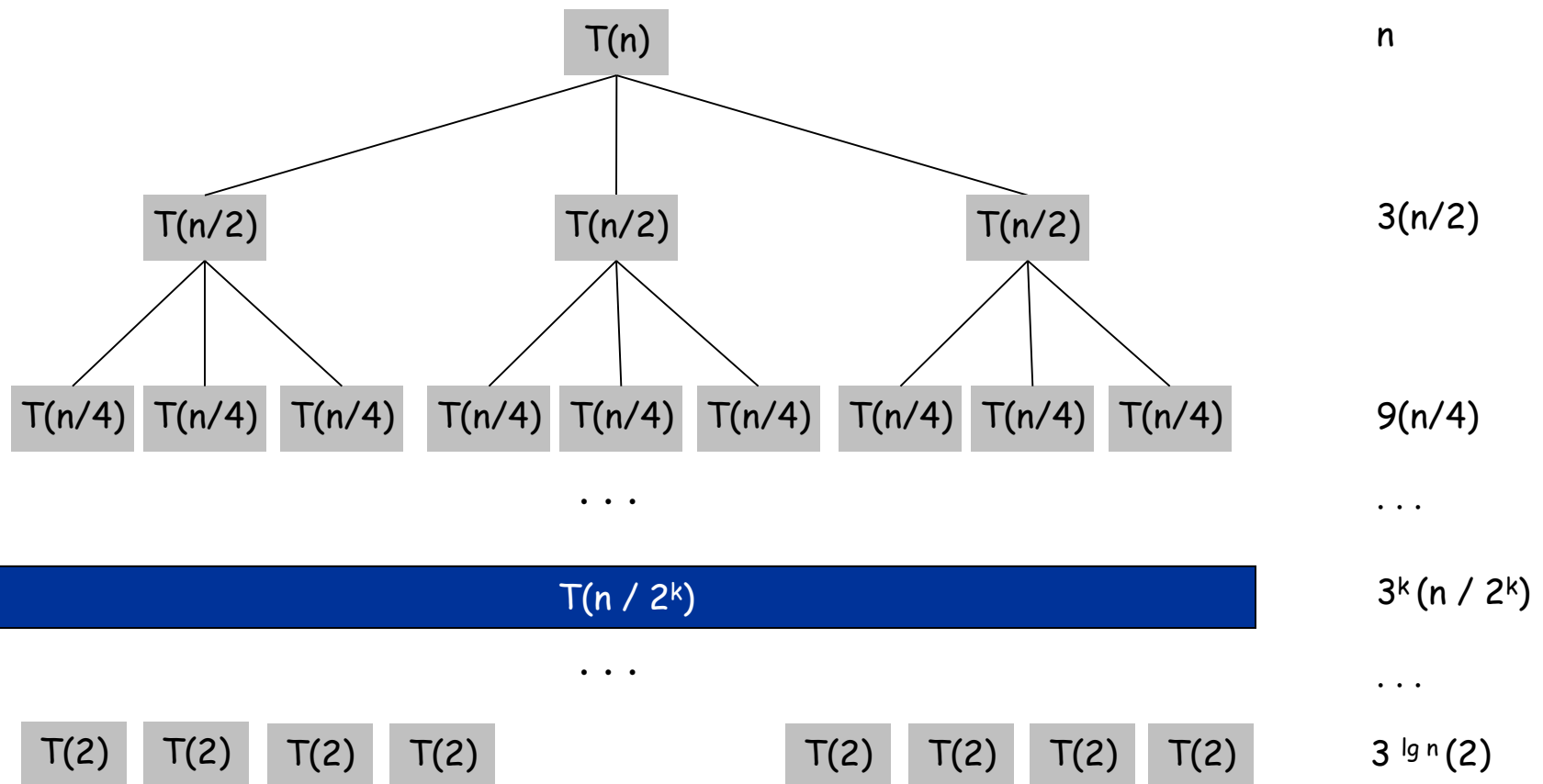
Theorem: [Karatsuba-Ofman, 1962]

3 multiplications of $n/2$ -bit numbers + additions, subtractions and shifts.

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\frac{3}{2}\right)^k = \frac{\left(\frac{3}{2}\right)^{1+\log_2 n} - 1}{\frac{3}{2} - 1} = 3n^{\log_2 3} - 2 = O(n^{1.59})$$



Summary: Divide-and-Conquer

- **Divide-and-conquer.**

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

- **Master theorem**

- **Problems**

- Merge Sort
- Closest pair
- Multiplication

**This weeks quiz is all
about solving recursions!**