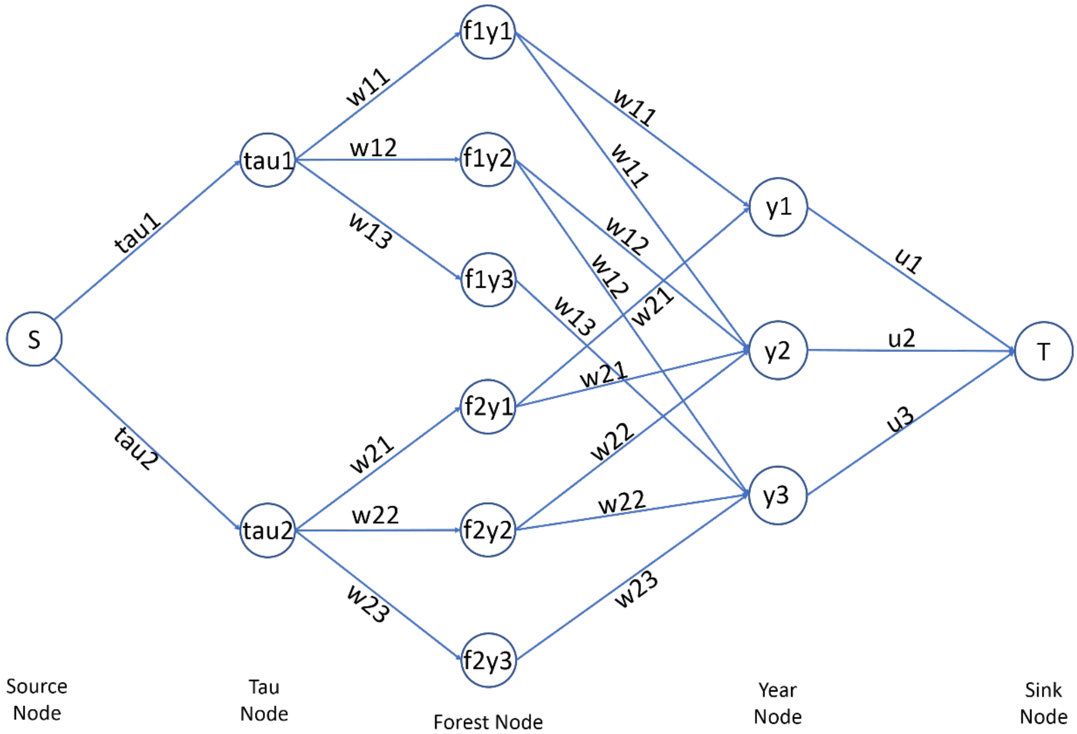
COMP2007 Assignment 4 Report

Jiashu Wu 460108049 jiwu0083

Question 1

1.1 Formulate the problem as a network flow problem

I will illustrate this Max Flow Algorithm using the following flow network.



Note:

1. In the flow network above, S is the source, and T is the sink.
2. In the flow network above, f_iy_j stands for node forest i year j.
3. Since $delta_1 = 2$, all the trees matured in the Year 1 can only be sold during Year 1 and Year 2. Hence, forest nodes f_1y_1 and f_2y_1 are connected to year node y_1 and y_2.
4. Since $delta_2 = 2$, all the trees matured in the Year 2 can only be sold during Year 2 and Year 3. Hence, forest nodes f_1y_2 and f_2y_2 are connected to year node y_2 and y_3.
5. Since $delta_3 = 1$, all the trees matured in the Year 3 can only be sold during Year 3. Hence, forest nodes of f_1y_3 and f_2y_3 are connected to year node y_3.

1.2 Argue the correctness of the algorithm

Note: The following theorem and properties will be used.

<u>Integrality Theorem</u>: If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

<u>Capacity Property of S-T flow</u>: The flow value through edge e will not exceed the capacity of edge e and will greater than or equal to zero, i.e. $0 \le f(e) \le c(e)$.

Conservation Property of S-T flow: For all nodes v except source node and sink node, the total value of incoming flows is equal to the total value of outgoing flows.

In the flow network constructed above, firstly, the edge between source node S and tau node tau_1 and tau_2 has capacity tau_1 and tau_2 respectively, which corresponds to the constraint that for each forest i the total number of trees sold in all Y years shouldn't exceed tau_i . Secondly, All edges between tau node tau_i and forest node f_iy_j and all edges between forest node f_iy_j and year nodes have capacity $w_{i,j}$, which correspond to the constraint that at year j, forest i will have $w_{i,j}$ number of trees become matured. Thirdly, all the forest nodes f_iy_j are connected to year node from y_j to $y_{(j + deltaj - 1)}$, which correspond to the constraint that for trees matured in year j, it can only be sold from year j to year (j + $delta_j$ - 1), and fourthly, all edges between year node y_i and sink node T have capacity u_i , which corresponds to the constraint that for each year i, we can only sell at most u_i trees in order to prevent the effect of overselling which will crash the market.

Based on the above demonstration, I will prove the following two things:

A. If f is the value of the max flow then there exists a schedule that sells f Christmas trees.

Since there exists a max flow with value f, and this flow satisfies the capacity property, the conservation property and the Integrality theorem, therefore, this flow is a valid s-t flow and satisfies all tree selling constraints depicted by the capacities of edges in the flow network. Therefore, this flow can be depicted by a tree selling schedule and the flow values of each edges between forest node f_iY_j and year nodes are the number of trees we need to harvest from forest i at year j and sell in that schedule. Since that flow is the max flow, thus the corresponding selling schedule is the best schedule.

B. If the maximum number of Christmas trees sold is f then there exists a flow of value f.

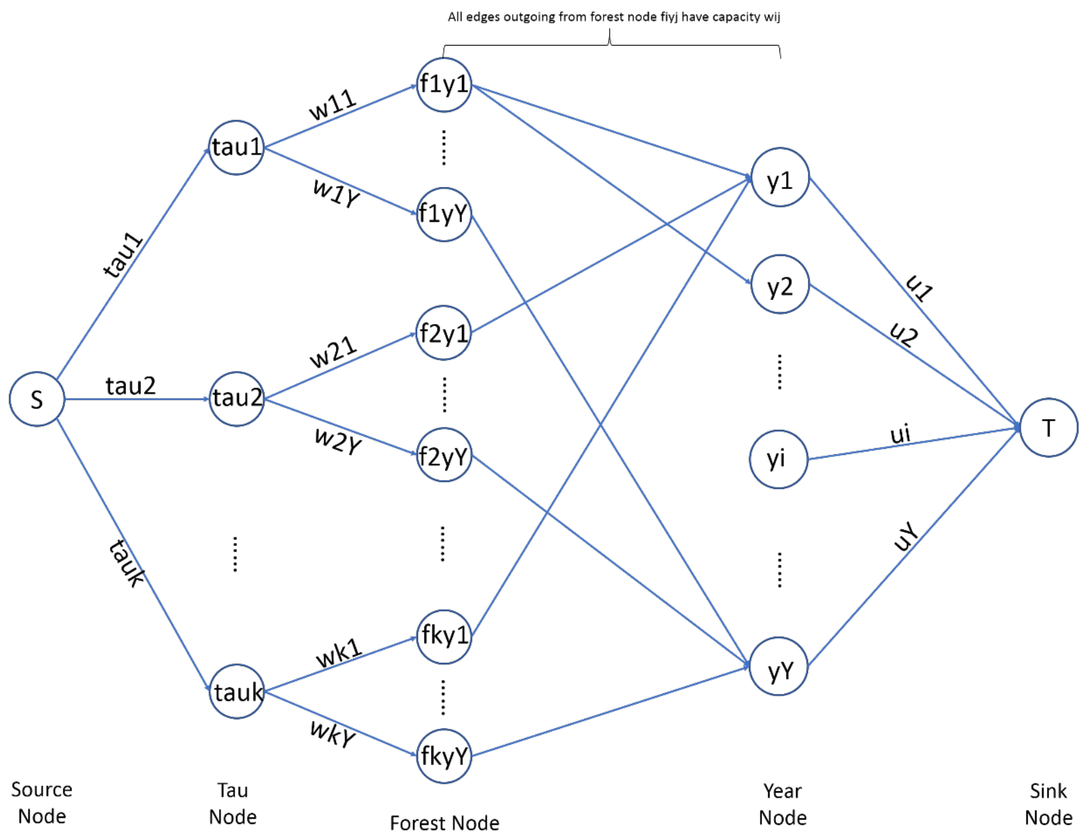
All the possible flows in this flow network satisfy the capacity property, the conservation property and the Integrality theorem. Since there is a selling schedule which sells maximum f trees, therefore, this schedule must satisfy all constraints, therefore, we can put the selling schedule into the corresponding edges in the flow network to construct a valid flow to depict this schedule. Therefore, there must exist a flow of value f.

Since both A and B holds, the maximum number of Christmas trees that can be sold is equal to the value of the max flow in the flow network.

Question 2

2.1 Formulate the problem as a network flow problem

I will illustrate this Max Flow Algorithm using the following flow network.



Note:

1. In the flow network above, S is the source, and T is the sink.
2. In the flow network above, $f_i y_j$ stands for node forest i year j.
3. All edges outgoing from forest node $f_i y_j$ has capacity $w_{i,j}$.
4. All the trees matured in the Year j can only be sold during Year j to Year j + $delta_j$ - 1. Hence, forest nodes $f_i y_j$ are connected to year node from y_j to $y_{j + deltaj - 1}$.

2.2 Argue the correctness of the algorithm

Note: The following theorem and properties will be used.

Integrality Theorem: If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

Capacity Property of S-T flow: The flow value through edge e will not exceed the capacity of edge e and will greater than or equal to zero, i.e. 0≤f(e)≤c(e).

Conservation Property of S-T flow: For all nodes v except source node and sink node, the total value of incoming flows is equal to the total value of outgoing flows.

In the flow network constructed above, firstly, the edge between source node S and tau node tau_i has capacity tau_i , which corresponds to the constraint that for each forest i the total number of trees sold in all Y years shouldn't exceed tau_i . Secondly, All edges between tau node tau_i and forest node f_iy_j and all edges between forest node f_iy_j and year nodes have capacity $w_{i,j}$, which correspond to the constraint that at year j, forest i will have $w_{i,j}$ number of trees become matured. Thirdly, all the forest nodes f_iy_j are connected to year node from y_j to $y_{(j + deltaj - 1)}$, which correspond to the constraint that for trees matured in year j, it can only be sold from year j to year (j + $delta_j$ - 1), and fourthly, all edges between year node y_i and sink node T have capacity u_i , which corresponds to the constraint that for each year i, we can only sell at most u_i trees in order to prevent the effect of overselling which will crash the market.

Based on the above demonstration, I will prove the following two things:

A. If f is the value of the max flow then there exists a schedule that sells f Christmas trees.

Since there exists a max flow with value f, and this flow satisfies the capacity property, the conservation property and the Integrality theorem, therefore, this flow is a valid s-t flow and satisfies all tree selling constraints depicted by the capacities of edges in the flow network. Therefore, this flow can be depicted by a tree selling schedule and the flow values of each edges between forest node f_iY_j and year nodes are the number of trees we need to harvest from forest i at year j and sell in that schedule. Since that flow is the max flow, thus the corresponding selling schedule is the best schedule.

B. If the maximum number of Christmas trees sold is f then there exists a flow of value f.

All the possible flows in this flow network satisfy the capacity property, the conservation property and the Integrality theorem. Since there is a selling schedule which sells maximum f trees, therefore, this schedule must satisfy all constraints, therefore, we can put the selling schedule into the corresponding edges in the flow network to construct a valid flow to depict this schedule. Therefore, there must exist a flow of value f.

Since both A and B holds, the maximum number of Christmas trees that can be sold is equal to the value of the max flow in the flow network.

2.3 Prove the upper bound of the time complexity

Loading the input data

Loading and storing the value of k and Y requires O(1) + O(1) = O(1) work.

Loading and storing the value of $delta_i$ into an array takes O(Y) time since there are Y delta values in total and each storing operation takes O(1) time.

Loading and storing the value of w_{ij} into an array takes O(k*Y) time since there are k*Y w_{ij} values in total and each storing operation takes O(1) time.

Loading and storing the value of u into an array takes O(Y) time since there are Y u values in total and each storing operation takes O(1) time.

Hence, loading and storing all data requires O(1) + O(Y) + O(k*Y) + O(Y) = O(k*Y) work.

Building the flow network

When building the flow network, we need to add Y edges between source node S and each tau node. We need to add k*Y edges between all tau nodes and all w_{ij} nodes since there are k forests and Y years in total. We need to add at most k*Y*Y edges between all forest nodes and all year nodes since for each forest node it can connect to at most Y year nodes and there are k*Y forest nodes in total. Finally, we need to add Y edges between all year nodes and the sink node t. Hence in total we need to add Y + k*Y + K*Y*Y + Y edges in total, and each add edge operation requires O(1) work. Therefore, in total building the flow network costs O(k*Y*Y) work.

Running the Ford-Fulkerson Algorithm

We assume without proof that the Ford-Fulkerson Algorithm has running time $O(m^2 logC)$, where *C* is the maximum flow in flow network *G*, in this case, is $\min(\sum_{i=1}^{k} taui\ ,\ \ \sum_{j=1}^{Y} uj\ ,$ $\sum_{i=1}^{k}\sum_{j=1}^{Y} Wi, j)$, and *m* is the number of edges in the flow network, in this case, is kY^2. Hence the Ford-Fulkerson Algorithm takes $O(k^2Y^4\log(\min(\sum_{i=1}^{k} taui\ ,\ \ \sum_{j=1}^{Y} uj\ ,$ $\sum_{i=1}^{k}\sum_{j=1}^{Y} Wi, j)))$ time.

Hence, the running time of the whole algorithm requires $O(kY)$ + $O(kY^2)$ + $O(k^2Y^4\log(\min(\sum_{i=1}^{k} taui\ ,\ \ \sum_{j=1}^{Y} uj,\ \sum_{i=1}^{k}\sum_{j=1}^{Y} Wi, j)))$ = $O(k^2Y^4\log(\min(\sum_{i=1}^{k} taui\ ,\ \ \sum_{j=1}^{Y} uj,\ \sum_{i=1}^{k}\sum_{j=1}^{Y} Wi, j)))$ work.