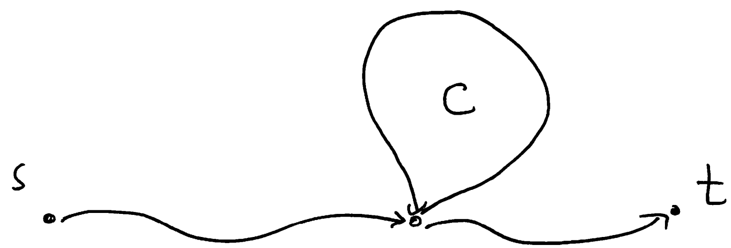
Shortest paths and negative weights

In some applications it is necessary to allow edges with negative weights

However, if we want distances to be well defined we cannot have a cycle with negative weight

E.g.,



If $w(C) > 0$ then drop cycle to get a better path

If $w(C) < 0$ then we decrease the weight by spinning around C

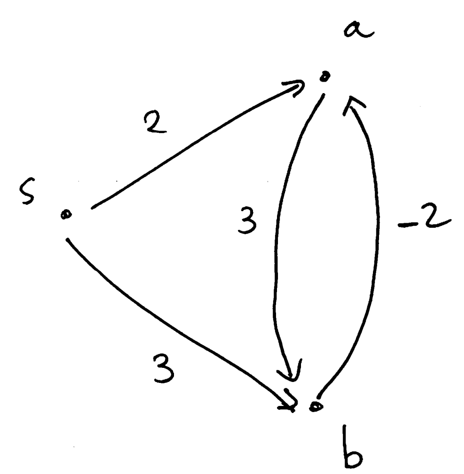
Input:

- Graph $G=(V,E)$ with $w: E \to \mathbb{R}$
- source s

Output:

- $dist(s,u)$ $\forall u \in V$, or
- determine that there is a negative cycle reachable from s.

Obs: Dijkstra fails to solve the problem

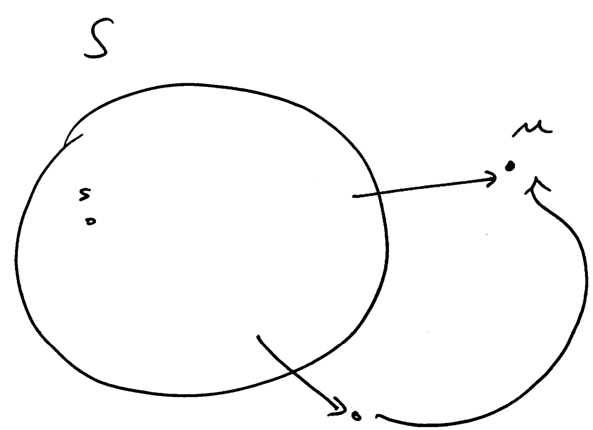


Dijkstra's output

dist(s, a) = 2 ← wrong!

dist(s, b) = 3

In the proof we used the fact that weights were positive



← this section has to be positive, which is not in the example

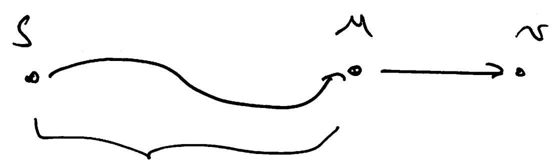
Dynamic Programming Solution

① Define subproblems

$M[v,i]$ = weight of shortest s-v path using $\leq i$ edges

② Derive recurrence

condition on the vertex that comes before v



an s-u path with $\leq i-1$ edges

$$\Rightarrow M[v,i] = w(u,v) + M[u,i-1]$$

③ Complexity analysis

\# DP states $= n^2$ ← space

each takes $= n$ time

total time $= O(n^3)$

Base case

$$M[v,0] = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{o.w} \end{cases}$$

Recursive case

$$M[v,i] = \min\left(M[v,i-1], \min_{u:(u,v)\in E}\left(M[u,i-1] + w(u,v)\right)\right)$$

③ Complexity Analysis

DP states $= n^2$

each takes $= O(n)$ time

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total time $= O(n^3)$

total space $= \Theta(n^2)$

Where is the solution?

$M[*, n-1]$?

what happened to negative cycles?

<u>Obs</u>: The graph has a negative cycle reachable from s iff

$\exists u : M[u, n-1] \neq M[u, n]$

$\boxed{\Leftarrow}$ Since $M[u,n-1] \neq M[u,n]$ then

$M[u,n] = M[u_{n-1}, n-1] + w(u_{n-1}, u)$

It must be that

$M[u_{n-1}, n-1] \neq M[u_{n-2}, n-2]$, so

$M[u_{n-1}, n-1] = M[u_{n-2}, n-2] + w(u_{n-2}, u_{n-1})$

$\vdots$

$s \to u_1 \to u_2 \; \dots \; \to u_n \to u$

there <u>must</u> be a negative cycle

$\Rightarrow$ If $M[u, n-1] = M[u,n] \; \forall u$

then $M[u, n-1] = M[u, \ell] \; \forall u \; \forall \ell \geq n$

contradicting $\exists$ neg-cycle

Improvements

- Time complexity analysis

  Filling all entries of the form $M[*, i]$ (for a fixed i) takes $O(n)$ time

  $\Rightarrow$ Filling all entries $M[*, *]$ takes $O(nm)$ time

- Space complexity

  To compute $M[*, i]$ we only need $M[*, i-1]$

  So just keep last two layers for $O(n)$ space

  ~~But how do we compute~~

- Even better keep track of a single layer

Bellman-Ford (V, E, ω, s)

$R[u] = \infty \quad \forall u \in V$
$R[s] = 0$
for $i = 1, \dots, n$
  for $(u, v) \in E$:
    if $R[u] + \omega(u,v) < R[v]$
      $R[v] = R[u] + \omega(u,v)$
      if $i == n$
        return "negative cycle"
return R

what the meaning of $R[u]$?

① $\exists$ s-u path with weight $R[u]$

② $R[u] \leq M[u, i]$ in iteration i