IT 5845
Mathematics for Artificial Intelligence

Lecture 3

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MSc in Artificial Intelligence

Chapter 1: Vectors

Learning Outcomes

By the end of the lecture, students will be able to;

- Explain what is meant by the span of a set of vectors both geometrically and algebraically.
- Determine the span of a set of vectors.
- Determine if a given vector is in the span of a set of vectors.
- Define linear independence.
- Determine whether a set of vectors is linearly dependent or linearly independent.
- Define Basis.

Spanning Sets

Linear Combination of Vectors

Definition

Given a nonempty finite set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \in \mathbb{R}^n$, a **linear combination** of these vectors is any vector of the form

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k,$$

where $a_1, a_2, \dots, a_k$ are scalars.

Special Set of Vectors

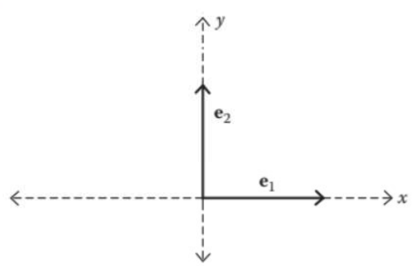
- In each space $\mathbb{R}^n$, there are special sets of vectors that play an important role in describing the space.
- For example, in $\mathbb{R}^2$ the vectors $\boldsymbol{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ have the significant property that *any* vector $\boldsymbol{v} = \begin{bmatrix}x\\y\end{bmatrix}$ can be written as a linear combination of vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$:

$$\boldsymbol{v} = \begin{bmatrix}x\\y\end{bmatrix} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2.$$

- In physics and engineering, the symbols $\boldsymbol{i}$ and $\boldsymbol{j}$ (or $\vec{i}$ and $\vec{j}$) are often used for $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively.

Special Set of Vectors (Continue...)

- If we extend (stretch) the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ using multiplication by positive and negative scalars, we get the usual x-axes and y-axes in the Euclidean plane.



Special Set of Vectors (Continue...)

- Similarly, any vector $\boldsymbol{w} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ in $\mathbb{R}^3$ can be expressed uniquely in terms of the vectors $\boldsymbol{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\boldsymbol{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$:

$$\boldsymbol{w} = \begin{bmatrix}x\\y\\z\end{bmatrix} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3.$$

- In applied courses, these three vectors are often denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ (or by $\vec{i}$, $\vec{j}$ and $\vec{k}$) respectively.

Special Set of Vectors (Continue...)

- Generalizing, any vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ can be written as a linear combination of the n vectors $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$:

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n.$$

- The special set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ will be used often in our analysis of $\mathbb{R}^n$. This set of vectors is a particular example of a spanning set for a particular space $\mathbb{R}^n$.
- The name spanning set is appropriate because such a set reaches across (spans) the entire space when all linear combinations of vectors in the set are formed.

Spanning Set

Definition

Given a nonempty set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \in \mathbb{R}^n$, the **span** of S, denoted by $span(S)$, is the set of all linear combinations of vectors from S:

$$span(S) = \{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k \mid a_i \in \mathbb{R}, \boldsymbol{v}_i \in S, i = 1, 2, \dots, k\}.$$

- It is also denoted by the notations $L(S)$ or $< S >$ or $[S]$.
- Note that some or all of the scalars may be zero.
- A nonempty set S of vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$ (or is a spanning set for $\mathbb{R}^n$) if every vector in $\mathbb{R}^n$ is an element of $span(S)$, that is, if every vector in $\mathbb{R}^n$ is a linear combination of vectors in S.
- Note that $span(S)$ is an infinite set of vectors unless $S = \{\boldsymbol{0}\}$, in which case $span(S) = S = \{\boldsymbol{0}\}$.

Exercise 6

Determine the span of the following vectors:

1. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Exercise 7

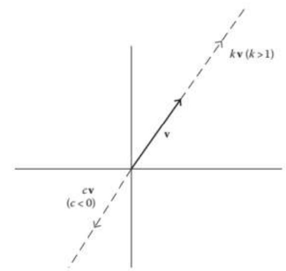
Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

Exercise 8

Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a spanning set of $\mathbb{R}^3$.

The Span of Non-Zero Vectors in $\mathbb{R}^2$

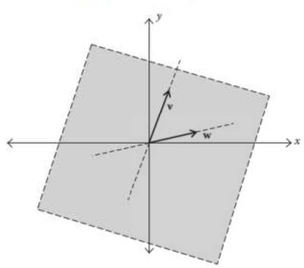
- Given a single nonzero vector $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, the span of $\{\boldsymbol{v}\}$ is the set of all scalar multiples of $\boldsymbol{v}$: $span(\{\boldsymbol{v}\}) = \left\{ a\boldsymbol{v} = \begin{bmatrix} ax \\ ay \end{bmatrix} : \ a \in \mathbb{R} \right\}$.
- Because $a = 0$ is a possibility, the origin is in the span and we can interpret the span of $\{\boldsymbol{v}\}$ as the set of all points on a straight line through the origin



- The slope of this line depends on the original coordinates of $\boldsymbol{v}$: Slope $= ay/ax = y/x$, assuming that $a \neq 0$ and $x \neq 0$. If $x = 0$, the span is just the y-axis, whereas if $y = 0$, the span is the x-axis.

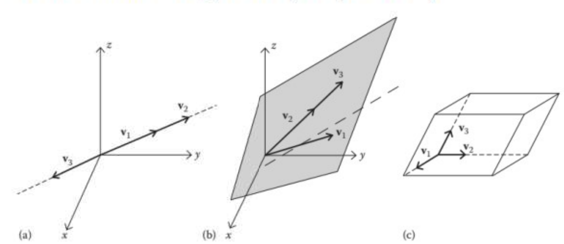
The span of linearly independent vectors in $\mathbb{R}^2$

- Suppose we have two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$.
- If one of these vectors is a scalar multiple of the other, then span$(\{\boldsymbol{v}, \boldsymbol{w}\})$ = span$(\{v\})$ = span$(\{w\})$, again a straight line through the origin. On the other hand, if the vectors do not lie on the same straight line through the origin, their span is all of $\mathbb{R}^2$.



The Span of Non-Zero Vectors in $\mathbb{R}^3$

- In $\mathbb{R}^3$ the span of a single nonzero vector is a line through the origin in a three-dimensional space.
- Given two nonzero vectors in $\mathbb{R}^3$, their span is either a line through the origin or a plane passing through the origin, depending on whether the vectors are scalar multiples of each other.
- Finally, three nonzero vectors in $\mathbb{R}^3$ span a line, a plane, or all of $\mathbb{R}^3$, depending on whether all three of the vectors lie on the same straight line through the origin, only two of the vectors are collinear, or no two vectors lie on the same straight line, respectively.



Linear Independence

Linear Independence

- Let us look at the set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ in $\mathbb{R}^3$, where $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -8 \\ 4 \\ 9 \end{bmatrix}$, and $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.
- If we analyze the relationships among the vectors in S, we might discover that $\boldsymbol{v}_2$ is a linear combination of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$:

$$\begin{bmatrix} -8 \\ 4 \\ 9 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$\boldsymbol{v}_2 = 2\boldsymbol{v}_1 - 3\boldsymbol{v}_3$$

- This algebraic relationship can also be written as $-2\boldsymbol{v}_1 + \boldsymbol{v}_2 + 3\boldsymbol{v}_3 = \boldsymbol{0}$.
- We say that this set of vectors S is linearly dependent because one vector can be expressed as a linear combination of other vectors in the set.

Linear Independence (Continue...)

Definition

A nonempty finite set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ in $\mathbb{R}^n$ is called **linearly independent** if the only way that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k = \boldsymbol{0},$$

where $a_1, a_2, \dots, a_k$ are scalars, is if $a_1 = a_2 = \cdots = a_k = 0$. Otherwise, S is called **linearly dependent**.

Remark

- Note that any set containing the zero vector cannot be linearly independent.
- If we speak somewhat loosely and say that vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ are linearly independent (or dependent), we mean that the set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is linearly independent (or dependent).

Exercise 9

Determine whether the following set of vectors is linearly independent in $\mathbb{R}^3$.

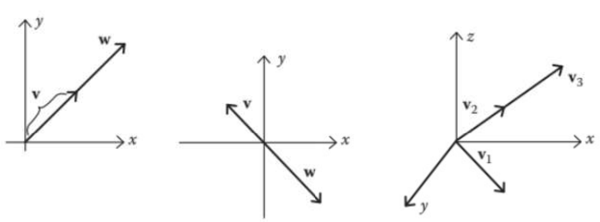
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

Examples of Linearly Dependent Sets of Vectors

- The vectors $\begin{bmatrix}1 & -1 & 1 & 2 & 1\end{bmatrix}^T$, $\begin{bmatrix}4 & -1 & 6 & 6 & 2\end{bmatrix}^T$, $\begin{bmatrix}-4 & -2 & -3 & -4 & -2\end{bmatrix}^T$ and $\begin{bmatrix}-2 & -1 & 1 & -2 & -2\end{bmatrix}^T$ are linearly dependent in $\mathbb{R}^5$.
- The vectors $\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$, $\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^T$, $\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$, $\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$ and $\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}^T$ linearly dependent in $\mathbb{R}^4$.

Linearly Independent Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Geometrically, in $\mathbb{R}^2$ or $\mathbb{R}^3$, a set of vectors is linearly independent if and only if no two vectors (viewed as emanating from the origin) lie on the same straight line.



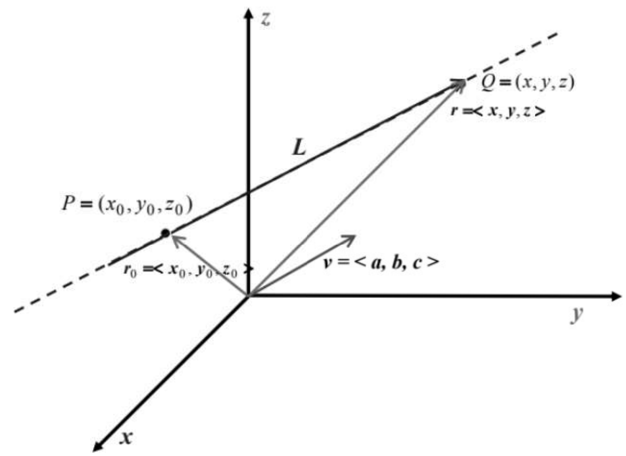
Spanning Sets and Linearly Independent Sets in $\mathbb{R}^3$

1. The set $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$ but is not linearly independent.
2. The set $B = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} \right\}$ is linearly independent but does not span $\mathbb{R}^3$.
3. The set $C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent and also spans $\mathbb{R}^3$.
4. The set $D = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent and does not span $\mathbb{R}^3$.

Vectors in Space Geometry

The Equation of a Straight Line Passing Through a Given Point and Parallel to a Given Vector

Consider the line L through the point $P(x_0, y_0, z_0)$ that is parallel to the vector $\boldsymbol{v}$.



Parametric Equations of a Line

The vector equation of a line in $\mathbb{R}^3$ is given by the equation

$$\boxed{\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v},}$$

where $\boldsymbol{r}_0 = \langle x_0, y_0, z_0\rangle$ is a vector whose components are made of the point (x_0, y_0, z_0) on the line L and $\boldsymbol{v} = \langle a, b, c\rangle$ are components of a vector that is parallel to the line L. The vector $\boldsymbol{v} = \langle a, b, c\rangle$ is called the direction vector for the line L and its components a, b, and c are called the direction numbers.

Symmetric Equations of a Line

The symmetric equations of a line L in $\mathbb{R}^3$ space are given by

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.}$$

where (x_0, y_0, z_0) is a point passing through the line and $\boldsymbol{v} = \langle a, b, c\rangle$ is a vector that the line is parallel to. The vector $\boldsymbol{v} = \langle a, b, c\rangle$ is called the direction vector for the line L and its components a, b, and c are called the direction numbers.

Exercise 12

Find a vector equation for the line through $(4, 6, -3)$ and parallel to $\langle 5, -10, 2 \rangle$.

Exercise 13

Find a vector equation for the line though $(2, -1, 8)$ and $(5, 6, -3)$.