IT 5845
Mathematics for Artificial Intelligence

Lecture 4

Dr. Thilini Piyatilake

Department of Computational Mathematics
University of Moratuwa

thilinisp@uom.lk

MSc in Artificial Intelligence

Chapter 2: Matrix Algebra

Learning Outcomes

By the end of the lecture, students will be able to;

- identify the basic definitions of matrices.
- perform the matrix arithmetic operations.
- apply arithmetic operations on matrices to solve real world problems.
- perform the matrix operations.
- obtain the determinant of a square matrices.
- describe some basic properties of determinants.
- find the inverse of a matrix.

Matrix Operations

Order of a Matrix

The **number of rows and columns** that a matrix has is called its order or its dimension.

Eg:

$$\begin{pmatrix} 9 & -1 & 4 & 19 \\ 8 & 15 & 20 & 5 \\ 19 & 4 & 4 & 10 \end{pmatrix}_{3\times 4} \qquad \Rightarrow order\ \ 3\times 4$$

General Representation of a Matrix

A **rectangular array** of numbers of the form

$$A = \begin{pmatrix} a_{11} & \cdots & \boxed{a_{1j}} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

is called a matrix, with m rows and n columns.

Equality of Matrices

For two matrices to be equal, they must have

1. the same order.
2. identical elements in the corresponding positions.

Column Matrix

A matrix which has just only one column is a **column matrix**.

Eg:-

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}_{3\times 1}$$

Row Matrix

A matrix which has just only one row is a **row matrix**.

Eg:- $\begin{pmatrix} 6 & 0 & 8 \end{pmatrix}_{1\times 3}$

Square Matrix

Any matrix in which the

$$\boxed{Number\ of\ Rows = Number\ of\ Columns,}$$

is called a **square matrix**.

Eg:- $\begin{pmatrix} 2 & 0 & 4 \\ 4 & 7 & 2 \\ 1 & 9 & 3 \end{pmatrix}_{3\times 3}$

Order of a Square Matrix

In square matrix if

$$\boxed{Number\ of\ Rows = Number\ of\ Columns = n,}$$

it is termed as n^{th} order matrix.

Zero Matrix

Any matrix in which every element is zero is a **zero matrix**.

Notation: $\mathbf{0}$

Eg:- $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$

Diagonal Matrix

A square matrix whose elements are zero, **except the principal (main) diagonal elements**, is a **diagonal matrix**.

Notation: $\boldsymbol{A} = diag\left[1, -4, \frac{1}{2}\right]_3$

Eg:-

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}_{3\times 3}$$

Identity Matrix

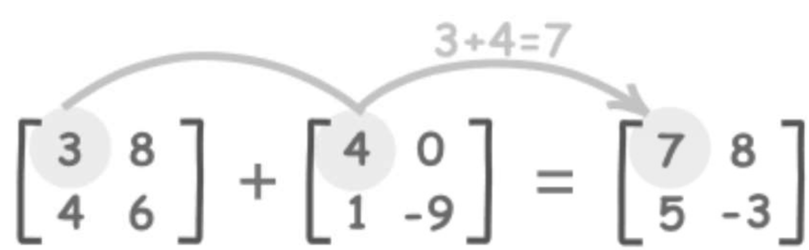
The diagonal matrix with all diagonal elements are 1 (or unity) is called the **identity or unit matrix** of order n.

$$\text{Notation: } \boldsymbol{I_n} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}_{n\times n} = diag\,[1, 1, \cdots, 1]_n$$

Matrix Operations

Addition of Matrices

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of the same order $m \times n$, then their sum $(\boldsymbol{A}+\boldsymbol{B})$ is defined to be the matrix of the order $m \times n$ obtained by adding the corresponding elements of $\boldsymbol{A}$ and $\boldsymbol{B}$.

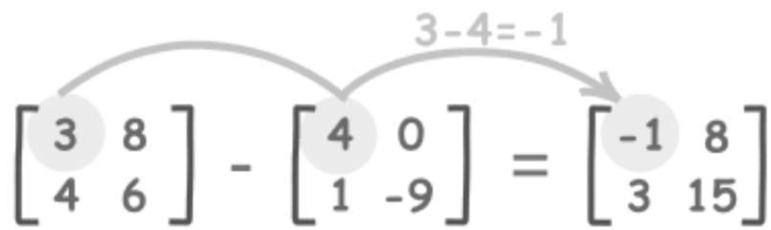


Properties of Matrix Addition

1. Matrix addition is commutative.
   $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
2. Matrix addition is associative.
   $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$
3. For any $\boldsymbol{A}$, $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$.
4. For any $\boldsymbol{A}$, there exist $-\boldsymbol{A}$ such that $\boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{0}$.

Subtraction of Matrices

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices of the same order $m \times n$, then their subtraction $(\boldsymbol{A} - \boldsymbol{B})$ is defined to be the matrix of the order $m \times n$ obtained by subtracting the corresponding elements of $\boldsymbol{A}$ and $\boldsymbol{B}$.



Scalar Multiplication

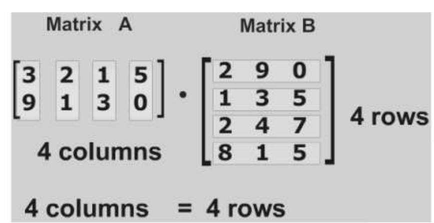
Let k be a scalar and $\boldsymbol{A}$ be the matrix of order $m \times n$. Then the order $m \times n$ matrix obtained by multiplying every element of matrix $\boldsymbol{A}$ by k is called the scalar multiple of $\boldsymbol{A}$ by k.

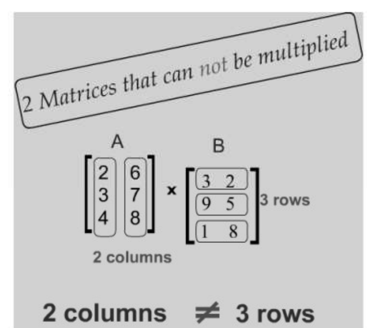
Eg:- Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$, find $4\boldsymbol{A}$.

$$4\boldsymbol{A} = 4\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 4\cdot 2 & 4\cdot 1 \\ 4\cdot 3 & 4(-2) \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 12 & -8 \end{pmatrix}$$

Matrix Multiplication

- You can multiply two matrices if, and only if, the number of columns in the first matrix **equals** the number of rows in the second matrix.

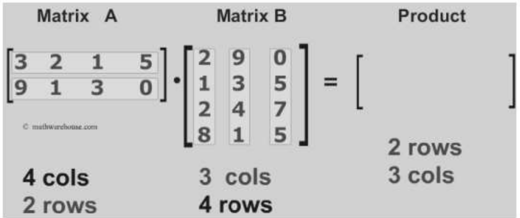




Product of Two Matrices (Continue...)

- What is the order of the resultant matrix?
  The resultant matrix order is

$$(\text{rows of first matrix}) \times (\text{columns of the second matrix}).$$



- How do we multiply two matrices?
  - Make sure that the number of columns in the first matrix equals the number of rows in the second matrix.
  - Multiply the elements of each row of the first matrix by the elements of each column in the second matrix.
  - Add the products.

Exercise 3

a. If $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \\ -6 & 1 \end{pmatrix}_{3\times 2}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}_{2\times 3}$.
   Find $\mathbf{AB}$.

b. If $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}_{1\times 3}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 4 \\ 2 & -2 & 0 \end{pmatrix}_{3\times 3}$.
   Find $\boldsymbol{AB}$.

Properties of a Matrix Multiplication

Theorem

Let $\boldsymbol{A}$ be an $m \times n$ matrix, and let $\boldsymbol{B}$ and $\boldsymbol{C}$ have sizes for which the indicated sums and products are defined.

1. **Associative law of multiplication**: $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
2. **Left distributive law**: $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$
3. **Right distributive law**: $(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A} = \boldsymbol{BA}+\boldsymbol{CA}$
4. $r(\boldsymbol{AB}) = (r\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(r\boldsymbol{B})$ for any scalar r.
5. **Identity for matrix multiplication**: $\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n$

Remark

- In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$.
- The cancellation laws do not hold for matrix multiplication. That is, if $\boldsymbol{AB} = \boldsymbol{AC}$, then it is **not** true in general that $\boldsymbol{B} = \boldsymbol{C}$.
- If a product $\boldsymbol{AB}$ is the zero matrix, you **cannot** conclude in general that either $\boldsymbol{A} = 0$ or $\boldsymbol{B} = 0$.

Transpose of a Matrix

The transpose of a matrix is one in which the rows and columns are interchanged. Transpose of $\boldsymbol{A}$ is denoted by A^T or A'.

$$[A]=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\\ 1 & 0 & 1\end{bmatrix}_{4\times3} \Rightarrow [A^T]=\begin{bmatrix}1 & 4 & 7 & 1\\ 2 & 5 & 8 & 0\\ 3 & 6 & 9 & 1\end{bmatrix}_{3\times4}$$

Properties of Transposition

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote matrices whose sizes are appropriate for the following sums and products.

1. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
2. $(\boldsymbol{A}+\mathbf{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
3. For any scalar r, $(r\boldsymbol{A})^T = r\boldsymbol{A}^T$
4. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

Determinants

Determinant of a Matrix

Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}_{n\times n}$ be a $n \times n$ square matrix.

The determinant of $\mathbf{A}$ is denoted by $\det\ \mathbf{A}$ or $|\mathbf{A}|$ and write,

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}.$$

Determinant of Order 2

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

In other words,

$$\begin{bmatrix} \text{Product of the elements along} \\ \text{the principal diagonal} \end{bmatrix} - \begin{bmatrix} \text{Product of the elements along} \\ \text{the off-diagonal} \end{bmatrix}.$$

Determinant of Order 3

To generalize the definition of the determinant to larger matrices, we'll use 2×2 determinants to rewrite the 3×3 determinant described above.

$$\begin{aligned} |\boldsymbol{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &+ a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

For brevity, write

$$|\boldsymbol{A}| = a_{11} \cdot \text{ det } \boldsymbol{A}_{11} - a_{12} \cdot \text{ det } \boldsymbol{A}_{12} + a_{13} \cdot \text{ det } \boldsymbol{A}_{13},$$

where $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{12}$, and $\boldsymbol{A}_{13}$ are obtained from $\boldsymbol{A}$ by deleting the first row and one of the three columns.

Determinant of Order n Matrix

Definition

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $\boldsymbol{A} = [a_{ij}]$ is the sum of n terms of the form a_{1j} det $\boldsymbol{A}_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12} \ldots a_{1n}$ are from the first row of $\boldsymbol{A}$. In symbols,

$$\begin{aligned} \text{det}\boldsymbol{A} &= a_{11} \text{ det } \boldsymbol{A}_{11} - a_{12} \text{ det } \boldsymbol{A}_{12} + \cdots + (-1)^{1+n} a_{1n} \text{ det } \boldsymbol{A}_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \text{ det } \boldsymbol{A}_{1j} \end{aligned}$$

Properties of Determinants

Row Operations

The secret of determinants lies in how they change when row operations are performed.

Theorem - Row Operations

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
2. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Column Operations

We can perform operations on the columns of a matrix in a way that is analogous to the row operations we have considered. We can identify the column operations have the same effects on determinants as row operations. Therefore, if A is an $n \times n$ matrix, then

$$\boxed{\det A^T = \det A.}$$

Multiplicative Property

If A and B are $n \times n$ matrices, then

$$\boxed{\det AB = (\det A)(\det B).}$$