

LECTURE 7

Interest Rate Models I: Short Rate Models

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5. Lecture V: Interest Rate Models I: Short Rate Models

The earliest interest rate models took as their starting point a stochastic model for the short rate, or instantaneous interest rate, r_t defined as the rate of interest for the (infinitesimal) interval $[t, t + dt]$:

$$(106) \quad r_t dt = \text{total interest gained in } [t, t + dt].$$

In practice, one takes the yield in a 1 month US Treasury bill, or a comparable sort-maturity bond, as a proxy for the sort rate.

Starting point of these sort rate models is a SDE for r_t :

$$(107) \quad dr_t = \alpha(r_t, t)dt + \sigma(r_t, t)dW_t,$$

for given coefficients $\alpha(r, t)$ and $\sigma(r, t)$.

Examples 5.1. Important examples of short rate models are:

(i) The *Vasicek model* (historically the first):

$$(108) \quad dr_t = \alpha(\theta - r_t)dt + \sigma dW_t.$$

Observe that the Vasicek-model is *mean reverting* (since it is simply an Ornstein-Uhlenbeck process: cf. Math. Methods I), which is economically reasonable (interest rates should fluctuate along a long term mean equilibrium rate, determined by economic equilibrium between demand and supply), and a common feature of all short rate models. A disadvantage of the Vasicek model is that interest rates can become negative, which is economically undesirable (why?), although it does seem to happen every once and then.

(ii) The *Cox, Ingersoll and Ross model*, or CIR-model:

$$(109) \quad dr_t = \alpha(\theta - r_t)dt + \sigma\sqrt{r_t} dW_t.$$

The CIR-model has the advantage that the short rate will stay positive (≥ 0), with probability 1.

(iii) The *Longstaff model*:

$$(110) \quad dr_t = \alpha(\theta - \sqrt{r_t})dt + \sigma\sqrt{r_t} dW_t,$$

a.k.a. the double square root process.

(iv) *Hull and White's generalized Vasicek model*, in which one allows the coefficients to be (deterministic) functions of time:

$$(111) \quad dr_t = \alpha(t)(\theta(t) - r_t)dt + \sigma(t)dW_t.$$

A disadvantage of the earlier models is that they have too little parameters to be able to closely fit the initial yield curve at a given time. Making the coefficients time-dependent (even one of them, usually θ) enables one to have an exact fit. One can similarly introduce time-dependent coefficients in the CIR or in Longstaff's model.

(iv) The *lognormal model*, (Black, Derman and Toy, Sandmann and Sondermann):

$$(112) \quad d \log r_t = \alpha(t)(\theta(t) - \log r_t)dt + \sigma(t)dW_t,$$

with deterministic time-dependent coefficients $\alpha(t)$, $\theta(t)$ and $\sigma(t)$. Interest rates clearly stay positive, and are mean-reverting, but prices of certain instruments can become infinite, which is a drawback.

(v) The *Ahn and Gao model*:

$$(113) \quad dr_t = \alpha(\theta - r_t)r_t dt + \sigma r_t^{3/2} dW_t,$$

where one could allow for deterministic time-dependent coefficients also.

The first aim of a short rate model (indeed, of any interest rate model) is to price zero-coupon bonds. A *zero-coupon bond* (also called a *discount bond*) is a bond which does not pay any coupons, but which pays its nominal-, or face-, value at maturity T . The face value will usually be normalized to 1 (of whatever currency we're working in: Sterling, \$, Euro, etc.). We will let

$$P_{t,T},$$

be the price at $t \leq T$ of a 0-coupon with maturity T and face value $P_{T,T} = 1$. We will suppose that there exist 0-coupon bonds of all possible maturities $T \geq 0$. This is of course not entirely realistic: first of all, there are only a finitely many maturities traded, and secondly, the longer-maturity bonds will pay coupons; however, these can be decomposed ('stripped') into a series of 0-coupon bonds with maturities corresponding to the coupon dates. Note, that having 0-coupon bonds of all maturities $T \geq 0$ means that we are dealing with a market with infinitely many (in practice, a very large number of) assets.

In a second stage, we would also like to price options on bonds, like simple calls and puts on zero-coupon bonds of a given maturity. The fact that interest rates are stochastic, and closely correlated with the pay-offs will make the pricing of such options more complicated than for options on simple stock.

The short rates themselves are not directly traded, only the (zero-) coupon bonds are. This makes the short rate models effectively incomplete: the basic risk-factor r_t is not a tradable, and we can only trade r_t , which is basically a European derivative on r_t , with a pay-off of 1 at maturity. In particular, we cannot hedge $P_{t,T}$ by buying or selling units of r_t , we can only hedge a bond of a given maturity by buying/selling bonds of another maturity. This will lead to the introduction of a *market price of risk*, which makes up for the difference in drift between the stochastic evolution of the interest rate with respect to the objective (or physical) probability, and the risk-neutral one. Its origin is probably most easily explained using the 'risk-less portfolio'

approach from PDE-style pricing, so we (uncharacteristically, for these lectures) do this approach first.

5.1. Derivation of the Bond price equation. . We suppose that the price $P_{t,T}$ is a function of time t and the short-rate r_t at t :

$$P_{t,T} = p(r_t, t; T).$$

(This can be a posteriori justified). An standard application of Ito's lemma then yields that

$$(114) \quad dP_{t,T} = a_{t,T}P_{t,T}dt + b_{t,T}P_{t,T}dW_t, \quad t < T,$$

with

$$(115) \quad b_{t,T} = \frac{\sigma(r_t, t)}{p(r_t, t)} \frac{\partial p}{\partial r},$$

and

$$(116) \quad a_{t,T} = \frac{1}{p(r_t, t)} \left(\frac{\partial p}{\partial t} + \alpha(r_t, t) \frac{\partial p}{\partial r} + \frac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 p}{\partial r^2} \right),$$

all derivatives of p evaluated in (r_t, t) .

Now consider two zero-coupon bonds having maturities $T_1 < T_2$. We try to make a locally risk-free portfolio

$$V_t = \Delta_{1,t}P_{t,T_1} - \Delta_{2,t}P_{t,T_2}.$$

Computing the change in value between t and $t + dt$ as

$$\begin{aligned} dV_t &= \Delta_{1,t} dP_{t,T_1} - \Delta_{2,t} dP_{t,T_2} = \\ &(\Delta_{1,t}a_{t,T_1}P_{t,T_1} - \Delta_{2,t}a_{t,T_2}P_{t,T_2}) dt + (\Delta_{1,t}b_{t,T_1}P_{t,T_1} - \Delta_{2,t}b_{t,T_2}P_{t,T_2}) dW_t, \end{aligned}$$

we see that the risk over $[t, t + dt]$ vanishes if:

$$\frac{\Delta_{1,t}}{\Delta_{2,t}} = \frac{b_{t,T_2}P_{t,T_2}}{b_{t,T_1}P_{t,T_1}}.$$

We can take, for example,

$$\Delta_{1,t} = b_{t,T_2}P_{t,T_2}, \quad \Delta_{2,t} = b_{t,T_1}P_{t,T_1}.$$

Since such a risk-free portfolio can only earn the risk-less rate, r_t , over $[t, t + dt]$, we must have, for these Δ , that

$$(\Delta_{1,t}a_{t,T_1} - \Delta_{2,t}a_{t,T_2}) dt = r_t (\Delta_{1,t}P_{t,T_1} - \Delta_{2,t}P_{t,T_2}) dt,$$

or

$$(a_{t,T_1}b_{t,T_2} - a_{t,T_2}b_{t,T_1}) P_{t,T_1}P_{t,T_2} = r_t (b_{t,T_2} - b_{t,T_1}) P_{t,T_1}P_{t,T_2}.$$

Re-arranging, this gives:

$$\frac{a_{t,T_1} - r_t}{b_{t,T_1}} = \frac{a_{t,T_2} - r_t}{b_{t,T_2}}.$$

In other words,

$$\frac{a_{t,T} - r_t}{b_{t,T}} = \frac{\text{return on } P_{t,T} - r_t}{\text{volatility of } P_{t,T}}$$

is *independent* of the maturity T and can therefore only be a function of r_t and t , say $q(r_t, t)$. This function $q = q(r, t)$ is called the *market price of risk*: using the standard deviation as a measure of financial risk, we have that

- $q(r, t) = \text{extra return per unit of risk an investor requires to hold the bond } P_{t,T} \text{ at time } t, \text{ if } r_t = r.$

(for return on $P_{t,s} = r_t + q(r_t, t) \cdot (\text{volatility of } P_{t,s})$.)

Inserting formulas (115) and (116) into the relation

$$\frac{a_{t,T} - r_t}{b_{t,T}} = q(r, t),$$

we find:

Proposition 5.2. *The bond pricing function $p(r, t; T)$ satisfies the following PDE:*

$$(117) \quad \frac{\partial p}{\partial t} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 p}{\partial r^2} + (\alpha(r, t) - q(r, t) \sigma(r, t)) \frac{\partial p}{\partial r} = r_t p, \quad t < T,$$

with final value

$$(118) \quad p(r, T; T) = 1.$$

So we have derived a boundary value problem for the bond-price, and all that is left is to solve it for the various models proposed. This can be done either by attacking the PDE directly, or by using Feynman and Kac. As an example we look at the extended Vasicek model of Hull and White.

5.2. Solving the Vasicek equation: analytic approach. We take $a(r, t) = \alpha(\theta - r)$ and $\sigma(r, t) = \sigma$, with α, θ and σ constants, so that

$$dr_t = \alpha(\theta - r_t)dt + \sigma dW_t.$$

For the market price of risk we simply choose a constant: $q(r, t) = q$ a constant. Note that in that case, $\alpha(r, t) - q(r, t)\sigma(r, t) = \alpha(\theta - r) - q\sigma = \alpha(\theta^* - r)$, with $\theta^* = \theta - q\sigma/\alpha$. The bond price equation becomes:

$$(119) \quad \frac{\partial P}{\partial t} + \alpha(\theta^* - r) \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} = rP, \quad \text{on } t < T,$$

which we have to solve with boundary condition (118). To do this analytically, we look for solutions having the special form:

$$(120) \quad e^{A(t;T) - rB(t;T)}.$$

To satisfy the boundary condition, we have to have

$$(121) \quad A(T; T) = 0, \quad B(T; T) = 0.$$

Substituting (120) into (119) gives:

$$(122) \quad \frac{\partial A}{\partial t} - \alpha\theta^*B + \frac{\sigma^2}{2}B^2 = r \cdot \left(\frac{\partial B}{\partial t} - \alpha B + 1 \right)$$

Note that, for fixed maturity T , A and B are functions of t only. Deriving both sides with respect to r gives:

$$(123) \quad \frac{\partial B}{\partial t} - \alpha B + 1 = 0,$$

and, upon substituting this again in (122):

$$(124) \quad \frac{\partial A}{\partial t} - \alpha\theta^*B + \frac{\sigma^2}{2}B^2 = 0.$$

General solution of (123):

$$Ce^{\alpha t} + \frac{1}{\alpha}.$$

By the second equation of (121), this has to be 0 for $t = s$. It follows that $C = e^{-\alpha s}/\alpha$, and therefore:

$$(125) \quad B(t, T) = \frac{1}{\alpha} (1 - e^{\alpha(t-T)}).$$

From this and (124) and (122), A can be found by integration:

$$(126) \quad A(t; T) = -\alpha\theta^* \int_t^T B(\tau, T) d\tau + \frac{\sigma^2}{2} \int_t^T B(\tau, T)^2 d\tau$$

Observe that these integrals are automatically 0 if $t = T$; the extra minus-sign is explained by the fact that differentiating an integral w.r.t. the lower bound of the integration domain gives *minus* the integrand.)

After some calculations one finds that $A(t; T)$ can be written as:

$$A(t; T) = \left(\theta^* - \frac{\sigma^2}{2\alpha^2} \right) (t-T) + \frac{1}{\alpha} (1 - e^{\alpha(t-T)}) \left(\theta^* - \frac{\sigma^2}{2\alpha^2} \right) - \frac{\sigma^2}{4\alpha^3} (1 - e^{\alpha(t-T)})^2.$$

Remembering (120) and (125), and introducing the quantity:

$$R_\infty = \theta^* - \frac{\sigma^2}{2\alpha^2},$$

we find that:

$$(127) \quad p^{\text{Vasicek}}(r, t; T) = \exp \left(R_\infty(t-s) + \frac{1}{\alpha} (1 - e^{\alpha(t-T)}) (R_\infty - r) - \frac{\sigma^2}{4\alpha^3} (1 - e^{\alpha(t-T)})^2 \right).$$

5.3. Solving the Vasicek equation: probabilistic approach. We know, by the Feynman-Kac theorem (see Math. Methods I), that the solution to (119) with boundary value $p(r, T; T) = F(r)$ is given by:

$$(128) \quad p(r, t, ; T) = \mathbb{E} \left(\exp \left(- \int_t^T \hat{r}_\tau d\tau \right) F(\hat{r}_T, T) \mid \hat{r}_t = r \right),$$

where the hatted, ‘risk-adjusted’ process follows the SDE:

$$d_t \hat{r}_t = \alpha(\theta^* - \hat{r})dt + \sigma dW_t.$$

This is again an Ornstein-Uhlenbeck SDE, whose solution we know is for times between t and T , with initial value r at t , is given by:

$$(129) \quad \hat{r}_\tau = \theta (1 - e^{-\alpha(\tau-t)}) + r e^{-\alpha(\tau-t)} + \sigma \int_t^\tau e^{\alpha(s-\tau)} dW_s.$$

In particular, we know that r_τ is Gaussian, whose mean and variance we computed in Math. Methods I. Applying (128) with $F(r) = 1$ gives

$$p(r, t; T) = \mathbb{E} \left(\exp \left(- \int_t^T \hat{r}_\tau d\tau \right) \mid \hat{r}_t = r \right),$$

and to be able to proceed, we have to analyze the integrated process

$$I_T = \int_t^T \hat{r}_\tau d\tau.$$

Inserting (129), and changing order of integration, we find that

$$\begin{aligned} I_T &= \int_t^T \theta (1 - e^{-\alpha(\tau-t)}) d\tau + \int_t^T r e^{-\alpha(\tau-t)} d\tau + \sigma \int_t^T \int_t^\tau e^{\alpha(s-\tau)} dW_s d\tau \\ &= \theta(T-t) - \frac{\theta}{\alpha}(1 - e^{-\alpha(T-t)}) + \frac{r}{\alpha}(1 - e^{-\alpha(T-t)}) + \sigma \int_t^T dW_s \int_s^T e^{\alpha(s-\tau)} d\tau \\ &= \theta(T-t) - \frac{\theta}{\alpha}(1 - e^{-\alpha(T-t)}) + \frac{r}{\alpha}(1 - e^{-\alpha(T-t)}) + \frac{\sigma}{\alpha} \int_t^T (1 - e^{\alpha(s-T)}) dW_s. \end{aligned}$$

It follows from the theory of Ito integrals with deterministic integrands that I_T is again Gaussian, with mean

$$\mu_I = \theta(T-t) - \frac{\theta}{\alpha}(1 - e^{-\alpha(T-t)}) + \frac{r}{\alpha}(1 - e^{-\alpha(T-t)}),$$

and variance

$$\begin{aligned} \sigma_I^2 &= \frac{\sigma^2}{\alpha^2} \mathbb{E} \left(\left(\int_t^T (1 - e^{\alpha(s-T)}) dW_s \right)^2 \right) \\ &= \frac{\sigma^2}{\alpha^2} \int_t^T (1 - 2e^{\alpha(s-T)} + e^{2\alpha(s-T)}) ds \\ &= \frac{\sigma^2}{\alpha^2} (T-t) - 2 \frac{\sigma^2}{\alpha^3} (1 - e^{-\alpha(T-t)}) + \frac{\sigma^2}{2\alpha^3} (1 - e^{-2\alpha(T-t)}). \end{aligned}$$

Now we know from Math Methods I (and this is also easily verified directly), that if $I \sim N(\mu_I, \sigma_I^2)$, then

$$\mathbb{E}(e^{-I}) = e^{-\mu_I + \sigma_I^2/2}.$$

Substituting the expressions we obtained for μ_I and σ_I^2 , and re-arranging, we find once more formula (127) for $p = p^{\text{Vasicek}} = \mathbb{E}(\exp(I_T))$.

The Vasicek model has three parameters, α, θ^* and σ with which to fit bond prices of all maturities, which there are in principle an infinite number, and one doesn't always get a good fit. This difficulty can be circumvented by letting the coefficients depend deterministically on t : this will allow us to fit at least any initial yield curve:

5.4. Hull and White's extended Vasicek model. We look at the simplest variant, with α and σ constant, and $\theta^* = \theta^*(t)$ a deterministic function of t . Solving the new bond price equation,

$$\frac{\partial p}{\partial t} + \alpha(\theta^*(t) - r) \frac{\partial p}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial r^2} = rp, \text{ on } t < T,$$

using for example the direct, analytical method, and looking for solutions of the same form (120), (121) as before, we find that $A(t; T)$ and $B(t; T)$ have to satisfy the same equations (123), (124), but the latter of course with $\theta^*(t)$. $B(t, T)$ will be given by the same function (125) as before, but (126) will have to be changed to:

$$\begin{aligned} (130) \quad A^{\text{HW}}(t; T) &= - \int_t^T \theta^*(\tau) B(\tau, T) d\tau + \frac{\sigma^2}{2} \int_t^T B(\tau; T)^2 d\tau \\ &= - \int_t^T \theta^*(\tau) B(\tau, T) d\tau + \frac{\sigma^2}{2\alpha^2} \left((t - T) - \frac{2}{\alpha} e^{\alpha(t-T)} + \frac{1}{2\alpha} e^{2\alpha(t-T)} + \frac{3}{2\alpha} \right). \end{aligned}$$

Hence,

$$(131) \quad p^{\text{HW}}(r, t; T) = \exp(A^{\text{HW}}(t; T) - rB(t; T))$$

where the suffix "HW" stands for "Hull and White". Making θ^* time-dependent may look like introducing an additional complication, but in fact gives an opportunity to do:

Exact Yield curve fitting: We suppose that σ and α are already determined from examining time series data for short term interest rates (a typical econometrical problem). Then we find θ^* by fitting at date $t = 0$ ('today') the theoretical prices $p^{\text{HW}}(r_0, 0; T)$, where r_0 is today's observed short rate, to the observed market prices $P_{0,T} = P_{0,T}^{\text{Market}}$ for all maturities $T > 0$. Taking logarithms, we find that $\theta^*(\tau)$ has to satisfy the integral equation

$$\begin{aligned} & - \int_0^T \theta^*(\tau) B(\tau; T) d\tau = \\ & \log P_{0,T} + rB(0; T) - \frac{1}{2} \sigma^2 \int_0^T B(\tau, T)^2 d\tau = \\ (132) \quad & \log P_{0,T} + \frac{r_0}{\alpha} (1 - e^{-\alpha T}) - \frac{\sigma^2}{2\alpha^2} \left(T + \frac{2}{\alpha} e^{-\alpha T} - \frac{1}{2\alpha} e^{-2\alpha T} - \frac{3}{2\alpha} \right). \end{aligned}$$

This equation can be solved for $\theta^*(T)$ by differentiating twice with respect to the maturity T :

$$\begin{aligned} \frac{\partial}{\partial T} \int_0^T \theta^*(\tau) B(\tau; T) d\tau &= \theta^*(T) B(T; T) + \int_0^T \theta^*(\tau) \frac{\partial B(\tau; T)}{\partial T} d\tau \\ &= \int_0^T \theta^*(\tau) e^{\alpha(\tau-T)} d\tau \\ &= e^{-\alpha T} \int_0^T \theta^*(\tau) e^{\alpha\tau} d\tau, \end{aligned}$$

whence

$$\frac{\partial^2}{\partial T^2} \int_0^T \theta^*(\tau) B(\tau; T) d\tau = \theta^*(T) - \alpha e^{-\alpha T} \int_0^T \theta^*(\tau) e^{\alpha\tau} d\tau.$$

It follows that

$$\theta^*(T) = \frac{\partial^2}{\partial T^2} \int_0^T \theta^*(\tau) B(\tau; T) d\tau - \alpha \frac{\partial}{\partial T} \int_0^T \theta^*(\tau) B(\tau; T) d\tau,$$

and substituting the right hand side of (132) one finds, after some algebra, that

$$(133) \quad \theta^*(T) = -\frac{\partial^2}{\partial T^2} \log P_{0,T} - \alpha \frac{\partial}{\partial T} \log P_{0,T} + \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha T}).$$

To use formula (133) in practice, we need to construct a smooth (that is, twice-differentiable) curve $P_{0,T}$ out of the finitely many bonds which are in fact quoted (corresponding to some finite sequence of maturities T_1, \dots, T_N). This is usually done by spline interpolation.

Finally we mention a problem with yield curve fitting. Suppose we calibrate with today's yield curve, and find a function $\theta_{now}^*(t)$. If we calibrate again in one week's time, say, we would in general find a *different* function $\theta_{then}^*(t)$, whereas this function should be the same, at least if the theory consistently describes reality. A slightly non-realistic aspect of the Vasicek model, and other one-factor models, is of course that there is only one risk-factor: W_t , driving the short rate, for the whole range of maturities. One could reasonably argue that at the long end of the yield curve (that is, for big T), other risks will come into play. This has led to the introduction of multi-factor models; for a review of the literature, see Musiela and Rutkowski, section 12.4. The HJM-models which we will study in the next lecture are typically also multi-factor.

5.5. Martingale approach to short rate models. Introduce, as before, the savings account B_t

$$(134) \quad B_t = e^{\int_0^t r_s ds}.$$

This is simply the amount of money in a bank account to which an initial deposit of 1, continuously re-invested at the short rate r_t , as grown

at time t (to see this, simply observe that $dB_t = r_t B_t dt$, and integrate). We are then dealing with a market consisting of the savings bond B_t , and infinitely many 0-coupons $P_{t,T}$, $T > t$. Extrapolating from the multi-asset markets treated in Lecture IV (1 risk factor, and infinitely many assets), we expect that there will be absence of arbitrage iff there is an E(quivalent) M(artingale) M(easure), \mathbb{Q} for the discounted assets $P_{t,T}/B_t$. That is, if $t < u \leq T$,

$$(135) \quad B_t^{-1} P_{t,T} = \mathbb{E}_{\mathbb{Q}} (B_u^{-1} P_{u,T} | \mathcal{F}_t).$$

In particular, taking $u = T$, and using $P_{T,T} = 1$, and (134), we find the pricing equation:

$$(136) \quad P_{t,T} = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r_s ds} | \mathcal{F}_t \right).$$

Observe that we obtain a similar equation (but with r_t replaced by \hat{r}_t) by applying the Feynman-Kac formula to the bond price equation (117): changing from r_t to \hat{r}_t can also be understood as replacing the SDE for r_t , $dr_t = \alpha_t dt + \sigma_t dW_t$ (where we have written $\alpha_t = \alpha(r_t, t)$, $\sigma_t = \sigma(r_t, t)$) by

$$dr_t = (\alpha_t - q_t \sigma_t) dt + \sigma_t d\widehat{W}_t, \quad q_t = q(r_t, t),$$

with $d\widehat{W}_t = q(r_t, t)dt + \sigma(r_t, t)dW_t$ being a Brownian motion with respect to some new probability measure \mathbb{Q} , obtained from Girsanov's theorem.

Conversely, it can be proved that on the Brownian \mathcal{F}_t^W , any EMM is of the Girsanov form:

$$d\mathbb{Q} = \exp \left(\int_0^t -\gamma_s dW_s - \frac{1}{2} \int_0^t \gamma_s^2 ds \right),$$

for some adapted process γ_t , with \widehat{W}_t , defined by $d\widehat{W}_t = \gamma_t dt + dW_t$, a Brownian motion with respect to \mathbb{Q} . With respect to \mathbb{Q} , r_t follows the SDE

$$dr_t = (\alpha_t - \gamma_t \sigma_t) dt + \sigma_t d\widehat{W}_t,$$

so that γ_t is the price of risk.

In the case of for example the Vasicek model, formula (136), together with

$$dr_t = \alpha(\theta^* - r_t)dt + \sigma d\widehat{W}_t,$$

\widehat{W}_t a \mathbb{Q} -Brownian motion, can be used to directly compute the 0-coupon prices, by the same arguments as in 5.3 above.

5.6. Exercises to Lecture VI.

Exercise 5.3. Write out the details of the computation leading to (114), (115), (116).

Exercise 5.4. Verify equation (133), and find a formula for $P_{t,T}$ in terms of today's market prices, $P_{0,T}$.

Exercise 5.5. A first order ODE of the form

$$y' = p(x)y^2 + q(x)y + r(x),$$

for an unknown function $y = y(x)$, with given functions $p(x), q(x)$ and $r(x)$, is called a *Riccati equation*. As usual, $y' = dy/dx$. Suppose we dispose of a *particular solution* $y_0 = y_0(x)$. Then we can transform the original equation into a first order *linear* equation, which (being first order) is explicitly solvable. Indeed, let $u = u(x)$ be defined by

$$y = y_0 + \frac{1}{u},$$

or, equivalently,

$$u(x) = \frac{1}{y(x) - y_0(x)}.$$

(Let us not worry about the possible singularity in points x for which $y(x) = y_0(x)$: this would have to be studied in any particular case to which we would apply this method).

a) Show that $u = u(x)$ satisfies the ODE

$$-u' = p(x) + 2p(x)y_0(x)u + q(x)u.$$

b) Consider from now the special case in which $p(x), q(x)$ and $r(x)$ are *constants*, p, q and r , respectively. Show that $y_0(x) = v_{\pm}$ are special solutions of

$$(137) \quad y' = py^2 + qy + r,$$

if v_{\pm} are the zeros of

$$(138) \quad pv^2 + qv + r = 0,$$

that is,

$$v_{\pm} = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}.$$

c) Take $y_0 = v_-$ as special solution. Show that $u = u(x)$ satisfies

$$u' = -p + du,$$

where $d := \sqrt{q^2 - 4pr}$ is the discriminant of the quadratic equation (138). Solve this equation by the usual method ("general solution = particular solution + general solution homogeneous equation") to find

$$u(x) = \frac{p}{d} + Ce^{dx},$$

with C an arbitrary constant. d) Conclude that the general solution of

(137) is given by

$$(139) \quad y(x) = \frac{(pv_- + d) + C'e^{dx}}{p + C'e^{dx}},$$

where C' is some constant ($C' = dv_-C$ if C is the constant from part c)).

e) Determine C' such that $y(0) = 0$, and show that the corresponding y is given by

$$(140) \quad y(x) = \frac{v_+(1 - e^{dx})}{1 - v_+e^{dx}},$$

where you will have to use that

$$pv_- + d = pv_+.$$

(verify this!).

f) Show that, if $y(x)$ is given by (140),

$$(141) \quad \int_0^x y(s)ds = x - \left(\frac{v_+ - 1}{v_+d} \right) \log(1 - v_+e^{dx}).$$

(Hint: Observe that

$$\frac{1 - e^{dx}}{1 - v_+e^{dx}} = 1 + (v_+ - 1) \frac{e^{dx}}{1 - v_+e^{dx}} = 1 - \left(\frac{v_+ - 1}{v_+d} \right) (\log(1 - v_+e^{dx}))'. \quad)$$

Exercise 5.6. We can use the results of the previous exercise to find the price of a zero coupon bond in the CIR model:

$$dr_t = \alpha(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

a) Let $P_{t,T} = p_T(r, t)$ be the price of a pure discount bond of maturity T . Show that, under the assumption that the market price of risk is a constant times \sqrt{r} (**Corrected!**), $p = p_T$ solves the boundary value problem

$$(142) \quad \frac{\partial p}{\partial t} + \alpha(\theta^* - r) \frac{\partial p}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 p}{\partial r^2} = rp, \quad t < T, \\ p(r, T) = 1,$$

where θ^* is the risk-adjusted mean rate.

b) Making the "Ansatz"¹²

$$(143) \quad p(r, t) = e^{A(\tau) - rB(\tau)},$$

¹²meaning trial solution

where $\tau := T - t$, and $A(\tau)$ and $B(\tau)$ are functions of τ only, show that the latter have to satisfy the following system of differential equations:

$$\begin{aligned}\frac{dA}{d\tau} &= -rB, \\ \frac{dB}{d\tau} &= -\frac{\sigma^2}{2}B^2 - \alpha B + 1,\end{aligned}$$

with initial conditions $A(0) = B(0) = 1$.

(N.B. The reason for introducing the new time variable τ is to have initial conditions at $\tau = 0$, instead of final conditions at $t = T$, simply because this is computationally more convenient).

c) You should recognize the differential equation for $B(\tau)$ as a Riccati equation. Use the results of the previous exercise on such equations to find $A(\tau)$ and $B(\tau)$, and to derive an explicit formula for the bond price in the CIR model. Check your answer against formulas which can be found in the literature.