Similarly, we can compute the no-arbitrage value at time t of the GLWB net liability from the insurer's perspective given by $N_{lw}(t, F_t)$ where

$$\begin{aligned} N_{lw}(t,F) &:= \tilde{\mathbb{E}}\left[\int_{\tau\wedge T_x}^{T_x} we^{-ru}\,du - \int_0^{\tau\wedge T_x} e^{-ru} m_w F_u\,du \,\middle|\, F_t = F, T_x > t\right] \\ &= \tilde{\mathbb{E}}\left[\int_{\tau\wedge T_{x+t}}^{T_{x+t}} we^{-ru}\,du - \int_0^{\tau\wedge T_{x+t}} e^{-ru} m_w F_u\,du \,\middle|\, F_0 = F\right]. \end{aligned}$$

Due to the independence of mortality and equity returns, we can show that

$$N_{lw}(t,F) = \frac{w}{r}\tilde{\mathbb{E}}\left[e^{-r\tau}\overline{Q}_{x+t}(\tau)\right] - \frac{w}{r}\tilde{\mathbb{E}}\left[\int_\tau^\infty e^{-ru} q_{x+t}(u)\,du\right] + m_w\tilde{\mathbb{E}}\left[\int_0^\tau e^{-ru}\overline{Q}_{x+t}(u)F_u\,du\right]$$

where $\overline{Q}$ is the survival function of the future lifetime T_{x+t}. Therefore, we find the approximation of N_{lw} by

$$N_{lw}(t,F) \approx \sum_{k=1}^{n} \frac{w\lambda_k}{(r+s_k)s_k} L(F; r+s_k) + \sum_{k=1}^{n} m_w \lambda_k D(F; r+s_k).$$

4.8 Actuarial Pricing

As alluded to earlier, no-arbitrage pricing is not directly used in the insurance industry for at least two reasons.

- The principle of no arbitrage pricing does not explicitly allow for expenses and profits, although one may forcefully add in these components under the risk-neutral valuation;

- No arbitrage pricing is based on the assumption that the underlying assets and financial derivatives are freely tradable and short-selling is allowed for any amount. An insurer typically transfers policyholders' purchase payments to competing third-party fund managers, who do not disclose proprietory information regarding the exact mix of their assets. Therefore, the opaqueness of underlying assets does not allow an investor to develop a replicating portfolio. In addition, an investor can buy investment guarantees from an insurer but cannot sell them to the insurer. Therefore, short-selling of investment guaranteed are possible either.

Nonetheless, the insurance industry has seen applications of the corresponding hedging theory, which provides a means of financial risk management of embedded options, as we shall demonstrate in Chapter 6.

In practice, an actual pricing model for variable annuity with guaranteed benefits is usually based on the projection of cash flows emerging from individual contracts from period to period under a range of economic scenarios. The end goal of each cash flow projection is to identify the profit or loss an insurer can expect from the contract over the projection period. This actuarial practice is commonly referred to as *profit testing*. The procedure allows an insurer to quantify its loss or gain under each set of assumptions including fees and charges. By testing various assumptions, an insurer can identify the "sweet spot" of fees and charges which meets specified profit measures with a certain level of confidence.

4.8.1 Mechanics of profit testing

We shall demonstrate the practice of profit testing with an example of single premium variable annuity (SPVA) with the GMDB, from which a policyholder's beneficiary is guaranteed to receive the greater of the subaccount value at death or the guarantee base with a combination of ratchet option and 5% annual roll-up option.

Since fees are charged on a daily basis, one may desire to project cash flows involving fee incomes at monthly intervals. However, an insurer may have hundreds and thousands of contracts and it is too costly and time consuming to run such projections for all contracts. As a compromise in practice, it is very common to project cash flows at monthly or quarterly intervals. In order for us to illustrate the mechanism of profit testing more clearly with limited space, we only project the cash flows at yearly intervals for the first seven years.

Readers should keep in mind that cash flow projections are often carried out at the level of a line of business. Even with the simple example of the SPVA, we assume that there are thousands of policyholders in the insurance pool for the contract with the same product specification. One may consider the following calculations as profit testing for the average of a large pool of contracts. We shall discuss the distinction between an individual model and an aggregate model in more detail in Section 5.4.

Actuarial assumptions

In order to carry out projections, the profit testing requires a set of assumptions on the following items driving cash flows, many of which are illustrated in Table 4.1 and Table 4.2.

- **Equity returns**: As all policyholders are offered a set of investment funds to choose from, the growth of their purchase payments are determined by financial returns of these investment funds. Hence, stochastic models are necessary to project the evolution of equity returns, interest rates, etc. This is typically done with a economic scenario generator developed internally by the insurer or externally by a third software vendor. Some regulators also provide pre-packaged scenarios for insurers to use for statutory pricing and reserving. In this example, we list percentage returns of the investment funds in Item A.

- **Fees and charges**: As discussed in Section 1.2, there are many types of fees and charges deducted from subaccounts for various purposes. For simplicity, we consider in this example an asset-value-based risk charges and fixed policy fees.

  - Risk charges: 1.5% of subaccount value per year. (Item B)
  - Policy fees: 30 per policy per year.

- **Surrenders**: Policyholders may volunteerily give up their policies prior to maturity. Lapse rates, given in Item F, provide estimates of likelihood that a policyholder surrenders year by year. As it takes time to recover initial expenses, an insurer typically tries to discourage early surrenders by levying high surrender charges in early years. As a result, surrender charges, as shown in Item I, form a declining schedule.

- **Annuitizations**: Policyholders may volunteerily convert their investments to an annuity prior to maturity. Such a process is irreversible. The annuitization rates, which are estimates of likelihood that a policyholder exits the contract by annuitization, are given in Item G.

- **Survival model**: In the case that a policyholder dies prior to maturity, a death benefit is provided to beneficiaries. Therefore, estimations of death benefit also require mortality rates, which are typically taken from an industrial life table. To make conservative assumptions, an insurer may incorporate a safety margin by multiplying standard mortality rates by some factor great than one. Item H shows an example of mortality rates by year.

- **Fund values**: As the insurer sets up a large fund from all policyholders' purchase payments, the aggregate nature ensures predicability of mortality and expenses, as demonstrated by the law of large numbers. As the cash flow projection is based on the average of similar contracts, we use the term fund value as opposed to account value often used for an individual contract.

- **Expenses**: There are several types of expenses associated with insurance policies – administrative expenses, commissions and claim expenses. Administrative expenses are occurred by an insurer throughout the term of the policy to cover overheads, staff salaries, etc. The initial administrative expense is typically higher than administrative expenses in subsequent years, as the former include underwriting expenses. In this example, we assume that

  - Initial administrative expenses: 100 per policy and 1% of subaccount value;
  - Renewal administrative expenses: 30 per policy per year;
  - Commission: 6.5% of first year's purchase payment;
  - Claim expneses (GMDB cost): 0.40% of subaccount value.

  Note, however, the renewal administrative expenses are estimated by the insurer's current conditions. We consider 2% inflation every year afterward on renewal expenses.

- **Interest on surplus**: From an insurer's point of view, the policyholder's purchase payment, fee incomes, and surrender charges are considered incoming cash flows, while expenses and death benefits are considered outgoing cash flows. As the opposing cash flows often do not offset each other, we expect surplus (profit) emerging at the end of each year, which are invested in liquid assets. Here, we assume a yield rate of 5.00% on profit for all years.

- **Statutory reserve**: In this example we consider cash flows from both an insurer's general account and seperate account. An insurer is expected to transfer to beneficiaries the balance of subaccounts (shown as seperate account on the insurer's balance sheet) at the end of the year of the policyholder's death. As we shall discuss in more details in Section 5.1, an insurer typically sets aside assets, known as reserves, to cover expected future payments. Since the insurer is expected to transfer subaccount balance for policies still inforce, we set the reserve to be the same as subaccount value after decrements by lapses and annuitizations.

- **Tax on surplus**: All profits are subject to federal income tax, which is assumed to be 37.00%.
- **Target surplus**: After profits are realized, an insurer may set aside additional assets from the profits to ensure long-term capital adequacy. Note that target surplus is the amount of capital held beyond statutory reserve. In this example, we assume target surplus to be 0.85% of statutory reserve for each year.

Profit testing

To illustrate the mechanism of profit testing more clearly, we seperate the procedure into several components. While all calculations are illustrated here in the format of an Excel spreadsheet, such exercise can be done in any other computational platform.

Section 1 - projected account values

Due to the equity-linking mechanism, most cash flows are dependent on projected account values and guarantee bases, which are themselves determined by projected economic scenarios of equity return rates and interest rates, etc. The calculations of account values are summerized in Table 4.1. Throughout the section, the abbreviation BOY stands for the beginning of year while EOY stands for the end of year. Items A–C and D–H are all specified in actuarial assumptions mentioned above, whereas Items I–V are determined by the following recursive relations. For the ease of presentation, we use the notation $X[t]$ to denote the t-th year for Item X for $t = 1, 2, 3, 4, 5$.

- Investment Income

$$I[t] = A[t] \times \left(H[t] + U[t] - \frac{1}{2} L[t] \right)$$

	Proj. Year	1	2	3	4	5	6	7
A	Earned Rate	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%
B	Risk Charges	1.50%	1.50%	1.50%	1.50%	1.50%	1.50%	1.50%
C	Credited Rate	4.75%	4.75%	4.75%	4.75%	4.75%	4.75%	4.75%
	Decrements							
D	Lapses by Year	3.00%	3.00%	3.00%	4.00%	4.00%	5.00%	6.00%
E	Annuitizations by Year	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
F	Mortality	0.41%	0.44%	0.48%	0.52%	0.56%	0.60%	0.64%
G	Surrender Charge	8.00%	7.00%	6.00%	5.00%	4.00%	3.00%	2.00%
H	Premium Collected	25,000	0	0	0	0	0	0
	Incomes							
I	Investment Income	1,559	1,562	1,564	1,566	1,551	1,535	1,503
J	Credited to Policyholder Account	1,185	1,187	1,189	1,190	1,178	1,167	1,142
K	Risk Charges	374	375	375	376	373	368	361
	Decrements							
L	Mortality	101	111	121	131	139	148	155
M	Lapses	775	776	777	1,037	1,026	1,270	1,492
N	Surrender Charge	62	54	47	52	41	38	30
O	Annuitizations	261	261	262	262	259	257	251
P	Current Inforce	95.59%	91.35%	87.25%	82.44%	77.85%	72.72%	67.16%
Q	Policyholder Fund Value (BOY)	25,000	25,048	25,087	25,116	24,876	24,630	24,122
R	– before Lapses & Anntzns (EOY)	26,084	26,124	26,155	26,175	25,915	25,649	25,109
S	– before Lapses after Anntzns (EOY)	25,823	25,863	25,893	25,913	25,656	25,392	24,858
T	– after Lapses & Anntzns (EOY)	25,048	25,087	25,116	24,876	24,630	24,122	23,366
U	Statutory Reserve (BOY)	0	25,048	25,087	25,116	24,876	24,630	24,122
V	Statutory Reserve (EOY)	25,048	25,087	25,116	24,876	24,630	24,122	23,366
W	GMDB Benefit Base	25,000	25,094	25,178	25,252	25,051	24,841	24,362
X	GMDB Benefit	101	0	0	1	1	1	2

Table 4.1: Section 1 – projected account values

- Credited to Policyholder Account

$$J[t] = C[t] \times \left(Q[t] - \frac{1}{2} \times L[t] \right)$$

- Risk Charges

$$K[t] = B[t] \times \left(Q[t] - \frac{1}{2} \times L[t] \right)$$

- Mortality

$$L[t] = Q[t] \times F[t]$$

- Lapses

$$M[t] = S[t] \times D[t]$$

- Surrender Charge
$$\mathsf{N}[t] = \mathsf{M}[t] \times \mathsf{G}[t]$$
- Annuitization
$$\mathsf{O}[t] = \mathsf{R}[t] \times \mathsf{E}[t]$$
- Current Inforce (as % of initial)
$$\mathsf{P}[t] = \mathsf{P}[t-1] \times (1 - \mathsf{D}[t] - \mathsf{E}[t] - \mathsf{F}[t])$$
- Policyholder Fund Value (BOY)
$$\mathsf{Q}[t] = \mathsf{T}[t-1] + \mathsf{H}[t]$$
- Policyholder Fund Value before Lapses & Annuitizations (EOY)
$$\mathsf{R}[t] = \mathsf{Q}[t] + \mathsf{J}[t] - \mathsf{L}[t]$$
- Policyholder Fund Value before Lapses & after Annuitizations(EOY)
$$\mathsf{S}[t] = \mathsf{R}[t] - \mathsf{O}[t]$$
- Policyholder Fund Value after Lapses & Annuitizations (EOY)
$$\mathsf{T}[t] = \mathsf{S}[t] - \mathsf{M}[t]$$
- Statutory Reserve (BOY)
$$\mathsf{U}[t] = \mathsf{V}[t-1]$$
- Statutory Reserve (EOY)
$$\mathsf{V}[t] = \mathsf{T}[t]$$
- GMDB Benefit Base (5% – roll-up rate)
$$W[t] = \mathsf{P}[t-1] \times \max\left(\mathsf{Q}[t], \mathsf{H}[1] \times (1+5\%)^{t}\right)$$
- GMDB Benefits
$$\mathsf{X}[t] = (W[t] - \mathsf{U}[t])_{+} \times \mathsf{F}[t]$$

	Proj. Year	1	2	3	4	5	6	7
	Revenues							
AA	Earned Rate	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%	6.25%
AB	Risk Charges	1.50%	1.50%	1.50%	1.50%	1.50%	1.50%	1.50%
AC	Credited Rate	4.75%	4.75%	4.75%	4.75%	4.75%	4.75%	4.75%
AD	Current Inforce	95.59%	91.35%	87.25%	82.44%	77.85%	72.72%	67.16%
AE	Premium Collected	25,000	0	0	0	0	0	0
AF	Investment Income	1,559	1,562	1,564	1,566	1,551	1,535	1,503
AG	Policy Fee Income	29	27	26	25	23	22	20
AH	**Total Revenues**	26,588	1,589	1,590	1,591	1,574	1,557	1,523
	Expenses							
AI	Administrative rate							
AJ	% of Premium	1.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
AK	Per Policy	100	30	30	30	30	30	30
AL	Commission	4.50%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
AN	Administrative expenses							
AO	Premium-based	250	0	0	0	0	0	0
AP	Per Policy	96	28	27	26	25	24	23
AQ	Commissions	1,000	0	0	0	0	0	0
AR	GMDB Cost	100	100	100	100	100	99	96
AS	**Total Expenses**	1,446	128	127	126	125	123	119
	Benefits							
AT	Death Claims	101	111	121	131	139	148	155
AU	Annuitizations	261	261	262	262	259	257	251
AV	Surrender Benefit	713	722	730	985	985	1,232	1,462
AW	Increase in Reserve	25,048	39	30	(239)	(246)	(507)	(755)
AX	GMDB Benefit	101	0	0	0	1	1	2
AY	**Total Benefits**	26,224	1,133	1,143	1,140	1,138	1,131	1,115
AZ	**Book Profit before Tax**	(1,082)	328	320	325	311	303	289

Table 4.2: Section 2 – projected cash flows from general and seperate accounts

Section 2 - projected cash flows

The calculations of cash flows emerging from the contract at the end of each year are set out in Table 4.2, which can be viewed as an income statement with three main components – (1) Revenues: the amount an insurer receives as gross income; (2) Expenses: the business costs an insurer incurs excluding the costs of insurance liabilities; (3) Benefits: the amount an insurer pays or prepares to set aside as the costs of insurance liabilities.

The key elements of cash flow projection are determined by recursive relations as follows. Numbers in brackets in Table 4.2 are considered negative.

- Policy Fee Income (30 – annual policy fee)

$$AG[t] = 30 \times AD[t-1]$$

- Total Revenues

$$AH[t] = AE[t] + AF[t] + AG[t]$$

- Premium-Based Administrative Expenses

$$AO[t] = AE[t] \times AJ[t]$$

- Per Policy Administrative Expenses (2% – inflation rate)

$$AP[t] = AK[t] \times AM[t] \times (1 + 2\%)^{t-1}$$

- Commissions

$$AQ[t] = AE[t] \times AL[t]$$

- GMDB Cost (0.4% of account value – GMDB cost)

$$AR[t] = T[t] \times 0.4\%$$

- Total Expenses

$$AS[t] = AO[t] + AP[t] + AQ[t] + AR[t]$$

- Death Claims

$$AT[t] = L[t]$$

- Annuitizations

$$AU[t] = O[t]$$

- Surrender Benefit

$$AV[t] = M[t] - N[t]$$

- Increase in Reserve

$$AW[t] = V[t] - V[t-1]$$

- GMDB Benefit

$$AX[t] = (W[t] - U[t])_+ \times F[t]$$

- Total Benefits

$$AY[t] = AT[t] + AU[t] + AV[t] + AW[t] + AX[t]$$

- Book Profit Before Tax

$$\mathsf{AZ[t]} = \mathsf{AH[t]} - \mathsf{AS[t]} - \mathsf{AY[t]}$$

Table 4.2 shows a common practice of cash flow projections involving both the insurer's general account and seperate account. For example, on the revenues side, policyholders have ownerships of Items AE and AF, which should be accounted for in a seperate account, whereas, on the benefits side, Items AU, AV, AW correspond to payments from the seperate account to policyholders or their beneficiaries or assets withheld for future payments. From an insurer's point of view, these are not real incomes and outgos in the same way as claims and expenses, but rather accounting transfers. The insurer does not take ownership of policyholders' funds, nor does it take any responsiblity for their losses. One should note confuse this with investment guarantees due to various guaranteed benefits, which are considered add-ons to a base contract and are genuine costs to the insurer and paid out of the general account.

If we only consider an insurer's general account, then the cash flows in and out of the seperate account can be cancelled. Observe that, on the revenues side,

$$\begin{aligned}
&\text{Premium Collected} + \text{Investment Income}\\
&= \text{Risk Charges and Surrender Charges (allocated to general account)}\\
&+\text{Premium and Credited Interest (allocated to seperate account)},
\end{aligned}$$

on the benefits side,

$$\begin{aligned}
&\text{Death Claim} + \text{Annuitization} + \text{Surrender Benefit} + \text{Increase in Reserve}\\
&= \text{Assets in seperate account}\\
&\text{GMDB Benefit} = \text{Assets in general account}.
\end{aligned}$$

Note that the death claim refers to the return of policyholders' fund values to beneficiaries, whereas the GMDB benefit records the actual cost for the insurer to match policyholders' total incomes to guaranteed minimums.

If the accounting transfers are removed from consideration, then we can obtain an alternative version of the income statement in Table 4.3. Observe that the book profits before tax in Table 4.3 still match those in Table 4.2, due to the cancellation of payments in and out the seperate account. Although it is less common to use the income statement in Table 4.3 in practice, its simplified calcuations reveal the mathematical structure of profit testing, which shall be compared with no-arbitrage pricing in the next section.

Projection Year	1	2	3	4	5	6	7
Revenues							
Surrender Charge	62	54	47	52	41	38	30
Policy Fee Income	29	27	26	25	23	22	20
Risk Charges	374	375	375	376	373	368	361
Total Revenues	26,588	1,589	1,590	1,591	1,574	1,557	1,523
Expenses							
Administrative rate							
% of Premium	1.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Per Policy	100	30	30	30	30	30	30
Commission	4.50%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Administrative expenses							
Premium-based	250	0	0	0	0	0	0
Per Policy	96	28	27	26	25	24	23
Commissions	1,000	0	0	0	0	0	0
GMDB Cost	100	100	100	100	100	99	96
Total Expenses	1,446	128	127	126	125	123	119
Benefits							
GMDB Benefit	101	0	0	0	1	1	2
Total Benefits	101	0	0	0	1	1	2
Book Profit Before Tax	(1,082)	328	320	325	311	303	289

Table 4.3: Section 2 – projected cash flows with general account only

Section 3: projected distributable earnings

The difference between book profit and distributable earning is due to taxation and withholding of additional capital required to support ongoing operations, which is beyond statutory reserves. Since it is truely the amount an insurer can distribute, distributable earnings are often used for measuring the profitability of the business.

The calculations of target surplus and distributable earnings are set out in Table 4.4 and recursive relations among various entries are described in more detail below.

- Taxes on Book Profit (37% – federal income tax rate)

$$\mathsf{BF}[\mathsf{t}] = \mathsf{BE}[\mathsf{t}] \times 37\%$$

	Proj. Year	1	2	3	4	5	6	7
BA	Total Revenues	26,588	1,589	1,590	1,591	1,574	1,557	1,523
BB	Total Expenses	1,446	128	127	126	125	123	119
BC	Total Benefit	26,224	1,133	1,143	1,140	1,138	1,131	1,115
BD	Book Profits after Tax	(682)	207	201	205	196	191	182
BE	Book Profit Before Tax	(1,082)	328	320	325	311	303	289
BF	Taxes on Book Profit	(400)	121	118	120	115	112	107
	Target Surplus							
BG	Increase in Target Surplus	213	0	0	(2)	(2)	(4)	(6)
BH	Target Surplus (BOY)	0	213	213	213	211	209	205
BI	Target Surplus (EOY)	213	213	213	211	209	205	199
BJ	After Tax Interest on Target Surplus	0	7	7	7	7	7	6
BK	Interest on Target Surplus	0	11	11	11	11	10	10
BL	Taxes on Interest on Target Surplus	0	4	4	4	4	4	4
BM	Distributable Earnings	(895)	213	208	214	205	202	195

Table 4.4: Projected Distributable Earnings after Tax and Target Surplus

- Book Profits after Tax

$$\mathsf{BD}[t] = \mathsf{BE}[t] - \mathsf{BF}[t]$$

- Target Surplus (BOY)

$$\mathsf{BI}[t] = \mathsf{BJ}[t-1]$$

- Target Surplus (EOY) (0.85% – target surplus rate)

$$\mathsf{BI}[t] = \mathsf{V}[t] \times 0.85\%$$

- Increase in Target Surplus

$$\mathsf{BG}[t] = \mathsf{BI}[t] - \mathsf{BH}[t]$$

- Interest on Target Surplus (5% – interest rate on surplus)

$$\mathsf{BK}[t] = \mathsf{BH}[t] \times 5\%$$

- Taxes on Interest on Target Surplus

$$\mathsf{BL}[t] = \mathsf{BK}[t] \times 37\%$$

- After Tax Interest on Target Surplus

$$\mathsf{BJ}[t] = \mathsf{BK}[t] - \mathsf{BL}[t]$$

- Distributable Earnings

$$\mathrm{BM}[t] = \mathrm{BD}[t] + \mathrm{BJ}[t] - \mathrm{BG}[t]$$

The bottom line of 4.4 shows the distributable earnings emerging at the end of each year under the current actuarial assumptions. Table 4.4 reveals a common feature of cash flow projections where a large deficit emerges in the first year. This is commonly referred to as *new business strain*. A careful exmination of Table 4.3 shows that expenses start off high in the first year and taper off in subsequent years whereas the insurer's genuine incomes tend to be flat over time. It takes a few years before the initial expenses are paid off gradually by the small but steady stream of incomes. This is why insurers often impose high surrender charges in the first few policy years to discourage early lapsation.

As income statements present the pattern of profits emerging from year to year, it is up to investors to interpret by various profit measures the financial condition of the companies which report the income statements. These profit measures allow investors to compare across companies and lines of business. Likewise, to determine whether or not a product line generates sufficient profits, an insurer often resort to simple metrics for measuring its profitability. There are a number of profit measures commonly used in insurance practice.

1. Net present value (NPV) – The difference between the present value of all distributable earnings/deficits. Since distributable earnings are considered incomes for shareholders, the interest rate for discounting surplus should be the yield rate required by shareholders on its capital to support the product line, which is commonly referred as the *hurdle rate*. Here we assume the hurdle rate to be $j = 8\%$ effective per year. Then the NPV is given by

   $$\text{NPV} = \sum_{t=1}^{7} \frac{\mathrm{BM}[t]}{(1+j)^t} = 56.29.$$

   A positive number indicates a net profit for the insurer, whereas a negative number suggests a net loss.

2. Internal rate of return (IRR) – The IRR is the discounting interest rate that makes the NPV of all distributable earnings zero. It reflects the

average annual yield rate realized for the investor. For example, we can determined the IRR denoted by i by solving the equation for i

$$\sum_{t=1}^{7} \frac{BM[t]}{(1+i)^t} = 0,$$

which implies $i = 10.24\%$. Since the IRR is higher than the hurdle rate, the product line is considered profitable.

3. Profit margin (PM) – Net present value of distributable earnings as a percentage of premiums. In the case of the SPVA example, there is only a single purchase payment and hence

$$\text{PM} = \frac{\text{NPV}}{H[1]} = 0.23\%.$$

One should keep in mind that the purchase payment is transferred to an insurer's seperate account and should not be considered as the insurer's revenue. Therefore, an alternative definition of the profit margin can be the NPV of distributable earnings as a precentage the present value of the insurer's total revenues (in Table 4.3).

The profit testing was originally developed for traditional life insurance produts which are mostly subject to mortality risk that can be diversified through a large pool of policyholders according to the law of large numbers. Therefore, the standard practice for pricing traditional products is to use "best estimates" for various actuarial assumptions including expenses, mortality rates, interest rates, etc. The profit testing allows an insurer to obtain ballpark estimates of its profits over long term. Actuarial assumptions used to project cash flows can be adjusted to test the impact of adverse experience to the profitability. As alluded to in Section 1.3, investment risk associated with equity-linked insurance products is not diversifiable. Hence it is impossible to use a single scenario to capture the uncertainty with equity returns. For instance, we assume the constant annualized equity return rate of 6.25% for all seven years in the example studied earlier. It begs the question of whether the profit still holds if the equity returns turn out differently over time. Hence, the profit testing technique is extended in the actuarial practice to allow for stochastic scenarios. As discussed in Section 3.4, a common approach to do this is to replace deterministic scenario in actuarial assumptions

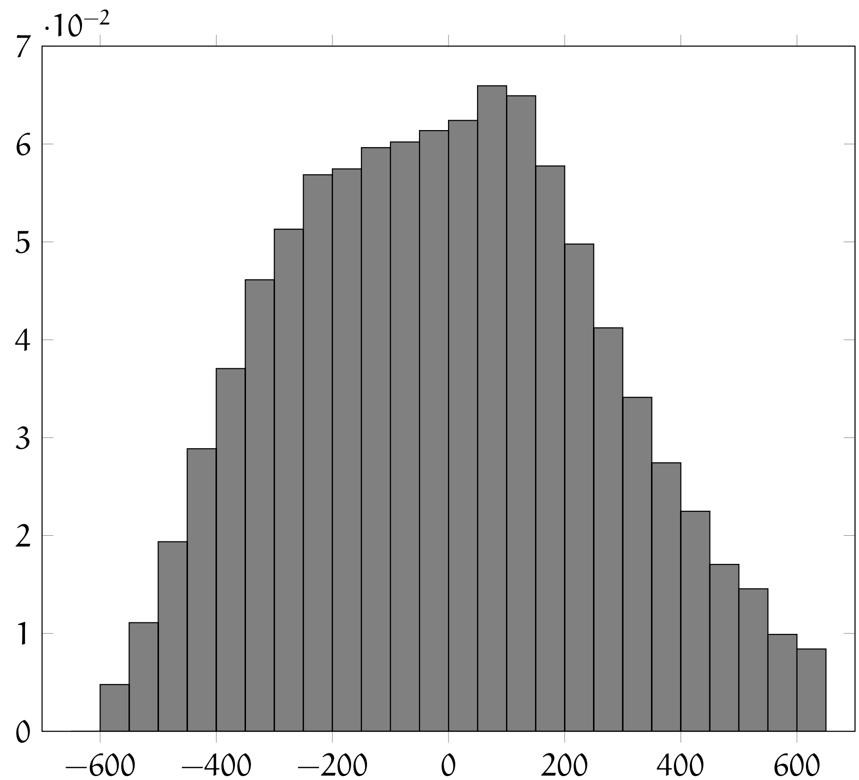

Figure 4.2: Histogram of NPV profit

by stochastic scenarios using Monte Carlo simulations. For example, simulated equity returns from a geometric Brownian motion model calibrated to market data can be used in place of the constant equity return assumption. The profit testing proceeds exactly as described above and a different value of the profit measure is determined for each economic scenario. Then the collection of all profit measures from all scenarios form an empirical distribution of the insurer's profit measure. For example, Figure 4.2 shows the hitogram (empirical density function) of the NPV of distributable earnings with annual equity returns generated from a geometric Brownian motion model with parameters $\mu = 0.060625$ and $\sigma = 0.10$.

Having developed a mechanism for identifying the distribution of profits, we can combine the stochastic profit testing with various premium principles. A common approach is to utilize the porfolio percentile premium principle introduced in Chapter 1. For example, we might be willing to write the contract if the NPV of distributable earnings turns out positive for 70% of scenarios. One cannot expect to solve for the exact fee rate explicitly. However, we can gradually adjust the fee rate as well as other actuarial assumptions until we reach the target NPV level with a specified probability.

4.8.2 Actuarial pricing vs. no-arbitrage pricing

In Table 4.3, we have shown how revenues are generated, expenses and benefits are taken out, the remaining are transformed into profits from year to year. It is in the calculation of profit measure that the overall profitability is assessed over the entire projection horizon. Therefore, it makes sense to investigate the attribution of profit measure to various components over time.

To get the big picture of profit testing and make a comparison with no-arbitrage pricing, let us bring out essential elements of the insurer's cash flows and formulate the mechanism in a mathematical model. While we present all recursive relations at yearly intervals in previoius sections, these ideas can be extended to projections with shorter time intervals. Consider the cash flow projection over integer T years at discrete intervals of length $\Delta t = 1/n$. Later on we intend to find its continuous-time analogue by shrinking Δt to zero. Denote policyholder fund values by $\{A_t : t = 0, \Delta t, \cdots\}$.

- Decrements: Let ${}_{\Delta t}q^d_{x+t}$, ${}_{\Delta t}q^l_{x+t}$, ${}_{\Delta t}q^a_{x+t}$ be the mortality rate, lapse rate and annuitization rate respectively over the period $[t, t+\Delta t]$. The policyholder fund value before decrements (EOY) is given by $A_t S_{t+\Delta t}/S_t(1-m\Delta t)$ *. Roughly speaking, the decrements to policyholder fund due to mortality, lapses and annuitizations (Items L, M, O) are determined by the fund value before decrements multiplied by ${}_{\Delta t}q^d_{x+t}$, ${}_{\Delta t}q^l_{x+t}$, ${}_{\Delta t}q^a_{x+t}$ respectively. We ignore the nuance of the order of decrements, as they make no difference in continous-time.

- Surrender charge (Item N): Let k_t be the annualized rate of surrender charge at time t as a percentage of account value (Item G). Then the total surrender charge over the period $[t, t+\Delta t]$ is given by $k_t \, {}_{\Delta t}q^l_{x+t} A_t$.

- Expenses: All expenses are either premium-based, per policy or account-value-based. Therefore, expenses can be formulated either as a deter-

*In Table 4.1, rider charge is shown as a portion of the investment return. Therefore, the more precise formulation of fund value should be $A_t(S_{t+\Delta t}/S_t - m\Delta t)$. However, the two expressions are equivalent with slightly different interpretation of the fee rate m.

ministic function of time or a time-varying percentage of policyholder fund value, both of which are commonly seen in other items. To provide concise formulation, we avoid identifying seperate expenses but instead use E_t to denote the annualized rate of expenses at time t.

- Risk charge (Item K): Let m be the annualized rate of fee as a percentage of account value. Then the total amount of risk charge over the period $[t, t+\Delta t]$ is $m\Delta t A_t$.

- GMDB benefit (Item AX): The GMDB benefit base (Item U) after decrements at time t, denoted by B_t, is given by

$$B_t = \max\left(A_t, B_{t-\Delta t}\left(1+\frac{\rho}{n}\right) {}_{\Delta t}p_{t-\Delta t}\right),$$

where the annualized roll-up rate is denoted by ρ. Hence the total GMDB benefit over the period $[t, t+\Delta t]$ is given by ${}_{\Delta t}q^{d}_{x+t}(B_t - A_t)_+$.

Policyholder fund values in Table 4.1 are determined recursively by investment returns, deductions due to benefit claims, expenses, charges, i.e.

$$\begin{aligned} A_{t+\Delta t} &= A_t \frac{S_{t+\Delta t}}{S_t}(1-m\Delta t)(1 - {}_{\Delta t}q^{l}_{x+t} - {}_{\Delta t}q^{a}_{x+t} - {}_{\Delta t}q^{d}_{x+t}) \\ &= A_t \frac{S_{t+\Delta t}}{S_t}(1-m\Delta t)\ {}_{\Delta t}p_{x+t}. \end{aligned} \tag{4.58}$$

A quick comparison of (1.12) and (4.58) shows that policyholder fund value in an aggregate model is in fact an individual account value multiplied by survival probability, i.e.

$$A_t = {}_tp_x F_t.$$

Similarly, one can show that the GMDB base in an aggregate model is also given by that in an individual model multiplied by survival probability, i.e.

$$B_t = {}_tp_x G_t, \qquad G_t = F_0\left(1+\frac{\rho}{n}\right)^{t/\Delta t}.$$

The profit emerging at the end of each period in Table 4.3 is determined by the following relationship

$$\begin{aligned} \text{Profit} &= \text{Surrender Charge} + \text{Policy Fee Income} + \text{Risk Charges} \\ &\quad - \text{Expenses} - \text{GMDB benefit}, \end{aligned}$$

which translates to

$$P_{t+\Delta t} \quad = \quad k_t\, {}_{\Delta t}q^l_t A_t + m\Delta t A_t - E_t \Delta t - {}_{\Delta t}q^d_{x+t}(B_t - A_t)_+$$

or equivalently,

$$P_{t+\Delta t} = k_t\, {}_tp_x\, {}_{\Delta t}q^l_t F_t + m\, {}_tp_x F_t \Delta t - E_t \Delta t - {}_tp_x\, {}_{\Delta t}q^d_{x+t}(G_t - F_t)_+.$$

Summing over all periods, we can determine the net present value of profits by

$$\mathfrak{P} = \sum_{k=1}^{nT} \left(1 + \frac{r}{n}\right)^k P_{k/n}, \tag{4.59}$$

where r is the annualized discounting rate/hurdle rate required by shareholders for their capital investments. Here we do not distinguish between book profit and distributable earning as their differences due to interests and taxes does not make significant difference in terms of mathematical modeling.

Let us now consider the continuous-time version of profit testing. Observe that as $\Delta t \to 0$,

$$\frac{{}_{\Delta t}q^d_{x+t-\Delta t}}{\Delta t} = \frac{\mathbb{P}(t - \Delta t < T_x < t)}{\Delta t} = \frac{\mathbb{P}(T_{x+t-\Delta t} < \Delta t)}{\Delta t \mathbb{P}(T_x > t - \Delta t)} \to {}_tp_x \mu^d_{x+t},$$

where μ^d_t is the force of mortality at the instant t. Similarly, one can show that ${}_{\Delta t}q^l_{x+t-\Delta}/\Delta t \to {}_tp_x \mu^l_{x+t}$ where μ^l_t is the force of lapsation at the instant t. Lapse rates may not be age-dependent, in which case one could model μ^l as a function of current time t alone. Therefore, letting $n \to \infty$ (i.e. $\Delta t \to 0$) in (4.59) gives the continous-time stochastic representation

$$\begin{aligned} \mathfrak{P} = \int_0^T e^{-rt} P_t \, dt \quad &= \quad \int_0^T e^{-rt} k_t \mu^l_{x+t}\, {}_tp_x F_t \, dt + m \int_0^T e^{-rt}\, {}_tp_x F_t \, dt \\ &\quad - \int_0^T e^{-rt} E_t \, dt - \int_0^T e^{-rt} \mu^d_{x+t}\, {}_tp_x (G_t - F_t)_+ \, dt. \end{aligned} \tag{4.60}$$

The percentile premium principle based on profit testing is to look for the fair fee rate m such that

$$\mathbb{P}(\mathfrak{P} > 0) > \alpha,$$

for some confident level $0 < \alpha < 1$. In other words, an insurer may want to make sure that with the probability of at least α the product is guaranteed to generate a profit of at least $V \geq 0$,

$$\mathbb{P}(\mathfrak{P} > V) > \alpha.$$

In the framework of no-arbitrage pricing, we ignore expenses, surrender charges and taxes. In the absense of these cash flows, we can easily recognize by comparing (4.41) and (4.60) that the net present value of profits is in fact the opposite of the net liability, i.e. $\mathfrak{P} = -L$. Although derived from entirely different theoretical basis, no-arbitrage pricing is very similar to equivalence premium principle, which is to find the fair fee rate m such that

$$\tilde{\mathbb{E}}[L] = 0.$$

In essence, the actuarial pricing practice is based on the quantile of L under the real-world measure, whereas the no-arbitrage pricing is based on the expectation of L under the risk-neutral measure.

4.9 Exercises

Section 4.1

1. Use a money market account earning the continously compounding interest rate of r per year and a T-period forward contract to create an arbitrage opporunity in the case where $S_0 < Fe^{-rT}$.

Section 4.2

1. Construct an arbitrage opportunity in the binomial tree model where

   (a) $u > d > e^r$;

   (b) $e^r > u > d$.

Section 4.6

1. We can prove the Black-Scholes formula for the European call option in Example 4.9 in two different ways.

   (a) Use the stochastic representation in (4.25) to prove (4.26).