Solving Linear Systems (Part 2)

(Elimination and Back Substitution)

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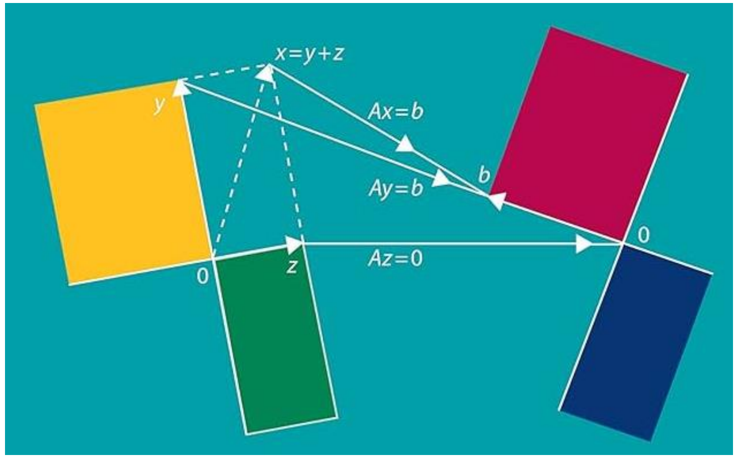

Matrix Operations (A to U and b to c)

First comes a matrix A (independent columns) that will require no row exchanges. We will apply elimination matrices E_{21} then E_{31} then E_{32}. A and $\boldsymbol{b}$ will change to U and $\boldsymbol{c}$.

The starting matrix is A
The first pivot is 2
The right side is $\boldsymbol{b}$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix} \tag{2}$$

E_{21} **multiplies equation 1 by 2 and subtracts from equation 2.** You see the new zero.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{E_{21}A} = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ \mathbf{0} & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \qquad \boldsymbol{E_{21}b} = \begin{bmatrix} 19 \\ \mathbf{17} \\ 50 \end{bmatrix} \tag{3}$$

Matrix Operations (A to U and b to c)

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad \begin{bmatrix} 19 \\ 17 \\ 31 \end{bmatrix} \tag{4}$$

Move now to column 2 and row 2 (the second pivot row). The pivot is 5, on the diagonal. To eliminate the 5 below it, multiply row 2 by the number 1 and subtract from row 3.

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \qquad c = \begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix} \tag{5}$$

$E_{32}E_{31}E_{21}A = U$ is triangular. $x = (4, 1, 2)$ solved $Ux = c$ on page 41 and $x = (4, 1, 2)$ solves $Ax = b$ here. Since U has $2, 5, 7$ on its diagonal we know that back substitution will succeed. The columns of U are independent (and therefore the columns of the original A were independent, as we will see). The matrices A and U have full rank.

Matrix Operations (A to U and b to c)

We can summarize the elimination steps when no row exchanges are involved.

Use the first equation to produce zeros in column 1 below the first pivot.
Use the new second equation to clear out column 2 below pivot 2 in row 2.
Continue to column 3. The expected result is an upper triangular matrix U.

Question

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix}$$

- Find X, Y and Z

Possible breakdown of Elimination

Elimination might fail. *Zero can appear in a pivot position.* Subtracting that zero from lower rows will not clear out the column below the unwanted zero. Here is an example:

Zero in pivot 2 from elimination in column 1

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 5 & 13 \end{bmatrix} = B$$

The cure is simple if it works. **Exchange row 2 with the zero for row 3 with the 5.** Then the second pivot is 5 and we can clear out the second column below that pivot. Elimination continues to U as normal after the row exchange by the matrix P.

Row exchange Successful

$$PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 5 & 13 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 13 \\ 0 & 0 & 6 \end{bmatrix}$$

For this small example, the row exchange is all we need. It produced U with nonzero pivots $2, 5, 6$. Normally there are more columns and rows to work on, before we reach U.

Question

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 3 & 17 \end{bmatrix}$$

- Find X, Y and Z

Possible breakdown of Elimination

Caution! That row exchange was a success. This is what we hope for, to reach U with no zeros on its main diagonal. (The pivots 2, 5, 6 are on the diagonal.) But a slightly different matrix A^* would lead to a bad situation: **no pivot is available in column 2.**

$$\begin{array}{l}\textbf{Dependent columns}\\ U^* \textbf{ is not invertible}\\ A^* \textbf{ is not invertible}\end{array} \quad \rightarrow \quad A^* = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 3 & 17 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 13 \end{bmatrix} = U^* \qquad (6)$$

At this point elimination is helpless in column 2. *No second pivot.* This misfortune tells us that **the matrix A^* did not have full rank.** Column 2 of U^* is in the same direction as column 1 of U^*. Column 2 of A^* is in the same direction as column 1 of A^*.

You see how dependent columns are systematically identified by elimination. They can't escape a zero in the pivot. Then there will be nonzero solutions X to $A^*X = 0$. The columns of U^* (and A^*) are not independent.

Elimination and Permutation

This chapter will go on to express the whole process using matrices. An elimination matrix E will act on $A\boldsymbol{x} = \boldsymbol{b}$. In case zero appears in a pivot position, use a permutation matrix P. The final result is an upper triangular U and a new right hand side $\boldsymbol{c}$. Then $U\boldsymbol{x} = \boldsymbol{c}$ is solved by back substitution.

In reality a computer takes those steps ($\boldsymbol{x} = A\backslash\boldsymbol{b}$ in MATLAB). But it is good to solve a few examples—*not too many*—by hand. You see the steps to $U\boldsymbol{x} = \boldsymbol{c}$ and then to the solution $\boldsymbol{x}$. This page contains a variety of examples, hopefully to show the way.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \boldsymbol{U}$$

Elimination and Permutation

Those elimination steps E_{21} and E_{31} and E_{32} produced zeros in positions $(2,1)$ and $(3,1)$ and $(3,2)$. The matrices E have -2 and $+1$ and -1 in those positions. The same steps E_{21}, E_{31}, E_{32} must be applied to the right hand side $\boldsymbol{b}$, to keep the equations correct.

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \to E_{21}\boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} \to E_{31}E_{21}\boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix} \to E_{32}E_{31}E_{21}\boldsymbol{b} = E\boldsymbol{b} = \boldsymbol{c} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Elimination and Permutation

There is a simple way to make sure that operations on the matrix A (left side of equations) are also executed on b (right side of equations). The good way is to **include b as an extra column with A**. The combination $\begin{bmatrix} A & b \end{bmatrix}$ is called an **augmented matrix**.

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} = \begin{bmatrix} U & c \end{bmatrix}. \qquad (7)$$

Now we include an example that requires a permutation matrix P. It will exchange equations and avoid zero in the pivot. The new matrix A needs P to improve column 2.

Exchange rows 2 and 3
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{P} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Those elimination steps E_{21} and E_{31} and E_{32} produced zeros in positions $(2, 1)$ and $(3, 1)$ and $(3, 2)$. The matrices E have -2 and $+1$ and -1 in those positions. The same steps E_{21}, E_{31}, E_{32} must be applied to the right hand side $\boldsymbol{b}$, to keep the equations correct.

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There is a simple way to make sure that operations on the matrix A (left side of equations) are also executed on $\boldsymbol{b}$ (right side of equations). The good way is to **include $\boldsymbol{b}$ as an extra column with A**. The combination $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ is called an **augmented matrix**.

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 4 & 9 & -3 & \mathbf{8} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ \mathbf{0} & 1 & 1 & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & 4 & \mathbf{8} \end{bmatrix} = \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix}. \qquad (7)$$

Now we include an example that requires a permutation matrix P. It will exchange equations and avoid zero in the pivot. The new matrix A needs P to improve column 2.

Exchange rows 2 and 3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{P} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} = U$$

That permutation P_{23} exchanged rows 2 and 3 when it was needed to avoid a zero pivot. But we could have exchanged rows 2 and 3 at the start. Then E_{21} and E_{31} change places.

Questions 2

Gaussian Elimination
vs
Gauss-Jordan elimination

x, y and z?

$$
\begin{aligned}
x + 3y + 2z &= 13\\
4x + 4y - 3z &= 3\\
5x + y + 2z &= 13
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
x + \quad 3y + 2z \quad &= 13\\
-8y - 11z \quad &= -49\\
45z/4 &= 135/4
\end{aligned}
$$

Upper Triangular Matrix

Reduced Row Echelon Form

- Converting augmented matrix into a reduced row echelon form is called Gauss–Jordan elimination.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

- We can read off the *x*, *y* and *z* values directly from this augmented matrix

Inverse Matrix

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, so that $\boldsymbol{A^{-1}}$ **times** $\boldsymbol{A}$ **equals** $\boldsymbol{I}$. Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}A\boldsymbol{x} = \boldsymbol{x}$. *But* A^{-1} *might not exist.*

The n by n matrix A needs n independent columns to be invertible. Then $A^{-1}A = I$.

What a matrix mostly does is to multiply a vector. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ by A^{-1} gives $A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$. ***This is*** $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. Numbers have inverses if they are not zero. Matrices are more complicated and interesting. The matrix A^{-1} is called "$\boldsymbol{A}$ **inverse**".

DEFINITION The matrix A is ***invertible*** if there exists a matrix A^{-1} that "inverts" A:

Two-sided inverse $\quad \boldsymbol{A^{-1}A = I} \quad$ and $\quad \boldsymbol{AA^{-1} = I}$. (4)

The Facts About Inverse Matrices

- Not all matrices have inverses. Columns must be independent.
- Notes:
  - The inverse exists if and only if elimination produces n pivots (row exchanges are allowed). Elimination solves Ax = b without explicitly using the matrix A^{-1}
  - The matrix A cannot have two different inverses.
  - If A is invertible, the one and only solution to Ax = b is $x = A^{-1}b$:

$$\boxed{\textit{Multiply} \quad Ax = b \quad \textit{by} \quad A^{-1}. \quad \textit{Then} \quad x = A^{-1}Ax = A^{-1}b.}$$

The Facts About Inverse Matrices

- Not all matrices have inverses. Columns must be independent.
- Notes:
  - The inverse exists if and only if elimination produces n pivots (row exchanges are allowed). Elimination solves Ax = b without explicitly using the matrix A^{-1}
  - The matrix A cannot have two different inverses.
  - If A is invertible, the one and only solution to Ax = b is $x = A^{-1}b$
  - A square matrix is invertible if and only if its columns are independent.
  - A 2 by 2 matrix is invertible if and only if the number ad — bc is not zero

$$\textbf{2 by 2 Inverse} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 3 Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $A\boldsymbol{x} = \mathbf{0}$) for the other three. The matrices are in the order $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{S}, \boldsymbol{T}$:

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The Inverse of a Product AB

If A and B are invertible (same size) then the inverse of AB is $B^{-1}A^{-1}$.

$$(AB)^{-1} = B^{-1}A^{-1} \qquad (AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I \qquad (7)$$

$B^{-1}A^{-1}$ illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same reverse order applies to three or more matrices:

Reverse order $$(ABC)^{-1} = C^{-1}B^{-1}A^{-1} \qquad (8)$$

Example 4 ***Inverse of an elimination matrix.*** If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2:

$$\boxed{\begin{array}{c} \textbf{\textit{E} subtracts} \\ \textbf{\textit{E}}^{-1} \textbf{ adds} \end{array} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. If $AC = I$ then for square matrices $CA = I$.

Example 5 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Non-invertible (singular) matrices

- If we cannot convert $(\mathbf{A}|\ \mathbf{I}\ )$ into the augmented matrix $(\mathrm{I}\ |\ \mathbf{A}^{-1})$ then matrix **A** is non-invertible (singular).

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 2 & 5 & 6 & 9 \\ -3 & 1 & 2 & 3 \\ 1 & 13 & -30 & -49 \end{pmatrix}$$

Non-invertible (singular) matrices

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{rrrr|rrrr} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 2 & 5 & 6 & 9 & 0 & 1 & 0 & 0 \\ -3 & 1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 13 & -30 & -49 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & -5 & 11 & 18 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 3 & 1 \end{array}\right)$$

$rref\,(\mathbf{A}) = \mathrm{R}$ has at least one row of zeros $\Leftrightarrow$ $\mathbf{A}$ is non-invertible.