SCS 1307
Probability & Statistics
Exam Problems with Solutions

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Problem Set 1: Sampling Distributions and Central Limit Theorem

Question 1: Vending Machine Problem

Question

A soft-drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 200ml and standard deviation of 15ml. What is the probability that the average amount dispensed in a random sample of size 36 is at least 204ml?

Solution

Let X be the amount of drink dispensed.
Given: $X \sim (\mu = 200\text{ml}, \sigma = 15\text{ml})$

Find: $P(\bar{X} > 204)$ where $n = 36$

By the Central Limit Theorem:

$$\bar{X} \sim N\left(200, \frac{225}{36}\right) = N(200, 6.25)$$

Calculation:

$$\begin{aligned} P(\bar{X} > 204) &= P\left(Z > \frac{204 - 200}{15/6}\right) \\ &= P(Z > 1.6) \\ &= 1 - P(Z < 1.6) \\ &= 1 - 0.9452 \\ &= \boxed{0.0548} \end{aligned}$$

Question 2: Poisson and Binomial Distributions

Question

In the Growmore Market Garden, plants are inspected for the presence of the deadly red angus leaf bug. The number of bugs per leaf is known to follow a Poisson distribution with mean one.

(a) What is the probability that any one leaf on a given plant will have been attacked (at least one bug is found on it)?

(b) A random sample of twelve plants are taken. For each plant 10 leaves are selected at random and inspected for these bugs. If more than eight leaves on any particular plant have been attacked, then the plant is destroyed. What is the probability that exactly two of these twelve plants are destroyed?

Solution

Part (a):
Let X be the number of bugs per leaf, where $X \sim \text{Poisson}(1)$

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - e^{-1} \\ &= 1 - 0.368 \\ &= \boxed{0.632} \end{aligned}$$

Part (b):
Let Y = the number of leaves that have been attacked on a plant
$Y \sim \text{Binomial}(10, 0.632)$

$$\begin{aligned} P(Y > 8) &= P(Y = 9) + P(Y = 10) \\ &= \binom{10}{9}(0.632)^9(0.368)^1 + \binom{10}{10}(0.632)^{10} \\ &= 0.069 \end{aligned}$$

The probability that any one plant is destroyed is 0.069.

Let R = the number of plants that are destroyed
$R \sim \text{Binomial}(12, 0.069)$

$$\begin{aligned} P(R = 2) &= \binom{12}{2}(0.069)^2(0.931)^{10} \\ &= \boxed{0.154} \end{aligned}$$

Question 3: Normal Distribution of Heights

Question

The heights of a particular species of plant follow a normal distribution with mean 21cm and standard deviation 9cm. A random sample of 10 plants is taken and the mean height calculated. Find the probability that this sample mean lies between 18cm and 27cm.

Solution

Let X be the height in cm of a plant.
Given: $X \sim N(21, 81)$ (where $\sigma^2 = 9^2 = 81$)

For sample size $n = 10$:

$$\bar{X} \sim N\left(21, \frac{81}{10}\right) = N(21, 8.1)$$

Standard deviation: $\sigma_{\bar{X}} = \sqrt{8.1} = 2.846 \approx 3$

Calculation:

$$\begin{aligned} P(18 < \bar{X} < 27) &= P\left(\frac{18-21}{3} < Z < \frac{27-21}{3}\right) \\ &= P(-1 < Z < 2) \\ &= P(Z < 2) - P(Z < -1) \\ &= 0.9772 - 0.1587 \\ &= \boxed{0.8185} \end{aligned}$$

Question 4: Sample Size Estimation

Question

If a large number of samples of size n are taken from Poisson(2.5) and approximately 5% of the sample means are less than 2.025, estimate n.

Solution

Given: $X \sim \text{Poisson}(2.5)$
Therefore: $E(X) = 2.5$ and $\text{Var}(X) = 2.5$

By the Central Limit Theorem:

$$\bar{X} \sim N\left(2.5, \frac{2.5}{n}\right)$$

Given condition: $P(\bar{X} < 2.025) = 0.05$

$$P\left(Z < \frac{2.025 - 2.5}{\sqrt{2.5/n}}\right) = 0.05$$
$$P\left(Z < \frac{-0.475}{\sqrt{2.5/n}}\right) = 0.05$$

From standard normal tables: $P(Z < -1.645) = 0.05$

Therefore:

$$\frac{0.475}{\sqrt{2.5/n}} = 1.645$$
$$\sqrt{2.5/n} = \frac{0.475}{1.645} = 0.2888$$
$$\frac{2.5}{n} = 0.0834$$
$$n = \frac{2.5}{0.0834}$$
$$n = 29.98 \approx \boxed{30}$$

Question 5: Hypergeometric Distribution

Question

Out of 60 applicants to a private university, 40 are male students. If 20 applicants are selected at random, find the probability that 10 are males.

Solution

This follows a **Hypergeometric distribution**.

Parameters:

- $N = 60$ (total population)
- $K = 40$ (males in population)
- $n = 20$ (sample size)
- $x = 10$ (males in sample)

Probability mass function:

$$P(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Calculation:

$$P(X = 10) = \frac{\binom{40}{10}\binom{20}{10}}{\binom{60}{20}}$$

This can be computed using statistical software or tables.

$$P(X = 10) = \boxed{0.1638}$$

Problem Set 2: Additional Probability Problems

Question 6: Poisson Distribution - Telephone Calls

Question

A telephone operator receives calls independently of one another. If the probability of receiving no calls in one hour is 0.0325, find the probability that the company will receive exactly 2 calls in an hour.

Solution

Let X be the number of calls received in an hour.
$X \sim \text{Poisson}(\lambda)$

Given: $P(X = 0) = 0.0325$

$$\begin{aligned} P(X = 0) &= \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = 0.0325 \\ -\lambda &= \ln(0.0325) \\ \lambda &= 3.42 \end{aligned}$$

Find: $P(X = 2)$

$$\begin{aligned} P(X = 2) &= \frac{e^{-3.42}(3.42)^2}{2!} \\ &= \frac{0.0325 \times 11.6964}{2} \\ &= \boxed{0.1912} \end{aligned}$$

Question 7: Difference of Sample Means

Question

Let $\bar{X}$ be the mean of a random sample with size 30 taken from a $N(106, 150)$ distribution and $\bar{Y}$ be the mean of a random sample of size 50 taken from a $N(103, 200)$ distribution.

(a) Write down the sampling distribution of $\bar{X}$ and $\bar{Y}$

(b) If X and Y are independent, what distribution does $(\bar{X} - \bar{Y})$ follow?

(c) Find the probability that $\bar{X}$ exceeds $\bar{Y}$ by at least 1.2

Solution

Part (a):

$$\bar{X} \sim N\left(106, \frac{150}{30}\right) = N(106, 5)$$
$$\bar{Y} \sim N\left(103, \frac{200}{50}\right) = N(103, 4)$$

Part (b):
Since $\bar{X}$ and $\bar{Y}$ are independent:

$$\begin{aligned}
(\bar{X} - \bar{Y}) &\sim N(\mu_{\bar{X}} - \mu_{\bar{Y}}, \sigma^2_{\bar{X}} + \sigma^2_{\bar{Y}}) \\
(\bar{X} - \bar{Y}) &\sim N(106 - 103, 5 + 4) \\
(\bar{X} - \bar{Y}) &\sim \boxed{N(3, 9)}
\end{aligned}$$

Part (c):
Find: $P((\bar{X} - \bar{Y}) > 1.2)$

$$\begin{aligned}
P((\bar{X} - \bar{Y}) > 1.2) &= P\left(Z > \frac{1.2 - 3}{3}\right) \\
&= P(Z > -0.6) \\
&= 1 - P(Z < -0.6) \\
&= 1 - 0.2743 \\
&= \boxed{0.7257}
\end{aligned}$$

Question 8: Geometric Distribution

Question

The probability that a birdwatcher sees an eagle on any given day is $\frac{1}{8}$. It is assumed that this probability is unaffected by whether he has seen an eagle on any other day. What is the probability that the birdwatcher first sees an eagle on the third day?

Solution

Let X = the day he sees an eagle for the first time.
$X \sim \text{Geometric}\left(p = \frac{1}{8}\right)$

Probability mass function:

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

Calculation:

$$\begin{aligned} P(X = 3) &= \left(\frac{7}{8}\right)^2 \times \frac{1}{8} \\ &= \frac{49}{64} \times \frac{1}{8} \\ &= \frac{49}{512} \\ &= \boxed{0.0957} \end{aligned}$$

Interpretation: The birdwatcher must not see an eagle on days 1 and 2, then see one on day 3.

Question 9: Fair Game Expected Value

Question

Three coins are thrown. If one head turns up, Rs 1.00 is paid. If two heads turn up, Rs 3.00 is paid, and if three heads turn up Rs 5.00 is paid. If the game is to be considered as fair, what should be the penalty if no heads turn up?

Solution

For a fair game, the expected value must be zero: $E(X) = 0$

Probability distribution:

Outcome	Probability	Payout
0 heads	$\frac{1}{8}$	$-x$ (penalty)
1 head	$\frac{3}{8}$	Rs 1.00
2 heads	$\frac{3}{8}$	Rs 3.00
3 heads	$\frac{1}{8}$	Rs 5.00

Expected value:

$$\begin{aligned} E(X) &= \frac{1}{8}(-x) + \frac{3}{8}(1) + \frac{3}{8}(3) + \frac{1}{8}(5) \\ 0 &= -\frac{x}{8} + \frac{3}{8} + \frac{9}{8} + \frac{5}{8} \\ 0 &= -\frac{x}{8} + \frac{17}{8} \\ \frac{x}{8} &= \frac{17}{8} \\ x &= \boxed{\text{Rs } 17.00} \end{aligned}$$

Answer: The penalty for no heads should be Rs 17.00 to make the game fair.

Question 10: Sampling Without Replacement

Question

Five identically shaped discs are in a bag; two of them are black and the rest are white. Discs are drawn at random from the bag in turn and not replaced. Let X be the number of discs drawn up to and including the first black one.

(i) List the values of X and the associated theoretical probabilities.

(ii) Calculate the mean value of X.

Solution

Setup: 5 discs total: 2 black, 3 white
Drawing without replacement
X = number of draws up to and including the first black disc

Part (i): Possible values: $X \in \{1, 2, 3, 4\}$

X	Calculation	$P(X = x)$
1	$\frac{2}{5}$	$\frac{2}{5} = 0.4$
2	$\frac{3}{5} \times \frac{2}{4}$	$\frac{6}{20} = 0.3$
3	$\frac{3}{5} \times \frac{2}{4} \times \frac{2}{3}$	$\frac{12}{60} = 0.2$
4	$\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2}$	$\frac{6}{60} = 0.1$

Part (ii): Mean value of X

$$\begin{aligned} E(X) &= \sum x \cdot P(X = x) \\ &= 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 \\ &= 0.4 + 0.6 + 0.6 + 0.4 \\ &= \boxed{2.0} \end{aligned}$$