SCS 1307
Probability & Statistics

by

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Independent Events

- Events A and B are said to be independent if one of the events can occur without being affected by the other.
- That is, if A is independent of B, then the probability of the occurrence of A is unchanged by the information as to whether or not B has occurred.
- That is two events A and B are independent if;

$$P(A \cap B) = P(A)\,P(B)$$

Property 1: If $P(A)>0$ and $P(B)>0$, then A and B are independent iff

$P(A \mid B) = P(A)$ and

$P(B \mid A) = P(B)$

Proof: If A and B are independent $P(A \cap B) = P(A)\,P(B)$

$$P(A \cap B) = P(A|B)\,P(B) = P(A)\,P(B)$$

Therefore $P(A \mid B) = P(A)$.

Now Consider $P(A \mid B) = P(A)$ is true

Then $P(A \mid B) = P(A \cap B) / P(B) \rightarrow P(A \cap B) = P(A)\,P(B)$

which means A and B are independent

Property 2

A and B are independent events implies that A and B′ are independent events

Independence of more than two events

- We illustrate the generalisation of the concept of independence to more than two events with reference to independence of three events.

Definition

- Three events A, B and C are mutually independent if and only if $P(A \cap B) = P(A)\,P(B)$,

$$P(A \cap C) = P(A)\,P(C)\ ,$$

$$P(B \cap C) = P(B)\,P(C) \text{ and}$$

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

Property

- If A, B and C are mutually independent events, then

  a). A and $B \cup C$ are independent events

  b). A and $B \cap C$ are independent events

  c). A, B and C' are mutually independent events

  d). A, B' and C' are mutually independent events

  etc.

Examples

- A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw. Check whether the A and B are independent.

Solution

Let A: “4 is obtained on the first throw”. Then $P(A) = 1/6$

Let B: “an odd number is obtained on the second die”. Then $P(B) = 3/6 = 1/2$

The result on the second throw is not affected in any way by the results on the first throw. Therefore A and B are independent events.

And also $P(A \cap B) = 3/36$ (show!!!)

That is $P(A \cap B) = 3/36 = 1/6 * 1/2 = P(A) * P(B)$

A and B are independent.

Examples…

- A bag contains 5 red counters and 7 black counters. A counter is drawn from the bag, the colour is noted and the counter is replaced. And a second counter is drawn. Let A be the event that the first counter is red and B be the event that the second counter is black. Are the events A and B independent? Find $P(A \cap B)$.

Solution…

- Let A: "the first counter is red". Then P(A)= 5/12

  Let B: "the second counter is black"

  since the first counter is replaced before the second draw is made, A and B are independent events.

  Now, P(B)= 7/12

  and $P(A \cap B) = P(A)P(B) = \left(\frac{5}{12}\right)\left(\frac{7}{12}\right) = \left(\frac{35}{144}\right)$

  Therefore A and B are independent.

Examples…

- Events A and B are such that $P(A)=1/3$, $P(A \cap B)=1/12$. If A and B are independent events, find

  a). $P(B)$ b). $P(A \cup B)$

Solution…

a). Since A and B are independent

$P(A \cap B) = P(A) * P(B) = 1/12$

$\Rightarrow P(B) = 1/4$

b).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

Examples…

- A fair die is thrown twice. Find the probability that
  - a). Neither throw results 4
  - b). At least one throw results 4

Solution…

a) A: obtain 4 in the first throw

B: obtain 4 in the second throw

$$P(nether\,throw\,is\,4) = P(\overline{A} \cap \overline{B})$$

$$= P(\overline{A}) \times P(\overline{B}) = \left(\frac{5}{6}\right) \times \left(\frac{5}{6}\right) = \frac{25}{36}$$

b)

$$P(at\,least\,one\,throw\,is\,4)$$

$$= 1 - P(neither\,result\,is\,4) = 1 - \frac{25}{36} = \frac{11}{36}$$

Examples…

- Two events A and B are such that $P(A)=1/4$, $P(A|B)=1/2$, $P(B|A)=2/3$

  a). Are the events A and B independent?

  b). Find $P(A \cap B)$

  c). Are A and B mutually exclusive?

  d). $P(B)$

Solution…

a). If A and B are independent, then P(A|B)=P(A).

Since P(A|B)= ½ and P(A)=1/4,

A and B are not independent.

b).

$$P(A \cap B) = P(B \mid A)P(A) = \left(\frac{2}{3}\right) \times \left(\frac{1}{4}\right) = \frac{1}{6}$$

Solution…

c). If A and B are mutually exclusive, then P(A $\cap$ B)=0.

Since P(A $\cap$ B) =1/6 $\neq$ 0 A and B are not mutually exclusive events.

d).

$$
\begin{aligned}
P(A \mid B)P(B) &= P(B \mid A)P(A) \\
\left(\frac{1}{2}\right)P(B) &= \left(\frac{2}{3}\right)\left(\frac{1}{4}\right) \\
\Rightarrow \qquad P(B) &= \frac{1}{3}
\end{aligned}
$$

Exercises

1) Toss a fair coin three times and observe the sequence of Heads and Tails. Define the following events

H1 = Heads on first toss

H2 =Heads on second toss

H3 =Heads on third toss

A =all tosses the same

a)Are the events H1, H2, H3 mutually independent ?

b)Are the events A, H1 and H2 mutually independent?