Foundations of Algorithm SCS1308

Dr. Dinuni Fernando
PhD

Senior Lecturer

The theory of NP-completeness

- Tractable and intractable problems able to solve in polynomial time or not
- NP-complete problems able to verify in polynomial time

Classifying problems

- Classify problems as tractable or intractable.
- Problem is *tractable* if there **exists at least one** polynomial bound algorithm that solves it
- An algorithm is *polynomial bound* if its worst case time complexity is bounded by a polynomial $p(n)$ in the size n of the problem

$$p(n) = a_n n^k + \ldots + a_1 n + a_0 \text{ where } k \text{ is a constant}$$

Intractability

- Dictionary definition : Difficulty to treat or work
- Problem in computer science is intractable if a computer has difficulty solving it. [Vague]

Polynomial-time algorithm

- A polynomial-time algorithm is one whose worst-case time complexity is bounded above by a polynomial function of its input size. That is, if n is the input size, there exists a polynomial p (n) such that

$$W(n) \in O(p(n))$$

Intractability

- Algorithms with the following worst-case time complexities are all polynomial-time.

$2n$ $3n^3 + 4n$ $5n + n^{10}$ $nlgn$

- Algorithms with the following worst-case time complexities are not polynomial-time.

-2^n $2^{0.01n}$ $2^{\sqrt{n}}$ $n!$

Intractability

$$nlgn < n^2$$

nlgn is not polynomial in n.

- It is bounded by a polynomial in n, which means that an algorithm with this time complexity satisfies the criterion to be called a polynomial-time algorithm.
- In computer science, a problem is called intractable if it is impossible to solve it with a polynomial-time algorithm. We stress that intractability is a property of a problem; it is not a property of any one algorithm for that problem.
- For a problem to be intractable, there must be no polynomial-time algorithm that solves it. Obtaining a nonpolynomial-time algorithm for a problem does not make it intractable.

Example 2^n vs n^{10}

- We can create extreme examples in which a non-polynomial-time algorithm is better than a polynomial-time algorithm for practical input sizes.
- For example, if n = 1, 000, 000.

$$2^{0.00001} = 1024 \text{ whereas } n^{10} = 10^{60}$$

Problems in Intractability

1. Problems for which polynomial-time algorithms have been found
2. Problems that have been proven to be intractable
3. Problems that have not been proven to be intractable, but for which polynomial-time algorithms have never been found

Problems for which polynomial-time algorithms have been found

- Any problem for which we found a polynomial-time algorithm falls in this first category.
- $\Theta(n \lg n)$ algorithms for sorting
- a $\Theta(\lg n)$ algorithm for searching a sorted array
- a $\Theta(n^{2.38})$ algorithm for matrix multiplication
- a $\Theta(n^3)$ algorithm for chained matrix multiplication, and so on.
- Because n is a measure of the amounts of data in the inputs to these algorithms, they are all polynomial-time.

Problems that have been proven to be intractable

- Two types of problems here.
- Type one problem - require a non-polynomial amount of output.
- The problem of determining all Hamiltonian Circuits. If there was an edge from every vertex to every other vertex, there would be $(n - 1)!$ such circuits. To solve the problem, an algorithm would have to output all of these circuits,
- Determining all possible circuits are nonpolynomial amount of output.

Problems that have been proven to be intractable

- Type two problem : when the type of intractability occurs when our requests are reasonable (that is, when we are not asking for a nonpolynomial amount of output).
- We can prove that the problem cannot be solved in polynomial time.
- Found relatively few such problems.
  - The first ones were undecidable problems (it can be proven that algorithms that solve them cannot exist)
  - Eg: The most well-known of these is the Halting problem. In this problem we take as input any algorithm and any input to that algorithm, and decide whether or not the algorithm will halt when applied to that input.
    - The output for a decision problem is a simple “yes” or “no” answer. Therefore, the amount of output requested is certainly reasonable.
  - Eg: Presburger Arithmetic, which was proven intractable by Fischer and Rabin in 1974.

Problems that have not been proven to be intractable, but for which polynomial-time algorithms have never been found

- This category includes any problem for which a polynomial-time algorithm has never been found, but yet no one has ever proven that such an algorithm is not possible.
- Eg: 0-1 Knapsack problem, the Traveling Salesperson problem, the Sum-of-Subsets problem, the m-Coloring problem for $m \geq 3$, the Hamiltonian Circuits problem, and the problem of abductive inference in a Bayesian network all fall into this category.
- There is a polynomial in n that bounds the number of times the basic operation is done when the instances are taken from some restricted subset. However, no such polynomial exists for the set of all instances. To show this, we need only find some infinite sequence of instances for which no polynomial in n bounds the number of times the basic operation is done.

Intractable problems

- Problem is *intractable* if it is not tractable.
- **1st Category: All** algorithms that solve the problem are not in polynomial bound.
- It has a worst case growth rate $f(n)$ which cannot be bound by a polynomial p(n) in the size n of the problem.
- For intractable problems the bounds are:

$$f(n) = c^n \text{ , or } n^{\log n} \text{, etc.}$$

Another set of intractable problems

- **2nd category**: Undecidable problems
  - Cannot give a “yes” or “no” answer
  - E.g., **Halting problem**
  - *No algorithm can be devised to solve the halting problem*

Halting problem

- Input: A string P and a string I. Consider P as a program and I as input to P.

- Output: 1 if P halts on I; 0 if P does not halt on I (infinite loop)

- **Theorem (Turing 1940): There is no program to solve the halting problem.** See next slide for proof.

Proof: Halting problem is undecidable

- Proof: To reach a contradiction, assume that there exists a program Halt(P, I) that solves the halting problem. Halt(P, I) returns true if and only if P halts on I. Otherwise, it returns false. Using Halt(P, I), we construct the following program Z:

```
program (string x)
begin
   If Halt (x, x) then
         while(1) printf ("ha ha ha ");
   Else exit(0)
end
```

Program x halts when given itself as input : returns true

Enters an infinite loop : contradicts the claim program doesn't halt.

returns false

Terminates immediately. Contradicting the claim that program does not halt.

Halting problem

```
program (string x)
begin
      If Halt (x, x) then
            while(1) printf (“ha ha ha ”);
      Else exit(0)
end
```

- Case 1: Program Z halts on input Z. By the correctness of Halt, Halt(Z, Z) returns true. Thus, program Z loops forever on input Z, printing “ha ha ha ….” Contradiction.

- Case 2: Program Z does not halt on input Z. Halt(Z, Z) returns false. Hence, program Z halts. Contradiction.

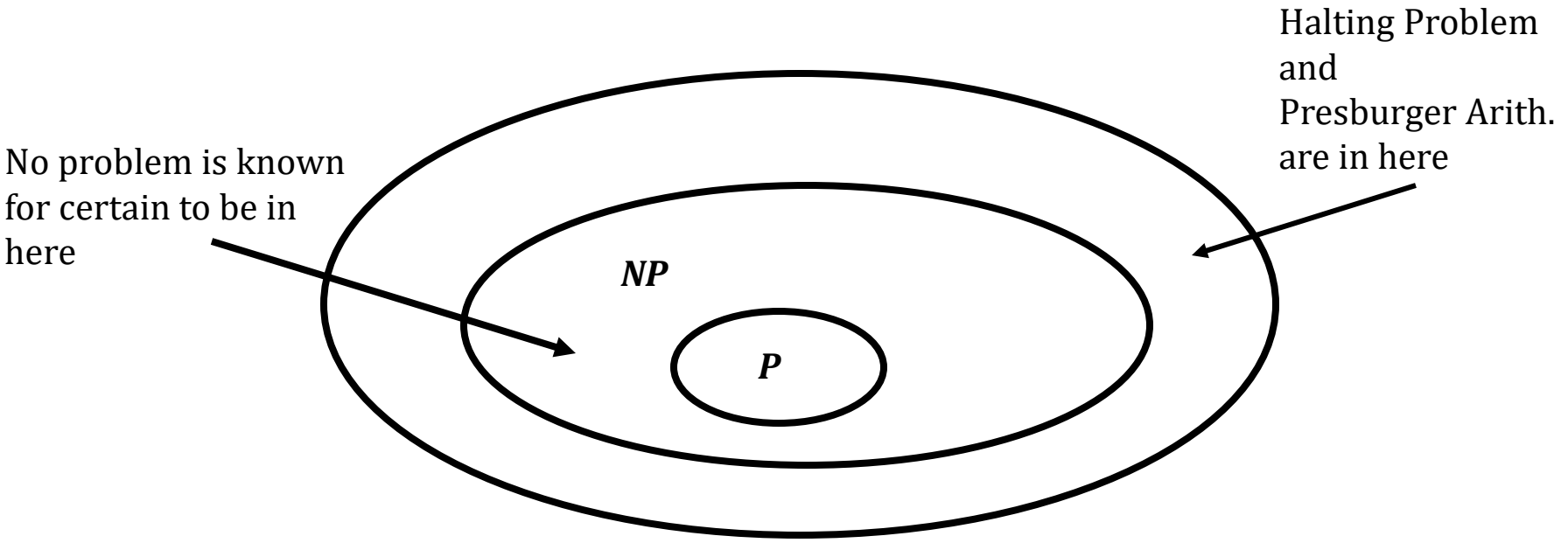
Why is this classification useful?

- If problem is intractable, no point in trying to find an *efficient* algorithm that solves the problem with polynomial time complexity in the worst case

- All algorithms will be too slow for **large inputs**.

Intractable problems

- Turing showed some problems are so hard that no algorithm can solve them (undecidable).
- Other researchers showed some decidable problems from automata, mathematical logic, etc. are intractable: Presburger arithmetic is doubly exponential.

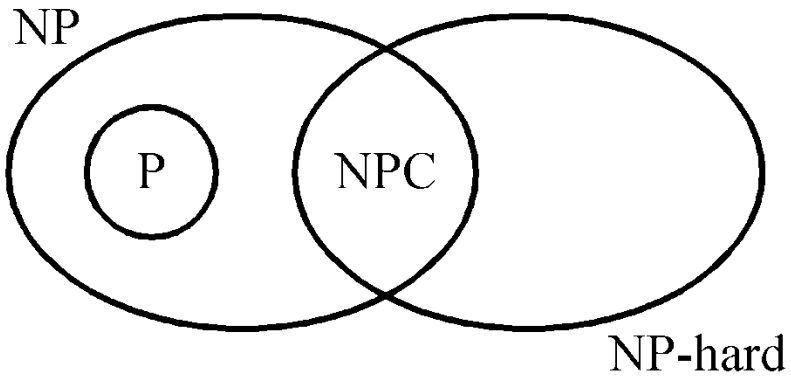


Problems Proven to be Intractable

- All Hamiltonian circuits: For a complete undirected graph, there are (n-1)! Circuits
- Halting problem: Undecidable
- Presburger Arithmetic
- …

Problems not proven to be intractable but no polynomial time algorithm

- 0-1knapsack
- Traveling salesperson
- Sum of subsets
- M-coloring for $m \geq 3$
- …



- NP is the class of decision problems (yes/no problems) for which a solution can be verified in polynomial time by a deterministic Turing machine.
- **P**: the class of problems which can be solved by a deterministic polynomial algorithm.
- NPC - are hardest problems in NP
- A problem X is NP-complete if:
  - X is in NP.
  - Every problem in NP can be reduced to X in polynomial time.
- NP Hard : problems are at least as hard as the hardest problems in NP but not necessarily belong to NP.

Coping with NP-Complete/NP-Hard Problems

- Rely on approximation algorithms, heuristics, etc.
- Sometimes we need to solve only a restricted version of the problem.
- If the restricted problem is tractable, design an algorithm for the restricted version

Nondeterministic algorithms

- A non-determinstic algorithm consists of

  phase 1: guessing

  phase 2: checking
- If the checking stage of a nondeterministic algorithm is of polynomial time-complexity, then this algorithm is called an NP (**nondeterministic polynomial**) algorithm.
- NP problems: (must be decision problems)
  - e.g. searching, MST, sorting
    satisfiability problem (SAT)
    traveling salesperson problem (TSP)

Nondeterministic operations and functions

- Choice(S) : arbitrarily chooses one of the elements in set S
- Failure : an unsuccessful completion
- Success : a successful completion
- Nonderministic searching algorithm:

```
j ← choice(1 : n)   /* guessing */
if A(j) = x then success  /* checking */
            else failure
```

- A nondeterministic algorithm terminates unsuccessfully iff there exist no set of choices leading to a success signal.

- The time required for *choice(1 : n)* is O(1)

Hard practical problems

- There are many practical problems for which *no one has yet* found a polynomial bound algorithm.
- Examples: 3-SAT, traveling salesperson, 0/1 knapsack, sum of subsets, graph coloring, bin packing etc.
- Most design automation problems such as testing and routing.
- Many OS, networks, database and graph problems.

Traveling Salesperson Problem

- Let a weighted, directed graph be given.
- A tour in such a graph is a path that starts at one vertex, ends at that vertex, and visits all the other vertices in the graph exactly once, and that the Traveling Salesperson Optimization problem is to determine a tour with minimal total weight on its edges.

- The Traveling Salesperson Decision problem is to determine for a given positive number d whether there is a tour having total weight no greater than d.

0-1 Knapsack problem

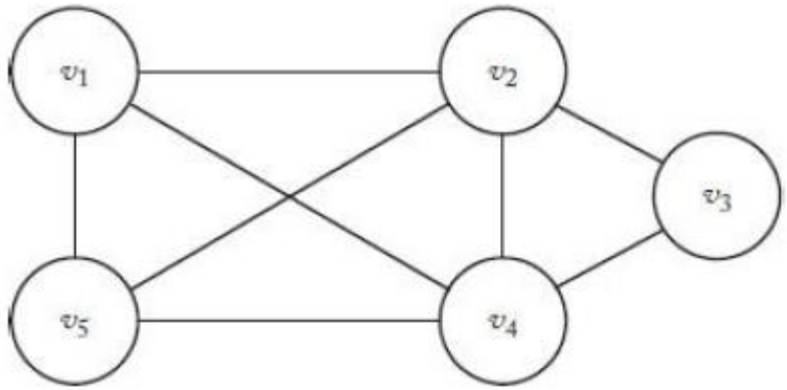
- **0-1 Knapsack Optimization problem** is to determine the maximum total profit of the items that can be placed in a knapsack given that each item has a weight and a profit, and that there is a maximum total weight W that can be carried in the sack.

- The **0-1 Knapsack Decision problem** is to determine, for a given profit P, whether it is possible to load the knapsack so as to keep the total weight no greater than W, while making the total profit at least equal to P.

Graph-Coloring Problem

- The Graph-Coloring Optimization problem is to determine the minimum number of colors needed to color a graph so that no two adjacent vertices are colored the same color.
- That number is called the chromatic number of the graph.
- The Graph-Coloring Decision problem is to determine, for an integer m, whether there is a coloring that uses at most m colors and that colors no two adjacent vertices the same color.

Clique Problem



- A clique in an undirected graph G = (V, E) is a subset W of V such that each vertex in W is adjacent to all the other vertices in W.
- {v2, v3, v4} is a clique, whereas {v1, v2, v3} is not a clique because v1 is not adjacent to v3. A maximal clique is a clique of maximal size. The only maximal clique in the graph in Figure 9.1 is {v1, v2, v4, v5}.
- The Clique Optimization problem is to determine the size of a maximal clique for a given graph.
- The Clique Decision problem is to determine, for a positive integer k, whether there is a clique containing at least k vertices.

Traveling Salesperson Decision problem

- Suppose, for some instance, answer was “yes”
  - For some graph and number d, a tour existed in which weigh was no greater than d.
  - We could write the algorithm that follows to verify whether what they produced was a tour with weight no greater than d.

```
Bool verify ( weighted digraph G, number d, claimed_tour S)
{
If(S is a tour && the weight of the edge in S is <= d)
        return true;
else
        return false;
}
```

- first checks to see whether S is indeed a tour. If it is, the algorithm then adds the weights on the tour.
- If the sum of the weights is no greater than d, it returns “true.”
- If weights exceeds d algorithm returns “false”.

Non-deterministic algorithm

1. Guessing (Nondeterministic) Stage: Given an instance of a problem, for some string S. The string can be thought of as a guess at a solution to the instance.

2. Verification (Deterministic) Stage: The instance and the string S are the input to this stage. This stage then proceeds in an ordinary deterministic manner either

(1) eventually halting with an output of “true,” which means that it has been verified that the answer for this instance is “yes,”

(2) halting with an output of “false,” or

(3) not halting at all (that is, going into an infinite loop). In these latter two cases, it has not been verified that the answer for this instance is “yes.”

Polynomial-time nondeterministic algorithm

- A polynomial-time nondeterministic algorithm is a nondeterministic algorithm whose verification stage is a polynomial-time algorithm.

- So what's set of NP
- NP is the set of all decision problems that can be solved by polynomial-time nondeterministic algorithms.

Satisfiability (SAT) problem

- CNF-Satisfiability Decision problem is to determine, for a given logical expression in CNF, whether there is some truth assignment that makes the expression true.

Conjunctive Normal Form (CNF)

- A literal is a variable or the negation of a var.
  - Example: The variable *x* is a literal, and its negation, $\neg x$, is a literal.
- A clause is a disjunction (an OR) of literals.
  - Example: $(x \vee y \vee \neg z)$ is a clause
- A formula is in Conjunctive Normal Form (CNF) if it is a conjunction (an AND) of clauses.
  - Example: $(x \vee \neg z) \wedge (y \vee z)$ is in CNF.
- A CNF formula is a conjunction of disjunctions, i.e., a product (AND) of sums (OR)

Example

- What is the answer to CNF –satisfiability ?

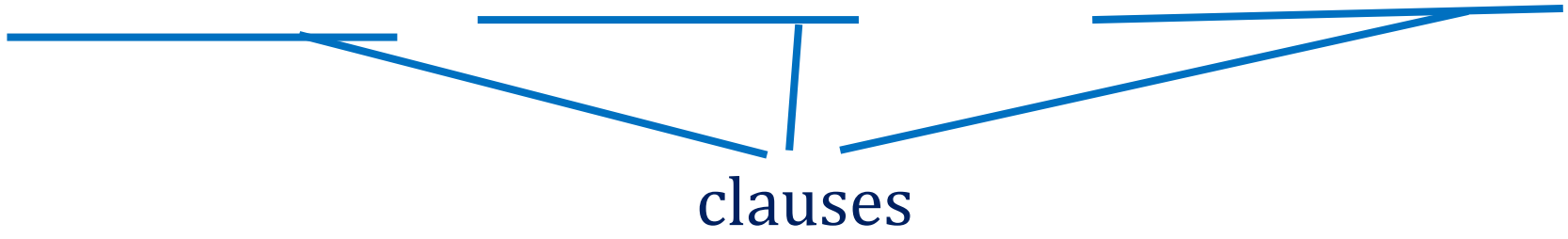
A. $(x_1 \vee x_2) \wedge (x_2 \vee \neg x_3) \wedge \neg x_2$

- X_1 = true
- X_2 = false
- X_3 = false

How about B. $(x_1 \vee x_2) \wedge \neg x_1 \wedge \neg x_2$

Definition: A CNF formula is a 3CNF-formula iff each clause has exactly 3 literals.

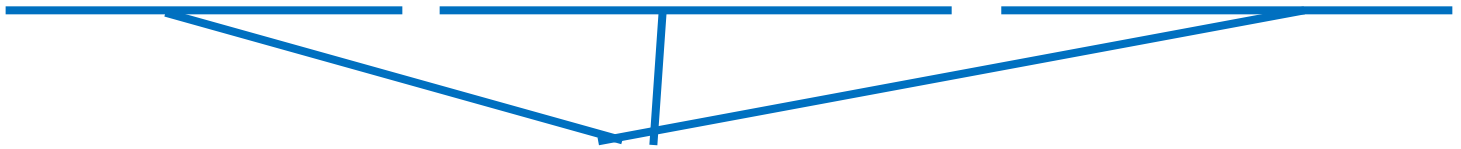
$$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_2 \vee x_5) \wedge \ldots \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$



- A literal is a variable or the negation of a var.
- A clause is a disjunction (an OR) of literals.
- A formula is in Conjunctive Normal Form (CNF) if it is a conjunction (an AND) of clauses.
- A CNF formula is a conjunction of disjunctions of literals.

Definition: A CNF formula is a 3CNF-formula iff each clause has exactly 3 literals.

$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_2 \vee x_5) \wedge \ldots \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$



clauses

YES $(x_1 \vee \neg x_2 \vee x_1)$

NO $(x_3 \vee x_1) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$

NO $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_4 \vee x_2 \vee x_1) \vee (x_3 \vee x_1 \vee \neg x_1)$

NO $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_3 \wedge \neg x_2 \wedge \neg x_1)$

3SAT = { ϕ | ϕ is a satisfiable 3cnf-formula }

Boolean Basics: Literals, Clauses, CNF

- Boolean function on n variables is a mapping $\{0,1\}^n \rightarrow \{0,1\}$
- Literal = Boolean variable or its negation
- Clause = disjunction of literals (no complementary pair)
- Conjunctive Normal Form (CNF) = conjunction of clauses, i.e., product-of-sums (Fact: Every Boolean function has a CNF representation)

- It is easy to write a polynomial-time algorithm that takes as input a logical expression in CNF and a set of truth assignments to the variables and verifies whether the expression is true for that assignment.
- Therefore, the problem is in NP.
- No one has ever found a polynomial-time algorithm for this problem, and no one has ever proven that it cannot be solved in polynomial time.
- So, we do not know if it is in P.
- In 1971, Stephen Cook published a paper proving that if CNF-Satisfiability is in P, Then P = NP

How are they handled?

- A variety of algorithms based on backtracking, branch and bound, dynamic programming, etc.

- None can be shown to be polynomial bound (exponential in the worst case)

Cook's theorem

- SAT is NP-complete
- 3-SAT is NP-complete (1-SAT or 2-SAT is P)
- It is the first NP-complete problem
- Every NP problem reduces to SAT
- NP = P iff the SAT problem is a P problem

NP Complete – Definition

- A problem B is called NP-complete if both of the following are true:

1. B is in NP.
2. For every other problem A in NP,

$$A \propto B$$

Theory of NP completeness

- The theory of NP-completeness enables showing that these problems are at least as hard as *NP-complete* problems
- Practical implication of knowing a problem is NP-complete is that it is **probably** intractable (whether it is or not has not been proved yet)
- So any algorithm that solves it will probably be very slow for large inputs

We will need to discuss

- Decision problems
- Converting optimization problems into decision problems
- The relationship between an optimization problem and its decision version
- The class P
- Verification algorithms
- The class NP
- The concept of polynomial transformations
- The class of NP-complete problems

Satisfiability (SAT) problem

- Is there a truth assignment to the n variables of a logical expression in CNF which makes the value of the expression true?
- The answer is yes, if all clauses evaluate to true
- Otherwise, the answer is “no”

SAT problem

- p=T, q=F, r=T and s=T is a truth assignment for:

$(p \vee q \vee s) \wedge (\neg q \vee r) \wedge (\neg p \vee r) \wedge (\neg r \vee s) \wedge (\neg p \vee \neg s \vee \neg q)$

- Note that if q=F then $\neg q$=T

- Each clause evaluates to true

A verification algorithm for SAT

1. Check that the certificate s is a string of exactly n characters which are T or F.
2. **while** (there are unchecked clauses) {
   select next clause
   if (clause evaluates to false) **return**(“no”) }
3. **return** (“yes”)

- Is verification algorithm polynomial bound?
- Satisfiability is in NP since there exists a polynomial bound verification algorithm for it

Cook's theorem

- **SAT (at least 3-SAT) problem is NP complete**
  - Cook proved that SAT is NP and every problem in NP reduces to SAT
  - First problem proved to be NP complete
  - Proof idea: encode the workings of a Nondeterministic Turing machine for an instance *I* of problem $X \in NP$ as a SAT formula so that the formula is satisfiable iff the nondeterministic Turing machine accepts the instance *I*

- After Cook's theorem, many NP-complete problems are found
  - E.g., $3\text{-SAT} \propto$ Clique, 3-SAT $\propto$ Hamiltonian Cycle Decision Problem, SAT $\propto$ 3-coloring, …
  - How to do this? See the following slides

- More NP-complete problems are found from NP complete problems that are not 3-SAT
  - E.g., Hamiltonian circuit $\propto$ Traveling Salesperson, Clique $\propto$ vertex cover …

Shortcut for NP-completeness Proofs

- **To prove a problem *L* is NP-complete:**
  - **Prove L ∈ NP.**
  - **Choose L' ∈ NPC, and show L’∝ L**
    - **Transitivity**

Reductions

For example, let’s discuss how to:

- Reduce 3-SAT to Clique
- Reduce Clique to Vertex Cover
- Reduce 3-SAT to Hamiltonian Cycle
- Reduce Hamiltonian Cycle to TSP

Clique

- Show clique is a NP-complete problem via reduction

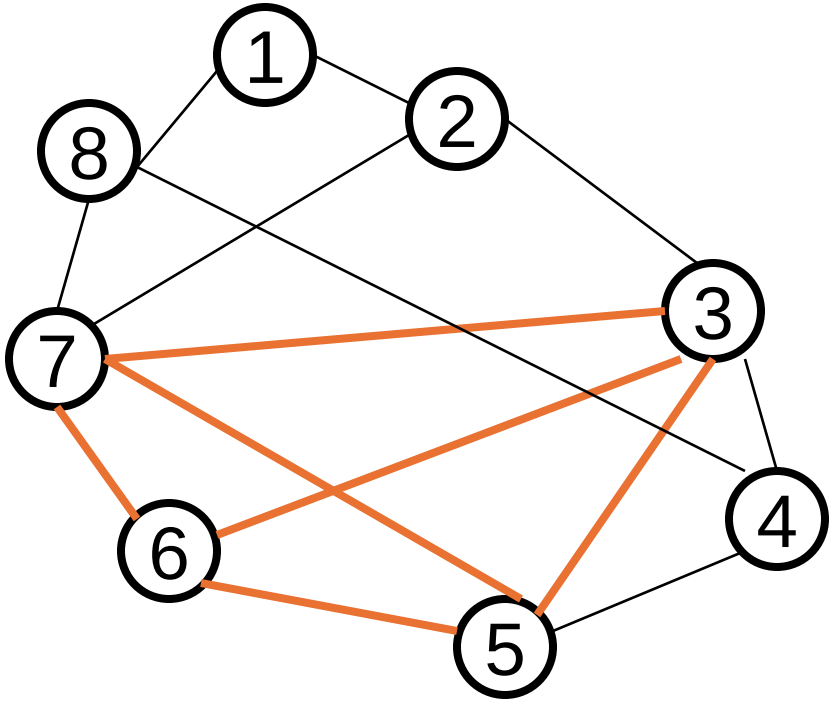
The Clique Problem

- **A *clique* is a complete undirected graph where every vertex is connected to every other vertex.**

CLIQUE

- Input: **An undirected graph *G* and a positive integer *k*.**
- Output: **YES iff a clique of size *k* exists in *G*.**

Clique example



- G contains a clique of 4 (with vertices 3, 5, 6, 7)
- The 4 people 3, 5, 6, 7 “know” (can work with each other) each other

The Clique Problem

- **Theorem**: CLIQUE is NP-complete.
- **Proof**:
- **Step 1**. CLIQUE $\in$ NP

  Given a certificate that contains a set of k vertices $V' \subseteq V$, we can check if V' forms a clique by checking for every pair of nodes u, v $\in$V' that (u,v) $\in$ E

- Clearly, this can be done in polynomial time.

The Reduction

Step 2. Selection

3-CNF-SAT which is NP-Complete.

Step 3. Mapping

For a formula $C_1 \wedge \ldots \wedge C_k$ such that $C_r = l_{1,r} \vee l_{2,r} \vee l_{3,r}$ we construct a graph G with vertices $v_{1,r}$ $v_{2,r}$ $v_{3,r}$ for r = 1,…, k, where $v_{i,r}$ represents the literal $l_{i,r}$

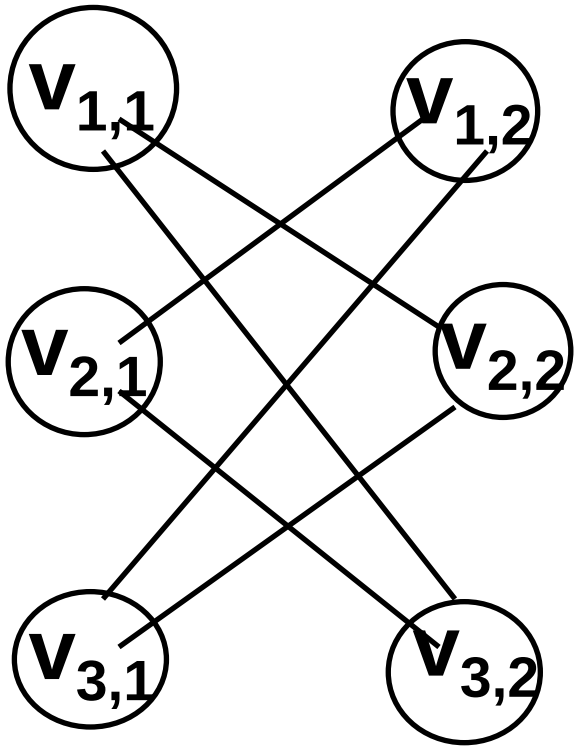
The Reduction

We put an edge between $v_{i,r}$ and $v_{j,s}$ if both of the following hold:

1. $r \neq s$ and

2. $l_{i,r}$ is not the negation of $l_{j,s}$.

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$$



k=2

The graph has 6 cliques of size 2

Step 4a. Yes for 3-Sat implies yes for clique

- **Assume formula satisfiable.**
- **With the satisfying assignment each clause contains at least 1 literal that is assigned 1.**
- **Since each literal from each clause is a vertex in the graph, if we pick out a literal that is assigned 1 from each of the k clauses, we get k vertices in the graph.**

Step 4a. Yes for 3-SAT implies yes for Clique

- This set of k vertices is a clique.
  - For any two vertices, the corresponding literals are from different clauses, and are both assigned 1, so they cannot be complements of a single variable
  - Thus, there is an edge between any two such vertices.

Step 4b. Yes for Clique implies yes for 3-Sat

- Assume G has a clique V' of size k
- No edge connects vertices in the same clause, so each of k triples has exactly one vertex in V'
- Assign 1 to each literal in V' without getting an inconsistent assignment (why?), and assign arbitrary values to the rest of the variables
- For this assignment, each clause is satisfied and thus the answer for 3-SAT is yes

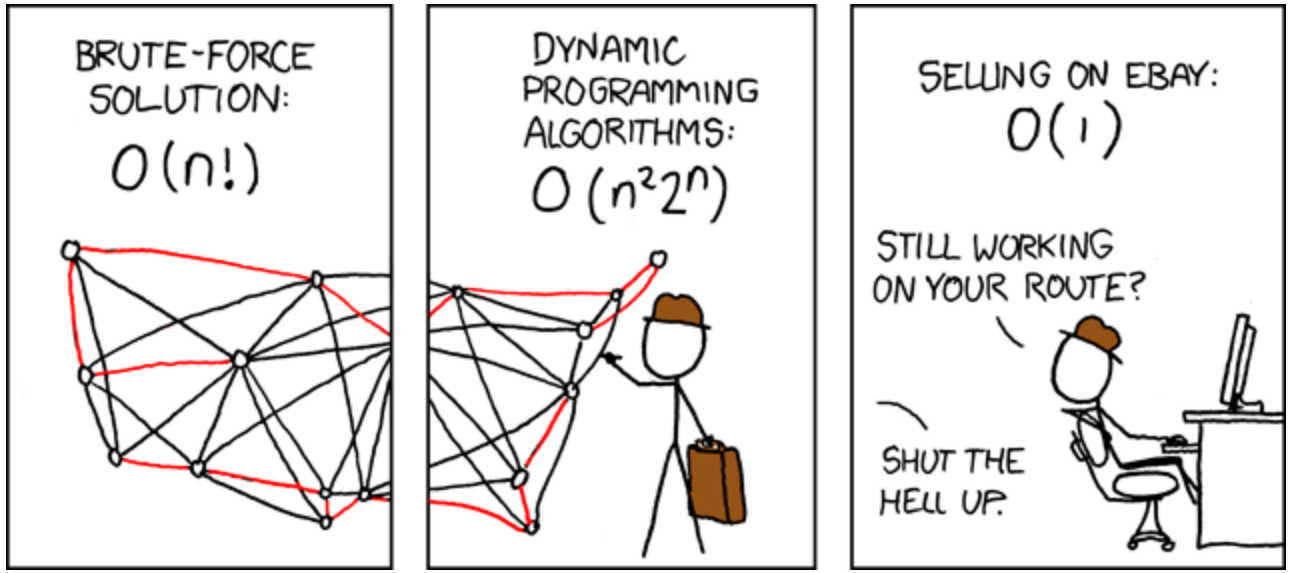
Step 5. Reduction is polynomial

- **Step 5. The reduction is polynomial.**
  - **The formula is read and 3k vertices are generated in O(k) steps. Then, each pair of literals from two different clauses is checked and an edge is added if the literals are not complimentary.**
  - **The reduction is $O(k^2)$**

Homework: Reduce 3SAT to Clique

$$F = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

Questions?


BRUTE-FORCE SOLUTION:
O(n!)
DYNAMIC PROGRAMMING ALGORITHMS:
O(n²2ⁿ)
SELLING ON EBAY:
O(1)
STILL WORKING ON YOUR ROUTE?
SHUT THE HELL UP.