Solving Linear Systems

(Linear Algebra)

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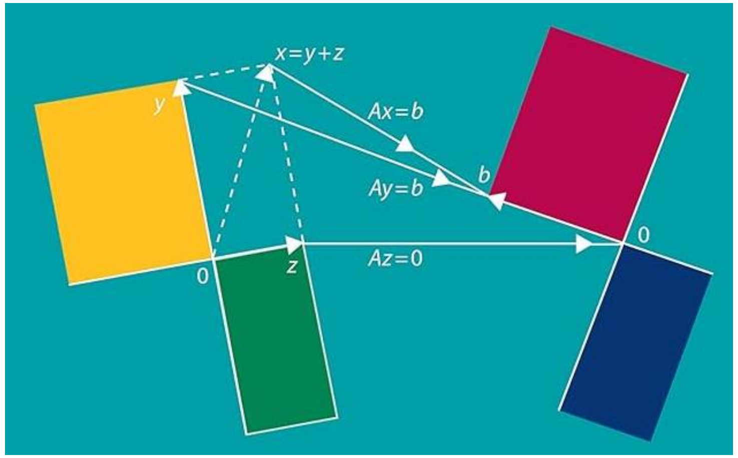


Column Way, Row Way, Matrix Way

Column way
Linear combination

$$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Row way
Two equations for c and d

$$\begin{aligned} v_1c + w_1d &= b_1 \\ v_2c + w_2d &= b_2 \end{aligned}$$

Matrix way
2 by 2 matrix

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Question 1

$$2x + y = 3$$
$$x - 2y = -1$$

Find (i) Row Picture
(ii) Column Picture
(iii) Matrix Form

Question 2

$$2x - y = 0$$
$$-x + 2y = 3$$

Find (i) Row Picture

(ii) Column Picture

(iii) Matrix Form

$$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

If the points $\boldsymbol{v}$ and $\boldsymbol{w}$ and the zero vector $\mathbf{0}$ are not on the same line, there is exactly one solution $\boldsymbol{c}, \boldsymbol{d}$. Then the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ exactly fill the xy plane. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **"linearly independent"**. The 2 by 2 matrix $A = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$ is "*invertible*".

Zero Vector Linearly Independent Invertible

Zero Vector

- Vector that has a zero magnitude and no direction.

- Example:
  - Suppose two people are pulling a rope from its two ends with equal force but in opposite directions. So, the net force applied to the rope will be a zero vector (null vector) as the two equal forces balance each other out because they are in opposite directions

Linearly Independent

- Two or more vectors are said to be linearly independent if none of them can be written as a linear combination of the others.

$$\begin{bmatrix}1\\1\\3\\3\end{bmatrix}\begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix}\begin{bmatrix}2\\5\\9\\6\end{bmatrix} \xrightarrow{\text{linear combination}} c_1\begin{bmatrix}1\\1\\3\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix} + c_3\begin{bmatrix}2\\5\\9\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Invertible

- A square matrix such that the product of the matrix and its inverse generates the identity matrix.

$$AB = BA = I_n$$

$$\Rightarrow B = A^{-1}$$

Where,

A is (nxn) invertible matrix.
B is (nxn) matrix called inverse of A.
I_n is (nxn) identity matrix.

Vectors in 3 - Dimensions

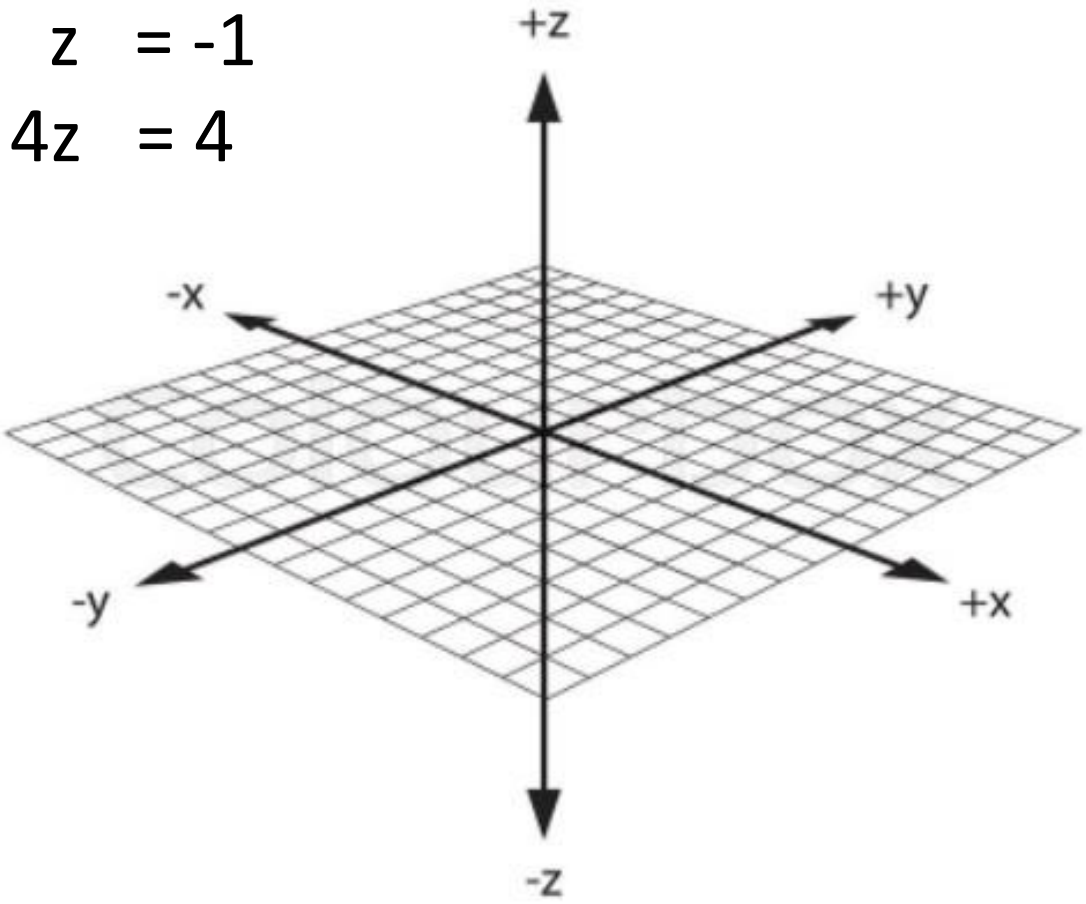
$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y - z &= -1 \\
-3y + 4z &= 4
\end{aligned}
$$

Find x, y and z

Vectors in 3 - Dimensions

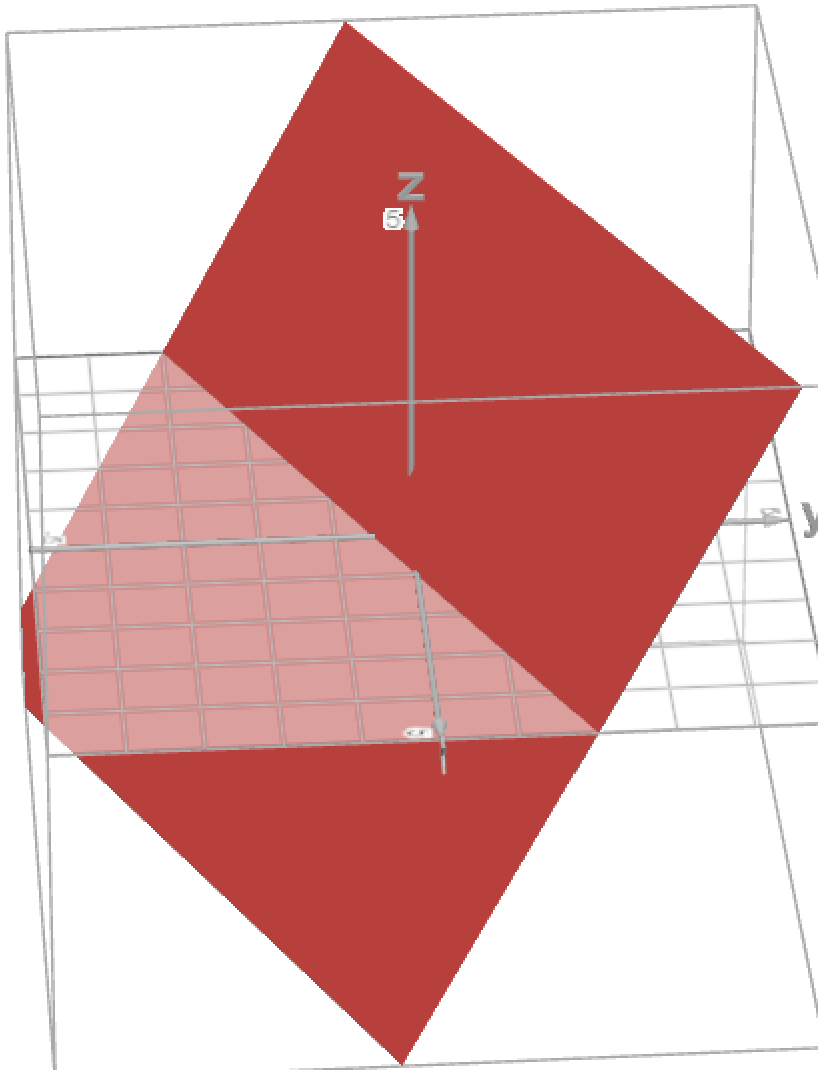
$$\begin{aligned} 2x &- y & &= 0 \\ -x &+ 2y &- z &= -1 \\ &-3y &+ 4z &= 4 \end{aligned}$$

Row Picture



Try: https://www.desmos.com/3d

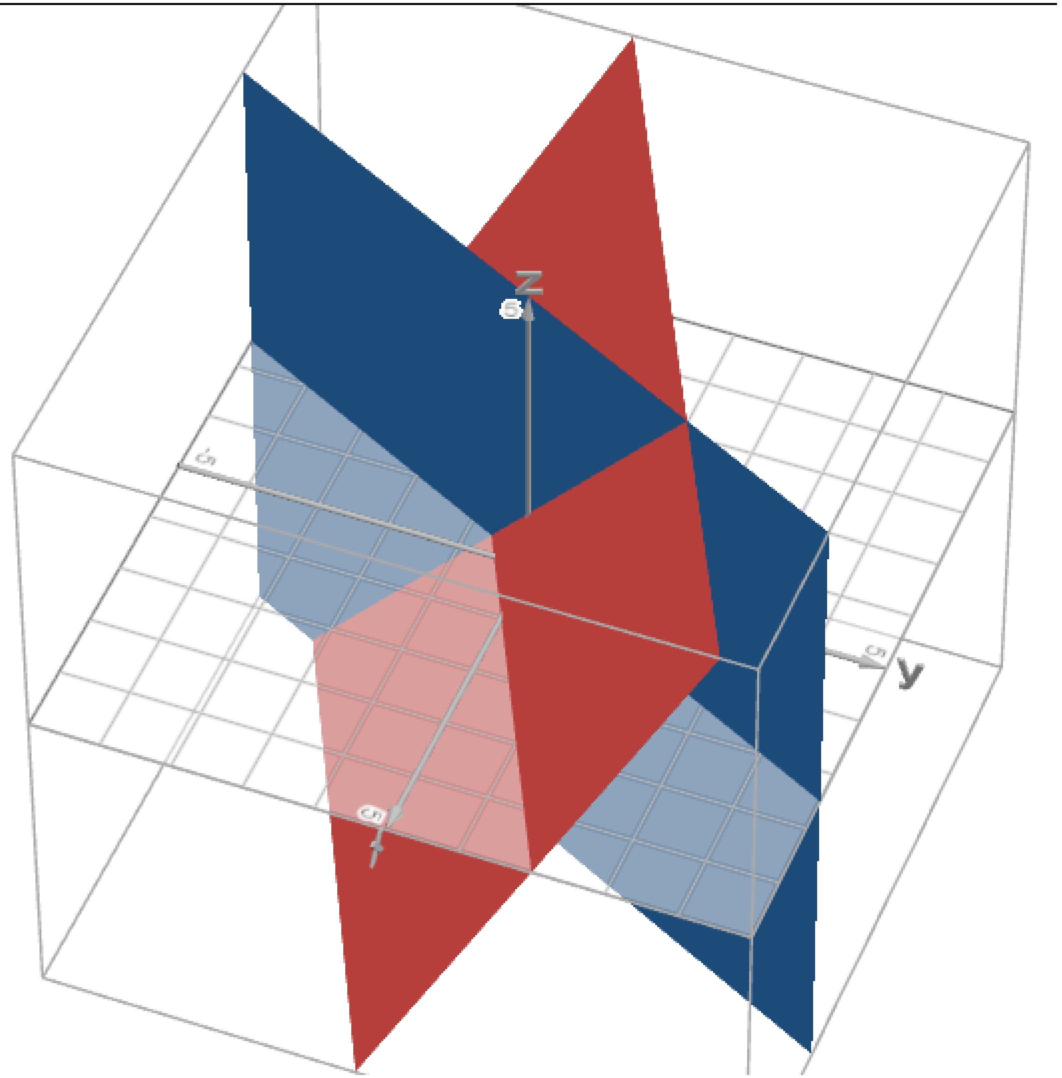
$$-x + 2y - z = -1$$



$$-x + 2y - z = -1$$

$$2x - y = 0$$

Extend to 3D

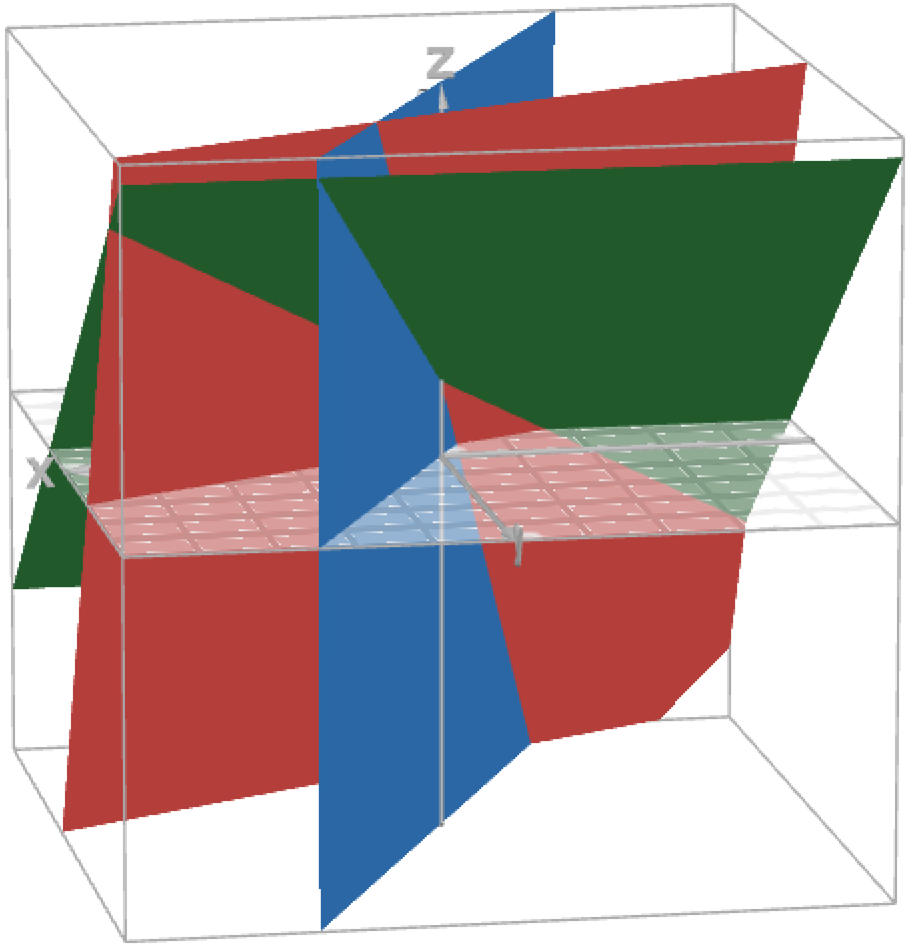


$$-x + 2y - z = -1$$

$$2x - y = 0$$

Extend to 3D

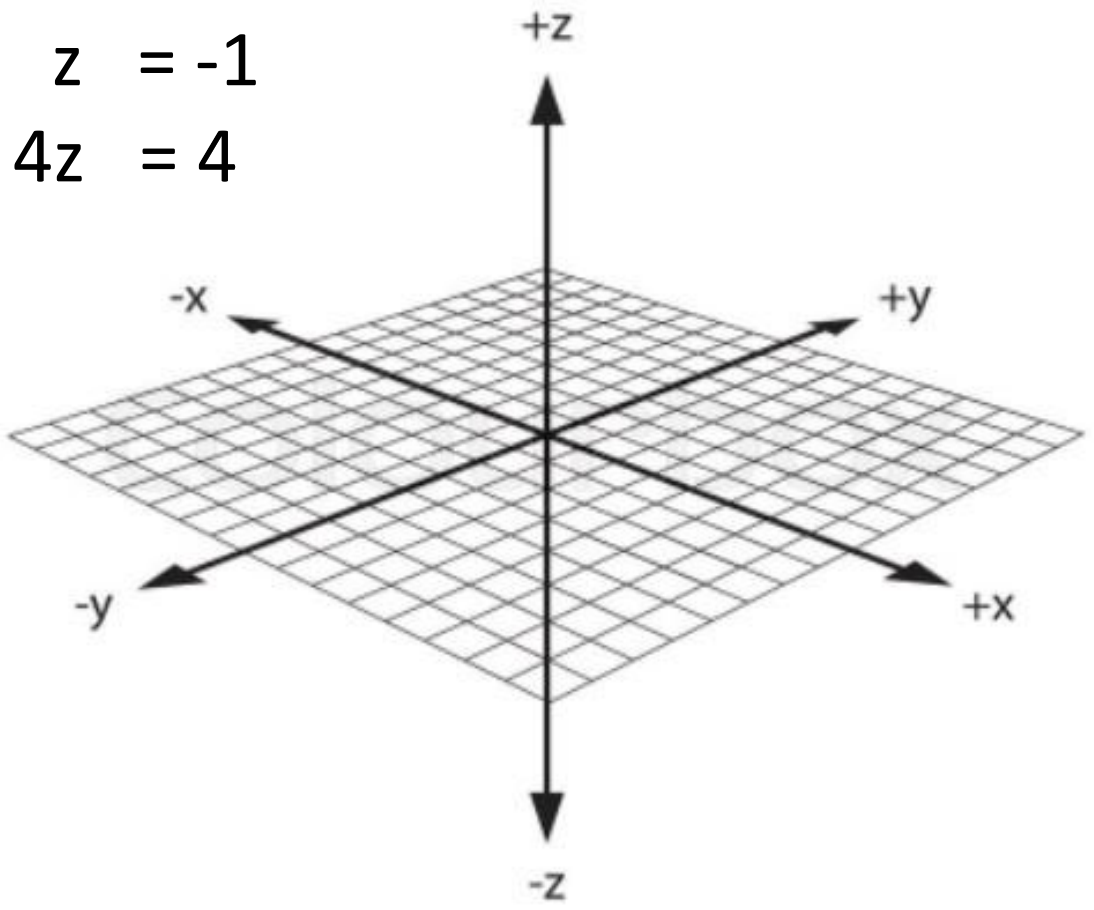
$$-3y + 4z = 4$$



Vectors in 3 - Dimensions

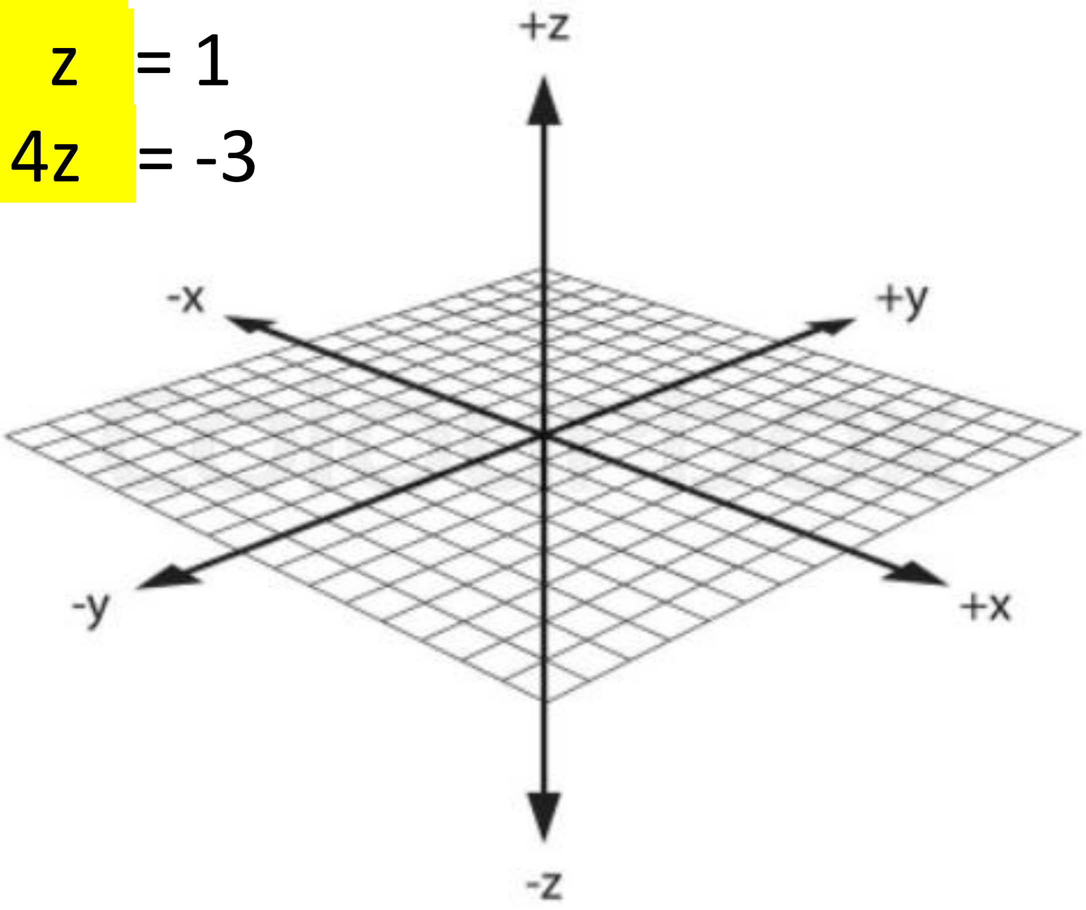
$$\begin{aligned} 2x &- y & &= 0 \\ -x &+ 2y &- z &= -1 \\ & -3y &+ 4z &= 4 \end{aligned}$$

Column Picture



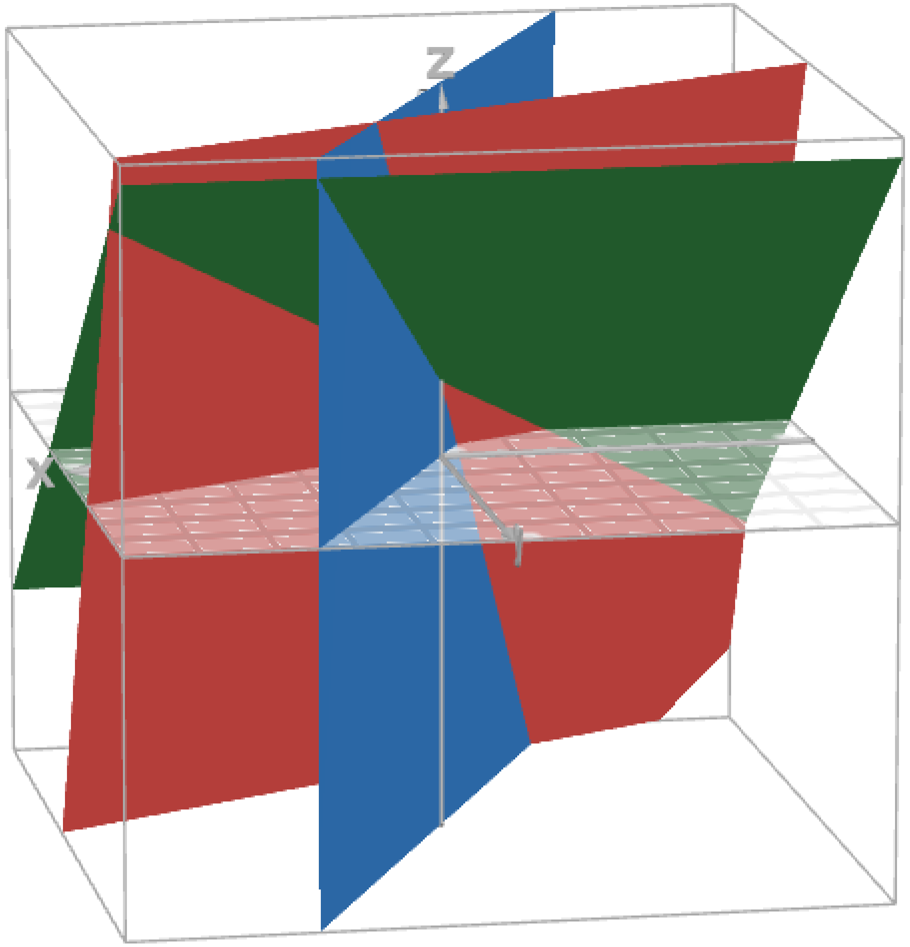
Vectors in 3 - Dimensions

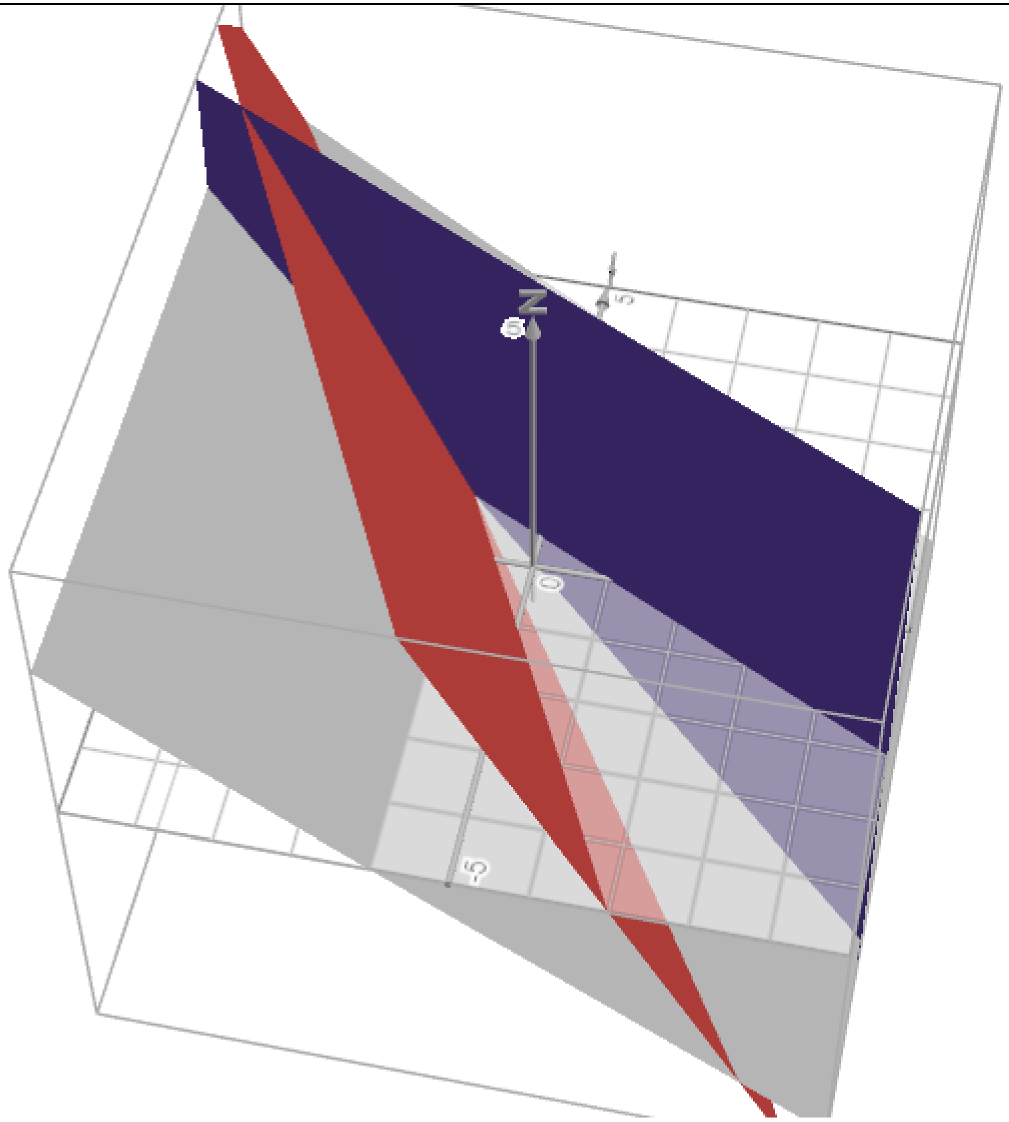
$$
\begin{aligned}
2x \; - \; y \quad\quad\quad\; &= 1 \\
-x + 2y - z &= 1 \\
-3y + 4z &= -3
\end{aligned}
$$



Column Picture

$x = ?, y = ?, z = ?$


Z
X
y


Z
5
0
-5

Matrix Form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$A.x = b$$

Matrices and Their Column Spaces

1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a **3 by 2 matrix**: $m = 3$ rows and $n = 2$ columns. Rank 2.

2 The 3 components of $A\boldsymbol{x}$ are dot products of the 3 rows of A with the vector $\boldsymbol{x}$:

Row at a time
A times $\boldsymbol{x}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 8 \\ 3 \cdot 7 + 4 \cdot 8 \\ 5 \cdot 7 + 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}.$$

3 $A\boldsymbol{x}$ is also a **combination of the columns of A** $\quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} \mathbf{7} \\ \mathbf{8} \end{bmatrix} = \mathbf{7} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \mathbf{8} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$

4 The **column space of A** contains **all combinations** $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ of the columns.

5 **Rank one matrices**: All columns of A (and all combinations $A\boldsymbol{x}$) are on **one line**.

Square Matrices and Types

- A typical matrix A is a rectangle of m times n numbers—m rows and n columns.
- If m equals n then A is a "square matrix".

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & -3 \\ 1 & 4 & 7 \\ -3 & 7 & 5 \end{bmatrix}$$

Identity matrix | **Diagonal** matrix | **Triangular** matrix | **Symmetric** matrix

Two ways to the same answer

The row picture of Ax will come from **dot products** of x with the **rows of** A.

The column picture will come from **linear combinations** of the **columns of** A.

We often think of the columns of A as vectors $a_1, a_2, \ldots, a_n$. Each of those n vectors is in m-dimensional space. In this example the a's have $m = 3$ components each:

$$\begin{array}{c} m = 3 \text{ rows} \\ n = 4 \text{ columns} \\ 3 \text{ by } 4 \text{ matrix } A \end{array} \qquad A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Row picture of $A\boldsymbol{x}$ Each row of A multiplies the column vector $\boldsymbol{x}$. Those multiplications *row times column* are dot products ! The first dot product comes from row 1 of A:

$$(\textbf{row 1}) \cdot \boldsymbol{x} = (-1, 1, 0, 0) \cdot (x_1, x_2, x_3, x_4) = \boldsymbol{x_2} - \boldsymbol{x_1}.$$

It takes m times n small multiplications to find the $m = 3$ dot products that go into $A\boldsymbol{x}$.

$$\begin{array}{c}\textbf{Three rows}\\ \textbf{Three dot}\\ \textbf{products}\end{array} \qquad A\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot \boldsymbol{x} \\ \text{row } 2 \cdot \boldsymbol{x} \\ \text{row } 3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x_2} - \boldsymbol{x_1} \\ \boldsymbol{x_3} - \boldsymbol{x_2} \\ \boldsymbol{x_4} - \boldsymbol{x_3} \end{bmatrix} \quad (1)$$

Notice well that each row of A has the same number of components as the vector $\boldsymbol{x}$. Four columns of A multiply x_1 to x_4. Otherwise multiplying $A\boldsymbol{x}$ would be impossible.

Column picture of Ax The matrix A times the vector x is **a combination of the columns of A**. The n columns are multiplied by the n numbers in x. Then add those column vectors $x_1 a_1, \ldots, x_n a_n$ to find the same vector Ax as in equation (1):

$$\boxed{Ax = x_1(\text{column } a_1) + x_2(\text{column } a_2) + x_3(\text{column } a_3) + x_4(\text{column } a_4)} \qquad (2)$$

This combination of n columns involves exactly the same multiplications as dot products of x with the m rows. But it is higher level! We have a vector equation instead of three dot products. You see the same result Ax in equation (1) above and equation (3) below.

Combination of columns

$$Ax = x_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} \qquad (3)$$

Independence, Dependence, and Column space

Example 1
Independent columns

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

Each column gives a new direction.

Their combinations fill 3D space $\mathbf{R}^3$.

Example 2
Dependent columns

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 6 & 0 & 6 \end{bmatrix}$$

Column 1 + column 2 = column 3

Their combinations don't fill 3D space

$$1 + 2 = 3$$
$$1 + 4 = 5$$
$$6 + 0 = 6$$

Independence, Dependence, and Column space

That plane is the column space of this matrix : Plane = all combinations of the columns.

Dependent columns in Example 2 $\boxed{\text{column 1} + \text{column 2} - \text{column 3 is } (0, 0, 0).}$

Example 3 $A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 15 & 20 \end{bmatrix}$ Now $\boldsymbol{a}_2$ is 3 times $\boldsymbol{a}_1$. And $\boldsymbol{a}_3$ is 4 times $\boldsymbol{a}_1$.

Every pair of columns is dependent.