Probability

* In instances where the result of an experiment cannot be predicted exactly, the probability is used to measure the possibility of occurrence of a certain result

Vocabulary used in probability

i. Random experiment

* If an experiment is repeated under the same condition and the set of possible outcomes can be predicted before conducting the experiment even though the exact result of the experiment cannot be predicted before conducting the experiment, it is known as a random experiment.

ii. Sample space

* The set of all possible outcomes of a random experiment is known as the sample space. It is denoted by 'S'.

iii. Sample point

* The elements of the sample space are known as sample points.
    - The sample points of the experiment of tossing a die are 1, 2, 3, 4, 5, 6.

iv. finite sample spaces

* If the number o elements of a sample space are finite, it is a finite sample space.
    - In the experiment of tossing a die, $S = \{1, 2, 3, 4, 5, 6\}$

v. Infinite sample spaces

* If the number of elements of a sample space is not finite, that sample space is an infinite sample space.

vi. Uncountable sample spaces

* Let us consider the experiment of tossing a coin till a head is obtained.
    - $S = \{H, TH, TTH, TTTH, \ldots\}$

vii. Continuous sample spaces

* If the elements of a sample space lie within a certain internal or number of intervals, this type of spaces are known as continuous sample spaces.
    - Heights of students in a school.

viii. Event

* Any subset of a sample space is known as an event.

ix. Null event

* The event which corresponds to null set is known as a null set, ϕ is also a subset of any sample space.

x. Complementary event

* The event consisting of all sample points which are not included in an event of the sample space of a random experiment is called the complementary event of that event.

xi. Simple events

* An event that cannot be dissociated into two or more events is called a simple event.

xii. Compound events

* If an event can be dissociated into two events or more, it is a compound event.

xiii. Event space

* The power set of the sample space, the set consisting of all the subsets of the sample space is called its event space.
    - If the no. of elements in the sample space is n, the no. of elements in the event space is 2^n.

xiv. Intersection of two events

* The event containing the elements common to any two events of the event space of a random experiment is known as the intersection. This is indicated by $A \cap B$. $A \cap B$ is the event that A or B occur simultaneously.

xv. Union of two events

* The event consisting of all the elements of any two events of the event space of a random experiment is called the union of the two events. It is indicated by $A \cup B$.

xvi. Mutually exclusive events.

* If any two events in the event space corresponding to the sample space of a random experiment are taken and if the events do not occur simultaneously, those events are known as mutually exclusive events.

xv. Equally likely events

* If the possibilities of number of events of a random experiment to occur are equal, these types of events are called equally likely events.

xvi. Exhaustive events

* If the union of number of events of a sample space of a random experiment is same as the sample space, these events are called exhaustive events.

Ways of Representing the Sample Space

* Consider a random exp iment of rolling 2 dice.

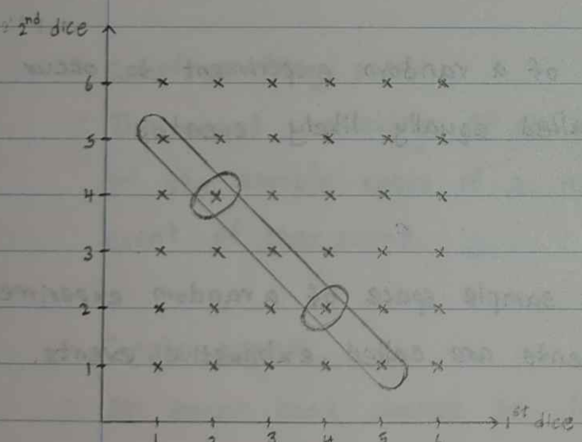
i. Rule method

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), \ldots, (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

ii. Roster method

$$S = \left\{ (x,y) \,\middle|\, \begin{matrix} x = 1,2,3,4,5,6 \\ y = 1,2,3,4,5,6 \end{matrix} \right\} \quad \therefore n(S) = 6 \times 6 = 36$$

iii. Cartesian method



$$A = \left\{ x+y=6 \,\middle|\, \begin{matrix} x = 1,2,3,4,5,6 \\ y = 1,2,3,4,5,6 \end{matrix} \right\}$$

$n(A) = 5$

$$B = \left\{ xy=8 \,\middle|\, \begin{matrix} x = 1,2,3,4,5,6 \\ y = 1,2,3,4,5,6 \end{matrix} \right\}$$

$n(B) = 2$

$A \cap B = (x+y=6)$ and $(xy=8)$

$A \cap B = \{(2,4), (4,2)\}$

$n(A \cap B) = 2$

$B \subset A$ (B is a proper subset of A)

$\therefore n(B \cap A) = n(B)$

* In addition to these methods, a tree diagram can also be used for experiments with smaller sample spaces.

Ex-1 $S = \{(H,H), (H,T), (T,T), (T,H)\}$

i. $A = \{(H,T), (T,T), (T,H)\}$

$B = \{(H,T), (T,T), (T,H)\}$

$C = \{(H,H)\}$

$D = \{(H,H), (T,T)\}$

ii. $A \cap B = \{(H,T), (T,T), (T,H)\}$

$B \cup D = \{(H,H), (H,T), (T,T), (T,H)\} = S$

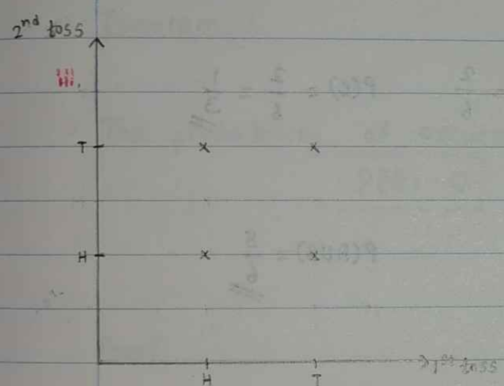
$C \cup D = \{(H,H), (T,T)\}$

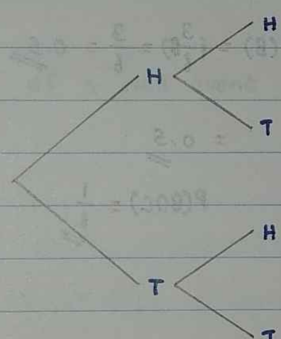
$C \cap D = \{(H,H)\}$

$\left.\begin{matrix} C \cup D \\ C \cap D \end{matrix}\right\}$ Since $C \subset D$; $C \cup D = D$, $C \cap D = C$

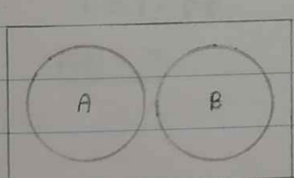
$A \cup D = \{(H,H), (H,T), (T,T), (T,H)\} = S$

iii.





When A and B are mutually exclusive

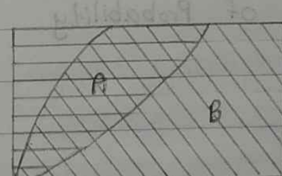


$A \cap B = \phi$

$P(A \cap B) = 0$

$\underline{\underline{P(A \cup B) = P(A) + P(B)}}$

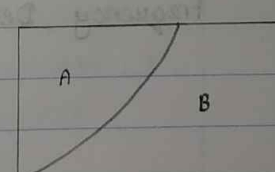
When A and B are mutually exhaustive



$A \cup B = S$

$\underline{\underline{P(A \cap B) = 1}}$

When A and B are mutually exclusive and exhaustive



$A \cup B = S$

$A \cap B = \phi$

$P(A \cup B) = P(A + P(B)$

$1 = P(A) + P(B)$

$\underline{\underline{P(B) = 1 - P(A)}}$

$\therefore$ A and B are complementary events.

Classical Definition of Probability

* If the sample space of a random experiment consists of N equally likely, mutually exclusive simple events and one of its events, event A consists of n_A simple events, the probability $P(A)$ of the event A is defined by;

$$\boxed{P(A) = \frac{n_A}{N}}$$

Ex-2

$S = \{1,2,3,4,5,6\}$

$A = \{2,4,6\}$

$B = \{2,3,5\}$

$C = \{5,6\}$

$A \cap B = \{2\}$

$B \cap C = \{5\}$

$A \cup B = \{2,3,4,5,6\}$

$B \cup C = \{2,3,5,6\}$

$A \cap B \cap C = \{\}$

$A \cup B \cup C = \{2,3,4,5,6\}$

$P(A) = \frac{3}{6} = 0.5$

$P(B) = \frac{3}{6} = 0.5$

$P(C) = \frac{2}{6} = \frac{1}{3}$

$P(A \cap B) = \frac{1}{6}$

$P(B \cap C) = \frac{1}{6}$

$P(A \cup B) = \frac{5}{6}$

$P(B \cup C) = \frac{4}{6} = \frac{2}{3}$

$P(A \cap B \cap C) = \frac{0}{6} = 0$

$P(A \cup B \cup C) = \frac{5}{6}$

Frequency Definition of Probability

Axioms

i. $P(A) \geqslant 0$; where A is any event

ii. $P(S) = 1$; where S is the sample space or in practical, a sure event

iii. If A and B are mutually exclusive events;

$$A \cap B = \phi$$

$$P(A \cap B) = 0$$

$$\underline{\underline{P(A \cup B) = P(A) + P(B)}}$$

Theorem 1

* The probability of occurrence of a null event is zero.

$$\boxed{P(\phi) = 0}$$

Proof

Since $S \cap \phi = \phi$;

Using Definitions

$$P(S \cup \phi) = P(S) + P(\phi)$$

$$P(S) = P(S) + P(\phi)$$

$$1 = 1 + P\phi$$

$$\underline{\underline{P(\phi) = 0}}$$

* If $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive events;

$$P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \dots + P(A_n)$$

$$\underline{\underline{P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)}}$$

Theorem 2

* If A is any event of the event space where A' is the complementary event of A;

$$\boxed{P(A') = 1 - P(A)}$$

Proof

Since $A \cap A' = \phi$;

$$P(A \cup A') = P(A) + P(A')$$

$$P(S) = P(A) + P(A')$$

$$1 = P(A) + P(A')$$

$$\underline{\underline{P(A') = 1 - P(A)}}$$

Theorem 3

* If A and B are any two events within the event space corresponding to the sample space of a random experiment;

$$\boxed{P(A) = P(A \cap B) + P(A \cap B')}$$

Proof

Since $(A \cap B) \cap (A \cap B') = \phi$;

$$P[(A \cap B) \cup (A \cap B')] = P(A \cap B) + P(A \cap B')$$

$$P[A \cap (B \cup B')] = P(A \cap B) + P(A \cap B')$$

$$P(A \cap S) = P(A \cap B) + P(A \cap B')$$

$$\underline{\underline{P(A) = P(A \cap B) + P(A \cap B')}}$$

Theorem 4

* If A and B are any two events in the event space corresponding to the sample space of a random experiment;

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

Proof

Since $A\cap(A'\cap B) = \phi$;

$$P[A\cup(A'\cap B)] = P(A) + P(A'\cap B)$$

$$P[(A\cup A')\cap(A\cup B)] = P(A) + P(B) - P(A\cap B)$$

$$P[S\cap(A\cup B)] = P(A) + P(B) - P(A\cap B)$$

$$\underline{\underline{P(A\cup B) = P(A) + P(B) - P(A\cap B)}}$$

Theorem 5

* When A and B are two events corresponding to a sample space of a random experiment;

$$\boxed{A\subseteq B;\ P(A)\leqslant P(B)}$$

Proof

Since $A\cap(A'\cap B) = \phi$

$$P[A\cup(A'\cap B)] = P(A) + P(A'\cap B)$$

$$P[(A\cup A')\cap(A\cup B)] = P(A) + P(A'\cap B)$$

$$P(A\cup B) = P(A) + P(A'\cap B) \text{ --- } (*)$$

If $A\subseteq B$;

$A\cup B = B$

$P(A'\cap B) \geqslant 0$

Using $(*)$;

$\underline{\underline{P(A)\leqslant P(B)}}$

Theorem 6

* If A, B and C are three events corresponding to a sample space of a random experiment;

$$\boxed{P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)}$$

Proof

$$
\begin{aligned}
L.H.S &= P(A\cup B\cup C) \quad [\text{Let } B\cup C = D] \\
&= P(A\cup D) \\
&= P(A) + P(D) - P(A\cap D) \\
&= P(A) + P(B\cup C) - P[A\cap(B\cup C)] \\
&= P(A) + P(B) + P(C) - P(B\cap C) - P[(A\cap B)\cup(A\cap C)] \\
&= P(A) + P(B) + P(C) - P(B\cap C) - [P(A\cap B) + P(A\cap C) - P(A\cap B\cap A\cap C)] \\
&= P(A) + P(B) + P(C) - P(B\cap C) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C) \\
&= P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C) \\
&= R.H.S
\end{aligned}
$$

$\boxed{\text{Ex-3}}$

i. $P(A\cap B) = P(A) + P(B) - P(A\cup B)$
$= 0.6 + 0.4 - 0.7$
$= 0.3$

ii. $P(A\cup B)' = 1 - P(A\cup B)$
$= 1 - 0.7$
$= 0.3$

iii. $P(A\cap B') = P(A) - P(A\cap B)$
$= 0.6 - 0.3$
$= 0.3$

iv. $P(A'\cup B') = P(A\cap B)'$
$= 1 - P(A\cap B)$
$= 0.7$

$\boxed{\text{Ex-4}}$

i. $P(A') = 0.5$

ii. $P(B) = P(A\cup B) - P(A) = 0.1$

iii. $P(A\cap B)' = 1$

Ex-6 a) $P(K) = \frac{4}{52} = \frac{1}{13}$

b) $P(F) = \frac{16}{52} = \frac{4}{13}$

c) $P(N) = \frac{36}{52} = \frac{9}{13}$

d) $P(H) = \frac{13}{52} = \frac{1}{4}$

e) $P(F \cap H) = \frac{4}{52} = \frac{1}{13}$

f) $P(F \cup S) = P(F) + P(S) - P(F \cap S)$

$= \frac{16}{52} + \frac{13}{52} - \frac{4}{52}$

$= \frac{25}{52}$

Ex-7 $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

$A = \{(3,5), (4,4), (4,5), (5,3), (5,4), (5,5)\}$

$B = \{(4,5), (5,4), (5,5)\}$

Since $B \subset A$;

$A \cup B = A$

$A \cap B = B$

$A' = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (5,1), (5,2)\}$

i. $P(A) = \frac{6}{25}$

ii. $P(B) = \frac{3}{25}$

iii. $P(A \cup B) = \frac{6}{25}$

iv. $P(A \cap B) = \frac{3}{25}$

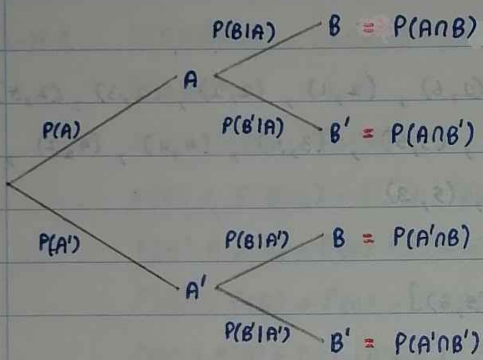
v. $P(A') = \frac{19}{25}$

Conditional Probability

* If A and B are two events in the event space corresponding to the sample space of a random experiment, then the probability of B, given that the event A has occurred, $P(B|A)$ is defined as;

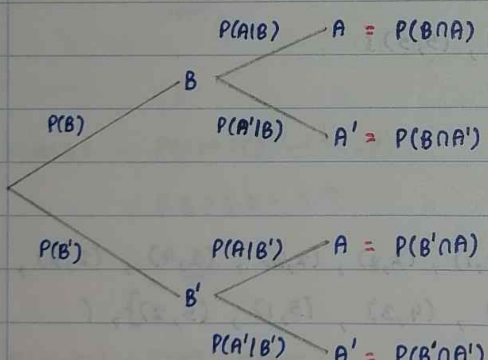
$$\boxed{P(B|A) = \frac{P(A\cap B)}{P(A)}} \; ; \text{ where } P(A) > 0$$

In general;
$P(B|A) \neq P(A|B)$



$\Longrightarrow P(A).P(B|A) = P(A\cap B)$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$



$\Longrightarrow P(B).P(A|B) = P(B\cap A)$

$$P(A|B) = \frac{P(B\cap A)}{P(B)}$$

$$\boxed{P(A'|B) = 1 - P(A|B)}$$

Proof

$$L.H.S = P(A'|B) = \frac{P(A'\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} - \frac{P(A\cap B)}{P(B)} = 1 - P(A|B) = R.H.S$$

Ex-8

$$S = \left\{ (x,y) \,\middle|\, \begin{matrix} x = 1,2,3,4,5,6 \\ y = 1,2,3,4,5,6 \end{matrix} \right\} \qquad n(S) = 36$$

$$A = \left\{ (x,y) \,\middle|\, \begin{matrix} x = 1,3,5 \\ y = 1,2,3,4\ 5,6 \end{matrix} \right\} \qquad P(A) = \frac{18}{36} = 0.5$$

$$B = \left\{ (x,y) \,\middle|\, \begin{matrix} x = 1,2,3,4,5,6 \\ y = 2,4,6 \end{matrix} \right\} \qquad P(B) = \frac{18}{36} = 0.5$$

$$A \cap B = \left\{ (x,y) \,\middle|\, \begin{matrix} x = 1,3,5 \\ y = 2,4,6 \end{matrix} \right\} \qquad P(A \cap B) = \frac{9}{36} = 0.25$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.25}{0.5} = \underline{\underline{0.5}}$$

Independency

* If the occurrence of A does not affect the occurrence of B, the two events are called independent events.

$P(B\|A) = P(B)$
$P(A \cap B) = P(A).P(B)$

Proof

If A and B are independent;

$$P(B|A) = P(B)$$

Generally $\Downarrow$ $\qquad$ $\Downarrow$ If independent

$$\frac{P(B \cap A)}{P(A)} = P(B)$$

$$\underline{\underline{P(A \cap B) = P(A).P(B)}}$$

* If $A_1, A_2, \dots, A_n$ are mutually independent events;

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1).P(A_2).P(A_3).\dots.P(A_n)$$

$$\underline{\underline{P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)}}$$

* If A and B are exhaustive events;

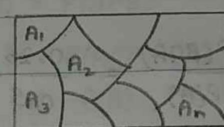
$$\boxed{A \cup B = S}$$
$$\boxed{P(A \cup B) = 1}$$

* If A and B are mutually exclusive and exhaustive events;

$$\boxed{P(A) + P(B) = 1}$$

* If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive and exhaustive events where $A_1, A_2, A_3, \ldots, A_n$ are known as partitions;

$$\boxed{P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) = 1}$$



De Morgan's Laws

i. $P(A' \cap B') = P(A \cup B)'$ $[1 - P(A \cup B)]$

ii. $P(A' \cup B') = P(A \cap B)'$ $[1 - P(A \cap B)]$

iii. $P(A' \cap B' \cap C') = P(A \cup B \cup C)'$ $[1 - P(A \cup B \cup C]$

iv. $P(A' \cup B' \cup C') = P(A \cap B \cap C)'$ $[1 - P(A \cap B \cap C]$

v. $P(A' \cap B')' = P(A \cup B)$

vi. $P(A' \cup B')' = P(A \cap B)$

vii. $P(A \cup B') = P(A' \cap B)'$ $[1 - P(A' \cap B) = 1 - \{P(B) - P(A \cap B)\}]$

viii. $P(A \cup B')' = P(A' \cap B)$ $[P(B) - P(A \cap B)]$

Date: __/__/__

(12) 2011 A/L

i. $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$= 1 - \frac{2}{5} + \frac{1}{3}$

$= \frac{14}{15}$

ii. $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{1}{3} \times \frac{15}{14}$

$= \frac{5}{14}$

iii. $P(A'|B') = \frac{P(A' \cap B')}{P(B')}$

$= \frac{P(A \cup B)'}{1 - P(B)}$

$= \frac{1 - (A \cup B)}{1 - P(B)}$

$= \frac{1 - 1}{1 - \frac{14}{15}}$

$= 0$

(18) 2013 A/L

$P(A \cup B) = P(A) + P(B) = \frac{1}{2}$ —— ①

$P(B \cup C) = P(B) + P(C) = \frac{1}{2}$ —— ②

$P(C \cup A) = P(C) + P(A) = \frac{2}{3}$ —— ③

① + ② + ③ ⇒

$2[P(A) + P(B) + P(C)] = \frac{5}{3}$

$2 \times 1 = \frac{5}{3}$

$2 = \frac{5}{3}$

But since $2 \neq \frac{5}{3}$, it is impossible to have the probabilities given.

(23) 2015 A/L

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A).P(B) - P(B).P(C) - P(A).P(C) + P(A).P(B).P(C)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A).P(B) - P(C)\,[P(B) + P(A) - P(A).P(B)]$$

$$P(A \cup B \cup C) = P(A) + P(B) - P(A).P(B) + P(C)\,[1 - P(B) - P(A) + P(A).P(B)]$$

$$P(C) = \frac{P(A \cup B \cup C) + P(A).P(B) - P(A) - P(B)}{1 - P(B) - P(A) + P(A).P(B)}$$

$$P(C) = \frac{3/4 + 1/4 \times 1/2 - 1/4 - 1/2}{1 - 1/2 - 1/4 + 1/4 \times 1/2}$$

$$P(C) = \frac{1}{3}$$

(29) 2017 A/L

$$P(A' \cup B') = P(A \cap B)'$$

$$\frac{5}{6} = 1 - P(A \cap B)$$

$$P(A \cap B) = \frac{1}{6}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\frac{1}{4} = \frac{1/6}{P(A)}$$

$$P(A) = \frac{2}{3}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{4}{5} = \frac{2}{3} + P(B) - \frac{1}{6}$$

$$P(B) = \frac{3}{10}$$

$$P(B) = 0.3$$

32. 2018 A/L

i. $$P(A|B') = \frac{P(A\cap B')}{P(B')}$$

$$= \frac{P(A) - P(A\cap B)}{1 - P(B)}$$

$$= \frac{1/3 - 1/6}{1 - 1/4}$$

$$= \frac{2}{9}$$

ii. $$P(A'|B') = \frac{P(A'\cap B')}{P(B')}$$

$$= \frac{P(A\cup B)'}{P(B')}$$

$$= \frac{1 - P(A\cup B)}{1 - P(B)}$$

$$= \frac{1 - [P(A) + P(B) - P(A\cap B)]}{1 - P(B)}$$

$$= \frac{1 - [1/3 + 1/4 - 1/6]}{1 - 1/4}$$

$$= \frac{7}{9}$$

iii. $$P(B'|A') = \frac{P(B'\cap A')}{P(A')}$$

$$= \frac{P(A\cup B)'}{P(A')}$$

$$= \frac{1 - P(A\cup B)}{1 - P(A)}$$

$$= \frac{1 - [P(A) + P(B) - P(A\cap B)]}{1 - P(A)}$$

$$= \frac{1 - \frac{5}{12}}{1 - 1/3}$$

$$= \frac{7}{8}$$

Four Theorems on Independency

Theorem 1

* If A and B are independent, A' and B' are also independent.

Proof

Data: $P(A \cap B) = P(A).P(B)$

Prove that: $P(A' \cap B') = P(A').P(B')$

Proof: $L.H.S = P(A' \cap B')$

$= P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A).P(B)$

$= [1 - P(A)] - P(B)[1 - P(A)]$

$= [1 - P(A)][1 - P(B)]$

$= P(A').P(B')$

$= R.H.S$

$\therefore$ If A and B are independent, A' and B' are also independent.

Date: ___/___/___

Theorem 2

* If A and B are independent, A and B' are also independent.

Proof

Data: $P(A \cap B) = P(A).P(B)$

Prove that: $P(A \cap B') = P(A).P(B')$

Proof: $L.H.S = P(A \cap B')$

$= P(A) - P(A \cap B)$

$= P(A) - P(A).P(B)$

$= P(A)[1 - P(B)]$

$= P(A).P(B')$

$= R.H.S$

$\therefore$ If A and B are independent A and B' are also independent.

Theorem 3

* If A' and B' are independent, A and B' are also independent.

Proof

Data: $P(A' \cap B') = P(A').P(B')$

Prove that: $P(A \cap B') = P(A).P(B')$

Proof: $P(A' \cap B') = P(A').P(B')$ (Let $B' = x$)

$P(A' \cap x) = P(A').P(x)$

$P(x) - P(A \cap x) = [1 - P(A)]\,P(x)$

$P(B') - P(A \cap B') = [1 - P(A)]\,P(B')$

$$P(B') - P(A\cap B') = P(B') - P(A).P(B')$$
$$P(A\cap B') = P(A).P(B')$$

$\therefore$ If A' and B' are independent, A and B' are also independent.

Theorem 4

* If A and B' are independent, A' and B are also independent.

Proof

Data: $P(A\cap B') = P(A).P(B')$

Prove that: $P(A'\cap B) = P(A').P(B)$

Proof:
$$P(A\cap B') = P(A).P(B')$$
$$P(A'\cup B)' = [1-P(A')][1-P(B)]$$
$$1-P(A'\cup B) = 1-P(B)-P(A')+P(A').P(B)$$
$$-[P(A')+P(B)-P(A'\cap B)] = P(A').P(B)-P(A')-P(B)$$
$$P(A'\cap B) = P(A').P(B)$$

$\therefore$ If A and B' are independent, A' and B are also independent.

Ex-10

$S = \left\{(x,y) \middle| \begin{array}{l} x = \text{head, tail} \\ y = \text{head, tail} \end{array}\right\}$ $\quad n(S) = 4$

$A = \left\{(x,y) \middle| \begin{array}{l} x = \text{head} \\ y = \text{head, tail} \end{array}\right\}$ $\quad P(A) = \frac{2}{4} = 0.5$

$B = \left\{(x,y) \middle| \begin{array}{l} x = \text{head, tail} \\ y = \text{head} \end{array}\right\}$ $\quad P(B) = \frac{2}{4} = 0.5$

$C = \left\{(x,y) \middle| \begin{array}{l} x = \text{tail} \\ y = \text{tail} \end{array}\right\}$ $\quad P(C) = \frac{1}{4} = 0.25$

$A\cap B = \left\{(x,y) \middle| \begin{array}{l} x = \text{head} \\ y = \text{head} \end{array}\right\}$ $\quad P(A\cap B) = \frac{1}{4} = 0.25$

If A and C are independent;

$P(A \cap C) = P(A).P(C)$

$0 = 0.5 \times 0.25$

$0 = 0.125$

Since it contradicts, A and C are not independent.

Ex-11

$S = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x = H, T \\ y = H, T \\ z = H, T \end{array} \right\}$ $\quad n(S) = 8$

$A = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x = H \\ y = H, T \\ z = H, T \end{array} \right\}$ $\quad P(A) = 4/8 = 0.5$

$B = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x = H, T \\ y = H \\ z = H, T \end{array} \right\}$ $\quad P(B) = 4/8 = 0.5$

$C = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x = H, T \\ y = H, T \\ z = H \end{array} \right\}$ $\quad P(C) = 4/8 = 0.5$

$A \cap B \cap C = \{(H,H,H)\}$

$P(A \cap B \cap C) = 1/8$

$P(A \cap B \cap C) = 0.125$

$P(A \cap B \cap C) = 0.5 \times 0.5 \times 0.5$

$P(A \cap B \cap C) = P(A).P(B).P(C)$

$\therefore$ A, B and C events are mutually independent.

Prove that $P(A_1|B) = \quad (A_1 \cap A_2)| \quad] + P[(A_1 \cap A_2')|B]$ if $A_1, A_2, B \in S$ and $P(B) \neq 0$

$$\text{R.H.S} = P[(A_1 \cap A_2)|B] + P[(A_1 \cap A_2')|B]$$

$$= \frac{P(A_1 \cap A_2 \cap B)}{P(B)} + \frac{P(A_1 \cap A_2' \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B \cap A_2) + P(A_1 \cap B \cap A_2')}{P(B)}$$ (Let $A_1 \cap B = x$)

$$= \frac{P(x \cap A_2) + P(x \cap A_2')}{P(B)}$$

Since A_2 and A_2' are mutually exclusive;

$$= \frac{P[(x \cap A_2) \cup (x \cap A_2')]}{P(B)}$$

$$= \frac{P[x \cap (A_2 \cup A_2')]}{P(B)}$$

$$= \frac{P(x \cap S)}{P(B)}$$

$$= \frac{P(x)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)}$$

$$= P(A_1|B)$$

$$= \text{R.H.S}$$

Multiplication Law of Probability

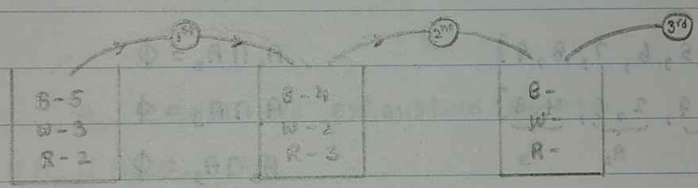
* If A_1, A_2 and A_3 are three consecutive events, then;

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) . P(A_2 | A_1) . P(A_3 | A_1 \cap A_2)$$

Proof

$$\begin{aligned} R.H.S &= P(A_1) . P(A_2 | A_1) . P(A_3 | A_1 \cap A_2) \\ &= P(A_1) \times \frac{P(A_1 \cap A_2)}{P(A_1)} \times \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \\ &= P(A_1 \cap A_2 \cap A_3) \\ &= L.H.S \end{aligned}$$

Ex-9



a) Let A_1 = 1^{st} selected ball being a black ball and A_2 = 2^{nd} selected ball being a black ball.

Applying multiplication law;

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1) . P(A_2 | A_1) \\ &= \frac{5}{10} \times \frac{5}{10} \\ &= 0.25 \end{aligned}$$

b) i.
$$\begin{aligned} P(B_1 \cap W_2 \cap R_3) &= P(B_1) . P(B_1 | W_2) . P(R_3 | B_1 \cap W_2) \\ &= \frac{5}{10} \times \frac{2}{10} \times \frac{2}{10} \\ &= 0.02 \end{aligned}$$

ii. $P(\text{At least 2 black balls}) = P(\text{Exactly 2 black balls}) + P(\text{Exactly 3 black balls})$

$= [P(B_1 \cap B_2 \cap B_3') + P(B_1 \cap B_2' \cap B_3) + P(B_1' \cap B_2 \cap B_3)] + P(B_1 \cap B_2 \cap B_3)$

$= \left(\frac{5}{10} \times \frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{5}{10} \times \frac{5}{10} \times \frac{4}{10}\right) + \left(\frac{5}{10} \times \frac{4}{10} \times \frac{6}{10}\right) + \left(\frac{5}{10} \times \frac{5}{10} \times \frac{5}{10}\right)$

$= \frac{125 + 100 + 120 + 125}{1000}$

$= 0.47$

iii. $P(\text{At most 2 white balls}) = 1 - P(W_1 \cap W_2 \cap W_3)$

$= 1 - \left(\frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}\right)$

$= 1 - 0.027$

$= 0.973$

Ex-12

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$S = \{\underbrace{1, 3, 5, 7, 9}_{A_1}, \underbrace{2, 6}_{A_2}, \underbrace{4, 8}_{A_3}\}$

$A_1 \cap A_2 = \phi$

$A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$

Since $S = A_1 \cup A_2 \cup A_3$, A_1, A_2 and A_3 are exhaustive events.

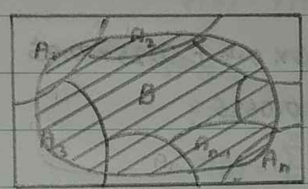
$\therefore$ A_1, A_2 and A_3 are exclusive events.

Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events, they are partitions.

Total Probability Law

* If $A_1, A_2, \dots, A_n$ are partitions (prior events) of an event space corresponding to the sample space of a random experiment such that $P(A_i) > 0$ $(i = 1, 2, \dots, n)$ and B is any arbitrary event (usually any selected posterior event) in the event space, then;

$$\boxed{\begin{aligned} P(B) &= P(A_1).P(B|A_1) + P(A_2).P(B|A_2) + \dots + P(A_n).P(B|A_n) \\ P(B) &= \sum_{i=1}^{n} P(A_i).P(B|A_i) \end{aligned}}$$



Proof

Since $A_1, A_2, \dots, A_n$ events are exhaustive events;

$$S = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$$

$$S \cap B = [A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n] \cap B$$

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \dots \cup (A_n \cap B)$$

$$P(B) = P[(A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \dots \cup (A_n \cup B)]$$

Since $A_1, A_2, \dots, A_n$ events are mutually exclusive;

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + \dots + P(A_n \cap B)$$

Applying multiplication law; (Since $A_i \cap B$ is consecutive)

$$P(B) = P(A_1).P(B|A_1) + P(A_2).P(B|A_2) + P(A_3).P(B|A_3) + \dots + P(A_n).P(B|A_n)$$

$$\underline{\underline{P(B) = \sum_{i=1}^{n} P(A_i).P(B|A_i)}}$$

Ex:13

Let A_1 = Being a 1st year student $\quad P(A_1) = 0.45$

A_2 = Being a 2nd year student $\quad P(A_2) = 0.3$

A_3 = Being a 3rd year student $\quad P(A_3) = 0.25$

B = Living outside Colombo $\quad P(B|A_1) = 0.15$ $\quad 0.85$

$P(B|A_2) = 0.4$ $\quad 0.6$

$P(B|A_3) = 0.25$ $\quad 0.75$

Applying total probability law;

$$P(B) = \sum_{i=1}^{3} P(A_i).P(B|A_i)$$

$$= 0.45 \times 0.15 + 0.3 \times 0.4 + 0.25 \times 0.25$$

$$= 0.0675 + 0.12 + 0.0625$$

$$= 0.25$$

$\therefore$ P(Living in Colombo) $= 1 - 0.25$

$= \underline{\underline{0.75}}$

Baye's Theorem

* If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events in the event space corre po ding to the sample space S of a random experiment and, B (where $P(B) \neq 0$) and A_j are any event of this event space, then;

$$\boxed{P(A_j|B) = \frac{P(A_j).P(B|A_j)}{\sum_{i=1}^{n} P(A_i).P(B|A_i)}}$$

Proof

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} \quad \text{—— ①}$$

$$P(A_j \cap B) = P(A_j) . P(B|A_j) \quad \text{—— ②}$$

② in ①;

$$P(A_j|B) = \frac{P(A_j) . P(B|A_j)}{P(B)} \quad \text{—— ⊛}$$

Since $A_1, A_2, \ldots, A_n$ events are exhaustive;

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$S \cap B = [A_1 \cup A_2 \cup \ldots \cup A_n] \cap B$$

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$$

$$P(B) = P[(A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)]$$

Since $A_1, A_2, \ldots, A_n$ events are mutually exclusive;

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$

Applying multiplication law;

$$P(B) = P(A_1) . P(B|A_1) + P(A_2) . P(B|A_2) + \ldots + P(A_n) . P(B|A_n)$$

$$P(B) = \sum_{i=1}^{n} P(A_i) . P(B|A_i) \quad \text{—— ⊛⊛}$$

⊛⊛ in ⊛;

$$P(A_j|B) = \frac{P(A_j) . P(B|A_j)}{\sum_{i=1}^{n} P(A_i) . P(B|A_i)}$$

Ex-14

Let A_1 = Having fever

A_2 = Having diarrhea

B = Having the disease

$P(A_1) = 0.4$

$P(A_2) = 0.6$

$P(B|A_1) = 0.2$

$P(B|A_2) = 0.8$

Applying total probability law;

$$P(B) = P(A_1) . P(B|A_1) + P(A_2) . P(B|A_2)$$

$$= 0.4 \times 0.2 + 0.6 \times 0.8$$

$$= 0.08 + 0.$$

$$= 0.56$$

Using Baye's theorem;

$$P(A_2|B) = \frac{P(A_2) . P(B|A_2)}{P(B)}$$

$$= \frac{0.6 \times 0.8}{0.56}$$

$$= \frac{6}{7}$$

Applications of Combinations and Permutations

In a class, there are 5 boys and 4 girls. Find the probability of selecting a group of 5 students with;

i. Exactly 2 boys and 3 girls

ii. Boys and girls

Total ways of selecting 5 students $[n(S)] = {}^9C_5$

$$= \frac{9\times 8\times 7\times 6\times 5}{5\times 4\times 3\times 2\times 1}$$

$$= 126$$

i. $n(\text{Exactly 2 boys and 3 girls}) = {}^5C_2 \times {}^4C_3$

$$= 10 \times 4$$

$$= 40$$

$\therefore$ Probability $= \frac{40}{126}$

$$= \frac{20}{63}$$

ii. Method I

$n(\text{Boys and girls}) = n(S) - [\underbrace{n(\text{all 5 boys})}_{{}^5C_5 = 1} + \underbrace{n(\text{all 5 girls})}_{\text{not possible}}]$

$$= 126 - 1$$

$$= 125$$

$\therefore$ Probability $= \frac{125}{126}$

Method II

$P(\text{Boys and girls}) = P(S) - [P(\text{All boys}) + P(\text{All girls})]$

$$= 1 - \frac{1}{126} - \frac{0}{126}$$

$$= \frac{125}{126}$$

Date: __/__/__

There are ten number cards numbered from 0 to 9 given. Find the probability of;

i. Making a 4 digit number

ii. Making an even number given that it's a 4 digit number

i. $$P(\text{Making a 4 digit number}) = \frac{n(\text{4 digit no.s})}{n(\text{any 4 digits})}$$

$$= \frac{\overset{\text{not zero}}{\boxed{\downarrow}}\,\square\,\square\,\square}{{}^{10}P_4}$$

$$= \frac{9\times 9\times 8\times 7}{10\times 9\times 8\times 7}$$

$$= \underline{\underline{0.9}}$$

ii. Let E = making an even number and A = making a 4 digit number.

$$P(E|A) = \frac{P(E\cap A)}{P(A)}$$

$$= \frac{\left(\overset{\text{not '0'}}{\boxed{\downarrow}}\,\square\,\square\,\overset{0,2,4,6,8}{\boxed{\downarrow}}\right)/{}^{10}P_4}{0.9}$$

$$= \frac{\left(\square\,\square\,\square\,\boxed{0} + \overset{\text{not '0'}}{\boxed{\downarrow}}\,\square\,\square\,\overset{2,4,6,8}{\boxed{\downarrow}}\right)/{}^{10}P_4}{0.9}$$

$$= \frac{(9\times 8\times 7 + 8\times 8\times 7\times 4)/(10\times 9\times 8\times 7)}{0.9}$$

$$= \underline{\underline{\frac{41}{81}}}$$