1.2 Lengths and Angles from Dot Products

1 The "dot product" of $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is $v \cdot w = (1)(4) + (2)(6) = 4 + 12 = 16$.

2 The length squared of $v = (1,3,2)$ is $v \cdot v = 1 + 9 + 4 = 14$. The length is $||v|| = \sqrt{14}$.

3 $v = (1,3,2)$ is **perpendicular** to $w = (4,-4,4)$ because $v \cdot w = 0$.

4 The angle $\theta = 45°$ between $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\cos\theta = \dfrac{v \cdot w}{||v||\ ||w||} = \dfrac{1}{(1)(\sqrt{2})}$.

5 All angles have $|\cos\theta| \le 1$. All vectors have **Schwarz inequality** $|v \cdot w| \le ||v||\ ||w||$ | **Triangle inequality** $||v+w|| \le ||v|| + ||w||$.

The most useful multiplication of vectors v and w is their **dot product** $v \cdot w$. We multiply the first components v_1w_1 and the second components v_2w_2 and so on. Then we **add those results** to get a single number $v \cdot w$:

$$\boxed{\text{The dot product of } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ is } v \cdot w = v_1w_1 + v_2w_2.} \qquad (1)$$

If the vectors are in n-dimensional space with n components each, then

$$\textbf{Dot product} \qquad \boxed{v \cdot w = v_1w_1 + v_2w_2 + \cdots + v_nw_n = w \cdot v} \qquad (2)$$

The dot product $v \cdot v$ tells us the squared length $||v||^2 = v_1^2 + \cdots + v_n^2$ **of a vector.** In two dimensions, this is the Pythagoras formula $a^2 + b^2 = c^2$ for a right triangle. The sides have $a^2 = v_1^2$ and $b^2 = v_2^2$. The hypotenuse has $||v||^2 = v_1^2 + v_2^2 = a^2 + b^2$.

To reach n dimensions, we can add one dimension at a time. Figure 1.2 shows $v = (1,2)$ in two dimensions and $w = (1,2,3)$ in three dimensions. Now the right triangle has sides $(1,2,0)$ and $(0,0,3)$. Those vectors add to w. The first side is in the xy plane, the second side goes up the perpendicular z axis . For this triangle in 3D with hypotenuse $w = (1,2,3)$, the law $a^2 + b^2 = c^2$ becomes $(1^2 + 2^2) + (3^2) = 14 = ||w||^2$.

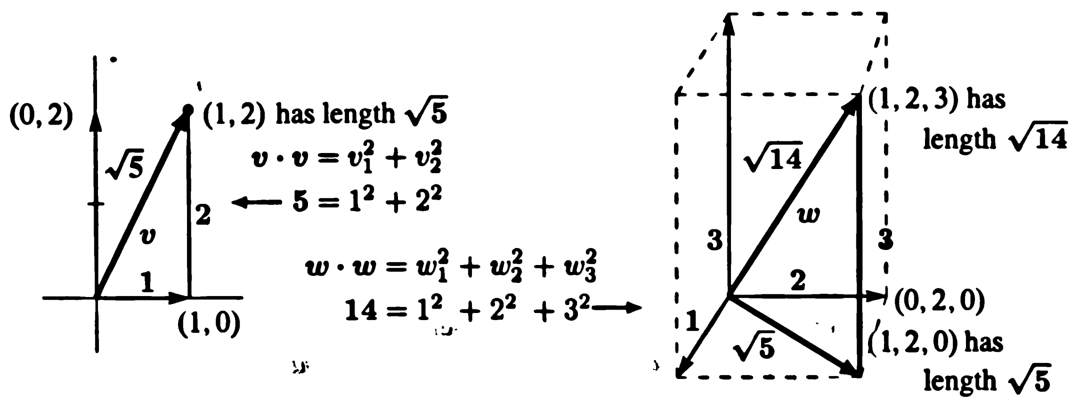


Figure 1.3: The length $\sqrt{v \cdot v} = \sqrt{5}$ in a plane and $\sqrt{w \cdot w} = \sqrt{14}$ in three dimensions.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus the vector $(1, 1, 1, 1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. That diagonal in n dimensions has length $\sqrt{n}$.

We use the words **unit vector** when the length of the vector is 1. Divide $\boldsymbol{v}$ by $||\boldsymbol{v}||$.

A unit vector $\boldsymbol{u}$ has length $||\boldsymbol{u}|| = 1$. If $\boldsymbol{v} \neq 0$ then $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{||\boldsymbol{v}||}$ is a unit vector.

Example 1 The standard unit vector along the x axis is written $\boldsymbol{i}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $\boldsymbol{u} = (\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \text{ Notice } \boldsymbol{i} \cdot \boldsymbol{u} = \cos\theta.$$

$\boldsymbol{u} = (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$ **is a unit vector because** $\boldsymbol{u} \cdot \boldsymbol{u} = \cos^2\boldsymbol{\theta} + \sin^2\boldsymbol{\theta} = \mathbf{1}$.

In four dimensions, one example of a unit vector is $\boldsymbol{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Or you could start with the vector $\boldsymbol{v} = (1, 5, 5, 7)$. Then $||\boldsymbol{v}||^2 = 1 + 25 + 25 + 49 = 100$. So $\boldsymbol{v}$ has length 10 and $\boldsymbol{u} = \boldsymbol{v}/10$ is a unit vector.

The word "**unit**" is always indicating that some measurement equals "one". The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we see that a "unit vector" has length = 1.

Perpendicular Vectors

Suppose the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 90°. Its cosine is zero. That produces a valuable test $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ for perpendicular vectors.

$$\boxed{\textbf{Perpendicular vectors have } \boldsymbol{v} \cdot \boldsymbol{w} = \mathbf{0}. \textbf{ Then } ||\boldsymbol{v} + \boldsymbol{w}||^2 = ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2.} \qquad (3)$$

This is the most important special case. It has brought us back to 90° angles and lengths $a^2 + b^2 = c^2$. The algebra for **perpendicular vectors** ($\boldsymbol{v} \cdot \boldsymbol{w} = 0 = \boldsymbol{w} \cdot \boldsymbol{v}$) is easy:

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2. \quad (4)$$

Two terms were zero. Please notice that $||\boldsymbol{v} - \boldsymbol{w}||^2$ is also equal to $||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2$.

Example 2 The vector $\boldsymbol{v} = (1, 1)$ is at a 45° angle with the x axis
The vector $\boldsymbol{w} = (1, -1)$ is at a $-45°$ angle with the x axis
The sum $\boldsymbol{v} + \boldsymbol{w}$ is $(2, 0)$. The difference $\boldsymbol{v} - \boldsymbol{w}$ is $(0, 2)$.

So the angle between $(1, 1)$ and $(1, -1)$ is 90°. Their dot product is $\boldsymbol{v} \cdot \boldsymbol{w} = 1 - 1 = 0$. This right triangle has $||\boldsymbol{v}||^2 = 2$ and $||\boldsymbol{w}||^2 = 2$ and $||\boldsymbol{v} - \boldsymbol{w}||^2 = ||\boldsymbol{v} + \boldsymbol{w}||^2 = 4$.

Example 3 The vectors $\boldsymbol{v} = (4,2)$ and $\boldsymbol{w} = (-1,2)$ have a *zero* dot product:

Dot product is zero
Vectors are perpendicular $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$

Put a weight of 4 at the point $x = -1$ (left of zero) and a weight of 2 at the point $x = 2$ (right of zero). The x axis will balance on the center point like a see-saw. The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4, 2)$. The vector of distances from the center is $(v_1, v_2) = (-1, 2)$. The weights times the distances, w_1v_1 and w_2v_2, give the "moments". The equation for the see-saw to balance is $\boldsymbol{w} \cdot \boldsymbol{v} = w_1v_1 + w_2v_2 = 0 =$ zero dot product.

Example 4 The unit vectors $\boldsymbol{v} = (1,0)$ and $\boldsymbol{u} = (\cos\theta, \sin\theta)$ have $\boldsymbol{v} \cdot \boldsymbol{u} = \cos\theta$. Now we are connecting the dot product to the angle between vectors.

Cosine of the angle $\boldsymbol{\theta}$ The cosine formula is easy to remember for unit vectors:

$$\boxed{\text{If } ||\boldsymbol{v}|| = 1 \text{ and } ||\boldsymbol{u}|| = 1, \text{ the angle } \theta \text{ between } \boldsymbol{v} \text{ and } \boldsymbol{u} \text{ has } \cos\boldsymbol{\theta} = \boldsymbol{v} \cdot \boldsymbol{u}.}$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular.* The angle between them is 90°. The clearest example of perpendicular vectors is $\boldsymbol{i} = (1,0)$ along the x axis and $\boldsymbol{j} = (0,1)$ up the y axis. Again the dot product is $\boldsymbol{i} \cdot \boldsymbol{j} = 0 + 0 = 0$. The cosine of 90° is zero.

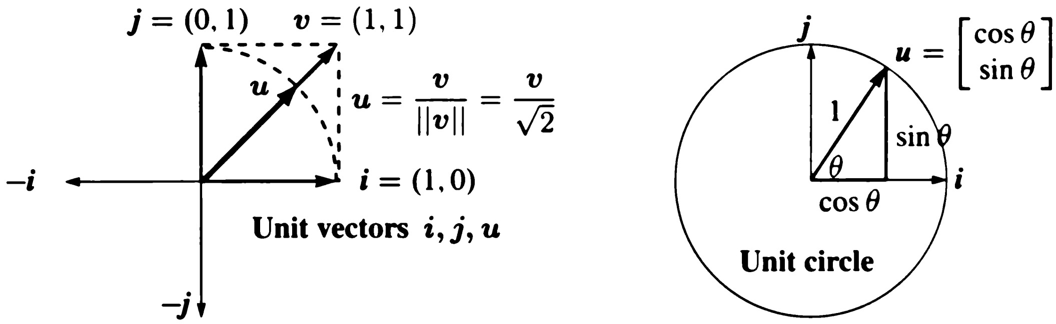


Figure 1.4: **Left:** The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ divides $\boldsymbol{v} = (1,1)$ by its length $||\boldsymbol{v}|| = \sqrt{2}$. **Right:** The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ with $\boldsymbol{i}$.

Example 5 Dot products enter in economics and business. We have three goods to buy. Their prices for each unit are (p_1, p_2, p_3)—this is the price vector $\boldsymbol{p}$. The quantities we buy are (q_1, q_2, q_3). *Buying q_1 units at the price p_1 brings in q_1p_1.* The total cost adds up quantities q times prices p. ***This is the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$ in three dimensions***:

$$\textit{Cost} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1p_1 + q_2p_2 + q_3p_3 = \textbf{\textit{dot product}}.$$

A zero dot product means that "the books balance". Total sales equal total purchases if $\boldsymbol{q} \cdot \boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). A supermarket with thousands of goods goes quickly into high dimensions.

Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

The Angle Between Two Vectors

We know that perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is 90°. Our next step is to connect all dot products to angles. The dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ finds the angle between any two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Example 6 The unit vectors $\boldsymbol{v} = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{w} = (\cos\beta, \sin\beta)$ have $\boldsymbol{v} \cdot \boldsymbol{w} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. In trigonometry this is the formula for $\cos(\beta - \alpha)$. Figure 1.5 shows that the angle between the unit vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\beta - \alpha$.

The dot product $\boldsymbol{w} \cdot \boldsymbol{v}$ ***equals*** $\boldsymbol{v} \cdot \boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

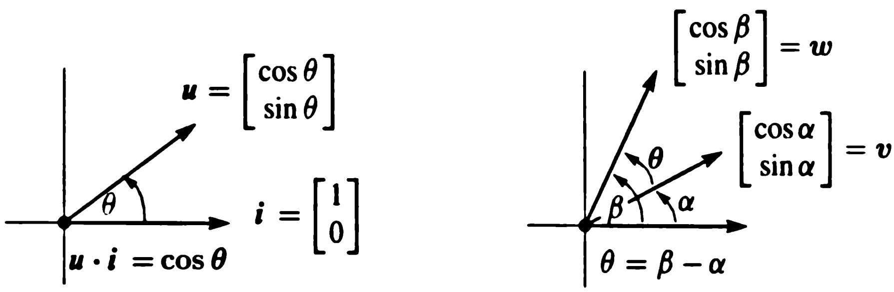


Figure 1.5: Unit vectors: $\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$. The angle between the vectors is θ.

Suppose $\boldsymbol{v} \cdot \boldsymbol{w}$ is **not zero**. It may be positive, it may be negative. The sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ immediately tells whether we are below or above a right angle. The angle is less than 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive. The angle is above 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative.

The borderline is where vectors are perpendicular to $\boldsymbol{v}$. On that dividing line between plus and minus, $\boldsymbol{w}_2 = (1, -3)$ is perpendicular to $\boldsymbol{v} = (3, 1)$. Their dot product is zero. Then $\boldsymbol{w}_3$ goes beyond a 90° angle with $\boldsymbol{v}$. The test becomes $\boldsymbol{v} \cdot \boldsymbol{w}_3 < 0$: *negative.*

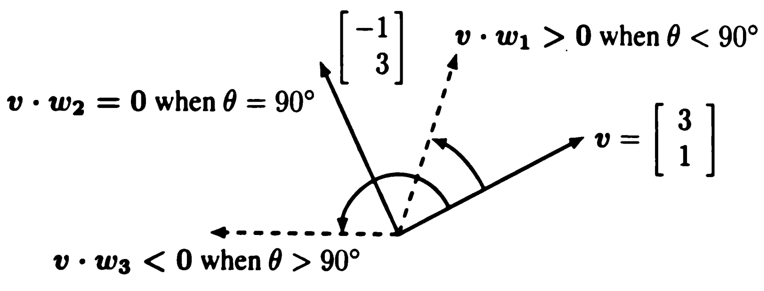


Figure 1.6: Small angle $\boldsymbol{v} \cdot \boldsymbol{w}_1 > 0$. Right angle $\boldsymbol{v} \cdot \boldsymbol{w}_2 = 0$. Large angle $\boldsymbol{v} \cdot \boldsymbol{w}_3 < 0$.

The dot product reveals the exact angle $\boldsymbol{\theta}$. To repeat: For unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$, ***the dot product*** $\boldsymbol{u} \cdot \boldsymbol{U}$ ***is the cosine of*** θ. This remains true in n dimensions.

Remember that $\cos\theta$ is never greater than 1. It is never less than -1. *The dot product of unit vectors is between* -1 *and* 1. **The cosine of** θ **is revealed by** $\boldsymbol{u} \cdot \boldsymbol{U}$.

What if $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are not unit vectors?* Divide by their lengths to get $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ and $\boldsymbol{U} = \boldsymbol{w}/\|\boldsymbol{w}\|$. Then the dot product of those unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ gives $\cos\theta$.

$$\boxed{\textbf{COSINE FORMULA} \quad \text{If } \boldsymbol{v} \text{ and } \boldsymbol{w} \text{ are nonzero vectors then } \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta.} \qquad (5)$$

Whatever the angle, this dot product of $\boldsymbol{v}/\|\boldsymbol{v}\|$ with $\boldsymbol{w}/\|\boldsymbol{w}\|$ never exceeds one. That is the ***"Schwarz inequality"*** $|\boldsymbol{v}\cdot\boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ for dot products—or more correctly the Cauchy-Schwarz-Bunyakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics).

Since $|\cos\theta|$ never exceeds 1, the cosine formula in (5) gives two great inequalities.

SCHWARZ INEQUALITY	$\lvert\boldsymbol{v}\cdot\boldsymbol{w}\rvert \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$
TRIANGLE INEQUALITY	$\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

The triangle inequality comes directly from the Schwarz inequality!

$$\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\,\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2. \quad (6)$$

The square root is $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. **Side 3 cannot exceed Side 1 + Side 2.**

Example 7 Find $\cos\theta$ for $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

The dot product is $\boldsymbol{v}\cdot\boldsymbol{w} = 4$. Both $\boldsymbol{v}$ and $\boldsymbol{w}$ have length $\sqrt{5}$. So $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| = 5$.

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

The Schwarz inequality is $4 < 5$. By the triangle inequality, side $3 = \|\boldsymbol{v}+\boldsymbol{w}\|$ is less than side 1 + side 2. With $\boldsymbol{v}+\boldsymbol{w} = (3,3)$ the three side lengths are $\sqrt{18} < \sqrt{5}+\sqrt{5}$. Square this inequality to get $18 < 20$. This confirms the triangle inequality.

Example 8 The dot product of $\boldsymbol{v} = (a,b)$ and $\boldsymbol{w} = (b,a)$ is $2ab$. Their lengths are $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = \sqrt{a^2+b^2}$. The Schwarz inequality $\boldsymbol{v}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ is $\boldsymbol{2ab \le a^2 + b^2}$.

Any numbers a^2 and b^2 have *geometric mean* $|ab| \le$ *arithmetic mean* $\frac{1}{2}\left(a^2+b^2\right)$. The proof is that $\frac{1}{2}\left(a^2+b^2\right) - ab = \frac{1}{2}(a-b)^2$ is a perfect square: never negative!

A Plane in 3 Dimensions

Suppose $\boldsymbol{n}$ is a unit vector with three components n_1, n_2, n_3. Look at all vectors $\boldsymbol{w}$ in $\mathbf{R}^3$ that are perpendicular to $\boldsymbol{n}$ (so $\boldsymbol{w}\cdot\boldsymbol{n} = 0$):

$$\boxed{\text{The vectors } \boldsymbol{w} \text{ with } \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ fill a 2-dimensional plane in } \mathbf{R}^3}$$

The whole plane is perpendicular to its "normal vector $\boldsymbol{n}$". The equation of a 2-dimensional plane in 3-dimensional space is $n_1w_1 + n_2w_2 + n_3w_3 = 0$. For the "$xy$ plane" the normal vector $\boldsymbol{n}$ going straight upward has components $0, 0, 1$. So the equation of the xy plane is just $w_3 = 0$ or $z = 0$, which we already knew.

■ WORKED EXAMPLES ■

1.2 A For the vectors $\boldsymbol{v} = (3,4)$ and $\boldsymbol{w} = (4,3)$ test the Schwarz inequality on $\boldsymbol{v} \cdot \boldsymbol{w}$ and the triangle inequality on $\|\boldsymbol{v} + \boldsymbol{w}\|$. Find $\cos\theta$ for the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution The dot product is $\boldsymbol{v} \cdot \boldsymbol{w} = (3)(4) + (4)(3) = 24$. The length of $\boldsymbol{v}$ is $\|\boldsymbol{v}\| = \sqrt{9+16} = 5$ and also $\|\boldsymbol{w}\| = 5$. The sum $\boldsymbol{v} + \boldsymbol{w} = (7,7)$ has length $7\sqrt{2} < 10$.

Schwarz inequality	$\lvert\boldsymbol{v} \cdot \boldsymbol{w}\rvert \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ is $24 < 25$.
Triangle inequality	$\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ is $7\sqrt{2} < 5 + 5$.
Cosine of angle	$\cos\theta = \dfrac{24}{25}$ Thin angle from $\boldsymbol{v} = (3,4)$ to $\boldsymbol{w} = (4,3)$

1.2 B Which $\boldsymbol{v}$ and $\boldsymbol{w}$ give *equality* $\lvert\boldsymbol{v} \cdot \boldsymbol{w}\rvert = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ and $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$?

Equality: One vector is a multiple of the other as in $\boldsymbol{w} = c\boldsymbol{v}$. Then the angle is $0°$ or $180°$. In this case $\lvert\cos\theta\rvert = 1$ and $\lvert\boldsymbol{v} \cdot \boldsymbol{w}\rvert$ *equals* $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$. If the angle is $0°$, as in $\boldsymbol{w} = 2\boldsymbol{v}$, then $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ (both sides give $3\|\boldsymbol{v}\|$). This $\boldsymbol{v}, 2\boldsymbol{v}, 3\boldsymbol{v}$ triangle is flat.

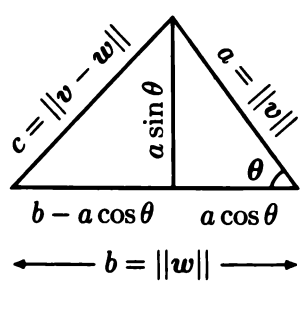
1.2 C Find a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = (3,4)$. Find a unit vector $\boldsymbol{U}$ that is perpendicular to $\boldsymbol{u}$. There are two possibilities for $\boldsymbol{U}$.

Solution For a unit vector $\boldsymbol{u}$, divide $\boldsymbol{v}$ by its length $\|\boldsymbol{v}\| = 5$. For a perpendicular vector $\boldsymbol{V}$ we can choose $(-4,3)$ or $(4,-3)$. For a *unit* vector $\boldsymbol{U}$, divide $\boldsymbol{V}$ by its length $\|\boldsymbol{V}\| = 5$.

1.2 D We want to explain the angle formula $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\| \cos\theta$. This is $ab\cos\theta$ in the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$.

Use Pythagoras in the left triangle !

$$\begin{aligned} c^2 &= (a\sin\theta)^2 + (b - a\cos\theta)^2 \\ &= a^2\sin^2\theta + b^2 - 2ab\cos\theta + a^2\cos^2\theta \\ &= \boldsymbol{a^2 + b^2 - 2ab\cos\theta} \end{aligned}$$



The picture shows the sides $a = \|\boldsymbol{v}\|, b = \|\boldsymbol{w}\|, c = \|\boldsymbol{v} - \boldsymbol{w}\|$ and two right triangles.

Compare with $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w}$ to match $\boldsymbol{v} \cdot \boldsymbol{w}$ with $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| \cos\theta$.