LINEAR ALGEBRA
TUTORIAL 10 – SINGULAR VALUE DECOMPOSITION

1. Let A = $\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
   (A) Determine V and the V^T.
   (B) Determine the singular values, σ and then Σ.
   (C) Determine U using $A = U\Sigma V^T \rightarrow AV = U\Sigma$ since V is orthogonal to V^T, we know $VV^T = I$.
   (D) Compute a single value decomposition of A.

2. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$
   (A) Determine U using $A = U\Sigma V^T \rightarrow AV = U\Sigma$ since V is orthogonal to V^T, we know $VV^T = I$.
   (B) Compute a single value decomposition of A.

3. Compute a single value decomposition of A = $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

4. Compute a single value decomposition of A = $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

2 Steps for Calculation of SVD

Here, we provide an algorithm to calculate a singular value decomposition of a matrix.

1. Compute $A^T A$ of a real $m \times n$ matrix A of rank r.

2. Compute the singular values of $A^T A$.

   Solve the characteristic equation $\Delta_{A^T A}(\lambda) = |A^T A - \lambda I| = 0$ of $A^T A$ for the eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A^T A$. These eigenvalues will be positive. Take their square roots to obtain $\sigma_1, \ldots, \sigma_r$ which are the singular values of A, that is,

$$\sigma_i = +\sqrt{\lambda_i}, \quad i = 1, \ldots, r. \tag{7}$$

3. Sort the singular values, possibly renaming them, so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$.

4. Construct the Σ matrix of size $m \times n$ such that $\Sigma_{ii} = \sigma_i$ for $i = 1, \ldots, r$, and $\Sigma_{ij} = 0$ when $i \neq j$.

5. Compute the eigenvectors of $A^T A$.

   Find a basis for $\text{Null}(A^T A - \lambda_i I)$. That is, solve $(A^T A - \lambda_i I)s_i = 0$ for s_i, an eigenvector of A corresponding to λ_i, for each eigenvalue λ_i. Since $A^T A$ is symmetric, its eigenvectors corresponding to different eigenvalues are already orthogonal (but likely not orthonormal). See Lemma 1.

6. Compute the (right singular) vectors $v_1, \ldots, v_r$ by normalizing each eigenvector s_i by multiplying it by $\frac{1}{||s_i||}$. That is, let

$$v_i = \frac{1}{||s_i||} s_i, \quad i = 1, \ldots, r. \tag{8}$$

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7. Construct the orthogonal matrix $V = [v_1 | \cdots | v_n]$.

8. Verify $V^T V = I$.

9. Compute the (left singular) vectors $u_1, \ldots, u_r$ as

$$Av_i = \sigma_i u_i \implies u_i = \frac{Av_i}{\sigma_i},\ i = 1 \ldots r. \tag{10}$$

   In this method, $u_1, \ldots, u_r$ are orthogonal by Lemma 5.

   Alternatively,

   (i) Note that $AA^T = U(\Sigma\Sigma^T)U^T$ suggests the vectors of U can be calculated as the eigenvectors of AA^T. In using this method, the vectors need to be normalized first. Namely, $u_i = \frac{1}{||s_i||} s_i$, where s_i is an eigenvector of AA^T.

   (ii) Since $\Delta_{A^T A}(\lambda) = \Delta_{AA^T}(\lambda)$ by Lemma 8, $\sigma_1, \ldots, \sigma_r$ are also the square roots of the eigenvalues of AA^T.

   If $m > r$, the additional $m - r$ vectors $u_{r+1}, \ldots, u_m$ need to be chosen as an orthonormal basis in $\text{Null}(A^T)$. Note that since $Av_i = \sigma_i u_i$ for $i = 1, \ldots,$ vectors $u_1, \ldots, u_r$ provide an orthonormal basis for $\text{Col}(A)$ while the vectors $u_{r+1}, \ldots, u_m$ provide an orthonormal basis for the left null space $\text{Null}(A^T)$. In particular,

$$\mathbb{R}^m = \text{Col}(A) \perp \text{Null}(A^T) = \text{span}\{u_1, \ldots, u_r\} \perp \text{span}\{u_{r+1}, \ldots, u_{r+(m-r)}\}. \tag{11}$$

10. Construct $U = [u_1 | \cdots | u_m]$.

11. Verify $U^T U = I$.

12. Verify $A = U\Sigma V^T$.

13. Construct the dyadic decomposition[1] of A, as described in Thm. 13:

$$A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + u_r \sigma_r v_r^T. \tag{12}$$