Bachelor of Science in Computer Science
University of Colombo School of Computing

Set Theory

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Week 01: Set Theory

- **Topics to be covered.**
  - Definition
  - Representation of Sets
  - Null set and Universal set
  - Set Identity
  - Set Operations
  - Venn Diagrams
  - Power Sets

Definition

- Set is one of the fundamental discrete structures in mathematics.
- Sets are used to group (usually similar) objects together.

Example1:A={‘Amal’, ‘Bimal’, ‘Dias’, ‘Ganesh’} is a set of names.
Example2:B={1,3,5,7,9} is the set of odd natural numbers between 1 and 10.
Example3:C={2,4,6,8,10} is the set of even natural numbers between 1 and 10.

Definition: Any **well-defined** collection of **distinct objects** is called a set. The elements of a set are called its **members**. The order of items in a set is not important.

A set has to be well-defined

- When defining a set, it's important to ensure that every element is described in a way that leaves no room for ambiguity or confusion.

Given any object there should be no confusion in identifying whether it is a member of the set or not.

- For example, consider the set of "positive integers less than 10." This is a well-defined set because it is clear which integers meet the criteria – 1, 2, 3, 4, 5, 6, 7, 8, and 9. There is no ambiguity about which numbers should be included.

A set has to be well-defined .....

- Consider the set of "interesting books." This is not a well-defined set because the criteria for what makes a book "interesting" can vary from person to person. Without a clear and universally agreed-upon definition of what constitutes an interesting book, different people might include different books in the set, leading to ambiguity.

Elements of a set

The number of elements in a set can be

- Empty
- Finite
- In-finite

Quiz

- Consider the following collection.

A={‘Amal’, ‘Bimal’, ’Amal’, ‘Dias’, ‘Ganesh’}

Is this collection of objects a set?

Ways of Describing Sets

- List the elements between braces: S = {a, b, c, d}

- Brace notation with ellipses:

$$A = \{1,2,\ldots\ldots,100\}$$
$$\mathbb{Z} = \{\ \ldots\ldots, -2, -1, 0\ , 1\ , 2, \ldots\ldots\ \}$$

- Verbal description:

  "A is the set of all integers from 1 to 100, inclusive"

- Set builder notation:

$$A = \{x \mid x \text{ is an integer}, 1 \leq x \leq 100\}$$
$$\mathbb{Q}^+ = \left\{x \in \mathbb{R} \mid\ x = \frac{p}{q}\ \text{ for some positive integers } p \text{ and } q\right\}.$$

Notation

Usually, sets are denoted with upper-case letters and the members of a set with lower-case letters.

$x \in A$ means that *x* is a member of the set A
$x \notin A$ means that *x* is not a member of the set A

The number of members in a set is called its **cardinality** denoted as **|A|.**

Null Set: (Empty set)

- Is the set with no elements.
- Notation: Φ, { }
- Note: {0} is not an empty set. Why?

Universal Set:

- Denoted by $\mathbb{U}$, universal set is the set of all elements under consideration.

Example:

Let

A={‘Amal’, ‘Bimal’, ‘Dias’, ‘Ganesh’}

If A contains few names from a given class, then $\mathbb{U}$ is the set of all names in the class.

Equality of Sets

- Two sets A and B are equal if and only if both A and B contain the same elements.

A={‘Amal’, ‘Bimal’, ‘Dias’, ‘Ganesh’}

B={‘Ganesh’ ,‘Bimal’, ‘Amal’, ‘Dias’}

A and B sets are equal

Some Special Sets

The Null Set or Empty Set

The Universal Set.

Singleton set: A set with one element is called a singleton set.

Example: $\{\phi\}$ is a singleton set with the empty set as its only element.

Subsets

Definition: The set A is a subset of B if and only if every element of A is also an element of B, and is denoted $A \subseteq B$.
That is; $A \subseteq B \iff \forall x\,[x \in A \rightarrow x \in B]$.

Definition: If $A \subseteq$ B but $A \neq B$ then we say A is a proper subset of B, denoted A $\subset$ B or $A \subsetneq B$.

$$\mathrm{A} \subset \mathrm{B} \Leftrightarrow \forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

Note:

- The assertion $x \in \emptyset$ is always false.
  Hence $\forall x\,[x \in \emptyset \rightarrow x \in B]$ is always true.
  Therefore, $\emptyset$ is a subset of every set.
- For any set A, A is always a subset of itself $(A \subseteq \mathrm{A})$.

Subsets

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That is; $A \subseteq B \iff \forall x\ [x \in A \rightarrow x \in B]$.

How to prove that $\emptyset \subseteq A$ for any set A
$\emptyset \subseteq A \iff \forall x\ [x \in \emptyset \rightarrow x \in B]$.

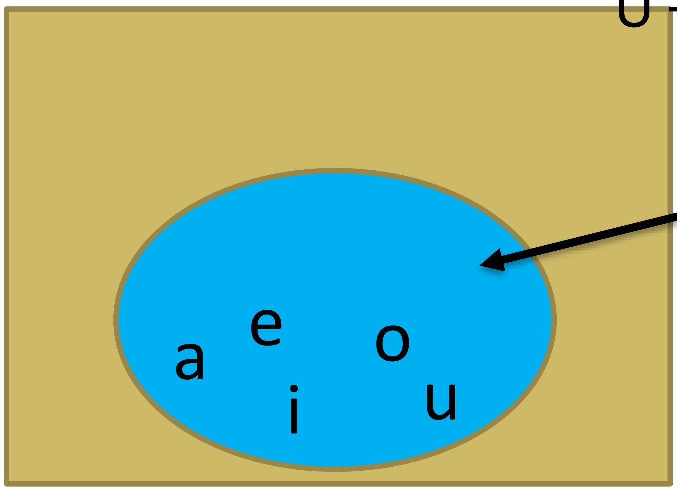
Note:

- The assertion $x \in \emptyset$ is always false.
  Hence $\forall x\ [x \in \emptyset \rightarrow x \in B]$ is always true.
  Therefore, $\emptyset$ is a subset of every set.
- For any set A, A is always a subset of itself $(A \subseteq A)$.

Venn Diagrams

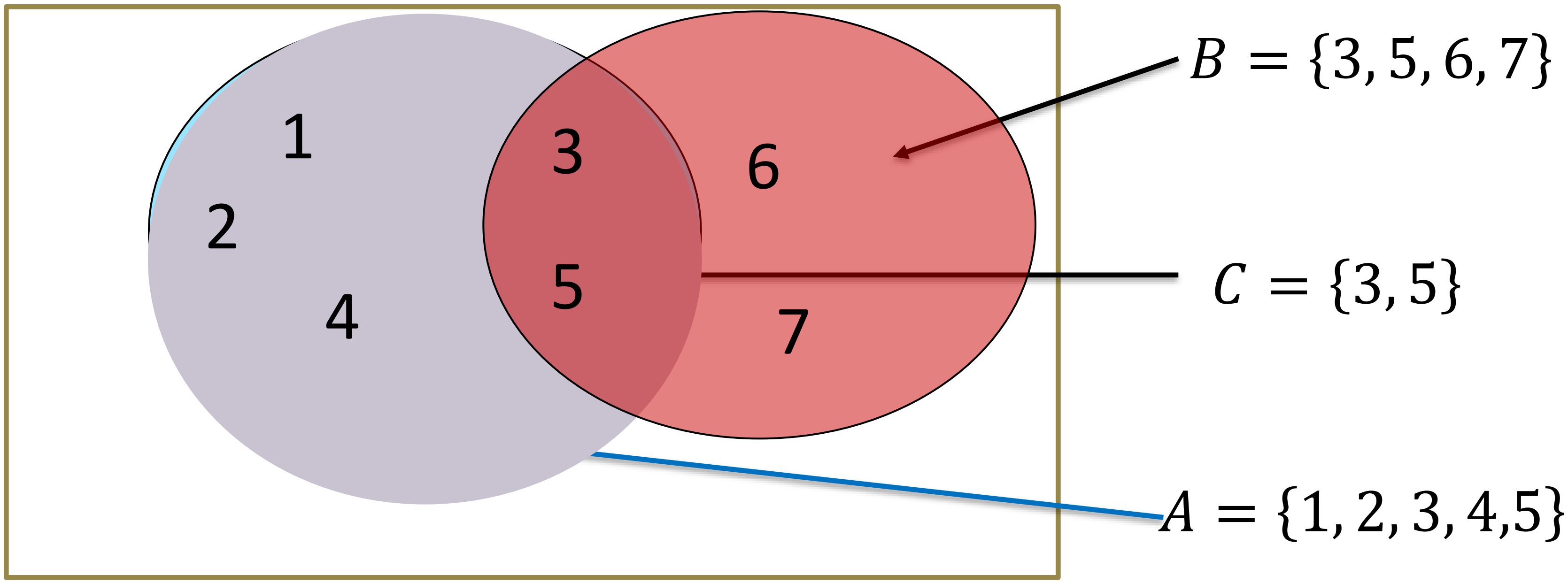
- A graphical way of representing sets(John Venn).
- The universal set U, which contains all the objects under consideration, is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.

U – The English Alphabet



V $=\{a, e, i, o, u\}$

Subsets



$$Then \quad C \subseteq A \;\; and \;\; C \subseteq B$$

But $A \nsubseteq B,\ B \nsubseteq C \ and \ A \nsubseteq C$.

Equality of two Sets

Definition: Two sets A and B are said to be equal if both sets have the same elements and is denoted by A = B. If sets 'A' and 'B' are not equal we write $A \neq B$.

$$A = B \;\Leftrightarrow\; A \subseteq B \;\; and \;\; B \subseteq A$$

$$\Leftrightarrow \; \big(\forall x (x \in A \;\rightarrow x \in B)\big) and \big(\forall x (x \in B \;\rightarrow x \in A)\big)$$

$$\Leftrightarrow \forall x (x \in A \;\leftrightarrow x \in B)$$

Power Set

Definition: The set of all subset of a set A, denoted P(A) or 2^A, is called the *power set* of A.

What is the power set of the set $\{a, b, c\}$?

$$\{\varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

What is the power set of the empty set? What is the power set of the set {∅}?

Theorem: If a set has n elements, then its power set has 2^n elements.

The Size of a Set

Definition: Let S be a set. If there are n elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S and is denoted by $|S|$.

Example: Let $A = \{a, b, c, d\}$. Then $|A| = 4$.
Because the null set has no elements, it follows that $|\emptyset| = 0$.

Definition: A set is said to be infinite if the number of element it has is not finite.

Example: The set of all natural numbers is infinite.

Truth Sets and Quantifiers

Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which P(x) is true. The truth set of P(x) is denoted by $\{x \in D \mid P(x)\}$.

Example: Find the truth sets of the predicates P(x) , Q(x), and R(x), where the domain is the set of integers and P(x) is "$|x| = 1$", Q(x) is "$x^2 = 2$", " and R(x) is "$|x| = x$".

The truth set of P = $\{x \in \mathbb{Z} \mid |x| = 1\} = \{-1, 1\}$.
The truth set of Q = $\{x \in \mathbb{Z} \mid x^2 = 2\} = \emptyset$.
The truth set of R = $\{x \in \mathbb{Z} \mid |x| = x\} = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N}$.

Note:
$\forall x\ P(x)$ is true over the domain U $\Leftrightarrow$ the truth set of $P = U$.
$\exists x\ P(x)$ is true over the domain U $\Leftrightarrow$ the truth set of $P \neq \emptyset$.

Set Operations

Two, or more, sets can be combined using different set operations such as Union, Intersection, Complement, etc.

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*.

The operators in set theory are defined in terms of the corresponding operator in propositional calculus.

As always there must be a universe U. All sets are assumed to be subsets of U.

Union & Intersection

Union: The union of two sets A and B, denoted by A∪B, is the set of all elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \ \vee x \in B\}$$

Intersection: The intersection of two sets A and B, denoted by A∩B, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \ \wedge x \in B\}$$

Disjoint Sets: Two sets A, B are called disjoint if $A \cap B = \emptyset$.

Complement

Definition: Let A and B be sets. The difference of A and B, denoted by A - B, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A, x \notin B\}$$

Definition: Let U be the universal set. The complement of the set A, denoted by $\overline{A}$ or A^c, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

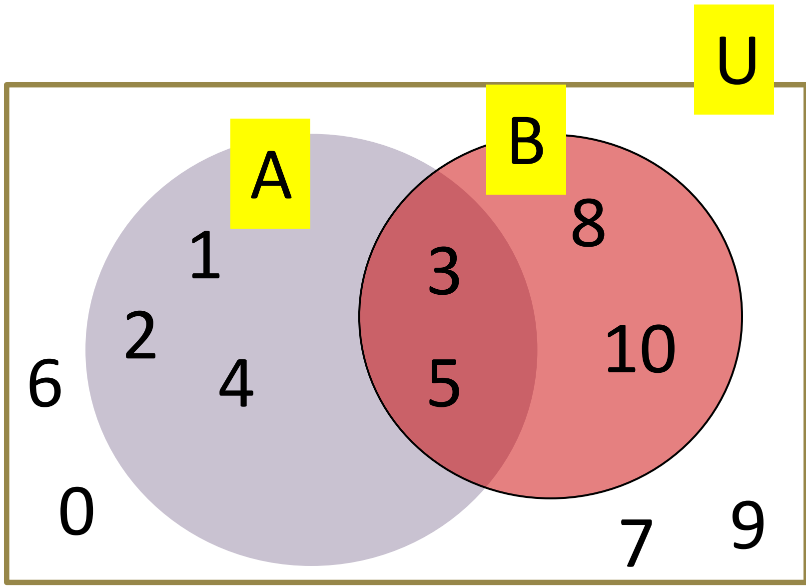
$$\overline{A} = U - A = \{x \in U \mid x \notin A\}$$

Definition: Let A, B be two sets. The symmetric difference of A and B, denoted by A ⊕ B, is the set $(A - B) \cup (B - A)$.

$$A \oplus B = (A - B) \cup (B - A) = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}.$$

Example

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 8, 10\}$.



Find the following sets:

1. $A \cap B = \{3, 5\}$
2. $A \cup B = \{1, 2, 3, 4, 5, 8, 10\}$
3. $A - B = \{1, 2, 4\}$
4. $B - A = \{8,10\}$
5. $A^C = \{0, 6, 7, 8, 9, 10\}$
6. $A \oplus B = \{1, 2, 4, 8, 10\}$.

Set Identities

Identity	Name
$A \cap U = A, \qquad A \cup \emptyset = A$	Identity laws
$A \cup U = U, \qquad A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A, \qquad A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A, \qquad A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C,$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{(A \cap B)} = \overline{A} \cup \overline{B}, \qquad \overline{(A \cup B)} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A, \qquad A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U, \qquad A \cap \overline{A} = \emptyset$	Complement laws

Prove that $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ (Usual method)

We will prove that the two sets $\overline{(A \cap B)}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

Let $x \in \overline{(A \cap B)}$
$\Longrightarrow \mathrm{x} \notin (A \cap B)$
$\Longrightarrow \neg\big(x \in (A \cap B)\big)$
$\Longrightarrow \neg\big((x \in A) \wedge (x \in B)\big)$
$\Longrightarrow \big(\neg(x \in A) \vee \neg(x \in B)\big)$
$\Longrightarrow x \notin A \vee x \notin B$
$\Longrightarrow x \in \overline{A} \vee x \in \overline{B}$
$\Longrightarrow x \in \overline{A} \cup \overline{B}$
$\therefore \overline{(A \cap B)} \subseteq \overline{A} \cup \overline{B}.$

Let $x \in \overline{A} \cup \overline{B}$
$\Longrightarrow \mathrm{x} \in \overline{A} \vee x \in \overline{B}$
$\Longrightarrow \big((x \notin A) \vee (x \notin B)\big)$
$\Longrightarrow \big(\neg(x \in A) \vee \neg(x \in B)\big)$
$\Longrightarrow \neg\big((x \in A) \wedge (x \in B)\big)$
$\Longrightarrow \neg\big(x \in (A \cap B)\big)$
$\Longrightarrow \mathrm{x} \notin (A \cap B)$
$\Longrightarrow x \in \overline{(A \cap B)}$
$\therefore \overline{A} \cup \overline{B} \subseteq \overline{(A \cap B)}.$

Therefore $\overline{(A \cap B)} = \overline{A} \cup \overline{B}.$

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (Set Builder Method)

$$\overline{(A \cup B)} = \{x \mid x \notin (A \cup B)\}$$

$$= \{x \mid \neg(x \in (A \cup B))\}$$

$$= \{x \mid \neg(x \in A \vee x \in B)\}$$

$$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$$

$$= \{x \mid (x \notin A) \wedge (x \notin B)\}$$

$$= \left\{x \mid \left(x \in \overline{A}\right) \wedge \left(x \in \overline{B}\right)\right\}$$

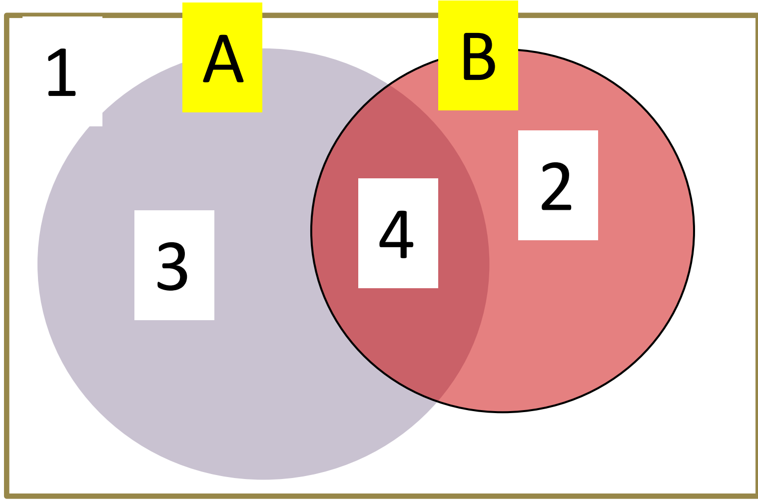
$$= \left\{x \mid x \in (\overline{A} \cap \overline{B})\right\}$$

$$= \overline{A} \cap \overline{B}$$

Prove that $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

A	B	A∩B	$\overline{(A \cap B)}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Hence $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.



Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ Using Membership table Method

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$(A \cap B)$	$(A \cap C)$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A \lor x \in (B \cap C)$

$\Leftrightarrow (x \in A) \lor (x \in B \land x \in C)$

$\Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$

$\Leftrightarrow x \in (A \cup B) \land x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$.

$\therefore \quad \forall x \left(x \in A \cup (B \cap C) \Leftrightarrow x \in (A \cup B) \cap (A \cup C)\right)$.

Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example

Let A, B, C be sets. Using the set identities, Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)} \quad \text{(by De Morgan's law)}$$

$$= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{(by De Morgan's law)}$$

$$= \overline{A} \cap (\overline{C} \cup \overline{B}) \quad \text{(by Commutative law)}$$

$$= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{(by Commutative law)}$$

Show that

$$(P - Q) \cap (R - Q) = (P \cap R) - Q$$

$x \in (P - Q) \cap (R - Q)$

$\Leftrightarrow x \in (P - Q) \;\wedge\; x \in (R - Q)$

$\Leftrightarrow (x \in P \wedge x \notin Q) \wedge (x \in R \wedge x \notin Q)$

$\Leftrightarrow x \in P \wedge x \notin Q \wedge x \in R \wedge x \notin Q$

$\Leftrightarrow x \in P \wedge x \in R \wedge x \notin Q$

$\Leftrightarrow (x \in P \wedge x \in R) \;\wedge x \notin Q$

$\Leftrightarrow x \in (P \cap R) \wedge x \notin Q$

$\Leftrightarrow x \in (P \cap R) - Q$

Therefore, $(P - Q) \cap (R - Q) = (P \cap R) - Q$.

Example

For any three subsets A, B, C of a universal set U, show that

1. $\mathrm{A} - \mathrm{B} = \mathrm{A} \cap \overline{\mathrm{B}}$;
2. $\overline{(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}} \cup \overline{\mathrm{C}}$.

1.

$x \in (\mathrm{A} - \mathrm{B})$

$\Leftrightarrow x \in A \wedge x \notin B$

$\Leftrightarrow x \in A \wedge x \in \bar{B}$

$\Leftrightarrow x \in (A \cap \bar{B})$

$\therefore \mathrm{A} - \mathrm{B} = \mathrm{A} \cap \overline{\mathrm{B}}$.

2.

$\overline{(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}$

$= \overline{((A \cap B) \cap C)}$ (∩ is associative)

$= \overline{(A \cap B)} \cup \bar{C}$ (De Morgan's Law)

$= (\bar{A} \cup \bar{B}) \cup \bar{C}$ (De Morgan's Law)

$= \overline{\mathrm{A}} \cup \overline{\mathrm{B}} \cup \overline{\mathrm{C}}$ (∪ is associative)

$\therefore \overline{(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}} \cup \overline{\mathrm{C}}$.

Exercise

Using set identities, show that $(A - B) - (B - C) = A - B$.

$$
\begin{aligned}
(A - B) - (B - C) &= (A - B) \cap \overline{(B - C)} && (\because X - Y = X \cap \overline{Y}) \\
&= (A \cap \bar{B}) \cap \overline{(B \cap \bar{C})} && (\because X - Y = X \cap \overline{Y}) \\
&= (A \cap \bar{B}) \cap (\bar{B} \cup C) && (\because \text{De Morgan's Law}) \\
&= [(A \cap \bar{B}) \cap \bar{B}] \cup [(A \cap \bar{B}) \cap C] && (\because \text{Distributive Law}) \\
&= [A \cap (\bar{B} \cap \bar{B})] \cup [A \cap (\bar{B} \cap C)] && (\because \text{Associative Law}) \\
&= (A \cap \bar{B}) \cup [A \cap (\bar{B} \cap C)] && (\because \text{Idempotent Law}) \\
&= A \cap [\bar{B} \cup (\bar{B} \cap C)] && (\because \text{Distributive Law}) \\
&= A \cap \bar{B} && (\because \text{Absorption Law}) \\
&= A - B && (\because A - B = A \cap \overline{B})
\end{aligned}
$$

Generalized Unions and Intersections

Let $A_1, A_2, \ldots, A_n$ be *n* sets. Then

- The union of the sets $A_1, A_2, \ldots, A_n$ is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i\}.$$

- The intersection of the sets $A_1, A_2, \ldots, A_n$ is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for all } i\}$$

Example

- For $i = 1, 2, 3, \ldots.$, let $A_i = \{i, i+1, i+2, \ldots\ldots\}$. Then find

$$\bigcup_{i=1}^{n} A_i \text{ and } \bigcap_{i=1}^{n} A_i .$$

$$\begin{aligned}
A_1 &= \{1, 2, 3, \cdots \cdots\} = \mathbb{Z}^+ \\
A_2 &= \quad \{2, 3, \cdots \cdots\} = \mathbb{Z}^+ - \{1\} \\
&\vdots \\
A_n &= \qquad \{n, n+1, \cdots \cdots\} = \mathbb{Z}^+ - \{1, 2, \cdots, n-1\}
\end{aligned}$$

$$\therefore \bigcup_{i=1}^{n} A_i = \mathbb{Z}^+ ,$$

$$\bigcap_{i=1}^{n} A_i = \{n, n+1, \cdots \cdots\} = \mathbb{Z}^+ - \{1, 2, \cdots, n-1\} .$$

Example

For $i = 1, 2, 3, \ldots.$, let $A_i = \{1, 2, 3, \ldots, i\}$. Then find

$$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i \text{ .}$$

$$A_1 = \{1\}$$
$$A_2 = \{1, 2\}$$
$$\vdots$$
$$A_n = \{1, 2, 3, \cdots, n\}$$
$$\vdots$$

$$\therefore \quad \bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$$

$$\bigcap_{i=1}^{\infty} A_i = \{1\}.$$

Computer Representation of Sets

If we store the elements of the set in an unordered fashion, the operations of computing the union, intersection, or difference of two sets would be time-consuming.

Assume that the universal set U is **finite**. Let $a_1, a_2, \cdots, a_n$ be an arbitrary ordering of the elements of U. Then any subset A of U can be represented with the bit string of length n, where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then
The subset of all odd integers in U is represented by the bit string of length 10 "1010101010".
the subset of integers not exceeding 5 in U is represented by the bit string "1111100000".

Set Operations on Bit Strings

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Then the bit strings representation for the sets $\mathrm{A} = \{1, 2, 3, 4, 5\}$ and $\mathrm{B} = \{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010.
Find the bit string representation for $A \cup B, A \cap B,$ and A^c.

Solution:
Representation for $A \cup B$ = (1111100000) ∨ (1010101010)
$= 1111101010.$

Representation for $A \cap B$ = (1111100000) ∧ (1010101010)
$= 1010100000.$

Representation for $A^c = U - A = (1111111111) - (1111100000)$
$= 0000011111.$