SCS1306 Linear Algebra

Tutorial

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Overview

- Lecture 1: Introduction to Linear Algebra; vectors, operations, and linear combinations.
- Lecture 2: Solving linear systems; row vs. column picture; independence and invertibility.
- Lecture 3: Row eliminations and back substitution.

What is Linear Algebra?

- Linear algebra studies linear equations, matrices, and vector spaces. It is widely used in math, science, and engineering (graphics, data science, machine learning, etc.).
- Linear algebra deals with **linear equations** and their representations through matrices.
- Linear equations follow the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$.

Practice Problem I

For each of the following equations, determine whether it is a linear equation. Justify your answer.

1. $3x - 4y + 2z = 0$
2. $x^2 + y = 7$
3. $\sqrt{x} + 2y - z = 1$
4. $xy + z = 5$
5. $0 \cdot x + 0 \cdot y = 0$

Solution I

For each of the following equations, determine whether it is a linear equation. Justify your answer.

1. $3x - 4y + 2z = 0$ **Linear**; Because all the variables x, y, z appear with an exponent 1.
2. $x^2 + y = 7$ **Non Linear**; The term x^2 is non-linear.
3. $\sqrt{x} + 2y - z = 1$ **Non Linear**; The term $\sqrt{x}$ is non-linear.
4. $xy + z = 5$ **Non Linear**; The term xy involves a product of two variables, which violates the condition for a linear equation.
5. $0 \cdot x + 0 \cdot y = 0$ **Linear**; It adheres to the definition of linear equations. However, the equation lacks meaning.

Vectors and Operations

- A **vector** $\vec{v} \in \mathbb{R}^n$ is an ordered list of numbers (a $n \times 1$ column). Equivalently, it can be seen as an arrow from the origin to the point $(v_1, \ldots, v_n)$. For example, $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an arrow from $(0, 0)$ to $(3, 1)$. A vector has a *direction* and a *magnitude*.
- The most important operation is taking **linear combinations** of vectors. Linear combinations are a combination of two vector operations: *vector addition* and *scalar multiplication*.

Vectors and Operations Contd.

- **Scalar multiplication:** For a real number c, $c\vec{v} = (cv_1, cv_2, \ldots, cv_n)$ scales each component by c. Geometrically, it stretches or shrinks the arrow by a factor $|c|$, and reverses direction if $c < 0$.
- **Vector addition:** $\vec{u} + \vec{v}$ adds components: e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Geometrically, add arrow $\vec{v}$ to the tip of arrow $\vec{u}$.
- A **linear combination** of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is any sum $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. This operation combines scalar multiplication and vector addition in one step.

System of Linear Equations

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables.
For example, consider the following system with two variables, x and y:

$$\begin{cases} 3x + y = 9 & \text{(Equation 1)} \\ x - y = -1 & \text{(Equation 2)} \end{cases}$$

The objective is to find values for the variables (in this case, x and y) that satisfy **all** equations in the system simultaneously.
A system can have:

- Exactly one unique solution (intersecting lines in a 2D scenario).
- No solution (parallel lines in a 2D scenario).
- Infinitely many solutions (coincident lines in a 2D scenario).

For the given example, the unique solution is $x = 2, y = 3$.

Solving Linear Systems

- Consider a system of linear equations. Three complementary viewpoints:
  - **Row picture:** Each equation is a geometric object. In $\mathbb{R}^2$, each equation is a line; in $\mathbb{R}^3$, a plane. Typically, in $\mathbb{R}^3$ three planes meet at a single point (the solution).
  - **Column picture:** We express $\vec{b}$ as a linear combination of the columns of coefficients. This is written in the form $x \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + y \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
  - **Matrix form:** The matrix represents the system as a product of a coefficient matrix and a variable vector equal to a constant vector. Matrix form is often written as $A\vec{x} = \vec{b}$, where A is the coefficient matrix, $\vec{x}$ is the variable vector, and $\vec{b}$ is a constant vector.

Practice Problem II

Find the following:

1. Row Picture
2. Column Picture
3. Matrix Form

$$\begin{cases} 4x - 2y = 8, \\ x + 5y = 11. \end{cases}$$

Solution II

$$\begin{cases} 4x - 2y = 8, \\ x + 5y = 11. \end{cases}$$

In the row picture, each equation is considered as a geometric object. In the above example in $\mathbb{R}^2$, each equation is a line.

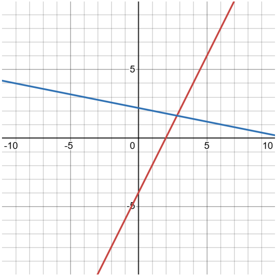

Figure: Row Picture. Created using Desmos

Solution II Contd.

$$\begin{cases} 4x - 2y = 8, \\ x + 5y = 11. \end{cases}$$

- Column Picture: $x \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$
- Matrix Form: $\begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$

Practice Problem III

Find the following:

1. Row Picture
2. Column Picture
3. Matrix Form

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

Solution III

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

In the above example in $\mathbb{R}^3$, each equation is a plane.

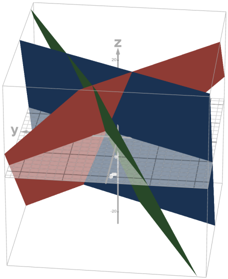

Figure: Row Picture. Created using Desmos 3D

Solution III Contd.

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

- Column Picture: $x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$
- Matrix Form: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

Linear Independence

- **Linear independence:** The vectors are linearly independent if the only way to get $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$ is the trivial combination $(c_1 = \cdots = c_k = 0)$. Otherwise, they are *dependent*.
- Note: the zero vector $\vec{0}$ can only appear in an independent set by the trivial combination (all coefficients zero). If any combination of non-zero coefficients yields $\mathbf{0}$, the vectors are dependent.

Practice Problem IV

Check whether each set of vectors is **linearly independent**.

- $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
  - Can you find constants a, b (not both zero) such that $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$?
- $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$
  - Can $\vec{v}_3$ be written as a combination of $\vec{v}_1$ and $\vec{v}_2$?

Solution IV

- $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
  - No: $\vec{v}_2 = 2 \cdot \vec{v}_1$, so they are **linearly dependent**.
  - A non-trivial combination: $2 \cdot \vec{v}_1 + (-1) \cdot \vec{v}_2 = \vec{0}$.
- $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - No scalar multiple of one gives the other. So, they are **linearly independent**.
  - Only solution to $a \cdot \vec{v}_1 + b \cdot \vec{v}_2 = \vec{0}$ is $a = 0,\ b = 0$.
- $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$
  - No: $\vec{v}_3 = 2 \cdot \vec{v}_1 + \vec{v}_2$, so they are **linearly dependent**.
  - A non-trivial combination: $2 \cdot \vec{v}_1 + \vec{v}_2 + (-1) \cdot \vec{v}_3 = \vec{0}$.

Span of Vectors

- Given vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$, their span is

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \left\{ c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \;\middle|\; c_i \in \mathbb{R} \right\}.$$

- Geometrically:
  - Two non-collinear vectors in $\mathbb{R}^2$ span the entire plane.
  - Three non-coplanar vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$.

Row Elimination

- Goal: transform $A\mathbf{x} = \mathbf{b}$ into an equivalent upper-triangular system.
- Use elementary row operations:
  1. Swap two rows.
  2. Multiply a row by a non-zero scalar.
  3. Add a scalar multiple of one row to another.
- Result: an augmented matrix in row echelon form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right).$$

- Each leading "pivot" isolates one variable.

Practice Problem V

Solve each system by elimination followed by back substitution.

(1)

$$\begin{cases} x + y + z = 6, \\ 2x + 3y + z = 10, \\ x - y + 2z = 5. \end{cases}$$

(2)

$$\begin{cases} x + y + z = 2, \\ 2x + 2y + 2z = 4, \\ x - y = 1. \end{cases}$$

Solution V

System:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 10 \\ 1 & -1 & 2 & 5 \end{array}\right)$$

- *Row elimination to upper-triangular:*

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 1 & -1 \end{array}\right)$$

$$R_3 \leftarrow R_3 + 2R_2 \quad \Longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -1 & -5 \end{array}\right)$$

- **Solution:** $(x, y, z) = (-2,\ 3,\ 5)$.

Solution V Contd.

System:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

- *Row elimination:*

$$R_2 \leftarrow R_2 - 2R_1 \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

$$R_3 \leftarrow R_3 - R_1 \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & -1 \end{array}\right)$$

$$R_3 \leftarrow -\frac{1}{2}R_3 \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

Solution V Contd.

- *Observe:* Row 2 is all zeros $\implies$ one equation dropped.
- *Back substitution:*

$$y + \tfrac{1}{2}z = \tfrac{1}{2} \implies y = \tfrac{1}{2} - \tfrac{1}{2}z,$$

$$x + y + z = 2 \implies x = 2 - y - z = 2 - \left(\tfrac{1}{2} - \tfrac{1}{2}z\right) - z.$$

$$x = \tfrac{3}{2} - \tfrac{1}{2}z.$$

- **Infinite solutions:**

$$(x, y, z) = \left(\tfrac{3}{2} - \tfrac{1}{2}z,\ \tfrac{1}{2} - \tfrac{1}{2}z,\ z\right), \quad z \in \mathbb{R}.$$

- **Explanation:** The second original row was a multiple of the first, so only two independent equations remain. Hence the coefficient columns are linearly dependent, and the system has infinitely many solutions.

Summary

- Key concepts: vectors, scalar multiplication, vector addition, linear combinations, and linear independence.
- Decomposing a system of linear equations using the row picture (intersecting lines/planes), the column picture (combining columns of A), and the matrix form ($A\vec{x} = \vec{b}$).
- Linear independence of vectors.
- Row eliminations and back-substitution.