

Discrete Maths - Kuppi 04

Set theory & Functions

Set Theory

Definition: Any well-defined collection of distinct objects is called a set. The elements of a set are called its members. The order of items in a set is not important

A set has to be well-defined

- Consider the set of "interesting books."

This is not a well-defined set because the criteria for what makes a book "interesting" can vary from person to person. Without a clear and universally agreed-upon definition of what constitutes an interesting book, different people might include different books in the set, leading to ambiguity.

The number of elements in a set can be:

- Empty
- Finite
- In-finite

Ways of describing sets

- elements within braces: $A = \{a, b, c, d\}$
- Brace notation with ellipses: $A = \{1, 2, 3, \dots, 100\}$
- Verbal description: "A is the set of integers from 1 to 100, inclusive"
- Set builder notation: $A = \{x \mid x \text{ is an integer, } 1 \leq x \leq 100\}$

Cardinality of a set:

The number of members in a set is called the **cardinality**, which is denoted by **|A|**.

Singleton set: a set with only one element is called a singleton set.

Subsets

The set A is a subset of B if and only if every element in A is also an element of B, and is denoted by $A \subseteq B \iff \forall x[x \in A \rightarrow x \in B]$

If $A \subseteq B$ but $A \neq B$ then, $A \subset B$ (A is a proper subset of B)

$$A \subset B \iff \forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

Equality of two sets

Definition: Two sets A and B are said to be equal if both sets have the same elements and is denoted by $A = B$. If sets 'A' and 'B' are not equal we write $A \neq B$

$$A = B \iff A \subseteq B \wedge B \subseteq A$$

$$\iff \forall x[x \in A \rightarrow x \in B]$$

Power set

The set of all subsets of set A, denoted $P(A)$ or 2^A , is called the **power set** of A.

power set of {a, b, c}:

$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

- If a set has n elements then its power set has 2^n elements.

Truth sets and quantifiers

Given a predicate of P , and a domain D , we define the truth set of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$

Union and Intersection

Union: The union of two sets A and B , denoted by $A \cup B$, is the set of all the elements that are in either A or B or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Intersection: The intersection of two sets A and B , denoted by $A \cap B$, is the set containing the elements those are in both A and B .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Disjoint Sets: Two sets A , B are called disjoint if $A \cap B = \emptyset$

Complement

Let A and B be sets. The difference of A and B , denoted by $A - B$, is the set of elements that are in A but not in B . **$A - B$ is also called the complement of B with respect to A .**

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

- A^c or A' (A bar) is the complement of A with respect to the universal set U .

Let A and B be two sets, **The symmetric difference** of A and B , denoted by $A \oplus B$ is the set **$(A - B) \cup (B - A)$**

$$A \oplus B = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

Set identities

TABLE 1 Set Identities.	
<i>Identity</i>	<i>Name</i>
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

examples:

show that;

1. $(A \cup B)^c = A^c \cap B^c$
 2. $A \oplus B = (A - B) \cup (B - A)$
 3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 4. $A - (B \cup C) = (A - B) \cap (A - C)$
-

Generalized Unions and Intersections

Let A_1, A_2, \dots, A_n be n sets. Then,

The union of all the n sets:

$$\bigcup_{k=1}^n A_k = A_1 \cup A_2 \cup \dots \cup A_n$$

Similarly the intersection of all the n sets:

$$\bigcap_{k=1}^n A_k = A_1 \cap A_2 \cap \dots \cap A_n$$

Computer representation of sets

Finite Universal Set U:

Suppose we have a universal set U with n elements. Each element of U is given a fixed index (e.g., u_1, u_2, \dots, u_n).

Bit Vector Representation:

Any subset $A \subseteq U$ can be represented as a binary string (or array of bits) of length n :

- **1** at position i means $u_i \in A$.
- **0** at position i means $u_i \notin A$.

Example:

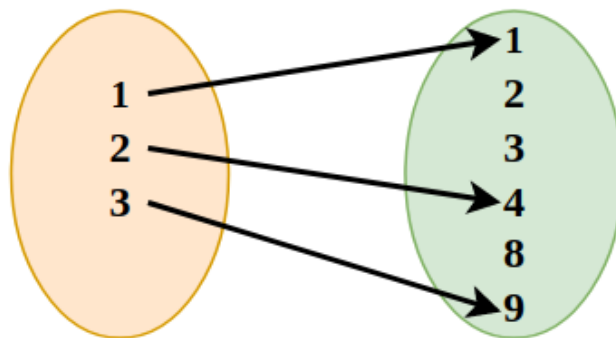
- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- Subset $A = \{u_1, u_4\}$
- Representation: $A = 10010$

So effectively:

$U = 11111, A = 10010$

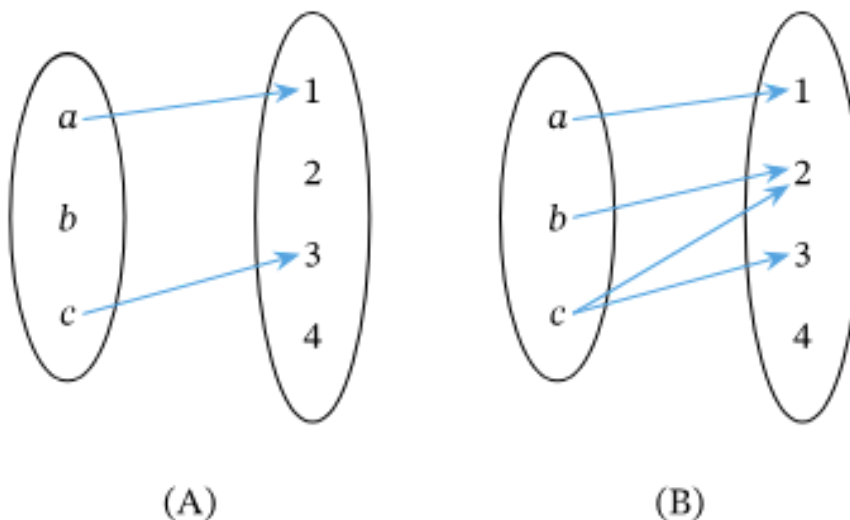
Functions

- A function is a mapping from elements in one set(Domain) to elements in another set (Co-Domain) satisfying the following two conditions.
 - All the elements of the domain should be mapped to some elements in the co-domain.
 - Same element in the domain cannot be mapped with multiple elements in the co-domain.



🌐 Domain and Range of a Relation - Definition, Types, and Examples - GeeksforGeeks

These are invalid mappings



🌐 Lesson Explainer: Identifying Functions | Nagwa

One-to-one Functions

Some functions never assign the same value to two different domain elements. These functions are said to be one-to-one.

Definition:

Let $f:A \rightarrow B$ be a function.

We say f is **one-to-one (injective)** if:

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

example:

Function:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3$$

check if the function is one-to-one

let there be two values x_1 and x_2 which map to the same value

then;

$$f(x_1) = f(x_2)$$

$$2x_1 + 3 = 2x_2 + 3$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

Hence, the function $f(x)$ is one-to-one

Onto Functions

An **onto function** (also called a **surjective function**) is a function in which **every element of the codomain has at least one element of the domain mapping to it**.

Formally, if $f: A \rightarrow B$, then f is **onto** if:

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y$$

In other words: **every element of the codomain B is “hit” by the function**.

example:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x - 5$$

Proof that it is onto:

- Take any $y \in \mathbb{R}$ (codomain).
- Solve $f(x) = y$:

$$2x - 5 = y$$

$$2x = y + 5$$

$$x = (y + 5)/2$$

- Since $x \in \mathbb{R}$, **every y in the codomain has a corresponding x in the domain.**

✓ Therefore, $f(x) = 2x - 5$ is **onto**.

One-to-one correspondence or Bijection

A function $f: A \rightarrow B$ is said to be a Bijection is and only if it is **one to one and onto**.

Example:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x + 2$$

Suppose $f(x_1) = f(x_2)$

then;

$$3x_1 + 2 = 3x_2 + 2$$

$$x_1 = x_2$$

Therefore, the function is One-to-one

Take any $y \in \mathbb{R}$.

Solve $f(x) = y$ for x :

$$3x + 2 = y \Rightarrow x = (y - 2)/3$$

- Since $x \in \mathbb{R}$ for any $y \in \mathbb{R}$, **every codomain element has a pre-image.**

✓ So f is **surjective** (onto).

Inverse Functions

An interesting property of bijections is that they have an **inverse function**.

Definition of Inverse Function:

Let $f: A \rightarrow B$ be a **bijection**. Then there exists a function

$$f^{-1}: B \rightarrow A$$

called the **inverse function** of f , such that for every $x \in A$ and $y \in B$:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

In other words, f^{-1} “reverses” the action of f , mapping each element of the codomain back to its **unique pre-image** in the domain.

Example 1:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 5x - 7$$

Check one-to-one:

$$f(x_1) = f(x_2)$$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7$$

$$\Rightarrow x_1 = x_2$$

✔ One-to-one

Check onto:

Take any $y \in \mathbb{R}$:

$$y = 5x - 7$$

$$\Rightarrow x = (y + 7)/5 \in \mathbb{R}$$

✔ Onto

Conclusion:

- f is bijective \rightarrow **invertible**
- Inverse: $f^{-1}(y) = y$

Example 2:

$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = \sin(x)$

Check one-to-one:

- $\sin(0) = 0$ and $\sin(\pi) = 0 \rightarrow$ different inputs map to the same output

✗ Not one-to-one

Check onto:

- Codomain is \mathbb{R}
- $\sin(x) \in [-1,1] \rightarrow$ cannot cover values like 2, -5, etc.

✗ Not onto

Conclusion:

- g is not bijective \rightarrow not invertible
- (It becomes invertible if we restrict the domain to $[-\pi/2, \pi/2]$ and codomain to $[-1,1]$)

Compositions of Functions

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. The **composition** of g and f , denoted by

$g \circ f: A \rightarrow C$

is defined by

$(g \circ f)(x) = g(f(x))$ for all $x \in A$

- In words: **first apply f to x , then apply g to the result.**
- The domain of $g \circ f$ is the domain of f , and the codomain is the codomain of g .

example:

$f(x) = 2x + 3$ and $g(x) = x - 3$

$f \circ g(x) = f(g(x)) = 2(g(x)) + 3 \rightarrow 2(x - 3) + 3$

$g \circ f(x) = g(f(x)) = (2x + 3) - 3 \rightarrow 2x$

Identity Function

The **identity function** is the function that leaves every element of its domain unchanged. In other words, it returns exactly the same value as the input. Formally, if A is a set, the identity function on A is the function

Definition:

Let A be a non-empty set. The **identity function** on A is the function,

$$I : A \rightarrow A$$

defined by

$$I_A(x) = x \text{ for all } x \in A.$$

Note:

For any function $f : A \rightarrow A$, the composition of f with its inverse f^{-1} (whenever the inverse exists) is always the **identity function** on A .

That is,

$$f \circ f^{-1} = f^{-1} \circ f = i_A$$

where $i_A(x) = x$ for all $x \in A$.

The Image and Inverse Image of a Subset of Domain and Co-Domain Respectively

For a given function $f : A \rightarrow B$

The image of S ($S \subseteq A$) under f (denoted by $f(S)$) is the subset of B consisting of the images of the elements of S .

$$f(S) = \{f(s) \mid s \in S\}$$

The inverse image of T ($T \subseteq B$) under f is the subset of A consisting of the pre-images of elements in T .

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

Example:

Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and define $f(1) = a$, $f(2) = b$, $f(3) = c$.

- If $E = \{1, 3\}$, then $f(E) = \{a, c\}$.
- If $F = \{a, b\}$, then $f^{-1}(F) = \{1, 2\}$.

Example Problem:

Let $f : A \rightarrow B$ be a function, and let $S, T \subseteq B$. Show that

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
2. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Also, give an example to show that in general,

$$f^{-1}(B - S) = A - f^{-1}(S).$$

Solution:

1. Proof that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

Step 1: Show $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$

- Let $a \in f^{-1}(S \cup T)$.
- By definition of inverse image: $f(a) \in S \cup T$.
- Then, $f(a) \in S$ or $f(a) \in T$.
- By definition of inverse image: $a \in f^{-1}(S)$ or $a \in f^{-1}(T)$.
- Then, $a \in f^{-1}(S) \cup f^{-1}(T)$.

Thus, $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

Step 2: Show $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$

- Let $a \in f^{-1}(S) \cup f^{-1}(T)$.
- Then $a \in f^{-1}(S)$ or $a \in f^{-1}(T)$.
- So $f(a) \in S$ or $f(a) \in T$.
- By definition of union: $f(a) \in S \cup T$.
- By definition of inverse image: $a \in f^{-1}(S \cup T)$.

Thus, $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

Conclusion:

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$$

2. Proof that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Step 1: Show $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$

- Let $a \in f^{-1}(S \cap T)$.
- Then $f(a) \in S \cap T$.
- By definition of intersection: $f(a) \in S$ and $f(a) \in T$.
- By definition of inverse image: $a \in f^{-1}(S)$ and $a \in f^{-1}(T)$.
- By definition of intersection: $a \in f^{-1}(S) \cap f^{-1}(T)$.

Step 2: Show $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$

- Let $a \in f^{-1}(S) \cap f^{-1}(T)$.
- Then $a \in f^{-1}(S)$ and $a \in f^{-1}(T)$.
- So, $f(a) \in S$ and $f(a) \in T$.
- By definition of intersection: $f(a) \in S \cap T$.
- By definition of inverse image: $a \in f^{-1}(S \cap T)$.

Conclusion:

$$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$$

3. Proof that $f^{-1}(B - S) = A - f^{-1}(S)$

Step 1: Show $f^{-1}(B - S) \subseteq A - f^{-1}(S)$

- Let $a \in f^{-1}(B - S)$.
- Then $f(a) \in B - S$.
- By definition of set difference: $f(a) \notin S$.
- By definition of inverse image: $a \notin f^{-1}(S)$.
- By definition of set difference: $a \in A - f^{-1}(S)$.

Step 2: Show $A - f^{-1}(S) \subseteq f^{-1}(B - S)$

- Let $a \in A - f^{-1}(S)$.
- Then, $a \notin f^{-1}(S) \rightarrow f(a) \notin S$.
- Since $f(a) \in B$ and $f(a) \notin S$, $f(a) \in B - S$.
- By definition of inverse image: $a \in f^{-1}(B - S)$.

Conclusion:

$$f^{-1}(B - S) = A - f^{-1}(S)$$

Special Functions

1. Floor Function ($\lfloor x \rfloor$)

- **Definition:** For any real number x , the **floor function** $\lfloor x \rfloor$ gives the **greatest integer less than or equal to x** .
- **Examples:**
 - $\lfloor 3.7 \rfloor = 3$
 - $\lfloor -2.1 \rfloor = -3$
 - $\lfloor 5 \rfloor = 5$

So basically, it **rounds down to the nearest integer**, but “down” means towards $-\infty$, not just reducing the decimal.

2. Ceiling Function ($\lceil x \rceil$)

- **Definition:** For any real number x , the **ceiling function** $\lceil x \rceil$ gives the **smallest integer greater than or equal to x** .
- **Examples:**
 - $\lceil 3.7 \rceil = 4$
 - $\lceil -2.1 \rceil = -2$
 - $\lceil 5 \rceil = 5$

So this one **rounds up to the nearest integer**, but “up” means towards $+\infty$.

- Floor and ceiling functions are onto but not one-to-one. Hence, they are not invertible.