SCS 1307
Probability & Statistics

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Random Variables

- Results of experiments are often summarized in terms of numerical values. For random experiments, the specific values cannot be predicted with certainty.

  Hence we talk about random variables.

- **Random variables**

  Let Ω be the outcome space corresponding to a random experiment.

A function $X: \Omega \rightarrow R$ is called a random variable

Discrete Random Variables

The random variable X is a discrete random variable if the values taken by the random variable contains only a finite number of sample points.

Example:

A coin is tossed three times, and the sequence of Heads and Tails is observed.

The outcome space is

Ω= {hhh, hht, hth, thh, htt, tht, tth, ttt}

Let X= number of Heads observed,

Y= number of Tails observed,

Z =X Y.

The functions X,Y and Z are conveniently specified by the table below.

	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	3	2	2	2	1	1	1	0
Y	0	1	1	1	2	2	2	3
Z	0	2	2	2	2	2	2	0

Therefore $\Omega_X = \Omega_Y = \{0,1,2,3\}$ and $\Omega_Z = \{0,2\}$.

Exercise

A coin is tossed until two successive tosses give the same result.

Let X = number of Heads observed and Y= total number of tosses made.

Give Ω_X, the sample space of X and Ω_Y , the sample space of Y.

Solution

Ω = {HH, TT, HTT, THH, HTHH, THTT, HTHTT, THTHH, …………..}

Ω_X ={0, 1, 2, ………..}

Ω_Y ={2, 3, 4, ………..}

Probability Distributions

Let X be a discrete random variable defined on the outcome space Ω. Let X be a value in ΩX, the sample space of X. We are interested in the event 'X takes the value x '. We are also interested in the probability associated with this event.

Definition

Let X be a random variable with sample space ΩX. The probability function of X is the function P_x defined by ,

$$P_x = \begin{cases} P(X=x) & \text{if } x \in \Omega X; \\ 0 & \text{Otherwise} \end{cases}$$

By the probability distribution of X we mean the way in which the probability mass is distributed over the values in Ω_X.

The probability function P_x is one way of specifying the probability distribution of X.

Example

Toss a coin three times and observe the sequence of Heads and tails.

Here Ω = {hhh, hht, hth, thh, htt, tht, tth, ttt}.

Suppose the results on the tosses of the coin are independent, and that any given toss falls Heads with probability 1/3.

Define X= Number of Heads observed and Y= number of Tails observed.

Solution

Ω = {hhh, hht, hth, thh, htt, tht, tth, ttt}.

X= Number of Heads observed and Y= number of Tails observed.

- Probabilities can be specified as follows.

ω	**HHH**	**HHT**	**HTH**	**THH**	**HTT**	**THT**	**TTH**	**TTT**
P(ω)	1/27	2/27	2/27	2/27	4/27	4/27	4/27	8/27
X(ω)	3	2	2	2	1	1	1	0
Y(ω)	0	1	1	1	2	2	2	3

Note that $P_x(0)$ =8/27, $P_x(1)$=12/27 , $P_y(2)$=12/27.

Functions of Random Variables

It is sometimes useful to define random variables as functions of other random variables.

Examples : If X is a random variable, then so are 2X+5, and X^2.

Example

A machine is used to cut rectangular shapes out of sheets of metal. Switch A has 2 settings which are used to select the length L of the rectangle (2cm or 3cm). Switch B has three settings which are used to select the width W of the rectangle(1cm, 2cm or 3cm). A mechanic tests the machine by selecting settings for switches A and B and making a rectangle. The length and width are observed. Suppose the settings are chosen randomly in a way so that in the Table below.

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
P(ω)	0.12	0.13	0.25	0.20	0.15	0.15

Example

a. Find pL, the probability distribution of length

b. Find pW, the probability distribution of width

c. ind pX, the probability distribution of perimeter X=2L+2W

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
P(ω)	0.12	0.13	0.25	0.20	0.15	0.15

Solution

a. Find pL, the probability distribution of length

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
L	2	2	2	3	3	3
P(ω)	0.12	0.13	0.25	0.20	0.15	0.15

Probability Distribution of length

L	**2**	**3**
P(L)	0.5	0.5

Solution

b. Find pW, the probability distribution of width

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
W	1	2	3	1	2	3
P(ω)	0.12	0.13	0.25	0.20	0.15	0.15

Probability Distribution of width

W	**1**	**2**	**3**
P(W)	0.32	0.28	0.4

Solution

c. Find pX, the probability distribution of perimeter

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
X	6	8	10	8	10	12
P(ω)	0.12	0.13	0.25	0.20	0.15	0.15

Probability Distribution of perimeter

X	**6**	**8**	**10**	**12**
P(X)	0.12	0.33	0.4	0.15

Expectation and Variance

When the range Ωx of a random variable X: $\Omega \rightarrow R$ contains only a few points it is easy to 'absorb' the entire probability distribution.

However, it is often the case that Ωx is very large.

In these cases it is useful to talk about ***expectation*** that can help to provide a measure of 'location' and a measure of 'spread '

Definition of Expectation

The ***expectation*** of a random variable X defined on an outcome space Ω is the value E(X) given by

$$E(X) = \sum_{\omega\in\Omega} X(\omega)P(\{\omega\})$$

This quantity is also referred to as the *mean* of X , or the *mean of the probability distribution* of X or the *expected value* of X.

Consider the following probability model:

(ω)	$\boldsymbol{\omega_1}$	$\boldsymbol{\omega_2}$	$\boldsymbol{\omega_3}$	$\boldsymbol{\omega_4}$	$\boldsymbol{\omega_5}$	$\boldsymbol{\omega_6}$
P({ω})	0.1	0.2	0.2	0.1	0.3	0.1
X(ω)	3	3	2	1	3	2

Calculate the expected value of X.

E(X)=0.1*3 + 0.2*3 + 0.2*2 + 0.1*1 + 0.3*3 + 0.1*2 = 2.5

Properties of Expectation

1. $E(X) = \sum_{X \in \Omega x} X \, P(x)$

2. E(aX + b) = a E(X) + b where a & b are constants

3. Let X and Y be random variables defined on the same outcome space Ω.

$$E(X + Y) = E(X) + E(Y)$$

Exercise

The random variable X has probability distribution given by

X	-2	-1	0	1	2
P(X)	0.3	0.1	0.2	0.1	0.3

(a) Find E(X)

(b) Find E(X^2)

Solution

The random variable X has probability distribution given by

X	-2	-1	0	1	2
X^2	4	1	0	1	4
P(X)	0.3	0.1	0.2	0.1	0.3

(a) E(X) = 0

(b) Find E(X^2) =2.6

Variance

The mean of the random variable X is the value of E(X). The variance of X is the quantity

$$\sigma_x^2 = \mathrm{E}[(X-\mu)^2] = \sum_{x\epsilon\Omega x}(X-\mu)^2 P(x).$$

Then σ_x is called the *standard deviation* of X. Both the standard deviation and the variance can be used as measures of the spread of a probability distribution, however the standard deviation is commonly used since it is a measurement in the same unit as X.

Properties of Variance

Property 1: V(X) = $E(X^2) - \mu^2$

Property 2: If c is any constant and X is a random variable then

Var(cX) = c^2 Var(X)

Property 3:

If X and Y are independent random variables,

Var(X+Y)= Var(X) + Var(Y) and

Var(X-Y) = Var(X) + Var(Y)

Exercise

Toss an unbiased coin once and win \$k if it is a Head and lose \$k if it is a Tail. Find the standard deviation of the profit X if the probability distribution of X is given by

X	-k	K
P(X)	0.5	0.5

Solution

X	-k	k
X^2	k^2	k^2
P(X)	0.5	0.5

E(X) = μ = 0

V(X) = V(X) = $E(X^2) - \mu^2$ = k^2

σ_x = k

Exercise

Suppose that a game is played with a single die assumed fair. In this game a player wins \$20 if a 2 turns up, \$40 if a 4 turns up; loses \$30 if a 6 turns up; while he neither wins or loses if any other face turns up. Find the expected sum of money won.