SCS1306 Linear Algebra

Tutorial

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Definition

Given an $n \times n$ matrix A, a scalar λ and a nonzero vector $\vec{x}$ such that:

$$A\vec{x} = \lambda\vec{x}$$

Then λ is an **eigenvalue**, and $\vec{x}$ is the corresponding **eigenvector**.

Geometric Definition

- After linear transformation A, if there are vectors $\vec{x}$, such that $\vec{x}$ does not change direction, those vectors are called **eigenvectors** of matrix A.
- The eigenvectors fall along the directions that are not affected by the linear transformation.
- The amounts each eigenvector stretches or compresses as a result of transformation A are called the **eigenvalues** of matrix A.

How to Find Eigenvalues

Eigenvectors are defined as:

$$A\vec{x} = \lambda\vec{x}$$

The left-hand side is a matrix-vector multiplication, and the right-hand side is a scalar-vector multiplication. To convert both sides into the same format, use:

$$\lambda\vec{x} = (\lambda I)\vec{x}, \; I \text{ is the identity matrix}$$

$$\begin{aligned} A\vec{x} &= (\lambda I)\vec{x} \\ A\vec{x} - (\lambda I)\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

How to Find Eigenvalues Contd.

$$(A - \lambda I)\vec{x} = \vec{0}$$

To find a non-trivial solution ($\vec{x} \neq \vec{0}$) for the above equation, the matrix $(A - \lambda I)$ must be singular. The determinant of a singular matrix is zero. Hence:

$$\det(A - \lambda I) = 0$$

This is called the *characteristic equation*. The roots of this equation, $\lambda_1, \lambda_2, \dots,$ are the eigenvalues.

How to Find Eigenvectors

After finding the roots of the characteristic equation (eigenvalues), for each eigenvalue λ, solve:

$$(A - \lambda I)\vec{x} = \vec{0}$$

This gives the eigenvector(s) corresponding to λ.

Example

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Find eigenvalues and eigenvectors.

$$
\begin{aligned}
(A - \lambda I) &= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \\
\det(A - \lambda I) &= (2-\lambda) \times (2-\lambda) - 1 \times 1 \\
&= (4 - 4\lambda + \lambda^2) - 1 \\
&= \lambda^2 - 4\lambda + 3 \\
&= (\lambda - 3)(\lambda - 1)
\end{aligned}
$$

Solution

Using the characteristic equation:

$$\det(A - \lambda I) = 0$$
$$(\lambda - 3)(\lambda - 1) = 0$$
$$\lambda = 1,\ \lambda = 3$$

solve $(A - \lambda I)\vec{x} = \vec{0}$ for each λ.
For $\lambda = 1$:

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

Assume $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix},\ x, y \in \mathbb{R}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution Contd.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0 \Rightarrow x = -y$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$$

Therefore, for $\lambda = 1$, the eigenvectors are of the form $\vec{x} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$.

When $y = 1$, $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a sample eigenvector.

For $\lambda = 3$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow \vec{x} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$$

Diagonalization

If A has n **linearly independent eigenvectors**, then:

$$A = PDP^{-1}$$

where D is diagonal with eigenvalues, P has eigenvectors as columns.

Determinant using Eigenvalues

Since D is a triangular matrix, the determinant $\det(D)$ is the product of the diagonal values, according to the properties of the determinant. Since the diagonal of D consists of eigenvalues, $\det(D)$ becomes the product of eigenvalues:

$$\det(D) = \lambda_1 \lambda_2 \dots \lambda_n$$

According to the previous slide:

$$\begin{aligned}
A &= PDP^{-1} \\
\det(A) &= \det(PDP^{-1}) \\
&= \det(P)\det(D)\det(P^{-1}) \quad (\because \ \det(AB) = \det(A)\det(B)) \\
&= \det(D) \quad (\because \ \det(P) = \frac{1}{\det(P^{-1})})
\end{aligned}$$

$$\therefore \det(A) = \det(D) = \lambda_1 \lambda_2 \dots \lambda_n$$

Things to Remember

- Not all matrices are diagonalizable
- The matrix A and the upper triangular form of A, U, don't have the same eigenvalues. Since the determinant is equal to the product of eigenvalues, the product of eigenvalues of A and U is equal.
- The trace of a matrix is the sum of diagonal elements $tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$. The trace is also equal to the sum of eigenvalues $tr(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

2.

Let $A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) Determine the eigenvalues of matrix A. **[6 marks]**

(b) Find the linearly independent eigenvectors corresponding to each eigenvalue obtained in part (a). **[12 marks]**

(c) If the matrix A is diagonalizable, determine the invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. **[4 marks]**

(d) If $P^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, determine A^2. **[4 marks]**

(e) Calculate the determinant $\det(A)$ and the trace $\mathrm{tr}(A)$ of matrix A using the eigenvalues. **[4 marks]**

2.

Let $A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) Determine the eigenvalues of matrix A. **[6 marks]**

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 6-\lambda & 2 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

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$$\therefore \begin{vmatrix} 6-\lambda & 2 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda+1)[(6-\lambda)(3-\lambda) - 2\times 2] = 0$$

$$(\lambda+1)(\lambda^2 - 9\lambda + 14) = 0$$

$$(\lambda+1)(\lambda-7)(\lambda-2) = 0$$

$\therefore$ The eigenvalues are $\lambda = -1,\ 7,\ 2$

(b) Find the linearly independent eigenvectors corresponding to each eigenvalue obtained in part (a). **[12 marks]**

$$(A - \lambda I)\vec{x} = \vec{0}$$

For $\lambda = -1$

$$[A - (-1)I]\vec{x} = \vec{0}$$

$$(A + I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 7 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

z is a free variable, $x, y = 0$

$$\therefore \vec{x} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}$$

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For $\lambda = 7$

$$(A - 7I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8z = 0$$

$$\therefore z = 0$$

The first and the second rows of the matrix are linearly dependent on each other; hence, it is sufficient to consider just one row.

$$-x + 2y = 0 \Rightarrow x = 2y$$

$$\therefore \vec{x} = \begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ y \in \mathbb{R}$$

For $\lambda = 2$

$$(A - 2I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3z = 0$$

$$\therefore z = 0$$

The first and the second rows of the matrix are linearly dependent on each other; hence, it is sufficient to consider just one row.

$$2x + y = 0 \Rightarrow y = -2x$$

$$\therefore \vec{x} = \begin{bmatrix} x \\ -2x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \ x \in \mathbb{R}$$

The eigenvectors take the following forms:

$$x\begin{bmatrix}1\\-2\\0\end{bmatrix} \quad y\begin{bmatrix}2\\1\\0\end{bmatrix} \quad z\begin{bmatrix}0\\0\\1\end{bmatrix},\ x,y,z\in\mathbb{R}$$

(c) If the matrix A is diagonalizable, determine the invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. **[4 marks]**

By substituting $x = y = z = 1$, we can obtain the following eigenvectors:

$$\begin{bmatrix}1\\-2\\0\end{bmatrix} \quad \begin{bmatrix}2\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

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The eigenvectors we obtained are linearly independent. Hence, we can construct the P matrix as follows:

$$P = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the first column corresponds to the eigenvalue $\lambda = 2$, the second column corresponds to $\lambda = 7$, and the third column corresponds to $\lambda = -1$. We must follow the same order when constructing the diagonalized matrix D.

$$\therefore D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) If $P^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, determine A^2. **[4 marks]**

Calculate P by calculating the inverse of P^{-1}.

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe the columns of P. Using the corresponding eigenvalues, obtain the D matrix.

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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$$
\begin{aligned}
A &= PDP^{-1} \\
A^2 &= PD^2P^{-1} \\
&= \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 40 & 18 & 0 \\ 18 & 13 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

(e) Calculate the determinant $\det(A)$ and the trace $\mathrm{tr}(A)$ of matrix A using the eigenvalues. **[4 marks]**

$$\begin{aligned}
\det(A) &= \text{Product of Eigenvalues} \\
&= 7 \cdot 2 \cdot (-1) \\
&= -14 \\
\mathrm{tr}(A) &= \text{Sum of Eigenvalues} \\
&= 7 + 2 + (-1) \\
&= 8
\end{aligned}$$

4.

(a) Using the **properties** of a determinant, find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

[6 Marks]

(b) Let A be a square matrix, and suppose that A has **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$. Prove that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly independent. **[4 Marks]**

(a) Using the **properties** of a determinant, find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

Multiply the 3rd row by 2.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 2 & 2 & 8 \end{bmatrix}$$

From the properties of the determinants: $\det(B) = 2 \cdot \det(A)$.

Subtract row 2 from row 3 of matrix B ($R_3 \leftarrow R_3 - R_2$).

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Since subtracting a row from another row won't change the determinant, $\det(C) = \det(B)$.

$$\begin{aligned} \det(C) &= 5 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ &= 5 \cdot (1 \times 2 - 2 \times 2) \\ &= -10 \\ \det(B) &= -10 \quad \because \det(B) = \det(C) \\ \det(A) &= -5 \quad \because \det(B) = 2 \cdot \det(A) \end{aligned}$$

(b) Let A be a square matrix, and suppose that A has **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$. Prove that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly independent. **[4 Marks]**

Proof by Contradiction

Assume that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are **not** linearly independent. Then there is a linear combination:

$$\vec{x}_j = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_{j-1}\vec{x}_{j-1} \tag{1}$$

Such that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{j-1}$ are linearly independent and $1 < j \leq k$.
Apply the matrix A to equation 1:

$$A\vec{x}_j = c_1A\vec{x}_1 + c_2A\vec{x}_2 + \cdots + c_{j-1}A\vec{x}_{j-1} \tag{2}$$

Using $A\vec{x}_i = \lambda_i\vec{x}_i$

$$A\vec{x}_j = c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 + \cdots + c_{j-1}\lambda_{j-1}\vec{x}_{j-1} \tag{3}$$

Multiply equation 1 by λ_j:

$$\lambda_j\vec{x}_j = c_1\lambda_j\vec{x}_1 + c_2\lambda_j\vec{x}_2 + \cdots + c_{j-1}\lambda_j\vec{x}_{j-1} \tag{4}$$

Using $A\vec{x}_j = \lambda_j\vec{x}_j$

$$A\vec{x}_j = c_1\lambda_j\vec{x}_1 + c_2\lambda_j\vec{x}_2 + \cdots + c_{j-1}\lambda_j\vec{x}_{j-1} \tag{5}$$

(3) - (5) and factor out $c_i x_i$:

$$c_1(\lambda_1 - \lambda_j)\vec{x}_1 + c_2(\lambda_2 - \lambda_j)\vec{x}_2 + \ldots + c_{j-1}(\lambda_{j-1} - \lambda_j)\vec{x}_{j-1} = \vec{0}$$

$$\sum_{i=1}^{j-1} c_i(\lambda_i - \lambda_j)\vec{x}_i = \vec{0}$$

$$\sum_{i=1}^{j-1} c_i(\lambda_i - \lambda_j)\vec{x_i} = \vec{0} \tag{6}$$

Since $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{j-1}$ are linearly independent, $c_i(\lambda_i - \lambda_j) = 0, \ \forall i \in \{1, j-1\}$. All the eigenvalues of A are **distinct**, hence $\lambda_i - \lambda_j \neq 0, \ \forall i \in \{1, j-1\}$. Therefore, for every i, $c_i = 0$. Using equation 1:

$$\begin{aligned}
\vec{x_j} &= c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_{j-1}\vec{x}_{j-1} \\
\vec{x_j} &= 0 \cdot \vec{x}_1 + 0 \cdot \vec{x}_2 + \cdots + 0 \cdot \vec{x}_{j-1} \\
\vec{x_j} &= \vec{0}
\end{aligned}$$

This is a contradiction since eigenvectors are non-zero vectors. Therefore, the initial assumption is wrong; the eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly independent.