

# Discrete Maths - Kuppi 04

## Set theory & Functions

### Set Theory

**Definition:** Any well-defined collection of distinct objects is called a set. The elements of a set are called its members. The order of items in a set is not important

#### A set has to be well-defined .....

- Consider the set of "interesting books."

This is not a well-defined set because the criteria for what makes a book "interesting" can vary from person to person. Without a clear and universally agreed-upon definition of what constitutes an interesting book, different people might include different books in the set, leading to ambiguity.

#### The number of elements in a set can be:

- Empty
- Finite
- In-finite

### Ways of describing sets

- elements within braces:  $A = \{a, b, c, d\}$
- Brace notation with ellipses:  $A = \{1, 2, 3, \dots, 100\}$
- Verbal description: "A is the set of integers from 1 to 100, inclusive"
- Set builder notation:  $A = \{x \mid x \text{ is an integer, } 1 \leq x \leq 100\}$

## **Cardinality of a set:**

The number of members in a set is called the **cardinality**, which is denoted by  $|A|$ .

**Singleton set:** a set with only one element is called a singleton set.

## **Subsets**

The set A is a subset of B if and only if every element in A is also an element of B, and is denoted by  $A \subseteq B \iff \forall x[x \in A \rightarrow x \in B]$

If  $A \subseteq B$  but  $A \neq B$  then,  $A \subset B$  (A is a proper subset of B)

$$A \subset B \iff \forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

## **Equality of two sets**

**Definition:** Two sets A and B are said to be equal if both sets have the same elements and is denoted by  $A = B$ . If sets 'A' and 'B' are not equal we write  $A \neq B$

$$A = B \iff A \subseteq B \wedge B \subseteq A$$

$$\iff \forall x[x \in A \rightarrow x \in B]$$

## **Power set**

The set of all subsets of set A, denoted  $P(A)$  or  $2^A$ , is called the **power set** of A.

power set of {a, b, c}:

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

- If a set has n elements then its power set has  $2^n$  elements.

# Truth sets and quantifiers

Given a predicate of  $P$ , and a domain  $D$ , we define the truth set of  $P$  to be the set of elements  $x$  in  $D$  for which  $P(x)$  is true. The truth set of  $P(x)$  is denoted by  $\{x \in D \mid P(x)\}$

## Union and Intersection

**Union:** The union of two sets  $A$  and  $B$ , denoted by  $A \cup B$ , is the set of all the elements that are in either  $A$  or  $B$  or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

**Intersection:** The intersection of two sets  $A$  and  $B$ , denoted by  $A \cap B$ , is the set containing the elements those are in both  $A$  and  $B$ .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

**Disjoint Sets:** Two sets  $A, B$  are called disjoint if  $A \cap B = \emptyset$

## Complement

Let  $A$  and  $B$  be sets. The difference of  $A$  and  $B$ , denoted by  $A - B$ , is the set of elements that are in  $A$  but not in  $B$ .  **$A - B$  is also called the complement of  $B$  with respect to  $A$ .**

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

- $A^c$  or  $A'$  (A bar) is the complement of  $A$  with respect to the universal set  $U$ .

Let  $A$  and  $B$  be two sets, **The symmetric difference** of  $A$  and  $B$ , denoted by  $A \oplus B$  is the set  $(A - B) \cup (B - A)$

$$A \oplus B = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

## Set identities

**TABLE 1** Set Identities.

| <i>Identity</i>  | <i>Name</i>         |
|--|---------------------|
| $A \cap U = A$<br>$A \cup \emptyset = A$   | Identity laws       |
| $A \cup U = U$<br>$A \cap \emptyset = \emptyset$   | Domination laws     |
| $A \cup A = A$<br>$A \cap A = A$   | Idempotent laws     |
| $\overline{(\overline{A})} = A$  | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$   | Commutative laws    |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$                               | Associative laws    |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$             | Distributive laws   |
| $\overline{A \cap B} = \overline{A} \cup \overline{B}$<br>$\overline{A \cup B} = \overline{A} \cap \overline{B}$ | De Morgan's laws    |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$   | Absorption laws     |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \emptyset$   | Complement laws     |

examples:

show that;

1.  $(A \cup B)^c = A^c \cap B^c$
  2.  $A \oplus B = (A - B) \cup (B - A)$
  3.  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  4.  $A - (B \cup C) = (A - B) \cap (A - C)$
- 

## Generalized Unions and Intersections

Let  $A_1, A_2, \dots, A_n$  be n sets. Then,

The union of all the n sets:

$$\bigcup_{k=1}^n A_k = A_1 \cup A_2 \cup \dots \cup A_n$$

Similarly the intersection of all the n sets:

$$\bigcap_{k=1}^n A_k = A_1 \cap A_2 \cap \dots \cap A_n$$

## Computer representation of sets

### Finite Universal Set U:

Suppose we have a universal set U with n elements. Each element of U is given a fixed index (e.g.,  $u_1, u_2, \dots, u_n$ ).

### Bit Vector Representation:

Any subset  $A \subseteq U$  can be represented as a binary string (or array of bits) of length n:

- **1** at position  $i$  means  $u_i \in A$ .
- **0** at position  $i$  means  $u_i \notin A$ .

Example:

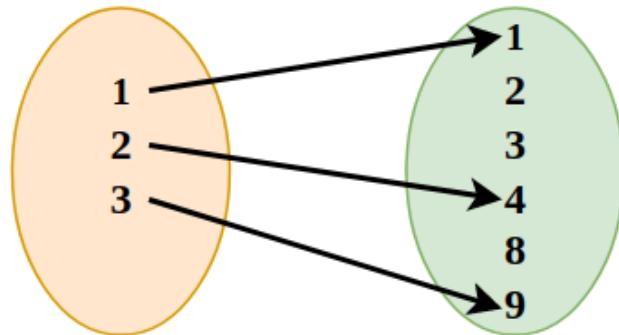
- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- Subset  $A = \{u_1, u_4\}$
- Representation:  $A = 10010$

So effectively:

$$U = 11111, A = 10010$$

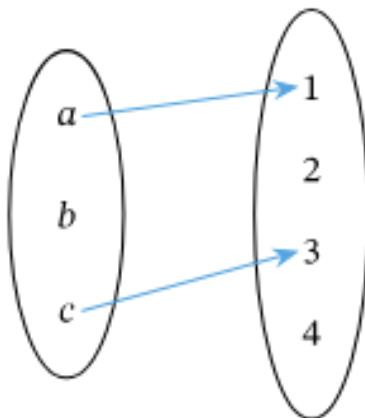
# Functions

- A function is a mapping from elements in one set(Domain) to elements in another set (Co-Domain) satisfying the following two conditions.
  - All the elements of the domain should be mapped to some elements in the co-domain.
  - Same element in the domain cannot be mapped with multiple elements in the co-domain.

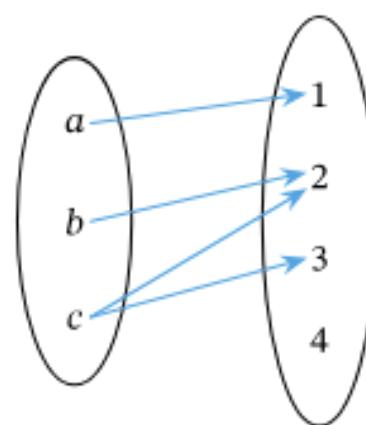


⊕ Domain and Range of a Relation - Definition, Types, and Examples - GeeksforGeeks

These are invalid mappings



(A)



(B)

⊕ Lesson Explainer: Identifying Functions | Nagwa

## One-to-one Functions

Some functions never assign the same value to two different domain elements. These functions are said to be one-to-one.

### Definition:

Let  $f:A \rightarrow B$  be a function.

We say  $f$  is **one-to-one (injective)** if:

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

### example:

Function:

$$f: R \rightarrow R, f(x) = 2x + 3$$

check is the function is one-to-one

let there be two values  $x_1$  and  $x_2$  which maps to the same value

then;

$$f(x_1) = f(x_2)$$

$$2x_1 + 3 = 2x_2 + 3$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

Hence, the function  $f(x)$  is one-to-one

## Onto Functions

An **onto function** (also called a **surjective function**) is a function in which **every element of the codomain has at least one element of the domain mapping to it**.

Formally, if  $f: A \rightarrow B$ , then  $f$  is **onto** if:

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y$$

In other words: **every element of the codomain B is “hit” by the function.**

### example:

$$f : R \rightarrow R, f(x) = 2x - 5$$

### Proof that it is onto:

- Take any  $y \in R$  (codomain).
- Solve  $f(x) = y$ :

$$2x - 5 = y$$

$$2x = y + 5$$

$$x = (y + 5)/2$$

- Since  $x \in R$ , **every y in the codomain has a corresponding x in the domain.**

✓ Therefore,  $f(x) = 2x - 5$  is **onto**.

## One-to-one correspondence or Bijection

A function  $f: A \rightarrow B$  is said to be a Bijection if and only if it is **one to one and onto**.

Example:

$$f : R \rightarrow R, f(x) = 3x + 2$$

|  |   |
|--|---|
| <p>Suppose <math>f(x_1) = f(x_2)</math><br/>then;<br/><math>3x_1 + 2 = 3x_2 + 2</math><br/><math>x_1 = x_2</math><br/><b>Therefore, the function is One-to-one</b></p> | <p>Take any <math>y \in R</math>.<br/>Solve <math>f(x) = y</math> for <math>x</math>:<br/><math>3x + 2 = y \Rightarrow x = (y - 2)/3</math></p> <ul style="list-style-type: none"><li>• Since <math>x \in R</math> for any <math>y \in R</math>, <b>every codomain element has a pre-image</b>.</li></ul> <p>✓ So <math>f</math> is <b>surjective</b> (onto).</p> |
|--|---|

# Inverse Functions

An interesting property of bijections is that they have an **inverse function**.

## Definition of Inverse Function:

Let  $f : A \rightarrow B$  be a **bijection**. Then there exists a function

$$f^{-1} : B \rightarrow A$$

called the **inverse function** of  $f$ , such that for every  $x \in A$  and  $y \in B$ :

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

In other words,  $f^{-1}$  “reverses” the action of  $f$ , mapping each element of the codomain back to its **unique pre-image** in the domain.

## Example 1:

$$f : R \rightarrow R, f(x) = 5x - 7$$

### Check one-to-one:

$$f(x_1) = f(x_2)$$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7$$

$$\Rightarrow x_1 = x_2$$

One-to-one

### Check onto:

Take any  $y \in R$ :

$$y = 5x - 7$$

$$\Rightarrow x = (y + 7)/5 \in R$$

Onto

## Conclusion:

- $f$  is bijective  $\rightarrow$  **invertible**
- Inverse:  $f^{-1}(y) = y$

### Example 2:

$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = \sin(x)$

#### Check one-to-one:

- $\sin(0) = 0$  and  $\sin(\pi) = 0 \rightarrow$  different inputs map to the same output

✗ Not one-to-one

#### Check onto:

- Codomain is  $\mathbb{R}$
- $\sin(x) \in [-1,1] \rightarrow$  cannot cover values like 2, -5, etc.

✗ Not onto

#### Conclusion:

- $g$  is not bijective  $\rightarrow$  not invertible
- (It becomes invertible if we restrict the domain to  $[-\pi/2, \pi/2]$  and codomain to  $[-1,1]$ )

## Compositions of Functions

Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions. The **composition** of  $g$  and  $f$ , denoted by

$g \circ f: A \rightarrow C$

is defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A$$

- In words: **first apply  $f$  to  $x$ , then apply  $g$  to the result.**
- The domain of  $g \circ f$  is the domain of  $f$ , and the codomain is the codomain of  $g$ .

example:

$$f(x) = 2x + 3 \quad \text{and } g(x) = x - 3$$

$$f \circ g (x) = f(g(x)) = 2(g(x)) + 3 \rightarrow 2(x - 3) + 3$$

$$g \circ f (x) = g(f(x)) = (2x + 3) - 3 \rightarrow 2x$$

## Identity Function

The **identity function** is the function that leaves every element of its domain unchanged. In other words, it returns exactly the same value as the input. Formally, if  $A$  is a set, the identity function on  $A$  is the function

#### Definition:

Let  $A$  be a non-empty set. The **identity function** on  $A$  is the function,

$$I : A \rightarrow A$$

defined by

$$I_A(x) = x \text{ for all } x \in A.$$

### Note:

For any function  $f : A \rightarrow A$ , the composition of  $f$  with its inverse  $f^{-1}$  (whenever the inverse exists) is always the **identity function** on  $A$ .

That is,

$$f \circ f^{-1} = f^{-1} \circ f = i_A$$

where  $i_A(x) = x$  for all  $x \in A$ .

## The Image and Inverse Image of a Subset of Domain and Co-Domain Respectively

For a given function  $f : A \rightarrow B$

The image of  $S$  ( $S \subseteq A$ ) under  $f$  (denoted by  $f(S)$ ) is the subset of  $B$  consisting of the images of the elements of  $S$ .

$$f(S) = \{f(s) \mid s \in S\}$$

The inverse image of  $T$  ( $T \subseteq B$ ) under  $f$  is the subset of  $A$  consisting of the pre-images of elements in  $T$ .

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

### Example:

Let  $A = \{1, 2, 3\}$ ,  $B = \{a, b, c\}$ , and define  $f(1) = a$ ,  $f(2) = b$ ,  $f(3) = c$ .

- If  $E = \{1, 3\}$ , then  $f(E) = \{a, c\}$ .
- If  $F = \{a, b\}$ , then  $f^{-1}(F) = \{1, 2\}$ .

### Example Problem:

Let  $f : A \rightarrow B$  be a function, and let  $S, T \subseteq B$ . Show that

1.  $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
2.  $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Also, give an example to show that in general,

$$f^{-1}(B - S) = A - f^{-1}(S).$$

Solution:

### 1. Proof that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

#### Step 1: Show $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$

- Let  $a \in f^{-1}(S \cup T)$ .
- By definition of inverse image:  $f(a) \in S \cup T$ .
- Then,  $f(a) \in S$  or  $f(a) \in T$ .
- By definition of inverse image:  $a \in f^{-1}(S)$  or  $a \in f^{-1}(T)$ .
- Then,  $a \in f^{-1}(S) \cup f^{-1}(T)$ .

Thus,  $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ .

#### Step 2: Show $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$

- Let  $a \in f^{-1}(S) \cup f^{-1}(T)$ .
- Then  $a \in f^{-1}(S)$  or  $a \in f^{-1}(T)$ .
- So  $f(a) \in S$  or  $f(a) \in T$ .
- By definition of union:  $f(a) \in S \cup T$ .
- By definition of inverse image:  $a \in f^{-1}(S \cup T)$ .

Thus,  $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$ .

### Conclusion:

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$$

### 2. Proof that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

#### Step 1: Show $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$

- Let  $a \in f^{-1}(S \cap T)$ .
- Then  $f(a) \in S \cap T$ .
- By definition of intersection:  $f(a) \in S$  and  $f(a) \in T$ .
- By definition of inverse image:  $a \in f^{-1}(S)$  and  $a \in f^{-1}(T)$ .
- By definition of intersection:  $a \in f^{-1}(S) \cap f^{-1}(T)$ .

#### Step 2: Show $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$

- Let  $a \in f^{-1}(S) \cap f^{-1}(T)$ .
- Then  $a \in f^{-1}(S)$  and  $a \in f^{-1}(T)$ .
- So,  $f(a) \in S$  and  $f(a) \in T$ .
- By definition of intersection:  $f(a) \in S \cap T$ .
- By definition of inverse image:  $a \in f^{-1}(S \cap T)$ .

### Conclusion:

$$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$$

### 3. Proof that $f^{-1}(B - S) = A - f^{-1}(S)$

#### Step 1: Show $f^{-1}(B - S) \subseteq A - f^{-1}(S)$

- Let  $a \in f^{-1}(B - S)$ .
- Then  $f(a) \in B - S$ .
- By definition of set difference:  $f(a) \notin S$ .
- By definition of inverse image:  $a \notin f^{-1}(S)$ .
- By definition of set difference:  $a \in A - f^{-1}(S)$ .

#### Step 2: Show $A - f^{-1}(S) \subseteq f^{-1}(B - S)$

- Let  $a \in A - f^{-1}(S)$ .
- Then,  $a \notin f^{-1}(S) \rightarrow f(a) \notin S$ .
- Since  $f(a) \in B$  and  $f(a) \notin S$ ,  $f(a) \in B - S$
- By definition of inverse image:  $a \in f^{-1}(B - S)$ .

#### Conclusion:

$$f^{-1}(B - S) = A - f^{-1}(S)$$

# Special Functions

## 1. Floor Function ( $\lfloor x \rfloor$ )

- **Definition:** For any real number  $x$ , the **floor function**  $\lfloor x \rfloor$  gives the **greatest integer less than or equal to  $x$** .
- **Examples:**
  - $\lfloor 3.7 \rfloor = 3$
  - $\lfloor -2.1 \rfloor = -3$
  - $\lfloor 5 \rfloor = 5$

So basically, **it rounds down to the nearest integer**, but “down” means towards  $-\infty$ , not just reducing the decimal.

## 2. Ceiling Function ( $\lceil x \rceil$ )

- **Definition:** For any real number  $x$ , the **ceiling function**  $\lceil x \rceil$  gives the **smallest integer greater than or equal to  $x$** .
- **Examples:**
  - $\lceil 3.7 \rceil = 4$
  - $\lceil -2.1 \rceil = -2$
  - $\lceil 5 \rceil = 5$

So this one **rounds up to the nearest integer**, but “up” means towards  $+\infty$ .

- Floor and ceiling functions are onto but not one-to-one. Hence, they are not invertible.