SCS1306 Linear Algebra

Tutorial

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July 30, 2025

Outline

Definition

The determinant is a scalar value that can be computed from the elements of a square matrix. It has some important properties:

- Determines whether the matrix is invertible (the determinant is nonzero if and only if the matrix is invertible).
- Determines the area/volume scaling factor in linear transformations.

Properties of Determinants

Let A and B be two square matrices.

- $\det(I) = 1$
- Exchanging two rows of A multiplies $det(A)$ by -1
- If row 1 of A has a combination $cv + dw$ then:

$$det \begin{bmatrix} cv + dw \\ row\ \ 2 \\ \dots \\ row\ \ n \end{bmatrix} = c \cdot det \begin{bmatrix} v \\ row\ \ 2 \\ \dots \\ row\ \ n \end{bmatrix} + d \cdot det \begin{bmatrix} w \\ row\ \ 2 \\ \dots \\ row\ \ n \end{bmatrix}$$

Properties of Determinants Contd.

- If two rows (or columns) are equal, $\det(A) = 0$
- Multiplying some row of A by a scalar k will result in the determinant $k \cdot det(A)$
- Substracting/Adding a multiple of one row from another leaves $det(A)$ unchanged:

$$
\begin{aligned}
det \begin{bmatrix} a & b \\ c - ta & d - tb \end{bmatrix} &= det \begin{bmatrix} a & b \\ c & d \end{bmatrix} - t \cdot det \begin{bmatrix} a & b \\ a & b \end{bmatrix} \\
&= det \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}
$$

Properties of Determinants Contd.

- If A has a row of zeros then $det(A) = 0$
- If A is triangular, then $det(A)$ is the product of the diagonal entries.
- If A is singular then $det(A) = 0$. If A is invertible then $det(A) \neq 0$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^\top) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$ if A is invertible

Definition

Given an $n \times n$ matrix A, a scalar λ and a nonzero vector $\vec{x}$ such that:

$$A\vec{x} = \lambda\vec{x}$$

Then λ is an **eigenvalue**, and $\vec{x}$ is the corresponding **eigenvector**.

Geometric Definition

- After linear transformation A, if there are vectors $\vec{x}$, such that $\vec{x}$ does not change direction, those vectors are called **eigenvectors** of matrix A.
- The eigenvectors fall along the directions that are not affected by the linear transformation.
- The amounts each eigenvector stretches or compresses as a result of transformation A are called the **eigenvalues** of matrix A.

How to Find Eigenvalues

Eigenvectors are defined as:

$$A\vec{x} = \lambda\vec{x}$$

The left-hand side is a matrix-vector multiplication, and the right-hand side is a scalar-vector multiplication. To convert both sides into the same format, use:

$$\lambda\vec{x} = (\lambda I)\vec{x}, \ I \text{ is the identity matrix}$$

$$\begin{aligned} A\vec{x} &= (\lambda I)\vec{x} \\ A\vec{x} - (\lambda I)\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

How to Find Eigenvalues Contd.

$$(A - \lambda I)\vec{x} = \vec{0}$$

To find a non-trivial solution ($\vec{x} \neq \vec{0}$) for the above equation, the matrix $(A - \lambda I)$ must be singular. The determinant of a singular matrix is zero. Hence:

$$\det(A - \lambda I) = 0$$

This is called the *characteristic equation*. The roots of this equation, $\lambda_1, \lambda_2, \ldots,$ are the eigenvalues.

How to Find Eigenvectors

After finding the roots of the characteristic equation (eigenvalues), for each eigenvalue λ, solve:

$$(A - \lambda I)\vec{x} = \vec{0}$$

This gives the eigenvector(s) corresponding to λ.

Example

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Find eigenvalues and eigenvectors.

$$\begin{aligned}
(A - \lambda I) &= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \\
\det(A - \lambda I) &= (2-\lambda) \times (2-\lambda) - 1 \times 1 \\
&= (4 - 4\lambda + \lambda^2) - 1 \\
&= \lambda^2 - 4\lambda + 3 \\
&= (\lambda - 3)(\lambda - 1)
\end{aligned}$$

Solution

Using the characteristic equation:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (\lambda - 3)(\lambda - 1) &= 0 \\ \lambda &= 1,\ \lambda = 3 \end{aligned}$$

solve $(A - \lambda I)\vec{x} = \vec{0}$ for each λ.
For $\lambda = 1$:

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

Assume $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix},\ x, y \in \mathbb{R}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution Contd.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0 \Rightarrow x = -y$$
$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$$

Therefore, for $\lambda = 1$, the eigenvectors are of the form $\vec{x} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$.

When $y = 1$, $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a sample eigenvector.

For $\lambda = 3$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow \vec{x} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; y \in \mathbb{R}$$

Diagonalization

If A has n **linearly independent eigenvectors**, then:

$$A = PDP^{-1}$$

where D is diagonal with eigenvalues, P has eigenvectors as columns.

Determinant using Eigenvalues

Since D is a triangular matrix, the determinant $\det(D)$ is the product of the diagonal values, according to the properties of the determinant. Since the diagonal of D consists of eigenvalues, $\det(D)$ becomes the product of eigenvalues:

$$\det(D) = \lambda_1 \lambda_2 \ldots \lambda_n$$

According to the previous slide:

$$\begin{aligned}
A &= PDP^{-1} \\
\det(A) &= \det(PDP^{-1}) \\
&= \det(P)\det(D)\det(P^{-1}) \quad (\because \ \det(AB) = \det(A)\det(B)) \\
&= \det(D) \quad (\because \ \det(P) = \frac{1}{\det(P^{-1})})
\end{aligned}$$

$$\therefore \det(A) = \det(D) = \lambda_1 \lambda_2 \ldots \lambda_n$$

Things to Remember

- Not all matrices are diagonalizable
- The matrix A and the upper triangular form of A, U, don't have the same eigenvalues. Since the determinant is equal to the product of eigenvalues, the product of eigenvalues of A and U is equal.
- The trace of a matrix is the sum of diagonal elements $tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$. The trace is also equal to the sum of eigenvalues $tr(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

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2.

Let $A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) Determine the eigenvalues of matrix A. **[6 marks]**

(b) Find the linearly independent eigenvectors corresponding to each eigenvalue obtained in part (a). **[12 marks]**

(c) If the matrix A is diagonalizable, determine the invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. **[4 marks]**

(d) If $P^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, determine A^2. **[4 marks]**

(e) Calculate the determinant $\det(A)$ and the trace $\mathrm{tr}(A)$ of matrix A using the eigenvalues. **[4 marks]**