Solving Linear Systems

(Elimination and Back Substitution)

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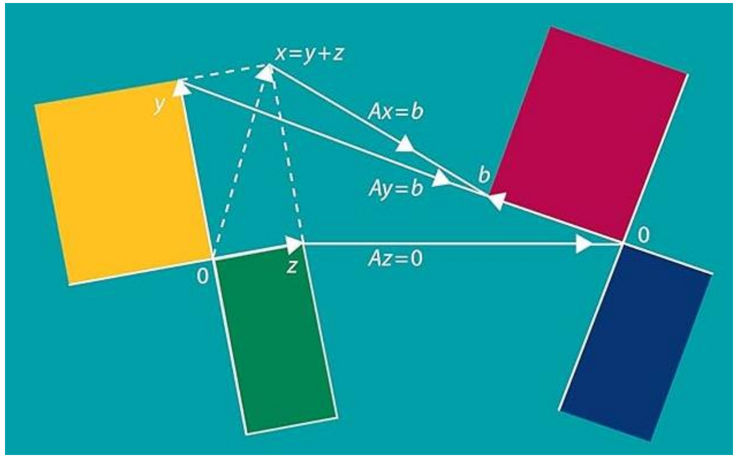


Matrix Multiplication

1 To multiply AB we need *row length for* A = *column length for* B.

2 The number in row i, column j of AB is **(row $\boldsymbol{i}$ of $\boldsymbol{A}$) $\cdot$ (column $\boldsymbol{j}$ of $\boldsymbol{B}$)**.

3 By columns: **$\boldsymbol{A}$ times column $\boldsymbol{j}$ of $\boldsymbol{B}$ produces column $\boldsymbol{j}$ of $\boldsymbol{AB}$.**

4 Usually AB is different from BA. But always $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$.

5 If A has r independent columns in C, then $\boldsymbol{A} = \boldsymbol{CR} = (\boldsymbol{m} \times \boldsymbol{r})(\boldsymbol{r} \times \boldsymbol{n})$.

Column j of AB equals A times column j of B

$$\textbf{If } B = \begin{bmatrix} b_1 \cdots b_p \end{bmatrix} \textbf{ then } AB = \begin{bmatrix} Ab_1 \cdots Ab_p \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Row way Dot Product

$$A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot \boldsymbol{b}_1 \\ \text{row } 2 \cdot \boldsymbol{b}_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} \mathbf{19} \\ \mathbf{43} \end{bmatrix}$$

Column way Dot Product

$$A\boldsymbol{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \mathbf{5} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mathbf{7} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} \mathbf{19} \\ \mathbf{43} \end{bmatrix}$$

AB and BA

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Multiply AB and BA and check whether those two multiplications give the same answer.

AB and BA

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

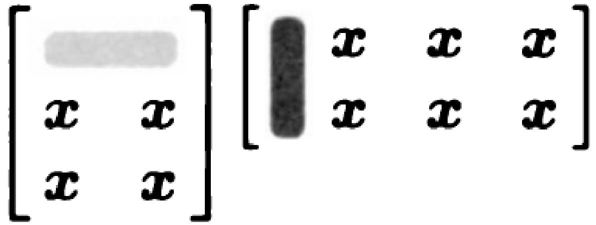
$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Matrix multiplication is not commutative.

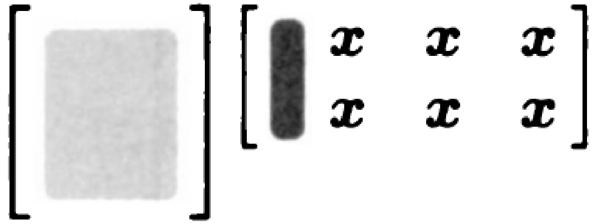
$$BA \neq AB$$

(3 by 2) (2 by 4) = (3 by 4)

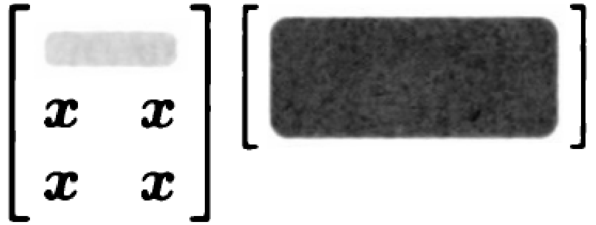
Four Ways to Multiply $AB = C$



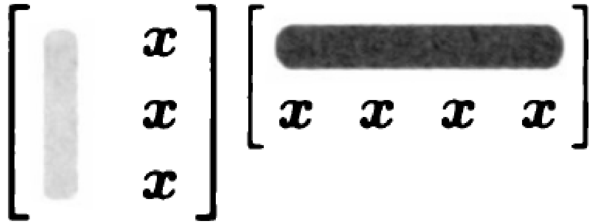
(**Row** $\boldsymbol{i}$ of A) $\cdot$ (**Column** $\boldsymbol{k}$ of B) = **Number** $\boldsymbol{C_{ik}}$
$i = 1$ to 3 $\quad$ $k = 1$ to 4 $\quad$ **12 numbers**



$\boldsymbol{A}$ times (**Column** $\boldsymbol{k}$ of B) = **Column** $\boldsymbol{k}$ **of** $\boldsymbol{C}$
$k = 1$ to 4 $\quad$ **4 columns**



(**Row** $\boldsymbol{i}$ of A) times $\boldsymbol{B}$ = **Row** $\boldsymbol{i}$ **of** $\boldsymbol{C}$
$i = 1$ to 3 $\quad$ **3 rows**



(**Column** $\boldsymbol{j}$ of A) (**Row** $\boldsymbol{j}$ of B) = **Rank 1 Matrix**
$j = 1$ to 2 $\quad$ **2 matrices**

Questions -1

Elimination and Back Substitution

Elimination and Back Substitution

- We consider n x n matrix.
- $Ax = b$ gives n equations and those equations have n unknowns.
- Often but not always there is one solution x for each b.

A has an inverse A^{-1} with $A^{-1}A = I$ and $AA^{-1} = I$

Multiplying $Ax = b$ by A^{-1} produces the symbolic solution $x = A^{-1}b$.

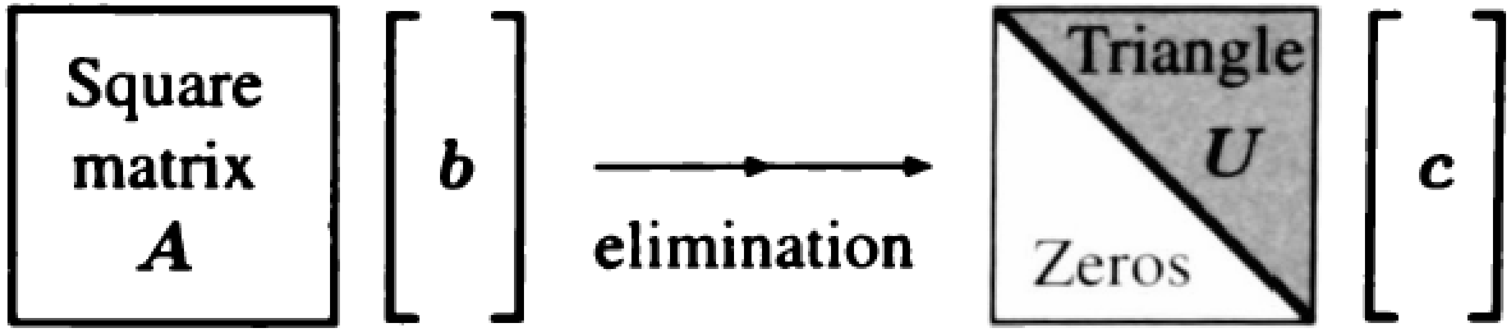
Do not compute A^{-1}

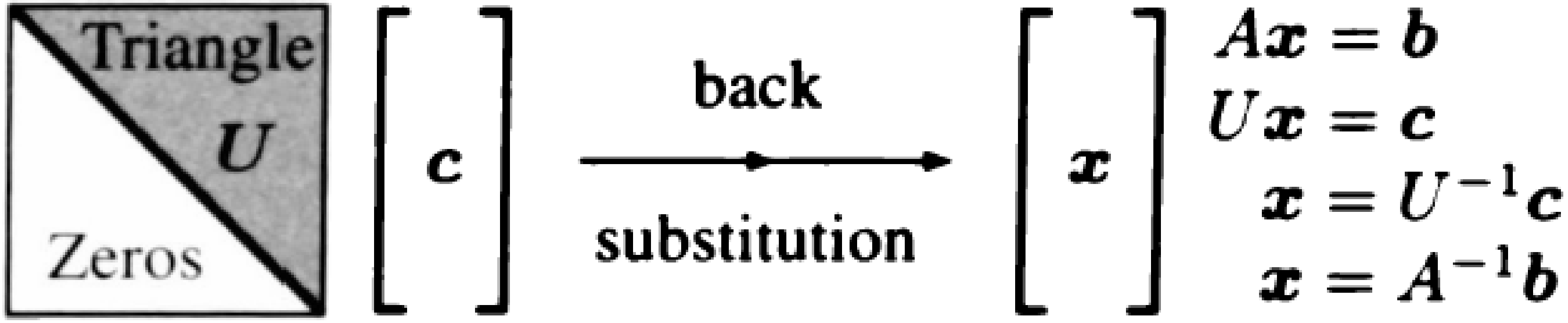
To describe all the steps we need matrices. This is the point of linear algebra! A simple elimination matrix E_{ij} produces a zero where row i meets column j ($i > j$). Overall, an elimination matrix E multiplies A to give $EA = U$. And we multiply U by an **inverse matrix** $L = E^{-1}$ to come back to A. Here are key matrices in this chapter:

Coefficient matrix $\boldsymbol{A}$	Upper triangular $\boldsymbol{U}$	Lower triangular $\boldsymbol{L}$
Elimination matrix $\boldsymbol{E_{ij}}$	Overall elimination $\boldsymbol{E}$	Inverse matrix $\boldsymbol{A^{-1}}$
Permutation matrix $\boldsymbol{P}$	Transpose matrix $\boldsymbol{A^{\mathrm{T}}}$	Symmetric matrix $\boldsymbol{S = S^{\mathrm{T}}}$

Our goal is to explain all the steps from $\boldsymbol{A}$ to $\boldsymbol{EA = U}$ to $\boldsymbol{A = E^{-1}U = LU}$ to $\boldsymbol{x}$. (If the steps fail, this signals that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution for most $\boldsymbol{b}$.) Every computer system has a code to find the triangular U and then the solution $\boldsymbol{x}$. Those codes are used so often that elimination adds up to the greatest cost in all of scientific computing.

We need to assume that A has independent columns





$Ax = b \rightarrow Ux = c$

Elimination and Back Substitution

1 Elimination subtracts ℓ_{ij} times row j from row i, leave a zero in row i.

2 $A\boldsymbol{x} = \boldsymbol{b}$ becomes $U\boldsymbol{x} = \boldsymbol{c}$ (or else $A\boldsymbol{x} = \boldsymbol{b}$ is proved to have no solution).

3 Then $U\boldsymbol{x} = \boldsymbol{c}$ is solved by **back substitution** because U is upper triangular.

Elimination and Back Substitution

- There may be
  - no vector that solves $Ax = b$,
  - there may be exactly one solution,
  - there may be infinitely many solution vectors
- Our job is to find all solutions

$$
\begin{tikzcd}
 & \text{Success} \\
\text{Elimination} \arrow[ur] \arrow[dr] & \\
 & \text{Failure}
\end{tikzcd}
$$

1 Exactly one solution to $A\boldsymbol{x} = \boldsymbol{b}$. In this case A has independent columns. The rank of A is 2. The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = 0$. **A has an *inverse matrix* A^{-1}.**

Example with one solution $(x, y) = (1, 1)$
Independent columns $(2, 4)$ and $(3, 2)$

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + 2y &= 6 \end{aligned} \qquad \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{2} \end{bmatrix}$$

2 No solution to $A\boldsymbol{x} = \boldsymbol{b}$. In this case $\boldsymbol{b}$ is not a combination of the columns of A. In other words $\boldsymbol{b}$ is not in the column space of A. The rank of A is 1.

Example with no solution
Dependent columns $(2, 4)$ and $(3, 6)$

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 15 \end{aligned} \qquad \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{6} \end{bmatrix}$$

Subtract 2 times the first equation from the second to get $0 = 3$. **No solution.**

Q1

$$X + 2Y + Z = 2$$

$$3X + 8Y + Z = 12$$

$$+ 4Y + Z = 2$$

Find X, Y and Z

Q1

$$2X + 3Y + 4Z = 19$$
$$4X + 11Y + 14Z = 12$$
$$2X + 8Y + 17Z = 50$$

Find X, Y and Z

Upper Triangular Matrix

$$\boldsymbol{Ux} = \boldsymbol{c} \quad \text{is} \quad \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{19} \\ \mathbf{17} \\ \mathbf{14} \end{bmatrix}$$

- 2, 5, 7 are pivots (on main diagonal) and should not equals to 0

Questions

(1)

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 2 \\ 4x & + & 9y & - & 3z & = & 8 \\ -2x & - & 3y & + & 7z & = & 10 \end{array}$$

(2)

$$\begin{array}{rcrcrcl} 2x & - & 3y & & & = & 3 \\ 4x & - & 5y & + & z & = & 7\;. \\ 2x & - & y & - & 3z & = & 5 \end{array}$$