SCS1306 Linear Algebra

Tutorial

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June 25, 2025

Overview

- Row eliminations.
- Back Substitutions
- Rank of a matrix.

Recap

Find the following:

1. Row Picture
2. Column Picture
3. Matrix Form

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

In the above example in $\mathbb{R}^3$, each equation is a plane.

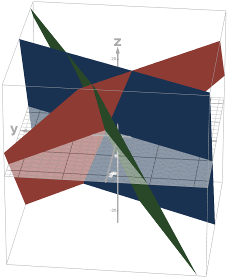

Figure: Row Picture. Created using Desmos 3D

Recap Contd.

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ -x + 2y - z = 4. \end{cases}$$

- Column Picture: $x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$
- Matrix Form: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

Row Elimination

- Goal: transform $A\mathbf{x} = \mathbf{b}$ into an equivalent upper-triangular system.
- Use elementary row operations:
  1. Swap two rows.
  2. Multiply a row by a non-zero scalar.
  3. Add a scalar multiple of one row to another.
- Result: an augmented matrix in row echelon form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right).$$

Elementary Row Operation I

Operation: Interchange two rows of a matrix.

Notation: $R_i \leftrightarrow R_j, i \neq j$

Example:

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Purpose: Row swapping is often used to position a non-zero pivot or to simplify row eliminations.

Elementary Row Operation II

Operation: Multiply a row by a non-zero scalar.

Notation: $R_i \leftarrow a \cdot R_i, a \in \mathbb{R} - \{0\}$

Example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{2} \cdot R_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 2 & -1 \end{bmatrix}$$

Purpose: This operation is often used to simplify row eliminations.

Elementary Row Operation III

Operation: Add a scalar multiple of one row to another row.

Notation: $R_i \leftarrow R_i + a \cdot R_j, i \neq j, a \in \mathbb{R} - \{0\}$

Example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + (-2) \cdot R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

Purpose: This operation is used to eliminate rows.

A system of linear equations can have:

- Exactly one unique solution (intersecting lines in a 2D scenario).

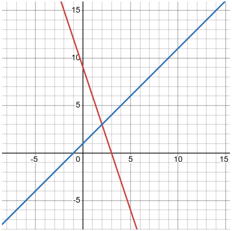

Figure: Intersecting lines. Created using Desmos

From Last Week...

A system of linear equations can have:

- No solution (parallel lines in a 2D scenario).

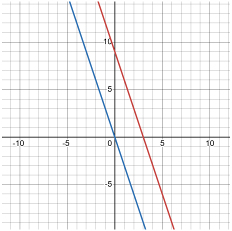

Figure: Parallel lines. Created using Desmos

From Last Week...

A system of linear equations can have:

- Infinitely many solutions (coincident lines in a 2D scenario).

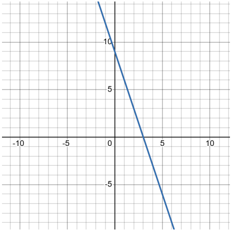

Figure: Coincident lines. Created using Desmos

Pratice Question I

Solve the following linear system with row elimination and back substitution:

$$\begin{cases} 2x + 3y - 4z = 7, \\ 3x + 4y - 2z = 9, \\ 5x + 7y - 6z = 20. \end{cases}$$

After row operations:

$$U = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 0 & -\frac{1}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The third row implies:

$$\begin{aligned} 0 \cdot x + 0 \cdot y + 0 \cdot z &= 4 \\ 0 &= 4 \end{aligned}$$

Which is impossible, so the system is inconsistent and has no solution.

Solution I

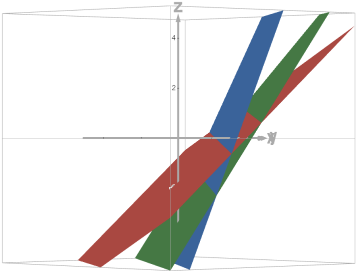

Figure: A system of linear equations with no solution. Created using Desmos 3D

Planes intersect pairwise and create 3 intersection lines. These 3 intersection lines are parallel to each other. Hence, there are no solutions that satisfy all 3 equations.

Past Paper Question I - SCS1306 2024

Consider the following linear system:

$$\begin{cases} 2x + y - z = 2, \\ 4x + 3y + 2z = -3, \\ 6x - 5y + 3z = -14. \end{cases}$$

1. Write the augmented matrix of the above linear system. **(5 Marks)**
2. Solve the linear system using row elimination with back substitution. **(10 Marks)**

Answer - SCS1306 2024

- Write the augmented matrix of the above linear system. **(5 Marks)**

  Matrix form: $\begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 2 \\ 6 & -5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -14 \end{bmatrix}$

  Augmented Matrix: The augmented matrix is obtained by appending the constant value vector to the coefficient matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 3 & 2 & -3 \\ 6 & -5 & 3 & -14 \end{array}\right]$$

Answer Contd. - SCS1306 2024

- Solve the linear system using row elimination with back substitution. **(10 Marks)**
  Row elimination

$$(1)\ R_1 \leftarrow \frac{1}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 4 & 3 & 2 & -3 \\ 6 & -5 & 3 & -14 \end{array}\right]$$

$$(2)\ R_2 \leftarrow R_2 + (-4)R_1, \quad R_3 \leftarrow R_3 + (-6)R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 4 & -7 \\ 0 & -8 & 6 & -20 \end{array}\right]$$

$$(3)\ R_3 \leftarrow R_3 + 8R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 38 & -76 \end{array}\right]$$

Answer Contd. - SCS1306 2024

- Solve the linear system using row elimination with back substitution. **(10 Marks)**
  Back-substitution

$$z = \frac{-76}{38} = -2$$
$$y + 4z = -7 \Rightarrow y = 1$$
$$x + \frac{1}{2}y - \frac{1}{2}z = 1 \Rightarrow x = -\frac{1}{2}$$

Final Solution: $\boxed{x = -\frac{1}{2},\ y = 1,\ z = -2}$

Past Paper Question II - SCS1211 2024

Let:

$$A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Find the upper triangular form of the matrix A. **(7 Marks)**
2. Hence, find the general solution to the following system of homogeneous equations: **(5 Marks)**

$$\begin{cases} 2x_2 + 3x_3 + 7x_4 & = 0 \\ x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 + 3x_2 + 4x_3 + 8x_4 & = 0 \\ x_4 & = 0 \end{cases}$$

Answer - SCS1211 2024

- Find the upper triangular form of the matrix A. **(7 Marks)**

$$(1)\ R_1 \leftrightarrow R_2 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2)\ R_3 \leftarrow R_3 + (-1)R_1 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2)\ R_3 \leftarrow R_3 + (-1)R_2 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the upper triangular form of matrix A.

Upper Triangular vs Row Echelon Form

A small clarification about the difference between upper-triangular form and row-echelon form:

A **square matrix** is called **upper triangular** if all the entries below the main diagonal are zero. Hence, the following matrix is upper triangular. The red color elements form the upper triangle.

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{2} & \mathbf{3} & \mathbf{7} \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

In the upper triangular form, the diagonal can contain zeros as long as everything below the diagonal is zero. Elements below the diagonal are colored in blue, and all of them are zero.

Upper Triangular vs Row Echelon Form Contd.

However, the above matrix is not in row-echelon form. For a matrix to be in **row-echelon form**, the matrix must fulfill the following conditions:

- All non-zero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

We can convert the previous matrix into row-echelon form by swapping R_3 and R_4.

$$R_3 \leftrightarrow R_4 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in both **upper triangular form** and **row-echelon form**.

Answer Contd. - SCS1211 2024

- Hence, find the general solution to the following system of homogeneous equations: **(5 Marks)**

$$\begin{cases} 2x_2 + 3x_3 + 7x_4 & = 0 \\ x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 + 3x_2 + 4x_3 + 8x_4 & = 0 \\ x_4 & = 0 \end{cases}$$

Step 1: Obtain equations from the upper triangular matrix.
Note that since the right-hand side of **all** the equations are **zero**, you do not have to row-eliminate the augmented matrix. Since, any operations on a set of zeros are zero, the answer would be the same (Try row-eliminating the augmented matrix of the above system to verify this statement).

Answer Contd. - SCS1211 2024

Using the **upper-triangular form** (or the **row-echelon form**) of A, obtain the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 0 \\
2x_2 + 3x_3 + 7x_4 &= 0 \\
x_4 &= 0
\end{aligned}
$$

Use $x_4 = 0$ in the other two equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \qquad (1) \\
2x_2 + 3x_3 &= 0 \qquad (2)
\end{aligned}
$$

Three variables, two equations $\Rightarrow$ The system has a free variable and infinitely many solutions.

Answer Contd. - SCS1211 2024

Step 2: Choose x_3 as the free variable and write x_1 and x_1 using x_3

$$\text{From (2): } x_2 = -\frac{3}{2}x_3$$

$$\text{Substitute into (1): } x_1 - \frac{3}{2}x_3 + x_3 = 0 \Rightarrow x_1 = \frac{1}{2}x_3$$

General solution of the system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot x_3 \\ -\frac{3}{2} \cdot x_3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot x_3 \\ -\frac{3}{2} \cdot x_3 \\ x_3 \\ 0 \cdot x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

Any real value of x_3 satisfies the system.
One free variable $\rightarrow$ infinitely many solutions

Rank of a Matrix

Definition:
The **rank** of a matrix is the maximum number of linearly independent rows or columns in the matrix.

Key Points:

- It is equal to the number of non-zero values (pivots) in the upper triangular form of the matrix.
- Rank is always $\leq$ the smaller of the number of rows or columns.

Notation: $\text{rank}(A)$

Example

Find the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \text{rank}(A) = 2$$

(The second row is a multiple of the first; only two rows are linearly independent.)

Past Paper Question III - SCS1211 2023 (Homework)

Let:

$$A = \begin{bmatrix} 1 & 2 & -2 & 2 & 3 \\ 1 & 2 & -1 & 3 & 5 \\ 1 & 2 & -3 & 1 & 1 \end{bmatrix}$$

1. Using elementary row operations, reduce A to the **row echelon** form. **(8 Marks)**
2. Find the rank of A. **(2 Marks)**
3. Solve the following system of linear equations: **(7 Marks)**

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 2x_4 & = 3 \\ x_1 + 2x_2 - x_3 + 3x_4 & = 5 \\ x_1 + 2x_2 - 3x_3 + x_4 & = 1 \end{cases}$$