SCS 1307
Probability & Statistics

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Conditional Probability

- If A and B are two events, where P(A) ≠ 0 and P(B) ≠ 0, then the probability of A given that B has already occurred is written as P(A | B).
- We read P(A | B) as **"probability of A, given B"**.
- This is the conditional probability of an event A, given event B, and, is defined as;

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad for \;\; P(B) > 0$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad for \;\; P(A) > 0$$

Conditional Probability

- This result is often written $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$

Clearly P(B | B) = 1, because we are certain that B is already occurred.

Example

1. Given that a heart is picked at random from a pack of 52 playing cards, find the probability that it is a pictured card.

2. When a die is thrown, an odd number occurs. What is the probability that the number is prime?

Answer 1: Let A : picture card and B : heart card

We require to calculate

P(picture card | heart)

That is;

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{3/52}{13/52} = \frac{3}{13}$$

Answer 2:

Let A : prime number and B: and odd number

we need to calculate P(prime | odd)

That is;

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{2/6}{3/6} = \frac{2}{3}$$

Example

A bag contains 10 counters, of which 7 are green and 3 are white. A counter is picked at random from the bag and its colour is noted. The counter is not replaced. A second counter is then picked out. Find the probability that

a) The first counter is green

b) The first counter is green and the second counter is white

c) The counters are of different colours

Answer

a) Let G_1: "the first counter is green"

$$P(G_1) = \frac{7}{10}$$

b) Let W_2:"the second counter picked is white"

Since there are 9 counters in the bag with 3 white; $P(W_2|G_1) = 3/9$

$$P(W_2 \cap G_1) = P(W_2|G_1)P(G_1) = \left(\frac{1}{3}\right)\times\left(\frac{7}{10}\right) \quad = \frac{7}{30}$$

c) We need $P(W_2 \cap G_1) + P(G_2 \cap W_1)$

Now $P(W_1) = \dfrac{3}{10}$ *and* $P(G_2|W_1) = \dfrac{7}{9}$

Therefore

$$P(G_2 \cap W_1) = P(G_2|W_1)P(W_1) = \left(\frac{7}{9}\right)\times\left(\frac{3}{10}\right) \quad = \frac{7}{30}$$

$$So \;\; P(W_2 \cap G_1) + P(G_2 \cap W_1) = \left(\frac{7}{30}\right)+\left(\frac{7}{30}\right) \quad = \frac{7}{15}$$

Exercises……

- A random experiment consists of tossing a fair coin three times and observing the sequence of Heads and Tails. Define the following events;

  A=event 'first toss is a Head' and B=event 'at least two heads'

a. Find P(A)

b. Find P(B)

c. Find P(A and B)

d. Find the conditional probability that the first toss is a Head, given that the outcome includes at least two Heads

Solution……

S ={hhh, hht, hth, thh, htt, tht, tth, ttt}

A= event ‘first toss is a Head’ ={hhh, hht, hth, htt}

B = event ‘at least two heads’ = {hhh, hht, hth, thh}

a. Find P(A) = 4/8

b. Find P(B) =4/8

c. Find P(A and B) = 3/8

d. Find the conditional probability that the first toss is a Head, given that the outcome includes at least two Heads

$P(A|B)= P(A\cap B)/P(B) = (3/8)/(4/8) =3/4$

A particular case of the Law of Total Probability

Let A and B be events. Observe that

1. $B \cap B' = \emptyset$ and 2. $A \cup B = \Omega$

From a Venn diagram we can easily see that

$A = (A \cap B) \cup (A \cap B')$ and

$(A \cap B) \cap (A \cap B') = \emptyset$.

Therefore $P(A) = P(A \cap B) + P(A \cap B')$ (by axiom 3)

Property

By using the definition of conditional probability

$$P(A) = P(A \mid B)\, P(B) + P(A \mid B')P(B')$$

This is a particular case of the Law of Total probability.

Total Probability

- Suppose $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive and exhaustive events and A is an arbitrary event . Then the total probability of event A for i=1,2,3,…,n is defined as;

- That is;

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_n)P(B_n)$$

- Proof:

Since $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive and exhaustive events it follows that $A \cap B_1$, $A \cap B_2, \ldots\ldots\ldots$, $A \cap B_n$ are mutually exclusive and $A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots\ldots\ldots \cup (A \cap B_n)$.

Therefore with repeated use of Axiom 3

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

Example

- A bag contains one fair coin, two doubled-headed coins and three doubled-tailed coins. A coin is selected at random and tossed. What is the probability that it lands on Heads?

Solution

Define H= event 'coin falls Heads', C_F=event 'fair coin selected' C_H= event 'doubled-headed coin selected',

C_T=event 'doubled-tailed coin selected'.

$$P(H)=P(H|C_F)P(C_F)+P(H|C_H)P(C_H)+P(H|C_T)P(C_T)$$

$$= 1/2 \times 1/6 + 1 \times 2/6 + 0 \times 3/6 = 5/12$$

Exercise

Ms. Ferrari parks her car in the open air overnight. The probability that it starts on the first attempt in the morning depends on whether or not it rained during the night. If it rained, her car starts first time with probability 0.6. If it did not rain, it starts first attempt with probability 0.9. The probability of rain during the night is 0.4. Find the probability that the car starts on the first attempt.

Solution

Let S =starts on the first attempt and R = it rained during the night

$P(S|R)=0.6$

$P(S|R')$ 0.9

$P(R) = 0.4.$

Probability that the car starts on the first attempt = $P(S)$

$$= P(S \cap R) + P(S \cap R')$$

$$= P(S|R)P(R) + P(S|R')P(R')$$

$$= 0.6 * 0.4 + 0.9 * 0.6$$

$$= 0.78$$

Bayes' Theorem

Suppose $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive and exhaustive events and B is an arbitrary event . Then for $i=1,2,3,\ldots,n$

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)} \qquad for \;\; P(B) > 0$$

Bayes' Theorem

Proof: The proof follows directly from the definition of conditional probability and the law of total probability.

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{k} P(B \mid A_j)P(A_j)}$$

- If A and B are two events, then;

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \qquad for \;\; P(B) > 0$$

Example

Suppose that a laboratory test to detect a certain disease has the following statistics. Let, A : test person has the disease, B: test result is positive.

It is known that $P(B|A)=0.99$ and $P(B|A^c)=0.005$ and 0.1% of the population actually has the disease. Find the probability that a person has the disease given that the test result is positive.

23

Solution

We have P(A)=0.001 then P(A^C)=0.999.

The desired probability is P(A|B) is;

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})}$$

$$= \frac{(0.99)(0.001)}{(0.99)(0.001)+(0.005)(0.999)} = 0.165$$

Example

A company producing electric relays has three manufacturing plants producing 50, 30 and 20 percent, respectively, of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are 0.02, 0.05, and 0.01 respectively.

a). If a relay is selected random from the output of the company, what is the probability that it is defective?

b). If a relay is selected random is found to be defective, what is the probability that it was manufactured by plant 2?