Lesson 7: Binary Relations and their Properties

Topics to be covered.

- Binary Relations
- Reflexive Relations
- Symmetric Relations
- Anti-symmetric Relations
- Transitive Relations
- Combining Relations
- n-ary Relations

Cartesian product

Definition: Let A and B be sets. The **Cartesian product** of A and B, denoted by $\boldsymbol{A \times B}$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}.$$

Example: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Definition: The Cartesian product of the sets $A_1, A_2, \dots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i for $i = 1,2, \dots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) | a_i \in A_i \text{ for } i = 1,2, \dots, n\}.$$

Binary Relations

Definition: Let A and B be sets. A **binary relation from A to B** is a **subset** of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Notation: Let R be a binary relation from a set A to a set B. Then we use the notation $a\ R\ b$ to denote that $(a, b) \in R$,

and $a \not R\ b$ to denote that $(a, b) \notin R$.

Moreover, when $(a, b) \in R$, a is said to be related to b by R.

Binary Relations on a set

Definition: A **binary relation** on the set A is a relation from A to A.

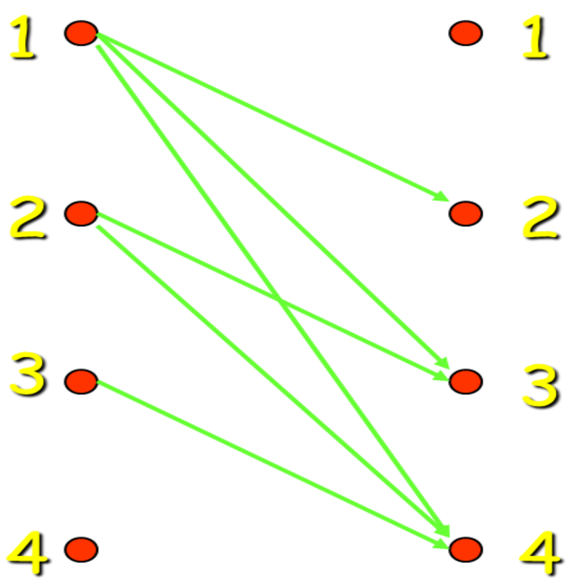
Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the binary relation $R = \{(a, b) \mid a \text{ divides } b\}$?

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Example -1

Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the binary relation on A given by $R = \{(a, b) \mid a < b\}$?

Solution: $R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$.



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations on a Set

How many different relations can we define on a set A with n elements?

- A relation on a set A is a subset of $A \times A$.
  How many elements are in $A \times A$?

- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

- Therefore, we can define 2^{n^2} different relations on A.

Properties of Relations

Definition: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

R is reflexive $\Leftrightarrow\ \forall a\ [(a, a) \in R]$.

Definition: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

R is symmetric $\Leftrightarrow\ \forall a\ \ \forall b\ [(a, b) \in R\ \rightarrow (b, a) \in R]$.

Definition: A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for all $a, b \in A$, is called anti-symmetric.

R is anti-symmetric $\Leftrightarrow\ \forall a\ \ \forall b\ \big[\big((a, b) \in R\ \wedge (b, a) \in R\big) \rightarrow a = b\big]$.

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

R is transitive $\Leftrightarrow\ \forall a\ \forall b\ \forall c\ \big[\big((a, b) \in R \wedge\ (b, c) \in R\big) \rightarrow (a, c) \in R\big]$.

Example -2

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{ (1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$,
$R_2 = \{ (1,1), (1,2), (2,1)\}$,
$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$,
$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$,
$R_5 = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$,
$R_6 = \{ (3,4)\}$.

1. Which of these relations are reflexive? $\boxed{R_3, \text{and } R_5}$
2. Which of the relations are symmetric? $\boxed{R_2, \text{and } R_3}$
3. Which of the relations are anti-symmetric? $\boxed{R_4, R_5 \text{ and } R_6}$
4. Which of the relations are transitive? $\boxed{R_4, R_5 \text{ and } R_6}$

Solution to Example -2 (R_1)

Consider the relation R_1 on $\{1, 2, 3, 4\}$ given by
$R_1 = \{ (1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$.

- R_1 is not reflexive, because $(3,3) \notin R_1$.
- R_1 is not symmetric, because $(3,4) \in R_1$ but $(4,3) \notin R_1$.
- R_1 is not anti-symmetric, because $(1,2) \in R_1$ and $(2,1) \in R_1$, but $1 \neq 2$.
- R_1 is not transitive, because $(3,4) \in R_1$ and $(4,1) \in R_1$, but $(3,1) \notin R_1$.

Solution to Example -2 (R_2)

Consider the relation R_2 on $\{1, 2, 3, 4\}$ given by
$R_2 = \{ (1,1), (1,2), (2,1)\}$.

- R_2 is not reflexive, because $(2,2) \notin R_2$.
- R_2 is symmetric, because whenever $(a, b) \in R_2$, we have $(b, a) \in R_2$.
- R_2 is not anti-symmetric, because $(1,2) \in R_2$ and $(2,1) \in R_2$, but $1 \neq 2$.
- R_2 is not transitive, because $(2,1) \in R_2$ and $(1,2) \in R_2$, but $(2,2) \notin R_2$.

Solution to Example -2 (R_5)

Consider the relation R_5 on $\{1, 2, 3, 4\}$ given by
$R_5 = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$.

- R_5 is reflexive, because $(1,1), (2,2), (3,3)$, and $(4,4) \in R_5$.

- R_5 is not symmetric, because $(1,4) \in R_5$ but $(4,1) \notin R_5$.

- R_5 is anti-symmetric, because there is no pair of elements a and b with $a \neq b$ such that both (a, b) and $(b, a) \in R_5$.

- R_5 is transitive, because we can verify that whenever $(a, b) \in R_5$ and $(b, c) \in R_5$, we have $(a, c) \in R_5$.

$(1,1)$ and $(1,2)$, $(1,1)$ and $(1,3)$, $(1,1)$ and $(1,4)$, $(1,2)$ and $(2,2)$, $(1,2)$ and $(2,3)$, $(1,2)$ and $(2,4)$, $(1,3)$ and $(3,3)$, $(1,3)$ and $(3,4)$, $(1,4)$ and $(4,4)$, $(2,2)$ and $(2,3)$, $(2,2)$ and $(2,4)$, $(2,3)$ and $(3,3)$, $(2,3)$ and $(3,4)$, $(2,4)$ and $(4,4)$, $(3,3)$ and $(3,4)$, $(3,4)$ and $(4,4)$ are the only such pairs.

Example -3

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_2 = \{(a, b) \mid a > b\}$,

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$,

$R_5 = \{(a, b) \mid a = b + 1\}$,

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

1. Which of these relations are reflexive?
2. Which of the relations are symmetric?
3. Which of the relations are anti-symmetric?
4. Which of the relations are transitive?

Solution to Example -3 (R_1)

Consider the relation R_1 on the set of integers given by:

$R_1 \ = \{(a, b) \mid a \leq b\}$.

- R_1 is reflexive, because a $\leq a$ for all $a \in \mathbb{Z}$.
- R_1 is not symmetric, because $1 \leq 2$ but $2 \nleq 1$.
- R_1 is anti-symmetric, because inequalities $a \leq b$ and $b \leq a$ imply that $a = b$.
- R_1 is transitive, because inequalities $a \leq b$ and $b \leq c$ imply that $a \leq c$.

Solution to Example -3 (R_5)

Consider the relation R_5 on the set of integers given by:

$R_5 \ = \{(a, b) \ |a = b + 1\}$.

- R_5 is not reflexive, because $1 \neq 1 + 1$ and hence $(1,1) \notin R_5$.
- R_5 is not symmetric, because $(2,1) \in R_5$ but $(1, 2) \notin R_5$.
- R_5 is anti-symmetric, because it is impossible to have that $a = b + 1$ and $b = a + 1$ for any $a, b \in \mathbb{Z}$.
- R_5 is not transitive, because $(3,2) \in R_5$ and $(2,1) \in R_5$, but $(3,1) \notin R_5$.

Solution to Example -3 (R_6)

Consider the relation R_6 on the set of integers:

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

- R_6 is not reflexive, because $(3,3) \notin R_6$.
- R_6 is symmetric, because for all $a, b \in \mathbb{Z}$, $a + b \leq 3$ implies $b + a \leq 3$.
- R_6 is not anti-symmetric, because $(1,2) \in R_6$ and $(2,1) \in R_6$ but $1 \neq 2$.
- R_6 is not transitive, because $(2,1) \in R_6$ and $(1,2) \in R_6$ but $(2,2) \notin R_6$.

Combining Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.
- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Example - 4: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$,

$R_1 \cap R_2 = \{(1,1)\}$,

$R_1 - R_2 = \{(2,2),(3,3)\}$,

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$.

Combining Relations

Example - 4: Let *A* and *B* be the set of all students and the set of all courses at UCSC, respectively. Suppose that R_1 consists of all ordered pairs (a, b), where *a* is a student who has taken course *b*, and R_2 consists of all ordered pairs (a, b), where *a* is a student who requires course *b* to graduate. What are the relations

- $R_1 \cup R_2 =$ $\{(a, b) | student\ a\ \ has\ taken\ course\ b\ \boldsymbol{or}\ needs\ course\ b\ to\ graduate\}$,
- $R_1 \cap R_2 =$ $\{(a, b) | student\ a\ \ has\ taken\ course\ b\ \boldsymbol{and}\ needs\ course\ b\ to\ graduate\}$
- $R_1\ \oplus R_2$
- $R_1\ - R_2 =$ $\{(a, b) | student\ a\ has\ taken\ course\ b\ but\ does\ not\ need\ it\ to\ graduate\}$
- $R_2\ - R_1 =$ $\{(a, b) |\ \ b\ is\ a\ course\ that\ a\ needs\ to\ graduate\ but\ has\ not\ taken.\}$

Example -5

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is,
$R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$.

What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

- $$\begin{aligned} R_1 \cup R_2 &= \{(x, y) \mid (x, y) \in R_1 \text{ or } (x, y) \in R_2\} \\ &= \{(x, y) \mid x < y \ or \ x > y\} \\ &= \{(x, y) \mid x \neq y\}. \end{aligned}$$

- $$\begin{aligned} R_1 \cap R_2 &= \{(x, y) \mid (x, y) \in R_1 \text{ and } (x, y) \in R_2\} \\ &= \{(x, y) \mid x < y \ and \ x > y\} = \varphi. \end{aligned}$$

- $$\begin{aligned} R_1 - R_2 &= \{(x, y) \mid (x, y) \in R_1 \text{ and } (x, y) \notin R_2\} \\ &= \{(x, y) \mid x < y \ and \ x \leq y\} = \{(x, y) \mid x < y\} = R_1. \end{aligned}$$

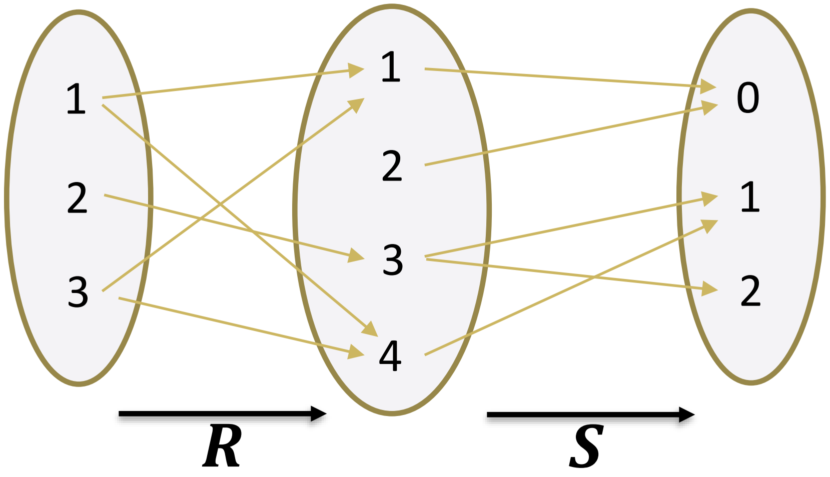
Combining Relations

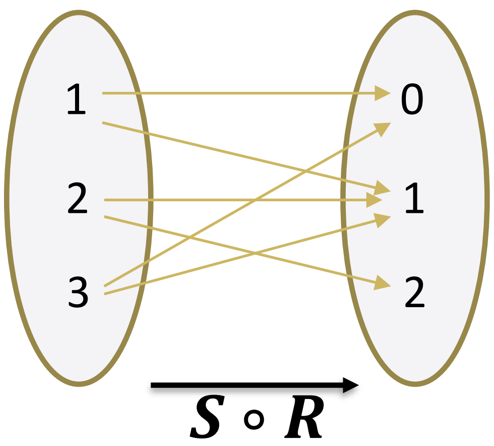
Definition: Let ***R*** be a relation from a set *A* to a set *B* and ***S*** a relation from *B* to a set *C*. The **composite of *R* and *S*** is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of *R* and *S* by $S \circ R$

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B\ [(a, b) \in R \,\&(b, c) \in S]\}.$$

Example -6

What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{ (1,1), (1,4), (2,3), (3,1), (3,4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{ (1,0), (2,0), (3,1), (3,2), (4,1)\}$?





$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}.$$

Combining Relations

Example -7: Let D and S be relations on $A = \{1, 2, 3, 4\}$.

$D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"

$S = \{(a, b) \mid a < b\}$ "a is smaller than b"

That is; $D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

Then $S \circ \mathrm{D} = \{(2,4), (3,3), (3,4), (4,2), (4,3), (4,4)\}$.

$$S \circ D = \left\{(a, c) \mid a, c \in A \; and \; \exists \, b \in A \text{ such that } \big((a, b) \in D \,\&(b, c) \in S\big)\right\}$$

$$= \{(a, c) \mid a, c \in A \; and \; \exists \, b \in A \text{ such that } (b = 5 - a \,\&b < c \;)\}$$

$$= \{(a, c) \mid a, c \in A \text{such that } c > 5 - a\}$$

$$= \{(a, c) \mid a, c \in A \text{ such that } a + c > 5\}$$

$$= \{(2,4), (3,3), (3,4), (4,2), (4,3), (4,4)\}.$$

Powers of a Relation

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

Definition: Let R be a relation on the set A. The powers $R^n, n = 1,2,3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Example -8: Let $R = \{ (1,1), (2,1), (3,2), (4,3)\}$.
Find the powers $R^n, n = 2,3,4, \ldots$.

$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$, $R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = \{(1,1), (2,1), (3,1), (4,1)\} = R^3$.

$R^n = R^3, \forall n \geq 4$.

Application

Theorem: The relation *R* on a set *A* is transitive if and only if $R^n \subseteq R, \ \forall n \in \mathbb{Z}^+$.

Proof: ($\Leftarrow$) Suppose that $R^n \subseteq R, \ \forall n \in \mathbb{Z}^+$. Then in particular $R^2 \subseteq R$.
Let $(a, b) \in R$ and $(b, c) \in R$. Then by composition, we have $(a, c) \in R^2$.
So, $(a, c) \in R$ since $R^2 \subseteq R$. Hence R is transitive.

($\Rightarrow$) Suppose that R is transitive. We shall show that $R^n \subseteq R, \ \forall n \in \mathbb{Z}^+$, using Mathematical induction. The result is trivially true when $n = 1$.
Assume that $R^k \subseteq R$, for some $k \in \mathbb{Z}^+$.
Assume that $(a, b) \in R^{k+1}$.
$\Longrightarrow \ \exists x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^k$ ($\because \ R^{k+1} = R^k \circ R$).
$\Longrightarrow \ \exists x \in A$ such that $(a, x) \in R$ and $(x, b) \in R$ ($\because \ R^k \subseteq R$ by our assumption).
$\Longrightarrow \ (a, b) \in R$ ($\because$ R is transitive)
Therefore, $R^{k+1} \subseteq R$, and the result is true for $n = k + 1$.
Hence by the mathematical induction principle, $R^n \subseteq R, \ \forall n \in \mathbb{Z}^+$.

Functions as Relations

- You might remember that a **function** f from a set *A* to a set *B* assigns a unique element of *B* to each element of *A*.
- The **graph** of f is the set of ordered pairs (a, b) such that $b = f(a)$.
- Since the graph of f is a subset of $A \times B$, it is a **relation** from *A* to *B*.
- Moreover, for each element ***a*** of *A*, there is exactly one ordered pair in the graph that has ***a*** as its first element.
- Conversely, if *R* is a relation from *A* to *B* such that every element in *A* is the first element of exactly one ordered pair of R, then a function can be defined with *R* as its graph.

n - ary Relations

We begin with the basic definition on which the theory of relational databases rests.

Definition: Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and *n* is called its degree.

Example -9: Let $R = \{(a, b, c) | a = 2b \wedge b = 2c \; with \; a, b, c \in \mathbb{Z}\}$.

The degree of R is 3, so its elements are triples.

Its domains are all equal to the set of integers.

$(4, 2, 1) \in R \;\; and \; (2, 4, 8) \notin R$.