Properties of Determinants

(Linear Algebra)

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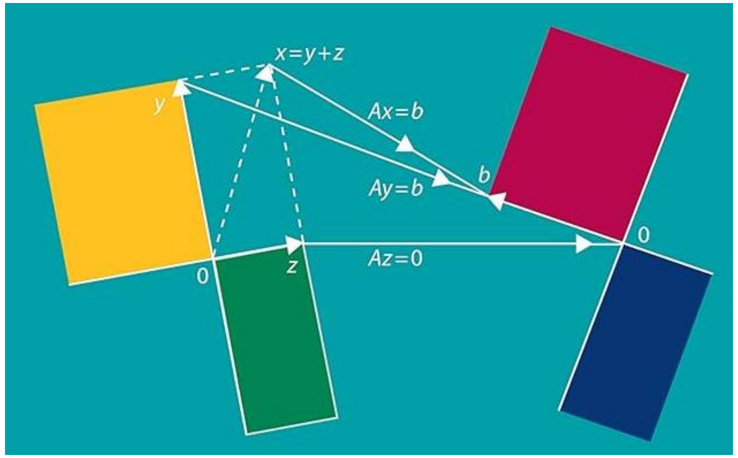


Properties of Determinants

1. The n by n identity matrix has $\det I = 1$.
2. Exchanging two rows of A reverses $\det A$ to $-\det A$.
3. If row 1 of A is a combination $cv + dw$, then add 2 determinants:

$$\det \begin{bmatrix} cv + dw \\ \text{row } 2 \\ \cdot\cdot \\ \text{row } n \end{bmatrix} = c \det \begin{bmatrix} v \\ \text{row } 2 \\ \cdot\cdot \\ \text{row } n \end{bmatrix} + d \det \begin{bmatrix} w \\ \text{row } 2 \\ \cdot\cdot \\ \text{row } n \end{bmatrix}$$

Determinant of a Matrix is a linear function of each row separately

If a few elements of a row or column are expressed as a sum of terms, then the determinant can be expressed as a sum of two or more determinants.

$$\begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

The elements of the first row represent the sum of terms, which can be split into two different determinants. Further, the new determinants also have the same second and third row, as the earlier determinant.

Properties of Determinants

- Derived Properties

Property 4 If two rows of A are equal then $det(A) = 0$.

Property 5 Subtracting a multiple of one row from another leaves $det(A)$ unchanged.

We can see this is true in the 2×2 case by noting

$$det \begin{pmatrix} a & b \\ c - ta & d - tb \end{pmatrix} = det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - t \left(det \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right)$$

$$= det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Source: https://www.math.utah.edu/~zwick/Classes/Fall2012_2270/Lectures/Lecture26.pdf

Properties of Determinants

- Derived Properties

Property 6 A matrix with a row of zeros has $det(A) = 0$.

If we multiply the row of zeros by t the matrix is unchanged, and so we must have $det(A) = t \times det(A)$ for all t, which is only true if $det(A) = 0$.

Property 7 If A is triangular then $det(A) = a_{11}a_{22}\cdots a_{nn}$, the product of the diagonal entries.

We can just use elimination to make all the non-diagonal entries 0, and then apply linearity n times.

Properties of Determinants

- Derived Properties

Property 8 If A is singular then $det(A) = 0$. If A is invertible then $det(A) \neq 0$.

We use elimination to go from A to U. Each step of elimination either leaves the determinant the same, or switches its sign. If A is singular than U has a zero on its diagonal, and so by property 7 the determinant is 0. On the other hand if A is invertible then U has pivots along its diagonal, and the product of non-zero terms is always non-zero.

Properties of Determinants

- Derived Properties

Property 8 If A is singular then $det(A) = 0$. If A is invertible then $det(A) \neq 0$.

The determinant of a square matrix tells us a lot. First of all, an invertible matrix has $\det A \neq 0$. A singular matrix has $\det A = 0$. When we come to eigenvalues λ and eigenvectors $\boldsymbol{x}$ with $A\boldsymbol{x} = \lambda\boldsymbol{x}$, we will write that eigenvalue equation as $(\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{x} = \mathbf{0}$. This means that $A - \lambda I$ is singular and $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}$. We have an equation for λ.

Overall, the formulas are useful for small matrices and also for special matrices. And the properties of determinants can make those formulas simpler. If the matrix is triangular or diagonal, we just multiply the diagonal entries to find the determinant:

Triangular matrix
Diagonal matrix

$$\det\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ & \boldsymbol{q} & \boldsymbol{r} \\ & & \boldsymbol{z} \end{bmatrix} = \det\begin{bmatrix} \boldsymbol{a} & & \\ & \boldsymbol{q} & \\ & & \boldsymbol{z} \end{bmatrix} = \boldsymbol{aqz} \qquad (1)$$

Properties of Determinants

- Derived Properties

Property 9 The determinant of AB is $det(A)det(B)$.

Property 10 The transpose A^T has the same determinant as A.

If we transpose A, the determinant formula gives the same result :

$$\textbf{Transpose the matrix} \qquad \det(A^{\mathrm{T}}) = \det(A) \tag{2}$$

If we multiply AB, we just multiply determinants (this is a wonderful fact) :

$$\textbf{Multiply two matrices} \qquad \det(AB) = (\det A)(\det B) \tag{3}$$

Cramer's Rule to solve Ax = b

- Cramer's rule gives us another way to solve Ax = b.
- Cramer's rule works better in theory than it works in practice.
- This is very useful to prove things about matrices and linear transformations.

Key idea

$$AM_1 = B_1 \qquad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1.$$

- Replace the first column of the identity matrix with the vector x

Cramer's Rule to solve Ax = b

This triangular M_1 has determinant x_1

Product rule $\boxed{(\det A)(x_1) = \det B_1 \quad \text{or} \quad x_1 = \det B_1/\det A.}$

To find x_2 and B_2, put the vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ into the *second columns* of I and A:

Same idea

$$AM_2 = B_2 \qquad \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{b} & \boldsymbol{a}_3 \end{bmatrix} = B_2.$$

Cramer's Rule to solve Ax = b

CRAMER'S RULE If det A is not zero, $Ax = b$ is solved by determinants:

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \cdots \qquad x_n = \frac{\det B_n}{\det A}$$

The matrix B_j has the jth column of A replaced by the vector b.

Question

Use Cramer's rule to solve for the vector **x**:

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

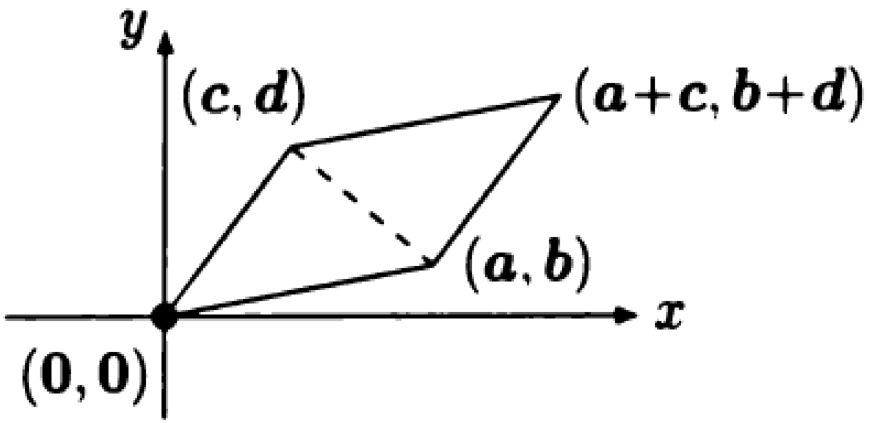
Areas and Volumes by Determinants

Determinants lead to good formulas for areas and volumes. The regions have straight sides but they are not square. A typical region is a parallelogram—or half a parallelogram which is a triangle. The problem is: **Find the area.** For a triangle, the area is $\frac{1}{2}bh$: half the base times the height. A parallelogram contains two triangles with equal area, so we omit the $\frac{1}{2}$. Then *parallelogram area = base times height*.

Those formulas are simple to remember. But they don't fit our problem, because we **are not given the base and height.** We only know **the positions of the corners.** For the triangle, suppose the corner points are $(0, 0)$ and (a, b) and (c, d). For the parallelogram (twice as large) the fourth corner will be $(a+c, b+d)$. Knowing a, b, c, d, *what is the area*?

Areas and Volumes by Determinants

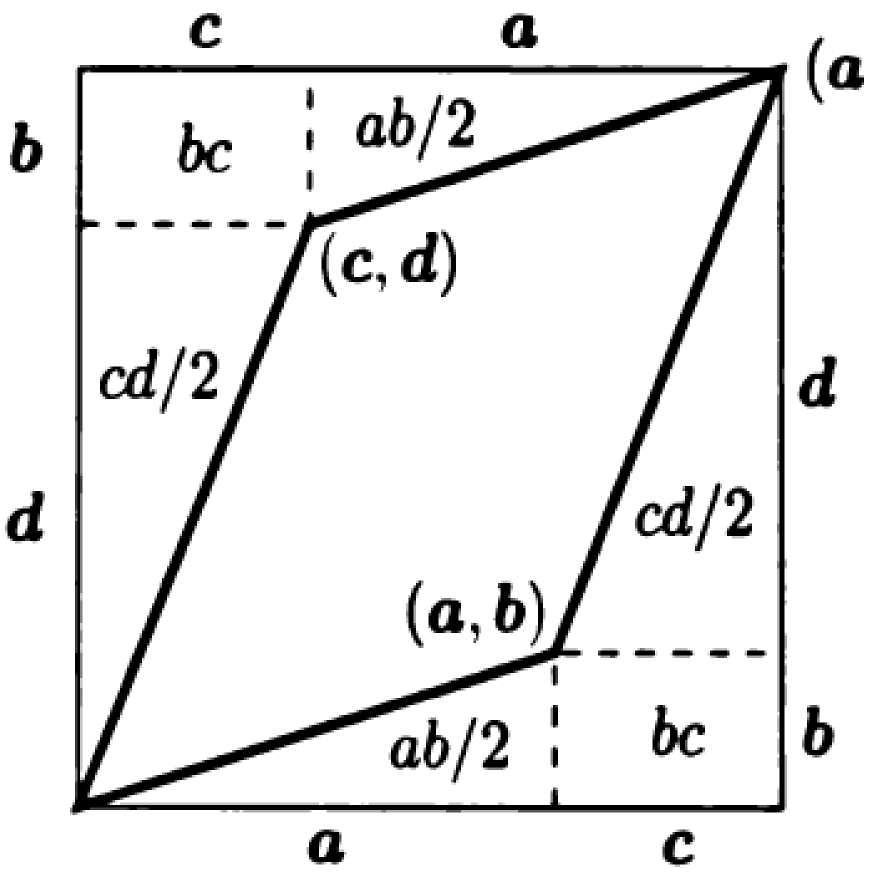
Triangle and parallelogram



The base could be the lowest side and its length is $\sqrt{a^2+b^2}$. To find the height, we could create a line down from (c, d) that is perpendicular to the baseline. The length h of that perpendicular line would involve more square roots. But the area itself does not involve square roots! Its beautiful formula $ad - bc$ is simple for $a = d = 1, b = c = 0$ (a square).

$$\boxed{\textbf{Area of parallelogram} = |\,\textbf{Determinant of matrix}\,| = \pm \begin{vmatrix} a & c \\ b & d \end{vmatrix} = |ad - bc|.} \quad (1)$$

Our first effort stays in a plane. For this case we use geometry. Figure 5.1 shows how adding pieces to a parallelogram can produce a rectangle. When we subtract the areas of those six pieces, we arrive at the correct parallelogram area $ad - bc$ (no square roots). The picture is not very elegant, but in two dimensions it succeeds.



$(a + c, b + d)$ = (base, height) of rectangle

Area of parallelogram

Rectangle minus 6 pieces

$$(a+c)\,(b+d) - 2bc - ab - cd$$
$$= ad - bc$$

Volume of a Box

$$\boxed{\textbf{Volume of box} = \left| \textbf{Determinant of matrix} \right| = \pm \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}}$$

Areas and Volumes by Determinants

1 A parallelogram in 2D starts from $(0,0)$ with sides $\boldsymbol{e}_1 = (a, b)$ and $\boldsymbol{e}_2 = (c, d)$.

2 Area of the parallelogram $= \left| \text{Determinant of the matrix } E = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{bmatrix} \right| = |ad - bc|$.

3 A tilted box in 3D starts with three edges $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ out from $(0,0,0)$.

4 Volume of a tilted box $= |\text{Determinant of the } 3 \times 3 \text{ edge matrix } E\,|$.

Areas and Volumes by Determinants

A box in n dimensions has n edges $e_1, e_2, \ldots, e_n$ going out from the origin. The parallelogram in two dimensions had two vectors $e_1 = (a, b)$ and $e_2 = (c, d)$. Those vectors e give two corners or n corners of the "box". In the 2-dimensional picture, the fourth corner was $e_1 + e_2$. In the n-dimensional picture, the other corners of the box would be sums of the e's. *The box is totally determined by the n edges in the matrix E*:

$$\textbf{Edge matrix} \quad E_2 = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \text{ and } E_n = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} \quad \begin{matrix}\textbf{Columns =} \\ \textbf{Box edges}\end{matrix}$$

- If the rows of A are a_1, a_2 , ... a_n , respectively, then $|\det(A)|$ is the n - dimensional volume (the generalization of area in R^2 and volume in R^3) of the parallelepiped having the vectors a_1 , a_2, ... , a_n as adjacent sides.