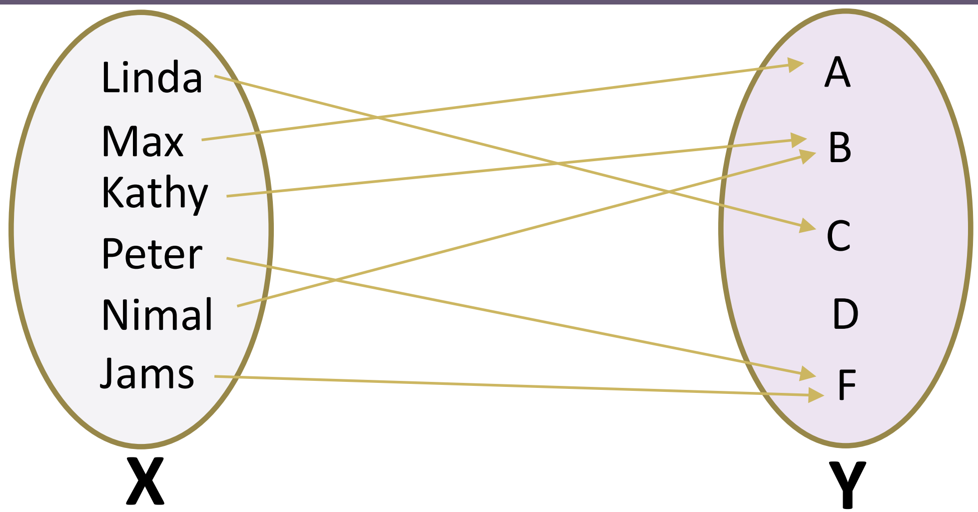
Lesson 6: Functions

- **Topics to be covered.**
  - **Defintion of a function**
  - **One-to-One Functions**
  - **Onto Functions**
  - **Inverse Functions**
  - **Composition of Functions**
  - **Some Important Functions**
  - **Partial Functions**

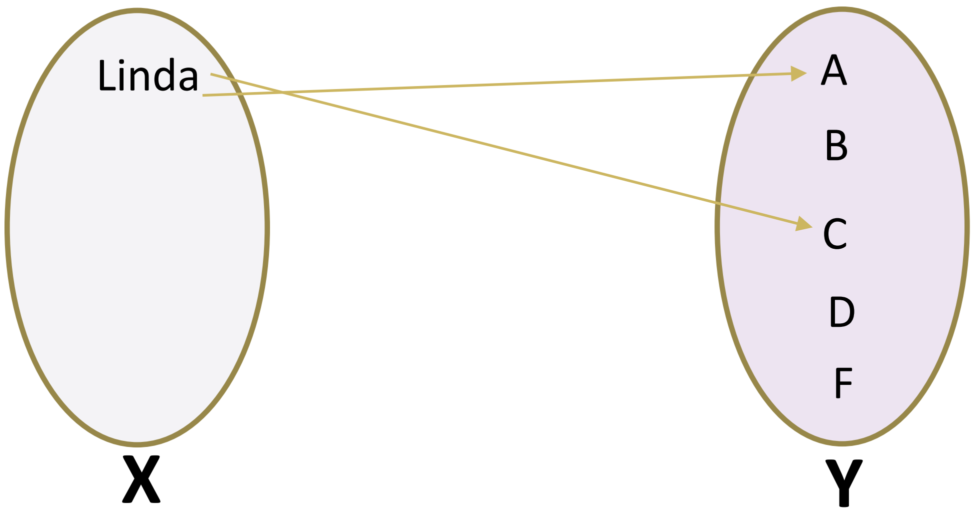
Functions



Definition: Let X and Y be nonempty sets. A function f from X to Y is an assignment of exactly one element of Y to each element of X. We write $f(a) = b$ if b is the unique element of Y assigned by the function f to the element a of X. If f is a function from X to Y, we write $f: X \longrightarrow Y$.
Functions are sometimes also called mappings or transformations.

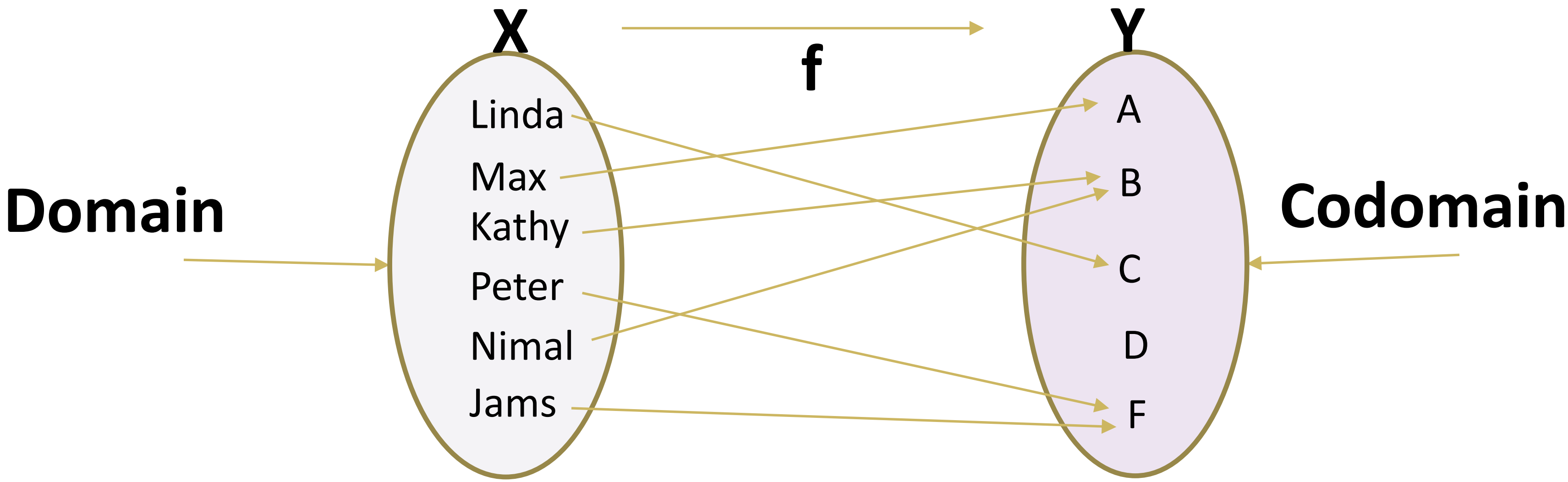
Rquirement of functions

- An element of the set X cannot be mapped with two different elements in y as shown in the following figure.



- Every element of the set x must be mapped with an element with Y subjected to the above condition.
- However there may be lements in y which are not assigned with elements of x from the function.

Terminology



If $f(a) = b$, we say that b is the **image** of *a* and *a* is a **pre-image** of *b*. The **range**, or image, of f denoted by Range(f) is the set of all images of elements of *X* and is .

$$Range(f) = \{y \in Y \mid y = f(x) \text{ } for \text{ } some \text{ } x \in X\}.$$

Range(f) $\subseteq$ Codomain(f)

Examples

Find the domain, codomain and range of the following functions:

1. Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$.
    - The domain of f is the set of all bit strings of length 2 or greater.
    - The codomain and range are the set $\{00,01,10,11\}$.

2. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2, for\ all\ x \in \mathbb{Z}$.
    - The domain of f is the set of all integers ($\mathbb{Z}$)
    - The codomain of f is the set of all integers ($\mathbb{Z}$)
    - The range of f is the set $\{n^2 \mid n \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, \cdots\cdots\}$.

One to one function

Some functions never assign the same co-domain value to two different domain elements. These functions are said to be one-to-one.

Definition: A function f is said to be one–to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all *a* and *b* in the domain of f.

Remark: We can express that f is one-to-one using quantifiers as follows:

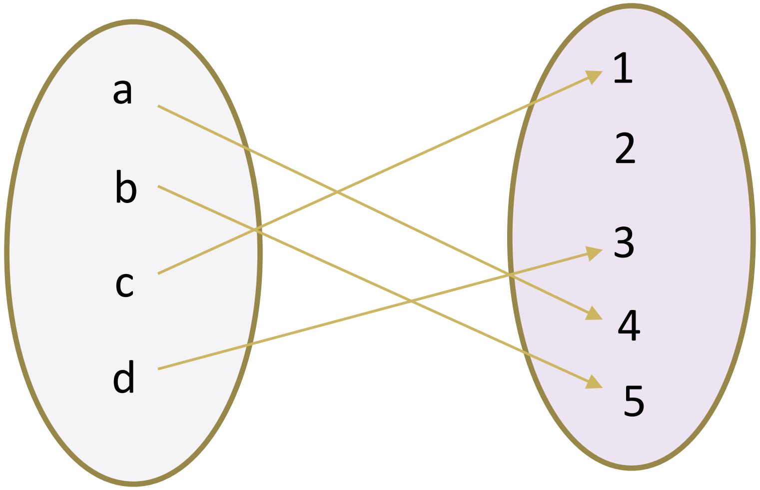
$$f \text{ is one} - \text{to} - \text{one} \iff \forall a, \forall b \, (f(a) = f(b) \rightarrow a = b)$$

$$\iff \forall a, \forall b \left(a \neq b \rightarrow f(a) \neq f(b)\right),$$

where the universe of discourse is the domain of the function.

Example

Determine whether the function f from $\{a, b, c, d\}$ to $\{1,2,3,4,5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1,\ and\ f(d) = 3$ is one-to-one.



The function f is **one to one** because f assignes different values to the elements of its domain.

Examples

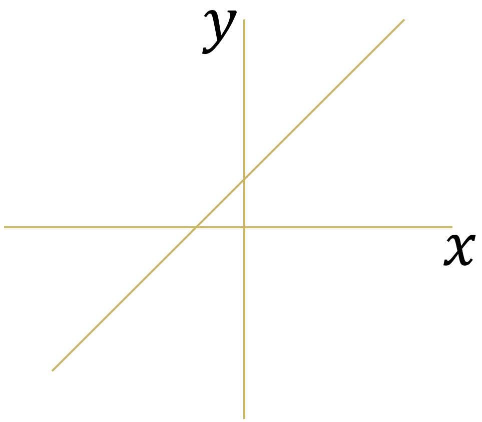
- Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

  Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

  $\Rightarrow \quad x_1 + 1 = x_2 + 1$

  $\Rightarrow \quad x_1 = x_2.$

  Hence, f is one to one.



- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

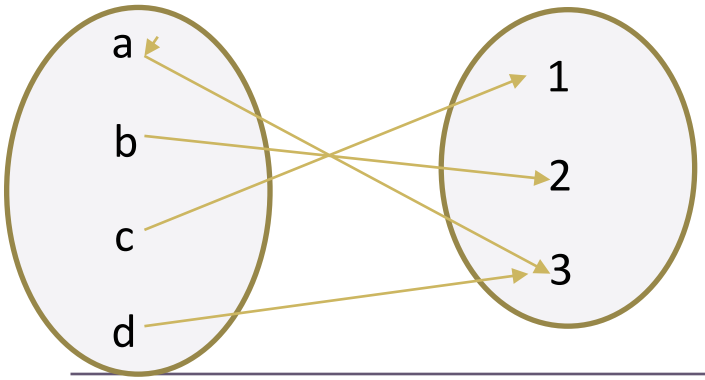
Since $f(-1) = f(1) = 1$, and $-1 \neq 1$, it follows that f is not one to one.

Onto Functions

Definition: A function f from A to B is called **onto**, or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.
That is;

$$f: A \rightarrow B \text{ is onto} \Leftrightarrow \forall y \in B\ \exists x \in A \text{ such that } f(x) = y.$$
$$\Leftrightarrow \quad f(A) = Range(f) = B = Co - domain(f)$$

Example: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1,$ and $f(d) = 3$. Is f an onto function?



Because all three elements of the codomain are images of elements in the domain, f is an onto function.

Examples

- Is the function $f(x) = 2x + 1$ from the set of real numbers to the set of real numbers onto?

Solution: Let $y \in \mathbb{R}$ be arbitrary.
Then $x = \frac{y-1}{2} \in \mathbb{R}$ and f(x) = y
$f(x) = f\left(\frac{y-1}{2}\right) = y - 1 + 1 = y$.
Hence, $\forall y \in \mathbb{R} \; \exists x \in \mathbb{R}$ such that $f(x) = y$.
Therefore f is onto.

$$f(x) = y$$
$$\Rightarrow \; 2x + 1 = y$$
$$\Rightarrow \; x = \frac{y-1}{2}$$

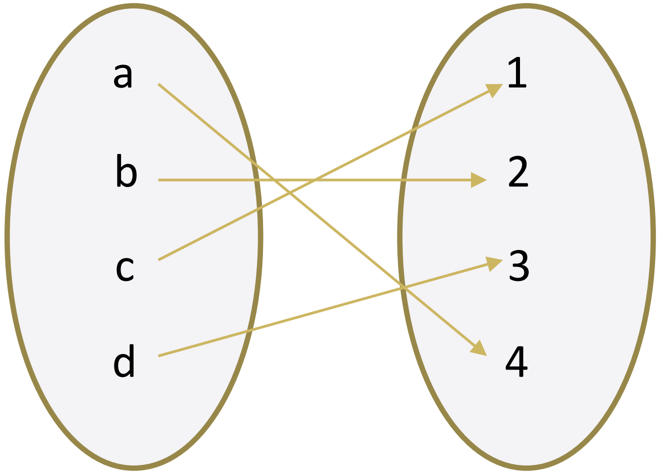
- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

The function f is not onto because there is no integer x with $x^2 = -1$.

One to one correspondence or Bijection

Definition: A function $f: A \longrightarrow B$ is a bijection, if and only if it is both one to one and onto.

Example:
Let f be the function from $\{a, b, c, d\}$ to $\{1,2,3,4\}$ with $f(a) = 4, f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?



The function f is one to one and onto.
Hence, f is a bijection.

Example

Consider the function $f\colon \mathbb{R}-\{3\} \longrightarrow \mathbb{R}-\{1\}$ defined by $f(x)=\left(\frac{x-2}{x-3}\right), \forall x\ \mathbb{R}-\{3\}$. Is f a bijection? Justify your answer.

Suppose that $f(x)=f(y)$ for some $x, y \in \mathbb{R}-\{3\}$.

$\Rightarrow \quad \frac{x-2}{x-3}=\frac{y-2}{y-3}$

$\Rightarrow \quad (x-2)(y-3)=(x-3)(y-2)$

$\Rightarrow xy-3x-2y+6=xy-2x-3y+6$

$\Rightarrow \quad x=y.$

$\therefore \quad f$ is one to one.

Let $y \in \mathbb{R}-\{1\}\ and\ f(x) = y$

Since $y \in \mathbb{R}-\{1\}\ y \neq 1.$

Since $f(x)=y$

$\Rightarrow \quad \frac{x-2}{x-3}=y$

$\Rightarrow \quad (x-2)=(x-3)y=xy-3y$

$\Rightarrow \quad x-xy=2-3y$

$\Rightarrow x(1-y)=2-3y$

$\Rightarrow \quad x=\frac{(2-3y)}{(1-y)} \in \mathbb{R}\ (\because y \neq 1).$

$\therefore \forall y \in \mathbb{R}-\{1\},\ \exists x \in \mathbb{R}-\{3\}\ s.t\ f(x)=y.$

Hence, f is onto

Since f is one to one and onto, f is a bijection.

Exercise

Give an example of a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ that is

- one-to-one but not onto:

  Define $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ by $f(n) = n + 1,\ \forall n \in \mathbb{Z}^+$.

  Then f is one to one but not onto.

There is no number in $\mathbb{Z}^+(domain)$ that maps to 1 in the codomain

- onto but not one-to-one:

  Define $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ by $f(n) = \begin{cases} 1 & if\ \ n = 1 \\ n - 1 & if\ n > 1 \end{cases}$.

  Then f is not one to one, but f is onto.

  f(2) = 2-1 = 1 = f(1)

Exercise

Give an example of a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ that is

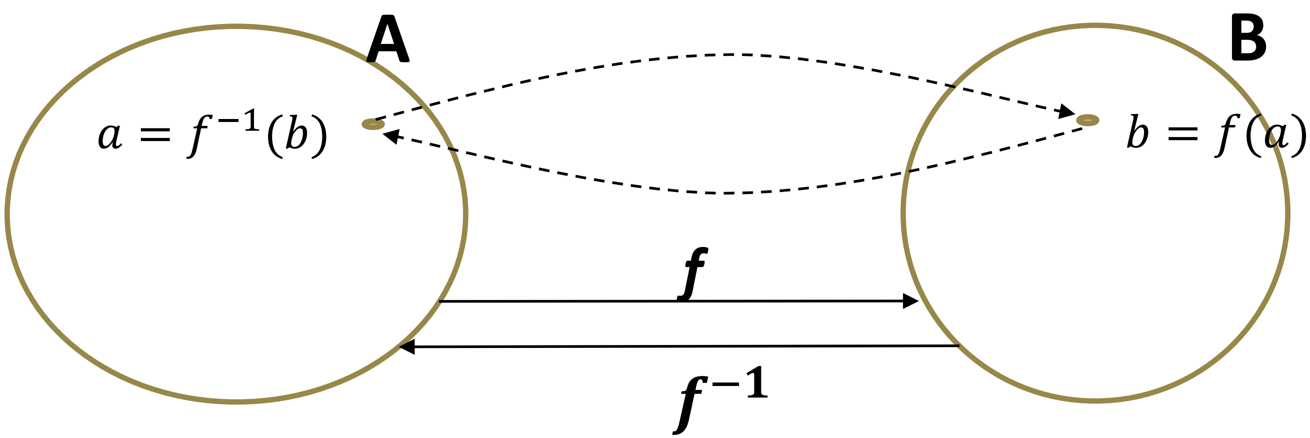
- both onto and one-to-one (but different from the identity function)

  Define $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ by $f(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$.

Inverse Functions

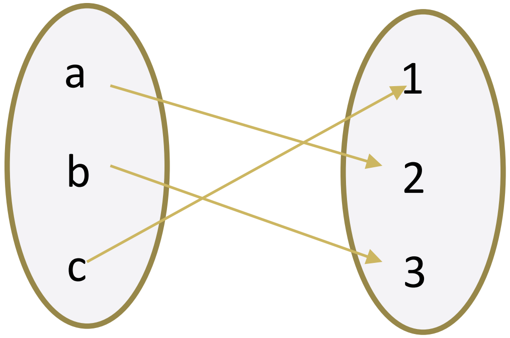
An interesting property of bijections is that they have an **inverse function**.

Definition: Let f be a bijection from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B an unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) =$ a when $f(a) = b$.

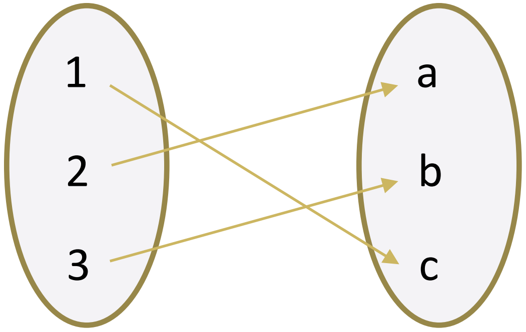


Example

Let f be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2, f(b) = 3,$ and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?



$\boldsymbol{f}: \boldsymbol{A} \longrightarrow \boldsymbol{B}$



$\boldsymbol{f^{-1}}: \boldsymbol{B} \longrightarrow \boldsymbol{A}$

Examples

Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

Because $f(-1) = f(1) = 1$, f is not one-to-one. If f has an inverse function thit would have to assign two elements 1 and -1 to 1. Hence, f is not invertible.

Examples

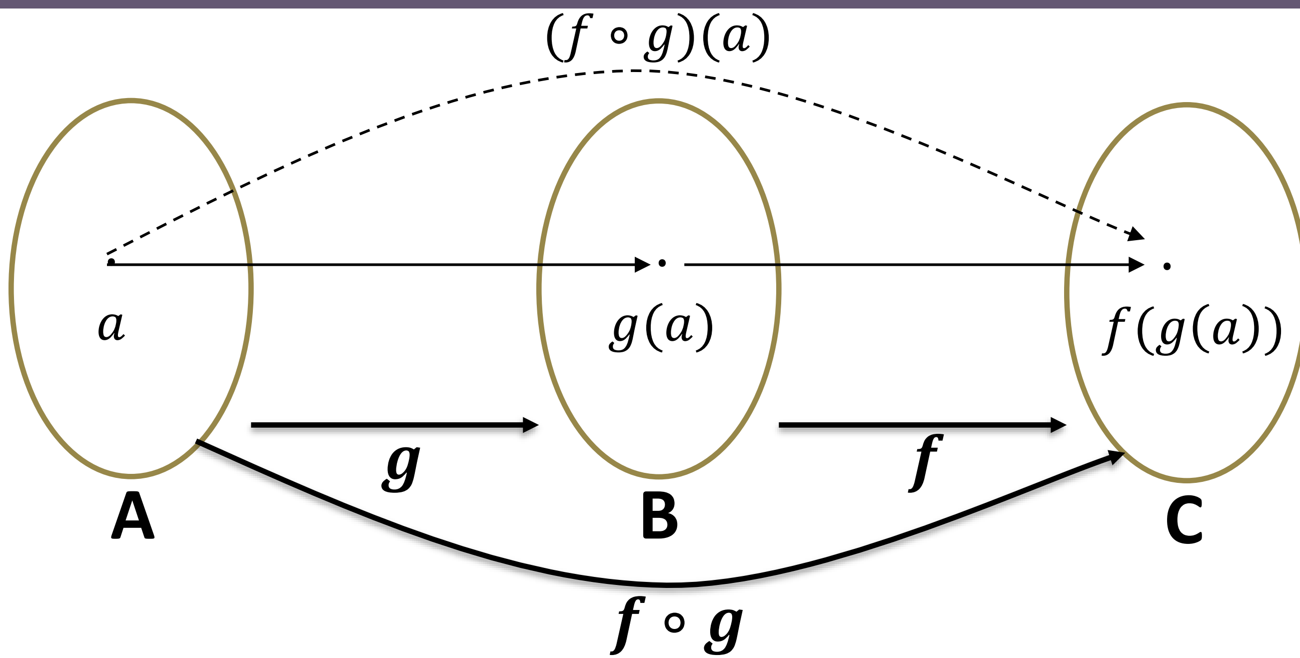
Let $f: \mathbb{R}^* \longrightarrow \mathbb{R}^*$ be the function defined by $f(x) = x^2$, where $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$ Is f invertible?

Let $f(x) = f(y)$, for $x, y \in \mathbb{R}^*$
$\Rightarrow x^2 = y^2$
$\Rightarrow (x-y)(x+y) = 0$
$\Rightarrow x = y, or\ x = -y$
$\Rightarrow x = y\ (\because x, y \geq 0)$
Hence, f is one to one.

Let $y \in \mathbb{R}^*$. Then $y \geq 0$.
$\Rightarrow \exists\, x \in \mathbb{R}^*$ such that $x = \sqrt{y}$
$\Rightarrow f(x) = x^2 = y$.
That is; $\forall y \in \mathbb{R}^*, \exists x \in \mathbb{R}^*$ s.t $f(x) = y$.
Hence, f is onto.

Since, f is one to one and onto f is invertible and
$f^{-1}(y) = \sqrt{y},\ \forall y \in \mathbb{R}^*$.

Compositions of Functions



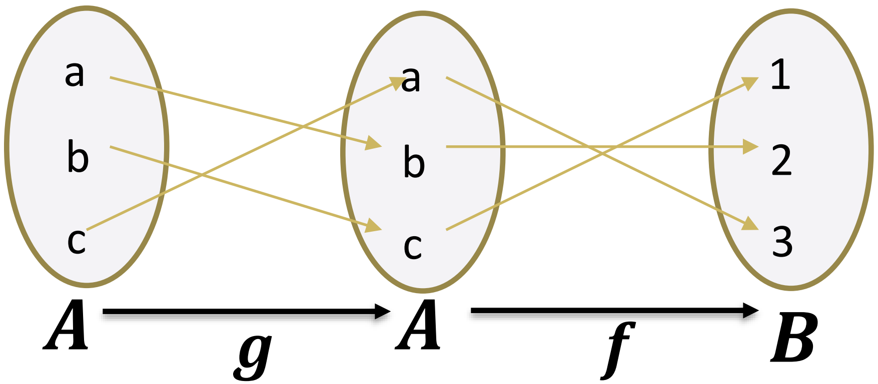
Definition: The **composition** of two functions $g: A \to B$ and $f: B \to C$, denoted by $f \circ g$, is defined by

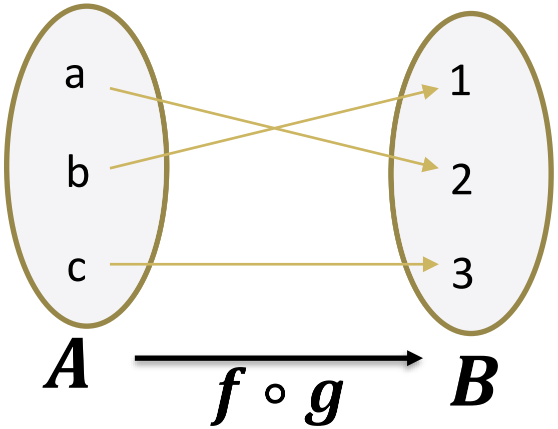
$$(f \circ g)(a) = f\big(g(a)\big) \text{ for all } a \in A.$$

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Example

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f ?





Note that g ∘ f is not defined, because the range of f=(1,2,3} is not a subset of the domain of g ={a,b,c}.

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x)$= 2x +3 and $g(x)$= 3x +2. What is the composition of f and g? What is the composition of g and f ?

Let $x \in \mathbb{Z}$ be arbitrary.

$$\begin{aligned}(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) &= \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})) \\ &= \boldsymbol{f}(\mathbf{3}\boldsymbol{x} + \mathbf{2}) \\ &= \mathbf{2}(\mathbf{3}\boldsymbol{x} + \mathbf{2}) + \mathbf{3} \\ &= \mathbf{6}\boldsymbol{x} + \mathbf{7}.\end{aligned}$$

Let $x \in \mathbb{Z}$ be arbitrary.

$$\begin{aligned}(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) &= \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) \\ &= \boldsymbol{g}(\mathbf{2}\boldsymbol{x} + \mathbf{3}) \\ &= \mathbf{3}(\mathbf{2}\boldsymbol{x} + \mathbf{3}) + \mathbf{2} \\ &= \mathbf{6}\boldsymbol{x} + \mathbf{11}.\end{aligned}$$

Note that even though f ∘ g and g ∘ f are defined for the functions f and g, $f \circ g$ and $g \circ f$ are not equal.

More Examples

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be both one to one functions. Show that $g \circ f: A \rightarrow C$ is one to one.

Solution: Assume that $(g \circ f)(a) = (g \circ f)(b)$ for $a, b \in A$.

$$\Longrightarrow g\big(f(a)\big) = g\big(f(b)\big) \text{ (}\because \text{ Def. of } g \circ f\text{)}$$

$$\Longrightarrow f(a) = f(b) \quad (\because g \; is \; 1-1)$$

$$\Longrightarrow a = b \quad (\because f \; is \; 1-1)$$

Hence, $\forall a, b \in A \quad [(g \circ f)(a) = (g \circ f)(b) \rightarrow a = b]$

Therefore $g \circ f$ is one-to-one.

Exercise: Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be both onto functions. Show that $g \circ f: A \rightarrow C$ is onto.

Identity Function

Definition: Let A be a non-empty set. The **identity function** on A is the function $i_A: A \to A$, where $i_A(x) = x$ for all $x \in A$. In other words, the identity function i_A is the function that assigns each element to itself.

The function i_A is one-to-one and onto, so it is a bijection.

When the composition of a function $f: A \to B$ and its inverse, if exists $f^{-1}: B \to A$ is formed, in either order, an identity function is obtained.

$$(f^{-1} \circ f)(a) = f^{-1}\big(f(a)\big) = a \text{ for all } a \in A, and$$
$$= i_A(a)$$
$$(f \circ f^{-1})(b) = f\big(f^{-1}(b)\big) = b \text{ for all } b \in B.$$
$$= i_B(b)$$

Hence $f^{-1} \circ f = i_A$, $f \circ f^{-1} = i_B$ and $(f^{-1})^{-1} = f$.

The Graphs of Functions

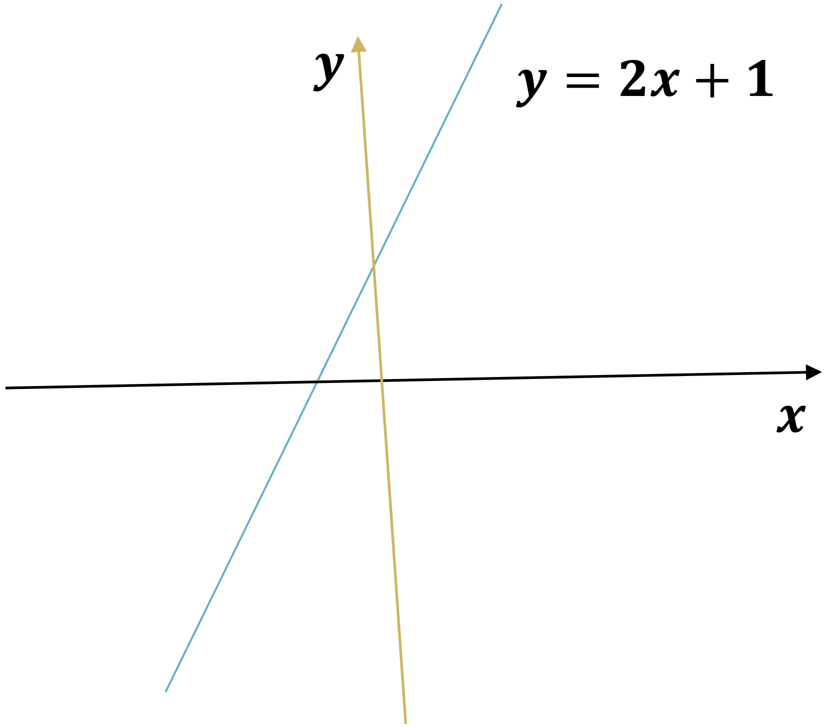
Definition: Let f be a function from the set A to the set B. The graph of the function f is denoted by $\mathcal{G}(f)$, and is the set of ordered pairs

$$\mathcal{G}(f) = \{(a, b) \mid a \in A \text{ and } f(a) = b\}.$$

Example: Display the graph of the function $f(x) = 2x + 1$ from the set of real numbers to the set of real numbers.

$$\mathcal{G}(f) = \{(x, y) \mid x \in \mathbb{R}, y = 2x + 1\}$$

$$= \{(x, 2x + 1) \mid x \in \mathbb{R}\}.$$



The Image and Inverse Image of a Subset of the Domain & the Codomain Respectively

Let $f: A \rightarrow B$ be a function and let $S \subseteq A$ and $\mathrm{T} \subseteq B$.

- The image of S under f is the subset of B that consists of the images of the elements of S, and is denoted by $f(S)$.

$$f(S) = \{\, t \mid \exists s \in S \left(t = f(s)\right)\} = \{f(s) \mid s \in S\}$$

Example -2

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2, \forall x \in \mathbb{R}$.
Find

1. $\mathrm{f}(\{-2, -1, 0, 1, 2\}) = \{0, 1, 4\}$
2. $\mathrm{f}(\mathbb{Z}) = \{f(n) | \, n \in \mathbb{Z}\} = \{n^2 | \, n \in \mathbb{Z}\} = \{n^2 | \, n \in \mathbb{N}\}$.
3. $\mathrm{f}(\mathbb{R}) = \{f(x) | \, x \in \mathbb{R}\} = \{x^2 | \, x \in \mathbb{R}\}$
4. $\mathrm{f}^{-1}(\{1\}) = \{-1, 1\}$
5. $\mathrm{f}^{-1}(\{\mathrm{x} \mid 0 < \mathrm{x} < 1\}) = \{x \mid -1 < x < 0 \text{ or } 0 < x < 1\}$.

Exercise -1

Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

1. $f(S \cup T) = f(S) \cup f(T)$
2. $f(S \cap T) \subseteq f(S) \cap f(T)$.

Solution for 1:

Let $y \in f(S \cup T)$.

$\Rightarrow\ \exists x \in (S \cup T)$ s.t $f(x) = y$.

$\Rightarrow\ \exists x \in S$ s.t $f(x) = y$ or

$\exists x \in T$ s.t $f(x) = y$

$\Rightarrow y \in f(S)$ or $y \in f(T)$

$\Rightarrow y \in f(S) \cup f(T)$

$\therefore f(S \cup T) \subseteq f(S) \cup f(T)$.

Let $y \in f(S) \cup f(T)$.

$\Rightarrow y \in f(S)$ or $y \in f(T)$

$\Rightarrow\ \exists x \in S$ s.t $f(x) = y$ or

$\exists\, x \in T$ s.t $f(x) = y$

$\Rightarrow\ \exists x \in (S \cup T)$ s.t $f(x) = y$

$\Rightarrow y \in f(S \cup T)$

$\therefore f(S) \cup f(T) \subseteq f(S \cup T)$.

Solution for 2:

Let $y \in \mathrm{f}(\mathrm{S} \cap \mathrm{T})$.

$\Rightarrow\ \exists x \in (\mathrm{S} \cap \mathrm{T})$ s.t $f(x) = y$.

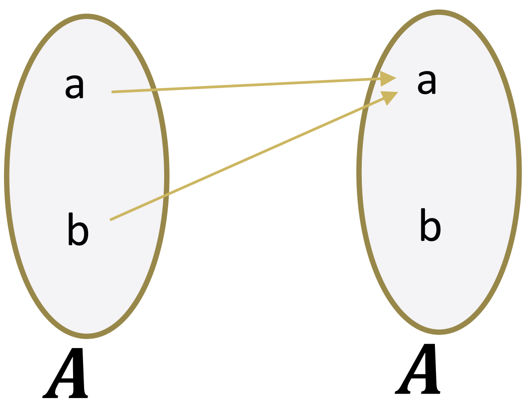
$\Rightarrow\ \exists x \in S$ s.t $f(x) = y$ and $\exists x \in T$ s.t $f(x) = y$

$\Rightarrow y \in f(S)$ and $y \in f(T)$

$\Rightarrow y \in \mathrm{f(S)} \cap \mathrm{f(T)}$

$\therefore \mathrm{f}(\mathrm{S} \cap \mathrm{T}) \subseteq \mathrm{f(S)} \cap \mathrm{f(T)}$.

Let f be the function defined by the following diagram:



Let $S = \{a\}$ and $T = \{b\}$. Then

$S \cap T = \varphi, f(S) = \{a\}$, and $f(T) = \{a\}$.

But $f(S \cap T) = \varphi$, and $f(S) \cap f(T) = \{a\}$.

Exercise -2

Let f be a **one-to-one** function from the set A to the set B. Let S and T be subsets of A. Show that $f(S \cap T) = f(S) \cap f(T)$.

Solution: We know from exercise -1 that $f(S \cap T) \subseteq f(S) \cap f(T)$ when f is a function from A to B.
Now we shall show that $f(S) \cap f(T) \subseteq f(S \cap T)$ when f is one-to-one.

Let $y \in f(S) \cap f(T)$

$\Rightarrow y \in f(S)$ and $y \in f(T)$

$\Rightarrow \exists\, s \in S$ s.t $y = f(s)$ and $\exists\, t \in T$ s.t $y = f(t)$

$\Rightarrow \exists\, s \in S$ and $\exists\, t \in T$ s.t $y = f(s) = f(t)$

$\Rightarrow \; s = t \text{ and } s \in (S \cap T) \text{ and } y = f(s) \quad (\because \; f \text{ is one} - \text{to} - \text{one})$

$\Rightarrow \; y \in f(S \cap T)$

$\therefore f(S) \cap f(T) \subseteq f(S \cap T)$

Example

Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

c) $f^{-1}(B - S) = A - f^{-1}(S)$.

a)

Let $x \in f^{-1}(S \cup T)$

$\Leftrightarrow f(x) \in (S \cup T)$

$\Leftrightarrow f(x) \in S$ or $f(x) \in \mathrm{T}$

$\Leftrightarrow x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\Leftrightarrow x \in (f^{-1}(S) \cup f^{-1}(T))$

$\therefore f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

c)

Let $x \in f^{-1}(B - S)$

$\Leftrightarrow f(x) \in (B - S)$

$\Leftrightarrow f(x) \in B$ and $f(x) \notin \mathrm{S}$

$\Leftrightarrow x \in f^{-1}(B)$ and $x \notin f^{-1}(S)$

$\Leftrightarrow x \in A$ and $x \notin f^{-1}(S)$

$\Leftrightarrow x \in A - f^{-1}(S)$.

$\therefore f^{-1}(B - S) = A - f^{-1}(S)$.

Special Functions: Floor Function

The **floor** function $\lfloor\ \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$ assigns to the real number x the largest integer that is less than or equal to x. The value of a floor function at x is denoted by $\lfloor x \rfloor$.

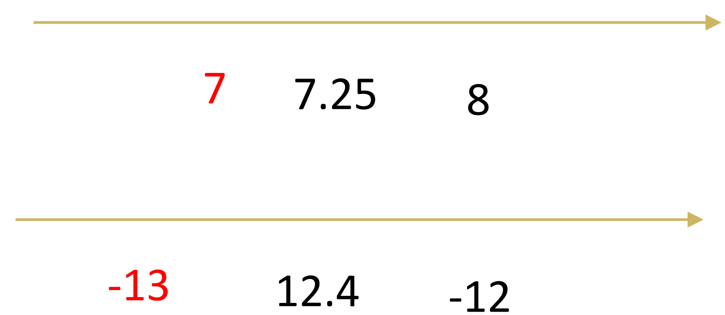
The floor function is often also called the greatest integer function.

$\lfloor 0.5 \rfloor = 0$

$\lfloor 7.25 \rfloor = 7$

$\lfloor -12.4 \rfloor = -13$

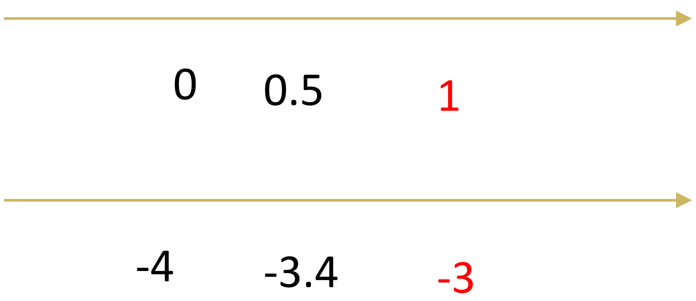
$\lfloor -10 \rfloor = -10$.



Special Functions: Ceiling Function

The **Ceiling** function $\lceil \quad \rceil : \mathbb{R} \longrightarrow \mathbb{Z}$ assigns to the real number x the smallest integer that is greater than or equal to x. The value of a ceiling function at x is denoted by $\lceil x \rceil$.

- $\lceil 0.5 \rceil = 1$
- $\lceil 7.5 \rceil = 8$
- $\lceil -3.4 \rceil = -3.$



Partial Functions

Definition: A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B. The sets A and B are called the domain and codomain of f, respectively. We say that f is undefined for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a total function.

Example: The function $f : Z \to R$ where $f(n) = \sqrt{n}$ is a partial function from Z to R where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

$$f(n) = \sqrt{n} \quad if \quad n \geq 0$$