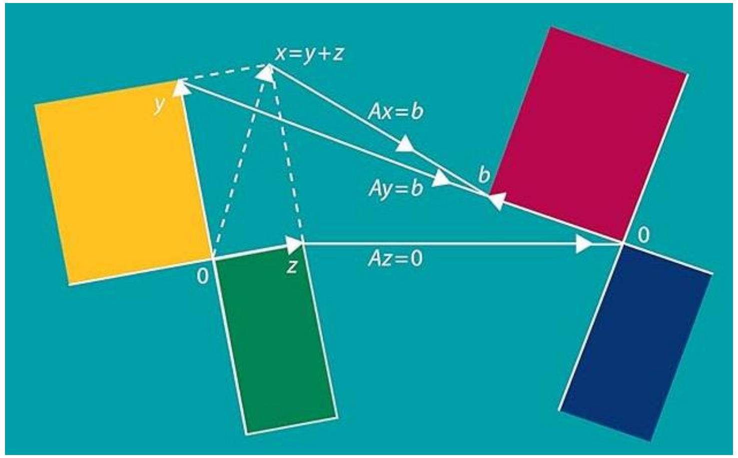
Transpose and Determinants of a Matrix

(Linear Algebra)

Randil Pushpananda, PhD
rpn@ucsc.cmb.ac.lk



Transpose of a Matrix

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ then the transpose of this matrix is $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

A column vector such as $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be written as the transpose of a row vector, that is $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$. This approach of writing a column vector $\mathbf{x}$ as the transpose of a row vector saves space. In this case, column vector $\mathbf{x}$ takes up n lines while $\mathbf{x} = (x_1 \ \cdots \ x_n)^T$ can be written on a single line.

Properties of Transpose Matrix

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of appropriate size so that the operations below can be carried out. We have the following properties (k is a scalar):

(a) $\left(\mathbf{A}^T\right)^T = \mathbf{A}$ (b) $(k\mathbf{A})^T = k\mathbf{A}^T$

(c) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ (d) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Proof

$$(\mathbf{A}+\mathbf{B})^T = \left[\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}^T \quad \begin{bmatrix}\text{adding the} \\ \text{corresponding} \\ \text{entries}\end{bmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \cdots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \cdots & a_{m2}+b_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \quad \begin{bmatrix}\text{taking the transpose,} \\ \text{that is interchanging} \\ \text{rows and columns}\end{bmatrix}$$

$$= \underbrace{\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}}_{=\mathbf{A}^T} + \underbrace{\begin{pmatrix} b_{11} & b_{21} & \cdots & b_{m1} \\ b_{12} & b_{22} & \cdots & b_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{mn} \end{pmatrix}}_{=\mathbf{B}^T} = \mathbf{A}^T + \mathbf{B}^T$$

Let $\mathbf{A}$ be an invertible (non-singular) matrix then the transpose of the matrix, $\mathbf{A}^T$, is also invertible and $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$.

- For an invertible matrix, you can change the order of the inverse and transpose.

$$\left(\mathbf{A}^{-1}\right)^T \mathbf{A}^T = \mathbf{I} \text{ and } \mathbf{A}^T \left(\mathbf{A}^{-1}\right)^T = \mathbf{I}$$

Examining the first of these $\left(\mathbf{A}^{-1}\right)^T \mathbf{A}^T$ we have

$$\begin{aligned}
\left(\mathbf{A}^{-1}\right)^T \mathbf{A}^T &= \left(\mathbf{A}\mathbf{A}^{-1}\right)^T && \left[\text{using Theorem } \mathbf{Y}^T\mathbf{X}^T = (\mathbf{XY})^T\right] \\
&= \mathbf{I}^T && \left[\text{remember } \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}\right] \\
&= \mathbf{I} && \left[\text{remember } \mathbf{I}^T = \mathbf{I} \text{ because } \mathbf{I} \text{ is the identity}\right]
\end{aligned}$$

Determinant

- The *determinant* is a number associated with a square matrix
- Determinant of a square matrix is a number.
- Can use this value to establish whether the *matrix has an inverse or not*, as well as *finding whether the linear system has a unique solution*
- The determinant is notated as det(A) or |A|

Determinant of a 2 by 2 Matrix

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$a \searrow d$ minus $c \nearrow b$

Determinant of a 2 by 2 Matrix

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1} \qquad \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbf{-1}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{ad - bc} \qquad \det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \boldsymbol{bc - ad}$$

- Determinants reverse sign when two rows are exchanged (or two columns are exchanged).
- The matrix has no inverse when its determinant is zero.
- det A = 0 means that the columns of A are not independent.

Determinant of a 3 by 3 Matrix

Start with the identity matrix ($\det I = 1$) and exchange two rows (then $\det = -1$). Exchanging again brings back $\det = +1$. You quickly have all six permutation matrices. Each row exchange will reverse the sign of the determinant ($+1$ to -1 or else -1 to $+1$):

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

$$\mathbf{det = +1} \qquad \mathbf{-1} \qquad \mathbf{+1} \qquad \mathbf{-1} \qquad \mathbf{+1} \qquad \mathbf{-1}$$

If I exchange two rows of A—say rows 1 and 2—then its determinant changes sign. This will carry over to all determinants: **Row exchange multiplies det A by -1.**

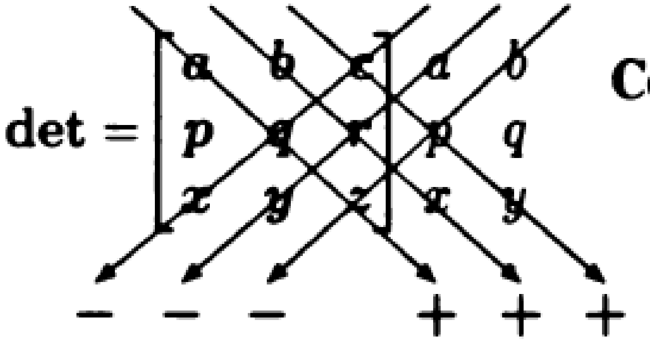
Determinant of a 3 by 3 Matrix

When you multiply a row by a number, this multiplies the determinant by that number. Suppose the three rows are $a\ b\ c$ and $p\ q\ r$ and $x\ y\ z$. Those nine numbers multiply ± 1.

$$\begin{bmatrix} a & & \\ & q & \\ & & z \end{bmatrix} \begin{bmatrix} & b & \\ p & & \\ & & z \end{bmatrix} \begin{bmatrix} & b & \\ & & r \\ x & & \end{bmatrix} \begin{bmatrix} & & c \\ & q & \\ x & & \end{bmatrix} \begin{bmatrix} & & c \\ p & & \\ & y & \end{bmatrix} \begin{bmatrix} a & & \\ & & r \\ & y & \end{bmatrix}$$

$$\det = +aqz \qquad -bpz \qquad +brx \qquad -cqx \qquad +cpy \qquad -ary$$

Finally we use the most powerful property we have. **The determinant of A is linear in each row separately.** As equation (4) will show, we can add those six determinants. To remember the plus and minus signs, I follow the arrows in this picture of the matrix.



Combine those 6 simple determinants into det A =

$$\underline{+\,aqz + brx + cpy - ary - bpz - cqx} \qquad (1)$$

Determinant of a 3 by 3 Matrix

When you multiply a row by a number, this multiplies the determinant by that number. Suppose the three rows are $a\ b\ c$ and $p\ q\ r$ and $x\ y\ z$. Those nine numbers multiply ± 1.

$$\begin{bmatrix} a & & \\ & q & \\ & & z \end{bmatrix} \begin{bmatrix} & b & \\ p & & \\ & & z \end{bmatrix} \begin{bmatrix} & b & \\ & & r \\ x & & \end{bmatrix} \begin{bmatrix} & & c \\ & q & \\ x & & \end{bmatrix} \begin{bmatrix} & & c \\ p & & \\ & y & \end{bmatrix} \begin{bmatrix} a & & \\ & & r \\ & y & \end{bmatrix}$$

$$\det = +aqz \qquad -bpz \qquad +brx \qquad -cqx \qquad +cpy \qquad -ary$$

Finally we use the most powerful property we have. **The determinant of A is linear in each row separately.** As equation (4) will show, we can add those six determinants. To remember the plus and minus signs, I follow the arrows in this picture of the matrix.

$$\det = \begin{array}{|ccc|cc} a & b & c & a & b \\ p & q & r & p & q \\ x & y & z & x & y \end{array}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

Combine those 6 simple determinants into $\det A =$

$$\underline{+\,aqz + brx + cpy - ary - bpz - cqx} \qquad (1)$$

Determinant of a 3 by 3 Matrix

$$A_1 = \begin{bmatrix} 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det A_1 = 2 \qquad \det A_2 = 4 - 1 = 3 \qquad \det A_3 = 8 - 2 - 2 = 4$$

Determinant of a 3 by 3 Matrix

Minors

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The *determinant* of the *remaining* matrix after *deleting* the *row* and *column* of an entry is called the **minor** of that entry.

Determinant of a 3 by 3 Matrix

Minor of entry *a*

$$\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$$

Minor of entry *b*

$$\det\begin{pmatrix} d & f \\ g & i \end{pmatrix}$$

Determinant of a 3 by 3 Matrix

Consider a square matrix **A**. Let a_{ij} be the entry in the ith row and jth column of matrix **A**. The **minor** M_{ij} of entry a_{ij} is the *determinant* of the remaining matrix after *deleting* the entries in the ith row and jth column.

$$\begin{array}{cc} & \text{jth column} \\ M_{ij} = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} & \text{ith row} \end{array}$$

Determinant of a 3 by 3 Matrix

Cofactor: Consider a square matrix A. Let a_{ij} be the entry in the i^{th} row and j^{th} column of matrix A. The cofactor c_{ij} of the entry a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of entry a_{ij}.

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Question

Determine the cofactor of 5 in $\begin{pmatrix} 3 & 5 & 7 \\ -1 & 2 & 3 \\ -4 & 4 & -9 \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \left[\text{remember that place signs are } \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \right]$$

$$\det(\mathbf{A}) = a\,(\text{cofactor of } a) + b\,(\text{cofactor of } b) + c\,(\text{cofactor of } c)$$

$$\det(\mathbf{A}) = a\left[\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right] - b\left[\det\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right] + c\left[\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right]$$

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \left[\text{remember that place signs are} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \right]$$

Middle Row

$$\det(\mathbf{A}) = d\,(\text{cofactor of } d) + e\,(\text{cofactor of } e) + f\,(\text{cofactor of } f)$$

Bottom Row

$$\det(\mathbf{A}) = g\,(\text{cofactor of } g) + h\,(\text{cofactor of } h) + i\,(\text{cofactor of } i)$$

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \left[\text{remember that place signs are } \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \right]$$

We can also expand along any of the columns to find the determinant of **A**. The formula for expanding along the first column is

$$\det(\mathbf{A}) = a\,(\text{cofactor of } a) + d\,(\text{cofactor of } d) + g\,(\text{cofactor of } g)$$

If any of the rows or columns contain zeros then we choose to expand along that row or column because it simplifies the arithmetic.

Question

Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Question

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} -1 & 5 & -2 \\ -6 & 6 & 0 \\ 3 & -7 & 1 \end{pmatrix}$. Find $\det(\mathbf{A})$.

Finding inverse of a Matrix

$$\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

Cofactor Matrix

Cofactor Matrix

Let **C** be the new matrix consisting of the cofactors of the general matrix **A**. If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{then} \quad \mathbf{C} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

where A is the cofactor of entry a, B is the cofactor of entry b, C is the cofactor of entry c ... The matrix **C** is called the **cofactor matrix** and it is used in finding the *inverse* of **A**. Note that bold **C** represents the cofactor matrix and plain C is the cofactor of the entry c.

Question

Find the cofactor matrix $\mathbf{C}$ of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 9 & 7 \\ -2 & 1 & 0 \end{pmatrix}$$

Adjoint

- We use the cofactor matrix to find the inverse of an invertible matrix. The final term we need to define is ‘adjoint’.

- Definition:

Let **A** be a square matrix then the matrix consisting of the cofactors of each entry in **A** is called the cofactor matrix and is normally denoted by **C**. The *transpose* of this cofactor matrix is called the **adjoint** of **A** and is denoted by *adj*(**A**). That is

$$adj(\mathbf{A}) = \mathbf{C}^T$$

Inverse of a matrix

- Theorem:

If $\det(\mathbf{A}) \neq 0$ (not zero) then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A})$$

$$\boxed{\textbf{Inverse matrix formula} \qquad A^{-1} = C^{\mathrm{T}} / \det A}$$

Question

Find the cofactor matrices $\mathbf{C}$ and $\mathbf{C}^T$ of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 3 & 7 \\ 6 & -1 & 0 \end{pmatrix}$

Also determine $\mathbf{A}^{-1}$.