SCS 1302 - Discrete Mathematics

Tutorial - 5

1. Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if
   a. $f(k) = \pm k$
   b. $f(k) = \sqrt{k^2 + 1}$
   c. $f(k) = \frac{1}{(n^2 - 4)}$
2. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a. the function that assigns to each nonnegative integer its last digit
   b. the function that assigns to a bit string the number of one bits in the string
   c. the function that assigns to a bit string the number of bits in the string
3. Find the domain and range of these functions.
   a. the function that assigns to each pair of positive integers the first integer of the pair
   b. the function that assigns to each positive integer its largest decimal digit
   c. the function that assigns to a bit string the number of ones minus the number of zeros in the string
   d. the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
4.
   a. Find these values.
      i. $\lfloor 1.1 \rfloor$
      ii. $\lceil 1.1 \rceil$
      iii. $\lfloor -0.1 \rfloor$
      iv. $\lceil -0.1 \rceil$
      v. $\lceil 2.99 \rceil$
      vi. $\lceil -2.99 \rceil$
      vii. $\lfloor \frac{1}{2} + \rfloor \frac{1}{2}] [$
      viii. $] \lfloor \frac{1}{2} [+] \frac{1}{2}] + \frac{1}{2}]$

b. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

i. $f(x) = 1$.

ii. $f(x) = 2x + 1$.

iii. $f(x) = \lceil x/5 \rceil$.

iv. $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

5. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

a. $f(k) = k - 1$

b. $f(k) = k^2 + 1$

c. $f(k) = k^3$

d. $f(k) = \lceil \frac{n}{2} \rceil$

6. Which functions in Exercise 5 are onto?

7. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

a. $f(m, k) = 2m - k$.

b. $f(m, k) = m^2 - k^2$.

c. $f(m, k) = m + k + 1$.

d. $f(m, k) = |m| - |k|$.

e. $f(m, k) = m^2 - 4$.

8. Find a bijection from $(-1, 1)$ onto $\mathbb{R}$.

9. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

a. $f(x) = -3x + 4$

b. $f(x) = -3x^2 + 7$

c. $f(x) = (x + 1)/(x + 2)$

d. $f(x) = x^5 + 1$

e. $f(x) = (x^2 + 1)/(x^2 + 2)$

10. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$.

a. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = 1/f(x)$ is strictly decreasing.

b. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

11.

a. Prove that a strictly increasing (decreasing) function from $\mathbb{R}$ to itself is one-to-one.

b. Give an example of an increasing (a decreasing) function from $\mathbb{R}$ to itself that is not one-to-one.

12. Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

13. Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

14. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

15. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

16. Suppose that g is a function from A to B and f is a function from B to C.

a. Show that if both f and g are onto functions, then $f \circ g$ is also onto.

b. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

17. Let $g : A \to B$ and $f : B \to C$ be two functions. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

18. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

19. Let f and g be functions given by $f\colon [1/2\,, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x^2 - 1/2}$, $x \in [1/2\,, \infty)$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^2/2(x^2 + 1)$, $x \in \mathbb{R}$.

a. Find $g \circ f$.

b. Show that $f \circ g$ does not exists.

20. Let f and g be functions given by,

$f(x) = \frac{1-7x}{x-5}$, $x \in (-\infty, 5) \cup (5, \infty)$ and $g(x) = \frac{5x+1}{x+7}$, $x \in (-\infty, -7) \cup (-7, \infty)$. Prove that

a. $-7 \notin \text{Rang}(f)$

b. $5 \notin \text{Rang}(g)$

c. $f \neq g$

d. $f \circ g = id_{(-\infty,-7)\cup(-7,\infty)}$

e. $g \circ f = id_{(-\infty,5)\cup(5,\infty)}$

f. $f \circ g \neq g \circ f$.

21. Suppose that f is a bijection from Y to Z and g is a bijection from X to Y, where X, Y, Z are non empty subsets of some universal set . Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

22. Show that if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 0$ if x is an integer.

23. Show that $\lfloor x + \frac{1}{2} \rfloor$ is the closest integer to the number x except when x is midway between two integers, when it is the larger of these two integers.

24.

a. Draw the graph of the function $f(x) = \lceil x \rceil + \lfloor x/2 \rfloor$ from $\mathbb{R}$ to $\mathbb{R}$.

b. Draw the graph of the function $f(x) = \lfloor \frac{x}{2} \rfloor$ from $\mathbb{R}$ to $\mathbb{R}$.

25. Prove or disprove each of these statements about the floor and ceiling functions.

a. $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.

b. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.

c. $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.

d. $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.

e. $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y.

26. For each of these partial functions, determine its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether it is a total function.

a. $f: \mathbb{Z} \to \mathbb{R}$, $f(k) = 1/k$

b. $f: \mathbb{Z} \to \mathbb{Z}$, $f(k) = \lceil k/2 \rceil$

c. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, $f(m, k) = m/k$

d. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f(m, k) = mk$

e. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f(m, k) = m - k$ if $m > k$