Vector Spaces

(Linear Algebra)

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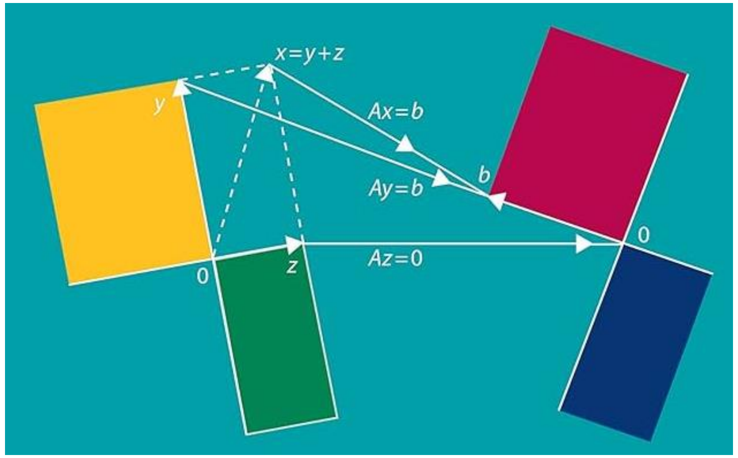

Basis and Spanning Set

To describe a vector in $\mathbb{R}^n$ we need a coordinate system. A basis is a coordinate system or framework which describes the Euclidean n-space.

For example, there are infinitely many vectors in the plane $\mathbb{R}^2$, but we can describe all of these by using the standard unit vectors $\mathbf{e}_1 = (1 \quad 0)^T$ in the x direction and $\mathbf{e}_2 = (0 \quad 1)^T$ in the y direction.

We can write a vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$*, but what does 2 and 3 represent?*

It indicates two units in the x direction and three units in the y direction which can be written as $2\mathbf{e}_1 + 3\mathbf{e}_2$.

Let $\mathbf{v} = (a \quad b \quad c)^T$ be any vector in $\mathbb{R}^3$. Write this vector $\mathbf{v}$ in terms of the unit vectors:

$$\mathbf{e}_1 = (1 \quad 0 \quad 0)^T, \quad \mathbf{e}_2 = (0 \quad 1 \quad 0)^T \text{ and } \mathbf{e}_3 = (0 \quad 0 \quad 1)^T$$

Vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ specify x, y and z directions respectively. (Illustrated in Fig. 2.26.)

Solution

We have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$$

That is, we can write the vector $\mathbf{v}$ as a linear combination of vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$.

Definition (2.25). Consider the n vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ in the n-space, $\mathbb{R}^n$. If every vector in $\mathbb{R}^n$ can be produced by a linear combination of these vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ then we say these vectors **span** or **generate** the n-space, $\mathbb{R}^n$.

This set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is called the **spanning set**. We also say that the set S spans the n-space or S spans $\mathbb{R}^n$.

For example, the standard unit vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span $\mathbb{R}^2$ because $k\mathbf{e}_1$ spans the x axis and $c\mathbf{e}_2$ spans the y axis. Hence, by introducing scalars, k and c, the linear combination, $k\mathbf{e}_1 + c\mathbf{e}_2$, of these vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can produce any vector in $\mathbb{R}^2$.

Basis

Given some vectors we can generate others by a linear combination. We need *just* enough vectors to build all other vectors from them through linear combination. This set of just enough vectors is called a basis.

An example is the standard unit vectors $\mathbf{e}_1 = (1 \;\; 0)^T$, $\mathbf{e}_2 = (0 \;\; 1)^T$ for $\mathbb{R}^2$. This is the basis which forms the x and y axes of $\mathbb{R}^2$ because $\mathbf{e}_1 = (1 \;\; 0)^T$ specifies the x direction and $\mathbf{e}_2 = (0 \;\; 1)^T$ specifies the y direction.

Each additional basis vector introduces a new direction.

Definition (2.26). Consider the n vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ in the n space, $\mathbb{R}^n$.

These vectors form a **basis** for $\mathbb{R}^n \Leftrightarrow$

(i) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ span $\mathbb{R}^n$ and

(ii) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ are linearly independent

Basis

We can write the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ and $\mathbf{v}_n$ as a set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. These are called the **basis vectors** – independent vectors which span $\mathbb{R}^n$. Any vector in $\mathbb{R}^n$ can be constructed from the basis vectors.

Bases (plural of basis) are the most efficient spanning sets. There are many sets of vectors that can span a space. However, in these sets some of the vectors might be redundant in spanning the space (because they can be 'made' from the other vectors in the set). A basis has no redundant vectors. This is exactly what is captured by demanding linear independence in the definition.

Vector Spaces

- Let V be a non-empty set of elements called vectors. We define two operations on the set V– vector addition and scalar multiplication. Scalars are real numbers.
- Let u, v and w be vectors in the set V. The set V is called a vector space if it satisfies the following 10 axioms.

1. The vector addition u + v is also in the vector space V. Generally in mathematics we say that we have closure under vector addition if this property holds.
2. Commutative law: u + v = v + u.

Vector Spaces

3. Associative law: (u + v) + w = u + (v + w).
4. Neutral element. There is a vector called the zero vector in V denoted by O which satisfies

$$\mathbf{u} + \mathbf{O} = \mathbf{u} \text{ for every vector } \mathbf{u} \text{ in } V$$

5. Additive inverse. For every vector u there is a vector –u (minus u) which satisfies the following:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{O}$$

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6. Let k be a real scalar then ku is also in V.
7. Associative law for scalar multiplication. Let *k* and *c* be real scalars then

$$k(c\mathbf{u}) = (kc)\mathbf{u}$$

8. Distributive law for vectors. Let *k* be a real scalar then

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\,\mathbf{v}$$

9. Distributive law for scalars. Let *k* and *c* be real scalars then

$$(k + c)\mathbf{u} = k\mathbf{u} + c\mathbf{u}$$

10. Identity element. For every vector **u** in *V* we have

$$1\mathbf{u} = \mathbf{u}$$

Vector Subspaces

Let V be a vector space and S be a non-empty subset of V. If the set S satisfies all 10 axioms of a vector space with respect to the same vector addition and scalar multiplication as V then S is also a vector space. We say S is a subspace of V.

Definition (3.4). A non-empty subset S of a vector space V is called a **subspace** of V if it is also a vector space with respect to the same vector addition and scalar multiplication as V.

We illustrate this in Fig. 3.3.

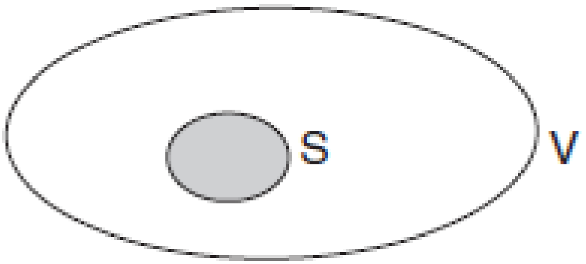

Figure 3.3

Note the difference between subspace and subset. A subset is merely a specific set of elements chosen from V. A subset must also satisfy the 10 axioms of vector space to be called a subspace.

Vector Subspaces

Proposition (3.5). Let S be a non-empty subset of a vector space V. Then S is subspace of V $\Leftrightarrow$:

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the set S then the vector addition $\mathbf{u}+\mathbf{v}$ is also in S.

(b) If $\mathbf{u}$ is a vector in S then for every scalar k we have, $k\mathbf{u}$ is also in S.

Note that this proposition means that we must have *closure* under both vector addition and scalar multiplication. This means that S is a subspace of V $\Leftrightarrow$ both the following are satisfied, as shown in Fig. 3.5.