Tutorial 11

Vector Spaces & Subspaces

Definition: A **Vector space** $(V,\oplus,\odot,\mathrm{F})$ over the field F is a nonempty set V together with two algebraic operations called vector addition and multiplication of vectors by scalars which satisfy the following conditions:

A. Vector addition ($\oplus$):

A.1 $x \oplus y \in V, \quad \forall x, y \in V$

A.2 $x \oplus y = y \oplus x, \ \forall\, x, y \in V$

A.3 $x \oplus (y \oplus z) = (x \oplus y) \oplus z, \ \forall\, x, y, z \in V$

A.4 $\exists\, 0_v \in V$ (zero vector) such that $x \oplus 0_v = x, \ \forall x \in V$

A.5 $\forall x \in V, \ \exists\, -x \in V$ (additive inverse of x) such that $x \oplus (-x) = 0_v$.

B. Multiplication by scalars:

B.1 $\propto \odot\, x \in V, \quad \forall \propto \in \mathrm{F}, \ \forall\, x \in V$

B.2 $\alpha \odot (\beta \odot x) = (\alpha\beta) \odot x, \ \forall\, \alpha, \beta \in \mathrm{F}, \ \forall\, x \in V$

B.3 $\exists\, 1 \in F$ (unit element of F) such that $1 \odot x = x, \forall x \in V$.

B.4 $\alpha \odot (x \oplus y) = \alpha \odot x \oplus \alpha \odot y, \ \forall \alpha \in \mathrm{F}, \ x, y \in V$

B.5 $(\alpha + \beta) \odot x = \alpha \odot x \oplus \beta \odot x, \ \forall \alpha, \beta \in \mathrm{F}, \ \forall x \in V$.

1. The set $\mathbb{R}$ of real numbers, with the usual addition and multiplication (i.e., $\oplus \equiv +$ and $\odot \equiv \cdot$) forms a vector space over $\mathbb{R}$.

2. Consider the set $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ of complex numbers itself is a vector space over $\mathbb{R}$. Where $i = \sqrt{-1}$.

3. Consider the set $S = \{(\mathrm{x}_1, \mathrm{y}_1) \colon \mathrm{x}_1, \mathrm{y}_1 \in \mathbb{R}\}$ with the following non-standard operations of addition and scalar multiplication: $(\mathrm{x}_1, \mathrm{y}_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + \mathrm{y}_2 - 1)$, and $\alpha \odot (x_1, y_1) = (\alpha x_1, \alpha y_1)$ Show that is a vector space with these operations. Hint: the zero vector is not (0, 0), but (1, 1).

Definition: A subspace of a vector space $(V, \oplus, \odot, \mathrm{F})$ is a nonempty subset Y of V such that $(Y, \oplus, \odot, \mathrm{F})$ is also a vector space.

Lemma 1.6.2.1 (Sub Space Test): Let V be a vector space over the field F, and let $W \subseteq V$ be a subset of V . Then W is a subspace of V if and only if

1. $\mathbf{0}_V \in W$ (the zero vector of V must be in W),
2. if $x, y \in W$, then $x + y \in W$ (i.e., W is closed under vector addition), and
3. if $x \in W$ and $c \in F$, then $cx \in W$ (i.e., W is closed under scalar multiplication).

Determine whether or not the following subsets of the vector space $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$:

a. $\{[a \quad b \quad c \quad d]^T : a + b = c + d\}$;

b. $\{[a \quad b \quad c \quad d]^T : a + b = 1\}$;

c. $\{[a \quad b \quad c \quad d]^T : a^2 + b^2 = 0\}$;

d. $\{[a \quad b \quad c \quad d]^T : a^2 + b^2 = 1\}$.

4. Are vectors $\mathrm{u} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$, $\mathrm{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and $\mathrm{w} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$ a linear combination of $\mathrm{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathrm{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathrm{u}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, and $\mathrm{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$?

5. Describe the span of the vectors $\mathrm{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

6. Let $\mathrm{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathrm{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, and $\mathrm{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Show that Span{u,v,w} = Span{u,v}.

Linear Transformation

7. Find the new vector *v* after rotating the $\mathrm{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ by $\theta = 60°$ degrees. Where v = Tu. Find T and hence find v.

8. Find the new vector *v* after scaling the $u = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ by magnitudes of 3, -2, 2 on x, y and z directions respectively. Where v = Tu. Find T and hence find v.
9. Find the reflection vectors of $u = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ on x and y axes respectively. Where v = Tu. Find T and hence find v.
10. Find the reflection vectors of $u = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$ on xy, yz and zx axes respectively. Where v = Tu. Find T and hence find v.