Lesson 3: Arguments

Topics to be covered.

- Valid Arguments in Propositional Logic
- Rules of Inference for propositional logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Combining Rules of Inference for Propositions and Quantified Statements

Consistency

- A set of statements is logically consistent if they can all be true at the same time. A set of statements is logically inconsistent if they cannot all be true at the same time.

Example

Example: The following statements are logically inconsistent.

All men have blonde hair.

I am a man.

I have brown hair.

- *Answer: These three beliefs are logically inconsistent. If the first two statements are true, the third must be false. If the third is true, the first or second must be false. They cannot all be simultaneously true.*

Valid Argument

An **argument** in propositional logic is a sequence of propositions. All proposition except the last in are called **premises** and the final proposition is called the **conclusion**.
An argument is said to be valid if the truth of all its premises implies that the conclusion is true.

An **argument form** in propositional logic is a collection of propositions involving **propositional variables**. An **argument form** is valid no matter which particular propositions are substituted for the propositional variables in its premises and conclusion, the conclusion is true if the premises are all true.

Example

Consider the following argument:
"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

Let p be the proposition "You have a current password." and let q be "You can log onto the network."
Then the argument has the form

$$p \to q$$
$$p$$
$$\overline{\quad\quad\quad\quad}$$
$$\therefore \quad q.$$

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

This is a valid argument

Rules of Inference

Rule of Inference	Name
p p → q ----------- ∴ q	Modus ponens
¬q p → q ---------- ∴ ¬p	Modus tollens
p → q q → r ------------ ∴ p → r	Hypothetical syllogism
p ∨ q ¬p ---------- ∴ q	Disjunctive syllogism

Rule of Inference	Name
p ---------- ∴ p ∨ q	Addition
p ∧ q --------- ∴ p	Simplification
p q ---------- ∴ p ∧ q	Conjunction
p ∨ q ¬ p ∨ r ------------ ∴ q ∨ r	Resolution Prove this is a valid argument?????

How to create a truth table with all possible truth values

p	**q**	**r**	**p ∨ q**	**¬p ∨ r**	**q ∨ r**
T					
T					
T					
T					
F					
F					
F					
F					

How to create a truth table with all possible truth values

p	q	r	p V q	¬p V r	q V r
T	T				
T	T				
T	F				
T	F				
F	T				
F	T				
F	F				
F	F				

How to create a truth table with all possible truth values

p	q	r	p V q	¬p V r	q V r
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

How to create a truth table with all possible truth values

p	q	r	p V q	¬p V r	q V r
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	F	T	T
F	F	F	F	T	F

Example of a Valid Argument

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Let p – " You send me an e-mail message", q – "I will finish writing the program",
r – " I will go to sleep early", and s - "I will wake up feeling refreshed."

The argument form is:

$$p \to q$$
$$\neg p \to r$$
$$r \to s$$
$$\overline{\qquad\qquad}$$
$$\therefore \neg q \to s$$

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Example

Show that the following argument is valid:

If today is a Friday, then I have a test in CS or a test in Mathematics. If my Mathematics lecturer is sick, then I will not have a test in Mathematics. Today is Friday and my Mathematics lecturer is sick.
Therefore, I have a test in CS.

Let p – "Today is a Friday", q – "I have a test in CS ",
r – " I have a test in Mathematics" s - "My Mathematics lecturer is sick."

The argument form is:

$$p \rightarrow (q \vee r)$$
$$s \rightarrow \neg r$$
$$p \wedge s$$
$$\overline{\therefore \quad q}$$

Example Continued

The argument form is:

$$p \to (q \vee r)$$
$$s \to \neg r$$
$$p \wedge s$$
$$\overline{\quad\quad\quad\quad}$$
$$\therefore \quad q$$

Step	Reason
1. $p \wedge s$	Premise
2. s	Simplification from (1)
3. p	Simplification from (1)
4. $s \to \neg r$	Premise
5. $\neg r$	Modus ponens from (2) & (4)
6. $p \to (q \vee r)$	Premise
7. $q \vee r$	Modus ponens from (3) & (6)
8. q	Resolution from (5) & (7)

Hence the argument is valid.

Exercise

Show that the following argument is valid:

Gary is intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

Fallacy of affirming the consequence

Is the following argument valid?
If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
Therefore, you did every problem in this book.

Let p and q be two propositions "You did every problem in this book" and "You learned discrete mathematics" respectively. Then this argument is of the form:

$$\begin{array}{l} p \rightarrow q \\ \quad q \\ \hline \therefore p \end{array}$$

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

This is **not a valid** argument, since there is a situation where both premises $p \rightarrow q$ & q are true but the conclusion p is false.

Fallacy of denying the antecedent

The following argument form is **not valid**

$$\begin{array}{l} p \rightarrow q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

p	q	$p \rightarrow q$	¬p	¬q
F	F	T	T	T
F	T	T	T	F
T	F	F	F	T
T	T	T	F	F

This is **not a valid** argument, since there is a situation where both premises $p \rightarrow q$ & $\neg p$ are true but the conclusion $\neg q$ is false.

Rules of Inference for Quantified Statements

Rule of Inference	Name
$\dfrac{\forall x\, P(x)}{\therefore P(c)}$	**Universal Instantiation**
$\dfrac{P(c) \text{ for an arbitary } c}{\therefore \forall x\, P(x)}$	**Universal generalization**
$\dfrac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$	**Existential instantiation**
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$	**Existential generalization**

Example

Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the exam has not read the book.”

Solution:
Let C(x) be “x is in this class,”
B(x) be “x has read the book,” and
P(x) be “x passed the first exam.”

Then the argument form is:

$$\begin{array}{l} \exists x\,(C(x) \land \neg B(x)) \\ \underline{\forall x\,(C(x) \rightarrow P(x))} \\ \therefore\ \exists x\,(P(x) \land \neg B(x)) \end{array}$$

Step	Reason
1. $\exists x\,(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x\,(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction of (6) and (7)
9. $\exists x\,(P(x) \land \neg B(x))$	Existential generalization from (8)

Example

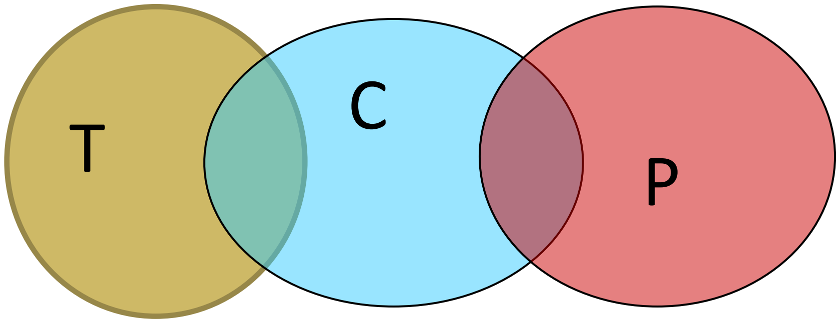
Test the validity of the following argument:

Some trigonometric functions are continuous.
Some continuous functions are periodic.
Therefore, Some trigonometric functions are periodic.

T(x) – x is a trigonometric function
C(x) – x is a continuous function
P(x) – x is a periodic function.

$\exists x\,(T(x)\ \wedge C(x))$
$\exists x\,(C(x)\ \wedge P(x))$

$\therefore\ \exists x\,(T(x)\ \wedge P(x))$



Under this interpretation conclusion is not true.
Hence, this argument is not valid.

Example Continued

Let a be an arbitrary person from the world (Universe).

$\forall x\ [B(x) \rightarrow I(x)]$
$\forall x\ [I(x) \rightarrow D(x)]$
$\neg\ \exists x\ [M(x) \wedge D(x)]$

$\therefore\ \ \forall x[B(x) \rightarrow \neg\ M(x)]$

Step	Reason
1. $\neg\ \exists x\ [M(x) \wedge D(x)]$	Premise
2. $\forall x\ \neg\ [D(x) \wedge M(x)]$	Equivalent to (1)
3. $\forall x\ [\neg\ D(x) \vee \neg M(x)]$	De Morgan's Law
4. $\forall x\ [D(x) \rightarrow \neg M(x)]$	Equivalent to (1)
5. $[B(a) \rightarrow I(a)]$	Universal Instantiation of 1st Premise
6. $[I(a) \rightarrow D(a)]$	Universal Instantiation of 2nd Premise
7. $[D(a) \rightarrow \neg\ M(a)]$	Universal Instantiation of (4)
8. $[B(a) \rightarrow D(a)]$	Hypothetical syllogism of (5) and (6)
9. $[B(a) \rightarrow \neg\ M(a)]$	Hypothetical syllogism of (8) and (7)
10. $\forall x[B(x) \rightarrow \neg\ M(x)]$	Universal generalization of (9).

Another Example

Test the validity of the argument:
Anyone performs well is either intelligent or a good actor.
If someone is intelligent, then he/she can count from 1 to 10.
Gary performs well.
Gary can only count from 1 to 3.
Therefore, not everyone is both intelligent and a good actor.

P(x): x performs well
I(x): x is intelligent
A(x): x is a good actor
C(x): x can count from 1 to 10

$$\forall x(P(x) \rightarrow (I(x) \vee A(x))$$
$$\forall x(I(x) \rightarrow C(x))$$
$$P(\text{Gary})$$
$$\neg C(\text{Gary})$$

$$\therefore \neg \forall x(I(x) \wedge A(x))$$

Solution

$\forall x(P(x) \rightarrow (I(x) \vee A(x)))$
$\forall x(I(x) \rightarrow C(x))$
$P(\mathrm{G})$
$\neg C(\mathrm{G})$

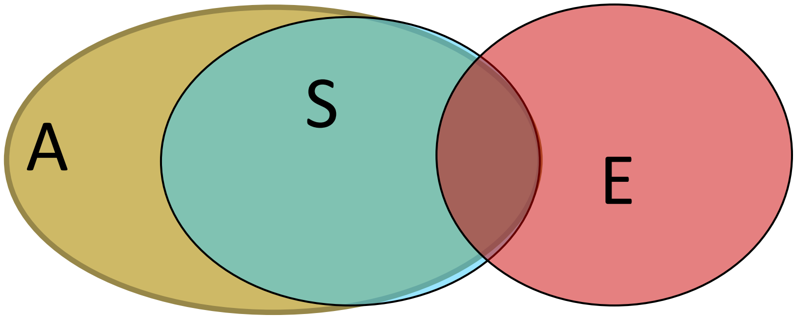
$\therefore \neg \forall x(I(x) \wedge A(x))$

Step	Reason
1. $\forall x\ (P(x) \rightarrow (I(x) \vee A(x)))$	Premise
2. $P(G) \rightarrow (I(G) \vee A(G))$	Universal instantiation from (1)
3. $P(G)$	Premise
4. $I(G) \vee A(G)$	Modus ponens from (2) and (3)
5. $\forall x\ (I(x) \rightarrow C(x))$	Premise
6. $I(G) \rightarrow C(G)$	Universal instantiation from (5)
7. $\neg\ C(\mathrm{G})$	Premise
8. $\neg\ I(G)$	Modus tollens from (6) and (7)
9. $\neg\ I(G) \vee \neg\ A(G)$	Addition from (8)
10. $\neg\ (I(G) \wedge A(G))$	De Morgan's Law
11. $\exists x \neg\ (I(x) \wedge A(x))$	Existential generalization from (10)
12. $\neg\ \forall x\ (I(x) \wedge A(x))$	Equivalent to (11).

Example

Test the validity of the following argument:
Some Scientists are not Engineers.
Some Astronauts are not Engineers.
Hence, some Scientists are not Astronauts.

Let E(x) – "x is an Engineer"
A(x) – "x is an Astronaut"
S(x) – "x is a Scientist"



Argument form:

$\exists x\,(S(x) \wedge \neg E(x))$

$\exists x\,(A(x) \wedge \neg E(x))$

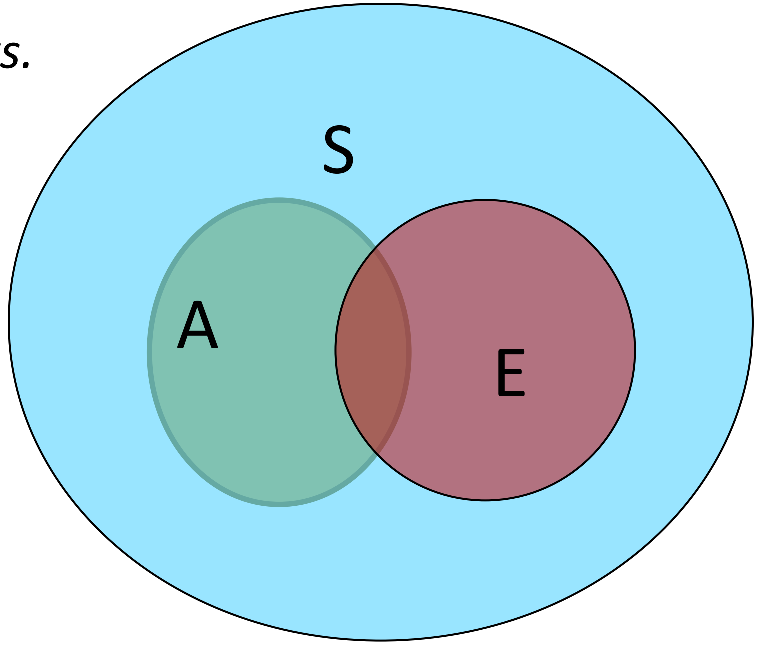
$\therefore\ \exists x\,(\neg A(x) \wedge S(x))$

Under this interpretation conclusion is not true.
Hence, this argument is not valid.

Example

Test the validity of the following argument:
All Astronauts are Scientists.
Some Astronauts are not Engineers.
Hence, some Engineers. are not Scientists.

Let E(x) – "x is an Engineer"
A(x) – "x is an Astronaut"
S(x) – "x is a Scientist"



Argument form:

$\forall x \left(A(x) \ \rightarrow S(x)\right)$

$\exists x \left(A(x) \ \wedge \neg\, E(x)\right)$

$\therefore\ \exists x \left(E(x) \ \wedge \neg\, S(x)\right)$

Under this interpretation conclusion is not true.
Hence, this argument is not valid.

Resolvent

- A variable or negation of a variable is called a literal.

- A clause (sum) is a disjunction of literals.

- For any two clauses C_1 and C_2, if there is a literal p_1 in C_1 that is complementary to the literal p_2 in C_2 then delete p_1 and p_2 from C_1 and C_2 respectively, construct the disjunction of the remaining clauses. The constructed clause is the **resolvent** of C_1 and C_2.

- **Example:** Let $C_1 = p \vee q \vee \neg s$ and $C_2 = \neg p \vee u$. Then the resolvent of C_1 and C_2 is $q \vee \neg s \vee u$.

Theorem

Theorem: Given the two clauses C_1 and C_2**,** a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.

$C_1 = \ \ p \lor \ C_1'$
$C_2 = \neg p \lor \ C_2'$

$\therefore \ \ C_1' \lor \ C_2'$

Modus Ponens

p		p
p → q	≡	$\neg p \lor q$
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∴ q		∴ q

Resolution Principle

Given a set S of clauses, a **reduction (resolution)** of C from S is a finite sequence $C_1, C_2, \cdots, C_k$ of clauses such that each C_i either a clause in S or resolvent of clauses preceding C_i; and $C_k = C$.
A deduction of $\square$ (empty clause) from S is called a **refutation** or a **proof** of S.

Modus Tollens

$\neg q$

$p \to q$

$\therefore \neg p$

$\equiv$

$\neg q$

$\neg p \lor q$

$\therefore \neg p$

Proof of Modus Tollens

$\neg q$

$\neg p \lor q$

p

$\therefore \square$

Example (Revisit)

Consider the valid argument:

$$p \to (q \lor r)$$
$$s \to \neg r$$
$$p \land s$$
$$\overline{\qquad\qquad}$$
$$\therefore\ q$$

$\equiv$

$$\neg p \lor q \lor r$$
$$\neg s \lor \neg r$$
$$p$$
$$s$$

$$\therefore q$$

Step	Reason
1. $\neg \mathbf{p} \lor \mathbf{q} \lor \mathbf{r}$	Premise
2. $\mathbf{p}$	Premise
3. $\mathbf{q} \lor \mathbf{r}$	Resolution from (1) & (2)
4. $\neg \boldsymbol{s} \lor \neg \boldsymbol{r}$	Premise
5.. $\neg \boldsymbol{s} \lor \boldsymbol{q}$	Resolution from (3) & (4)
6. s	Premise
7. q	Resolution from (5) & (6)

Applications of Resolution Principle:

1. This is very much used in the logic programing language PROLOG.
2. It is used in artificial intelligence applications.