Computer Systems

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Forms of Boolean Algebraic Expressions

Boolean Expressions

- There can be multiple boolean expressions for a single operational behavior (single truth table).
- Thus, there can be multiple circuit structures with different logic gate arrangements for the same behavior.
- Simple and minimal circuit structure is always preferred.
    - Efficiency
    - Cost

Simple Expression

- Boolean algebra helps to simplify expressions.
  - $F1 = \bar{x}\bar{y}z + \bar{x}yz + x\bar{y}$
  - $F2 = x\bar{y} + \bar{x}z$

x	y	z	$F2$	$F1$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0

- You will get two circuits if you draw circuits for F1 and F2.

Expression to Truth Table

- When a boolean expression is given, we can formulate the truth table for the expresssion.
- Identify the variables in the expression
  - If there are n variables, then there will be 2^n combinations of values
  - 2^n rows in the turth table
- Derive the resulting value for each value combination in the truth table rows.

Expression to Truth Table. (Cont.)

- Ex.

$$F(x, y, z) = x.y.\bar{z} + \bar{y}.z$$

  - Three variables in $F(x, y, z)$
  - There will be 2^3 combinations
  - There will be 2^3 rows in the truth table

Expression to Truth Table. (Cont.)

- Ex.

$$F(x, y, z) = x.y.\bar{z} + \bar{y}.z$$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Truth Table to Expression

- When a truth table is given, we can formulate the boolean algebraic expression to represent it.
- There are two forms ($Canonical$ forms) in presenting the expression.
  - Sum of *Minterms* (Standard Products)
  - Product of *Maxterms* (Standard Sums)

Minterm

- In a three variable boolean expression, we can combine 3 variables with an AND operator.
- There are 8 possible such arrangements.
  - $\bar{x}.\bar{y}.\bar{z}$
  - $\bar{x}.\bar{y}.z$
  - ....
  - $x.y.z$
- Each of these AND terms is called ***Minterm*** (or Standard Product).
- E.g. - $(x.y)$ is **not** a ***Minterm*** in this context.

Maxterm

- In a three variable boolean expression, we can combine 3 variables with an OR operator.
- There are 8 possible such arrangements.
  - $\bar{x} + \bar{y} + \bar{z}$
  - $\bar{x} + \bar{y} + z$
  - ……………
  - $x + y + z$
- Each of these OR terms is called ***Maxterm*** (or Standard Sum)
- E.g. - $(\bar{y} + z)$ is not a ***Maxterm*** in this context.

Thank You..

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Canonical Forms & Standard Forms

[Sum – Product] in Mathematics

A **product** is the result of multiplying, or an expression that identifies factors to be multiplied.

A **sum** is the aggregate of two or more numbers or particulars as determined by or as if by the mathematical process of addition.

Sum of Minterms

- Disjunction of terms where each term is a conjunction of literals.
- The OR operations are performed on the terms that are made by AND operations.
- Ex.

$$F = \bar{x}.y.\bar{z} + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.\bar{z} + x.y.z$$

- Each term in the expression is referred as ***Minterm*** or ***Standard Product***.

Product of Maxterms

- Conjunction of terms where each term is a disjunction of literals.
- The AND operations are performed on the terms that are made by OR operations.
- Ex.

$$F = (x + y + z).(x + y + \bar{z}).(x + \bar{y} + \bar{z})$$

- Each term in the expresssion is referred as ***Maxterm*** or ***Standard Sum***.

Standard Forms

- In ***Cannonical*** Form, each term must contain all the variables in the truth table.
- But ***Standard Form*** is an alternative way to express Boolean function.
- A term in an expression that is expressed in a standard form doesn’t need to contain all the variables in the truth table.
- There are two standard forms.
    - Sum of Products
    - Product of Sums

Standard Form – Sum of Products

- OR operation is performed over several terms where each term is made up by performing AND operation over several literals.
- Ex.

$$F = \bar{x}.y + \bar{y}.\bar{z} + x.\bar{y}.z + y$$ ✓

$$F = \bar{x}.y + \bar{y}.(x + \bar{z}) + x.\bar{y}.z + y$$ ✗

Standard Form - Product of Sums

- AND operation is performed over several terms where each term is made up by performing OR operation over several literals.
- Ex.

$$F = y.(x + z).(\bar{x} + \bar{y} + \bar{z})$$

$$F = y.(x + z.\bar{y}).(\bar{x} + \bar{y} + \bar{z})$$

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Truth Table to Canonical Forms

A Row of a Truth Table

- A row of a truth table represents a possible state of the expression.
- The expression cannot take two states simultaniously.
- Ex. Consider the 3rd row
  - It says the expressions results '1' when $x = 0$ and $\mathrm{y} = 1$ and $\mathrm{z} = 0$
  - Note the Conjuntion (AND)
  - Thus, it can be formalized in to $\bar{x}.y.\bar{z}$

x	y	z	F
0	0	0	0
0	0	1	0
0	**1**	**0**	**1**
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Truth Table to Expression

- All such rows that result ‘1’ say the expression will be ‘1’ in either case
- The expression will be ‘1’ when the variables are set as rows 3 **or** 5 **or** 6 or 7 **or** 8.

- Note the Disjunction (OR)
- Thus, it can be formalized as follows:

$$\boldsymbol{F = \bar{x}.y.\bar{z} + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.\bar{z} + x.y.z}$$

Expression in “**Sum of Minterms**” Form

x	y	z	F
0	0	0	0
0	0	1	0
0	**1**	**0**	**1**
0	1	1	0
1	**0**	**0**	**1**
1	**0**	**1**	**1**
1	**1**	**0**	**1**
1	**1**	**1**	**1**

Terms in Disjunctive Form

- If we take the 3rd row, we understood that it represents $\mathrm{F}_3 = \bar{x}.y.\bar{z}$
- When $x = 0$ and $\mathrm{y} = 1$ and $\mathrm{z} = 0$, F_3 results 1
- If one of $\bar{x}$ or y or $\bar{z}$ becomes 0, then F_3 will result 0.
- $\bar{x} = 0 \quad OR \quad y = 0 \quad OR \quad \bar{z} = 0$ will result the complement of F_3
- Therefore, complement of F_3 can be written in disjuctive form.
- $\overline{F_3} = x + \bar{y} + z$
- $\therefore F_3 = \overline{(x + \bar{y} + z)}$

De Morgan's Theorem

x	y	z	F
0	0	0	0
0	0	1	0
0	**1**	**0**	**1**
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

De Morgan's Theorem

- We have proved that,

$$\bar{x}.y.\bar{z} = \overline{(x + \bar{y} + z)}$$

- Similarly, we can prove that,

$$\overline{(x.\bar{y}.z)} = \bar{x} + y + \bar{z}$$

- This is a theorem in Boolean Algebra and applicable for any number of variables.

Expression in Conjunctive Form

- We derived the exprssion of the truth table as follows:

$$F = \bar{x}.y.\bar{z} + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.\bar{z} + x.y.z$$

- This means when at least one of those terms become 1 then F will result 1.
- It implies when all the terms in F are **negated** at the same time, it will result 0.
- $\overline{(\bar{x}.y.\bar{z})}.\overline{(x.\bar{y}.\bar{z})}.\overline{(x.\bar{y}.z)}.\overline{(x.y.\bar{z})}.\overline{(x.y.z\ )} = \bar{F}$

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

PoM - Representation

- Similarly, $\bar{F}$ can be written as $\bar{F} = \bar{x}.\bar{y}.\bar{z} + \bar{x}.\bar{y}.z + \bar{x}.y.z$
- When all the term in $\bar{F}$ are negated, then it will result $\bar{\bar{F}}$
- $\bar{\bar{F}} = \overline{(\bar{x}.\bar{y}.\bar{z})}.\overline{(\bar{x}.\bar{y}.z)}.\overline{(\bar{x}.y.z)}$
- As we studied in the slide "De Morgan's Theorem", each term in $\bar{\bar{F}}$ can be written in disjunctive form.
- $\bar{\bar{F}} = (x + y + z).(x + y + \bar{z}).(x + \bar{y} + \bar{z})$ where $\bar{\bar{F}} = F$
- $F = (x + y + z).(x + y + \bar{z}).(x + \bar{y} + \bar{z})$

Expression in "**Product of Maxterms**" Form

x	y	z	F
0	**0**	**0**	**0**
0	**0**	**1**	**0**
0	1	0	1
0	**1**	**1**	**0**
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Truth Table to Expression - Notes

- For a given truth table, there can be a logical expression in two cannonical forms,
  - Sum of Minterms
  - Product of Maxterms
- Conversion of expressions in SoM to PoM and vice versa can be done.

Thank You..

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Canonical Forms to Truth Table

SoM Expression to Truth Table

- Each Minterm is associated with a row
  - Find the row corresponds to each Minterm
  - Fill the result column with ‘1’
- Fill all the result column for all the remaining rows with ‘0’
- Ex. $F = \bar{x}.\bar{y} + x\bar{y} + x.y$

x	y	F
0	0	1
0	1	0
1	0	1
1	1	1

PoM Expression to Truth Table

- **Intution**: If a single Maxterm becomes zero then the entire expression would result zero.
- Literals in a Maxterm are combined with OR operation. Ex. $(x + y + z)$.
- All literals in a term are inverted and combined with AND operation $(\bar{x}.\bar{y}.\bar{z})$, would give the logical complement of the same term (De Morgan's Thrm).
- Then we can map each product term to a row in the truth table.

PoM Expression to Truth Table (Cont.)

- Negate each literal in a Maxterm.
- Assuming logical AND over all literals, select the row.
  - Fill the result column with ‘0’
- Fill the result column for all the remaing rows with ‘1’.

PoS Expression to Truth Table (Ex.)

- $F = (x + y + z).(x + y + \bar{z}).(x + \bar{y} + \bar{z})$
- $\bar{F} = (\bar{x}.\bar{y}.\bar{z}) + (\bar{x}.\bar{y}.z) + (\bar{x}.y.z)$

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Exercises

- Derive the logical expression for the given truth table in
  - Sum of Minterm form
  - Product of Maxterms form

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Exercise – Sum of Minterms

- Sum of Minterms

$$F = \bar{x}.\bar{y}.\bar{z} + \bar{x}.\bar{y}.z + \bar{x}.y.\bar{z} + x.\bar{y}.\bar{z} + x.y.\bar{z} + x.y.z$$

$$F(x,y,z) = \sum(0,1,2,4,6,7)$$

Note that row index starts from 0

	x	y	z	F
$\bar{x}.\bar{y}.\bar{z}$	0	0	0	1
$\bar{x}.\bar{y}.z$	0	0	1	1
$\bar{x}.y.\bar{z}$	0	1	0	1
	0	1	1	0
$x.\bar{y}.\bar{z}$	1	0	0	1
	1	0	1	0
$x.y.\bar{z}$	1	1	0	1
$x.y.z$	1	1	1	1

Exercise – Product of Maxterms

- Product of Maxterms

$$G = (x + \bar{y} + \bar{z}).(\bar{x} + y + \bar{z})$$

$$G(x, y, z) = \prod(3,5)$$

Note that row index starts from 0

	x	y	z	F
	0	0	0	1
	0	0	1	1
	0	1	0	1
$x + \bar{y} + \bar{z}$	0	1	1	0
	1	0	0	1
$\bar{x} + y + \bar{z}$	1	0	1	0
	1	1	0	1
	1	1	1	1

Homework..!

- Practice SoM, PoM
  - Truth table to SoM and PoM
  - When given SoM or PoM, construct the truth table
  - Conversion of SoM to PoM and PoM to SoM
- Be familiar with De Morgan's Theorem
- Circuit construction
  - Construct the circuit when the expression is given
  - Construct the expression when the circuit is given

Thank You..!