SCS 1307
Probability & Statistics
Tutorial Solutions

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Contents

1 Tutorial 1: Basic Probability

Question 1: Card Selection

Question

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card is:

(a) a club or a diamond

(b) a club or a king

Solution

Part (a): A club or a diamond
In a standard deck:

- Number of clubs = 13
- Number of diamonds = 13
- These are mutually exclusive events (a card cannot be both)

$$\begin{aligned} P(\text{Club or Diamond}) &= P(\text{Club}) + P(\text{Diamond}) \\ &= \frac{13}{52} + \frac{13}{52} \\ &= \frac{26}{52} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

Part (b): A club or a king

- Number of clubs = 13
- Number of kings = 4
- Number of cards that are both club AND king = 1 (King of clubs)

Using the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned} P(\text{Club or King}) &= P(\text{Club}) + P(\text{King}) - P(\text{Club and King}) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \boxed{\frac{4}{13}} \end{aligned}$$

Question 2: Two Dice

Question

Two dice are thrown together. Find the probability of obtaining:

(a) two fours

(b) a three and a number greater than 3

Solution

Total possible outcomes when throwing two dice = $6 \times 6 = 36$

Part (a): Two fours
Only one outcome satisfies this: (4, 4)

$$P(\text{Two fours}) = \frac{1}{36} = \boxed{0.0278}$$

Part (b): A three and a number greater than 3
Favorable outcomes:

- First die shows 3, second die shows 4, 5, or 6: (3,4), (3,5), (3,6) $\rightarrow$ 3 outcomes
- First die shows 4, 5, or 6, second die shows 3: (4,3), (5,3), (6,3) $\rightarrow$ 3 outcomes

Total favorable outcomes = 6

$$\begin{aligned} P(3 \text{ and number} > 3) &= \frac{6}{36} \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

Question 3: Events A and B

Question

Events A and B are such that $P(A) = \frac{19}{30}$, $P(B) = \frac{2}{5}$, and $P(A \cup B) = \frac{4}{5}$. Find $P(A \cap B)$.

Solution

Using the addition rule for probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Rearranging to solve for $P(A \cap B)$:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\begin{aligned} P(A \cap B) &= \frac{19}{30} + \frac{2}{5} - \frac{4}{5} \\ &= \frac{19}{30} + \frac{12}{30} - \frac{24}{30} \\ &= \frac{19 + 12 - 24}{30} \\ &= \frac{7}{30} \\ &= \boxed{\frac{7}{30}} \end{aligned}$$

Question 4: Die Thrown Twice

Question

A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

Since the two throws are independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

- $P(4 \text{ on first throw}) = \frac{1}{6}$
- $P(\text{Odd on second throw}) = \frac{3}{6} = \frac{1}{2}$ (odd numbers: 1, 3, 5)

$$\begin{aligned} P(4 \text{ first and odd second}) &= \frac{1}{6} \times \frac{1}{2} \\ &= \boxed{\frac{1}{12}} \end{aligned}$$

Question 5: Conditional Probability - Tennis

Question

The probability that it will be sunny tomorrow is $\frac{1}{3}$. If it is sunny, the probability that Susan plays tennis is $\frac{4}{5}$. If it is not sunny, the probability that Susan plays tennis is $\frac{2}{5}$. Find the probability that Susan plays tennis tomorrow.

Solution

Let:

- S = event that it is sunny
- T = event that Susan plays tennis

Given:

- $P(S) = \frac{1}{3}$, therefore $P(S') = 1 - \frac{1}{3} = \frac{2}{3}$
- $P(T|S) = \frac{4}{5}$
- $P(T|S') = \frac{2}{5}$

Using the **Law of Total Probability**:

$$P(T) = P(T|S) \cdot P(S) + P(T|S') \cdot P(S')$$

$$\begin{aligned} P(T) &= \frac{4}{5} \times \frac{1}{3} + \frac{2}{5} \times \frac{2}{3} \\ &= \frac{4}{15} + \frac{4}{15} \\ &= \frac{8}{15} \\ &= \boxed{\frac{8}{15}} \end{aligned}$$

Question 6: Marbles Without Replacement

Question

Suppose a bag contains two red and one blue marble. Select two marbles in order without replacement and observe their colours. Find:

(i) the probability that the first marble drawn is red

(ii) the probability that the second marble drawn is blue

(iii) the probability that one of the marbles drawn is blue

Solution

Setup: Bag contains 2 Red (R) and 1 Blue (B) marble

Part (i): First marble is red

$$P(\text{First is Red}) = \frac{2}{3} = \boxed{\frac{2}{3}}$$

Part (ii): Second marble is blue
Two scenarios:

- First Red, then Blue: $P(R_1 \cap B_2) = \frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$
- First Blue, then... (impossible - only 1 blue): $P(B_1 \cap B_2) = 0$

$$P(\text{Second is Blue}) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

Part (iii): One of the marbles is blue
This is the complement of "both marbles are red":

$$\begin{aligned} P(\text{At least one Blue}) &= 1 - P(\text{Both Red}) \\ &= 1 - \left(\frac{2}{3} \times \frac{1}{2}\right) \\ &= 1 - \frac{1}{3} \\ &= \boxed{\frac{2}{3}} \end{aligned}$$

Alternative method: Direct calculation

- RB: $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$
- BR: $\frac{1}{3} \times \frac{2}{2} = \frac{1}{3}$
- Total: $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

Question 7: Tetrahedral Dice

Question

Two tetrahedral dice, with faces labelled 1, 2, 3 and 4, are thrown and the number on which each lands is noted. The 'score' is the sum of these two numbers. Find the probability that:

(i) the score is even, given at least one die lands on a 3

(ii) at least one die lands on a 3, given that the score is even

Solution

Total possible outcomes = $4 \times 4 = 16$

Part (i): $P(\text{Score is even}|\text{At least one 3})$
Let A = score is even, B = at least one die shows 3
Outcomes where at least one die shows 3:

$$(3,1),\ (3,2),\ (3,3),\ (3,4),\ (1,3),\ (2,3),\ (4,3) \rightarrow 7 \text{ outcomes}$$

Among these, which have even scores?

- (3,1): sum = 4
- (3,3): sum = 6
- (1,3): sum = 4

3 outcomes satisfy both conditions.

$$P(\text{Even}|\text{At least one 3}) = \frac{3}{7} = \boxed{\frac{3}{7}}$$

Part (ii): $P(\text{At least one 3}|\text{Score is even})$
Outcomes where score is even:

$$\begin{array}{ll} \text{Sum} = 2: & (1,1) \\ \text{Sum} = 4: & (1,3),\ (2,2),\ (3,1) \\ \text{Sum} = 6: & (2,4),\ (3,3),\ (4,2) \\ \text{Sum} = 8: & (4,4) \end{array}$$

Total even-score outcomes = 8
Among these, which have at least one 3? (1,3), (3,1), (3,3) $\rightarrow$ 3 outcomes

$$P(\text{At least one 3}|\text{Even}) = \frac{3}{8} = \boxed{\frac{3}{8}}$$

Question 8: Restaurant Orders

Question

In a famous restaurant customers may order any combination of chips, peas and salad to accompany the main course. Given:

- $P(\text{Salad}) = 0.45$
- $P(\text{Peas and Chips}) = 0.19$
- $P(\text{Salad and Peas}) = 0.15$
- $P(\text{Salad and Chips}) = 0.25$
- $P(\text{Salad or Peas}) = 0.6$
- $P(\text{Salad or Chips}) = 0.84$
- $P(\text{Salad or Chips or Peas}) = 0.9$

Find the probability that a customer chooses:

(i) peas

(ii) chips

(iii) all three

(iv) none of these

Solution

Let S = Salad, P = Peas, C = Chips

Part (i): Find $P(P)$
Using: $P(S \cup P) = P(S) + P(P) - P(S \cap P)$

$$\begin{aligned} 0.6 &= 0.45 + P(P) - 0.15 \\ P(P) &= 0.6 - 0.45 + 0.15 \\ P(P) &= \boxed{0.30} \end{aligned}$$

Part (ii): Find $P(C)$
Using: $P(S \cup C) = P(S) + P(C) - P(S \cap C)$

$$\begin{aligned} 0.84 &= 0.45 + P(C) - 0.25 \\ P(C) &= 0.84 - 0.45 + 0.25 \\ P(C) &= \boxed{0.64} \end{aligned}$$

Part (iii): Find $P(S \cap P \cap C)$ (all three)
Using the inclusion-exclusion principle:

$$P(S \cup P \cup C) = P(S) + P(P) + P(C) - P(S \cap P) - P(S \cap C) - P(P \cap C) + P(S \cap P \cap C)$$

$$\begin{aligned} 0.9 &= 0.45 + 0.30 + 0.64 - 0.15 - 0.25 - 0.19 + P(S \cap P \cap C) \\ 0.9 &= 0.80 + P(S \cap P \cap C) \\ P(S \cap P \cap C) &= \boxed{0.10} \end{aligned}$$

Part (iv): None of these

$$\begin{aligned} P(\text{None}) &= 1 - P(S \cup P \cup C) \\ &= 1 - 0.9 \\ &= \boxed{0.10} \end{aligned}$$

2 Tutorial 2: Independence and Conditional Probability

Question 1: Independent Events

Question

Events A and B are such that $P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{4}$. If A and B are independent events, find $P(A \cap B)$, $P(A \cup B')$ and $P(A' \cup B')$.

Solution

Given: $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{4}$, A and B are independent

Find $P(A \cap B)$:
For independent events: $P(A \cap B) = P(A) \times P(B)$

$$P(A \cap B) = \frac{2}{5} \times \frac{1}{4} = \boxed{\frac{2}{20} = \frac{1}{10}}$$

Find $P(A \cup B')$:
First, $P(B') = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$
Since A and B are independent, A and B' are also independent:

$$P(A \cap B') = P(A) \times P(B') = \frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}$$

$$\begin{aligned} P(A \cup B') &= P(A) + P(B') - P(A \cap B') \\ &= \frac{2}{5} + \frac{3}{4} - \frac{3}{10} \\ &= \frac{8}{20} + \frac{15}{20} - \frac{6}{20} \\ &= \boxed{\frac{17}{20}} \end{aligned}$$

Find $P(A' \cup B')$:
Using De Morgan's Law: $P(A' \cup B') = 1 - P(A \cap B)$

$$P(A' \cup B') = 1 - \frac{1}{10} = \boxed{\frac{9}{10}}$$

Question 2: Conditional Probability

Question

Events A and B are such that $P(A) = \frac{2}{3}$, $P(A|B) = \frac{2}{3}$, $P(B) = \frac{1}{4}$. Find $P(B|A)$ and $P(A \cap B)$.

Solution

Find $P(A \cap B)$:
Using the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{2}{3} = \frac{P(A \cap B)}{1/4}$$
$$P(A \cap B) = \frac{2}{3} \times \frac{1}{4}$$
$$= \boxed{\frac{1}{6}}$$

Find $P(B|A)$:
Using Bayes' theorem:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{1/6}{2/3}$$
$$= \frac{1}{6} \times \frac{3}{2}$$
$$= \boxed{\frac{1}{4}}$$

Question 3: At Least Once

Question

Find the probability of a 4 turning up at least once in two tosses of a fair die.

Solution

Method 1: Complement Rule

$$P(\text{At least one 4}) = 1 - P(\text{No 4 in both tosses})$$

- $P(\text{Not 4 on first toss}) = \frac{5}{6}$
- $P(\text{Not 4 on second toss}) = \frac{5}{6}$

$$\begin{aligned} P(\text{At least one 4}) &= 1 - \left(\frac{5}{6}\right)^2 \\ &= 1 - \frac{25}{36} \\ &= \boxed{\frac{11}{36}} \end{aligned}$$

Method 2: Direct Calculation
Favorable outcomes:

- (4, not 4): $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$
- (not 4, 4): $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$
- (4, 4): $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$$P(\text{At least one 4}) = \frac{5}{36} + \frac{5}{36} + \frac{1}{36} = \frac{11}{36}$$

Question 4: Die Tossed Twice

Question

A fair die is tossed twice. Find the probability of getting a 4, 5 or 6 on the first toss and a 1, 2, 3 or 4 on the second toss.

Solution

Since the tosses are independent:

- $P(4, 5, \text{or } 6 \text{ on first toss}) = \frac{3}{6} = \frac{1}{2}$
- $P(1, 2, 3, \text{or } 4 \text{ on second toss}) = \frac{4}{6} = \frac{2}{3}$

$$\begin{aligned} P(\text{Both events}) &= \frac{1}{2} \times \frac{2}{3} \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

Question 5: Two Aces

Question

Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is:

(a) replaced

(b) not replaced

Solution

Part (a): With replacement
Each draw is independent:

- $P(\text{First ace}) = \frac{4}{52}$
- $P(\text{Second ace}) = \frac{4}{52}$

$$\begin{aligned} P(\text{Both aces}) &= \frac{4}{52} \times \frac{4}{52} \\ &= \frac{16}{2704} \\ &= \boxed{\frac{1}{169}} \end{aligned}$$

Part (b): Without replacement

- $P(\text{First ace}) = \frac{4}{52}$
- $P(\text{Second ace}|\text{First ace}) = \frac{3}{51}$

$$\begin{aligned} P(\text{Both aces}) &= \frac{4}{52} \times \frac{3}{51} \\ &= \frac{12}{2652} \\ &= \boxed{\frac{1}{221}} \end{aligned}$$

Question 6: Colored Balls

Question

A ball is drawn at random from a box containing 6 red balls, 4 white balls, and 5 blue balls. Determine the probability that it is:

(a) red

(b) white

(c) blue

(d) not red

(e) red or white

Solution

Total balls = $6 + 4 + 5 = 15$

(a) $P(\text{Red}) = \frac{6}{15} = \boxed{\frac{2}{5}}$

(b) $P(\text{White}) = \frac{4}{15} = \boxed{\frac{4}{15}}$

(c) $P(\text{Blue}) = \frac{5}{15} = \boxed{\frac{1}{3}}$

(d) $P(\text{Not Red}) = 1 - P(\text{Red}) = 1 - \frac{6}{15} = \boxed{\frac{9}{15} = \frac{3}{5}}$

(e) $P(\text{Red or White}) = \frac{6}{15} + \frac{4}{15} = \boxed{\frac{10}{15} = \frac{2}{3}}$

Question 7: Independent Events Problem

Question

If A and B are two independent events with $P(A) < P(B)$, the probability that both A and B occur is $\frac{6}{25}$, and $P(A|B) + P(B|A) = 1$, find the value of $P(A)$.

Solution

Given:

- A and B are independent
- $P(A \cap B) = \frac{6}{25}$
- $P(A|B) + P(B|A) = 1$
- $P(A) < P(B)$

For independent events:

$$P(A \cap B) = P(A) \times P(B) = \frac{6}{25}$$

Also, for independent events:

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

Using the condition:

$$\begin{aligned} P(A) + P(B) &= 1 \\ P(B) &= 1 - P(A) \end{aligned}$$

Substituting into the first equation:

$$\begin{aligned} P(A) \times (1 - P(A)) &= \frac{6}{25} \\ P(A) - P(A)^2 &= \frac{6}{25} \\ P(A)^2 - P(A) + \frac{6}{25} &= 0 \\ 25P(A)^2 - 25P(A) + 6 &= 0 \end{aligned}$$

Using the quadratic formula:

$$P(A) = \frac{25 \pm \sqrt{625 - 600}}{50} = \frac{25 \pm 5}{50}$$

$P(A) = \frac{30}{50} = \frac{3}{5}$ or $P(A) = \frac{20}{50} = \frac{2}{5}$
Since $P(A) < P(B)$ and $P(A) + P(B) = 1$:

- If $P(A) = \frac{2}{5}$, then $P(B) = \frac{3}{5}$ (satisfies $P(A) < P(B)$)
- If $P(A) = \frac{3}{5}$, then $P(B) = \frac{2}{5}$ (violates $P(A) < P(B)$)

Therefore: $P(A) = \boxed{\frac{2}{5}}$

Question 8: Three Shooters

Question

The probabilities that three men hit a target are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$ respectively, and each shoots once at the target.

(a) Find the probability that all three men hit the target.

(b) Find the probability that exactly one of them will hit the target.

Solution

Let A, B, C be the events that men 1, 2, 3 hit the target respectively.
Given:

- $P(A) = \frac{1}{3}$, so $P(A') = \frac{2}{3}$
- $P(B) = \frac{1}{4}$, so $P(B') = \frac{3}{4}$
- $P(C) = \frac{1}{6}$, so $P(C') = \frac{5}{6}$

Part (a): All three hit

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \times P(B) \times P(C) \\ &= \frac{1}{3} \times \frac{1}{4} \times \frac{1}{6} \\ &= \boxed{\frac{1}{72}} \end{aligned}$$

Part (b): Exactly one hits
Three scenarios:

1. Only A hits: $P(A \cap B' \cap C') = \frac{1}{3} \times \frac{3}{4} \times \frac{5}{6} = \frac{15}{72}$
2. Only B hits: $P(A' \cap B \cap C') = \frac{2}{3} \times \frac{1}{4} \times \frac{5}{6} = \frac{10}{72}$
3. Only C hits: $P(A' \cap B' \cap C) = \frac{2}{3} \times \frac{3}{4} \times \frac{1}{6} = \frac{6}{72}$

$$\begin{aligned} P(\text{Exactly one hits}) &= \frac{15}{72} + \frac{10}{72} + \frac{6}{72} \\ &= \frac{31}{72} \\ &= \boxed{\frac{31}{72}} \end{aligned}$$

3 Tutorial 3: Discrete Distributions

Question 1: Binomial Distribution

Question

A coin is biased so that the probability of obtaining a head is $\frac{2}{3}$. The coin is tossed four times. Find the probability of obtaining exactly two heads.

Solution

Let X = number of heads obtained.
$X \sim \text{Binomial}\left(n = 4, p = \frac{2}{3}\right)$
Probability mass function: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
For exactly 2 heads:

$$\begin{aligned} P(X = 2) &= \binom{4}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 \\ &= 6 \times \frac{4}{9} \times \frac{1}{9} \\ &= 6 \times \frac{4}{81} \\ &= \frac{24}{81} \\ &= \boxed{\frac{8}{27}} \end{aligned}$$

Question 2: Quality Control

Question

Suppose that 1% of the items made by a certain machine are defective. To keep a check on the quality of the output, a batch of ten items is inspected occasionally. What is the probability that the next ten items inspected include more than one defective?

Solution

Let X = number of defective items in a batch of 10.
$X \sim \text{Binomial}(n = 10, p = 0.01)$
Find: $P(X > 1)$
Using the complement: $P(X > 1) = 1 - P(X \le 1) = 1 - [P(X = 0) + P(X = 1)]$
Calculate $P(X = 0)$:

$$\begin{aligned} P(X = 0) &= \binom{10}{0}(0.01)^0(0.99)^{10} \\ &= (0.99)^{10} \\ &= 0.9044 \end{aligned}$$

Calculate $P(X = 1)$:

$$\begin{aligned} P(X = 1) &= \binom{10}{1}(0.01)^1(0.99)^9 \\ &= 10 \times 0.01 \times 0.9135 \\ &= 0.0914 \end{aligned}$$

Final answer:

$$\begin{aligned} P(X > 1) &= 1 - (0.9044 + 0.0914) \\ &= 1 - 0.9958 \\ &= \boxed{0.0042} \end{aligned}$$

Question 3: Hypergeometric Distribution

Question

A group of three will be selected out of 5 boys and 7 girls. What is the probability that exactly 2 girls are in the selected group?

Solution

This follows a **Hypergeometric distribution**.
Parameters:

- Total population: $N = 5 + 7 = 12$
- Number of girls: $K = 7$
- Sample size: $n = 3$
- Desired girls in sample: $x = 2$

Formula: $P(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

$$\begin{aligned} P(X = 2) &= \frac{\binom{7}{2}\binom{5}{1}}{\binom{12}{3}} \\ &= \frac{21 \times 5}{220} \\ &= \frac{105}{220} \\ &= \boxed{\frac{21}{44}} \end{aligned}$$

Question 4: Probability Distribution

Question

The number of phone calls, X, received per day by a person has the following probability distribution:

x	0	1	2	3	4	≥ 5
$P(X = x)$	0.24	0.35	$2k$	k	0.05	0

(i) Find the value of k.

(ii) Find the value of $E(X)$.

Solution

Part (i): Find k

Since probabilities must sum to 1: $\sum P(X = x) = 1$

$$\begin{aligned} 0.24 + 0.35 + 2k + k + 0.05 + 0 &= 1 \\ 0.64 + 3k &= 1 \\ 3k &= 0.36 \\ k &= \boxed{0.12} \end{aligned}$$

Part (ii): Find $E(X)$

Complete probability distribution:

x	0	1	2	3	4
$P(X = x)$	0.24	0.35	0.24	0.12	0.05

$$\begin{aligned} E(X) &= \sum x \cdot P(X = x) \\ &= 0(0.24) + 1(0.35) + 2(0.24) + 3(0.12) + 4(0.05) \\ &= 0 + 0.35 + 0.48 + 0.36 + 0.20 \\ &= \boxed{1.39} \end{aligned}$$

Question 5: Binomial Approximation to Poisson

Question

Screws are sold in packets of 15. Faulty screws occur randomly. A large number of packets were checked for faulty screws and the mean number of faulty screws per packet is found to be 1.2.

(i) Show that the variance of the number of faulty screws in a packet is 1.104.

(ii) Find the probability that a packet contains at most 2 faulty screws.

(iii) A person buys 8 packets of screws at random. Find the probability that there are exactly 7 packets in which there is at least 1 faulty screw.

Solution

Let X = number of faulty screws in a packet.
Since faulty screws occur randomly with small probability and $n = 15$: $X \sim \text{Binomial}(n = 15, p)$
Given: $E(X) = np = 1.2$, so $p = \frac{1.2}{15} = 0.08$

Part (i): Show variance = 1.104
For binomial distribution:

$$\begin{aligned}\text{Var}(X) &= np(1-p)\\ &= 15 \times 0.08 \times 0.92\\ &= 1.2 \times 0.92\\ &= 1.104\end{aligned}$$

Part (ii): $P(X \leq 2)$
We can use Poisson approximation since n is large and p is small: $X \approx \text{Poisson}(\lambda = 1.2)$

$$\begin{aligned}P(X \leq 2) &= P(X=0) + P(X=1) + P(X=2)\\ &= e^{-1.2}\left[\frac{(1.2)^0}{0!} + \frac{(1.2)^1}{1!} + \frac{(1.2)^2}{2!}\right]\\ &= 0.3012\,[1 + 1.2 + 0.72]\\ &= 0.3012 \times 2.92\\ &= \boxed{0.8795}\end{aligned}$$

Part (iii): 7 out of 8 packets have at least 1 faulty
First, find $P(\text{at least 1 faulty in a packet})$:

$$\begin{aligned}P(X \geq 1) &= 1 - P(X=0)\\ &= 1 - e^{-1.2}\\ &= 1 - 0.3012\\ &= 0.6988\end{aligned}$$

Let Y = number of packets (out of 8) with at least 1 faulty screw. $Y \sim \text{Binomial}(n = 8, p = 0.6988)$

$$\begin{aligned}P(Y = 7) &= \binom{8}{7}(0.6988)^7(0.3012)^1\\ &= 8 \times 0.0743 \times 0.3012\\ &= \boxed{0.1790}\end{aligned}$$

Question 6: Germination Problem

Question

The germination percentage of a certain seed from plant nursery A is known to be 85% and from plant nursery B is known to be 75%. A farmer purchases 50% of seeds from nursery A and 50% from nursery B and plants them on his land.

(i) What is the probability that a randomly taken seed will be germinated?

(ii) It is found that a seed is germinated, what is the probability that it has been purchased from plant nursery B?

Solution

Let:

- A = seed from nursery A
- B = seed from nursery B
- G = seed germinates

Given:

- $P(A) = 0.5$, $P(B) = 0.5$
- $P(G|A) = 0.85$, $P(G|B) = 0.75$

Part (i): $P(G)$
Using the Law of Total Probability:

$$\begin{aligned} P(G) &= P(G|A) \cdot P(A) + P(G|B) \cdot P(B) \\ &= 0.85 \times 0.5 + 0.75 \times 0.5 \\ &= 0.425 + 0.375 \\ &= \boxed{0.80} \end{aligned}$$

Part (ii): $P(B|G)$ (Bayes' Theorem)

$$\begin{aligned} P(B|G) &= \frac{P(G|B) \cdot P(B)}{P(G)} \\ &= \frac{0.75 \times 0.5}{0.80} \\ &= \frac{0.375}{0.80} \\ &= \boxed{0.46875 \approx 0.469} \end{aligned}$$

Question 7: Continuous Random Variable

Question

The continuous random variable X has probability density function: $f(x) = \begin{cases} kx(5-x) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$ where k is a constant.

(i) Find the value of k

(ii) Find the value of $E(X)$

Solution

Part (i): Find k

For a valid pdf: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\begin{aligned}
\int_0^4 kx(5-x)\,dx &= 1 \\
k\int_0^4 (5x-x^2)\,dx &= 1 \\
k\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^4 &= 1 \\
k\left[\frac{5(16)}{2}-\frac{64}{3}\right] &= 1 \\
k\left[40-\frac{64}{3}\right] &= 1 \\
k\left[\frac{120-64}{3}\right] &= 1 \\
k\times\frac{56}{3} &= 1 \\
k &= \boxed{\frac{3}{56}}
\end{aligned}$$

Part (ii): Find $E(X)$

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x\cdot f(x)\,dx \\
&= \int_0^4 x\cdot\frac{3}{56}x(5-x)\,dx \\
&= \frac{3}{56}\int_0^4 x^2(5-x)\,dx \\
&= \frac{3}{56}\int_0^4 (5x^2-x^3)\,dx \\
&= \frac{3}{56}\left[\frac{5x^3}{3}-\frac{x^4}{4}\right]_0^4 \\
&= \frac{3}{56}\left[\frac{5(64)}{3}-\frac{256}{4}\right] \\
&= \frac{3}{56}\left[\frac{320}{3}-64\right] \\
&= \frac{3}{56}\times\frac{320-192}{3} \\
&= \frac{3}{56}\times\frac{128}{3} \\
&= \frac{128}{56} \\
&= \boxed{\frac{16}{7}\approx 2.286}
\end{aligned}$$

Question 8: Independence Test

Question

In a college, 60% of students study Mathematics (Event A), and 50% study Physics (Event B). It is also known that 30% of the students study both Mathematics and Physics. Determine whether events A and B are independent.

Solution

Given:

- $P(A) = 0.60$
- $P(B) = 0.50$
- $P(A \cap B) = 0.30$

Test for independence:
Events A and B are independent if and only if: $P(A \cap B) = P(A) \times P(B)$
Check:

$$\begin{aligned} P(A) \times P(B) &= 0.60 \times 0.50 \\ &= 0.30 \end{aligned}$$

Since $P(A \cap B) = 0.30 = P(A) \times P(B)$:
$\boxed{\text{Events A and B are INDEPENDENT}}$
Interpretation: Studying Mathematics does not affect the probability of studying Physics, and vice versa.

4 Tutorial 4: Normal Distribution and Applications

Question 1: Basic Normal Distribution

Question

The random variable X has a normal distribution with mean 20 and variance 16. Find the probability that $X > 22$.

Solution

Given: $X \sim N(20, 16)$, where $\mu = 20$ and $\sigma^2 = 16$
Therefore: $\sigma = 4$
Standardize: $Z = \frac{X-\mu}{\sigma} = \frac{X-20}{4}$

$$\begin{aligned} P(X > 22) &= P\left(Z > \frac{22-20}{4}\right) \\ &= P(Z > 0.5) \\ &= 1 - P(Z \leq 0.5) \\ &= 1 - 0.6915 \\ &= \boxed{0.3085} \end{aligned}$$

Question 2: Heights of Boys

Question

The heights of boys at a particular age follow a normal distribution with mean 150.3 cm and standard deviation 5 cm. Find the probability that a boy picked at random from this age group has a height:

(i) less than 153 cm

(ii) more than 144 cm

Solution

Given: $X \sim N(150.3, 25)$, where $\mu = 150.3$ and $\sigma = 5$

Part (i): $P(X < 153)$

$$\begin{aligned} P(X < 153) &= P\left(Z < \frac{153 - 150.3}{5}\right) \\ &= P(Z < 0.54) \\ &= \boxed{0.7054} \end{aligned}$$

Part (ii): $P(X > 144)$

$$\begin{aligned} P(X > 144) &= P\left(Z > \frac{144 - 150.3}{5}\right) \\ &= P(Z > -1.26) \\ &= 1 - P(Z < -1.26) \\ &= 1 - 0.1038 \\ &= \boxed{0.8962} \end{aligned}$$

Question 3: Fair Game (Repeated)

Question

Three coins are thrown. If one head turns up, Rs 1.00 is paid. If two heads turn up, Rs 3.00 is paid, and if three heads turn up Rs 5.00 is paid. If the game is to be considered as fair, what should be the penalty if no heads turn up?

Solution

This is the same as Tutorial 2, Question 9 from the exam handout.
For a fair game: $E(X) = 0$
Probability distribution:

Heads	Probability	Payout
0	$\frac{1}{8}$	$-x$
1	$\frac{3}{8}$	Rs 1.00
2	$\frac{3}{8}$	Rs 3.00
3	$\frac{1}{8}$	Rs 5.00

$$\begin{aligned} E(X) &= \frac{1}{8}(-x) + \frac{3}{8}(1) + \frac{3}{8}(3) + \frac{1}{8}(5) = 0 \\ -\frac{x}{8} + \frac{17}{8} &= 0 \\ x &= \boxed{\text{Rs } 17.00} \end{aligned}$$

Question 4: Examination Marks

Question

Marks of 100 students in an examination are normally distributed with mean of 65 and variance 225.

(i) Given the pass mark is 50, estimate the number of students who have passed the examination.

(ii) If a student obtained a mark 'x' or above they are eligible for a merit certificate. If there were 8% of students who obtained merit certificates, find the value of x.

Solution

Given: $X \sim N(65, 225)$, where $\mu = 65$ and $\sigma = 15$
Total students = 100

Part (i): Number who passed (mark ≥ 50)

$$\begin{aligned} P(X \geq 50) &= P\left(Z \geq \frac{50-65}{15}\right) \\ &= P(Z \geq -1) \\ &= 1 - P(Z < -1) \\ &= 1 - 0.1587 \\ &= 0.8413 \end{aligned}$$

Number of students: $100 \times 0.8413 = \boxed{84 \text{ students}}$

Part (ii): Find x for top 8%
We need: $P(X \geq x) = 0.08$, which means $P(X < x) = 0.92$
From standard normal tables: $P(Z < 1.405) = 0.92$

$$\begin{aligned} \frac{x-65}{15} &= 1.405 \\ x - 65 &= 21.075 \\ x &= \boxed{86.08 \approx 86} \end{aligned}$$

Question 5: Discrete Random Variable

Question

A discrete random variable X can assume values 10 and 20 only. If $E(X) = 16$, write the p.d.f. (probability distribution function) of X in table form.

Solution

Let $P(X = 10) = p$ and $P(X = 20) = 1 - p$
Using the expected value:

$$\begin{aligned} E(X) &= 10p + 20(1 - p) = 16 \\ 10p + 20 - 20p &= 16 \\ -10p &= -4 \\ p &= 0.4 \end{aligned}$$

Therefore: $P(X = 20) = 1 - 0.4 = 0.6$
Probability distribution:

x	10	20
$P(X = x)$	0.4	0.6

Question 6: Expected Value Game

Question

Toss an unbiased coin until either two heads or two tails (no need to be consecutively) have been occurred. You have to pay Rs 6.00 to start and receive Rs 5.00 for every head observed. Find the expected amount you get.

Solution

Let X = number of heads when game ends.
Possible outcomes:

- **HH:** 2 heads, $P = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
- **HTH:** 2 heads (ends), $P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
- **HTT:** 1 head (ends), $P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
- **THH:** 2 heads (ends), $P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
- **THT:** 1 head (ends), $P = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
- **TT:** 0 heads, $P = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Probability distribution:

X	$P(X)$
0	$\frac{1}{4}$
1	$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$
2	$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

Expected number of heads:

$$
\begin{aligned}
E(X) &= 0 \times \frac{1}{4} + 1 \times \frac{1}{4} + 2 \times \frac{1}{2} \\
&= 0 + 0.25 + 1 \\
&= 1.25
\end{aligned}
$$

Expected amount received: $5 \times 1.25 =$ Rs 6.25
Net expected amount: Expected gain $= 6.25 - 6.00 = \boxed{\text{Rs } 0.25}$

Question 7: Vending Machine (CLT)

Question

A soft-drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 200ml and standard deviation of 15ml. What is the probability that the average amount dispensed in a random sample of size 36 is at least 204 ml?

Solution

This is the same as Question 1 from the exam handout.
Given: $\mu = 200\text{ml}$, $\sigma = 15\text{ml}$, $n = 36$
By the Central Limit Theorem: $\bar{X} \sim N\left(200, \frac{15^2}{36}\right) = N\left(200, \frac{225}{36}\right)$
Standard error: $\sigma_{\bar{X}} = \frac{15}{6} = 2.5$

$$\begin{aligned}
P(\bar{X} \geq 204) &= P\left(Z \geq \frac{204 - 200}{2.5}\right) \\
&= P(Z \geq 1.6) \\
&= 1 - P(Z < 1.6) \\
&= 1 - 0.9452 \\
&= \boxed{0.0548}
\end{aligned}$$

Question 8: Finding a Value from Probability

Question

If X is normally distributed with mean 3 and standard deviation 0.5, and $P(3 < X < C) = 0.4656$, find the value of C.

Solution

Given: $X \sim N(3, 0.25)$, where $\mu = 3$ and $\sigma = 0.5$
Find: C such that $P(3 < X < C) = 0.4656$
Since the normal distribution is symmetric about the mean: $P(X < 3) = 0.5$
Therefore:

$$\begin{aligned}
P(X < C) &= P(X < 3) + P(3 < X < C) \\
&= 0.5 + 0.4656 \\
&= 0.9656
\end{aligned}$$

From standard normal tables: $P(Z < 1.82) = 0.9656$
Standardizing: $\frac{C-3}{0.5} = 1.82$

$$\begin{aligned}
C - 3 &= 1.82 \times 0.5 \\
C - 3 &= 0.91 \\
C &= \boxed{3.91}
\end{aligned}$$

Verification:

$$\begin{aligned}
P(3 < X < 3.91) &= P(0 < Z < 1.82) \\
&= P(Z < 1.82) - P(Z < 0) \\
&= 0.9656 - 0.5 \\
&= 0.4656
\end{aligned}$$

5 Summary of Key Formulas

Important Note

Probability Rules

Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Multiplication Rule (Independent): $P(A \cap B) = P(A) \times P(B)$
Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
Law of Total Probability: $P(B) = \sum_i P(B|A_i)P(A_i)$
Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Important Note

Discrete Distributions

Binomial Distribution: $X \sim \text{Bin}(n, p)$ $P(X = x) = \binom{n}{x}p^x(1-p)^{n-x}$ $E(X) = np, \quad \text{Var}(X) = np(1-p)$
Poisson Distribution: $X \sim \text{Po}(\lambda)$ $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $E(X) = \lambda, \quad \text{Var}(X) = \lambda$
Geometric Distribution: $X \sim \text{Geo}(p)$ $P(X = k) = (1-p)^{k-1}p$ $E(X) = \frac{1}{p}$
Hypergeometric Distribution: $P(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

Important Note

Continuous Distributions

Normal Distribution: $X \sim N(\mu, \sigma^2)$
Standardization: $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$
Central Limit Theorem: $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ for large n
Properties:

- If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent: $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ $X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Important Note

Expected Value and Variance

For discrete random variables: $E(X) = \sum_i x_i P(X = x_i)$ $\text{Var}(X) = E(X^2) - [E(X)]^2$
For continuous random variables: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$ $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - [E(X)]^2$
Properties:

- $E(aX + b) = aE(X) + b$
- $\text{Var}(aX + b) = a^2\text{Var}(X)$
- If X and Y are independent: $E(XY) = E(X)E(Y)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ (if independent)

6 Important Notes and Tips

Important Note

Common Mistakes to Avoid

1. **Independence vs. Mutual Exclusivity:**
   - Mutually exclusive: $P(A \cap B) = 0$
   - Independent: $P(A \cap B) = P(A) \times P(B)$
   - Events cannot be both (unless one has zero probability)
2. **Conditional Probability:**
   - $P(A|B) \neq P(B|A)$ in general
   - Always identify which event is the condition
3. **With/Without Replacement:**
   - With replacement: probabilities stay the same
   - Without replacement: probabilities change after each draw
4. **Binomial vs. Poisson:**
   - Use Binomial when: n is fixed, p is known
   - Use Poisson when: events occur randomly over time/space
   - Poisson approximates Binomial when n is large and p is small
5. **Normal Distribution:**
   - Always standardize before using tables
   - Remember: $P(X > a) = 1 - P(X \leq a)$
   - Standard deviation is the square root of variance

Important Note

Problem-Solving Strategy

1. **Identify the type of problem:**
   - Basic probability (cards, dice, etc.)
   - Conditional probability
   - Distribution problem (binomial, Poisson, normal, etc.)
2. **Define your random variables clearly:**
   - What does X represent?
   - What are the possible values?
3. **Identify the distribution:**
   - What are the parameters (n, p, μ, σ)?
   - Are events independent?
4. **Apply the appropriate formula:**
   - Write out the formula first
   - Substitute values carefully
5. **Check your answer:**
   - Does the probability lie between 0 and 1?
   - Does the answer make intuitive sense?
   - Do all probabilities sum to 1 (for discrete distributions)?

Important Note

Standard Normal Table Usage

Key Points:

- Tables typically give $P(Z < z)$ for positive z
- For negative values: $P(Z < -z) = 1 - P(Z < z)$ by symmetry
- $P(Z > z) = 1 - P(Z < z)$
- $P(a < Z < b) = P(Z < b) - P(Z < a)$

Common Values to Remember:

z	$P(Z < z)$
1.28	0.90
1.645	0.95
1.96	0.975
2.33	0.99

Conclusion

This comprehensive tutorial solution set covers all major topics in SCS 1307 Probability & Statistics:

- **Tutorial 1:** Basic probability, addition and multiplication rules, conditional probability
- **Tutorial 2:** Independence, conditional probability, Bayes' theorem
- **Tutorial 3:** Discrete distributions (Binomial, Poisson, Hypergeometric, Geometric), expected value
- **Tutorial 4:** Normal distribution, Central Limit Theorem, standardization

Study Tips for Success

- Practice problems regularly
- Understand the concepts, not just memorize formulas
- Draw diagrams when possible (tree diagrams, Venn diagrams)
- Check your work and verify answers make sense
- Review formulas and their conditions of applicability
- Work through past exam questions

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SCS 1307 - Probability & Statistics

Good luck with your studies!