1

ASSIGNMENT 2

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1 Linear forms 2.11

Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

1)
$$(3,-5)$$
 $\mathbf{x} = 20$ and $(6,-10)$ $\mathbf{x} = 40$

2)
$$(1,-3)$$
 x = 7 and $(3,-3)$ **x** = 15

2 Solution

1) Given (3,-5) $\mathbf{x} = 20$ and (6,-10) $\mathbf{x} = 40$. The above equations can be expressed as a matrix equation.

$$\begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix} \tag{1}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 3 & -5 & 20 \\ 6 & -10 & 40 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 3 & -5 & 20 \\ 0 & 0 & 0 \end{pmatrix}$$
 (2)

The rank of

$$\begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} = 1 \tag{3}$$

and the rank of

$$\begin{pmatrix} 3 & -5 & 20 \\ 6 & -10 & 40 \end{pmatrix} = 1 \tag{4}$$

from 2.

$$\therefore rank \begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} = rank \begin{pmatrix} 3 & -5 & 20 \\ 6 & -10 & 40 \end{pmatrix}$$
 (5)
$$= 1 < dim \begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} = 2$$
 (6)

 \implies 1 has infinitely many solutions. Fig2.1 shows that the lines are the same.

2) Given (1,-3) $\mathbf{x} = 7$ and (3,-3) $\mathbf{x} = 15$. The above equations can be expressed as a matrix equation.

$$\begin{pmatrix} 1 & -3 \\ 3 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 15 \end{pmatrix} \tag{7}$$

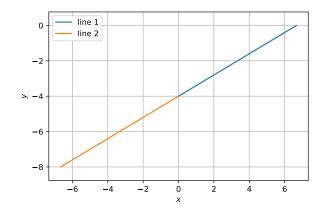


Fig. 2.1. Lines coincide:infinitely many solutions

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -3 & 7 \\ 3 & -3 & 15 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & 8 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & 8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -3 & 7 \\ 0 & 6 & -6 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} 1 & -3 & 7 \\ 0 & 6 & -6 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 6 & -6 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 6 & -6 \end{pmatrix} \stackrel{R_2 \leftarrow \frac{R_2}{6}}{\longleftrightarrow} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{pmatrix} \tag{11}$$

$$\Longrightarrow \mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \tag{12}$$

is a solution of 7. The rank of

$$\begin{pmatrix} 1 & -3 \\ 3 & -3 \end{pmatrix} = 2 \tag{13}$$

and the rank of

$$\begin{pmatrix} 1 & -3 & 7 \\ 3 & -3 & 15 \end{pmatrix} = 2 \tag{14}$$

from 11.

$$\therefore rank \begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} = rank \begin{pmatrix} 3 & -5 & 20 \\ 6 & -10 & 40 \end{pmatrix}$$
 (15)
= $dim \begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} = 2$ (16)

 \implies 7 has a unique solution, $\mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ Fig2.2 shows that the lines intersect only at one point.

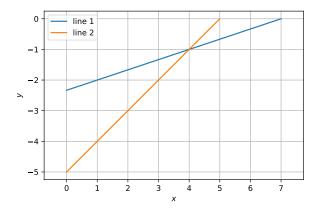


Fig. 2.2. Lines intersecting only at one point:unique solution