§13 Basis for a Topology

For each of the examples in the preceding section, we were able to specify the topology by describing the entire collection $\mathcal{T}$ of open sets. Usually this is too difficult. In most cases, one specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition. If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the ***topology $\mathcal{T}$ generated by*** $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

We will check shortly that the collection $\mathcal{T}$ is indeed a topology on X. But first let us consider some examples.

EXAMPLE 1. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathcal{B}$ satisfies both conditions for a basis. The second condition is illustrated in Figure 13.1. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U.

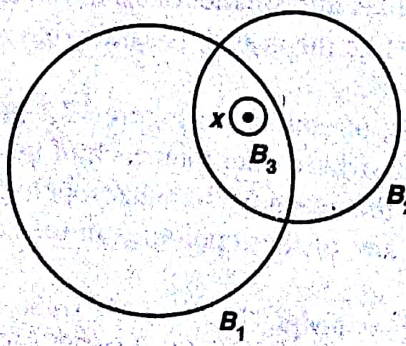

Figure 13.1

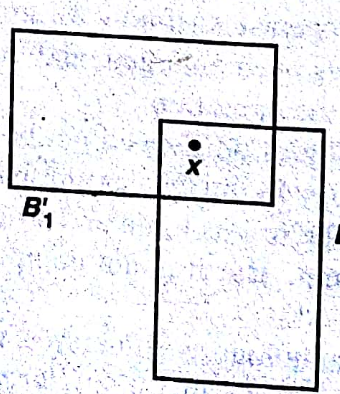

Figure 13.2

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EXAMPLE 2. Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both conditions for a basis. The second condition is illustrated in Figure 13.2; in this case, the condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty). As we shall see later, the basis $\mathcal{B}'$ generates the same topology on the plane as the basis $\mathcal{B}$ given in the preceding example.

EXAMPLE 3. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

Let us check now that the collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is, in fact, a topology on X. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in $\mathcal{T}$, since for each $x \in X$ there is some basis element B containing x and contained in X. Now let us take an indexed family $\{U_\alpha\}_{\alpha \in J}$, of elements of $\mathcal{T}$ and show that

$$U = \bigcup_{\alpha \in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in U$, there is an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition.

Now let us take *two* elements U_1 and U_2 of $\mathcal{T}$ and show that $U_1 \cap U_2$ belongs to $\mathcal{T}$. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis element B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. See Figure 13.3. Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to $\mathcal{T}$, by definition.

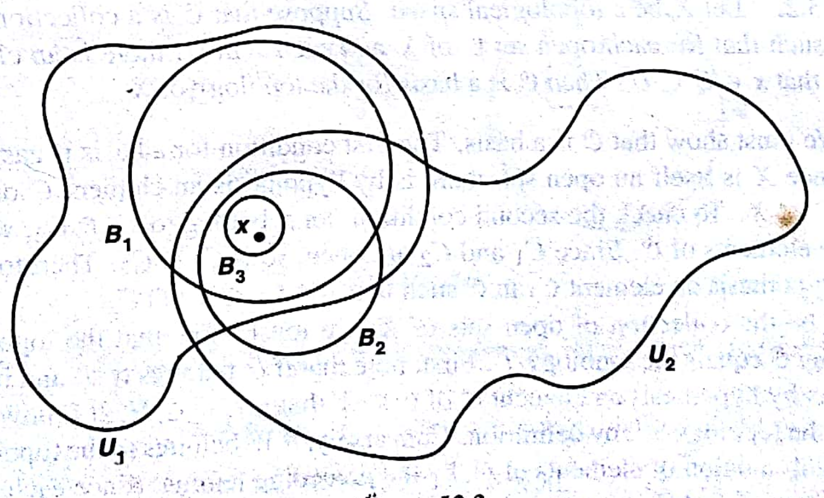

Figure 13.3

Finally, we show by induction that any finite intersection $U_1 \cap \cdots \cap U_n$ of elements of $\mathcal{T}$ is in $\mathcal{T}$. This fact is trivial for $n = 1$; we suppose it true for $n - 1$ and prove it for n. Now

$$(U_1 \cap \cdots \cap U_n) = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n.$$

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By hypothesis, $U_1 \cap \cdots \cap U_{n-1}$ belongs to $\mathcal{T}$; by the result just proved, the intersection of $U_1 \cap \cdots \cap U_{n-1}$ and U_n also belongs to $\mathcal{T}$.

Thus we have checked that collection of open sets generated by a basis $\mathcal{B}$ is, in fact, a topology.

Another way of describing the topology generated by a basis is given in the following lemma:

Lemma 13.1. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. ∎

This lemma states that every open set U in X can be expressed as a union of basis elements. This expression for U is not, however, unique. Thus the use of the term "basis" in topology differs drastically from its use in linear algebra, where the equation expressing a given vector as a linear combination of basis vectors *is* unique.

We have described in two different ways how to go from a basis to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a basis generating it. Here is one way of obtaining a basis for a given topology; we shall use it frequently.

Lemma 13.2. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. We must show that $\mathcal{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$. To check the second condition, let x belong to $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let $\mathcal{T}$ be the collection of open sets of X; we must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$. First, note that if U belongs to $\mathcal{T}$ and if $x \in U$, then there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology $\mathcal{T}'$, by definition. Conversely, if W belongs to the topology $\mathcal{T}'$, then W equals a union of elements of $\mathcal{C}$, by the preceding lemma. Since each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology, W also belongs to $\mathcal{T}$. ∎

When topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than another. One such criterion is the following:

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Lemma 13.3. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:*

(1) *$\mathcal{T}'$ is finer than $\mathcal{T}$.*

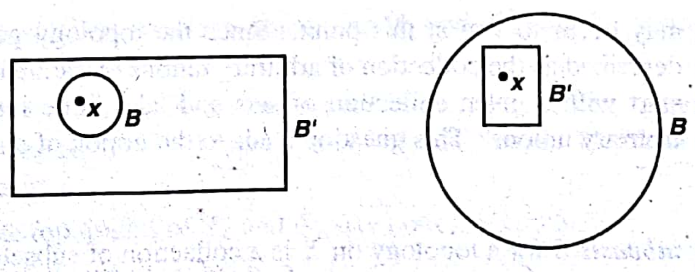
(2) *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. (2) $\Rightarrow$ (1). Given an element U of $\mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition (2) tells us there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}'$, by definition.

(1) $\Rightarrow$ (2). We are given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. ∎

Some students find this condition hard to remember. "Which way does the inclusion *go*?" they ask. It may be easier to remember if you recall the analogy between a topological space and a truckload full of gravel. Think of the pebbles as the basis elements of the topology; after the pebbles are smashed to dust, the dust particles are the basis elements of the new topology. The new topology is finer than the old one, and each dust particle was contained inside a pebble, as the criterion states.

EXAMPLE 4. One can now see that the collection $\mathcal{B}$ of all circular regions in the plane generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions; Figure 13.4 illustrates the proof. We shall treat this example more formally when we study metric spaces.



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