⊛ The Metric Topology :- One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set.

Definition :- A metric on a set X is a function

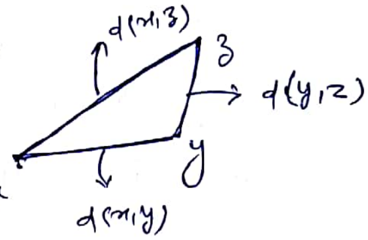
$$d : X \times X \longrightarrow R$$

having the following properties:

(i.) $d(x,y) \geq 0$ for all $x, y \in X$, equality holds iff $x = y$.

(ii.) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(iii) $d(x,y) + d(y,z) \geq d(x,z)$, $\forall$ $x, y, z \in X$.
(Triangle inequality)

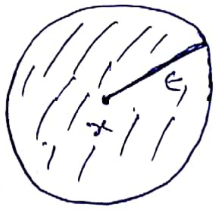


For a given metric d on X, the number $d(x,y)$ is often called the distance between x and y in the metric d.

Given $\epsilon > 0$, consider the set.

$$B_d(x, \epsilon) = \{ y \in X : d(x,y) < \epsilon \}$$

of all points y whose distance from x is less than ϵ.



It is called the ϵ-ball centered at x.

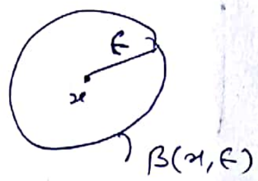
{{ Sometimes, we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion is there. }}

Definition :- If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, form a basis for a topology on X. The topology generated by this basis is called the <u>metric topology</u> <u>induced by d</u>.

Proof :- First, we will show that such type of collection form a basis.

The first condition for a basis is trivial, since $x \in B(x,\epsilon)$, for any $\epsilon>0$.

Before checking the second condition for a basis, we show that if y is the point of the basis element $B(x,\epsilon)$, then there is a basis element $B(y,\delta)$, centered at y that is contained in $B(x,\epsilon)$.



Define $\delta = \epsilon - d(x,y)$, +ve number.

Then $B(y,\delta) \subset B(x,\epsilon)$,

for if $z \in B(y,\delta)$, then $d(y,z) < \delta$

$$\Rightarrow d(y,z) < \epsilon - d(x,y)$$

So, $d(x,z) \le d(x,y) + d(y,z)$

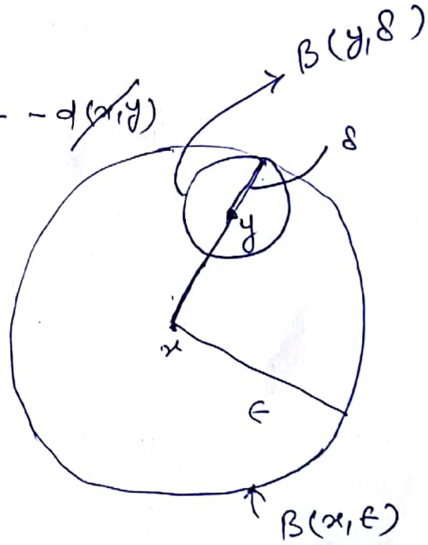
$$< \cancel{d(x,y)} + \epsilon - \cancel{d(x,y)}$$

$$= \epsilon$$

$$\Rightarrow \boxed{d(x,z) < \epsilon}$$

$$\Rightarrow z \in B(x,\epsilon)$$

$$\Rightarrow \boxed{B(y,\delta) \subset B(x,\epsilon)}$$



Now, to check the second condition for a basis,

Let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$.

We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y,\delta_1) \subset B_1$ and $B(y,\delta_2) \subset B_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$

$$\Rightarrow \boxed{B(y,\delta) \subset B_1 \cap B_2}$$

Thus, collection of all ϵ-balls $B_d(x,\epsilon)$, $x \in X$, $\epsilon > 0$ form a basis for a topology on X.

Thus, we can rephrase the definition of the metric topology as follows :→

Imp.

$\boxed{\text{A set } U \text{ is open in the metric topology induced by } d \text{ iff. for each } y \in U, \text{ there is a } \delta > 0 \text{ such that } B_d(y,\delta) \subset U.}$

Clearly, the later condition implies that U is open.

Conversely, if U is open and $y \in U$

$\Rightarrow$ then $\exists$ a basis element ~~$B_d(x,\epsilon)$~~ $B = B_d(x, \epsilon)$ such that

$y \in B = B_d(x,\epsilon) \subset U$

$B_d(y,\delta) = \{z : d(y,z) < \delta\}$

$\Rightarrow$ $\exists$ a basis element $B_d(y,\delta)$, centered at y such that

$B_d(y,\delta) \subset B_d(x,\epsilon) \subset U$

$\Rightarrow$ $\boxed{B_d(y,\delta) \subset U}$

Example 1 :→ Given a set X, define.

$$d(x,y) = 1 \quad \text{if } x \neq y$$

$$d(x,y) = 0 \quad \text{if } x = y.$$

It is trivial to check that d is a metric.

The topology it induces is the discrete topology.

because any subset of X will satisfy the property for openness. Suppose take any subset U of X → for eg.

then for each $x \in U$, ~~$B(x,1)$~~ $\exists$ $1 > 0$ such that

$B_d(x,1) \subset U$

Actually, $B_d(x,1) = \{x\}$ alone.

or $B_d(x,\frac{1}{2}) = \{x\}$

or $B_d(x,\frac{1}{3}) = \{x\}$

So, all the subsets are open or belong to the metric topology induced by d.

$\Rightarrow$ the induced topology is discrete topology.//