Topology

Topological Space :→

Let X be a set and τ be the collection of subsets of X, satisfying the following three conditions:

(1) $\phi \in \tau$ and $X \in \tau$

(2.) If $G_1 \in \tau$ and $G_2 \in \tau_2$, then $G_1 \cap G_2 \in \tau_2$

{i.e, The intersection of the elements of any finite subcollection of τ is in τ.}

(3) If $G_\lambda \in \tau$ for every $\lambda \in \Lambda$, where Λ is an arbitrary set, then

$$\cup \{G_\lambda : \lambda \in \Lambda\} \in \tau.$$

{i.e, the union of the elements of any subcollection of τ is in τ}.

Then τ is called a topology for X, the members of τ are called τ-open (or simply open) sets and the pair (X, τ) is called the topological space.

The elements of X will be called points of the space.

" Properly speaking, a topological space is an ordered pair (X, τ) consisting of a set X and a topology τ on X, but we often ~~omit~~ omit specific mention of τ if no confusion is there."

In terms of open sets, the conditions ①, ② and ③ can be restated as follows:

" A topological space is a set X, together with a collection of subsets of X, called open sets, such that,

(1.) ϕ and X are both open.

(2.) finite intersection of open sets is open.

(3) The union of arbitrary collection of open sets is open (or arbitrary union of open sets is open).

Imp.

"Briefly speaking, a topology for X is a collection of subsets of X containing ϕ, X and closed under finite intersections and arbitrary unions."

Example:- Let $X = \{a, b, c\}$ and consider the following collection of subsets of X.

$\tau_1 = \{\phi, X\} \longrightarrow$ Topology for X

$\tau_2 = \{\phi, \{a\}, \{b, c\}, X\} \longrightarrow$ Topology "

$\tau_3 = \{\phi, \{a\}, \{b\}, X\} \longrightarrow$ Not topology on X

$\tau_4 = \{\phi, \{a\}, X\} \longrightarrow$ Topology

$\tau_5 = \{\phi, \{a\}, \{b\}, \{a,b\}, X\} \longrightarrow$ Topology

$\tau_6 = \{\phi, \{a,b\}, X\} \longrightarrow$ Topology

$\tau_7 = \{\phi, \{b\}, \{a,b\}, X\} \longrightarrow$ Topology

$\tau_8 = \{\{a\}, \{b,c\}, X\} \longrightarrow$ Not topology on X

$\tau_9 = \{\phi, \{a\}, \{b\}, \{a,b\}\} \longrightarrow$ Not topology

$\tau_{10} = \{\phi, \{a,b\}, \{b,c\}, X\} \longrightarrow$ Not topology

$\tau_{11} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, X\} \longrightarrow$ Not topology.

$\tau_{12} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, X\}$ $\longrightarrow$ topology

Let us verify these axioms for τ_7.

(1.) $\phi, X \in \tau_7$

(2.) $\phi \cap \{b\} = \phi \cap \{a,b\} = \phi \cap X = \phi \in \tau_7$

$b \cap \{a,b\} = \{b\} \cap X = \{b\} \in \tau_7$

$\{a,b\} \cap X = \{a,b\} \in \tau_7$

(3.) $\phi \cup \{b\} = \{b\} \in \tau_7$, $\phi \cup \{a,b\} = \{a,b\} \in \tau_7$

$\phi \cup X = X \in \tau_7$, $\{b\} \cup \{a,b\} = \{a,b\} \in \tau_7$

$\{b\} \cup X = X \in \tau_7$, $\{a,b\} \cup X = X \in \tau_7$

$\{b\} \cup \{a,b\} \cup X = X \in \tau_7$

From the above example, you can see that even a three element set has many different topologies. But not every collection of subsets of X is a topology on X.

(Indiscrete Topology)

Example :- Let X be any set.

Then the collection $I = \{\phi, X\}$ consisting of empty set and whole space is always a topology on X. We shall call it, the indiscrete topology (or the trivial topology).

The pair (X, I) is called an indiscrete topological space.

Example :- Discrete Topology

Let X be any set.

Let D be the collection of all subsets of X, then D is a topology on X, called the discrete topology.

The pair (X, D) is called the discrete topological space.

Proof :- Since $\phi \subset X$, $X \subseteq X \Rightarrow \phi \in D$, $X \in D$

so, condition ① is satisfied.

Condition ② and ③ are also easy to verify.

Example :- Let X be any set.

Let τ be the collection of all those subsets of X whose complements are finite together with the empty set, i.e. a subset A of X belongs to τ iff A is empty or A' is finite.

Show that τ is a topology for X called the Co-finite topology or the finite complement topology.

Proof. ① Since $X' = \phi$ (which is finite)

$\Rightarrow X \in \tau$, Also $\phi \in \tau$ by definition.

(2) Let $G_1, G_2, \dots G_n$ are nonempty elements of τ

then we will show that $\bigcap_{i=1}^{n} G_i \in \tau$.

Since $G_1, G_2, \dots G_n \in \tau$

$\Rightarrow G_1', G_2', \dots G_n'$ are finite.

$\Rightarrow G_1' \cup G_2' \cup \dots \cup G_n'$ is finite,

finite union of finite sets. and therefore finite.

$\Rightarrow (G_1 \cap G_2 \cap \dots \cap G_n)'$ is finite. [De-Morgan Law]

$\Rightarrow G_1 \cap G_2 \cap \dots \cap G_n \in \tau$.

$$\boxed{\text{i.e. } \bigcap_{i=1}^{n} G_i \in \tau}$$

(3.) $G_\lambda \in \tau \quad \forall \lambda \in \Lambda$

$\Rightarrow G_\lambda'$ is finite. $\forall \lambda \in \Lambda$

$\Rightarrow \cap \{G_\lambda' : \lambda \in \Lambda\}$ is finite.

$\Rightarrow [\cup \{G_\lambda : \lambda \in \Lambda\}]'$ is finite. [By De-Morgan Law]

$\Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \in \tau$.

Example → Let X be any set.

Let τ consists of all those subset of X whose complements are ~~cou~~ countable sets together with the empty set.

~~So, that~~ Show that, τ is a topology on X, called the co-countable topology.

Try yourself

term finite will be replaced by countable.

Gate 2007 Suppose $X = \{\alpha, \beta, \delta\}$. Let $\tau_1 = \{\phi, X, \{\alpha\}, \{\alpha, \beta\}\}$ and $\tau_2 = \{\phi, X, \{\alpha\}, \{\beta, \delta\}\}$. Then

(A) both $\tau_1 \cap \tau_2$ and $\tau_1 \cup \tau_2$ are topologies

(B) neither $\tau_1 \cap \tau_2$ nor $\tau_1 \cup \tau_2$ is a topology

(C) $\tau_1 \cup \tau_2$ is a Topology but $\tau_1 \cap \tau_2$ is not a topology.

(D) $\tau_1 \cap \tau_2$ is a topology but $\tau_1 \cup \tau_2$ is not a topology.

Solution. $\tau_1 \cap \tau_2 = \{\phi, X, \{\alpha\}\}$

so, $\tau_1 \cap \tau_2$ is a topology on X.

$\tau_1 \cup \tau_2 = \{\phi, X, \{\alpha\}, \{\alpha, \beta\}, \{\beta, \delta\}\}$

Since. $\{\alpha, \beta\} \cap \{\beta, \delta\} = \{\beta\} \notin \tau_1 \cup \tau_2$

so, condition (2) is not satisfies, so, $\tau_1 \cup \tau_2$ is not a topology on X.

Gate 2009. Let $\tau_1 = \{G \subseteq R : G \text{ is finite or } R \sim G \text{ is finite.}\}$ and $\tau_2 = \{G \subseteq R : G \text{ is Countable or } R \sim G \text{ is Countable}\}$. Then

(A) neither τ_1 nor τ_2 is a topology on R.

(B) τ_1 is a topology on R but τ_2 is not a topology on R.

(C) τ_2 〃 〃 〃 τ_1 〃 〃

(D) Both τ_1 and τ_2 are topologies on R.

Scanned by CamScanner