Comparison of Topologies :→

Definition :→ Let τ_1 and τ_2 be two topologies for a set X. We say that τ_1 is coarser (or weaker or smaller) than τ_2 or that τ_2 is finer (or stronger or larger) than τ_1 iff $\tau_1 \subseteq \tau_2$, i.e., iff every τ_1-open set is τ_2 open.

If τ_2 properly contains τ_1, we say that τ_2 is strictly finer than τ_1, or τ_1 is strictly coarser than τ_2.

⊛ If either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, we say that the topologies τ_1 and τ_2 are comparable.

If $\tau_1 \nsubseteq \tau_2$ and $\tau_2 \nsubseteq \tau_1$, then we say that τ_1 and τ_2 are not comparable.

Example :→ For any set X, the indiscrete topology I is the coarsest topology and the discrete topology D is the finest topology.

Exp :→ Find three mutually non-comparable topologies for the set $X = \{a, b, c\}$.

Solution :→
$$\tau_1 = \{\phi, \{a\}, X\}$$
$$\tau_2 = \{\phi, \{b\}, X\}$$
$$\tau_3 = \{\phi, \{c\}, X\}$$

Then the topologies τ_1, τ_2 and τ_3 are mutually non-comparable.

Exp :→ Let $X = \{a, b, c\}$ and let
$$\tau_1 = \{\phi, \{a\}, X\}$$
$$\tau_2 = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$$
$$\tau_3 = \{\phi, \{b\}, \{b,c\}, X\}$$

• Then, we can see that, τ_1, τ_2 and τ_3 are all topologies for X, and τ_2 is finer than τ_1 or τ_1 is coarser than τ_2.

The topologies τ_1 and τ_3 are not comparable since $\tau_1 \nsubseteq \tau_3$ and $\tau_3 \nsubseteq \tau_1$. Similarly, the topologies τ_2 and τ_3 are not comparable.

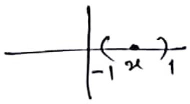
The usual topology for R. ⇢

Let U consists of ϕ and all those subsets G of R having the property that to each $x \in G$, $\exists\ \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset G$.

Show that U is a topology for R, called the usual topology or standard topology.

Proof.

(1) $\phi \in U$ by definition. Also $R \in U$, since to each $x \in R$
$(x-1, x+1) \subset R$



$\boxed{\text{Infact, for any } \epsilon > 0,\ (x-\epsilon, x+\epsilon) \subset R}$

(2) Let $G_1, G_2 \in U$, if $G_1 \cap G_2 = \phi$, there is nothing to prove.

If $G_1 \cap G_2 \neq \phi$, let $x \in G_1 \cap G_2$.

$\Rightarrow x \in G_1$ and $x \in G_2$

Hence $\exists\ \epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$(x-\epsilon_1, x+\epsilon_1) \subset G_1$ and $(x-\epsilon_2, x+\epsilon_2) \subset G_2$

Take $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $\epsilon > 0$ and $(x-\epsilon, x+\epsilon) \subset G_1 \cap G_2$

Hence $\boxed{G_1 \cap G_2 \in U}$

(3) Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of members of U, and $x \in \cup \{G_\lambda ; \lambda \in \Lambda\}$.

$\Rightarrow x \in G_\lambda$ for some $\lambda \in \Lambda$

Since $G_\lambda \in U$, $\exists\ \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset G_\lambda$.

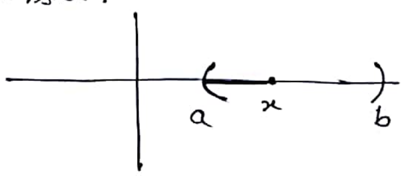
But then $(x-\epsilon, x+\epsilon) \subset \cup\{G_\lambda ; \lambda \in \Lambda\}$.

$\Rightarrow \cup\{G_\lambda ; \lambda \in \Lambda\} \in U$.

$\boxed{\text{Hence } U \text{ is a topology on } R.}$

Example:- Every open interval on R is U-open set.

Let (a, b) be any open interval on R and let $x \in (a, b)$.



Take $\epsilon = \min\{x-a, b-x\}$.

Then it is easy to see that

$(x-\epsilon, x+\epsilon) \subset (a, b)$,

Hence (a, b) is a U-open set.

Example :- Let R be the set of all real numbers and let S consists of subsets of R defined as follows:

(i) $\phi \in S$

(ii) a non-empty subset G of R belongs to S iff to each $p \in G$, $\exists$ a right half open interval $[a,b)$, $a,b \in R$, $a<b$ such that $p \in [a,b) \subset G$.

Show that S is a topology on R, called the lower limit topology on R. This topology is also called the right half open interval topology or RHO topology on R.

Prove Yourself

Similarly, the upper limit topology on R consists of ϕ and all those subsets G of R having the property that to each $p \in G$, $\exists$ a left half open interval $(a,b]$ such that

$$p \in (a,b] \in G.$$

Example :- Consider the usual topology U on R and the lower limit topology S on R and show that S is finer than U.

Solution :- We have to show that every U-open set is S-open.

Let $G \in U$.

To show, $G \in S$, let p be any point in G.

Since, G is U-open, $\exists$ an open interval (a,b) with p as mid point such that

$$p \in (a,b) \subset G$$

Now $p \in (a,b) \Rightarrow [p,b) \subset (a,b)$

Thus to each $p \in G$, $\exists$ half open interval $[p,b)$ such that

$$p \in [p,b) \subset G.$$

Hence $G \in S$. Therefore $U \subset S$, $\Rightarrow$ $\boxed{S \text{ is finer than } U}$

Intersection and Union of Topologies.

Let $X = \{a, b, c\}$

Consider two topology τ_1 and τ_2 on X defined as follows.

$\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$

Then $\tau_1 \cup \tau_2 = \{\phi, \{a\}, \{b\}, X\}$.

which is not a topology on X.

Thus, the union of topologies is not necessarily a topology on X. However, the intersection of any collection of topologies is a topology on X.

Imp Let $\{\tau_\lambda : \lambda \in \Lambda\}$, where Λ is an arbitrary set, be a collection of topologies on X. Then the intersection $\cap\{\tau_\lambda : \lambda \in \Lambda\}$ is also a topology on X.

Proof. Here $\{\tau_\lambda : \lambda \in \Lambda\}$ is a collection of topologies on X. We have to show that $\cap\{\tau_\lambda : \lambda \in \Lambda\}$ is also a topology on X.

Let $\Lambda \neq \phi$.

(1) Since τ_λ is a topology $\forall \lambda \in \Lambda$,

$\Rightarrow \phi, X \in \tau_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow \phi, X \in \cap\{\tau_\lambda : \lambda \in \Lambda\}$

(2) Let $G_1, G_2 \in \cap\{\tau_\lambda : \lambda \in \Lambda\}$

$\Rightarrow G_1, G_2 \in \tau_\lambda \quad \forall \lambda \in \Lambda$

Since τ_λ is a topology on X $\forall \lambda \in \Lambda$

$\Rightarrow G_1 \cap G_2 \in \tau_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow G_1 \cap G_2 \in \cap\{\tau_\lambda : \lambda \in \Lambda\}$

(3) Let $G_\alpha \in \cap\{\tau_\lambda : \lambda \in \Lambda\}$ for $\alpha \in \Delta$, where Δ is an arbitrary set. Then $G_\alpha \in \tau_\lambda \ \forall \lambda \in \Lambda$ and $\forall \alpha \in \Delta$.

Since each τ_λ is a Topology on X,

$\Rightarrow \cup\{G_\alpha : \alpha \in \Delta\} \in \tau_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow \cup\{G_\alpha : \alpha \in \Delta\} \in \cap\{\tau_\lambda : \lambda \in \Lambda\}$

Thus $\cap\{\tau_\lambda : \lambda \in \Delta\}$ is a topology on X.

Result:- (1.) Show that for any family of topologies for X there exist a unique largest topology which is smaller than each member of the family. (9)

$\boxed{\text{Try Yourself}}$

(2.) Show that for any collection of topologies on X, there exist a unique smallest topology larger than each member of the collection.

$\boxed{\text{Try Yourself}}$

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Let $\tau_1 = \{G \subseteq \mathbb{R} : G \text{ is finite or } \mathbb{R} - G \text{ is finite.}\}$

and $\tau_2 = \{G \subseteq \mathbb{R} : G \text{ is countable or } \mathbb{R} - G \text{ is countable}\}$

Then (A) neither τ_1 nor τ_2 is a topology on $\mathbb{R}$ ✓

(B) τ_1 is a topology on $\mathbb{R}$ but τ_2 is not a topology on $\mathbb{R}$.

(C) τ_2 " " " τ_1 " " ;

(D) Both τ_1 and τ_2 are topologies on $\mathbb{R}$.

Ans. τ_1 is not a topology on $\mathbb{R}$ because it is not closed under arbitrary union,

$\{1\}, \{2\}, \{3\}, \ldots \in \tau_1$

but their union will be $\mathbb{N}$, set of natural no.

which is not finite, and its complement is also not finite.

$\Rightarrow \mathbb{N} \notin \tau_1$.

τ_2 is also not a topology on $\mathbb{R}$ because it is not closed under arbitrary union,

for every $r \in [0,1]$, $\{r\} \in \tau_2$

but $\bigcup_{r \in [0,1]} r = [0,1] \notin \tau_2$

uncountable and its complement is also uncountable.