Basis for a Topology :→

Till now, in each example of a Topological space, we were able to specify the topology by describing the entire collection τ of open sets.

Usually, this is too difficult. for some sets.

In most cases, we specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition :→ Let X be a set.

A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X (called basis elements.) if it satisfies the following two properties:

(1.) For each $x \in X$, there is atleast one basis element B containing x.

(2.) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 ~~containg x~~ containing x such that $\boxed{B_3 \subset B_1 \cap B_2}$

If $\mathcal{B}$ satisfies these two conditions, then we define the topology τ generated by $\mathcal{B}$ as follows :-

A subset U of X is said to be open in X (i.e., to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $\boxed{x \in B \text{ and } B \subset U.}$

$\boxed{\text{Note that each basis element is itself an element of } \tau.}$

$\boxed{B_1 \in \mathcal{B} \Rightarrow B_1 \in \tau}$

Now, we will check that the collection τ is indeed a topology on X.

So, let us check now that the collection τ, generated by the basis $\mathcal{B}$, is infact a topology on X.

If U is the empty set, then it satisfies the defining condition of openness vacuously.

Also, X is in τ, since for each $x \in X$, there is some basis element B containing x and contained in X. (by definition)

~~Now~~

Now, Consider $U_1 \in \tau$ and $U_2 \in \tau_2$

then we want to show that $\boxed{U_1 \cap U_2 \in \tau}$

$U_1 \cap U_2$ will belong to τ if for each $x \in U_1 \cap U_2$, there exist a basis element B such that $\boxed{x \in B \subset U_1 \cap U_2}$

Now. Take $x \in U_1 \cap U_2$

$\Rightarrow x \in U_1$ and $x \in U_2$

$\Rightarrow \exists B_1$ such that $x \in B_1 \subset U_1$ and

$\exists B_2$ such that $x \in B_2 \subset U_2$

Since U_1, U_2 are open i.e. $U_1, U_2 \in \tau$

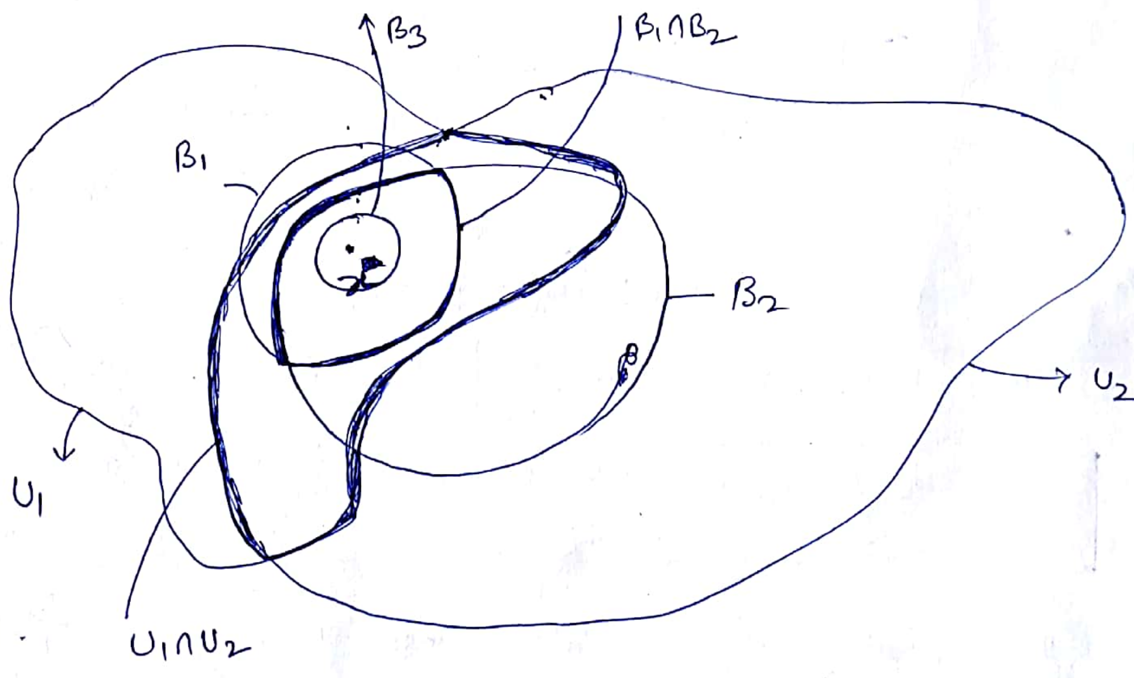
$\Rightarrow x \in B_1 \cap B_2$

So, the second condition for a basis enables us to choose a basis B_3 containing x and contained in $B_1 \cap B_2$, i.e.

$$\boxed{x \in B_3 \subset B_1 \cap B_2}$$

$\Rightarrow x \in B_3$ and $B_3 \subset U_1 \cap U_2$ (see figure for more clarity)

$\Rightarrow U_1 \cap U_2 \in \tau$, by definition.



Finally, we can show that (by induction) that any <u>finite intersection</u> $U_1 \cap U_2 \cap \cdots \cap U_n$ of elements of τ is in τ.

This fact is trivial for $n=1$.;

~~Now~~ Suppose, it is true for ~~[illegible]~~ $n-1$ and prove it for n.

Now
$$(U_1 \cap U_2 \cdots \cap U_n) = (U_1 \cap U_2 \cap \cdots \cap U_{n-1}) \cap U_n$$

By hypothesis, $U_1 \cap U_2 \cap \cdots \cap U_{n-1} \in \tau$

then by the result just proved, the intersection of $U_1 \cap U_2 \cdots \cap U_{n-1}$ and U_n will also belong to τ.

$\boxed{\text{Thus, we have proved that the intersection of elements of any subcollection of } \tau \text{ is also in } \tau.}$

Now ~~Let $\{U_\alpha : \alpha \in \Lambda\}$~~

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of elements of τ.

arbitrary set (may be finite, infinite, countable or uncountable)

We want to show that $\boxed{\cup\{U_\lambda : \lambda \in \Lambda\} \in \tau.}$

Take $x \in \cup\{U_\lambda : \lambda \in \Lambda\}$

$\Rightarrow$ $x \in U_\lambda$ for some $\lambda \in \Lambda$

$\Rightarrow$ there exist a basis element B such that

$x \in B \subset U_\lambda$ [since U_λ is open i.e. $U_\lambda \in \tau$]

$\Rightarrow$ $x \in B$ and $B \subset \cup\{U_\lambda : \lambda \in \Lambda\}$

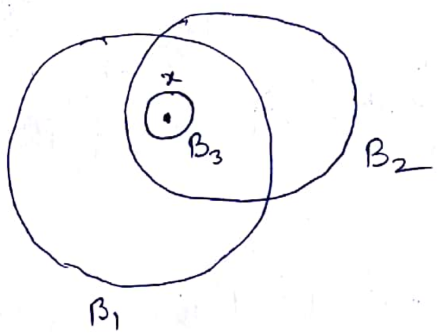
$\Rightarrow$ $\boxed{\cup\{U_\lambda : \lambda \in \Lambda\} \in \tau}$ i.e. open.

$\boxed{\text{Thus, we have shown that the union of elements of any arbitrary collection of } \tau \text{ is in } \tau.}$

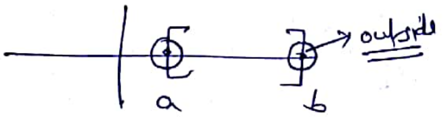
Thus, the collection of open sets generated by a basis $\mathcal{B}$ is infact a topology. //

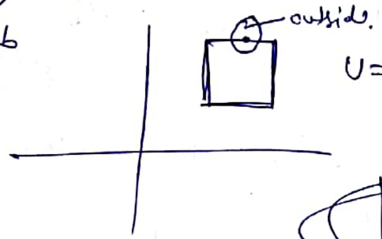
Example 1 :- If X is any set.
Then the collection of all one-point subsets of X is a basis for the discrete topology on X.

Example 2 :- Let $\mathcal{B}$ is the collection of all circular regions (interior of circles) in the plane. Then $\mathcal{B}$ satisfies both the conditions for a basis. The second condition is illustrated in below figure.

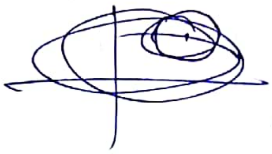


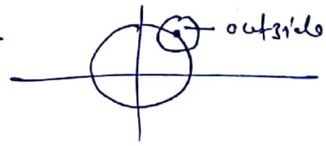
In the topology, generated by $\mathcal{B}$, a subset U of the plane is open (i.e., belong to τ) if every x in U lies in some circular region (elements of $\mathcal{B}$, basis) contained in U.



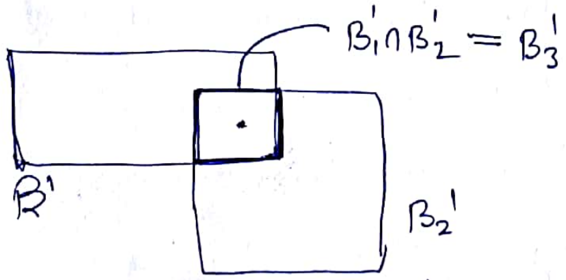


$U = \{(x,y) : x^2+y^2 \leq 1\}$
then it will not belong to τ, topology.





Example 3 :- Let $\mathcal{B}'$ is the collection of all rectangular regions (interior to rectangles) in the plane, when the rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both the conditions for a basis. The second condition is illustrated in below figure.



In this case, the condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty).

(i) Now, another way of describing the topology generated by a basis is given in the following lemma.

Lemma 1 :→ Let X be a set.

Let $\mathcal{B}$ be a basis for a topology τ on X.

Then τ equals the collection of all unions of elements of $\mathcal{B}$.

Proof :→ Let $\mathcal{B}$ be a basis.

then the elements of $\mathcal{B}$ are also the elements of τ.

Since, τ is a topology, so, the union of elements of $\mathcal{B}$ will be in τ.

Conversely, let $U \in \tau$.

Choose, for each $x \in U$, ~~such~~ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$.

Then $U = \bigcup_{x \in U} B_x$.

Thus we have shown

$\tau \subseteq$ set of all union of elements of $\mathcal{B}$.

and set of all union of elements of $\mathcal{B}$ $\subseteq \tau$.

So, U equals a union of elements of $\mathcal{B}$.

Proved//

Remark. :→ This lemma states that every open set U in X can be express as a a union of basis elements. This expression of U is not, however, unique.

Thus, the use of the term "basis" in topology differ drastically from its use in Linear Algebra, where the ~~expression~~ equation expressing a given vector as a linear combination of basis vectors is unique.

(*) So, we have described in two different ways, how to go from a basis to the topology it generates.

Sometimes, we need to go in the reverse direction, from a topology to a basis generating it.

Lemma 2 :- Let (X, τ) be a topological space. Suppose that $\mathcal{C}$ is the collection of open sets of X (i.e. $\mathcal{C}$ is the collection of elements of τ) such that for each open set U of X (i.e. for each $U \in \tau$) and for each $x \in U$, there is an element C of $\mathcal{C}$ such that $\boxed{x \in C \subset U}$.

Then $\mathcal{C}$ is the basis for the topology of X.

Proof :- First of all, we must show that, $\mathcal{C}$ is a basis.

The first condition for a basis is easy.

Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathcal{C}$ such that $\boxed{x \in C \subset X}$.

Now check for second condition.

Let $x \in C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$.

Since, C_1 and C_2 are open (i.e. $C_1, C_2 \in \tau$),

so $C_1 \cap C_2 \in \tau$ (i.e. $C_1 \cap C_2$ is open).

Therefore, there exist, by hypothesis, an element C_3 in $\mathcal{C}$ such that

$$\boxed{x \in C_3 \subset C_1 \cap C_2}.$$

Thus $\mathcal{C}$ is a basis for X.

Now we will show that, the topology τ' generated by $\mathcal{C}$ is equal to the topology τ.

Let $U \in \tau$

$\Rightarrow$ for each $x \in U$, $\exists$ an element $C \in \mathcal{C}$ such that $x \in C \subset U$. (by hypothesis).

$\Rightarrow$ $U \in \tau'$ (by definition).

Conversely, if $w \in \tau'$

$\Rightarrow$ w is equal to a union of elements of $\mathcal{C}$ (by previous Lemma)

Since, each element of $\mathcal{C}$ belongs to τ, and τ is a topology.

$\Rightarrow$ $w \in \tau$

so, $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$

$$\Rightarrow \boxed{\tau = \tau'}$$

(*) When topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether or not one topology is finer than other.

One such criterion is the following:

Lemma 3 :→ Let $\mathcal{B}$ and $\mathcal{B}'$ be the bases for the topologies τ and τ', respectively, on X. Then the following are equivalent.

(i.) τ' is finer than τ.

(ii.) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B$$

Prove yourself

Proof :→ (2.) $\Rightarrow$ (1.) we want to show that $\tau \subseteq \tau'$

Let $U \in \tau$ [we want to show that $U \in \tau'$]

Let $x \in U$.

Since $\mathcal{B}$ generates τ, $\exists$ an element $B \in \mathcal{B}$ such that

$$x \in B \subset U.$$

From condition (2.), $\exists$ an element $B' \in \mathcal{B}'$ such that

$$x \in B' \subset B.$$

$\Rightarrow$ $x \in B' \subset U$ $\Rightarrow$ $U \in \tau'$, by definition.

$\Rightarrow$ $\tau \subseteq \tau'$

$\Rightarrow$ τ' is finer than τ.

Now ① $\Rightarrow$ ②

i.e. let τ' is finer than τ

i.e. $\boxed{\tau \subseteq \tau'}$

Let $x \in X$ and $B \in \mathcal{B}$, with $x \in B$

Now $\mathcal{B} \in \tau$ (by definition)

and $\tau \subseteq \tau'$ (by condition ①):

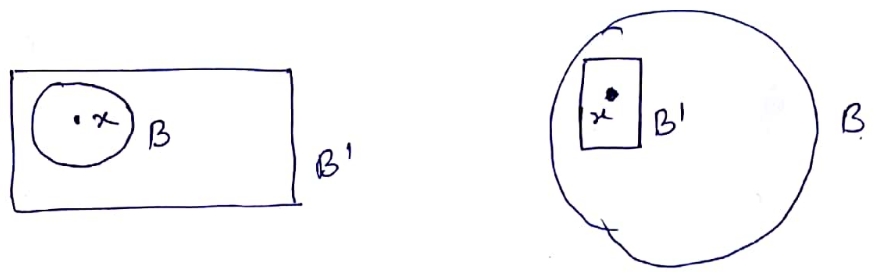
$\Rightarrow$ $B \in \tau'$ and $x \in B$, so

Since τ' is generated by $\mathcal{B}'$, there is an element

$B' \in \mathcal{B}'$ such that, $\boxed{x \in B' \subset B}$

//Proved//

Example. Now, we can see that, the collection $\mathcal{B}$ of all circular regions in the plane generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions.



Definition :- Let $\mathcal{B}$ is the collection of all open intervals in the real line, $(a,b) = \{x :\ a < x < b\}$,

the topology generated by $\mathcal{B}$ is called the standard topology or usual topology.

⊛ If $\mathcal{B}'$ is the collection of all half-open intervals of the form

$$[a,b) = \{x ;\ a \leq x < b\},\quad a < b,$$

the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$.