Газизов Булат

МФЗ-24-2

ДЗ №1 ТФКП

1.04.

• 1. $\frac{1}{1-i} = \frac{1(1+i)}{(1-i)(1+i)} = \frac{1+i}{1+1} = \frac{1+i}{2} = \frac{1}{2} + \frac{i}{2}$ $\quad Re\,z = \frac{1}{2}$

$Im\,z = \frac{1}{2}$

• 2. $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^3 = \left(\frac{(1-i)^2}{1+1}\right)^3 = \left(\frac{(1-i)^2}{2}\right)^3 = \left(\frac{1-2i+i^2}{2}\right)^3 =$

$= \left(\frac{-2i}{2}\right)^3 = \frac{-8\cdot(-1)\cdot i}{8} = i$ $\quad Re\,z = 0$

$Im\,z = 1.$

• 3. $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = \frac{1}{8} - 3\,\frac{1}{4}\cdot\frac{\sqrt{3}}{2}i + \frac{3}{2}\cdot\frac{3}{4}i + \frac{3\sqrt{3}}{8}i =$

$= \frac{1}{8} - \frac{3\sqrt{3}}{8}i + \frac{9}{8}i - \frac{9}{8} + \frac{3\sqrt{3}}{8}i = \frac{1}{8} - \frac{9}{8} = -\frac{8}{8} = -1$ $\quad Re\,z = -1$

$Im\,z = 0$

• 4. $\left(\frac{i^5+2}{i^{19}+1}\right)^2 = \left(\frac{i^4\cdot i+2}{i^{18}\cdot i+1}\right)^2 = \left(\frac{i+2}{(-1)^9\cdot i+1}\right)^2 = \left(\frac{i+2}{1-i}\right)^2 = \left(\frac{(i+2)(1+i)}{(1-i)(1+i)}\right)^2 =$

$$= \left(\frac{(i+2)(1+i)}{2}\right)^2 = \left(\frac{i+2-1+2i}{2}\right)^2 = \left(\frac{3i+1}{2}\right)^2 = \frac{-9+6i+1}{4} = \frac{-8+6i}{4} =$$

$= -2 + \frac{3}{2}i$

$\mathrm{Re}\, z = -2$

$\mathrm{Im}\, z = \frac{3}{2}$

• 5. $$\frac{(1+i)^5}{(1-i)^3} = \frac{(1+i)^5(1+i)^3}{2^3} = \frac{(1+i)^8}{8} = \frac{((1+i)^2)^4}{8} =$$

$$= \frac{(1+2i-1)^4}{8} = \frac{(2i)^4}{8} = \frac{16\cdot((-1)^2)^2}{8} = 2\cdot 1 = 2$$

$\mathrm{Re}\, z = 2$

$\mathrm{Im} = 0$

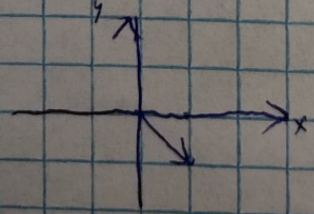
1.06

• 1. i $|z| = 1$ $\arg z = \frac{\pi}{2} + 2\pi n$ $n \in \mathbb{Z}$.

• 2 -3 $|z| = 3$ $\arg z = \pi + 2\pi n$ $n \in \mathbb{Z}$

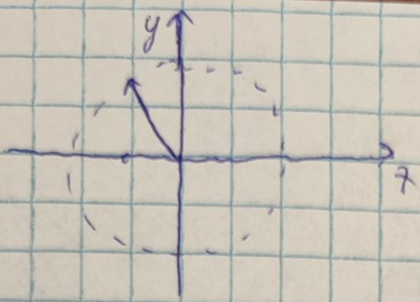
• 3 $1 + i^{123} = 1 + i^{122}\cdot i = 1 + (i^2)^{61}\cdot i = 1 + (-1)^{61}\cdot i = 1 - i$

$|z| = \sqrt{2}$



$\arg z = -\frac{\pi}{4} + 2\pi n$ $n \in \mathbb{Z}$

•4 $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$



$|z|=1$

$\arg = \frac{2\pi}{3}+2\pi n \quad n\in Z$

•5 $\frac{1-i}{1+i}=\frac{(1-i)^2}{2}=\frac{1-2i-1}{2}=\frac{-2i}{2}=-i$

$|z|=1$

$\arg z=-\frac{\pi}{2}+2\pi n$

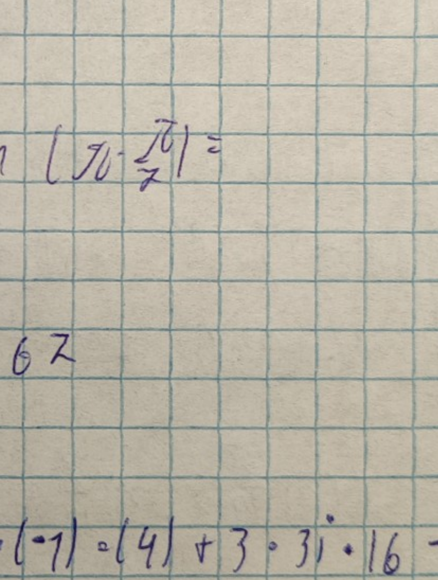
•6 $-\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}=\cos(\pi-\frac{\pi}{7})+i\sin(\pi-\frac{\pi}{7})=$

$=\cos(\frac{6\pi}{7})+i\sin(\frac{6\pi}{7})$

$|z|=1 \qquad \arg z=\frac{6\pi}{7}+2\pi n \quad n\in Z$

•7 $(-4+3i)^3=(3i-4)^3=27\cdot(-1)\cdot i-3\cdot 9\cdot(-1)\cdot(4)+3\cdot 3i\cdot 16-64=$

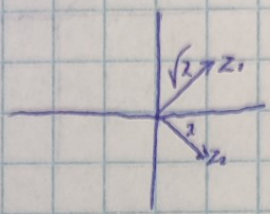
$=-27i+108+144i-64=44+117i$



$|z|=\sqrt{44^2+117^2}=125$

$\arg z=\operatorname{arctg}\frac{117}{44}+2\pi n \quad n\in Z$

8 $\underset{z_1}{(1+i)^8}\;\underset{z_2}{(1-i\sqrt{3})^{-6}}$; $z^n = r^n(\cos\varphi + i\sin\varphi)$.



$|z|_1 = \sqrt{2}$ $\qquad |z_2| = 2$

$(1+i)^8(1-i\sqrt{3})^{-6} = \sqrt{2}\left(\cos 8\frac{\pi}{4} + i\sin 8\frac{\pi}{4}\right)\cdot 2\left(\cos(-6)\left(-\frac{\pi}{3}\right) + i\sin(-6)\left(-\frac{\pi}{3}\right)\right)$

$= (\sqrt{2})^8(\cos 2\pi + i\sin 2\pi)\cdot 2^{-6}(\cos 2\pi + i\sin(2\pi)) =$

$= \frac{(\sqrt{2})^8}{2^6} = \frac{2^4}{2^6} = \frac{1}{2^2} = \frac{1}{4}$ $\qquad |z| = \frac{1}{4}$

$\arg z = 2\pi n \quad n \in Z$.

9. $1 + \cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = \cos 2\overset{=0}{\pi} + i\sin 2\overset{=0}{\pi} + \cos\frac{\pi}{7} + \sin\frac{\pi}{7} =$

$\begin{cases} \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \end{cases}$

$= 2\cos\frac{\pi}{14}\cdot\cos\frac{\pi}{14} + 2\sin\frac{\pi}{14}\cos\frac{\pi}{14}\cdot i =$

$= 2\cos\frac{\pi}{14}\left(\cos\frac{\pi}{14} + i\sin\frac{\pi}{14}\right)$

$|z| = 2\cos\frac{\pi}{14}$

$\arg z = \frac{\pi}{14} + 2\pi n \quad n \in Z$.

Газизов (лист 2)

1.51.

1. $\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}} \cdot \sin\frac{n\theta}{2}$.

$\sin\theta + \sin 2\theta + \ldots \sin n\theta = \operatorname{Im}\left\{ e^{i\theta} + e^{2i\theta} + \ldots e^{in\theta} \right\}$ $\quad b_1 = e^{i\theta}$, $q = e^{i\theta}$

$S = \operatorname{Im}\left\{\frac{e^{i\theta}\left(e^{in\theta} - 1\right)}{\left(e^{i\theta} - 1\right)}\right\} = \operatorname{Im} \frac{e^{i\theta}\left(e^{in\theta} - e^{i0}\right)}{\left(e^{i\theta} - e^{i0}\right)} =$

$= \operatorname{Im} \frac{e^{i\theta}\, e^{\frac{in\theta}{2}}\left(e^{\frac{in\theta}{2}} - e^{-\frac{in\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}\right)} = \frac{e^{i\theta} \cdot e^{\frac{in\theta}{2}}}{e^{\frac{i\theta}{2}}} \cdot \frac{2i\sin\left(\frac{n\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} =$

$\operatorname{Im}\left\{ e^{i\frac{\theta}{2}} \cdot e^{in\theta} \ldots\ldots \right\} = \operatorname{Im}\left\{ e^{\frac{i\theta}{2}(n+1)} \cdot \frac{2i\sin\left(\frac{n\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} \right\} =$

$= \sin\frac{n+1}{2}\theta \cdot \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$ ч.т.д.

1.52

1. $\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$

$$\cos\theta + \ldots + \cos(2n-1)\theta = Re\left\{e^{i\theta} + e^{3i\theta} + \ldots + e^{i(2n-1)\theta}\right\}$$

$$b_1 = e^{i\theta};\ q = e^{2i\theta} \Rightarrow \sum = Re\left\{e^{i\theta}\left(\frac{e^{2in\theta} - e^{i\theta}}{e^{2i\theta} - e^{i\theta}}\right)\right\} =$$

$$= Re\left\{e^{i\theta}\,\frac{e^{in\theta}(e^{in\theta} - e^{-in\theta})}{e^{i\theta}(e^{i\theta} - e^{-i\theta})}\right\} = Re\left\{\frac{e^{i\theta}\cdot e^{in\theta}}{e^{i\theta}}\,\frac{2i\sin n\theta}{2i\sin\theta}\right\} =$$

$$= Re\left\{e^{in\theta}\,\frac{\sin n\theta}{\sin\theta}\right\} = \frac{\cos n\theta \cdot \sin n\theta}{\sin\theta} = \frac{\frac{1}{2}\sin 2n}{\sin\theta} =$$

$$= \frac{\sin 2n\theta}{2\sin\theta}$$ ч.т.д.

2. $\sin\theta - \sin 3\theta + \ldots + (-1)^{n+1}\sin(2n-1)\theta = (-1)^{n+1}\frac{\sin 2n\theta}{2\cos\theta}$

$$e^{i\theta} - e^{3i\theta} + e^{5i\theta} - e^{7i\theta} \ldots (-1)^{n+1} e^{(2n-1)i\theta};\quad \left\{\begin{array}{l} b_1 = e^{i\theta} \\ q = \frac{-e^{3i\theta}}{e^{i\theta}} = -e^{2i\theta} \end{array}\right\}$$

$$= (-1)^{n+1}\, Im\left\{\frac{e^{i\theta}(-e^{2in\theta} - e^{i\theta})}{(-e^{2i\theta} - e^{i\theta})}\right\} = \ldots \quad e^{i\theta}(-1)$$

$$= \ldots \frac{e^{i\theta}(-1)e^{in\theta}(e^{in\theta} + e^{-in\theta})}{e^{i\theta}(e^{i\theta} + e^{-i\theta})} = Im\left\{\frac{e^{i\theta}\cdot e^{in}}{e^{i\theta}}\cdot\frac{2\cos n\theta}{2\cos\theta}\right\}\cdot(-1)^{n+1} =$$

$= (-1)^{n+1} \cdot \cdot \frac{\cos n\theta \cdot \sin n\theta}{\cos\theta} = \frac{\sin 2n\theta}{2\cos\theta} \cdot (-1)^{n+1}$ ч.т.д.

1.53. Док-ть: $z^n = 1$ имеет ровно n различных решений $z_k = e^{\frac{2k\pi i}{n}}$ $k = 0, 1 \dots n-1$.

В общем случае $z^n = w$.

$|z| = r$ $|w| = R$

$\arg z = \varphi$ $\arg w = \psi$

Тогда

$$r = \sqrt[n]{R}$$

$\varphi = \frac{\psi}{n}$ $\Rightarrow n\varphi = \psi$ (учитывая период 2π) $\Rightarrow$

$\Rightarrow$ $n\varphi = \psi \pmod{2\pi} \Rightarrow$

$\Rightarrow \varphi_0 = \frac{\psi}{n}, \ \varphi_1 = \frac{\psi + 2\pi}{n} \dots \varphi_n = \frac{\psi + 2\pi(n-1)}{n}$

В нашем случае $w = 1$, $|w| = 1$ $\Rightarrow$

$\arg w = 0$

$\Rightarrow$ решениями уравнения $z^n = w$ являются

$e^{\frac{2k\pi i}{n}}$ где $k = 0, 1 \dots n-1$ ч.т.д.

1.6e.

$$-1 \quad 1 + 2\varepsilon + 3\varepsilon^2 + \ldots + n\varepsilon^{n-1} = \frac{n}{\varepsilon - 1}; \quad \left\{\varepsilon = \sqrt[n]{1} \Rightarrow \varepsilon^n = 1\right\}$$

Заметим, что $1 + 2\varepsilon + 3\varepsilon^2 + n\varepsilon^{n-1} = (\varepsilon + \varepsilon^2 + \varepsilon^3 + \ldots + \varepsilon^n)' =$

$$= S_n' = \left(\frac{\varepsilon(1-\varepsilon^n)}{1-\varepsilon}\right)';$$

Дифференцируем: $\left(\frac{\varepsilon(1-\varepsilon^n)}{1-\varepsilon}\right)' = \left(\frac{\varepsilon - \varepsilon^{n+1}}{1-\varepsilon}\right)' =$

$$= \frac{[1-(n+1)\varepsilon^n](1-\varepsilon) + (\varepsilon - \varepsilon^{n+1})}{(1-\varepsilon)^2} = \frac{1 - \varepsilon - (n+1)\varepsilon^n + (n+1)\varepsilon^{n+1} + \varepsilon - \varepsilon^{n+1}}{(1-\varepsilon)^2} = \frac{1-(n+1)\varepsilon^n + (n+1)\varepsilon^{n+1} - \varepsilon^{n+1}}{(1-\varepsilon)^2} = \frac{1-(n+1)\varepsilon^n + n\varepsilon^{n+1}}{(1-\varepsilon)^2}$$

$$= \frac{1 - n\varepsilon^n - \varepsilon^n + n\varepsilon^{n+1}}{(1-\varepsilon)^2} \quad \left\{\varepsilon^n = 1\right\} = \frac{1 - n - 1 + n\varepsilon^{n+1}}{(1-\varepsilon)^2} =$$

$$= \frac{n\varepsilon^{n+1} - n}{(1-\varepsilon)^2} = \frac{n(\varepsilon \cdot \overset{=1}{\varepsilon^n} - 1)}{\underset{=(\varepsilon-1)^2}{(1-\varepsilon)^2}} = \frac{n\cancel{(\varepsilon - 1)}}{(\varepsilon - 1)^{\cancel{2}}} = \frac{n}{\varepsilon - 1} \quad \text{ч.т.д.}$$

Газизов лист 3.

1.21.

• 4. $|1+z| < |1-z|$

$|z+1| < |1-z|$

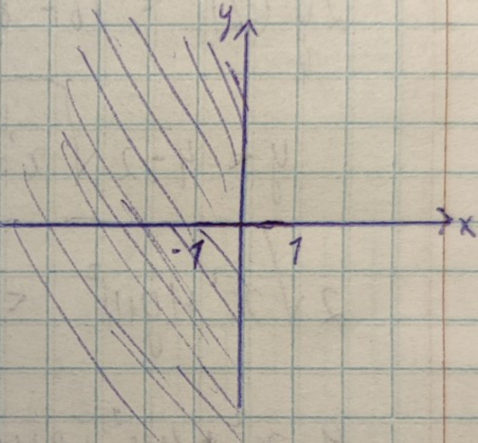
$|x+iy+1| < |1-x-iy|$

$\sqrt{(x+1)^2+(y)^2} < \sqrt{(1-x)^2+(-y)^2}$

$\sqrt{(x+1)^2+(y)^2} < \sqrt{(x-1)^2+y^2}$

$\sqrt{(x+1)^2+y^2} < \sqrt{(x-1)^2+y^2}$

$(|z_1| < |z_2|)$



• 7. $|z-i|+|z+i| < 4$

$|x+iy-i|+|x+iy+i| < 4$

$|x+i(y-1)|+|x+i(y+1)| < 4$

$\sqrt{x^2+(y-1)^2}+\sqrt{x^2+(y+1)^2} < 4$

$\sqrt{x^2+(y-1)^2} < 4-\sqrt{x^2+(y+1)^2}$

$$x^2+(y-1)^2 < 16-8\sqrt{x^2+(y+1)^2} + x^2+(y+1)^2$$

$$(y-1)^2 < 16-8\sqrt{x^2+(y+1)^2} + (y+1)^2$$

$$(y-1)^2-(y+1)^2 < 16-8\sqrt{x^2+(y+1)^2}$$

$$(y-1+y+1)(y-1-y-1) < 16-8\sqrt{x^2+(y+1)^2}$$

$$2y(-2) < 16-8\sqrt{x^2+(y+1)^2}$$

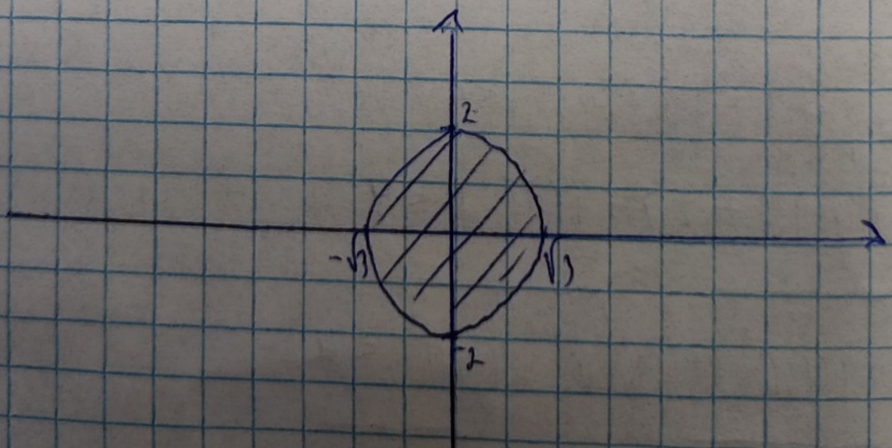
$$y < 4-2\sqrt{x^2+(y+1)^2}$$

$$2\sqrt{x^2+(y+1)^2} < 4-y$$

$$4x^2+4y^2+8y+4 < 16-8y+y^2$$

$$4x^2+3y^2 < 12 \quad |:12$$

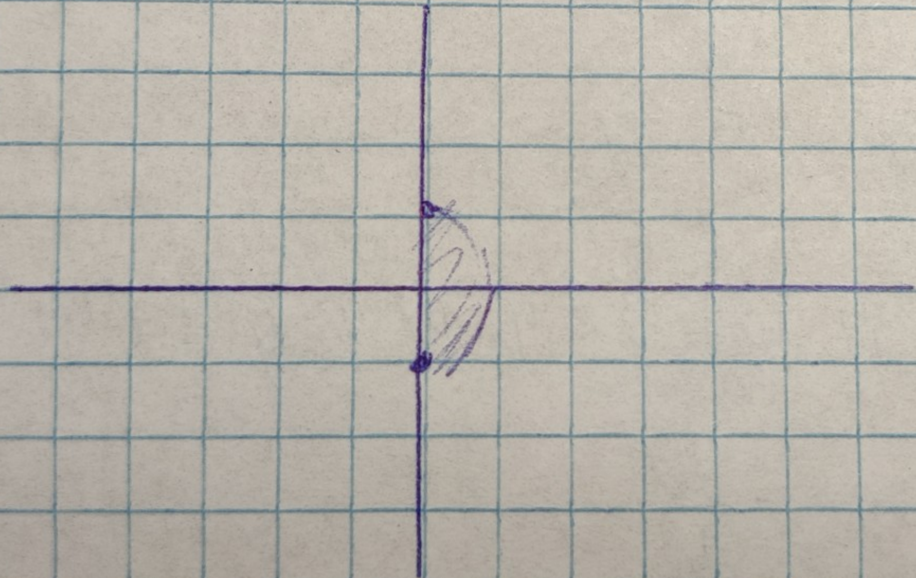
$$\frac{x^2}{3}+\frac{y^2}{4} < 1$$



5. $0 < \arg \frac{i-z}{z+i} < \frac{\pi}{2}$

$$\arg \overbrace{\frac{i-z}{z+i}}^{\varphi_0} = \arg \overbrace{(i-z)}^{\varphi_1} - \arg \overbrace{(z+i)}^{\varphi_2} = \varphi_1 - \varphi_2$$

$\varphi_0 = \varphi_1 - \varphi_2$



8.51.

1 $\operatorname{Re} f = x^3 + 6x^2y - 3xy^2 - 2y^3$ $f(0) = 0$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2 \Leftarrow \frac{\partial v}{\partial y}$

$v = \int 3x^2 + 12xy - 3y^2 \, dy = 3x^2y + 6xy^2 - y^3 + C(x)$

$\frac{\partial u}{\partial y} = 6x^2 - 6xy - 6y = -\frac{\partial v}{\partial x}$

$6xy + 6y^2 + C'(x) = -(6x^2 - 6xy - 6y^2)$

$\underline{6xy} + \underline{6y^2} + C'(x) = -6x^2 + \underline{6xy} + \underline{6y^2}$

$C'(x) = -6x^2$

$C(x) = -\int -6x^2 dx = -2x^3 + C$

$v = 3x^2y + 6xy^2 - y^3 - 2x^3 + C$

$f(z) = x^3 + 6x^2y - 3xy^2 - 2y^3 + i(3x^2y + 6xy^2 - y^3 - 2x^3 + C)$

$f(0) = 0 \Rightarrow C = 0$

$\cancel{f(z) = (x^3 + 3x^2y - 3xy^2 - y^3 i) + (2i)(x^3 - 3x^2y^2 - 3xy^2}$

$f(z) = (x^3 + 3x^2yi - 3xy^2 - y^3 i) + 6x^2y + 6xy^2 - 2x^3 i - 2y^3 =$

$= (x^3 + 3x^2yi - 3xy^2 - y^3 i) + (-2i)(x^3 + 3x^2yi - 3xy^2 - y^3 i) =$

$= (1 - 2i) z^3$

Газизов лист 4.

2. $\operatorname{Re} f = e^x(x\cos y - y\sin y) \qquad f(0)=0$

$$\frac{\partial u}{\partial x} = e^x x\cos y + e^x\cos y - y\sin y e^x \Leftarrow \frac{\partial v}{\partial y}$$

$$v = \int e^x x\cos y + e^x\cos y - y\sin y e^x = e^x x\sin y + e^x\sin y -$$

$$- e^x\int y\sin y\,dy = \left| \begin{matrix} u=y \\ dv=\sin y \\ v=-\cos y \end{matrix} \right| \overset{\dots}{=} -e^x\left(-\cos y\cdot y + \int\cos y\,dy\right) =$$

$$= e^x x\sin y + e^x\sin y + e^x\cos y\, y - \sin y e^x =$$

$$= e^x x\sin y + e^x y\cos y + C(x)$$

$$\frac{\partial u}{\partial y} = (e^x x\cos y - y\sin y e^x)' = -e^x x\sin y - (\sin y e^x + y\cos y e^x) =$$

$$= -e^x x\sin y - \sin y e^x - y\cos y e^x \Leftarrow -\frac{\partial v}{\partial x}$$

$$v' = e^x x\sin y + e^x\sin y + e^x y\cos y + C'(x)$$

$$e^x x\sin y + \sin y e^x + y\cos y e^x = e^x x\sin y + e^x\sin y + e^x y\cos y + C'(x)$$

$$\Rightarrow C'(x) = 0$$

$$f(z) = e^x(x\cos y - y\sin y) + i(e^x(x\sin y + y\cos y)) + C. =$$

$$= e^x x(\cos y + i\sin y) + ie^x y(\cos y + i\sin y) =$$

$= e^x x e^{iy} + i e^x y e^{iy} = e^{x+iy}(x+iy) = e^z z + C$

$f(0) = 0 \Rightarrow C = 0$

$f(z) = z e^z$

5. $|f| = (x^2+y^2) e^x$

$w = \ln f = \ln(|f| \cdot e^{i \arg f}) = \underbrace{\ln|f|}_{u} + i \underbrace{\arg f}_{v}$

$\frac{du}{dx} = (\ln((x^2+y^2)e^x))' = \ln(x^2+y^2) + x = \frac{2x}{x^2+y^2} + 1 = \frac{dv}{dy}$

$v = \int \left(\frac{2x}{x^2+y^2} + 1\right) dy = 2x \int \frac{1}{x^2(1+\frac{y^2}{x^2})} + 1 \, dy = \frac{2x}{x} \cdot \operatorname{arctg}\left(\frac{y}{x}\right) + y =$

$= 2 \operatorname{arctg}\left(\frac{y}{x}\right) + y + C(x)$

$\frac{du}{dy} = \frac{2y}{x^2+y^2} = -\frac{dv}{dx}$

$\frac{dv}{dx} = \frac{-2y}{x^2+y^2} + C'(x)$

$\frac{2y}{x^2+y^2} = \frac{2y}{x^2+y^2} + C'(x) \Rightarrow C(x) = const$

$\ln f = \ln(x^2+y^2) + x + i\left(y + 2\operatorname{arctan}\left(\frac{y}{x}\right) + C\right).$

$(x+iy) + \ln(x^2+y^2) + i\,2\operatorname{arctan}\left(\frac{y}{x}\right) + C$

$f = \exp\left((x+iy) + \ln(x^2+y^2) + i\operatorname{arctg}\left(\frac{y}{x}\right) + C\right) =$

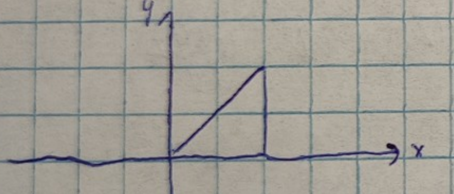
$= e^{z} \cdot z^{2} \cdot e^{i(\text{const})}$

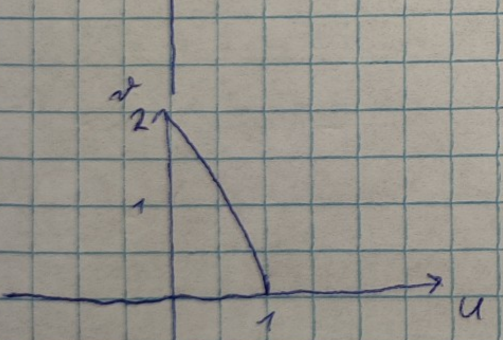
9.16

1 $\quad C: z = it + 1 \qquad 0 \le t \le 1 \qquad w = z^2$

$w = (x+iy)^2 = x^2 + 2ixy - y^2$

$\underbrace{(x^2+y^2)}_{u} + i\underbrace{(2xy)}_{v}$





I $\quad y = 0 \quad x = t \quad t \in [0; 1]$

II $\quad u = t^2 \quad v = 0$

II $\quad x = 1 \quad y = t \quad t \in [0; 1].$

$u = 1 - t^2 \quad v = 2t$

$t^2 = 1 - u$

$t = \sqrt{1-u} \quad v = 2\sqrt{1-u}$

$l = \int_0^1 \sqrt{1 + \left(\frac{d}{dx} 2\sqrt{1-x}\right)^2}\, dx =$

≈ 2.2956

III $\quad y = x$

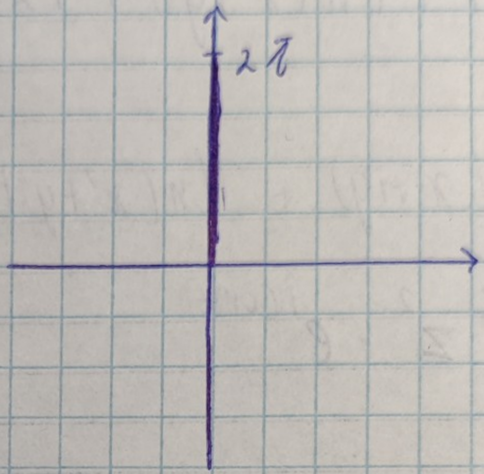
2 $\quad z = it \quad 0 \le t \le 2\pi \qquad w = e^z.$

$$w = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

$x + iy = it$

$x + i(y+t) = 0$

$\Rightarrow$ прямая от $(0;0)$ до $(0;2\pi)$



$l = 2\pi.$

$|w| = e^x$

$\arg w = y$

$u = \ln|z| \qquad v = \arg z$