Inverse Trig Basics

3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.6, but focused there on the arcsine and arccosine functions. Here we complete the study of how all six inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.

Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The graphs of all six basic inverse trigonometric functions are shown in Figure 3.39. We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.6) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

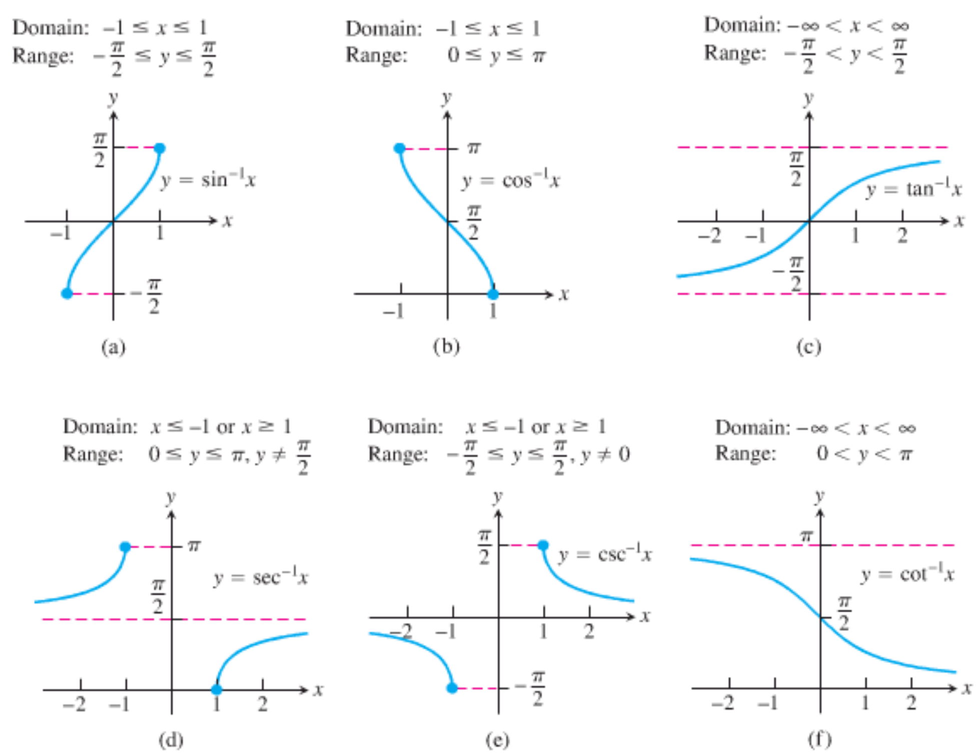


FIGURE 3.39 Graphs of the six basic inverse trigonometric functions.

The arctangent of x is a radian angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x. The angles belong to the restricted domains of the tangent and cotangent functions.

DEFINITION

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

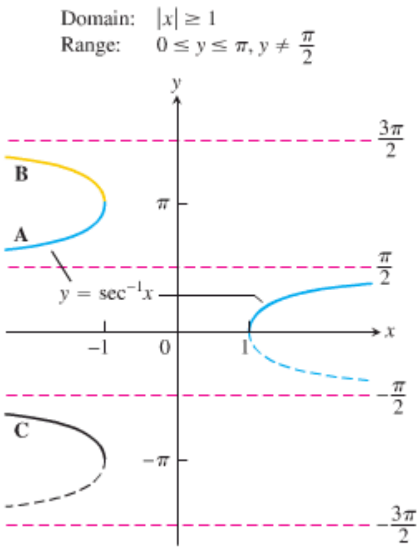


FIGURE 3.40 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.39c). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 3.39f). Notice from Figure 3.39c that the graph of the arctangent function has two horizontal asymptotes; one at $y = \pi/2$ and the other at $y = -\pi/2$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 3.39d and 3.39e.

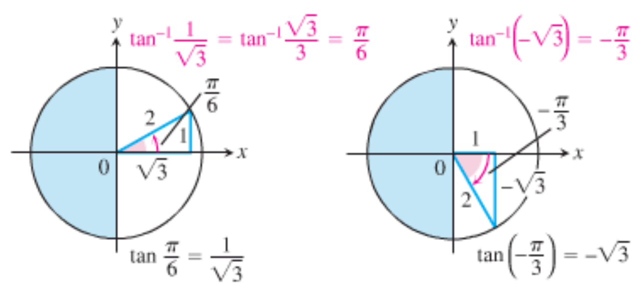
Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.40). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{1}$$

by applying Equation (5) in Section 1.6.

EXAMPLE 1 The accompanying figures show two values of $\tan^{-1} x$.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■