Integration Using the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. In Section 4.8 we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by

$$\int f(x)\,dx.$$

Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation. When finding the indefinite integral of a function f, remember that it always includes an arbitrary constant C.

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C. \tag{1}$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n \, du = \frac{u^{n+1}}{n + 1} + C,$$

which suggests that the simpler expression du can be substituted for $(du/dx)\,dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx}\,dx.$$

EXAMPLE 1 Find the integral $\displaystyle\int (x^3 + x)^5 (3x^2 + 1)\,dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\begin{aligned}
\int (x^3 + x)^5 (3x^2 + 1)\,dx &= \int u^5\,du && \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx. \\
&= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\
&= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u. \quad ■
\end{aligned}$$

EXAMPLE 2 Find $\int \sqrt{2x+1}\, dx$.

Solution The integral does not fit the formula

$$\int u^n \, du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx}\, dx = 2\, dx$$

is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by a factor of $1/2$ in front of the integral sign. So we write

$$\begin{aligned}
\int \sqrt{2x+1}\, dx &= \frac{1}{2}\int \sqrt{\underbrace{2x+1}_{u}} \cdot \underbrace{2\, dx}_{du} \\
&= \frac{1}{2}\int u^{1/2}\, du && \text{Let } u = 2x+1,\ du = 2\,dx. \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
&= \frac{1}{3}(2x+1)^{3/2} + C && \text{Substitute } 2x+1 \text{ for } u. \quad \blacksquare
\end{aligned}$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$