Sigma Notation and Limits of Finite Sums

In estimating with finite sums in Section 5.1, we encountered sums with many terms (up to 1000 in Table 5.1, for instance). In this section we introduce a more convenient notation for sums with a large number of terms. After describing the notation and stating several of its properties, we look at what happens to a finite sum approximation as the number of terms approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n .$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) $\quad \displaystyle\sum_{k=1}^{n} a_k \quad$ a_k is a formula for the kth term.

The index k starts at $k = 1$.

Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

EXAMPLE 2 Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k = 0$ or $k = 1$, but we can start with any integer.

$$\textit{Starting with } k = 0: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{4}(2k + 1)$$

$$\textit{Starting with } k = 1: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5}(2k - 1)$$

$$\textit{Starting with } k = 2: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6}(2k - 3)$$

$$\textit{Starting with } k = -3: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1}(2k + 7)$$

■

When we have a sum such as

$$\sum_{k=1}^{3}(k + k^2)$$

we can rearrange its terms,

$$\begin{aligned}\sum_{k=1}^{3}(k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms.} \\ &= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2.\end{aligned}$$

Four such rules are given below. A proof that they are valid can be obtained using mathematical induction (see Appendix 2).

Algebra Rules for Finite Sums

1. *Sum Rule:* $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:* $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:* $\displaystyle\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule:* $\displaystyle\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\displaystyle\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$ Difference Rule and Constant Multiple Rule

(b) $\displaystyle\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1) \cdot a_k = -1 \cdot \sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$ Constant Multiple Rule

(c) $\displaystyle\sum_{k=1}^{3}(k+4) = \sum_{k=1}^{3}k + \sum_{k=1}^{3}4$ Sum Rule

$= (1+2+3) + (3 \cdot 4)$ Constant Value Rule

$= 6 + 12 = 18$

(d) $\displaystyle\sum_{k=1}^{n}\frac{1}{n} = n \cdot \frac{1}{n} = 1$ Constant Value Rule ($1/n$ is constant) ■

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first n integers.

You Try It

EXAMPLE 4 Show that the sum of the first n integers is

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Solution The formula tells us that the sum of the first 4 integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n-1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the n columns together we get n terms, each equal to $n + 1$, for a total of $n(n+1)$. Since this is twice the desired quantity, the sum of the first n integers is $(n)(n+1)/2$. ■

Formulas for the sums of the squares and cubes of the first n integers are proved using mathematical induction (see Appendix 2). We state them here.

The first n squares: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

The first n cubes: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$