# Related Rates Problems

Related rates problems are story problems that involve 2 or more rates that are happening at the same time and are therefore related to each other.

### Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use $t$ for time. Assume that all variables are differentiable functions of $t$.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to $t$*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

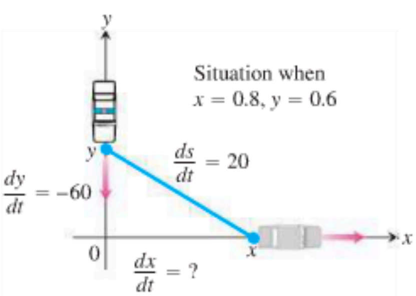


**FIGURE 3.45** The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

**EXAMPLE 3** A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

**Solution** We picture the car and cruiser in the coordinate plane, using the positive $x$-axis as the eastbound highway and the positive $y$-axis as the southbound highway (Figure 3.45). We let $t$ represent time and set

$$x = \text{position of car at time } t$$
$$y = \text{position of cruiser at time } t$$
$$s = \text{distance between car and cruiser at time } t.$$

We assume that $x$, $y$, and $s$ are differentiable functions of $t$.

We want to find $dx/dt$ when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that $dy/dt$ is negative because $y$ is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for $dx/dt$.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

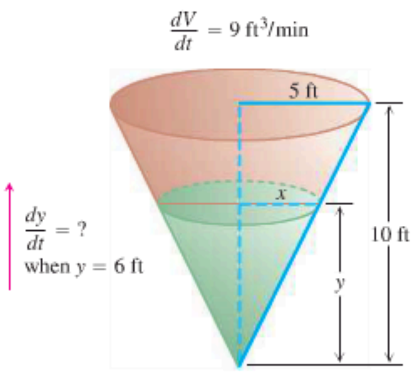


**FIGURE 3.43** The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

**EXAMPLE 1** Water runs into a conical tank at the rate of $9\ \text{ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

**Solution** Figure 3.43 shows a partially filled conical tank. The variables in the problem are

$V =$ volume ($\text{ft}^3$) of the water in the tank at time $t$ (min)

$x =$ radius (ft) of the surface of the water at time $t$

$y =$ depth (ft) of the water in the tank at time $t$.

We assume that $V$, $x$, and $y$ are differentiable functions of $t$. The constants are the dimensions of the tank. We are asked for $dy/dt$ when

$$y = 6\ \text{ft} \qquad \text{and} \qquad \frac{dV}{dt} = 9\ \text{ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves $x$ as well as $V$ and $y$. Because no information is given about $x$ and $dx/dt$ at the time in question, we need to eliminate $x$. The similar triangles in Figure 3.43 give us a way to express $x$ in terms of $y$:

$$\frac{x}{y} = \frac{5}{10} \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore, find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for $dy/dt$.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■