Improper Integrals – Type II

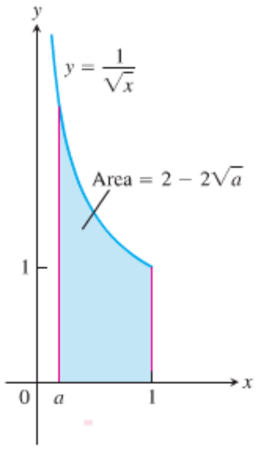


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.12b). First we find the area of the portion from a to 1 (Figure 8.16).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

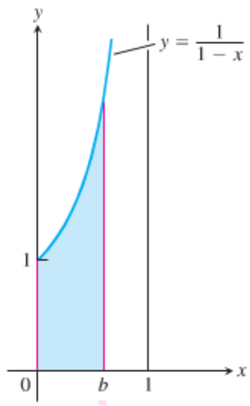


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.17). We evaluate the integral as

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx &= \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b \\ &= \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

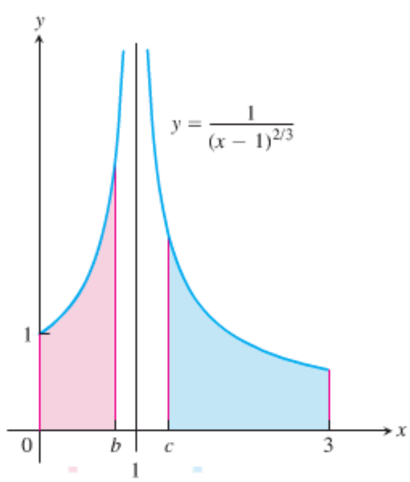


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b \\ &= \lim_{b\to 1^-}[3(b-1)^{1/3}+3] = 3 \\ \int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3 \\ &= \lim_{c\to 1^+}\Big[3(3-1)^{1/3} - 3(c-1)^{1/3}\Big] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

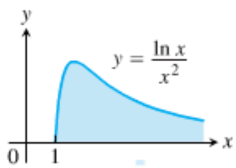
■

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION: TYPE I

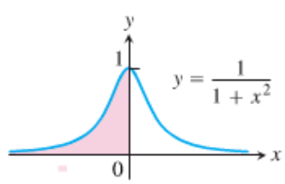
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}\,dx$$



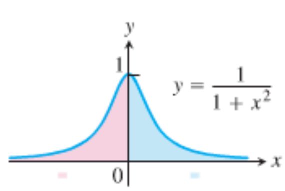
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^{0} \frac{dx}{1+x^2}$$



3. Both limits

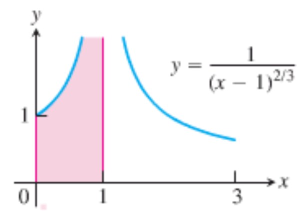
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty}\int_b^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty}\int_0^{c} \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE: TYPE II

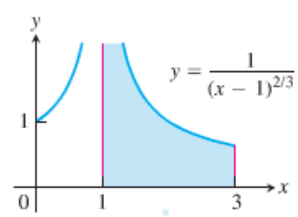
4. Upper endpoint

$$\int_0^{1} \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-}\int_0^{b} \frac{dx}{(x-1)^{2/3}}$$



5. Lower endpoint

$$\int_1^{3} \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+}\int_d^{3} \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^{3} \frac{dx}{(x-1)^{2/3}} = \int_0^{1} \frac{dx}{(x-1)^{2/3}} + \int_1^{3} \frac{dx}{(x-1)^{2/3}}$$