The Derivative as a Rate of Change

In Section 2.1 we introduced average and instantaneous rates of change. In this section we study further applications in which derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.14) is

$$\Delta s = f(t + \Delta t) - f(t),$$

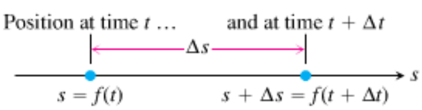


FIGURE 3.14 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

DEFINITION **Velocity** (**instantaneous velocity**) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Besides telling how fast an object is moving along the horizontal line in Figure 3.14, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 3.15 represent position along the line over time; they do not portray the path of motion, which lies along the s-axis.

If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 2 Figure 3.16 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line (as opposed to showing a position function $s = f(t)$ such as in Figure 3.15). In the graph of the velocity function, it's not the slope of the curve that tells us if the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Looking at Figure 3.16, we see that the particle moves forward for the first 3 sec (when the velocity is positive), moves backward for the next 2 sec (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ sec (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ sec, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ sec, speeding up during the final second of the forward motion indicated in the velocity graph. ■

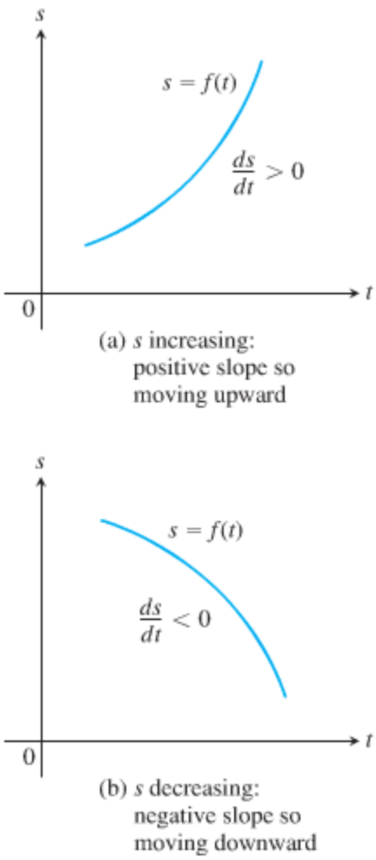


FIGURE 3.15 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of the Earth all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

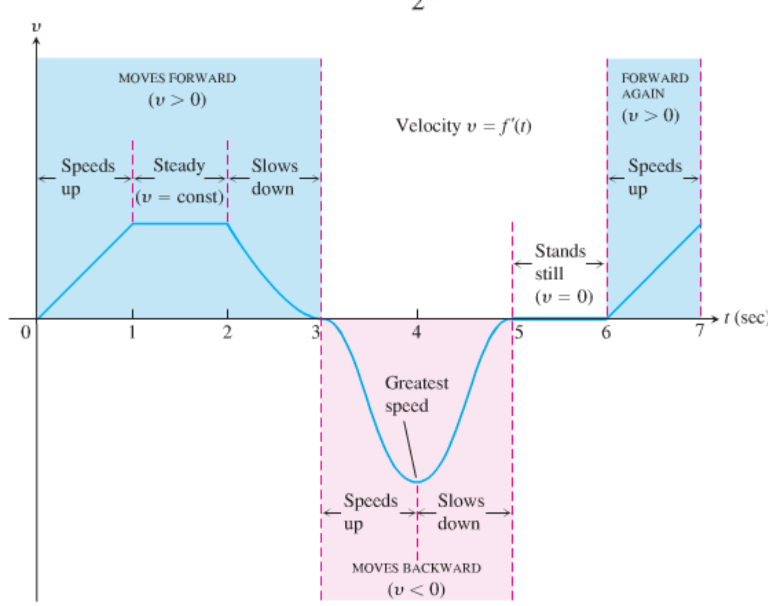


FIGURE 3.16 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

where s is the distance fallen and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s. With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately 32 ft/sec^2 (feet per second squared) in English units, and $g = 9.8\ \text{m/sec}^2$ (meters per second squared) in metric units. (These gravitational constants depend on the distance from Earth's center of mass, and are slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 32\ \text{ft/sec}^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

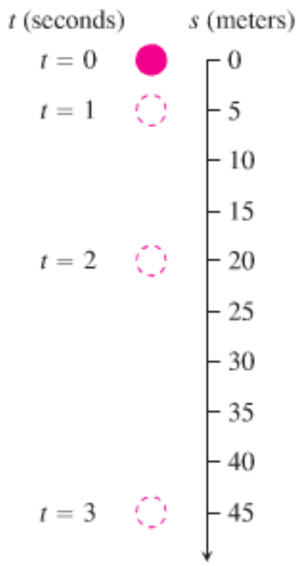


FIGURE 3.17 A ball bearing falling from rest (Example 3).

EXAMPLE 3 Figure 3.17 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

(a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6\ \text{m}.$$

(b) At any time t, *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 2$, the velocity is

$$v(2) = 19.6\ \text{m/sec}$$

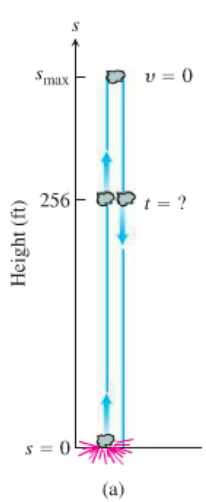
in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{speed} = |v(2)| = 19.6\ \text{m/sec}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8\ \text{m/sec}^2.$$

At $t = 2$, the acceleration is $9.8\ \text{m/sec}^2$. ■



EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.18a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

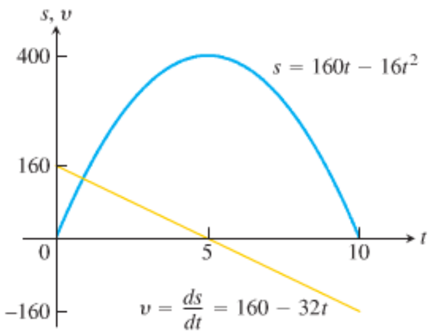
(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution



(b)

FIGURE 3.18 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

(a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{\max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Figure 3.18b.

(b) To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t = 2 \text{ sec}, t &= 8 \text{ sec}. \end{aligned}$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec}.$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

(c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. As the rock rises, it slows down; as it falls, it speeds up.

(d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

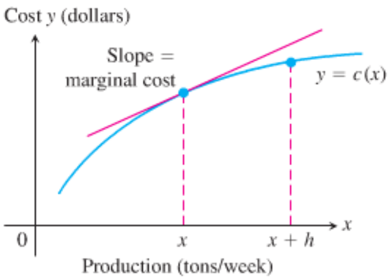


FIGURE 3.19 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx.

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where δ represents *fixed costs* such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs* such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

EXAMPLE 5 Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

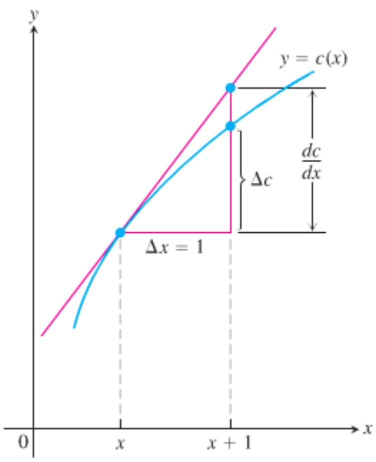


FIGURE 3.20 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 15x\right) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. ■

EXAMPLE 6 To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ■

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

EXAMPLE 7 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 3.21a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 3.21b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth skin peas. ■

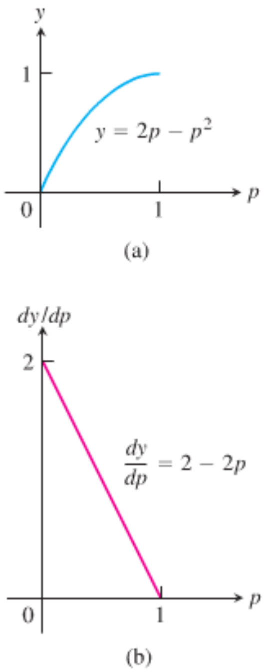


FIGURE 3.21 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas in the next generation. (b) The graph of dy/dp (Example 7).