## Integrable and Nonintegrable Functions

Not every function defined over the closed interval $[a, b]$ is integrable there, even if the function is bounded. That is, the Riemann sums for some functions may not converge to the same limiting value, or to any value at all. A full development of exactly which functions defined over $[a, b]$ are integrable requires advanced mathematical analysis, but fortunately most functions that commonly occur in applications are integrable. In particular, every *continuous* function over $[a, b]$ is integrable over this interval, and so is every function having no more than a finite number of jump discontinuities on $[a, b]$. (The latter are called *piecewise-continuous functions,* and they are defined in Additional Exercises 11–18 at the end of this chapter.) The following theorem, which is proved in more advanced courses, establishes these results.

**THEOREM 1—Integrability of Continuous Functions** If a function $f$ is continuous over the interval $[a, b]$, or if $f$ has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and $f$ is integrable over $[a, b]$.

The idea behind Theorem 1 for continuous functions is given in Exercises 86 and 87. Briefly, when $f$ is continuous we can choose each $c_k$ so that $f(c_k)$ gives the maximum value of $f$ on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose $c_k$ to give the minimum value of $f$ on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition $P$ tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number $J$ in the definition of the definite integral exists, and the continuous function $f$ is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the $x$-axis cannot be approximated well by increasingly thin rectangles. The next example shows a function that is not integrable over a closed interval.

**EXAMPLE 1** The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there is both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the $x$-axis to be approximated by rectangles, no matter how thin they are. We show, in fact, that upper sum approximations and lower sum approximations converge to different limiting values.

If we pick a partition $P$ of $[0, 1]$ and choose $c_k$ to be the point giving the maximum value for $f$ on $[x_{k-1}, x_k]$ then the corresponding Riemann sum is

$$U = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (1)\,\Delta x_k = 1,$$

since each subinterval $[x_{k-1}, x_k]$ contains a rational number where $f(c_k) = 1$. Note that the lengths of the intervals in the partition sum to 1, $\sum_{k=1}^{n} \Delta x_k = 1$. So each such Riemann sum equals 1, and a limit of Riemann sums using these choices equals 1.

On the other hand, if we pick $c_k$ to be the point giving the minimum value for $f$ on $[x_{k-1}, x_k]$, then the Riemann sum is

$$L = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (0)\,\Delta x_k = 0,$$

since each subinterval $[x_{k-1}, x_k]$ contains an irrational number $c_k$ where $f(c_k) = 0$. The limit of Riemann sums using these choices equals zero. Since the limit depends on the choices of $c_k$, the function $f$ is not integrable. ■