Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration (see Chapter 8). Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

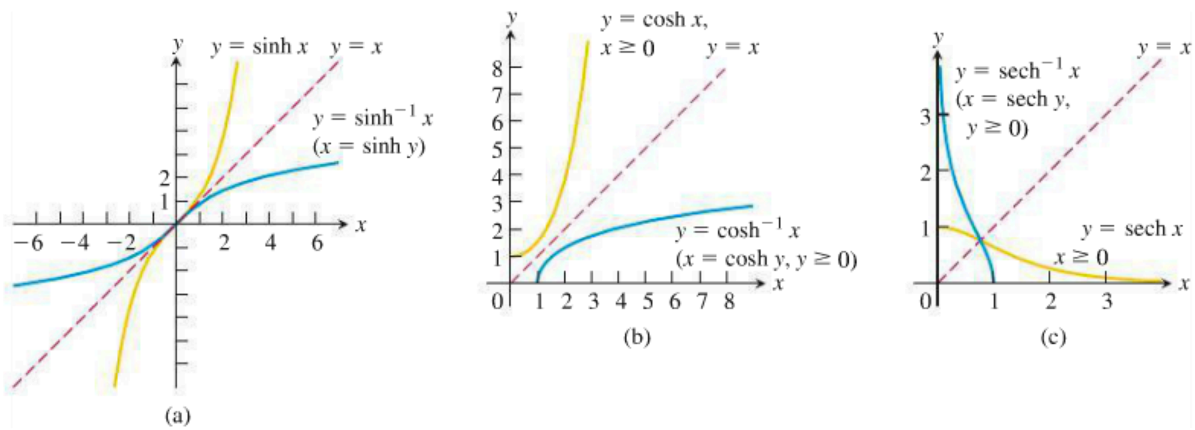

FIGURE 7.8 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure 7.8a.

The function $y = \cosh x$ is not one-to-one because its graph in Table 7.4 does not pass the horizontal line test. The restricted function $y = \cosh x$, $x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure 7.8b.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 7.8c.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 7.9.

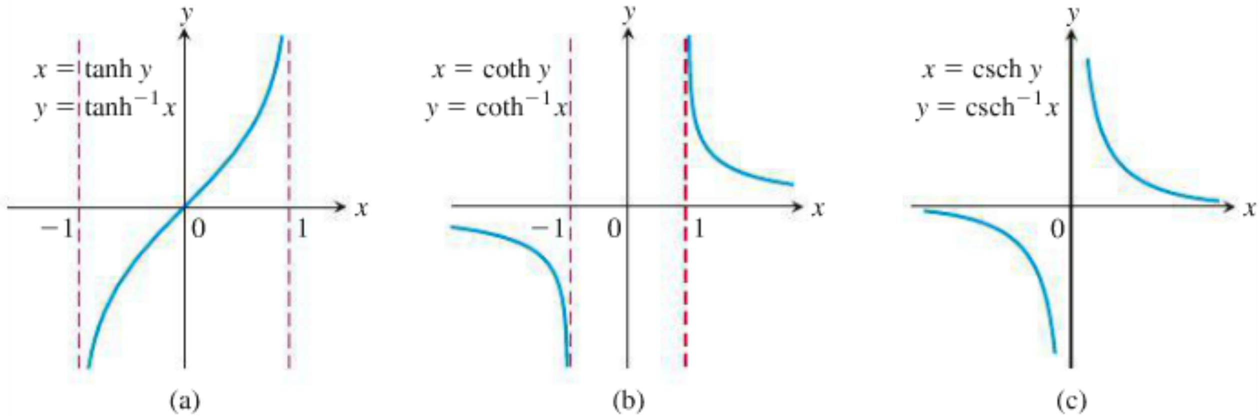

FIGURE 7.9 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Derivatives of Inverse Hyperbolic Functions

An important use of inverse hyperbolic functions lies in antiderivatives that reverse the derivative formulas in Table 7.9.

TABLE 7.9 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 + u^2}}\dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2 - 1}}\dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1 - u^2}\dfrac{du}{dx},$	$\lvert u \rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1 - u^2}\dfrac{du}{dx},$	$\lvert u \rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{1 - u^2}}\dfrac{du}{dx},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = -\dfrac{1}{\lvert u \rvert\sqrt{1 + u^2}}\dfrac{du}{dx},$	$u \neq 0$

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. (See Figure 7.9a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas.

With appropriate substitutions, the derivative formulas in Table 7.9 lead to the integration formulas in Table 7.10. Each of the formulas in Table 7.10 can be verified by differentiating the expression on the right-hand side.

TABLE 7.10 Integrals leading to inverse hyperbolic functions

1.	$\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$	
4.	$\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C,$	$u \neq 0$ and $a > 0$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.10}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

■