Fundamental Theorem, Part 1

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

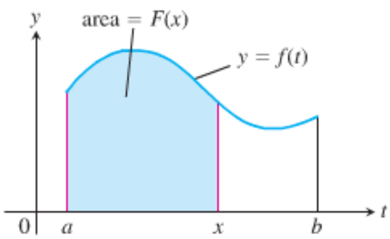

FIGURE 5.18 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x (Figure 5.18). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

Equation (1) gives a way to define new functions (as we will see in Section 7.2), but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x, it asserts that

$$\frac{d}{dx}F(x) = f(x).$$

To gain some insight into why this result holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of the derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x + h) - F(x)}{h}.$$

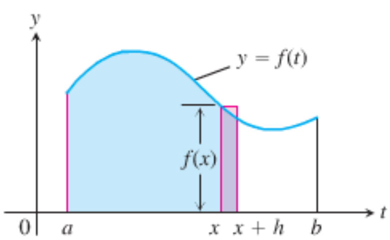

FIGURE 5.19 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$ (Figure 5.19). If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h, which can be seen from Figure 5.19. That is,

$$F(x + h) - F(x) \approx hf(x).$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$ **(b)** $y = \int_x^5 3t \sin t\, dt$

(c) $y = \int_1^{x^2} \cos t\, dt$ **(d)** $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\, dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\, dt = x^3 + 1 \qquad \text{Eq. (2) with } f(t) = t^3 + 1$$

(b) $$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\, dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right) && \text{Table 5.4, Rule 1} \\ &= -\frac{d}{dx}\int_5^x 3t \sin t\, dt \\ &= -3x \sin x && \text{Eq. (2) with } f(t) = 3t \sin t \end{aligned}$$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du}\int_1^u \cos t\, dt\right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ &= \cos(x^2) \cdot 2x \\ &= 2x \cos x^2 \end{aligned}$$

(d) $$\begin{aligned} \frac{d}{dx}\int_{1+3x^2}^4 \frac{1}{2 + e^t}\, dt &= \frac{d}{dx}\left(-\int_4^{1+3x^2} \frac{1}{2 + e^t}\, dt\right) && \text{Rule 1} \\ &= -\frac{d}{dx}\int_4^{1+3x^2} \frac{1}{2 + e^t}\, dt \\ &= -\frac{1}{2 + e^{(1+3x^2)}} \frac{d}{dx}(1 + 3x^2) && \text{Eq. (2) and the Chain Rule} \\ &= -\frac{6x}{2 + e^{(1+3x^2)}} \end{aligned}$$

■

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x + h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b). Thus,

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right] \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \qquad \text{Table 5.4, Rule 5} \end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h} \int_x^{x+h} f(t)\,dt = f(c). \tag{4}$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x). \tag{5}$$

In conclusion, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \\ &= \lim_{h \to 0} f(c) \qquad \text{Eq. (4)} \\ &= f(x). \qquad \text{Eq. (5)} \end{aligned}$$

If $x = a$ or b, then the limit of Equation (3) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. Then Theorem 1 in Section 3.2 shows that F is continuous for every point in $[a, b]$. This concludes the proof. ■

The Relationship between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t)\,dt = F(x) - F(a),$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) for every continuous function f.