Derivatives of Exponentials and Logs Text

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Derivative of the Natural Logarithm Function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$

Alternate Derivation Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$y = \ln x$$

$$e^y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Differentiate implicitly}$$

$$e^y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad e^y = x$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula for positive functions $u(x)$:

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \qquad (2)$$

EXAMPLE 3 We use Equation (2) to find derivatives.

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3}\cdot\frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3}\cdot 2x = \frac{2x}{x^2 + 3}. \qquad \blacksquare$$

Notice the remarkable occurrence in Example 3a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant b, provided that $bx > 0$:

$$\frac{d}{dx}\ln bx = \frac{1}{bx}\cdot\frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}. \tag{3}$$

If $x < 0$ and $b < 0$, then $bx > 0$ and Equation (3) still applies. In particular, if $x < 0$ and $b = -1$ we get

$$\frac{d}{dx}\ln(-x) = \frac{1}{x} \qquad \text{for } x < 0.$$

Since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have the following important result.

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0} \tag{4}$$

The Derivatives of a^u and $\log_a u$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}$, which was established in Section 1.6:

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx} \\ &= a^x \ln a . \end{aligned}$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a .$$

This equation shows why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x .$$

With the Chain Rule, we get a more general form for the derivative of a general exponential function.

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
>
> $$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{5}$$

EXAMPLE 5 We illustrate using Equation (5).

(a) $\dfrac{d}{dx} 3^x = 3^x \ln 3$ — Eq. (5) with $a = 3$, $u = x$

(b) $\dfrac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \dfrac{d}{dx}(-x) = -3^{-x} \ln 3$ — Eq. (5) with $a = 3$, $u = -x$

The Derivative of $\log_a u$

To find the derivative of $\log_a u$ for an arbitrary base ($a > 0$, $a \neq 1$), we start with the change-of-base formula for logarithms (reviewed in Section 1.6) and express $\log_a u$ in terms of natural logarithms,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}
\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\
&= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{$\ln a$ is a constant.} \\
&= \frac{1}{\ln a}\cdot\frac{1}{x} \\
&= \frac{1}{x\ln a}.
\end{aligned}$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives the following formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}. \tag{7}$$