Separable Differential Equations

Separable Differential Equations

Exponential change is modeled by a differential equation of the form $dy/dx = ky$ for some nonzero constant k. More generally, suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

where f is a function of *both* the independent and dependent variables. A **solution** of the equation is a differentiable function $y = y(x)$ defined on an interval of x-values (perhaps infinite) such that

$$\frac{d}{dx}\,y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into the differential equation, the resulting equation is true for all x in the solution interval. The **general solution** is a solution $y(x)$ that contains all possible solutions and it always contains an arbitrary constant.

Equation (3) is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

g is a function of x;
H is a function of y.

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

$H(y) = \dfrac{1}{h(y)}$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{4}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

The justification that we can simply integrate both sides in Equation (4) is based on the Substitution Rule (Section 5.5):

$$\begin{aligned}\int h(y)\,dy &= \int h(y(x))\,\frac{dy}{dx}\,dx \\ &= \int h(y(x))\,\frac{g(x)}{h(y(x))}\,dx \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)} \\ &= \int g(x)\,dx.\end{aligned}$$

You Try It

EXAMPLE 1 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\begin{aligned}\frac{dy}{dx} &= (1 + y)e^x \\ dy &= (1 + y)e^x\,dx && \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx. \\ \frac{dy}{1 + y} &= e^x\,dx && \text{Divide by } (1 + y). \\ \int \frac{dy}{1 + y} &= \int e^x\,dx && \text{Integrate both sides.} \\ \ln(1 + y) &= e^x + C && C \text{ represents the combined constants of integration.}\end{aligned}$$

The last equation gives y as an implicit function of x. ■

EXAMPLE 2 Solve the equation $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

Solution We change to differential form, separate the variables, and integrate:

$$\begin{aligned} y(x+1)\,dy &= x(y^2+1)\,dx \\ \frac{y\,dy}{y^2+1} &= \frac{x\,dx}{x+1} && x \neq -1 \\ \int \frac{y\,dy}{1+y^2} &= \int \left(1 - \frac{1}{x+1}\right) dx && \text{Divide } x \text{ by } x+1. \\ \frac{1}{2}\ln\left(1+y^2\right) &= x - \ln|x+1| + C. \end{aligned}$$

The last equation gives the solution y as an implicit function of x. ■

The initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0 e^{kt}$ expresses exponential change. We now present several examples of such change.