

Limits of Finite Sums

The finite sum approximations we considered in Section 5.1 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and their number grows to infinity.

EXAMPLE 5 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x -axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.4a.)

Solution We compute a lower sum approximation using n rectangles of equal width $\Delta x = (1 - 0)/n$, and then we see what happens as $n \rightarrow \infty$. We start by subdividing $[0, 1]$ into n equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$\left[f\left(\frac{1}{n}\right)\right]\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{k}{n}\right)\right]\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\left(\frac{1}{n}\right).$$

We write this in sigma notation and simplify,

$$\begin{aligned}\sum_{k=1}^n f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) &= \sum_{k=1}^n \left(1 - \left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right) \\&= \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\&= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} && \text{Difference Rule} \\&= n \cdot \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^n k^2 && \text{Constant Value and Constant Multiple Rules} \\&= 1 - \left(\frac{1}{n^3}\right) \frac{(n)(n+1)(2n+1)}{6} && \text{Sum of the First } n \text{ Squares} \\&= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}. && \text{Numerator expanded}\end{aligned}$$

We have obtained an expression for the lower sum that holds for any n . Taking the limit of this expression as $n \rightarrow \infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3} \right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^n f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region R as this limiting value. In Section 5.3 we study the limits of such finite approximations in a general setting. ■

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral studied in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.8, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \dots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < \dots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n , so that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

The set

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **k th subinterval of P** is $[x_{k-1}, x_k]$, for k an integer between 1 and n .

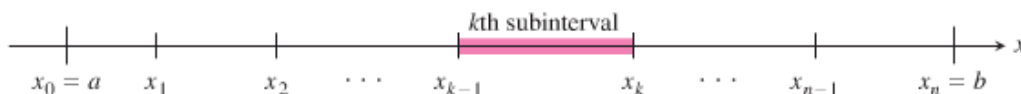
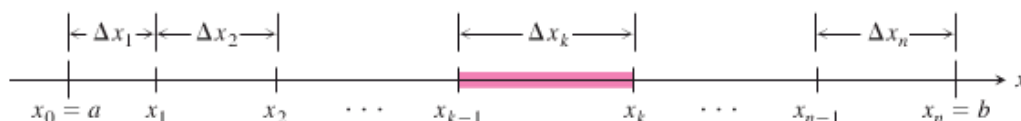


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1 , the width of the second $[x_1, x_2]$ is denoted Δx_2 , and the width of the k th subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



In each subinterval we select some point. The point chosen in the k th subinterval $[x_{k-1}, x_k]$ is called c_k . Then on each subinterval we stand a vertical rectangle that stretches from the x -axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x -axis, depending on whether $f(c_k)$ is positive or negative, or on the x -axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k . When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x -axis to the negative number $f(c_k)$.

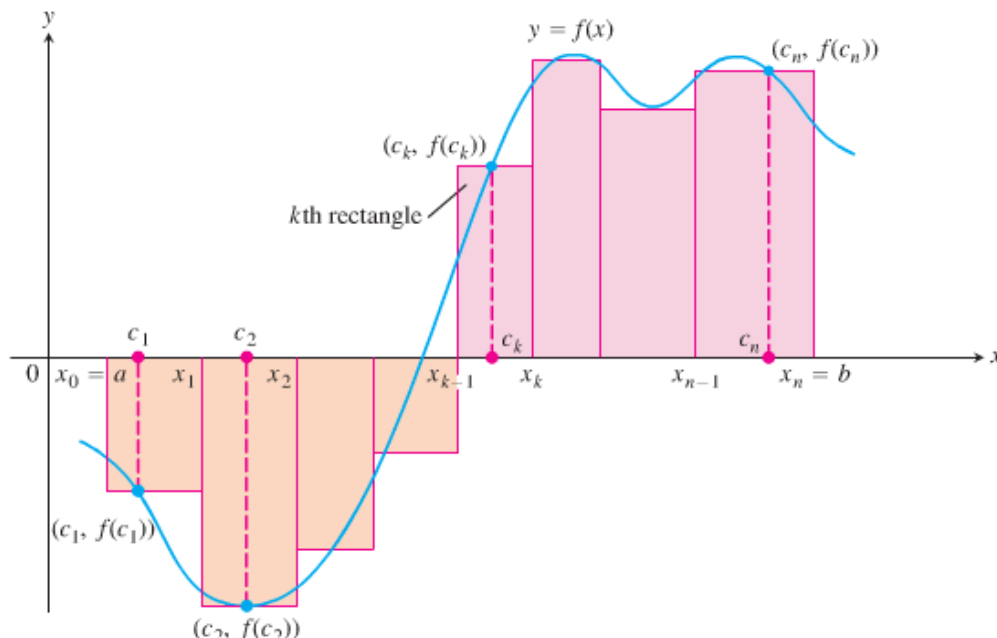


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x -axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

Finally we sum all these products to get

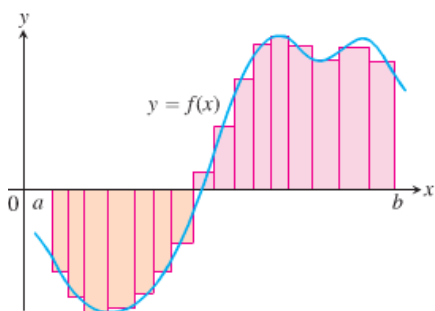
$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k.$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$** . There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

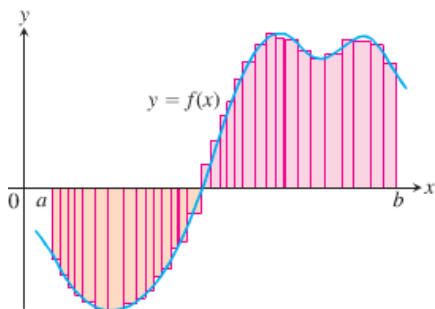
$$S_n = \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n . When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P , written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width. Let's look at an example of these ideas.



(a)



(b)

FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x -axis with increasing accuracy.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P : $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x -axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.

In each subinterval we select some point. The point chosen in the k th subinterval $[x_{k-1}, x_k]$ is called c_k . Then on each subinterval we stand a vertical rectangle that stretches from the x -axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x -axis, depending on whether $f(c_k)$ is positive or negative, or on the x -axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k . When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x -axis to the negative number $f(c_k)$.