# Parts with Trig Text

**Integration by Parts Formula**

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of $u$ and $v$, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for $u$ and $dv$. The next examples illustrate the technique. To avoid mistakes, we always list our choices for $u$ and $dv$, then we add to the list our calculated new terms $du$ and $v$, and finally we apply the formula in Equation (2).

**EXAMPLE 1** Find

$$\int x \cos x\,dx.$$

**Solution** We use the formula $\displaystyle\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= \cos x\,dx, \\ du &= dx, & v &= \sin x. \end{aligned} \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C. \qquad \blacksquare$$

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

**EXAMPLE 4** Evaluate

$$\int e^x \cos x \, dx.$$

**Solution** Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

**EXAMPLE 8** Evaluate

$$\int x^3 \sin x \, dx.$$

**Solution** With $f(x) = x^3$ and $g(x) = \sin x$, we list:

| $f(x)$ and its derivatives | | $g(x)$ and its integrals |
|---|---|---|
| $x^3$ | $(+)$ | $\sin x$ |
| $3x^2$ | $(-)$ | $-\cos x$ |
| $6x$ | $(+)$ | $-\sin x$ |
| $6$ | $(-)$ | $\cos x$ |
| $0$ | | $\sin x$ |

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \quad \blacksquare$$