G

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Contents

1 Improper Integrals

1.1 Infinite Intervals

$$\begin{aligned}
\int_a^\infty f(x)dx &= \lim_{b\to\infty}\int_a^b f(x)dx \\
\int_{-\infty}^b f(x)dx &= \lim_{a\to-\infty}\int_a^b f(x)dx \\
\int_{-\infty}^\infty f(x)dx &= \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx
\end{aligned} \tag{1}$$

1.2 convergent, divergent

The integrals with infinite intervals can be convergent or divergent, they are convergent if the limit exists, or divergent if the limit does not exist.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^n}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^n}dx \\
&= \lim_{b\to\infty}\int_1^b x^{-n}dx \\
&= \lim_{b\to\infty}[\frac{x^{-n+1}}{-n+1}] \\
&= \lim_{b\to\infty}[\frac{b^{-n+1}}{1-n} - \frac{1^{-n+1}}{1-n}] \\
&= \frac{1}{1-n}\lim_{b\to\infty}[\frac{1}{b^{n-1}} - 1]
\end{aligned} \tag{2}$$

So, $\int_1^\infty \frac{1}{x^n}dx$ converges if $n > 1$, and diverges if $n \leq 1$.

Example:

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$$

$$\lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}} dx + \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}} dx$$

$$\int \frac{1}{1+u^2} du \qquad\qquad u = e^x = du = e^x dx$$

$$\tan^{-1} e^x$$

$$\lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}} dx + \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}} dx \tag{3}$$

$$\lim_{a\to-\infty} [\tan^{-1} e^x]_a^0 + \lim_{b\to\infty} [\tan^{-1} e^x]_0^b$$

$$\tan^{-1} 1 - \tan^{-1} 0 + \tan^{-1} \infty - \tan^{-1} 1$$

$$\frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}$$

$$\frac{\pi}{2}$$

1.3 infinite discontinuities

Example:

$$\begin{gathered} \int_0^9 \frac{1}{\sqrt{x}} dx \\ \lim_{c\to 0^+} \int_c^9 \frac{1}{\sqrt{x}} dx \end{gathered} \tag{4}$$

1. for b: $\lim_{c\to b^-} \int_a^c f(x)dx$
2. for a: $\lim_{c\to a^+} \int_c^b f(x)dx$
3. for c: $\lim_{d\to c^+} \int_a^d f(x)dx + \lim_{e\to c^-} \int_e^b f(x)dx$

1.4 comparison test for convergence or divergence

If $f(x) \geq g(x)$ on $[a,\infty]$, then if $\int_a^\infty f(x)dx$ is convergent, $\int_a^\infty g(x)dx$ is convergent, and if $\int_a^\infty g(x)dx$ is divergent, $\int_a^\infty f(x)dx$ is divergent.

Example:

$$\int_1^\infty \frac{1}{x+\sin^2 x}dx \text{ and } \int_1^\infty \frac{1}{1+x}dx$$

$$\sin x \leq 1$$
$$\sin^2 x \leq 1$$
$$x + \sin^2 x \leq 1 + x$$
$$\frac{1}{x+\sin^2 x} \geq \frac{1}{1+x}$$

$$\lim_{b\to\infty} \int_1^b \frac{1}{1+x}dx$$
$$\lim_{b\to\infty} [\ln|1+x|]_1^b$$
$$\lim_{b\to\infty} [\ln(1+b) - \ln 2] = \infty$$

So both equations are divergent.