ISE - 364/464: Introduction to Machine Learning
Placement Exam

The goal of this placement exam is to ensure that you have enough background knowledge in calculus, linear algebra, probability & statistics, and python coding to take this somewhat math-intensive class. If you can accomplish both "Minimum" and "Intermediate" sections then you should have a solid background to feel relatively confident throughout the semester. If you are taking the 364 version of this class and are able to finish the questions in the "Minimum" section but struggle with the questions in the "Intermediate" section, you can still take the class and should be okay, but be sure to study and refresh your memory from your math classes to not fall behind. If you are taking the 464 version of this class and are able to finish the questions in the "Minimum" section but struggle with the questions in the "Intermediate" section, you may struggle with parts of the homework assignments as there will be more difficult problems for graduate students. If this is the case I would strongly recommend you consider your workload and possibly think about taking this section at a later time or even swapping to the 364 section, as this class may be much more demanding on you. Lastly, if you are in either 364 or 464 versions and find yourself struggling with the questions in the "Minimum" background section, you should properly learn the prerequisite mathematics required for this class and come back next year!

Grading: This "Exam" will not be graded (woohoo!); however, everyone should still submit their write-up of the solutions to Coursesite by E.O.D. 09/08/2024. I strongly encourage all to try their best to answer all of the questions herein, and properly evaluate your level of comfort and performance with the above-mentioned considerations in mind. Please be honest with yourself, as I do not want this class to be a miserable experience for you. This exam is not meant to be too difficult, but may take a few hours to go through, which is totally fine! Grab a coffee, find a quiet place, and see what you can do. We will discuss this exam further during our first week of class and you should have until September 6th to drop the class without receiving a W if you feel you need more background preparation.

Submitting: Only electronic submissions on Coursesite are accepted.

1 Minimum Background Test

Calculus

1. If $y = \frac{4}{5}x^5 + 15x^2 - x + 3$, what is the derivative of y with respect to x?

2. Let $f(x, y) = 4x\cos(2y)e^{-x}$.

a) What is the partial derivative of f with respect to x?

b) What is the partial derivative of f with respect to y?

Linear Algebra

1. Consider the matrix A and the vectors x and b below:

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

a) What is the inner product of the vectors x and b (This is also referred to as the dot-product and is written as $x^\top b$)?

b) What is the product Ax?

c) Is A invertable? If so, give the inverse, and if not, explain why.

d) Given the vector of variables $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, solve the linear system $Ay = b$ for y.

Probability & Statistics

1. Consider a sample of data $D = \{1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1\}$ obtained by flipping a coin 12 times. Further, let the random variable $X = 0$ when the coin lands up heads and $X = 1$ when the coin lands up tails.

a) What is the sample mean of this data?

b) What is the *sample* variance of this data? What is the *population* variance of this data?

c) What is the probability of observing this exact data D, assuming that the probability of obtaining a single heads or a single tails is equal (50% for heads and 50% for tails).

d) Notice that the probability of obtaining this outcome would be greater if the values of $\mathbb{P}(X = 0)$ and $\mathbb{P}(X = 1)$ were not equal to 0.5. What is the value that maximizes the probability of the sample D? Justify your answer.

Python Coding

If you do not already have python on your computer, I highly recommend following the link below to set up Anaconda on your system, which will allow you to access python through a variety of ways (Spyder, PyCharm, etc.; I myself use Spyder):
https://docs.anaconda.com/anaconda/install/
Then, use Python, Numpy, and Matplotlib or Seaborn (which can easily be installed with Anaconda) to answer the following questions.

1. Using Numpy, sample 500 datapoints from a Gaussian distribution with a mean of $(0, 0)$ and a covariance matrix of $\Sigma = I_{2\times 2}$ (the 2×2 identity matrix).

a) Plot the sample points.

b) Change the mean to $(-1, 1)$ and plot the samples points again.

c) Change the covariance matrix to $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and plot the samples points again.

d) Change the covariance matrix to $\Sigma = \begin{bmatrix} 1 & -1 \\ -0.5 & 1 \end{bmatrix}$ and plot the samples points again.

2. Using Numpy, generate a 5×5 matrix A of values from a uniform distribution with the range of $[0, 50]$.

a) Compute the transpose of the matrix.

b) Compute the inverse of the matrix if possible.

c) Create a 1D Numpy array x of length 5 with values generated according to a Gaussian distribution with mean 10 and variance 2 (this will be a vector in $\mathbb{R}^5$). Using Numpy, compute the matrix-vector product Ax. Then compute the matrix-vector product $x^\top A$. Then compute the matrix-vector product $A^\top x$.

2 Intermediate Background Test

Probability & Statistics

1. Write down the probability density function (PDF) of an n-dimensional Gaussian random vector $x \in \mathbb{R}^n$ with a mean μ and covariance matrix Σ.

2. (True or False) Let X and Y be two continuous random variables and let $\mathbb{E}$ and Var denote the expectation and variance of a random variable, respectively, and let α be a real number. Identify which statements are true. Those that are false, give the correct statement.

a) $\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X]$.

b) $\mathbb{E}[X + Y] = X + Y$.

c) If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

d) $\mathrm{Var}[\alpha X] = \alpha \mathrm{Var}[X]$.

d) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

3. Variance is defined as $\mathrm{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$. Prove that $\mathrm{Var}[X] = \mathbb{E}(X^2) + \mathbb{E}(X)^2$.

Calculus

1. Write down the general form of the natural logarithm (ln) of the Gaussian PDF that you wrote down from question 1 in the "Probability & Statistics" subsection of the "Intermediate Background Test" section.

2. Consider the function $f(x) = 5x^2 - 25 + 3\lambda x$, where λ is a real number.

a) What value of x is a stationary point (derivative of 0) of this function?

b) Is the point you computed a minimum, maximum, or saddle point of f? Justify your answer mathematically.

Linear Algebra

1. Recall that the dot product of two vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ can be written as $a^\top b = \|a\|_2 \cdot \|b\|_2 \cos(\theta)$, where $\|\cdot\|_2$ denotes the ℓ_2 euclidean vector norm and θ is the angle between the two vectors. Prove that $a^\top b < 0$ when the angle between the two vectors is obtuse (i.e., $\theta \in [90°, 180°]$).

2. Prove that the inverse of an invertable and symmetric matrix is also symmetric. That is, if A is symmetric and invertable with the inverse A^{-1}, show that $\left(A^{-1}\right)^\top = A^{-1}$.