Robotics

Path Planning

Path finding vs. trajectory optimization, local vs. global, Dijkstra, Probabilistic Roadmaps, Rapidly Exploring Random Trees, non-holonomic systems, car system equation, path-finding for non-holonomic systems, control-based sampling, Dubins curves

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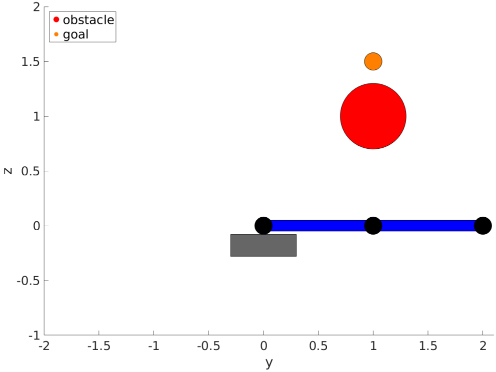
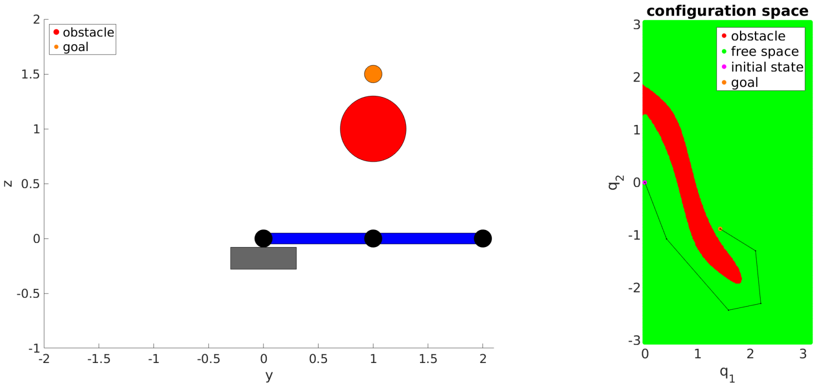
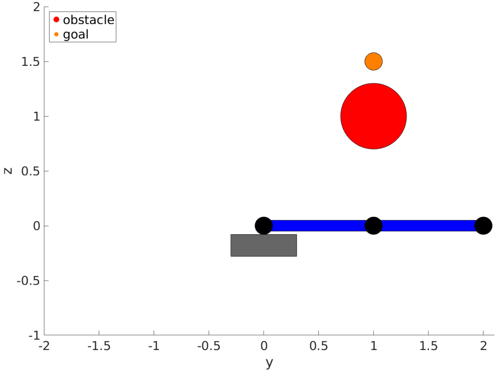
Path finding examples

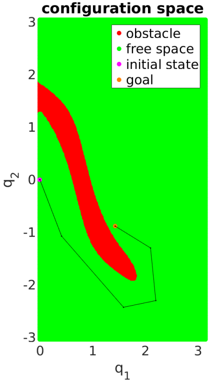


Path finding examples

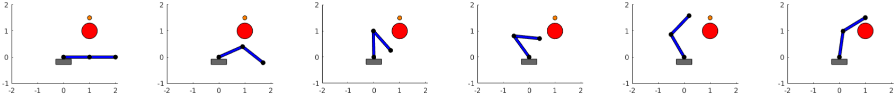


Path finding examples

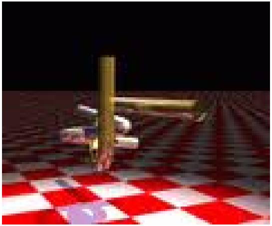




Trajectory from initial state to goal:

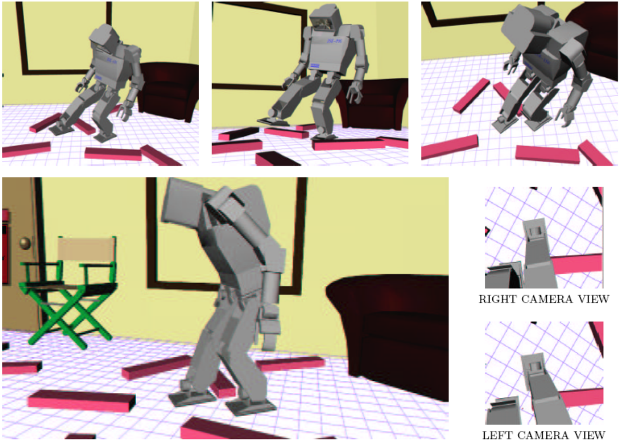


Path finding examples



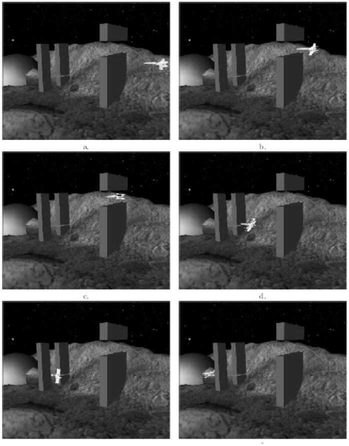
Alpha-Puzzle, solved with James Kuffner's RRTs

Path finding examples



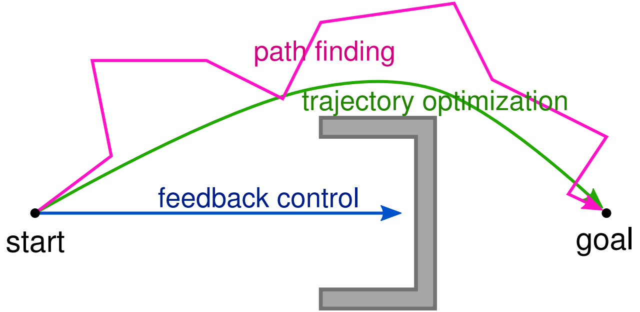
J. Kuffner, K. Nishiwaki, S. Kagami, M. Inaba, and H. Inoue. Footstep Planning Among Obstacles for Biped Robots. Proc. IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS), 2001.

Path finding examples



S. M. LaValle and J. J. Kuffner. Randomized Kinodynamic Planning. International Journal of Robotics Research, 20(5):378–400, May 2001.

Feedback control, path finding, trajectory optim.



- Feedback Control: E.g., $q_{t+1} = q_t + J^\sharp(y^* - \phi(q_t))$
- Trajectory Optimization: $\text{argmin}_{q_{0:T}} f(q_{0:T})$
- Path Finding: Find some $q_{0:T}$ with only valid configurations

Outline

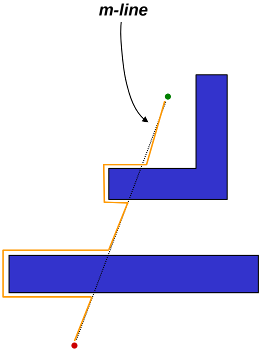
- **Really briefly:** Heuristics & Discretization (slides from Howie Choset's CMU lectures)
  - Bugs algorithm
  - Potentials to guide feedback control
  - Dijkstra

- **Sample-based Path Finding**
  - Probabilistic Roadmaps
  - Rapidly Exploring Random Trees

- **Non-holonomic systems**

Background

A better bug?



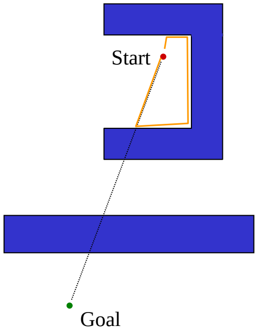
"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

A better bug?



"Bug 2" Algorithm

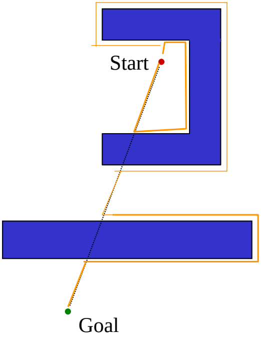
1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

NO! How do we fix this?

A better bug?



"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again ***closer to the goal***.

3) Leave the obstacle and continue toward the goal

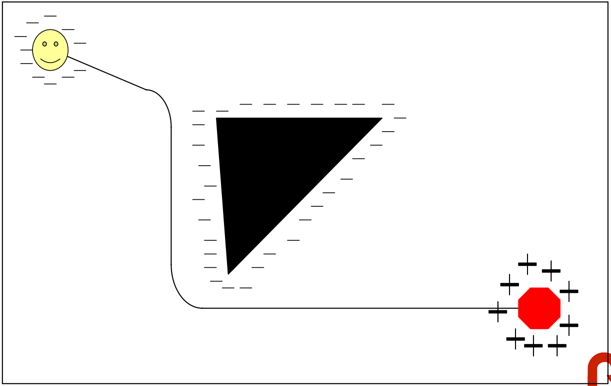
Better or worse than Bug1?

BUG algorithms – conclusions

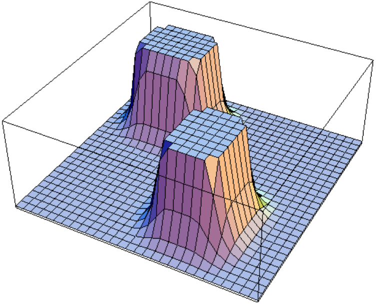
- Other variants: TangentBug, VisBug, RoverBug, WedgeBug, . . .
- only 2D! (TangentBug has extension to 3D)
- Guaranteed convergence
- Still active research:
  K. Taylor and S.M. LaValle: *I-Bug: An Intensity-Based Bug Algorithm*

⇒ Useful for minimalistic, robust 2D goal reaching
– not useful for finding paths in joint space

Start-Goal Algorithm: Potential Functions



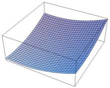
Repulsive Potential



Total Potential Function

$$U(q) = U_{\text{att}}(q) + U_{\text{rep}}(q)$$

$$F(q) = -\nabla U(q)$$



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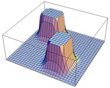

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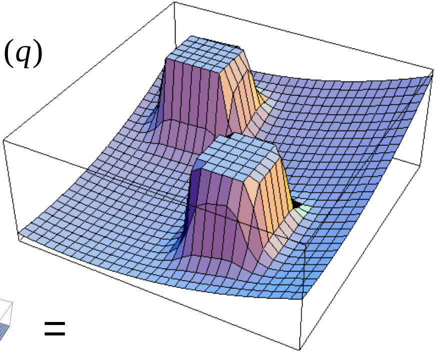
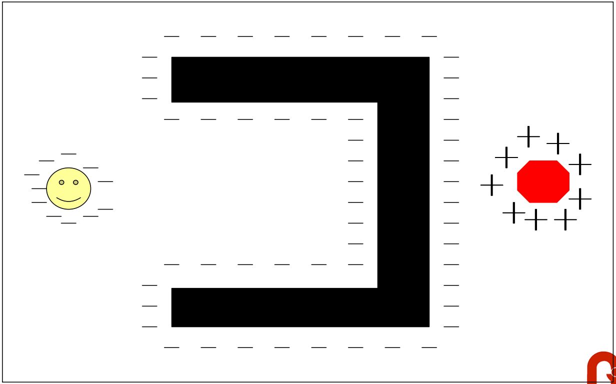


Local Minimum Problem with the Charge Analogy



Potential fields – conclusions

- Very simple, therefore popular
- In our framework: Combining a goal (endeffector) task variable, with a constraint (collision avoidance) task variable; then using inv. kinematics is *exactly* the same as "Potential Fields"

$\Rightarrow$ Does not solve locality problem of feedback control.

The Wavefront in Action (Part 1)

- Starting with the goal, set all adjacent cells with "0" to the current cell + 1
  - 4-Point Connectivity or 8-Point Connectivity?
  - Your Choice. We'll use 8-Point Connectivity in our example

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0’s are adjacent to cells with values >= 2
    - 0’s will only remain when regions are unreachable

7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	2
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 3)

- Repeat again...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Part 4)

- And again...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

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The Wavefront in Action (Part 5)

- And again until...

7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	7	7	7	7	7
4	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	7	6	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront in Action (Done)

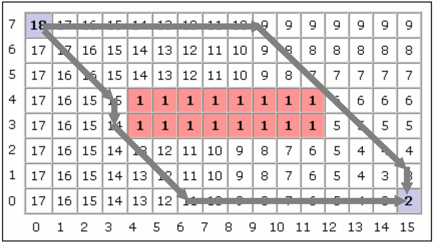
- You’re done
  - Remember, 0’s should only remain if unreachable regions exist

7	**18**	17	16	15	14	13	12	11	10	9	9	9	9	9	9	9
6	17	17	16	15	14	13	12	11	10	9	8	8	8	8	8	8
5	17	16	16	15	14	13	12	11	10	9	8	7	7	7	7	7
4	17	16	15	15	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	6	6	6	6
3	17	16	15	14	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	5	5	5	5
2	17	16	15	14	13	12	11	10	9	8	7	6	5	4	4	4
1	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	3
0	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	**2**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The Wavefront, Now What?

- To find the shortest path, according to your metric, simply always move toward a cell with a lower number
  - The numbers generated by the Wavefront planner are roughly proportional to their distance from the goal

Two possible shortest paths shown



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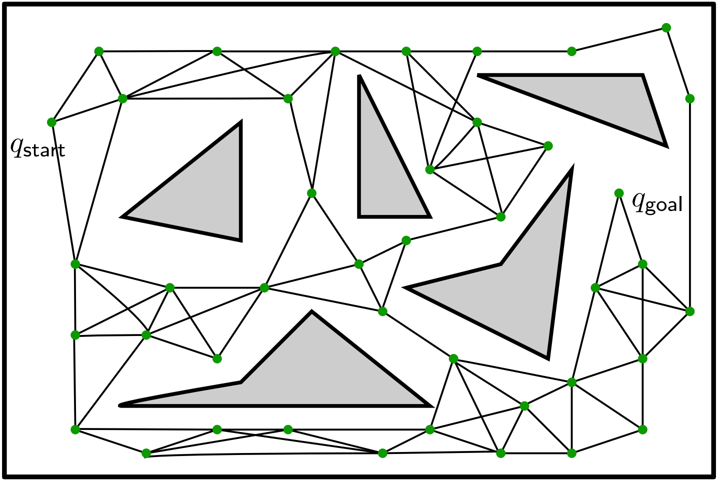
Dijkstra Algorithm

- Is efficient in **discrete domains**
  - Given start and goal node in an arbitrary graph
  - Incrementally label nodes with their distance-from-start
- Produces optimal (shortest) paths
- Applying this to continuous domains requires discretization
  - Grid-like discretization in high-dimensions is daunting! *(curse of dimensionality)*
  - What are other ways to “discretize” space more efficiently?

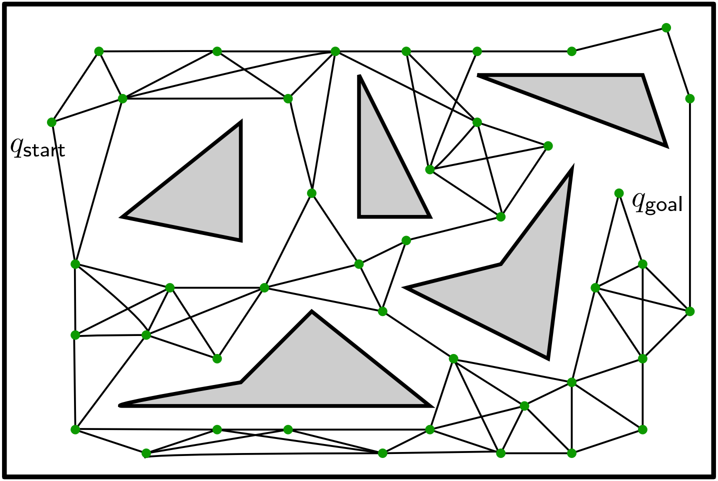
Sample-based Path Finding

Probabilistic Road Maps



[Kavraki, Svetska, Latombe, Overmars, 95]

Probabilistic Road Maps

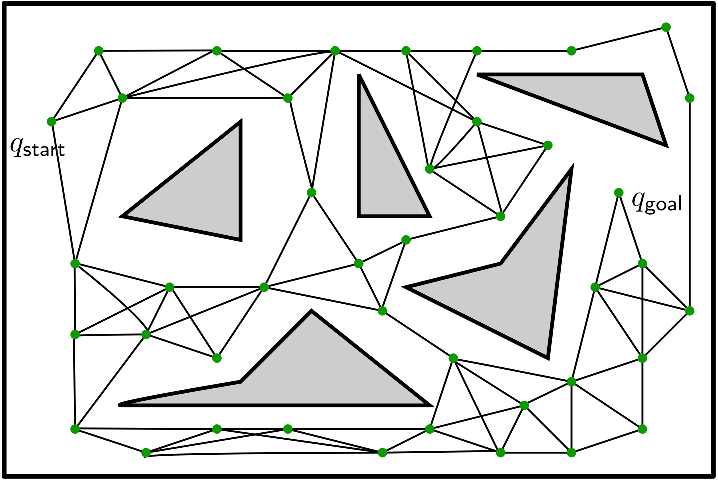


[Kavraki, Svetska, Latombe, Overmars, 95]

$q \in \mathbb{R}^n$ describes configuration

Q_{free} is the set of configurations without collision

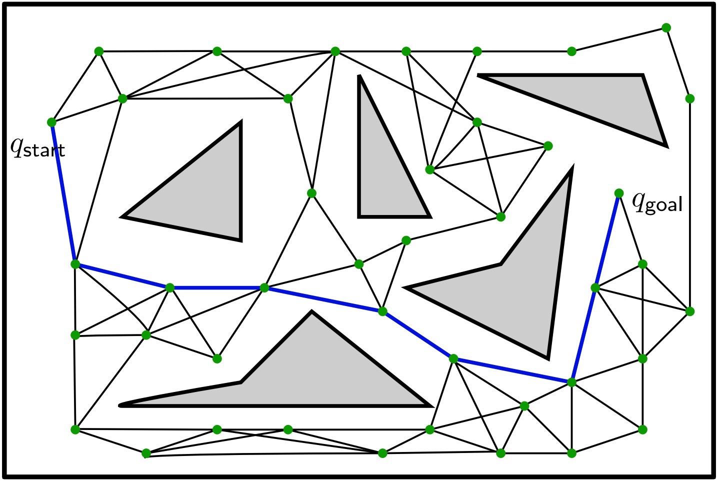
Probabilistic Road Maps



[Kavraki, Svetska, Latombe,Overmars, 95]

- Probabilistic Road Maps generate a graph $G = (V, E)$ of configurations
  - such that configurations along each edges are $\in Q_{\text{free}}$

Probabilistic Road Maps



Given the graph, use (e.g.) Dijkstra to find path from q_{start} to q_{goal}.

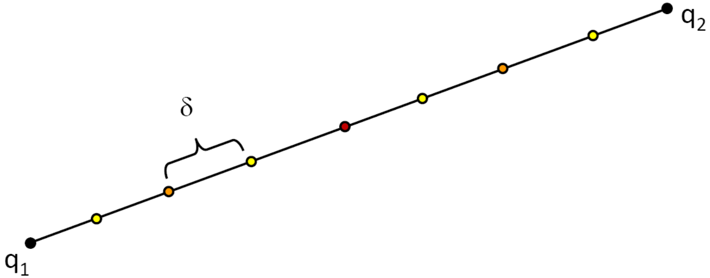
Probabilistic Road Maps – generation

Input: number n of samples, number k number of nearest neighbors
Output: PRM $G = (V, E)$

initialize $V = \emptyset$, $E = \emptyset$
while $|V| < n$ **do** *// find n collision free points q_i*
    $q \leftarrow$ random sample from Q
    if $q \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q\}$
end while
for all $q \in V$ **do** *// check if near points can be connected*
    $N_q \leftarrow k$ nearest neighbors of q in V
    for all $q' \in N_q$ **do**
        if path$(q, q') \in Q_{\text{free}}$ **then** $E \leftarrow E \cup \{(q, q')\}$
    end for
end for

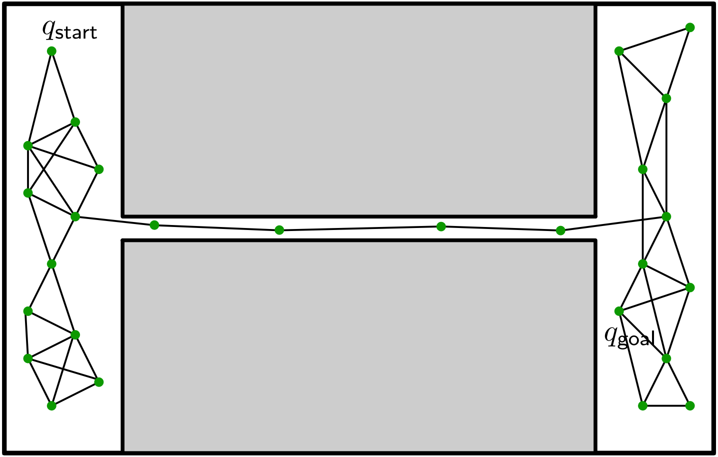
where path(q, q') is a local planner (easiest: straight line)

Local Planner



tests collisions up to a specified resolution δ

Problem: Narrow Passages



The smaller the gap (clearance ϱ) the more unlikely to sample such points.

PRM theory

(for uniform sampling in d-dim space)

- Let $a, b \in Q_{\text{free}}$ and γ a path in Q_{free} connecting a and b.

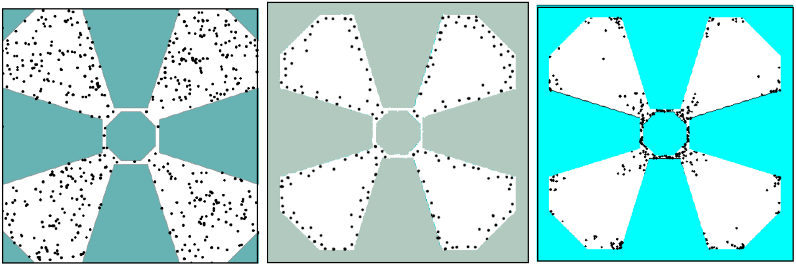
  Then the probability that PRM found the path after n samples is

$$P(\text{PRM-success} \mid n) \geq 1 - \frac{2|\gamma|}{\varrho} \; e^{-\sigma \varrho^d n}$$

  $\sigma = \frac{|B_1|}{2^d |Q_{\text{free}}|}$
  $\varrho =$ clearance of γ (distance to obstacles)
  (roughly: the exponential term are "volume ratios")

- This result is called *probabilistic complete* (one can achieve any probability with high enough n)
- For a given success probability, n needs to be exponential in d

Other PRM sampling strategies

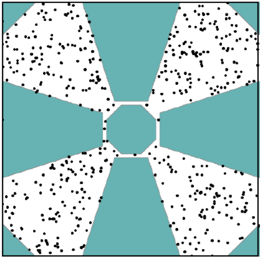


uniform　　Gaussian　　Bridge

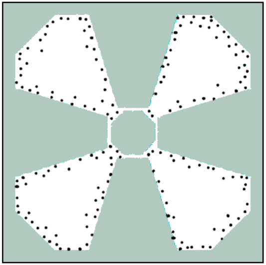
illustration from O. Brock's lecture

Gaussian: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; if $q_1 \in Q_{\text{free}}$ and $q_2 \notin Q_{\text{free}}$, add q_1 (or vice versa).

Other PRM sampling strategies



uniform



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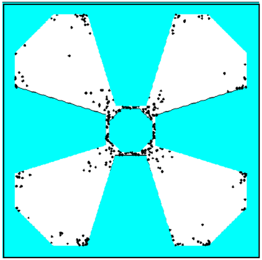

Bridge

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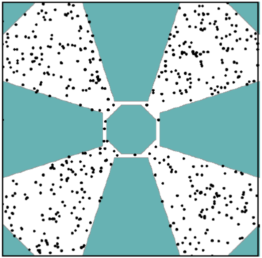
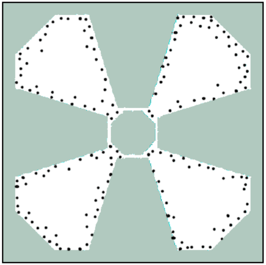
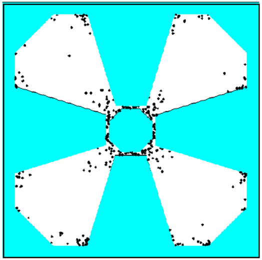
Gaussian: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; if $q_1 \in Q_{\text{free}}$ and $q_2 \notin Q_{\text{free}}$, add q_1 (or vice versa).

Bridge: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; $q_3 = (q_1 + q_2)/2$; if $q_1, q_2 \notin Q_{\text{free}}$ and $q_3 \in Q_{\text{free}}$, add q_3.

Other PRM sampling strategies


uniform


Gaussian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Bridge: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; $q_3 = (q_1 + q_2)/2$; if $q_1, q_2 \notin Q_{\text{free}}$ and $q_3 \in Q_{\text{free}}$, add q_3.

- Sampling strategy can be made more intelligence: "utility-based sampling"
- Connection sampling (once earlier sampling has produced connected components)

Probabilistic Roadmaps – conclusions

- Pros:
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees
  - Good to answer many queries in an *unchanged* environment

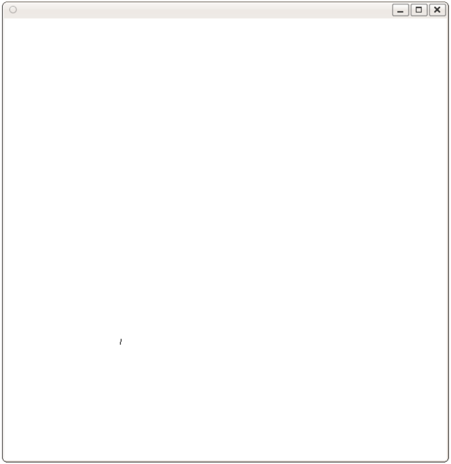
- Cons:
  - Precomputation of exhaustive roadmap takes a long time
    (but not necessary for “Lazy PRMs”)

Rapidly Exploring Random Trees

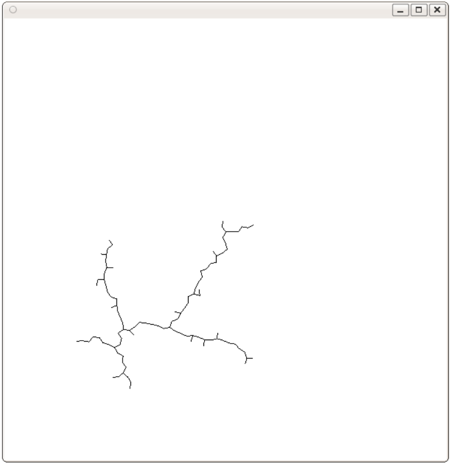
2 motivations:

- Single Query path finding: Focus computational efforts on paths for specific $(q_{\text{start}}, q_{\text{goal}})$
- Use actually controllable DoFs to incrementally explore the search space: *control-based* path finding.

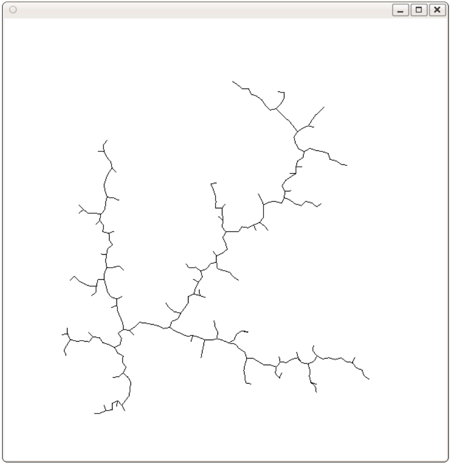
  (Ensures that RRTs can be extended to handling differential constraints.)



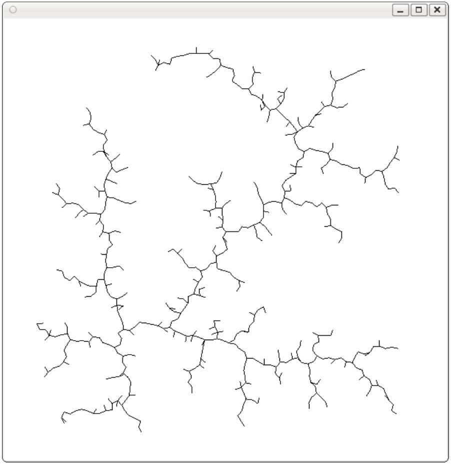
$$n = 1$$



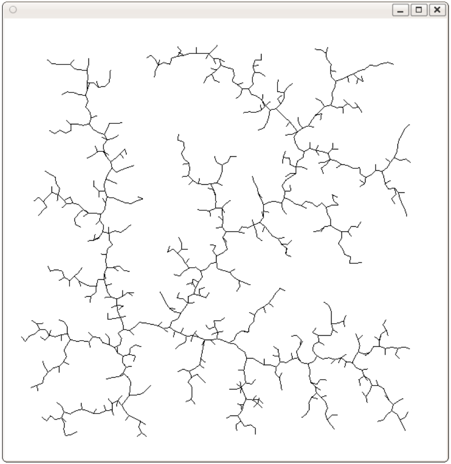
$n = 100$



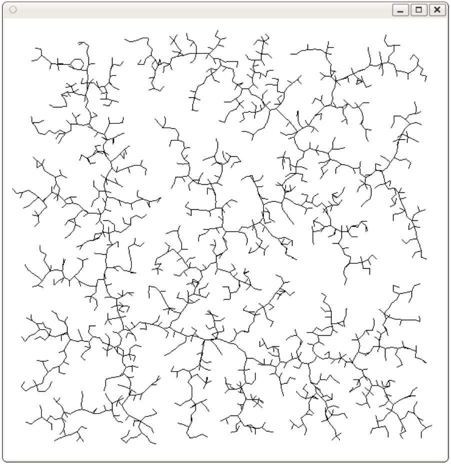
$n = 300$



$n = 600$



$$n = 1000$$



$$n = 2000$$

Rapidly Exploring Random Trees

Simplest RRT with straight line local planner and step size α

Input: q_{start}, number n of nodes, stepsize α
Output: tree $T = (V, E)$

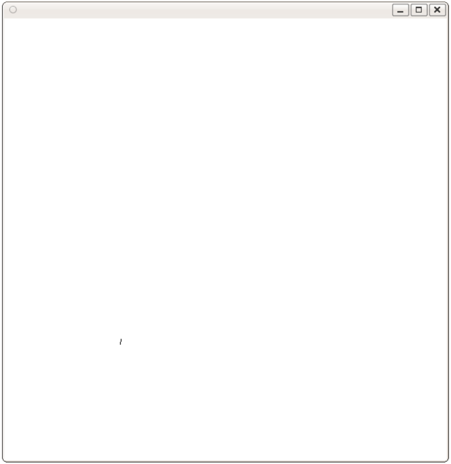
1: initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
2: **for** $i = 0 : n$ **do**
3: $q_{\text{target}} \leftarrow$ random sample from Q
4: $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
5: $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
6: **if** $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
7: **end for**

Rapidly Exploring Random Trees

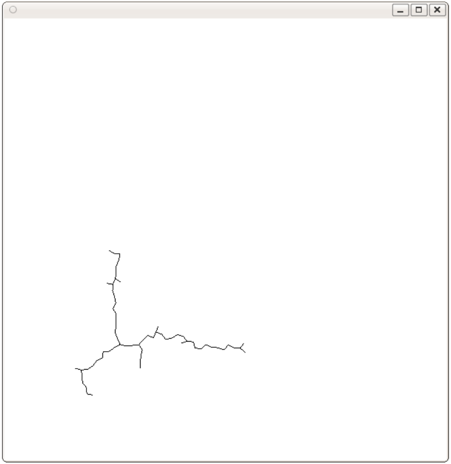
RRT growing directedly towards the goal

Input: q_{start}, q_{goal}, number n of nodes, stepsize α, β
Output: tree $T = (V, E)$

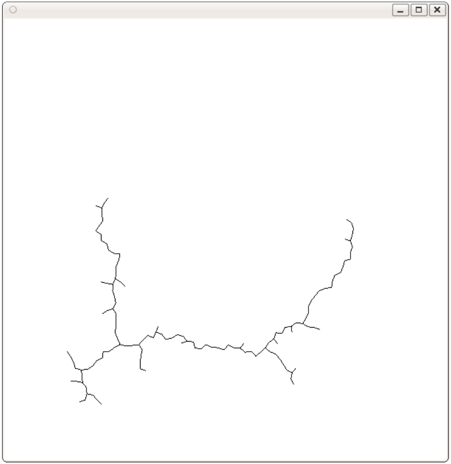
initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
for $i = 0 : n$ **do**
    if $\text{rand}(0,1) < \beta$ **then** $q_{\text{target}} \leftarrow q_{\text{goal}}$
    else $q_{\text{target}} \leftarrow$ random sample from Q
    $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
    $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
    if $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
end for



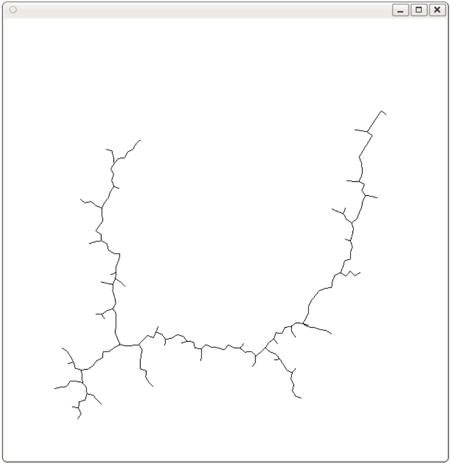
$$n = 1$$



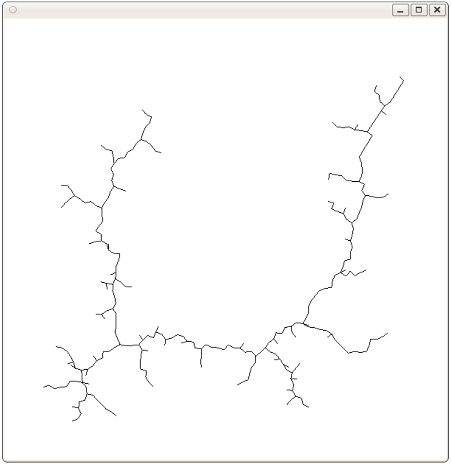
$n = 100$



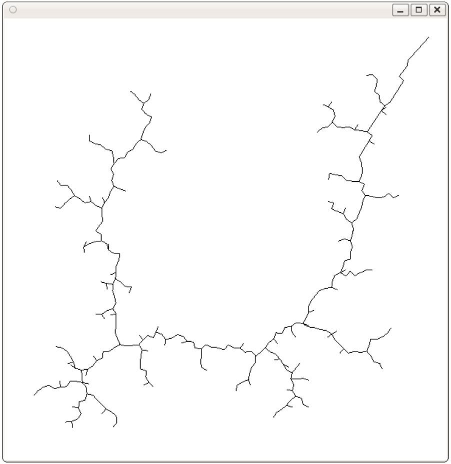
$n = 200$



$$n = 300$$



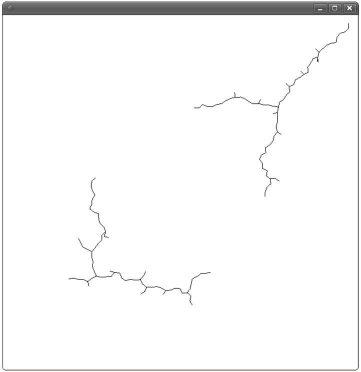
$$n = 400$$



$$n = 500$$

Bi-directional search

- grow two trees starting from q_{start} and q_{goal}



let one tree grow towards the other
(e.g., "choose q_{new} of T_1 as q_{target} of T_2")

Summary: RRTs

- Pros (shared with PRMs):
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees

- Pros (beyond PRMs):
  - Focus computation on single query $(q_{\text{start}}, q_{\text{goal}})$ problem
  - Trees from multiple queries can be merged to a roadmap
  - Can be extended to differential constraints (nonholonomic systems)

- To keep in mind (shared with PRMs):
  - The metric (for nearest neighbor selection) is sometimes critical
  - The local planner may be non-trivial

References

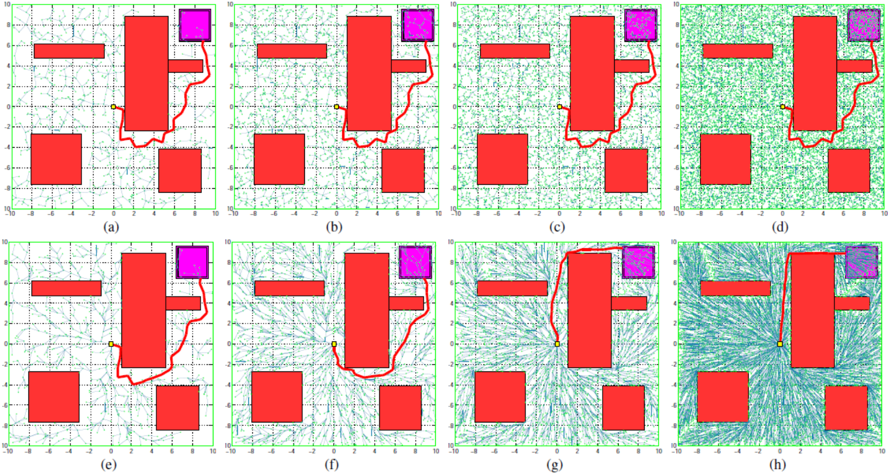
Steven M. LaValle: *Planning Algorithms*, `http://planning.cs.uiuc.edu/`.

Choset et. al.: *Principles of Motion Planning*, MIT Press.

Latombe's "motion planning" lecture, `http://robotics.stanford.edu/~latombe/cs326/2007/schedule.htm`

RRT*

Sertac Karaman & Emilio Frazzoli: Incremental sampling-based algorithms for optimal motion planning, arXiv 1005.0416 (2010).



RRT*

Sertac Karaman & Emilio Frazzoli: Incremental sampling-based algorithms for optimal motion planning, arXiv 1005.0416 (2010).

Algorithm 4: $\texttt{Extend}_{RRT^*}(G, x)$

$V' \leftarrow V$; $E' \leftarrow E$;
$x_{\text{nearest}} \leftarrow \texttt{Nearest}(G, x)$;
$x_{\text{new}} \leftarrow \texttt{Steer}(x_{\text{nearest}}, x)$;
if $\texttt{ObstacleFree}(x_{\text{nearest}}, x_{\text{new}})$ **then**
    $V' \leftarrow V' \cup \{x_{\text{new}}\}$;
    $x_{\text{min}} \leftarrow x_{\text{nearest}}$;
    $X_{\text{near}} \leftarrow \texttt{Near}(G, x_{\text{new}}, |V|)$;
    for *all* $x_{\text{near}} \in X_{\text{near}}$ **do**
        if $\texttt{ObstacleFree}(x_{\text{near}}, x_{\text{new}})$ **then**
            $c' \leftarrow \texttt{Cost}(x_{\text{near}}) + c(\texttt{Line}(x_{\text{near}}, x_{\text{new}}))$;
            if $c' < \texttt{Cost}(x_{\text{new}})$ **then**
                $x_{\text{min}} \leftarrow x_{\text{near}}$;
    $E' \leftarrow E' \cup \{(x_{\text{min}}, x_{\text{new}})\}$;
    for *all* $x_{near} \in X_{\text{near}} \setminus \{x_{\text{min}}\}$ **do**
        if $\texttt{ObstacleFree}(x_{\text{new}}, x_{\text{near}})$ *and* $\texttt{Cost}(x_{\text{near}}) > \texttt{Cost}(x_{\text{new}}) + c(\texttt{Line}(x_{\text{new}}, x_{\text{near}}))$ **then**
            $x_{\text{parent}} \leftarrow \texttt{Parent}(x_{\text{near}})$;
            $E' \leftarrow E' \setminus \{(x_{\text{parent}}, x_{\text{near}})\}$;
            $E' \leftarrow E' \cup \{(x_{\text{new}}, x_{\text{near}})\}$;
return $G' = (V', E')$

Non-holonomic systems

Non-holonomic systems

- We define a **nonholonomic system** as one with **differential constraints**:

$$\dim(u_t) < \dim(x_t)$$

$\Rightarrow$ *Not all degrees of freedom are directly controllable*

- Dynamic systems are an example!

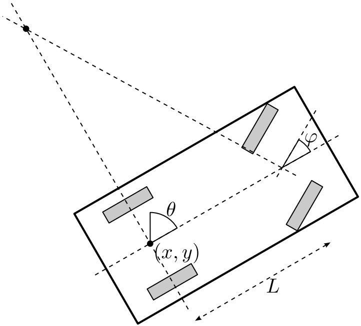
- General definition of a differential constraint:
  For any given state x, let U_x be the tangent space that is generated by controls:

$$U_x = \{\dot{x} \;:\; \dot{x} = f(x, u),\; u \in U\}$$

(non-holonomic $\iff \dim(U_x) < \dim(x)$)

The elements of U_x are elements of T_x subject to additional *differential constraints*.

Car example



$$\dot{x} = v\ \cos\theta$$
$$\dot{y} = v\ \sin\theta$$
$$\dot{\theta} = (v/L)\ \tan\varphi$$
$$|\varphi| < \Phi$$

State $q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ **Controls** $u = \begin{pmatrix} v \\ \varphi \end{pmatrix}$

Sytem equation $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v\ \cos\theta \\ v\ \sin\theta \\ (v/L)\ \tan\varphi \end{pmatrix}$

Car example

- The car is a *non-holonomic* system: not all DoFs are controlled, $\dim(u) < \dim(q)$
  We have the *differential constraint* $\dot{q}$:

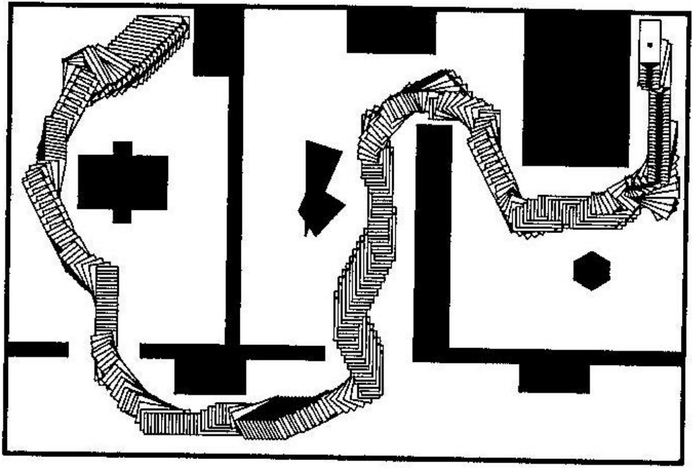
  $$\dot{x} \sin\theta - \dot{y} \cos\theta = 0$$

  "A car cannot move directly lateral."

- Analogy to dynamic systems: Just like a car cannot instantly move sidewards, a dynamic system cannot instantly change its position q: the current change in position is *constrained* by the current velocity $\dot{q}$.

Path finding for a non-holonomic system

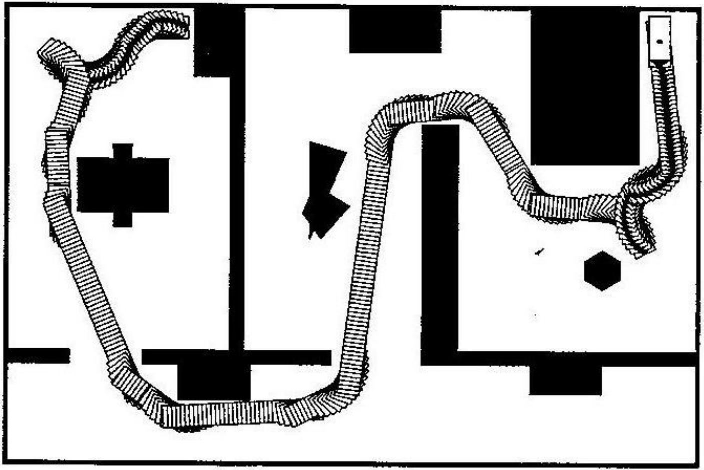
Could a car follow this trajectory?



This is generated with a PRM in the state space $q = (x, y, \theta)$ *ignoring the differential constraint*.

Path finding with a non-holonomic system

This is a solution we would like to have:



The path respects **differential constraints.**

Each step in the path corresponds to setting certain controls.

Control-based sampling to grow a tree

- Control-based sampling: fulfils none of the nice exploration properties of RRTs, but fulfils the differential constraints:

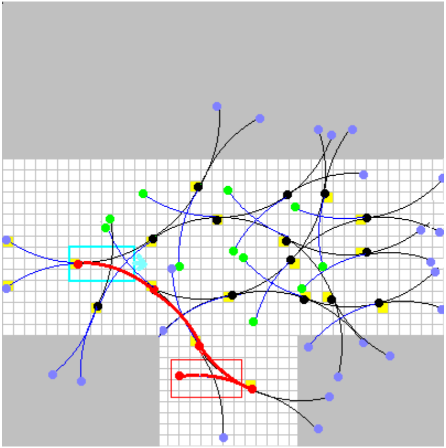
  1) Select a $q \in T$ from tree of current configurations

  2) Pick control vector u at random

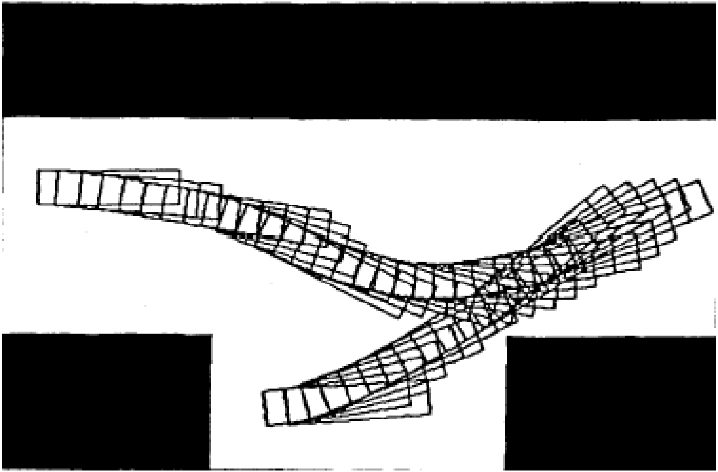
  3) Integrate equation of motion over short duration (picked at random or not)

  4) If the motion is collision-free, add the endpoint to the tree

Control-based sampling for the car

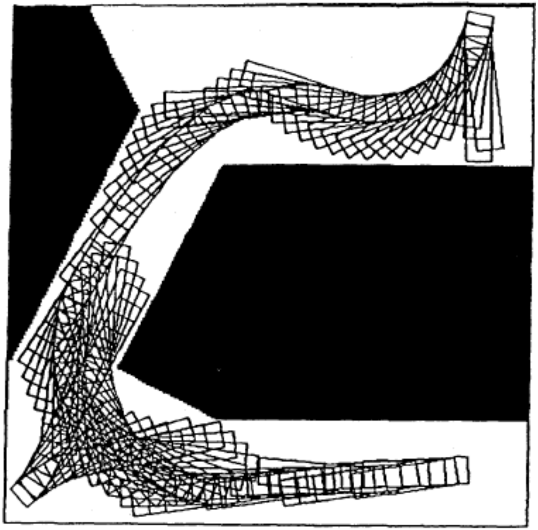


1) Select a $q \in T$
2) Pick v, ϕ, and τ
3) Integrate motion from q
4) Add result if collision-free

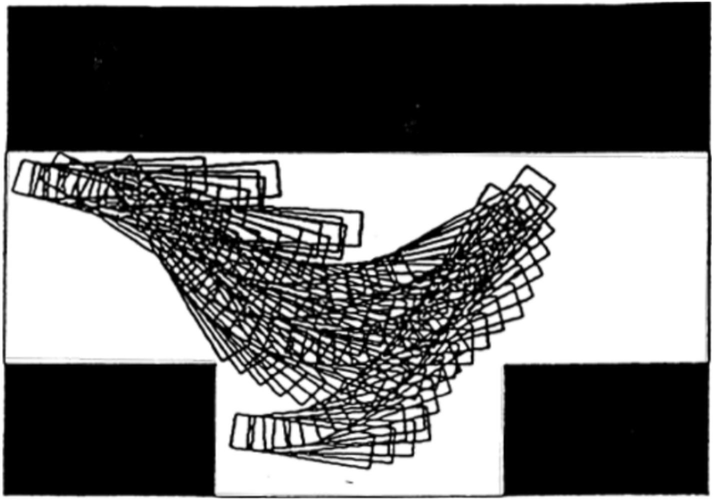


J. Barraquand and J.C. Latombe. Nonholonomic Multibody Robots: Controllability and Motion Planning in the Presence of Obstacles. Algorithmica, 10:121-155, 1993.

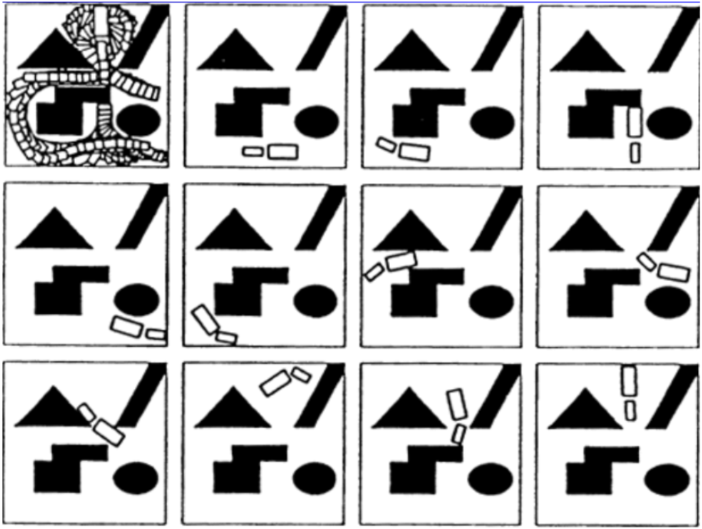
car parking



car parking



parking with only left-steering



with a trailer

Better control-based exploration: RRTs revisited

- RRTs with differential constraints:

Input: q_{start}, number k of nodes, time interval τ
Output: tree $T = (V, E)$

```
1: initialize V = {q_start}, E = ∅
2: for i = 0 : k do
3:     q_target ← random sample from Q
4:     q_near ← nearest neighbor of q_target in V
5:     use local planner to compute controls u that steer q_near towards q_target
6:     q_new ← q_near + ∫_{t=0}^{τ} q̇(q, u) dt
7:     if q_new ∈ Q_free then V ← V ∪ {q_new}, E ← E ∪ {(q_near, q_new)}
8: end for
```

- Crucial questions:
- How meassure *near* in nonholonomic systems?
- How find controls u to steer towards target?

Configuration state metrics

Standard/Naive metrics:

- Comparing two 2D rotations/orientations $\theta_1, \theta_2 \in SO(2)$:
  a) Euclidean metric between $e^{i\theta_1}$ and $e^{i\theta_2}$
  b) $d(\theta_1, \theta_2) = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}$
- Comparing two configurations $(x, y, \theta)_{1,2}$ in $\mathbb{R}^2$:
  Eucledian metric on $(x, y, e^{i\theta})$
- Comparing two 3D rotations/orientations $r_1, r_2 \in SO(3)$:
  Represent both orientations as unit-length quaternions $r_1, r_2 \in \mathbb{R}^4$:
  $$d(r_1, r_2) = \min\{|r_1 - r_2|, |r_1 + r_2|\}$$
  where $|\cdot|$ is the Euclidean metric.
  (Recall that r_1 and $-r_1$ represent exactly the same rotation.)

Configuration state metrics

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  where $|\cdot|$ is the Euclidean metric.
  (Recall that r_1 and $-r_1$ represent exactly the same rotation.)

- **Ideal metric:**
  Optimal cost-to-go between two states x_1 and x_2:
- Use optimal trajectory cost as metric
- This is as hard to compute as the original problem, of course!!
  $\rightarrow$ Approximate, e.g., by neglecting obstacles.

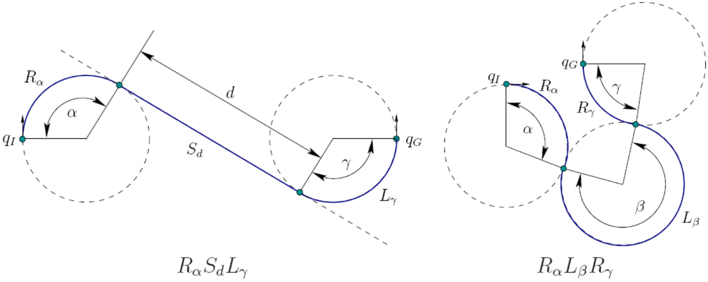
Side story: Dubins curves

- Dubins car: constant velocity, and steer $\varphi \in [-\Phi, \Phi]$

- Neglecting obstacles, there are only **six** types of trajectories that connect any configuration $\in \mathbb{R}^2 \times \mathbb{S}^1$:
$$\{LRL, RLR, LSL, LSR, RSL, RSR\}$$

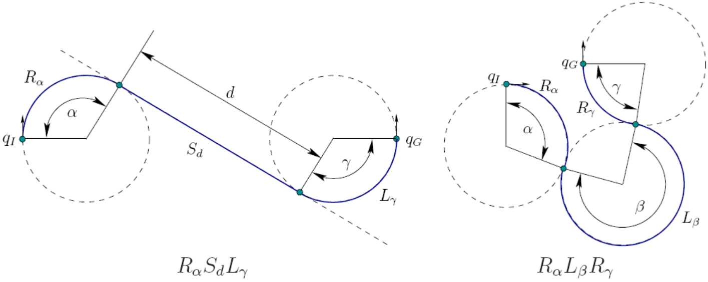
- annotating durations of each phase:
$$\{L_\alpha R_\beta L_\gamma, , R_\alpha L_\beta R_\gamma, L_\alpha S_d L_\gamma, L_\alpha S_d R_\gamma, R_\alpha S_d L_\gamma, R_\alpha S_d R_\gamma\}$$
with $\alpha \in [0, 2\pi), \beta \in (\pi, 2\pi), d \geq 0$

Side story: Dubins curves



$\rightarrow$ By testing all six types of trajectories for (q_1, q_2) we can define a Dubins metric for the RRT – and use the Dubins curves as controls!

Side story: Dubins curves



$\rightarrow$ By testing all six types of trajectories for (q_1, q_2) we can define a Dubins metric for the RRT – and use the Dubins curves as controls!

- **Reeds-Shepp curves** are an extension for cars which can drive back. (includes 46 types of trajectories, good metric for use in RRTs for cars)