

## **\*\*Understanding Reliability in Technical Systems\*\***

Reliability refers to the ability of a technical item?be it a product, machinery, or service?to perform as required within a specified operational context and time frame. We increasingly rely on the functioning of various technical systems in daily life, expecting them to work consistently without failures, which can lead to dire consequences or customer dissatisfaction. Although definitions of reliability differ across industries, this book adopts a broad perspective that encompasses hardware, software, and user interfaces. The reliability evaluation involves comparing predicted performance against required performance, determining whether an item can be deemed reliable based on customer and supplier expectations, as illustrated in the provided diagram.

**\*\*Threats, Reliability, and Metrics Overview\*\*** Deliberate threats, including physical attacks like arson and cyberattacks, are executed by threat actors exploiting specific vulnerabilities. Natural events such as earthquakes represent threats without human actors. RAM, denoting reliability, availability, and maintainability, extends to RAMS by including safety. Reliability encompasses metrics derived from random variable analyses of time-to-failure and repair time. Key reliability metrics are: 1. Mean Time To Failure (MTTF) 2. Failure frequency (failures per time unit) 3. Survivor probability (absence of failure in a time interval) 4. Availability at a given time. For instance, average availability is calculated as uptime divided by total operational time. Reliability analysis branches into hardware, software, and human reliability, focusing mainly on hardware with physical and systems approaches.

## **\*\*Reliability Approaches in Technical Systems\*\***

The reliability of technical items can be approached physically or systematically. In the physical approach, the strength (S) and load (L) are modeled as random variables, and failure occurs when load exceeds strength. The survival probability (R) is defined as the likelihood that strength is greater than load, mathematically expressed as  $R = P(S > L)$ . The time to failure (T) is the earliest time when strength is less than load,  $T = \text{minimum time such that } S(t) < L(t)$ . Conversely, the systems approach uses the probability distribution function  $F(t)$  of time-to-failure T without detailed modeling of load and strength. Reliability metrics, including the survivor probability and mean time-to-failure, derive directly from  $F(t)$ . This approach, akin to actuarial assessments, is focused on systems reliability and emphasizes model simplicity and realism to ensure practical relevance in reliability analyses.

## **\*\*Understanding System Reliability through Block Diagrams\*\***

This section highlights the significance of modeling system structures for reliability studies, emphasizing the complexity paradigm in contrast to traditional approaches. Although this book focuses on simple systems, the concepts can extend to complicated systems. Reliability Block Diagrams (RBDs) serve as a key tool for representing system functions, comprising blocks (or components) that can either be functioning or failed. Each block is denoted by a binary state variable, which is defined as: - 1: Block is functioning - 0: Block has failed An RBD with 'n' blocks indicates the system function operates successfully if certain blocks are active, showing connections between components to track reliability effectively.

### **\*\*Reliability Analysis Methods: FMECA and FTA\*\***

Failure Mode, Effects, and Criticality Analysis (FMECA) ranks potential system failures based on failure rates and severity, represented in a risk matrix. The Risk Priority Number (RPN), calculated by multiplying severity (S), occurrence (O), and detection (D) ratings ( $RPN = S \times O \times D$ ), helps prioritize design concerns. While FMECA is most effective during the design phase for identifying weaknesses, it has limitations, such as insufficient consideration of human errors and redundant systems. Fault Tree Analysis (FTA) presents logical relationships between system faults and their causes, aiding in risk assessment across various industries. FTA can provide qualitative and quantitative insights into potential system failures, determined by environmental factors and component faults.

### **### Summary**

The text discusses an event tree analysis (ETA) for the initiating event of a blockage in the gas outlet line, featuring critical outcomes based on the functionality of three protection systems. The consequences range from severe (rupture or explosion of a separator leading to total installation loss) to non-critical (controlled shutdown resulting in production downtime). ETA provides different insights compared to fault tree analysis (FTA), which focuses on failure causes of the protection systems. It also contrasts fault trees with reliability block diagrams (RBDs), stating that both methods yield similar results when only using simple gates. The text emphasizes the importance of analyzing fault trees to uncover potential causes of failures, while indicating that RBDs are structured based on the functions of components. It is recommended to begin with a fault tree during analysis for a more comprehensive understanding of risks.

### Python Equations

1. **\*\*Event Tree Outcomes\*\***: ``python def event\_tree(outcomes): """ Evaluates outcomes based on the functioning of three protection systems (PSDs, PSVs, Rupture disc). Outcomes: 1: Rupture or explosion 2: Gas flowing from rupture disc 3: Gas relieved to flare 4:

```

Controlled shutdown      """    if (PSDs and PSVs and rupture_disc_open):
return 4 # Controlled shutdown elif (not PSVs):      return 1 #
Rupture or explosion    elif (rupture_disc_open):      return 2 #
Gas flowing from rupture disc else:      return 3 # Gas relieved
to flare ``` 2. **Fault Tree to RBD Relationship**: ```python def
fault_tree_to_rbd(fault_tree):      """    Converts a fault tree into an
RBD structure by replacing gates with series or parallel components.
"""    if gate_type == 'or':      return
"series_structure(components)"    elif gate_type == 'and':
return "parallel_structure(components)" ``` 3. **Probability of System
Failure**: ```python def system_failure_probability(failure_rates):
"""    Calculates the overall probability of system failure based on
component failure rates.      """    return 1 - (1 - failure_rates[0]) *
(1 - failure_rates[1]) * ... * (1 - failure_rates[n]) ``` This concise
distillation of the original text clarifies the essential elements while
maintaining the key concepts and context.

```

## **\*\*Structure Function in Reliability Systems\*\***

In a reliability system with  $(n)$  components, each component  $(i)$  can be in a functioning (1) or failed (0) state, represented by a state vector  $(\mathbf{x} = (x_1, x_2, \dots, x_n))$ . The structure's functioning state is represented by a structure function  $(\phi(\mathbf{x}))$ , which outputs 1 if the structure functions and 0 if it fails. For a series structure, all components must function, leading to:  $(\phi(\mathbf{x}) = x_1 \times x_2 \times \dots \times x_n)$ . For a parallel structure, at least one component must function, given by:  $(\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2) \dots (1 - x_n))$ . In Boolean algebra, the "and" operation corresponds to multiplication, while the "or" operation corresponds to the logical maximum: - And:  $(x_1 \cdot x_2 = \min(x_1, x_2))$  - Or:  $(x_1 \vee x_2 = x_1 + x_2 - x_1 \cdot x_2)$ . This framework is vital for analyzing the reliability of electronic and mechanical systems.

## **### Concise Rewrite**

```

##### Structure Function Consider a system with (n) components, each
having two states: functioning (1) or failed (0). The state of component
(i) is represented as: ```python x_i = 1 # functioning x_i = 0 #
failed ``` The overall state vector of the system is  $(\mathbf{x} = (x_1, x_2, \dots, x_n))$ . The structure's functionality can be
described by the binary function: ```python phi(x) = 1 # if the
structure is functioning phi(x) = 0 # if failed ``` ##### 4.6.1 Series
Structure In a series configuration, the system works only if all
components function. The structure function is: ```python def
phi_series(x):      return prod(x) # Equivalent to multiplication of all
x_i ``` This can also be expressed as: ```python phi_series(x) = min(x)

```

# Minimum value of  $x_i$  ```` ##### 4.6.2 Parallel Structure For a parallel structure, the system functions if at least one component is functioning. The structure function is: ``python def phi\_parallel(x): return 1 - prod(1 - x) # Product of (1- $x_i$ ) terms ```` Alternatively, it can be expressed using logical symbols: ``python phi\_parallel(x) = max(x) # Maximum value of  $x_i$  ```` For two components: ``python phi\_parallel(x1, x2) = x1 + x2 - (x1 \* x2) # Using binary values ```` ##### Boolean Algebra Boolean algebra deals with binary variables (1 for true, 0 for false). The operations are defined as: ``python # 'and' operation x1\_and\_x2 = x1 \* x2 # Product is minimum # 'or' operation x1\_or\_x2 = x1 + x2 - (x1 \* x2) # Sum minus product for binary values ```` ##### 4.6.3 k-out-of-n Structure A  $(k \text{ out of } n)$  structure functions if at least  $k$  of the  $n$  components are functioning. A series structure is a  $(1 \text{ out of } n)$  configuration, while a parallel structure is a  $(n \text{ out of } n)$  configuration. ### Summary This condensed rewrite presents the structure function clearly, defining key equations using Python-like syntax to improve clarity and reduce redundancy while retaining essential concepts regarding series, parallel configurations, and Boolean algebra.

## **\*\*KooN:G Structure Overview\*\***

A KooN:G structure operates if at least  $k$  out of  $n$  components are functional. In this context, a series structure is modeled as  $n$  out of  $n$  functioning (nooN:G), while a parallel setup corresponds to at least one functioning (1ooN:G). To denote KooN structures simply, the "good" reference is omitted. The structure function for a KooN structure can be expressed as: "1 if the sum of functioning components is greater than or equal to  $k$ , otherwise 0." For example, a 2oo3 structure allows one component to fail; a three-engine airplane needing two engines running exemplifies this. The structure function can also be formulated as: "sum of pairs of functioning components minus twice the product of all components being functional."

## **The structure function for a k-out-of-n (koo) system is defined as:**

``python def phi(x, n, k): return 1 if sum(x) >= k else 0 ```` For a 2oo3 system, which allows one component failure, it can be expressed as: ``python def phi\_2oo3(x): return (x[0] \* x[1] + x[0] \* x[2] + x[1] \* x[2] - 2 \* x[0] \* x[1] \* x[2]) ```` A truth table for the 2oo3 system shows it functions (outputs 1) in the last four combinations of states where at least two components are operational. This structure is commonly utilized in safety systems, like gas detectors, requiring a signal from at least two detectors for an action. The 2oo3 configuration minimizes false alarms while maintaining reliability.

**In a 2-out-of-3 (2oo3) structure, at least two components must signal an**

**event (e.g., gas detection) for an alarm. The structure function is given by:**

```
python def phi(x1, x2, x3): return (x1 * x2 + x1 * x3 + x2 * x3 -
2 * x1 * x2 * x3)`` This configuration reduces the likelihood of a
false alarm since all three detectors rarely fail concurrently. A truth
table summarizes possible states, showing the system functions (state 1)
in the last four combinations of three binary inputs (0 or 1) and fails
(state 0) in the first four. It's crucial to identify single points of
failure, defined as components whose failure leads to the overall failure
of the system.
```

**Redundancy increases reliability, as shown by the relation:**

$[\phi(x \text{ ? } y) \text{ ? } \phi(x) \text{ ? } \phi(y)]$  This indicates better structures are achieved through component-level redundancy rather than system-level redundancy. This principle can be complex with multiple failure modes, like in fire detection systems. For a system of  $(n)$  components:

1. Let  $(C = \{1, 2, \dots, n\})$ .
2. Define minimal path set and cut set for reliability analysis.

In Python, the redundancy relationship can be represented as:

```
python def redundancy(phi_x, phi_y): return
max(phi_x, phi_y) # Represents phi(x ? y) ? phi(x) ? phi(y) def
component_set(n): return list(range(1, n + 1)) # Represents C = {1,
2, ..., n}`` This succinctly summarizes the concepts of redundancy and
component structures.
```

**A modular decomposition of a coherent structure  $(C, ?)$  consists of disjoint modules  $(A_i, ?i)$ , for  $i = 1, ?, r$ , organized by ? such that:**

- 1)  $C = ?(A_i)$  for  $i \neq j$  ( $A_i \text{ ? } A_j = ?$ )
- 2)  $?(x) = ?[?1(xA1), ?2(xA2), ?, ?r(xAr)]$

Prime modules cannot be further partitioned. Bayesian networks (BNs), introduced by Judea Pearl in 1985, are directed acyclic graphs (DAGs) used in reliability analysis. Nodes represent states, and directed arcs (edges) indicate cause-effect relationships. For example, an arc from A to B (written as  $?A, B?$ ) suggests B depends on A's state. In Python:

```
python def modular_decomposition(C, A): return union(A) #
C is the union of modules A_i def state_function(x): return
omega([chi_i(xA_i) for i in range(r)]) # ?(x) as the function of modules
``
```

**In case of a reactor core meltdown, the LPCI is essential for injecting water to cool the core, activated when two of three pressure transmitters detect low pressure. If LPCI fails, a meltdown occurs, needing at least two of four LPIPs for proper function.**

Key tasks include:

1. Create a Reliability Block Diagram (RBD) for LPCI.
2. Identify minimal cut sets, defining their order.
3. Develop a Bayesian Network (BN) for LPCI failure.

For a related chemical reactor system,

three flow transmitters (2oo3 configuration) and three pressure transmitters (also 2oo3) are used to trigger shutdown valves (1oo2 configuration) when high flow or pressure is detected. In Python:

```

python def LPCI_activation(pressure_transmitters):
    return sum(pressure_transmitters) >= 2
def flow_shutdown(flow_transmitters):
    return sum(flow_transmitters) >= 2
def pressure_shutdown(pressure_transmitters):
    return sum(pressure_transmitters) >= 2
def shutdown_valves(flow_or_pressure):
    return sum(flow_or_pressure) >= 1
    """ These functions model the
    activation logic for safety systems in reliability analysis.
  
```

**The empirical survivor function estimates probabilities for future experiments, but continuous functions are more commonly used in reliability studies. Given a continuously distributed time-to-failure variable  $(T)$  with probability density function  $(f(t))$  and cumulative distribution function  $(F(t))$ :**

1. The cumulative distribution function is defined as:  $F(t) = \Pr(T \leq t) = \int_0^t f(u) du$
2. The probability density function is the derivative of the cumulative distribution function:  $f(t) = \lim_{\Delta t \rightarrow 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$

In Python, these can be represented as:

```

python def F(t, f):
    return integrate.quad(f, 0, t)[0]
def f(t, F_delta):
    return (F(t + F_delta) - F(t)) / F_delta
    """ Here,  $(F(t))$  calculates the
    cumulative probability up to time  $(t)$  using integration, and  $(f(t))$ 
    computes the probability density based on the change in  $(F)$  over a
    small increment  $(\Delta t)$ .
  
```

**The probability density function (PDF)  $(f(t))$  quantifies the likelihood of failure within time interval  $(t, t + \Delta t]$ . It satisfies two conditions:  $(f(t) \geq 0)$  and  $(\int_0^{\infty} f(t) dt = 1)$  for  $(t \geq 0)$ . The cumulative distribution function (CDF)  $(F(t))$  is non-decreasing, where  $(0 \leq F(t) \leq 1)$ . The probability that failure occurs in interval  $(t_1, t_2]$  can be computed as  $(F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(u) du)$ .**

In Python:

```

python def pdf(t):
    return max(0, f_value) # replace
    f_value with actual formula
def cdf(t):
    return integrate.quad(pdf,
    0, t)[0] # returns the integral value from 0 to t
def prob_failure(t1, t2):
    return cdf(t2) - cdf(t1)
    """ This summarization maintains the
    core principles of the PDF and CDF while providing a clear representation
    in Python code.
  
```

**This section reviews various parametric time-to-failure distributions, including exponential, gamma, Weibull, normal, and lognormal distributions, along with their properties.**

**\*\*Exponential Distribution:\*\*** - Probability density function (PDF):  
`python def f(t, lambd): return lambd * exp(-lambd * t) if t > 0`  
 else 0  
 - Survivor function: `python def R(t, lambd): return exp(-lambd * t)`  
 - Mean time to failure (MTTF): `python MTTF = 1 / lambd`  
 - Variance: `python var_T = 1 / lambd**2`  
 The probability of surviving the MTTF is approximately 36.8%. The failure rate function is constant: `python def z(t, lambd): return lambd`  
 With a median time-to-failure: `python def t_m(lambd): return 0.693 / lambd`  
 For a scaled time-to-failure,  $(T_1 \sim \text{exp}(\lambda/a))$  and MTTF scales linearly. Approximation for small  $(\lambda t)$ :  
`python approx = 1 - lambd * t`

**Let  $\lambda$  be the failure rate for plant  $i$  ( $i=1,2$ ). The survival function for a randomly selected item is given by:**

`python R(t) = p * exp(- $\lambda_1$  * t) + (1 - p) * exp(- $\lambda_2$  * t)`  
 The MTTF can be calculated as: `python MTTF = (p *  $\lambda_1$  * exp(- $\lambda_1$  * t) + (1 - p) *  $\lambda_2$  * exp(- $\lambda_2$  * t)) / (p * exp(- $\lambda_1$  * t) + (1 - p) * exp(- $\lambda_2$  * t))`

For items with a stepwise constant failure rate, where the failure when not running is given by probability  $p$  and constant failure rates are  $\lambda_1$  (running) and  $\lambda_2$  (standby), the average failure rate is: `python  $\lambda_t = \lambda_d + p * \lambda_r + (1 - p) * \lambda_s$`  where  $\lambda_t = \lambda$  (start failures per time unit). This framework captures the failure dynamics in mixed item populations and stepwise operation.

**The Chi-square distribution is crucial in statistics, relating to standard normal variables  $(U_i)$ . If  $(X = \sum_{i=1}^n U_i^2)$ , then  $(X)$  follows a chi-square distribution with  $(n)$  degrees of freedom. Its probability density function is:**

`python def chi_square_pdf(x, n): from scipy.special import gamma`  
 return  $(1 / (x^{(n/2 - 1)} * (2^{(n/2)} * \text{gamma}(n/2)))) * (\text{np.exp}(-x / 2))$  for  $x > 0$   
 With mean  $(E(X) = n)$  and variance  $(\text{var}(X) = 2n)$ . The gamma distribution models failure rate  $(\lambda)$  for a product, with the density of  $(\lambda)$  given by: `python def gamma_density(lam, k, alpha): from scipy.special import gamma`  
 return  $(\alpha^k * \text{lam}^{(k - 1)} * \text{np.exp}(-\alpha * \text{lam})) / \text{gamma}(k)$  for  $\text{lam} > 0, \alpha > 0, k > 0$   
 The Weibull distribution, defined by parameters  $(\alpha)$  and  $(\theta)$ , has cumulative distribution function (CDF): `python def weibull_cdf(t, alpha, theta): return 1 - \text{np.exp}(-(t / \theta)^alpha)` for  $t > 0$   
 And probability density function (PDF): `python def weibull_pdf(t, alpha, theta): return (\alpha / \theta) * (t / \theta)^{(\alpha - 1)} * \text{np.exp}(-(t / \theta)^alpha)` for  $t > 0$   
 Weibull distribution with  $(\alpha = 1)$  shows exponential behavior.

**The Gumbel distribution under consideration provides the failure**

### characteristics for a random variable $(Y)$ :

1. **Distribution Function**:  $(F_Y(y) = 1 - e^{-e^y})$  2. **Probability Density**:  $(f_Y(y) = e^y e^{-e^y})$  3. **Failure Rate**:  $(z_Y(y) = e^y)$  The expected value is  $(E(T) = \mu - \alpha \gamma)$ , where  $(\gamma \approx 0.5772)$  is Euler's constant and  $(\mu)$  and  $(\alpha)$  are known parameters. For a time-to-failure distribution truncated at zero, the survivor function is:  $[R_T(t) = \frac{P(T > t)}{P(T > 0)} = \exp\left(-e^{-\frac{t - \mu}{\alpha}}\right) \text{ for } t > 0]$  Introducing parameters  $(\beta = e^{-\frac{\mu}{\alpha}})$  and  $(\eta = \frac{1}{\alpha})$ :  $[R_T(t) = \exp\left(-\beta(e^{\eta t} - 1)\right)]$  The time-to-failure  $(T)$  is a function of the random depths caused by corrosion, leading to a simplified survival function for high  $(n)$  approximating:  $[R(t) \approx \exp\left(-\beta(e^{\eta t} - 1)\right)]$  Covariates like temperature or voltage can significantly affect reliability, with a covariate vector  $(\mathbf{s} = (s_1, s_2, \dots, s_k))$  representing multiple influencing factors.

### The paragraph presents various reliability distributions and models. The extreme value distributions include the Gumbel and Weibull forms:

1. Gumbel Distribution (Y): - Cumulative Distribution:  $(F_Y(y) = e^{-e^y})$  for  $(y < \infty)$  - Standardized form:  $(f_Y(y) = e^y \cdot e^{-e^y})$  2. Weibull Distribution (T): - Cumulative Distribution:  $(F_T(t) = 1 - e^{-((t - \theta)/\eta)^\beta})$  for  $(t \geq \theta)$  - Standardized form:  $(F_Y(y) = 1 - e^{-y^\beta})$  for  $(y > 0)$  For time-to-failure due to pitting corrosion, where failure occurs when the maximum pit depth equals the wall thickness  $(\theta)$ , the time,  $(T)$ , is given by  $(T = \min(T_1, T_2, \dots, T_n))$ . The distribution of the depth  $(D_i)$  of pits is right-truncated exponential, and the failure time distribution can be approximated as:  $(R(t) \approx e^{-\beta(e^{\eta t} - 1)})$  Covariates like temperature or pressure influence the reliability models such as accelerated failure time, which scales time-to-failure based on deviations from baseline conditions.

### The time-to-penetration distribution function, $(F_{T_i}(t))$ , is expressed as:

-  $(F_{T_i}(t) = \Pr(T_i \leq t) = 1 - F_{D_i}(\theta - \frac{t e^{\eta t}}{k} - 1)^k)$  for  $(0 \leq t \leq k\theta)$ . The survivor function,  $(R(t))$ , is given by:  $(R(t) = \Pr(T > t) = [1 - F_{T_i}(t)]^n \approx e^{-n F_{T_i}(t)})$ . For large  $(n)$ , substituting leads to:  $(R(t) \approx e^{-n e^{\eta t/k} - 1} e^{\eta t/\theta})$ . Introducing  $(\beta = \frac{n}{e^{\eta \theta} - 1})$  and  $(\rho = \frac{\eta}{k})$ , we find:  $(R(t) \approx e^{-\beta(e^{\rho t} - 1)})$ . This implies that time-to-failure due to



pitting corrosion has a truncated Gumbel distribution. Reliability can be affected by covariates, which may include temperature or pressure. The accelerated failure time (AFT) model illustrates how covariates scale the time-to-failure relative to a baseline, expressed as:  $R(t|s) = R_0[h(s)t]$  and its related probability density function. For constant failure rates:  $\lambda_s = h(s) \lambda_0$ .

**We analyze field data from identical items used in various conditions to assess reliability and identify influential covariates. Key covariates impacting a shutdown valve include fluid corrosiveness, erosiveness, flow rate, pressure, proof-testing principle, and frequency.**

Three reliability models are utilized: **Accelerated Failure Time (AFT) Model**, **Arrhenius Model**, and **Proportional Hazards Model**. 1.

**AFT Model**: A new covariate vector  $(s = (s_1, s_2, \dots, s_k))$  affects reliability through a scaling factor  $h(s)$ :  
 - Time-to-failure:  $T \sim \frac{T_0}{h(s)}$  - Survivor function:  $R(t|s) = R_0[h(s)t]$  - Probability density function:  $\frac{dR(t|s)}{dt} = h(s) f_0[h(s)t]$  - Failure rate function:  $z(t|s) = h(s) z_0[h(s)t]$  - Mean Time To Failure (MTTF):  $MTTF_s = \frac{MTTF_0}{h(s)}$   
 2. **Arrhenius Model**: For reaction rates,  $k = A_0 e^{-E_a/(kT)}$ , where  $T$  is temperature in Kelvin,  $A_0$  is a scaling factor, and  $E_a$  is activation energy. For failure times, it adapts to:  $L(T) = A e^{-E_a/(kT)}$ , linking failure time to temperature.

**The survivor function of time-to-failure  $R(t|\mathbf{s})$  with covariate vector  $\mathbf{s}$  is given by:**

1. Survivor Function:  $R(t|\mathbf{s}) = \Pr(T > t | \mathbf{s}) = R_0[h(\mathbf{s})t]$  2. Probability Density Function:  $\frac{dR(t|\mathbf{s})}{dt} = h(\mathbf{s}) f_0[h(\mathbf{s})t]$  3. Failure Rate Function:  $f(t|\mathbf{s}) \propto z(t|\mathbf{s})$  and  $z(t|\mathbf{s}) = h(\mathbf{s}) z_0[h(\mathbf{s})t]$  4. Mean Time To Failure (MTTF) at covariate  $\mathbf{s}$ :  $MTTF_{\mathbf{s}} = \frac{MTTF_0}{h(\mathbf{s})}$  For constant failure rates, the relationship between  $\lambda_{\mathbf{s}} = h(\mathbf{s}) \lambda_0$  and  $\lambda_0$  is:  $\lambda_{\mathbf{s}} = h(\mathbf{s}) \lambda_0$  The acceleration factor between two covariate vectors  $\mathbf{s}_1, \mathbf{s}_2$  is:  $AF(\mathbf{s}_1, \mathbf{s}_2) = \frac{g(\mathbf{s}_2)}{g(\mathbf{s}_1)} = \frac{MTTF_1}{MTTF_2}$  The Arrhenius model for time-to-failure is expressed as:  $L(\tau) = A \exp\left(-\frac{E_a}{k\tau}\right)$  With acceleration factor:  $A(\tau_1, \tau_2) = \exp\left(\frac{E_a}{k} \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)\right)$  For the Weibull distribution, assuming constant shape parameter:  $A(\tau_1, \tau_2) = \frac{\theta_1}{\theta_2}$

**The Arrhenius model describes the temperature dependence of chemical**

## reaction rates and times-to-failure for materials like semiconductors.

### The reaction rate is given by:

$\log(r(T)) = \log(A) - (E / (k * T))$ . Here,  $T$  is the temperature in Kelvin,  $r(T)$  is the reaction rate,  $A$  is a constant,  $E$  is the activation energy, and  $k$  is the Boltzmann constant. The time-to-failure model is expressed as:  $L(T) = A * \exp(E / (k * T))$ , where  $A$  is a constant related to material properties. The acceleration factor between two temperatures  $T_1$  and  $T_2$  is:  $A(T_1, T_2) = (E / L(T_1))(1/T_1 - 1/T_2)$ . For constant failure rates, MTTF( $T$ ) serves as  $L(T)$ , with survivor functions derived from this relationship. Proportional hazards models further segment failure rates into time-dependent and stress-dependent components.

### In testing a semiconductor at two temperatures $(T_1 < T_2)$ , the acceleration factor due to temperature change is expressed as:

$$A(T_1, T_2) = \frac{E_a}{k} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \exp \left( \frac{E_a}{k} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \right)$$

This factor is zero when the temperatures are equal, positive when  $(T_1 < T_2)$ , and negative otherwise. For items with a constant failure rate, the Mean Time To Failure (MTTF) at temperatures can be expressed as:  $R(t | T) = \exp(-\lambda t)$

Where  $(\lambda = \frac{1}{\text{MTTF}(T)})$ . Under constant failure rates, the relationship between failure rates at different temperatures leads to:  $\lambda_2 = A(T_1, T_2) \lambda_1$  In the Weibull distribution case, the acceleration factor becomes:

$$A(T_1, T_2) = \left( \frac{\theta_1}{\theta_2} \right)^{1/\beta} \text{Proportional hazards models express the failure rate } (z(t | \mathbf{s})) \text{ as: } (z(t | \mathbf{s}) = z_0(t) g(\mathbf{s})) \text{ This separates the time-dependent effects from stress-related influences.}$$

### Proportional Hazards (PH) models are widely used in reliability analysis to incorporate covariates into failure rate estimations. The failure rate $(z(t | \mathbf{s}))$ can be expressed as:

$(z(t | \mathbf{s}) = z_0(t) \cdot g(\mathbf{s}))$  where  $(z_0(t))$  is time-dependent, and  $(g(\mathbf{s}))$  is stress-dependent. Two continuous distributions relevant in reliability are the Uniform Distribution and the Beta Distribution. 1. **Uniform Distribution**: A random variable  $(X)$  is uniformly distributed over an interval  $([a, b])$  if:  $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$  The expected value is  $E(X) = \frac{a+b}{2}$  and variance is  $\text{Var}(X) = \frac{(b-a)^2}{12}$ . 2. **Beta Distribution**: A random variable  $(X)$  follows a beta distribution with parameters  $(\alpha)$  and  $(\beta)$  over  $([0, 1])$  if:  $f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$  for  $(0 \leq x \leq 1)$

1\). The mean and variance are:  $E(X) = \frac{\alpha}{\alpha + \beta}$   
 $Var(X) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$ . When  $(\alpha = \beta = 1)$ , the beta distribution  
equals the uniform distribution  $(\text{beta}(1, 1) = \text{unif}(0, 1))$ .  
\).

**This section covers two continuous distributions: the uniform and beta distributions, which serve various purposes in reliability analysis but are rarely used as time-to-failure distributions.**

1. **\*\*Uniform Distribution\*\***: A random variable  $(X)$  is uniformly distributed over an interval  $([a, b])$  denoted as  $(X \sim \text{unif}(a, b))$ . The probability density function is:  $f_X(x) = \frac{1}{b-a}$  for  $(a \leq x \leq b)$ , else 0. The expected value is  $(E(X) = \frac{a + b}{2})$  and the variance is  $(\text{var}(X) = \frac{(b-a)^2}{12})$ . 2. **\*\*Beta Distribution\*\***: A random variable  $(X)$  has a beta distribution  $(X \sim \text{beta}(\alpha, \beta))$  when its density is given by:  $f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$  for  $(0 \leq x \leq 1)$ , else 0. The mean is  $(E(X) = \frac{\alpha}{\alpha + \beta})$  and the variance is  $(\text{var}(X) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)})$ . Next, three discrete distributions are introduced: binomial, geometric, and negative binomial, typically observed under the "binomial situation," which entails  $(n)$  independent trials with two outcomes and constant probability  $(p)$ . The binomial probability mass function is:  $\Pr(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ ,  $\text{quad } (x = 0, 1, \dots, n)$ . \]

**For a continuous uniform distribution  $(X \sim \text{unif}(a, b))$ , the expected value is**

$E(X) = \frac{a + b}{2}$  and the variance is  $\text{var}(X) = \frac{(b - a)^2}{12}$ . The Beta distribution, denoted  $(X \sim \text{beta}(\alpha, \beta))$  for  $(0 \leq x \leq 1)$ , has a probability density function defined as:  $f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$ . Its mean and variance are given by:  $E(X) = \frac{\alpha}{\alpha + \beta}$  and  $\text{var}(X) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$ . In discrete distributions, such as the binomial distribution, if  $(X)$  represents the number of successes in  $(n)$  independent Bernoulli trials with success probability  $(p)$ , then its probability mass function is:  $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$  for  $(x = 0, 1, \dots, n)$ . Here,  $\binom{n}{x}$  represents the binomial coefficient  $(\frac{n!}{x!(n-x)!})$ .

**The event E occurs in a time interval  $(t, t + \Delta t]$  with probability  $\Delta t +$**

$o(\lambda t)$ , where  $\lambda$  is a positive constant. The likelihood of more than one event occurring in this interval is negligible ( $o(\lambda t)$ ), and events in disjoint intervals are independent. Let  $N(t)$  represent the number of occurrences of  $E$  in  $(0, t]$ . This process follows a homogeneous Poisson process (HPP) with rate  $\lambda$ , indicating a constant event rate over time. The probability of  $E$  occurring exactly  $n$  times in  $(0, t]$  is given by:

$\Pr(N(t) = n) = (\lambda t)^n * e^{-(\lambda t)} / n!$  for  $n = 0, 1, 2, \dots$ . We can also observe the same pattern for intervals  $(s, s+t]$ . For a very short interval  $(t, t + \lambda t]$  where at most one event can occur:  $\Pr(N(\lambda t) = 1) = \lambda t$ . The mean number of events in  $(0, t]$  is  $E[N(t)] = \lambda t$ , where  $o(\lambda t)$  denotes negligible effects as  $\lambda t$  approaches zero.

**In this section, "increasing" and "decreasing" refer to "nondecreasing" and "nonincreasing."**

**\*\*Uniform Distribution\*\*:** For a uniform distribution over  $(0, b]$ , the probability density function (PDF) is given by:  $f(t) = \frac{1}{b}$  for  $0 < t \leq b$ ,  $F(t) = \frac{t}{b}$  for  $0 < t \leq b$ . The failure rate function  $z(t) = \frac{1 - \frac{t}{b}}{b - t}$  shows it is strictly increasing, classifying it as IFR (Increasing Failure Rate). **\*\*Exponential Distribution\*\*:** For an exponential distribution,  $f(t) = \lambda e^{-\lambda t}$  and  $z(t) = \lambda$  (constant). Thus, it belongs to both IFR and DFR (Decreasing Failure Rate) families. **\*\*Weibull Distribution\*\*:** Given by  $F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\alpha}$ , it is IFR for  $\alpha \geq 1$  and DFR for  $\alpha \leq 1$ . **\*\*Gamma Distribution\*\*:** Defined as  $f(t) = \frac{\lambda}{\Gamma(\alpha)} (\lambda t)^{\alpha-1} e^{-\lambda t}$ . It is IFR when  $\alpha \geq 1$  and DFR when  $\alpha \leq 1$ . The document also introduces IFRA (Increasing Failure Rate Average) and DFRA distributions, stating IFRA if  $-\log R(t)$  increases with  $t$ , and vice versa for DFRA. Additionally, it defines NBU (New Better Than Used) and NWU (New Worse Than Used) distributions based on reliability comparisons over time, leading to NBUE and NWUE classifications concerning mean residual lifetime. The text highlights that the families of distributions are interconnected.

**\*\*Independent Times-to-Failure and Negative Binomial Distribution\*\*** Let  $(T_1)$  and  $(T_2)$  be independent failure times with respective failure rate functions  $z_1(t)$  and  $z_2(t)$ . The probability that  $(T_1)$  occurs before  $(T_2)$ , given the minimum time is  $t$ , is calculated as:  $\Pr(T_1 < T_2 | \min(T_1, T_2) = t) = \frac{z_1(t)}{z_1(t) + z_2(t)}$ . For a negative binomial random variable  $(Z_r)$  with distribution parameters  $(p)$  and  $(r)$ , where  $(r = 1)$  indicates  $(Z_1)$ , the expected value and variance are given by:  $E(Z_r) = \frac{r(1-p)}{p}$  and  $\text{Var}(Z_r) = \frac{r(1-p)}{p^2}$ .

$= \frac{r(1-p)}{p^2}$ ]. This results confirms  $(E(Z_r) = rE(Z_1))$  and  $(\text{Var}(Z_r) = r\text{Var}(Z_1))$ , showing the realistic scalability of these metrics for independent distributions. Additionally, if  $(Z_1, Z_2, \dots, Z_n)$  are independent with parameters  $(p, r_i)$ , then the total variable  $(Z = \sum_{i=1}^n Z_i)$  follows a negative binomial distribution with parameters  $(p, \sum_{i=1}^n r_i)$ .

### **\*\*Geometric and Negative Binomial Distributions\*\***

If  $(X_1, X_2, \dots, X_r)$  are independent geometric random variables with parameter  $(p)$ , their sum  $(Z_r = X_1 + X_2 + \dots + X_r)$  follows a negative binomial distribution with parameters  $(p)$  and  $(r)$ . Furthermore, if  $(Z_1, Z_2, \dots, Z_n)$  are independent random variables with a negative binomial distribution characterized by parameters  $(p)$  and  $(r_i)$ , then the total  $(Z = Z_1 + Z_2 + \dots + Z_n)$  also exhibits a negative binomial distribution with parameters  $(p)$  and the sum of the  $(r_i)$ , which is  $(r = \sum_{i=1}^n r_i)$ . Lastly, if  $(X)$  follows a uniform distribution on the interval  $[0,1]$ , the transformed variable  $(T = -\frac{1}{\lambda} \log(1 - X))$  has an exponential distribution with parameter  $(\lambda)$ .

### **\*\*Reliability of 2oo3 Structures and System Analysis\*\***

The 2oo3 structure relies on three components, ensuring system reliability through a specific equation. If component reliabilities are independent and equal, the system reliability simplifies to three times reliability squared minus two times reliability cubed. The structure function indicates how the system is affected by individual component failures. For a gas leakage alarm system, the reliability incorporates a 2oo3 voting unit and must include multiple detectors functioning correctly before raising an alarm. The system's reliability is represented by the product of individual component probabilities reflecting their reliability. Using pivotal decomposition, the reliability function exhibits linearity concerning a critical component, defining conditions for system failure. For nonrepairable systems, reliability calculations consider the survivor functions of series and parallel configurations, dictated by individual component failure rates, leading to expressions for overall system performance. Systems with uniform failure rates yield straightforward survivor functions, showcasing the interaction between component failure mechanisms.

### **\*\*Mean Residual Life and Failure Rates of Structural Systems\*\***

The mean residual life of a structure at time  $(t)$  can be represented as  $(MRLS(t) = \frac{1}{4} - e^{-\lambda t})$  and approaches  $($

$\frac{1}{\lambda}$  as  $t$  approaches infinity. In nonrepairable systems with constant failure rates  $\lambda_1$  and  $\lambda_2$ , the survivor function is given by  $R_S(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$ . The mean time-to-failure is determined by  $MTTF_S = \int_0^\infty R_S(t) dt$ . For a 2-out-of-3 structure, the survivor function can be defined as  $R_S(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$ , and the corresponding failure rate is calculated using  $z_S(t) = \frac{-R'_S(t)}{R_S(t)}$ , indicating that the mean time-to-failure reflects the structure's reliability dynamics over time.

### **\*\*Failure Rate Functions in Reliability Structures\*\***

Figure 6.3 shows the behavior of the failure rate function  $z_S(t)$  depending on the parameters  $\gamma_1$  and  $\gamma_2$ , constrained such that  $\gamma_1 + \gamma_2 = 1$ . When  $\gamma_1 \neq \gamma_2$ ,  $z_S(t)$  increases to a maximum at time  $t_0$  before decreasing to  $\min\{\gamma_1, \gamma_2\}$ . In a parallel configuration of two Weibull components with a life distribution characterized by shape parameter  $\gamma$  and scale parameter  $\eta$ , the survivor function  $R_S(t)$  can be expressed with a transformation. The mean time-to-failure ( $MTTF_S$ ) for this structure is derived using specific gamma functions. For a 2 out of 3 (2oo3) structure with constant failure rate  $\lambda$ , the survivor function  $R_S(t)$  yields a distinct failure rate function, and the  $MTTF_S$  is calculated to be shorter than that of a single component. A summary comparison of MTTF across different structures reveals insights on reliability.

### **\*\*System Reliability and Failure Analysis\*\***

This section discusses the reliability of a system with two items. Item 1 must not fail during the interval  $(0, t]$ , while Item 2 activates upon Item 1's failure in  $(t, t + d]$ . The system's survival probability,  $R_S(t)$ , combines the probabilities of two scenarios: Item 1 failing within  $(t, t + d]$  and the successful functioning of Item 2. The expression for  $R_S(t)$  is:  $R_S(t) = e^{-\gamma_1 t} + \int_0^t (1 - p) * \gamma_1 * e^{-\gamma_0 \tau} * e^{-\gamma_2 (t-\tau)} * e^{-\gamma_1 \tau} d\tau$ . For  $(\gamma_1 + \gamma_0 - \gamma_2) \neq 0$ , this simplifies to a sum involving the exponential terms. Otherwise, it holds for the equality case. The mean time to system failure is provided by  $MTTFS$ , offering insights into reliability by assuming independence, which is elaborated with redundancy concepts and Markov models in later sections. The subsequent section addresses reliability assessment in single repairable items.

### **\*\*Production Availability, Punctuality, and Failure Rates in Reliability Engineering\*\***

Production availability  $(A_{100})$  measures full production within a time interval  $(t_1, t_2)$  as the ratio of hours with full production to total

hours:  $(A_{100} = \frac{\text{Hours at full production}}{t_2 - t_1})$ .  
 Reduced capacity availability, like 80% ( $A_{80}$ ), is defined similarly.  
 Punctuality in transport is defined as the number of on-time flights divided by total scheduled flights, expressed as a percentage. The failure rate function ( $f(t)$ ) outlines the probability of failure over time, defined as  $(\frac{\Pr(t < T \leq t + \Delta t \mid T > t)}{\Delta t / R(t)})$ . The Rate of Occurrence of Failures (ROCOF) approximates mean failures in small intervals and can be estimated for repairable items as  $(\text{ROCOF} \approx \frac{1}{\text{Mean Up-Time} + \text{Mean Down-Time}})$ .  
 The Vesely failure rate relates failure probability and availability in both repairable and nonrepairable contexts, revealing the item's reliability over time.

### **\*\*Repairable 2003 System Analysis\*\***

In a repairable 2003 system, three components with identical failure rate ( $\lambda$ ) and repair rate ( $\mu$ ) contribute to reliability through three minimal cut sets. The average unavailability ( $A$ ) of a component is given by the ratio of mean downtime (MDT) to the sum of mean uptime (MUT) and MDT per failure rate:  $(A = \frac{\text{MDT}}{\lambda} = \frac{\text{MUT} + \text{MDT}}{\lambda + \mu})$ . For the minimal cut set, this results in  $(A_{\text{MCPS}} = \frac{2}{\lambda + \mu})$ . The frequency of MCPS failures due to a component is  $(w_{\text{MCPS}} = A_{\text{MCPS}} \mu = \frac{\lambda}{\lambda + \mu})$  and system failures is  $(w_S = 3A_S)$ , with mean uptimes and downtimes calculated similarly. There's a noted discrepancy in independence among cut sets due to shared components, affecting accuracy. Additionally, for a parallel structure of three components, distinct repair strategies yield varying probabilities for repair completion before a defined downtimes, thus impacting overall system availability ( $A$ ).

### **\*\*Reliability Analysis of Fault Trees Using Approximation Methods\*\***

The TOP event probability ( $Q_0(t)$ ) for a fault tree with ( $n$ ) basic events is expressed as  $(Q_0(t) = 1 - \prod_{i=1}^n (1 - q_i(t)))$ . The probability that a minimal cut parallel structure (MCPS) fails, assuming its basic events are independent, is given as  $(\psi_j(t) = \prod_{i \in C_j} q_i(t))$ . The TOP event probability can be approximated using  $(Q_0(t) \leq 1 - \prod_{j=1}^k \psi_j(t))$ . For minimal cut structures with small ( $q_i(t)$ ), an upper bound approximation is:  $(Q_0(t) \approx 1 - \sum_{j=1}^k \psi_j(t))$ . The inclusion-exclusion principle allows for precise calculation of ( $Q_0$ ) through  $(Q_0 = W_1 - W_2 + W_3 - \dots)$ . An alternative approximation uses the rare event theory, wherein probabilities of negating events are simplified. Kinetic tree theory (KTT) is an earlier quantitative FTA method that laid the groundwork for these approaches.

### **\*\*Probabilistic Analysis of Fault Trees in Bridge Structures\*\***

This section details the probability of a fault tree TOP event for a bridge structure, defined as:  $Q_0 = W_1 - W_2 + W_3 - W_4$  where  $W_1$  is the sum of various component failure probabilities (e.g.,  $q_1, q_2, q_4, q_5$ ),  $W_2$  reflects probabilities of pairs of failures, and  $W_3$  and  $W_4$  represent failures across multiple components. Approximations for  $Q_0$  can be made using the inclusion-exclusion principle and bounds can be established. For example, using equal probabilities ( $q_i = 0.05$ ), bounds of  $0.5218\% \leq Q_0 \leq 0.5220\%$  were calculated. The approach is rooted in kinetic tree theory and could leverage binary decision diagrams to streamline evaluations.

### **\*\*Fault Tree and Event Tree Analysis using BDDs\*\***

Fault trees and their associated Binary Decision Diagrams (BDDs) represent logical relationships between events, where the top event only occurs if specific basic events (A and B) occur (both in state 1). BDDs, which can depict any Boolean function, must have leaves labeled 0 or 1, nodes each with two children, and a single root node. A reduced, ordered BDD (ROBDD) optimizes storage by eliminating duplicate nodes and redundant substructures. Event trees quantify event sequences, often modeled as a homogeneous Poisson process, with conditional probabilities assigned to safety functions. The calculation of end consequences involves multiplying the initiating event frequency by the probabilities along the sequence. Bayesian networks (BNs) provide probabilistic evaluations for systems modeled with binary states, facilitating robust reliability analysis.

### **\*\*Independence and Probabilities in Bayesian Networks\*\***

When the power supply ( $C = 1$ ) is functioning, two pumps (A and B) act independently; thus, the probability of both working together is the product of their individual probabilities. Given that  $\Pr(C = 1) = 0.95$  and  $\Pr(A = 1 | C = 1) = \Pr(B = 1 | C = 1) = 0.90$ , we can find that  $\Pr(A = 1) = 0.77$  and  $\Pr(B = 1) = 0.77$ . However,  $\Pr(A = 1 \cap B = 1) = 0.77$  reveals that A and B are not independent since their intersection is not equal to the product of their individual probabilities. Observing  $A = 0$  raises the chance of B also failing. Bayesian inference allows for updating beliefs using observed data and Bayes' theorem, supporting analysis even with dependent components. Monte Carlo simulation can model systems with uncertain failure times to obtain empirical results.

### **\*\*Bayesian Networks and Monte Carlo Simulation Overview\*\***

This section discusses the calculation of probabilities in Bayesian networks (BNs) and introduces Monte Carlo simulation. In BNs, events A and B have given probabilities. For an and-gate, the probability of both



events occurring ( $Q_0$ ) is calculated as the product of their probabilities. For an or-gate,  $Q_0$  considers the sum of probabilities minus their joint occurrence. A parent-child relationship is illustrated where node M's probability distribution is derived from A and B. Monte Carlo simulation, named after Monaco, involves generating random samples from defined distributions (e.g., Weibull) to estimate system failure distributions. Random numbers are generated using pseudo-random generators, resembling uniform distribution, for simulations and analyses in software like R.

### **\*\*Monte Carlo Simulation in Reliability Analysis\*\***

Monte Carlo simulation is a probabilistic technique for generating random samples to estimate numerical outcomes, particularly for reliability analysis of systems with multiple components. When analyzing a system with independent, non-repairable components having time-to-failure modeled by Weibull distributions, the simulation generates random failure times. From these, the system's time-to-failure distribution can be estimated through repeated simulations. Pseudo-random numbers are drawn from uniform distributions to simulate lifetimes according to specified distributions. The process involves generating random variables from target distributions, which helps analyze complicated systems effectively by aggregating various failure and repair scenarios. This method calculates key metrics such as observed availability and total failures over the simulation period. Monte Carlo simulations provide insights for system design and decision-making by accommodating numerous operational variations and decision rules, facilitating comprehensive reliability assessments.

### **\*\*Generating Random Variables for Reliability Analysis\*\***

In reliability analysis, Monte Carlo simulations are used to generate pseudo-random numbers from life distributions, such as exponential or Weibull. A random variable  $(T)$  with a distribution function  $(F_T(t))$ , which is strictly increasing, can be transformed to obtain a uniform distribution  $(Y)$  between 0 and 1 using the inverse function:  $(T = F_T^{-1}(Y))$ . In R, to simulate pseudo-random numbers from a Weibull distribution, the command is `rweibull(n, shape, scale)`. For complex systems, simulating lifespans includes tracking components through failures and repairs, ultimately calculating metrics like availability and system throughput. Decision rules guide responses to failures, requiring extensive input data and potentially thousands of simulations for accuracy, especially in multicomponent systems. Simulations can be executed using programming tools like R or spreadsheet applications.

### **\*\*Production Availability Simulation of Two Items\*\*    This simulation**

**evaluates a two-item production system operating on January 1, 2020. When both items function, item 1 contributes 60% and item 2 40% to total output. Times-to-failure (denoted as  $t_1$  and  $t_2$ ) are independent and follow a Weibull distribution. Upon failure of item 1 at  $t_1$ , its downtime is lognormally distributed, causing item 2 to increase output to 60%. Following repair time  $d_1$ , item 1 resumes operation with decreased load on item 2. The simulation accounts for conditional time-to-failure distributions, periodic maintenance, and increased loading on item 1 if item 2 fails. Results provide metrics like downtime and resource use, with repeated trials to ascertain average performance.**

### **\*\*Reliability Analysis of Items A and B in System S\*\***

Items A and B have failure metrics characterized as follows: Item A has a mean failure rate of 100 failures per  $10^6$  hours ( $MTTF_A = 10,000$  hours), and Item B has a mean time-to-first-failure of 100 days ( $MTTF_B = 2,400$  hours). The system S functions when at least one of each item is operational. (a) Reliability at  $MTTF_A$  and  $MTTF_B$  can be computed using their respective failure rates, highlighting that the system's reliability is higher at the point with the longer MTTF. (b) Mean time-to-failure for S, denoted  $MTTF_S$ , incorporates both items' timelines. (c) The probability of survival at  $MTTF_S$  indicates the system's robustness, showing the likelihood of operational continuity. (d) To enhance reliability, adding either item A or item B is evaluated. The option yielding higher reliability should be analyzed by comparing the survivor functions at times  $MTTF_A$  and  $MTTF_B$ . In subsequent analysis (6.9), the reliability of an offshore wind turbine's generator is assessed through a fault tree, requiring a reliability block diagram (RBD), structure function determination, identification of minimal cut sets, and evaluation of unreliability at 10,000 hours, potentially necessitating approximation.

### **\*\*DIM Calculation Options in Reliability Metrics\*\***

In reliability analysis, two methods for calculating importance metrics, DIM1 and DIM2, are used. For DIM1, all changes (???) are equal across events, simplifying the equation to express DIM1 as a sum of Birnbaum's importance metrics across basic events, ensuring the total equals 1. DIM1 reflects the same ranking as Birnbaum's but is scaled to sum to 1. For DIM2, changing ratios across events (???/??) also yield a simplified form reflecting criticality importance metrics, providing similar rankings to DIM1 but calculated differently; DIM2 also sums to 1. Both methods exhibit additive properties, allowing aggregation of metrics from multiple events.

### **\*\*System Failure and Component Importance Metrics\*\* The occurrence of**

system failure ( $S^*(t, t+dt)$ ) can be influenced by the failure of critical components ( $B_i^*(t, t+dt)$ ). The conditional probability that component  $i$  causes the system failure is given by the formula involving the probabilities of both events. Birnbaum's importance metric ( $IB(i|t)$ ) evaluates the criticality of component  $i$  at time  $t$ , and when nonrepairable, it can be expressed as  $IB(i|t)$  multiplied by the probability density function of the component's failure time. The Barlow-Proshan metric ( $IBP(i)$ ) quantifies the importance of a nonrepairable component, integrating  $IB(i|t)$  over time, while for repairable components, it uses the rate of failure ( $w_i(t)$ ). In systems of independent components,  $IBP(i)$  represents the fraction of system failures due to component  $i$ .

### **\*\*Reliability Importance and System Structures\*\***

This section addresses the analysis of a 2oo3:G structure with independent components. It confirms that if the component reliabilities ( $p_1, p_2, p_3$ ) are ordered ( $p_1 \geq p_2 \geq p_3$ ), then: (a) for  $p_1 \geq 0.5$ , importance indices  $IB(1) \geq IB(2) \geq IB(3)$ , and (b) for  $p_1 \leq 0.5$ ,  $IB(1) \leq IB(2) \leq IB(3)$ . The system reliability, expressed as  $p_S(t) = 1 - Q_0(t)$ , relates to component reliability  $p_i(t) = 1 - q_i(t)$  through the derivative relationship: the change in system reliability with respect to  $Q_0$  equals the change in component reliability with respect to  $q_i$ . Additional tasks involve analyzing the system structure functions and computing reliability importance using metrics such as Birnbaum's and Fussell-Vesely's for given component reliabilities in specified figures.

**\*\*Types and Effects of Dependent Failures in Systems\*\*** In a system with  $(n)$  dependent components, several failure dependencies can occur: 1)

**\*\*Cascading or Domino Effect\*\***: One component's failure increases the likelihood of others failing, creating a chain reaction. 2) **\*\*Common-Cause Failures\*\***: Multiple components fail simultaneously due to shared shock (e.g., lightning) or stress (e.g., humidity). 3) **\*\*Common Mode Failures\*\***: Several components fail in the same manner due to a shared cause, possibly at different times. 4) **\*\*Negative Dependence\*\***: The failure of one component can improve the operational environment of others. Cascading failures, commonly seen in power systems and networks, may arise from various events (e.g., storms, errors) and are examined using models like Markov methods and Monte Carlo simulations. Tightly coupled systems are particularly at risk, with characteristics such as rigid processes and rapid disturbance propagation.

### **\*\*Analysis of Common-Cause Failures (CCFs) in Redundant Systems\*\***

CCFs are critical in redundant systems, particularly in k-out-of-n (k/n) configurations, where simultaneous failures can compromise function. A CCF is defined as failures resulting from a shared cause that leads to multiple components failing concurrently, causing structural failure. However, simultaneous failures that do not lead to system failure are termed Multiple Failures with Shared Causes (MFSC). CCFs arise from shared causes (e.g., environmental factors) and coupling factors (e.g., shared design or procedures), making them susceptible to identical failures. To combat CCFs, defenses like physical separation, system diversity, robustness, and simplicity are employed to ensure independent functioning and minimize risk.

## **\*\*Overview of CCF Models and Probability Problems\*\***

Several SINTEF reports provide theoretical and practical insights into common cause failure (CCF) models. Hokstad and Rausand (2008) offer a comprehensive review of these models and techniques for parameter estimation. Key problems include: 1. For independent events E1 and E2, show that E1 and E2's failure (E2\*) are also independent. 2. Prove if the probability of A given B equals the probability of A, then the probability of B given A equals the probability of B. 3. Calculate the probability of two heads in three coin tosses under various conditions. 4. Discuss if a more likely event A results in a more likely event B. 5. Given certain probabilities, find the intersection of two events ( $\Pr(A \cap B)$ ). 6. Explore CCF concepts, including definitions, root causes, and explanations for CCF occurrences. 7. Analyze vulnerability to CCFs in different structural configurations and challenge the realism of using a uniform beta factor across structures.

## **\*\*Reliability Problem Set Overview\*\***

This set contains various reliability problems focusing on independent events, probabilities, and common cause failures (CCF). 1.

**\*\*Independence of Outcomes\*\***: If events E1 and E2 are independent, show E1 and E2 failed (E2\*) are also independent. 2. **\*\*Conditional**

**Probability\*\***: Prove  $\Pr(A | B) = \Pr(A)$  implies  $\Pr(B | A) = \Pr(B)$ . 3.

**\*\*Coin Toss Probabilities\*\***: Calculate the probability of getting exactly two heads under different conditions based on initial toss outcomes. 4.

**\*\*Event Likelihood\*\***: Address whether event A's occurrence increases B's likelihood reciprocally. 5. **\*\*Joint Probability Calculation\*\***: Given

$\Pr(A^*) = 0.35$  and  $\Pr(B | A) = 0.55$ , find  $\Pr(A \text{ and } B)$ . 6. **\*\*CCF**

**Concepts\*\***: Explore criticisms of the CCF definition, root causes, coupling factors, and nature of CCFs as either systematic or random failures. 7. **\*\*Structure Vulnerability\*\***: Compare vulnerabilities of

1oo4:G, 2oo4:G, and 3oo4:G to CCFs and assess beta-factor modeling realism across these structures. 8. **\*\*Beta-Factor Model and MTTF\*\***: In a

2oo3:G structure with constant failure rate  $\lambda$ , identify the beta value

for maximum Mean Time To Failure (MTTF) and explain the MTTF curve shape in relation to ?.

### **\*\*Maintenance Types and Reliability Assessment\*\***

Maintenance includes localization, isolation, disassembly, interchange, reassembly, alignment, and checkout. Preventive maintenance (PM) aims to reduce failure probabilities through regular tasks like inspection, lubrication, and part replacement. PM types are: 1. **\*\*Age-based\*\***: Maintenance performed at a specified age (e.g., time or kilometers). 2. **\*\*Clock-based\*\***: Scheduled at calendar dates, easier to manage. 3. **\*\*Condition-based\*\***: Triggered by performance variables (e.g., temperature). 4. **\*\*Opportunity-based\*\***: Performed during related maintenance tasks. 5. **\*\*Overhaul\*\***: Comprehensive maintenance during low-demand periods. Predictive maintenance anticipates failures using collected data. Maintenance classifications (DIN 31051) include servicing, inspection, repair, and improvement. Repair types are outlined as perfect (as-good-as-new), imperfect (inferior), or as-bad-as-old (pre-failure state). Condition monitoring involves systematic data collection to plan cost-effective maintenance actions. Downtime is categorized into planned (scheduled) and unplanned (random failures), with measures including mean downtime (MDT) and mean time to repair (MTTR). Various distributions (exponential, normal, lognormal) model downtime behaviors.

### **\*\*Automobile Maintenance and Downtime Management\*\***

Automobile service involves scheduled maintenance based on time or distance, often aided by on-board indicators. Maintenance tasks can include oil changes, fluid checks, and component lubrication. Proof tests for safety valves ensure functionality during critical failures, while modifications alter system functions without being maintenance tasks. Repair tasks can be classified as perfect (like new), imperfect (functional but lesser), or as-bad-as-old. Condition monitoring supports reliability through systematic data evaluation. Downtime is categorized into unplanned (due to failures) and planned (preventive maintenance). Mean downtime (MDT) encompasses detection and repair, while the mean time to recovery (MTTR) typically is shorter. Downtime distributions, including exponential, normal, and lognormal, model repair times, essential for reliability assessments. The mean downtime of structures is calculated based on failure rates and is affected by the configuration (series or parallel).

### **\*\*Overview of Downtime Distributions in Reliability Assessment\*\***

Downtime (MDT) is the average time an item remains non-functional after failure, influenced by factors like accessibility and maintenance

resources. MDT generally exceeds Mean Time to Repair (MTTR), incorporating detection, logistics, and testing. Mean Uptime (MUT) equals Mean Time to Failure (MTTF), while Mean Time Between Failures (MTBF) is the average time between failures. Common downtime distributions include exponential, normal, and lognormal. The exponential distribution is defined by repair rate ( $\lambda$ ), where MDT is  $1/\lambda$ , and the probability of downtime exceeding a certain duration is calculated using the formula  $\Pr(D > d) = e^{-\lambda d}$ . The normal distribution assumes the sum of many independent elements, while the lognormal distribution reflects that repair rates decrease with prolonged downtimes. For series structures, mean downtime approximates:  $MDT \approx (\sum \lambda_i * MDT_i) / (\sum \lambda_i)$ . In parallel structures, MTDS can vary based on maintenance strategy and requires stochastic process analysis. For complex systems, Monte Carlo simulations offer reliable estimates.

**\*\*Total Productive Maintenance (TPM) Overview\*\*** Total Productive Maintenance (TPM) aims to enhance equipment effectiveness by reducing six major losses: equipment failure, setup delays, idling, reduced speed, defective products, and yield losses. These contribute to Overall Equipment Effectiveness (OEE), calculated as: - Operational availability = available time / net available time - Performance rate = net operating time / operating time - Quality rate = (processed products - rejected products) / processed products OEE is a key indicator of productivity and quality, with  $\geq 85\%$  deemed world class. TPM fosters teamwork among maintenance, production, and engineering staff, enhancing workers' skills and communication about equipment issues. It mirrors total quality management (TQM) by requiring management commitment and employee empowerment, while focusing on long-term benefits.

### **\*\*Analysis of Failure Times in Items\*\***

Example 10.1 presents failure times recorded from a single item over 410 days, with seven failures noted. The failure times and interoccurrence times suggest that as the calendar time progresses, failures occur more frequently, indicating item deterioration, thus termed a "sad item." Conversely, items where failures decrease over time are termed "happy items." The number of failures, denoted as  $N(t)$ , plotted against time ( $t$ ), shows that when the item is sad,  $N(t)$  forms a convex curve. In contrast, a happy item's curve is concave, while a steady item presents  $N(t)$  as approximately linear.

### **\*\*Reliability Analysis: HPPs and Failure Time Distribution\*\***

In an HPP (Homogeneous Poisson Process) with rate  $\lambda$ , the mean number of

failures in time interval  $(0, t]$  is  $E[N_C(t)] = p \cdot t$ . Given one failure in  $(0, t_0]$ , the time  $T_1$  of failure follows a uniform distribution in  $(0, t_0]$  with expected value  $E(T_1 | N(t_0) = 1) = t_0/2$ . Wald's equation states that for independent variables  $X_i$ ,  $E(\sum X_i) = E(N)E(X)$ . In a Compound Poisson process,  $Z(t)$  represents cumulative consequences, leading to  $E[Z(t)] = \lambda t$ . If cumulative loss exceeds a threshold  $c$ , the probability of failure is  $\Pr(T_c > t) = \Pr(Z(t) > c)$ . The mean time to failure can be derived using the relation involving integrals of Poisson probabilities and the distribution function  $F_V(c)$ .

### **\*\*Wald's Equation and Compound Poisson Processes\*\***

Wald's equation states that for independent and identically distributed random variables  $(X_i)$  with a finite mean  $(E(X))$ , and a stochastic integer variable  $(N)$  independent of  $(X)$ , the expected sum is given by: the expected value of  $(N)$  multiplied by the expected value of  $(X)$  (i.e.,  $E(\sum_{i=1}^N X_i) = E(N) \cdot E(X)$ ). In the context of a Compound Poisson process where  $(N(t))$  follows a Poisson distribution with rate  $(\lambda)$ , the cumulative consequence at time  $(t)$  is  $(Z(t) = \sum_{i=1}^{N(t)} X_i)$ . The mean and variance of  $(Z(t))$  are  $(E[Z(t)] = \lambda t)$  and  $(\text{var}[Z(t)] = \lambda(\nu^2 + \tau^2)t)$ , respectively. When failures occur upon  $(Z(t) > c)$ , the probability of no failure by time  $(t)$  is expressed in terms of the cumulative random variables  $(V_i)$ . The mean time to total failure  $(E(T_c))$  involves integrating the distribution related to  $(Z(t))$  and can be derived through exponential variables.

### **\*\*Probability Distribution and Renewal Process Insights\*\***

The probability density function of  $\tau$  can be approximated using the inverse Laplace transform, although this process can be complex. By the strong law of large numbers, as the number of trials  $(n)$  approaches infinity,  $\tau$  converges to the mean  $\bar{\tau}$ . The central limit theorem indicates that the standardized form of  $\tau$  is normally distributed. For a renewal process with interoccurrence times having an IFR distribution, the probability that the  $n$ th failure occurs before time  $\tau$  is bounded. Specifically, for large  $n$ , the number of events is approximately linear, and the probability distributions can be estimated using normal approximations. The fundamental renewal equation relates the expected number of renewals to these probabilities, although exact solutions for the renewal function  $M(\tau)$  are challenging to find and often require approximations.

### **\*\*Interoccurrence Times in Renewal Processes\*\***

In a renewal process with interoccurrence times following an Increasing

Failure Rate (IFR) distribution with mean time-to-failure (MTTF) denoted as  $\theta$ , the survivor function  $R_T(t)$  exceeds a threshold when  $(t < \theta)$ . Specifically,  $R_T(t) \approx e^{-t/\theta}$ . For independent random variables  $(U_1, U_2, \dots, U_n)$  with an exponential distribution rate of  $(1/\theta)$ , the sum follows a gamma distribution. The probability of the  $n$ th failure occurring before time  $(t)$  is bounded by:  $F_n(t) \approx 1 - \sum_{j=0}^{n-1} \frac{(t/\theta)^j e^{-t/\theta}}{j!}$ . As time  $(t)$  grows, the expected number of renewals  $(N(t))$  approaches  $(t/\theta)$ , confirming linear behavior. The renewal function  $(W(t))$  estimates the number of renewals via:  $W(t) = F_T(t) + \int_0^t W(t-x) dF_T(x)$ . For large  $(t)$ , the average length of intervals  $(\theta \approx t/W(t))$ , leading to the elementary renewal result. This shows the expected failures approach  $(t/\theta)$  as  $(t)$  becomes large.

### **\*\*Age and Remaining Lifetime in Renewal Processes\*\***

The age  $(Z(t))$  of an item at time  $(t)$  is defined as: - If the number of renewals  $(N(t) = 0)$ :  $Z(t) = t$  - If  $(N(t) > 0)$ :  $Z(t) = t - S_{N(t)}$  The remaining lifetime  $(Y(t))$  is calculated as:  $Y(t) = S_{N(t)+1} - t$  where  $(S_n)$  is the total time of the first  $(n)$  renewals. The remaining lifetime can also relate to the first failure  $(T)$  using the conditional probability:  $\Pr(Y(t) > y) = \frac{\Pr(T > y+t)}{\Pr(T > t)}$  The mean remaining lifetime  $(E[Y(t)])$  involves an integral:  $E[Y(t)] = \frac{1}{\Pr(T > t)} \int_t^\infty \Pr(T > u) du$  For exponentially distributed  $(T)$  with rate  $(\lambda)$ ,  $E[Y(t)] = \frac{1}{\lambda}$ . At large  $(t)$ , both  $(Y(t))$  and  $(Z(t))$  converge to the same limiting distribution, with the mean given by:  $\lim_{t \rightarrow \infty} E[Y(t)] = \frac{\sigma^2 + \mu^2}{2\mu}$

### **\*\*Superimposed and Delayed Renewal Processes\*\***

Renewal processes occur when items are replaced or restored after failure, leading to a superimposed renewal process (SRP) formed from multiple distinct failure processes. While the SRP is generally not a renewal process, Drenick (1960) showed that an infinite number of independent stationary renewal processes can exhibit a homogeneous Poisson process (HPP). For example, in a two-item series, the first failure results in a superimposed renewal process that is not fully restored, indicating imperfect repairs. Moreover, delayed renewal processes differ in their first failure time distribution compared to subsequent ones, allowing for extensions of earlier results. A stationary renewal process, where the first period has a unique distribution, offers insights into long-term behavior. It has properties indicating the average state, remaining life distribution, and stationary increments over time.

### **\*\*Renewal Processes and Their Variants\*\*    Renewal processes model events**



over time with cycles defined by interoccurrence times. Each cycle has a reward associated with it, represented by random variables. The total reward by time  $t$  is the sum of rewards from completed cycles, and its expected value relates to the expected number of cycles. In delayed renewal processes, the initial cycle may differ from subsequent ones, which can affect properties like distribution functions. A stationary renewal process, a subtype, maintains constant characteristics over time and follows a specific distribution for its initial cycle. Alternating renewal processes consider items that alternate between operational and failed states, denoted by their respective time-to-failure sequences.

### **\*\*Understanding Stationary and Alternating Renewal Processes\*\***

The function  $f_e(t)$  reflects the same distribution as  $f_T(t)$  and describes the density of a stationary renewal process where the remaining life of an item is examined. The properties of this process show that  $W_S(t)$  (the renewal function) is proportional to time over the mean interoccurrence time, and the probability of remaining life  $\Pr(Y_S(t) \leq y)$  derives from its density. Alternating renewal processes combine functions of up-times and down-times, leading to a convolution of their distributions. The mean time between renewals combines mean time-to-failure (MTTF) and mean downtime (MDT). The average availability of the item approaches  $\frac{MTTF}{MTTF + MDT}$  as time progresses.

### **\*\*Stationary Renewal Processes and Availability in Alternating Systems\*\***

A homogeneous Poisson process (HPP) is a stationary renewal process due to the memoryless property of the exponential distribution, satisfying properties of renewal functions. In a gamma-distributed renewal process, the distribution function is given by  $FT(t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$  and the mean interoccurrence time is  $E(T) = 2/\lambda$ . Over time, this leads to a stationary renewal process with a renewal function  $WS(t) = \lambda t / 2$ . In an alternating renewal process, where items alternate between up and downtime, the average time between renewals is  $\mu T = MTTF + MDT$ . The probability of item availability at time  $t$  is derived from the sum of functioning states, leading to  $A(t) = 1 - FU(t) + \int_0^t A(t-x) dFT(x)$ . For parallel items, the structure fails when all components are down, leading to  $MDT_S = \sum \mu_i^{-1}$  and  $MTTF_S = MTBF_S - MDT_S$ .

### **\*\*Mean Number of Failures and Repairs\*\***

This section discusses renewal processes related to repairs and failures. For repairs, the mean number of completed repairs in time  $(0, t]$  is

expressed using a renewal function, with the Laplace transform given by  $W1^*(s) = (fU^*(s) * fD^*(s)) / (s[1 - fU^*(s) * fD^*(s)])$ . The probability density function for repair times,  $T$ , is determined by convolution:  $fT(t) = \int_0^t (fU(t-x) * fD(x) dx)$ . For failures, the delayed renewal process defines a similar mean,  $W2^*(s)$ . Availability  $A^*(s)$  at a given time is represented as  $A^*(s) = (fU^*(s) * fD^*(s)) / (s[1 - fU^*(s)])$ . Ultimately, the mean number of repairs, failures, and availability can be calculated for any life and repair time distributions.

### **\*\*Reliability Models in Conditional ROCOF Analysis\*\***

In Chan and Shaw's model, the conditional Rate Of Occurrence of Failure (ROCOF) is represented in intervals as follows: before the first failure, it equals the initial ROCOF; after the first failure, it decreases by a proportional factor ( $\alpha$ ) based on previous states. This leads to the equation:  $w_C(t) = w_1(t) - \alpha \sum_{i=1}^{\infty} (1 - \alpha)^i * w_1(S_N(t)-i)$ , indicating an arithmetic reduction model (ARI $\alpha$ ) with infinite memory. In contrast, ARI1 limits reduction to the most recent wear, with:  $w_C(t) = w_1(t) - \alpha * w_1(S_N(t))$ . The efficiency index ( $\alpha$ ) highlights the effectiveness of repairs: 0 (no effect), 1 (optimal repair), and negative values implying detrimental effects. Malik's age reduction models extend these by reducing virtual age, focusing on efficiency to maintain reliability across varying repair strategies.

### **\*\*Wear Intensity and Repair Action Models\*\***

The ARI models describe wear intensity during repairs. The ARI $\alpha$  model, with infinite memory, reduces total accumulated wear by a percentage ( $\alpha$ ) with each repair. In contrast, the ARI1 model accounts for wear only since the last repair, implying shortened memory. Doyen and Gaudoin (2011) expand this to ARIm models, where wear reduction considers only the last  $m$  repairs. The conditional Rate of Occurrence of Failures (ROCOF) equation is given by:  $w_C(t) = w_1(t) - \alpha \sum_{i=1}^m (1 - \alpha)^i * w_C(S_N(t)-i)$ , with minimal wear intensity as  $w_{min}(t) = (1 - \alpha)^m * w_1(t)$ . Malik's age reduction models similarly propose that repair actions diminish the age of the item. The Trend Renewal Process, defined by Lindqvist (1998), reveals how ROCOF varies with failure distributions, demonstrating different cases such as constant or linearly increasing rates influencing outcome.

### **\*\*Optimal Maintenance Policy and Trend Renewal Process\*\***

Shin et al. (1996) establish an optimal maintenance policy, defining the conditional Rate of Occurrence of Failures (ROCOF) as  $w_C(t) = w_1(t - \alpha S_N(t))$ . The minimal wear intensity is given by  $w_{min}(t) = w_1((1 - \alpha)^m * t)$ . Doyen and Gaudoin (2002) refer to this as an arithmetic reduction of age with memory (ARA1). The Trend Renewal Process (TRP), created by

Lindqvist (1998), uses an NHPP with failure times  $S_1, S_2, \dots$ , and has a transformed occurrence time  $W(S_i)$  that forms a renewal process with underlying distribution  $F(t)$ . The conditional ROCOF for TRP is  $w_C(\text{TRP})(t) = z(W(t) - W(S_N(t-))) * w(t)$ , where  $z(t)$  is the failure rate function. An example with linear ROCOF and Weibull distribution shows specific cases and properties of this process and contributes insights into selecting appropriate models for repairable items.

## **\*\*Branching Poisson Process Models\*\***

This section discusses branching Poisson processes. If failure intervals are independent and identically distributed, we view this as a renewal process and apply methodologies from Chapter 14. For dependent intervals, alternative approaches are necessary, such as those outlined by Crowder et al. (1991). The included problems involve calculations and properties of homogeneous Poisson processes (HPP). For instance, one problem asks to determine the expectation of the product of two counts at different times, while another shows that the sum of two independent HPPs results in a new HPP with the combined rate. Additional inquiries address the joint distribution of occurrence times and conditions for the number of events relative to time.

## **\*\*Reliability Process Problems Overview\*\***

This section presents several problems related to homogeneous and non-homogeneous Poisson processes (HPPs and NHPPs). 1. **\*\*Expectation Calculation\*\***: Determine the expectation of the product  $E[N(t)N(t+s)]$  where  $N(t)$  is a HPP. 2. **\*\*Conditional Probability\*\***: For a HPP with rate  $\lambda$ , verify the conditional probability  $Pr(N(t) = k | N(s) = n) = \frac{(n-k)!}{n!} \left(\frac{t}{s}\right)^k (1 - \frac{t}{s})^{n-k}$  for conditions on  $t$  and  $s$ . 3. **\*\*First Occurrence Time\*\***: Show that for the first occurrence time  $T_1$  of a HPP,  $Pr(T_1 \leq s | N(t) = 1) = \frac{s}{t}$ , and derive  $E(T_1)$  and the standard deviation. 4. **\*\*Sum of Independent HPPs\*\***: Show that the combined process  $N(t) = N_1(t) + N_2(t)$  also represents an HPP with rate  $\lambda_1 + \lambda_2$ . 5. **\*\*Mean Value Representation\*\***: Prove that the expected value  $E[N(t)] = \sum_{n=0}^{\infty} Pr(N(t) > n)$ . 6. **\*\*Joint Distribution of Event Times\*\***: Present the joint distribution of  $(S_1, S_2, \dots, S_n)$  given  $N(t) = n$  as  $f_{S_1, \dots, S_n | N(t)=n}(s_1, \dots, s_n) = \frac{n!}{t^n}$  for ordered times. 7. **\*\*Renewal Process Relationships\*\***: Investigate the relationships between  $N(t)$  and  $S_r$ . 8. **\*\*NHPP Analysis\*\***: Analyze a non-homogeneous Poisson process with rate  $w(t) = \lambda \frac{t+1}{t}$  for  $t \geq 0$  and sketch its functions. This section encapsulates essential concepts and calculations concerning Poisson processes within reliability theory.

## **\*\*Summary of Independent HPPs and Counting Processes\*\***

This section discusses key concepts in counting and renewal processes. Two independent homogeneous Poisson processes (HPPs) with rates  $\lambda_1$  and  $\lambda_2$  can be combined to form a new HPP,  $N(t)$ , with a rate of  $\lambda_1 + \lambda_2$ . The mean value  $E[N(t)]$  of a counting process can be expressed as: the sum of the probabilities that  $N(t)$  is greater than or equal to  $n$  for all  $n$ , or as the sum of probabilities that  $N(t)$  exceeds  $n$ . For an HPP with rate  $\lambda$ , the joint probability density function for arrival times of  $n$  events can be defined. The superposition of independent HPPs is a renewal process, whereas the superposition of independent renewal processes is generally not. The section also presents different mathematical models for counting processes and their implications on event rates related to time.

### **\*\*Reliability Analysis of Repairable Items\*\***

The analysis assumes items are restored to "as-good-as-new" after failure, with a constant failure rate of  $\lambda = 0.0005$  failures per hour. This denotes a Poisson counting process. The Mean Time To Failure (MTTF) is 2000 hours, while the Mean Time Between Failures (MTBF) is 2006 hours. Data from Table 10.2 provides sequences of failure times ( $S_1$  to  $S_{10}$ ) for various items. Plots of failures over time indicate trends and the expected number of failures. Probability calculations for experiencing  $k$  failures in a timeframe and observations of crossing probabilities further aid understanding. In a separate case, with a Weibull distribution (shape 4, scale 500), empirical methods estimate  $E[N(t)]$ . For average availability with a mean downtime of 6 hours, the required assumptions for reliability formulas should be documented for clarity.

### **\*\*Understanding Markov Chains in Reliability Engineering\*\***

This chapter discusses Markov chains, a specific stochastic process used to model systems with multiple states and transitions. Each Markov chain, represented as  $\{X(t), t \geq 0\}$ , possesses the Markov property, meaning that future states depend only on the current state, not past states. The state space (denoted as  $S$ ) can be finite or countably infinite. Continuous-time Markov chains are of primary interest for reliability modeling. The Kolmogorov equations are utilized to determine the probability distribution  $P(t)$  across the states, where  $P_i(t)$  signifies the probability of being in state  $i$  at time  $t$ . As time approaches infinity, this distribution settles into a steady state, allowing assessment of system performance measures like availability and failure rates. Examples highlight how systems with various components yield numerous states. The chapter concludes with a discussion on generalized Markov processes that can model more complex systems.

### **\*\*Steady State Probabilities in Markov Processes\*\***

In Markov processes, steady state probabilities ( $P_0, P_1, \dots$ ) represent

long-term behavior as time approaches infinity. From the equations involving state probabilities, we derive the function  $P_0(t) = (\lambda/(\lambda + \mu)) - [(1/(\lambda + \mu)) * e^{-(\lambda + \mu)t}]$  and  $P_1(t) = (\lambda/(\lambda + \mu)) * e^{-(\lambda + \mu)t}$ , where  $P_1(t)$  indicates component availability. The limiting availability is given by  $P_1 = \lambda/(\lambda + \mu)$ , while  $MTTF = 1/\lambda$  and  $MTTR = 1/\mu$  lead to  $P_1 = MTTF / (MTTF + MTTR)$ . For steady states, the equation  $P * A = 0$  must be satisfied, where  $A$  is the transition matrix. The initial state does not affect these probabilities, which can be numerically solved using tools like R.

### **\*\*Steady State Probabilities and System Performance Metrics\*\***

This section presents the steady state probabilities for a system of independent generators, which are governed by the equations: 1. Negative sum of rates multiplied by the state probabilities, balanced by arrival rates, fulfilling:  $-(\lambda_1 + \lambda_2) * P_0 + \lambda_2 * P_1 + \lambda_1 * P_2 = 0$   $-\lambda_2 * P_0 - (\lambda_2 + \lambda_1) * P_1 + \lambda_1 * P_3 = 0$   $-\lambda_1 * P_0 - (\lambda_1 + \lambda_2) * P_2 + \lambda_2 * P_3 = 0$   $-P_0 + P_1 + P_2 + P_3 = 1$  The probabilities  $P_0, P_1, P_2, P_3$  represent system states (functioning or failed), where  $q_i$  denotes unavailability and  $p_i$  indicates availability for components. The overall average system availability, defined as the proportion of time the system functions ( $A_s$ ), is calculated from  $P_j$  for functioning states and the frequency of failures:  $1 - A_s = \sum P_j$  (failed states). Additionally, key metrics such as mean duration of a system failure (linked to frequency) and mean time between failures are included, enabling analysis of system reliability across given operational data.

### **\*\*Steady State Reliability Analysis\*\***

This section presents steady-state equations and solutions for a system with independent components, expressed through transition properties and performance metrics. The four steady state equations reflect component interactions based on mean time-to-failure (MTTF) and mean time-to-repair (MTTR). Solutions yield steady state probabilities  $(P_0, P_1, P_2, P_3)$  as functions of parameters such as  $(\lambda)$  (failure rates) and  $(\mu)$  (repair rates). Key metrics include system availability  $(A_s)$  (proportion of time functioning) and unavailability, with frequency of failures  $(\omega_F)$  equal to the visit frequency of the failed state. The average time between failures (MTBF) is defined by these frequencies, and results can be generalized for parallel structures of components, guiding reliability assessments in system engineering.

**\*\* (System Reliability: Availability and Failure Metrics) \*\*** In system reliability analysis, transitions between states dictate arrival and departure frequencies. The frequency of arrivals into state  $(j)$  is determined by summing the product of preceding state probabilities and

transition rates. In steady-state, the visit frequency to state  $(j)$  corresponds to the departure frequency, establishing balance:  $(\nu_j = P_j \cdot \alpha_j)$ . The mean sojourn time in state  $(j)$  is inversely proportional to the transition rate. The average availability  $(A_s)$  reflects the proportion of time the system is operational, while unavailability is measured by the sum of failure states. The frequency of failures  $(\omega_F)$  is linked to transitions from functional to failed states. Key metrics like Mean Time Between Failures (MTBF) relate directly to failure frequency, while mean functioning time before failure (up-time) distinguishes failure rates induced by operational transitions. Parallel and series configurations affect availability and unavailability uniquely, with specific formulas that generalize across multiple components.

### **\*\*Parallel Systems with Cold Standby and Fault Tree Analysis\*\***

In a cold standby system, item A operates while item B remains inactive until A fails. Activation of B occurs with a probability of  $(1 - p)$ , where  $p$  includes potential startup failures. The state probabilities can be described by a linear equation system, with steady-state probabilities determined as follows:  $P_0 + P_3 + P_4 = 1$ , and probabilities  $P_3$  and  $P_4$  are derived from transition rates involving failure and operational constants. The mean time to failure for the system (MTTFS) is based on these probabilities. Fault tree analysis can be integrated by establishing minimal cut sets, where the system unavailability approximates to  $Q_0 \approx 1 -$  the product of individual component unavailabilities. Independent component behavior leads to expected failure frequencies and mean duration metrics for cut sets, accounting for failure and repair rates.

### **\*\*Markov Analysis in Fault Tree Analysis\*\***

Markov analysis aids fault tree analysis by approximating the probability of a system failure (TOP event) using cut sets of basic events. The unavailability of the system,  $Q_0(t)$ , can be estimated as 1 minus the product of the average unavailability of minimal cut sets. Average unavailability for a component is defined as its repair rate divided by the sum of repair and failure rates. The failure frequency for a cut set can be approximated using the product of component failure rates. In a series configuration of cut sets, the total failure frequency is the sum of individual frequencies. The mean time between system failures (MTBF) and mean time to repair (MTTR) relate to system availability, calculated as MTBF divided by the total of MTBF plus MTTR. These formulas are essential for fault tree analysis, including time-dependent solutions described by the Kolmogorov forward equations.

### **\*\*System Reliability and Time-Dependent Solutions\*\***

A system's failure frequency can be approximated by the sum of the frequencies of its independent minimal cut parallel structures, although this is not always accurate due to dependencies and downtime effects. The expected frequency of system failures is given by the total of individual frequencies. The Mean Time Between Failures (MTBF) is calculated as the inverse of the failure frequency, while the mean system downtime (MTTR) relates each structure's downtime to its frequency. System availability is defined as the ratio of MTBF to the total time (MTBF + MTTR). In time-dependent analyses, the distribution of the system state at time  $t$ , denoted as  $P(t)$ , can be derived using Kolmogorov forward equations or through Laplace transforms, which simplify the process of solving linear differential equations. The survival probability  $R(t)$  utilizes the initial state probability vector and considers functioning and failed states for effective reliability assessment.

### **\*\*Time-Dependent Solutions in Markov Processes\*\***

This section discusses the time-dependent solutions for Markov processes via Kolmogorov's equations. Given the initial probability distribution  $P(0)$ , the distribution at time  $t$ , represented as  $P(t)$ , can be calculated using the matrix exponential:  $P(t) = P(0) \cdot e^{tA}$ , where  $A$  is the transition rate matrix. For systems with absorbing states, a vector  $C$  is defined to determine the survival probability  $R(t) = P(0) \cdot e^{tA} \cdot C$ . Additionally, Laplace transforms can be utilized to simplify the set of linear differential equations, where the transform of the derivative  $\mathcal{L}[P_j'(t)]$  leads to a relation  $P_j^*(s) \cdot A = sP_j^*(s) - P_j(0)$ . Solving this allows for finding  $P_j(t)$  through inverse transforms.

### **\*\*Solving the Kolmogorov Equations with Laplace Transforms\*\***

The Kolmogorov equations can yield the state probabilities over time, expressed as  $P(t) = P(0) \cdot e^{tA}$ . For systems with absorbing states, a column vector  $C$  indicates functioning states (1) and failed states (0), where  $C = [0, 1, 1]^T$  signifies two working states. The survival probability  $R(t)$  can be computed using a series expansion of the transition matrix  $A$ . An efficient approximation is  $P(t) \approx P(0) \cdot (I + tA/n)^n$  for large  $n$ . Additionally, employing Laplace transforms simplifies solving the underlying linear differential equations. The state probabilities  $P_j(t)$  can be derived through inverses of  $P_j^*(s)$ , which relate to the initial probability distribution at  $t=0$ .

**\*\*Simulation of Markov Processes for Reliability Analysis\*\*** In reliability analysis, if no failure occurs before scheduled maintenance, the state remains unchanged, with survival probability approximated as:

(1 - failure rate of item 1 \* time increment) \* (1 - failure rate of item 2 \* time increment). Conversely, if a failure of item 1 happens prematurely, the system state transitions to a lower state with probabilities relying on the failure ratings. Similar computations apply when item 2 fails. The essence is analyzing transition rates and states' behavior under stochastic events. Markov processes utilize these transitions to simulate system histories, calculate expected times in states, and estimate probabilities of failures through repeated Monte Carlo simulations. Adjustments in transition distributions can be made without altering the foundational code structure. Accuracy of outcomes relies on a sufficient number of simulated histories, as specified in the literature.

### **\*\*Simulation of a Markov Process\*\***

A Markov chain consists of system states and transitions triggered by stochastic events. To simulate a Markov chain, track the current state and treat transitions as competing events. Each transition from state  $i$  to  $j$  has a constant rate  $(a_{i,j})$ , with the transition time  $T_{i,j}$  following an exponential distribution. The earliest triggered transition determines the next state. Simulated histories may end when a specific state is reached or after a set time. For accurate results, Monte Carlo simulations must run multiple histories to ensure statistical reliability. Transition rates can be easily adjusted without altering the simulation structure. However, more complex systems may require intricate coding. Implementing the pseudocode in Python allows for efficient analysis of systems like those with components A, B1, and B2, calculating factors such as mean time to failure (MTTF) and failure probabilities.

### **\*\*System State Transitions and Simulation\*\***

The system has four states: - State 3: All components functional (as-good-as-new). - State 2: All functional except component B1 (degraded state). - State 1: Components B1 and B2 failed (system failed). - State 0: Both states implying failure of component A. Only states where A fails are considered; other states are unreachable. Each component has a constant failure rate:  $\lambda_A$  for A, and  $\lambda_B$  for B1 and B2. The exponential distribution's memoryless property means transition times from state 2 to 0 rely solely on  $\lambda_A$ . Consequently, transitions from states 3 to 0 and 2 to 0 occur at rate  $\lambda_A$ , whereas transitions from states 3 to 2 and 2 to 1 occur at  $\lambda_B$ . Pseudocode for simulating the system's history and conducting a Monte Carlo simulation allows estimating MTTF and state probabilities. Variations in time-to-failure distributions (e.g., Weibull) can easily be integrated. Furthermore, Markov analysis can be executed with R packages designed for discrete and continuous time Markov processes.



### **\*\*Markov Chain Simulation and Time-to-Failure Analysis\*\***

This text describes a procedure for simulating the time-to-failure (TTF) of a system using a Markov chain model. The function ``GetOneHistory(??, ??)`` initializes TTF and states, drawing from exponential distributions for failure events of components A and B until the system reaches a terminal state (either failure of A or B). The function returns the accumulated TTF and final state. The ``SystemMonteCarlo(?, ??, ??)`` function repeats this for N histories, calculating the mean TTF and the probabilities of ending in states 0 and 1. The simulation can be adapted for different time-to-failure distributions, like Weibull. Furthermore, R packages like ``markovchain`` support Markov analysis.

### **\*\*Parallel Structure Reliability Analysis\*\***

This section discusses a parallel structure of four identical components, each with failure rate  $\lambda$  and repair rate  $\mu$ , functioning at time  $t = 0$ . It outlines tasks to establish state transition diagrams and equations, compute the Mean Time To Failure (MTTF), and inquire about a general formula for n components. Additionally, the reliability of a pitch system, consisting of hydraulic cylinders, pitch systems, accumulators, pump, and filter, is analyzed. Each item has constant failure rates and a uniform repair rate  $\mu$ . The system states and transition rate matrix A for the pitch system are defined. Lastly, the system's behavior is described with redundant components A, B, C, D, and E, where each can experience increased failure rates during repairs and undergo renewal after total failure.

**\*\*Reliability Analysis of Production and Pumping Systems\*\*** This analysis discusses a production system with two identical channels that operates 24/7, each capable of states: 100%, 50%, and 0% capacity. The failure rates are constant, with the 100% capacity failure rate being 0.00024 failures/hour and the 50% capacity failure rate being 0.0018 failures/hour. External shocks occur at a rate of 0.000005 failures/hour. A channel moves to 50% capacity with a 60% chance upon failure, and both channels shut down when capacity falls to 50% or below. Repair times vary from 20 to 30 hours, and the system takes 48 hours to restart. The pumping system has three pumps, where two operate, and one stands by. Each pump has a mean time to failure (MTTF) of 550 hours with repairs taking 10 hours. The presence of common cause failures is modeled with a beta-factor of 0.12. The heating system uses three burners, and the standby burner has a 98% reliability upon activation, with each burner having a constant failure rate of 0.0025 per hour.

### **\*\*Transition Rate Matrix and Reliability Analysis for a Redundant**

## **System\*\***

The transition rate matrix  $\mathbf{A}$  corresponds to state changes between inspections in a redundant system of two identical items (1 and 2). The items have a Weibull-distributed time-to-failure with shape parameter  $\alpha = 2.25$  and scale parameter  $\theta = 10,000$  hours. Each item undergoes corrective maintenance with downtime  $d_c = 1,000$  hours and preventive maintenance with downtime  $d_p = 500$  hours when the operational age reaches  $a = 7,500$  hours. Modeling with a multiphase Markov process assesses unavailability (probability of being in state 4) with and without preventive maintenance (PM) tasks. Perfect diagnosis is assumed, but with a 90% accuracy for states 2 and 3, and a 10% misdiagnosis to state 1. The state transition diagram and system states should use minimal classifications, and further modifications will allow for calculating the system's surviving function and availability over time.

## **\*\*System Reliability Analysis and Maintenance Optimization\*\***

Define the relevant system states with minimal states for two inspections and create a state transition diagram. Develop the transition rate matrix  $A$ . Adapt the provided PDMP to account for maintenance tasks duration. Evaluate system survival and availability over time, considering constant repair rates ( $\lambda_A = 10^{-4}$ ,  $\lambda_B = 5 \times 10^{-3}$ ,  $\mu_A = \mu_B = 10^{-1}$  per hour) for items A and B. Calculate Mean Time To Failure (MTTF) and Mean Time Between Failures (MTBF). Modify the analysis for constant repair durations of 100 hours. Implement the adaptations in a Python or R script and compare results with previous metrics.

## **\*\*Time-Based Preventive Maintenance Strategies\*\***

This section discusses time-based preventive maintenance (PM) strategies specifically related to replacements that restore items to an "as-good-as-new" state. Two primary strategies are identified:

- \*\*Age Replacement:\*\*** Items are replaced either upon failure or when a predetermined operational age ( $t_0$ ) is reached, whichever occurs first. This avoids unnecessary preventive replacements shortly after a corrective replacement but results in uncertainty regarding future PM tasks as they depend on failure occurrences.
- \*\*Block Replacement:\*\*** Items are replaced at fixed intervals, irrespective of failures, allowing for predictable scheduling. However, this can lead to replacing items that have been recently corrected. In age replacement, the replacement period (TR) is defined as the lesser of  $t_0$  or the time-to-failure (T). The mean length of a replacement period is expressed by the integral of the item's failure density, plus the probability of failing after age  $t_0$ , signifying an expected value that approaches the mean time-to-failure (MTTF) as  $t_0$  increases. Cost considerations include planned replacement costs ( $c$ ) and additional expenses ( $k$ ) incurred from unplanned failures.

## **\*\*Cost Analysis of Block Replacement Strategy without Spares\*\***

This section analyzes the average costs associated with a block replacement strategy for items with a Weibull time-to-failure distribution, characterized by parameters  $\beta = 3$  and  $\eta = 0.1$ . The average cost over a replacement interval, when no spares are available ( $m = 0$ ), can be expressed as:  $\text{Cost} = c + k * F(t_0) + k_u * (\text{integral from } 0 \text{ to } t_0 \text{ of } F(t) dt)$ . Here,  $c$  is the cost per time unit,  $k$  represents the replacement cost, and  $k_u$  is the cost when the item is unavailable. As the replacement interval ( $t_0$ ) approaches infinity, costs stabilize, mimicking a scenario where no replacement occurs. Cost functions are illustrated in Figure 12.4 and 12.5 for various parameter values.

**\*\*P-F Interval Approach in Reliability Management\*\*** The P-F interval approach in reliability-centered maintenance examines the inspection and replacement strategy for items subjected to random shocks, modeled as a homogeneous Poisson process. Time between shocks follows an exponential distribution with mean  $1/\lambda$ . The P-F interval, denoted by the time from detectable potential failure (P) to critical failure (F), is a random variable. Preventive and corrective replacement costs are denoted as  $c_P$  and  $c_C$ , with inspection cost  $c_I$ . The optimal inspection interval  $\tau$  should balance costs and is based on the estimated P-F interval, which is subjectively determined. The mean total cost function for the interval considers the probabilities of replacement and inspection effectiveness. Additionally, models extend to scenarios with non-perfect inspections, random repair times, and various degradation metrics, informed by degradation indicators and RUL distributions that guide maintenance decisions.

**\*\*Understanding Remaining Useful Lifetime (RUL)\*\*** Remaining Useful Lifetime (RUL) measures an item's remaining operational time from a specified moment until it becomes unusable. RUL is treated as a random variable with a probability distribution based on a degradation model. It can be estimated using regression methods, linking current conditions to RUL via a function  $RUL(t_i) = f(t_i, y_i, u_i)$  where  $t_i$  is time,  $y_i$  is the condition measurement, and  $u_i$  describes the operational context. There are two prognostic approaches: data-driven (without probabilistic modeling and focusing on mean estimates) and probabilistic (fitting historical data to a chosen degradation model). To define RUL statistically, one needs the state variable  $X(t)$ , unacceptable states, and observed conditions over time.

## **\*\*Condition-Based Maintenance Overview\*\***

Condition-Based Maintenance (CBM) aims to optimize maintenance strategies by monitoring and modeling item degradation. Four strategies exist: corrective replacements, age-based replacements, block replacements, and ideal replacements where the item is replaced just before failure. The asymptotic cost per time unit for corrective replacements is calculated as the sum of fixed costs divided by the Mean Time To Failure (MTTF). For age and block replacements, optimal timing minimizes costs, which are generally lower than corrective maintenance costs. However, the ideal strategy remains unattainable due to uncertainties in monitoring and prediction. The CBM framework involves defining the degraded state, identifying post-maintenance states, and choosing an appropriate monitoring approach (continuous or inspection-based). Common assumptions include perfect monitoring and that preventive tasks return the item to a near-new state, with potential cost variations depending on degradation levels.

### **\*\*Time Intervals and Maintenance Strategies in Reliability Engineering\*\***

The time between renewals is represented as  $TR = T_m + \tau$ , with the mean value  $E(TR) = E(T_m) + \tau$ . The mean downtime during failures is considered when  $T \leq T_m + \tau$ , determined by the indicator function for the failure event. The probability that the last failure occurs beyond the scheduled maintenance interval is  $Pr(T \leq T_m + \tau)$ . For a gamma process, this probability and expected time calculations utilize integral density functions. Inspection-based monitoring involves deterministic inspection dates to assess item states. This system functions as a Markov process with transition matrices representing degradation states. The cumulative maintenance cost between inspections accounts for preventive and corrective maintenance, integrating costs over a time interval influenced by item state transitions and inspection-triggered interventions.

### **\*\*Estimation of Probabilities in Safety-Instrumented Systems (SIS)\*\***

Monte Carlo simulation is a common method for estimating probabilities, supported by various modeling software. In a safety-instrumented system (SIS) with redundant actuators ( $n$  channels), each channel may fail through two modes modeled in series. A logic solver is paired with the actuators, and partial inspections are made at intervals  $\tau$ . Maintenance for failures of type  $a$  items occurs, while type  $b$  is undetectable. The failure of type  $c$  is instantly detected, with a constant failure rate ( $\lambda_c$ ). The degradation of items  $a$  follows a discrete state Markov process, transitioning through  $k$  states to the failed state, where the transition matrix is defined, leading to a systematic maintenance matrix  $B$ . Overall system availability combines the individual availabilities of components, allowing for the evaluation of maintenance costs against safety limits in operational contexts.

## **\*\*Survival Probability and Overhaul Impact on Reliability\*\***

To determine the survival probability  $R(t)$  at  $t = 2000$  hours with a failure rate  $\beta = 5 \times 10^{-8} \text{ hours}^{-2}$ , the following equation models the total failure impact over time intervals between overhauls:  $z(t) = \beta t - \alpha k \tau$ , where  $k$  is the number of completed overhauls and  $\tau$  is the overhaul interval. The term  $\alpha k \tau$  represents the reduction in failure rate from overhauls. For  $t = k\tau$ , calculate  $R(t)$  just before the  $k$ -th overhaul. The conditional probability that the item functions before overhaul  $k + 1$  given it functioned before  $k$  must also be assessed. Additionally, using the age replacement strategy, the mean time between failures  $E(Y_i)$  for an item with (a) an exponential distribution failure rate  $\lambda$ , and (b) a gamma distribution with parameters  $(2, \lambda)$  provides insights into item reliability and expected operation time between failures.

## **\*\*Simulation and Parameter Estimation of Degradation Models\*\***

An item is inspected every 15 months ( $p = 15$ ) with 35 total inspections ( $n = 35$ ), starting from a perfect state. The deterministic degradation over time is modeled as: Degradation,  $X(t) = 0.001 * t + 0.001 * t^2$ . Observation,  $Y(t) = X(t) + \epsilon(t)$ , where  $\epsilon(t)$  is Gaussian noise (mean = 0, variance = 100). For parameter estimation: 1. Generate a timeline of inspections, compute actual degradation, and simulate measurement noise via Monte Carlo methods. 2. Estimate degradation parameters using the least squares method, visually assess model predictions against actual data, and compare empirical cumulative distributions for reaching a specified degradation level ( $\epsilon = 5000$ ). 3. Examine how variations in inspection frequency and quantity affect estimation accuracy. Additionally, for a different degradation scenario with  $p = 15$  months and  $n = 6$  inspections, simulate random increments of degradation using gamma or exponential distributions.

## **\*\*Safety Instrumented Systems Overview\*\***

Safety Instrumented Systems (SIS) are crucial for maintaining safe operations in equipment under control (EUC). They detect predefined process deviations via sensors, actuate necessary components, and are not supposed to activate spuriously. Failures can be classified as "fail to function" (FTF) or spurious trip (ST). SIS components include electric power sources, user interfaces, and hydraulic systems. Regular and diagnostic self-testing is essential to identify hidden failures. Proof testing is performed periodically to verify operational integrity, balancing safety and economic considerations. Failures are categorized: dangerous (subdivided into dangerous undetected and dangerous detected)

or safe (subdivided into safe undetected and safe detected). The safety unavailability of a tested safety item is assessed based on the presence of dangerous undetected failures during defined intervals.

### **\*\*Safety Unavailability and Probability of Failure on Demand (PFD)\*\***

The safety unavailability  $A^*(t)$  indicates the probability of a dangerous undetected (DU) failure occurring by time  $t$ , expressed as  $A^*(t) = F(t)$ , where  $F(t)$  is the distribution function of DU failures. It is discontinuous at intervals of  $n\tau$  (where  $n = 1, 2, \dots$ ). The average PFD over time is calculated as  $PFD = \frac{1}{\tau} \int_0^{\tau} F(t) dt$  or alternatively as  $PFD = 1 - \frac{1}{\tau} \int_0^{\tau} R(t) dt$ , where  $R(t) = 1 - F(t)$ . The mean downtime,  $E(D_1)$ , during a test interval is expressed as  $E(D_1) = \int_0^{\tau} F(t) dt$ . The PFD represents the mean proportion of time the item is non-functional as a safety barrier and is also termed mean fractional dead time (MFDT). In examples with constant failure rates, the PFD can be further approximated using series expansions.

**\*\*Staggered Testing and Safety Unavailability\*\*** Staggered testing involves two independent items with constant failure rates that are tested at different times. Item 1 is tested at intervals of  $0, \tau, 2\tau$ , while Item 2 is tested at  $t_0, \tau + t_0, 2\tau + t_0$ . The unavailability for Item 1 ( $q_1(t)$ ) in the first test interval is given by 1 minus the exponential of negative failure rate ( $\lambda_{DU,1}$ ) multiplied by time ( $t$ ). For Item 2,  $q_2(t)$  adjusts based on the test timing. The combined system unavailability ( $q_s(t)$ ) is simply the product of  $q_1(t)$  and  $q_2(t)$ . In scenarios with significant repair times, like a downhole safety valve, the total downtime and restoration risks must also be assessed. Safety unavailability (SU) represents the probability a system fails to perform when needed, categorized into known (NSU) and unknown risks (PFD), among others.

### **\*\*Testing Valve Actuators and Common Cause Failures\*\***

Valve actuators can detect issues in electrical cables, solenoids, and shutdown valves through partial stroke testing, which achieves nearly 100% coverage for cables and solenoid valves, but finds only some valve failure modes. Common cause failures (CCFs) can affect multiple items simultaneously or sequentially, necessitating explicit modeling to improve reliability assessments. The failure rate for dangerous undetected failures is expressed as the sum of independent and common cause components. The common cause factor ( $\beta$ ) represents the percentage

of failures caused by CCFs. A higher diagnostic coverage reduces  $\lambda$ . For parallel redundant structures, the probability of failure on demand (PFD) due to CCFs can be roughly calculated as  $\lambda \tau^2 / 2$ , where  $\lambda$  is the failure rate and  $\tau$  is the time period under consideration.

### **\*\*Reliability and CCF Models in Voting Configurations\*\***

In reliability scenarios with common cause failures (CCFs), the probability of failure for k-out-of-n voting configurations (PFD<sub>k</sub>) approximates to  $(\beta \lambda \tau) / 2$  when  $\lambda \tau$  is small. This suggests that the result is consistent across various voting types and largely unaffected by component numbers ( $n \geq 2$ ). The standard IEC 61508 endorses the beta-factor model, but criticisms led to the introduction of a multiple beta-factor (MBF) model to enhance realism. Additionally, reliability data can differ between sources, impacting CCF evaluations. The IEC 61508 standard outlines a rigorous safety lifecycle involving conceptualization, risk analysis, and safety requirements, relevant in the context of safety instrumented systems (SISs) and their subsystems.

### **\*\*Safety Integrity Levels (SIL) Overview\*\***

Safety Integrity Level (SIL) is defined in IEC 61508 as the ability of a safety-related system to perform required safety functions under specified conditions. SIL is categorized into four levels based on the average probability of failure (PFD) for low and high demand operational modes. In low demand mode, SIL classifications range from  $< 10^{-5}$  (SIL 4) to  $< 10^{-1}$  (SIL 1), while in high demand mode, classifications go from  $< 10^{-9}$  (SIL 4) to  $< 10^{-5}$  (SIL 1). Selection of SIL for Safety Instrumented Functions (SIFs) requires assessing the frequency of demands and potential consequences of critical events. The IEC 61508 outlines a lifecycle approach for identifying SIFs, establishing SILs, and implementing safety functions through various phases: system definition, risk acceptance criteria, hazard analysis, quantitative risk assessment, non-SIS protection evaluations, and SIL determinations, culminating in system risk evaluation and verification.

**\*\*Critical Situations in Low Demand Safety Instrumented Functions (SIFs)\*\*** For a Safety Instrumented Function (SIF) in low-demand mode, process demands follow a homogeneous Poisson process (HPP) with a rate of  $\lambda$  demands per hour. The probability of failure on demand (PFD) leads to critical situations occurring as an HPP with a rate of  $\lambda_c = \lambda \cdot \text{PFD}$ . The expression for the probability of  $n$  critical situations in time  $t$  is given as  $(\lambda \cdot \text{PFD} \cdot t)^n \cdot e^{-(\lambda \cdot \text{PFD} \cdot t)} / n!$ . The mean time between critical situations (MTBC) is  $1 / (\lambda \cdot \text{PFD})$ . To assess necessary Safety Integrity Levels (SIL), one must evaluate demand frequency and potential

consequences. Compliance with IEC 61508 involves defining systems, risk acceptance criteria, conducting hazard and risk assessments, and determining SILs while ensuring the design adheres to specified reliability requirements.

### **\*\*Assessment of Safety Instrumented Systems (SIS) Reliability\*\***

The PDS method, developed by SINTEF, quantifies the reliability, safety unavailability, and life cycle cost of Safety Instrumented Systems (SIS), ensuring compatibility with IEC 61508 standards. In low-demand conditions, the average Probability of Failure on Demand (PFD) over test intervals can be calculated using integrals across states of failure. The system can be modeled as a time-homogeneous continuous-time Markov chain with transition matrices to reflect state changes and repairs after failures. The stationary distributions of the system's states before and after tests provide insights into failure risks. The average PFD and mean time between failures are expressed in terms of state probabilities and transition behavior, enabling an effective reliability analysis.

**\*\*Markov Approach in Safety Systems\*\*** This approach models a safety system subjected to periodic testing, with states divided into functioning (B) and failed (F). The average probability of failure on demand (PFD) during the test interval  $n$  is calculated as the integral of the failure state probability over the testing period. The system state follows a continuous-time Markov chain with a transition probability matrix  $A$ . If a failure is detected, it transitions to a post-test state based on a repair matrix  $R$ , which accounts for repair strategy and potential maintenance failures. The stationary distribution of the states before and after testing diverges, impacting long-term failure rates, given by  $\frac{1}{\text{MTBF}_{\text{DU}}} = \frac{\tau}{\pi_F}$ . For instance, a safety valve example shows how transition matrices dictate probabilities across different failure scenarios and repairs.

**\*\*Long-Term Average Probability of Failure on Demand (PFD) Analysis\*\*** As the number of test intervals increases, the probability of the system being in state  $i$  approaches a long-term average, allowing the calculation of the average PFD. Equation (13.46) for PFD is derived from integrals involving the rate of failures  $\lambda_i$  and the state transition rates for a system transitioning through states 0 to 3, where states 0 and 1 are absorbing (critical failures). The matrix defined  $Q$  accounts for failures from shocks and degradation. Various repair policies influence the PFD outcomes. For a perfect repair after each test, the average PFD integrates the probabilities of staying in functional states and



**transitions to failure states. Imperfect repairs introduce additional complexity to calculations, which can yield significantly varied PFD results. Further calculations and implications are explored in Lindqvist and Amundrustad (1998).**

### **\*\*Beta-Factor Model and PFD Analysis\*\***

The beta-factor model efficiently incorporates Common Cause Failures (CCFs), yet it raises concerns regarding its effects. Reducing the beta (?) value can lower the independent failure rate, but this relationship is questionable due to its simplistic assumptions. In configurations like 1oo4:G and 2oo4:G, under identical conditions, both yield similar Probability of Failure on Demand (PFD), which may not reflect realistic operational scenarios depending on component failure distributions. For a 2oo3:G fire detector system with distinct failure rates ( $\lambda_{DU,1}$ ,  $\lambda_{DU,2}$ ,  $\lambda_{DU,3}$ ), the PFD could be calculated; for small failure rates, approximations are viable. Evaluating 2oo4:G versus 2oo3:G structures highlights that the former might result in fewer spurious trips. Lastly, a pressure sensor system with various configurations will involve calculating PFD concerning dangerous undetected failures (DU) and assessing false alarm probabilities, guiding installation choices.

### **\*\*PFD Calculation for Parallel Systems\*\***

To derive the Probability of Failure on Demand (PFD) for a system with  $n$  identical components that have constant failure rates, start by modeling the system behavior over a life cycle including regular testing (every  $\tau$ ) and possible repairs. For the parallel structure PFD, use  $\lambda$  (failure rate) and  $\beta$  (common cause failure) in the formulation. 1. The PFD for the parallel structure,  $PFD_{\parallel}$ , can be expressed as a function of  $\lambda$ ,  $\beta$ , and  $\tau$ , incorporating the effects of testing and common cause failures. 2. For optimal staggering delay  $t_0$ , balance failure rates and testing intervals. For equal failure rates,  $t_0$  equals half of  $\tau$  (i.e.,  $t_0 = \tau/2$ ). 3. Specific calculations for PFD can be illustrated: for  $\lambda = 5 \times 10^{-7}$  failures per hour and  $\tau = 3$  months, sketch the PFD variations for  $n = 2$  and  $n = 3$  across different  $\beta$  values. Analyze and quantify the differences in PFD when  $\beta = 0$  and  $\beta = 0.20$ . This gives insights into system reliability and the impact of testing and common cause failures in parallel systems.

### **\*\*Testing Regimes and Safety Valves in Offshore Production\*\***

The probability of failure on demand (PFD) for a subsystem varies based on different testing methods: (i) staggered testing can yield higher PFD due to unknown timing; (ii) simultaneous testing can reduce detection failures; (iii) staggered testing (one month apart) balances workload but

may expose potential undetected failures. The DHSV is crucial for preventing blowouts, subjected to monthly tests with a mean repair time of 9 hours. It possesses failure modes: Fail to close (FTC), Leakage while closed (LCP), Fail to open (FTO), and Premature Closure (PC), with critical failures (FTC, LCP) detected only during tests. The mean time between FTC failures is calculable; testing impacts operational shutdown time and overall safety. For the gas detector with a hidden failure rate of  $1.8e-6$ , the mean time-to-failure and PFD need assessment based on regular testing every 6 months. A 2-out-of-3 logic configuration enhances reliability but necessitates independent detection evaluation.

## **\*\*Basic Concepts of Statistical Analysis\*\***

This section introduces key statistical terminology. A population consists of similar items or events of interest, such as all valves in a plant or all mobile phones of a brand. To analyze a population, we define a random variable ( $X$ ) and establish a probabilistic model ( $M$ ), which can be parametric with an unknown parameter ( $\theta$ ). For discrete variables, the model uses a conditional probability mass function for  $X$  given  $\theta$ . For continuous variables, a probability density function applies. Due to practical constraints, we often work with a sample—a subset of the population—collected through a defined procedure. Inference allows us to make conclusions about the population based on sample data, leveraging joint distributions derived from independent experiments conducted on sample items.

## **\*\*Statistical Modeling and Inference for Random Variables\*\***

To analyze a population, we define a random variable ( $X$ ) and establish a probabilistic model ( $M$ ), which can be parametric, nonparametric, or semiparametric. Initially, we assume ( $M$ ) is parametric with an unknown population parameter ( $\theta$ ). For a discrete ( $X$ ), use the conditional probability mass function ( $\Pr(X = x \mid \theta)$ ); for continuous ( $X$ ), use the probability density function ( $f(x \mid \theta)$ ). Due to costs and time, we study a random sample from the population through independent experiments, yielding a dataset  $(x_1, x_2, \dots, x_n)$ . The joint distribution of this dataset is given by the product of individual densities, expressed as  $f(x_1, x_2, \dots, x_n \mid \theta) = f(x_1 \mid \theta) \times f(x_2 \mid \theta) \times \dots \times f(x_n \mid \theta)$ . Inference utilizes sample data to infer population characteristics.

## **\*\*Understanding Datasets and Survival Times in Reliability Analysis\*\***

This section discusses a dataset of independent items with non-negative time-to-failure, denoted as  $T$ . Each item's failure time ( $T_i$ ) is represented by a sample of ( $n$ ) observations, which are independently

and identically distributed, with distribution function  $(F_T(t))$  and density function  $(f_T(t))$ . Observations may be censored, meaning they might stop before failure occurs, due to various factors like equipment issues or time limits. We observe the survival time until either a failure or censoring occurs, expressed as the minimum of the failure time and censoring time:  $(\min(T_i, C_i))$ . The survival data consists of pairs  $((t_i, \delta_i))$ , where  $(\delta_i = 1)$  if the item fails before censoring and  $(0)$  otherwise. It's assumed that all survival times begin from  $(t_i = 0)$  for uniformity in analysis.

## **\*\*Entering Survival Data and Censoring Types in R\*\***

Survival times can be entered into R as a spreadsheet, CSV file, or manual vectors. Two vectors are created: one for survival times (e.g., `survtime <- c(17.88, 28.92, 33.00, 41.52, 42.12, 45.60)`) and one for the corresponding status (1 for failure, 0 for censoring). Load the 'survival' package using `library(survival)`, and prepare data with `my.surv <- Surv(survtime, status)`. Censoring types include: 1. **Type I**: Data before a fixed time  $t$ ; some items survive, making failure observations random. 2. **Type II**: Test stops after a set number  $r$  of failures, leaving remaining items unobserved. 3. **Type III**: Combines type I and II, stopping the test at the first occurrence of  $t$  or  $r$  failures. 4. **Type IV (Random Censoring)**: Observed survival time is the minimum of failure time and censoring time, often seen in real operational data. Non-informative censoring assumes independence between failure times and the censoring process, while informative censoring ties removal from study to performance issues.

## **\*\*Exploratory Data Analysis Overview\*\***

Exploratory Data Analysis (EDA) is a crucial initial step in data analysis, involving the calculation of sample statistics (mean, median, standard deviation) and data visualization (histograms, Q-Q plots). EDA helps identify data structures, anomalies, and assumptions, gaining prominence after Tukey's 1977 work. A complete dataset is denoted  $(t_1, t_2, \dots, t_n)$  with all survival times as correct observations, typically sorted as  $(t(1) \leq t(2) \leq \dots \leq t(n))$ . Ties occur when multiple identical survival times exist, marked by their multiplicity  $(d_i)$ . Key metrics include: - **Mean**:  $(\bar{t} = \frac{1}{n} \sum_{i=1}^n t_i)$ ; e.g.,  $(\bar{t} = 68.08)$ . - **Median**: Depending on  $(n)$ 's parity, it's the middle value or average of two middle values. - **Variance**:  $(s^2 = \frac{1}{n-1} \sum_{i=1}^n (t_i - \bar{t})^2)$ . - **Standard Deviation**:  $(s = \sqrt{s^2})$ . Quantiles are computed as  $(t_p)$  where  $(F_T(t_p) = p)$ , approximated from ordered datasets as  $(t_{(\lceil np \rceil)})$ .

## **\*\*Statistical Analysis of Survival Time Data\*\***

The `survtime` dataset, which can be ordered using R's `sort` function, can be read from a text file or directly from formats like Excel or CSV. Ties occur when multiple survival times share the same value, affecting their representation. The key sample metrics include:

- **Mean**: Calculated as the sum of data values divided by the number of values.
- **Median**: The middle value; for an even count, it's the average of the two central values.
- **Variance**: Measures data dispersion around the mean, defined as the average of squared differences from the mean.
- **Standard Deviation**: The square root of variance, indicating dispersion in the same unit as the data.
- **Quantiles**: Values that segment the data at specified probabilities (e.g., lower and upper quartiles).
- **Skewness**: Describes data asymmetry; positive indicates a left skew, while negative indicates a right skew.
- **Kurtosis**: Measures the tail thickness of the distribution; normal has zero kurtosis.

These metrics can be computed in R using commands like `mean()`, `median()`, `var()`, `sd()`, `quantile()`, `skewness()`, and `kurtosis()`.

## **\*\*Summary of Median, Variance, Quantiles, Skewness, and Kurtosis in Data Analysis\*\***

The median of a dataset is the middle value: for an odd count of values ( $n = 2k + 1$ ), it's the  $(k + 1)$ th value; for an even count ( $n = 2k$ ), it's the average of the  $k$ -th and  $(k + 1)$ th values. In Table 14.1, with  $n = 22$ , the median is  $(t(11) + t(12))/2 = 61.68$ . Variance measures data dispersion around the mean, calculated as the sum of squared differences between each value and the mean, divided by  $(n - 1)$ . The standard deviation is its square root. The quantile of order  $p$  is where the cumulative distribution function equals  $p$ . Skewness (1) indicates asymmetry, while kurtosis (2) describes the tail shape of the distribution. Histograms and density plots visually represent data distribution, with R commands facilitating analysis.

## **\*\*Visualizing Probability Distributions and Survival Functions\*\***

This section discusses the analysis of survival times using R. The kurtosis of the dataset is 2.555. Histograms graphically display frequency distributions; users can specify either absolute or relative counts of data within intervals. The command ``hist(survtime$V1, breaks=3, freq=F)`` generates a histogram, with relative frequency enabled by changing ``freq=F`` to ``freq=T``. A sample density plot can be created using ``d <- density(survtime)`` followed by ``plot(d)``, which provides a smooth estimate of the distribution. The empirical survivor function  $\hat{R}_n(t)$  estimates the survival probability  $R(t) = P(T > t)$  using  $\hat{R}_n(t) = \frac{N(t)}{n}$ , where  $N(t)$  is the number of items surviving past time  $t$ .

## **\*\*Estimating Parameters in a Binomial Model\*\* In a series of $n$**

independent Bernoulli trials with success probability  $p$ , let  $X$  represent the number of successes, which follows a binomial distribution defined as follows: the probability of obtaining  $x$  successes is given by  $p^x$  multiplied by  $(1-p)$  raised to the power of  $(n-x)$ , divided by  $x$  factorial for  $x$  values ranging from 0 to  $n$ . The mean ( $E(X)$ ) is  $n$  times  $p$ , and the variance ( $Var(X)$ ) is  $n$  times  $p$  times  $(1-p)$ . To estimate  $p$ , a common method is to use the relative frequency of successes:  $\hat{p} = X/n$ , which is unbiased. Various methods for parameter estimation include the Method of Moments, Maximum Likelihood Estimation, and Bayesian estimation.

### **\*\*Noncentral Moments and Method of Moments Estimation\*\***

To estimate noncentral population moments ( $\mu_{1,nc}, \mu_{2,nc}, \dots, \mu_{k,nc}$ ) and sample moments ( $m_{1,nc}, m_{2,nc}, \dots, m_{k,nc}$ ), we follow these steps: First, calculate the first  $k$  noncentral population moments which depend on parameters  $\theta_1, \theta_2, \dots, \theta_k$ . Next, compute the first  $k$  noncentral sample moments. Using the equations  $\mu_{i,nc} = m_{i,nc}$  (for  $i = 1, 2, \dots, k$ ), we solve for parameters  $\theta$  to estimate  $(\theta_1, \theta_2, \dots, \theta_k)$ . The first sample moment converges to the first population moment as  $n$  approaches infinity.

**\*\*Example 14.3\*\*** shows how to estimate the failure rate  $\lambda$  for an exponential distribution using the first moment:  $\hat{\lambda} = n / \sum T_i$ , where  $\sum T_i$  is the total operation time. **\*\*Example 14.4\*\*** illustrates estimating parameters  $\alpha$  and  $\lambda$  for gamma distribution using two equations from sample moments. Code snippets enable computation of estimates from a dataset.

**\*\*Method of Moments Estimation (MME)\*\*** The method of moments is used to estimate parameters by solving a system of equations derived from equating population moments to sample moments. For a distribution characterized by a parameter set  $(\theta)$ , the MME estimate can be expressed as  $(\hat{\theta})$ . For example, in the exponential distribution case, if the times-to-failure  $(T_i)$  are assumed to be independent and follow  $(\exp(-\lambda t))$ , the MME for  $(\lambda)$  is calculated as the number of items divided by the total failure time. In a gamma distribution scenario with parameters  $(\alpha)$  and  $(\lambda)$ , the MME solutions for  $(\alpha)$  and  $(\lambda)$  stem from equating the first and second population moments to their sample counterparts. MMEs are easy to compute and consistent but may lack uniqueness and efficiency.

### **\*\*Beta Function and Maximum Likelihood Estimation\*\***

The beta function,  $B(a, b)$ , can be expressed as the ratio of gamma functions:  $B(a, b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a + b)}$ . In R, use `beta(a, b)` for the beta function and `factorial(7)` for

factorial calculations. The maximum likelihood estimate (MLE) of a parameter  $\theta$  is defined as the value that maximizes the likelihood function  $L(\theta | \text{data})$ . Formally, it is represented as  $\hat{\theta} = \text{arg max}_{\theta} L(\theta | \text{data})$ , indicating that  $\hat{\theta}$  is the parameter value making the observed data most likely.

### **\*\*Key Results on Continuous Life Distributions\*\***

Three important results on continuous life distributions are summarized.

1) For a strictly increasing life distribution  $F(t)$  between 0 and 1, the derivative of the inverse hazard function at  $v = F(t)$  is related to the failure rate  $z(t)$ : The change in  $H(v)$  with respect to  $v$  at  $F(t)$  is equal to  $(1 - F(t)) / f(t)$ , where  $f(t)$  is the probability density function. 2) If  $F(t)$  is continuous and increasing, it is increasing failure rate (IFR) if the failure rate  $z(t)$  is non-decreasing in  $t$ . It is decreasing failure rate (DFR) if the hazard function is concave. 3) For  $n$  observations of  $F(t)$ , the empirical scaled total-time-on-test (TTT) transform can be estimated as the integral of observed lifetimes against the cumulative distribution. These results facilitate the understanding of the underlying failure distributions, which can be visually assessed through TTT plots. For instance, if the TTT curve is concave, the life distribution is IFR; if convex, it is DFR. Example analysis of ball bearing failure data suggests a Weibull model with parameters  $\alpha = 2.10$  and  $\lambda = 0.0122$ , showing an increasing failure rate.

### **\*\*Analysis of Life Distributions through TTT Plots\*\***

TTT plots help assess life distributions based on the shape of the TTT transform,  $H_F^{-1}(v)$ . A concave curve indicates an increasing failure rate (IFR), while a convex curve indicates a decreasing failure rate (DFR). A curve that is first convex and then concave suggests a bathtub-shaped failure rate. For example, TTT data for ball bearings shows an increasing failure rate, with a Weibull distribution fitted, yielding parameters: shape  $\alpha = 2.10$  and scale  $\lambda = 0.0122$ . The optimal replacement age in the age replacement problem can be determined using the scaled TTT transform, which involves minimizing costs  $C(t_0) = (c + kF(t_0)) / H_F^{-1}(1)$ .

### **\*\*Total-Time-on-Test (TTT) Transform and Censored Datasets\*\***

To create a scaled TTT transform, draw the TTT plot on a 1x1 coordinate system and identify the point  $(-c/k, 0)$ . Draw a tangent from this point to find the optimal replacement value,  $v_0$ , which corresponds to where the tangent meets the TTT curve. If  $v_0$  equals 1,  $t_0$  is infinite, indicating

no preventive replacements. For incomplete datasets with random censoring, use the Kaplan-Meier estimator to calculate the empirical distribution function. The TTT transform remains valid for various distribution functions, and the TTT plot is generated by plotting the calculated points, which aligns with that of complete datasets when no data is censored.

## **\*\*Comparison of Nonparametric Estimation Techniques in Reliability Analysis\*\***

Sections 14.5-14.7 discuss nonparametric techniques for estimating survival functions applicable to both complete and censored data. The empirical survivor function aligns with the Kaplan-Meier estimate in complete datasets, while Kaplan-Meier and Nelson-Aalen yield similar results, differing mainly in sensitivity to data variation: Kaplan-Meier is sensitive in early and mid phases, while Nelson-Aalen is not sensitive early on. The TTT transform provides distinct information, sensitive primarily in mid phases. To analyze data affected by covariates, the proportional hazards (PH) model presents the failure rate function as the product of a baseline failure rate and a covariate-dependent factor. The Cox model, a PH variant, demonstrates a linear relationship in log-failure rates, allowing efficient parameter estimation without specifying the baseline. For example, in exponential distributions, failure rates can be modified based on environmental covariates.

## **\*\*Estimating Exponential of Rate Parameter in Poisson Processes\*\***

For a set of independent and identically distributed Poisson random variables  $(X_1, X_2, \dots, X_n)$  with an unknown rate  $(\lambda)$ :

(a) The Maximum Likelihood Estimator (MLE) for  $(e^{-\lambda})$  can be derived from the likelihood function. (b) An unbiased estimator for  $(e^{-\lambda})$  is also required. In a homogeneous Poisson process with rate  $(\lambda)$ , where  $(N(t))$  counts events in a time interval  $(t)$ :

(a) Given  $(t = 2)$  years and observing 7 events, the estimate for  $(\lambda)$  can be calculated as  $(7/2)$ . (b) A 90% confidence interval for  $(\lambda)$  can then be obtained based on the observed data. For a Poisson-distributed random variable  $(X)$  with  $(\lambda)$ :

(a) An exact 90% confidence interval for  $(\lambda)$  when  $(X = 6)$  involves Poisson distribution methods, while an approximate 90% confidence interval using the normal approximation to  $(N(\lambda, \lambda))$  is also provided. (b) Repeat the process for  $(X = 14)$ .

## **\*\*Poisson and Chi-squared Distributions in Reliability Analysis\*\***

This section discusses relationships between the Poisson distribution with parameter  $\lambda$  and the Chi-squared distribution with  $2\lambda$  degrees of freedom. It establishes that the cumulative distribution function (CDF)

of the Poisson distribution,  $Po(x | \lambda)$ , equals 1 minus the CDF of the Chi-squared distribution evaluated at  $(x + 1) \cdot 2\lambda$ . The equations for  $\lambda_1(X)$  and  $\lambda_2(X)$  involve upper percentiles of the Chi-squared distribution using  $z_{\alpha/2}$ . Historical failure times of a pressure transmitter suggest a constant failure rate, leading to the empirical failure rate estimation and survivor function analysis. Additionally, Kaplan-Meier and Nelson-Aalen estimates for non-starred time-to-failure data are to be computed and graphically represented.

## **\*\*Empirical Distribution and Reliability Analysis of Sensor Data\*\***

This passage references two sets of datasets for reliability analysis and tasks related to the empirical distribution function (EDF) and failure rates. It discusses creating the EDF analytically, scripting and plotting it, and fitting an exponential probability density function to the datasets with an unknown parameter,  $\lambda$ . The text also prompts verification of a constant failure rate assumption, empirical cumulative distribution determination, failure rate estimation methods, survivor function comparisons, and calculations of Mean Time To Failure (MTTF). Finally, it suggests plotting survivor functions to identify critical time horizons where the survivor probability exceeds 0.9.

## **\*\*Bayesian Modeling and Its Interpretations\*\***

This chapter introduces Bayesian modeling and data analysis using simple examples with a single parameter,  $\lambda$ , and a random variable,  $X$ . While simple cases can be solved manually, complex models require computer assistance, with an overview of computerized methods provided. Bayesian analysis reflects the analyst's belief about specific situations, differing from classical and frequentist perspectives of probability. Probability interpretations include: 1. **\*\*Classical:\*\*** Probability of event  $A$  calculated as count of favorable outcomes divided by total outcomes (e.g.,  $A$ =odd numbers when rolling a die,  $\Pr(A) = 3/6 = 1/2$ ). 2. **\*\*Frequentist:\*\*** Estimates probability based on the limit of the frequency of event  $A$  in  $n$  experiments as  $n$  approaches infinity. 3. **\*\*Subjective:\*\*** Probability reflects personal belief about  $A$ , influenced by knowledge and experience, with  $\lambda$  viewed as a random variable. Bayes' formula updates beliefs based on new evidence: - For discrete outcomes:  $\Pr(A)$  is expressed as the sum of probabilities involving mutually exclusive events. - For continuous outcomes: The likelihood of parameter  $\lambda$  given observed data  $x$  is computed using density functions, expressing how evidence impacts belief about the parameter.

## **\*\*Three Interpretations of Probability in Reliability Analysis\*\***

Probability has three primary interpretations: classical, frequentist, and subjective. 1. **\*\*Classical View\*\***: Probability ( $\Pr(A)$ ) of event  $A$



is the ratio of fulfilling outcomes to total equally likely outcomes in sample space  $S$ . For example, rolling a die gives  $\Pr(A) = \text{number of favorable outcomes} / \text{total outcomes}$  (e.g.,  $\Pr(\text{"odd number"}) = 3/6 = 1/2$ ).

2. **Frequentist View**: Probability is determined by the limiting frequency of event  $A$  occurring over numerous identical experiments, using probabilistic models when necessary (e.g., for time to failure  $T$ ). This view remains somewhat objective, but model choice can introduce subjectivity. 3. **Subjective View**: Probability reflects an analyst's degree of belief, based on personal knowledge or expert opinion. Bayesian probability treats parameters, like  $\theta$ , as random variables with probability density  $p(\theta)$ , accommodating a broader range of cases than classical and frequentist approaches. These perspectives are essential for making reliable predictions in system designs.

### **Frequentist vs. Bayesian Data Analysis**

Frequentist data analysis begins with a defined parametric model of observed data, represented by a probability density function or mass function with an unknown parameter,  $\theta$ . It processes a set of  $n$  independent observations  $(x_1, x_2, \dots, x_n)$  to estimate  $\theta$  without prior knowledge of its value. Conversely, Bayesian data analysis incorporates prior beliefs (prior distribution  $p(\theta)$ ) about  $\theta$ , a likelihood function  $L(\theta|d)$  for the observed data  $d$ , and results in a posterior distribution  $p(\theta|d)$  that updates this belief post-observation. Thus, while frequentists disregard initial parameter information, Bayesians integrate it to refine their inferences through methods like point or interval estimates.

### **Likelihood Functions and Posterior Distributions in Bayesian Analysis**

This section discusses likelihood functions and how to derive posterior distributions using Bayes' theorem. For a binomially distributed variable  $X$ , the likelihood is given by the formula:  $L(\theta | x) = C(n, x) \cdot \theta^x \cdot (1-\theta)^{n-x}$ , showing its maximal value at the MLE of  $\theta = 0.3$  for  $n=10$  and  $x=3$ . The posterior distribution combines prior beliefs,  $p(\theta)$ , with likelihood data:  $p(\theta | x) \propto L(\theta | x) \cdot p(\theta)$ . Common prior distributions include beta for parameters in  $[0, 1]$ , where  $E(\theta) = r/(r+s)$  and standard deviation can be adjusted. Similarly, in exponential models, gamma distributions are used for parameters taking positive values, yielding a posterior  $p(\theta | t_1) \propto L(\theta | t_1) \cdot p(\theta)$ , leading to updated beliefs on the failure rate  $\theta$ .

### **Binomial and Exponential Models in Bayesian Analysis**

The binomial model for a random variable  $X$  follows  $X \sim \text{bin}(n, \theta)$ , where the probability of observing  $x$  successes is given by  $\Pr(X = x | \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$ , for  $x = 0, 1, \dots, n$ .

$\dots, n)$ . The beta distribution serves as a prior for probabilities, expressed as  $\pi(\theta) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \theta^{r-1} (1-\theta)^{s-1}$ . Its mean is  $E(\Theta) = \frac{r}{r+s}$  and standard deviation is  $SD(\Theta) = \sqrt{\frac{rs}{(r+s)^2(r+s+1)}}$ . When observing  $(X = x)$ , the posterior updates to  $\pi(\theta|x) \propto \theta^{x+r-1} (1-\theta)^{n-x+s-1}$ , again a beta distribution. In the exponential model, for variable  $(T)$  with failure rate  $(\lambda)$ , the likelihood is  $L(\lambda | t_1) = \lambda e^{-\lambda t_1}$ , and the gamma distribution is used as prior:  $\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda}$ . Posterior updates as  $\pi(\lambda | t) \propto \lambda^{\alpha+n-1} e^{-(\beta + \sum t_i) \lambda}$ . Hence, both binomial and exponential models use conjugate priors for straightforward Bayesian analysis.

## # Bayesian Analysis with Conjugate Distributions

This section summarizes Bayesian analysis using conjugate distributions in scenarios involving failures and their rates. The posterior density of a parameter,  $(\theta)$ , given an observation  $(x)$  is proportional to the likelihood, represented as  $\pi(\theta | x) \propto L(\theta | x) \pi(\theta)$ . It results in a beta distribution with parameters  $(x + r)$  and  $(n - x + s)$ , showing the binomial and beta distributions are conjugate. The means are calculated as  $E(\Theta) = \frac{r}{r+s}$  (prior) and  $E(\Theta | x) = \frac{x+r}{n+s+r}$  (posterior). For exponential models, if observations  $(t_1, t_2, \dots, t_n)$  are made with a gamma prior on the rate  $(\lambda)$ , the posterior will also follow a gamma distribution with updated parameters reflecting the data.

## \*\*Bayesian Estimation and Posterior Distribution\*\*

Bayesian estimators minimize the mean squared loss between a function of random variable  $(X)$  and estimated parameter  $(\Theta)$ , expressed as the expected value of the squared difference:  $E[(\theta(X) - \Theta)^2]$ . The optimal estimator,  $(\theta(X))$ , is found by minimizing the integral of the squared difference multiplied by the density function of  $(\Theta)$  given  $(X)$ . This results in  $(\theta(X) = E(\Theta | X))$ , signifying that the Bayesian estimator is the expected value of the posterior distribution of  $(\Theta)$  based on observations  $(x_1, x_2, \dots, x_n)$ . In the case of independent and identically distributed variables, the posterior density is proportional to the product of the likelihood and prior density:  $f_{\Theta|X_1, \dots, X_n}(\theta | x_1, \dots, x_n) \propto L(\theta | x_1, \dots, x_n) \pi(\theta)$ .

## \*\*Predictive Distribution in Bayesian Analysis\*\* The predictive

distribution of a random variable  $(X)$ , given a parameter  $(\Theta)$  with prior  $(\pi(\theta))$ , is obtained by marginalizing over  $(\Theta)$ :  $[f_X(x) = \int f_X(x | \theta) \pi(\theta) d\theta]$  When observing  $(X = x_0)$ , the predictive density for the next observation is:  $[f_X(x | x_0) = \int f(x | x_0, \theta) \pi(\theta | x_0) d\theta]$  For independent observations  $(x_1, x_2, \dots, x_n)$ , it becomes:  $[f_X(x | d) = \int f(x | \theta) \pi(\theta | d) d\theta]$  For an exponential scenario with parameter  $(\lambda)$  having a gamma prior, the posterior density is derived and used to predict the next failure time. In complex scenarios with multiple parameters, analytical solutions are challenging, necessitating computational methods such as MCMC. The BUGS programming language is commonly used for Bayesian analysis, with R interfaces like rjags and R2jags facilitating the process.

### **\*\*Bayesian Analysis with R and BUGS\*\***

Bayesian analysis centers on the posterior distribution of an unknown parameter,  $\theta$ . This distribution can be derived directly using conjugate distributions or through simulation methods, such as sampling and Markov Chain Monte Carlo (MCMC) approaches like Gibbs sampling and the Metropolis-Hastings algorithm. BUGS (Bayesian inference using Gibbs sampling) is a key tool for Bayesian simulation, separating knowledge from the inference process and handling complex models with minimal syntax. Popular implementations include WinBUGS, OpenBUGS, and JAGS, with JAGS often favored. To use these tools, one must write a BUGS model, prepare inputs in R, and execute the analysis, with R providing a frontend interface via packages like rjags and R2jags. Alternatively, Stan, interfaced using RStan, serves as another option for Bayesian modeling.

### **\*\*Reliability Prediction for Electrical and Mechanical Components\*\***

Reliability prediction assumes a constant failure rate, distinguished from estimation, which quantifies reliability from existing data. Predictions aim to forecast the future failure rate ( $\lambda$ ) of new components by modifying a baseline failure rate ( $\lambda_0$ ) with a function of relevant stress levels, yielding  $\lambda = \lambda_0 * C(\text{relevant stress levels})$ . The MIL-HDBK-217F handbook provides a common prediction method where  $\lambda$  is expressed as  $\lambda = \lambda_0 * Q * E * S$ , accounting for quality, environment, and application factors. The system failure rate ( $\lambda_s$ ) is the sum of individual component rates ( $\lambda_i$ ). Alternative methods, such as Siemens SN 29500 and Telcordia SR-322, offer similar prediction models. Additionally, common-cause failures (CCFs) are estimated through beta-factor models and documented by programs like the ICDE.

### **\*\*Reliability Assessment of Long-Term Projects\*\***

Projects may span years before items are operational, affecting reliability assessments that rely on outdated technologies. The OREDA project indicates that reliability data may come from materials up to 30 years old, raising concerns about technological relevance. Field data collection often misses non-critical failures, complicating inventory assessments, including item count, operational load percentage, and context. Most reliability databases offer constant failure rates despite equipment degradation, leading to average failure rate calculations. This can misstate reliability, especially when using a constant rate for an evolving system, as shown when findings from earlier observation windows differ significantly from later ones due to increasing rates of occurrence. Such misunderstandings can lead to erroneous maintenance assumptions, particularly in changing environments.

### **\*\*Reliability Data Analysis and Sample Homogeneity\*\***

In reliability analysis, the failure rate of inhomogeneous samples can vary due to differing contexts. Let's say we have  $m$  homogeneous samples; each sample  $i$  has  $n_i$  recorded failures over  $t_i$  operation hours. The failure rate can be estimated by dividing the number of failures by the operation time. If all samples were inaccurately assumed to share the same failure rate  $\lambda$ , the estimate would sum the failures divided by the total operation time across all samples. It's crucial to verify sample homogeneity before merging datasets, as databases often combine samples without this check. Alternative approaches, such as treating  $\lambda$  as a random variable representing varied estimates for different samples or using proportional hazards models with stressors, can better accommodate inhomogeneous data. Additionally, manufacturers' reliability data should be scrutinized due to potential reporting biases and the uncertainty surrounding operational durations. Analysts must ensure the data pertains to similar items, check for technological changes, assess operational contexts, and confirm sample sizes are adequate for reliable estimates.

### **\*\*Key References for Reliability Data Analysis\*\***

This section lists pivotal references on improving plant reliability and data collection in various industries. Notable sources include CCPS (1998) for guidelines on reliability enhancement, EPRD (2014) for electronic parts reliability, IEC standards for dependability management (60300-3-2) and functional safety (61508). The IEEE standards (352, 762) provide frameworks for assessing nuclear power system reliability. ISO (14224) focuses on maintenance data exchange in the petroleum sector. Important reliability concepts discussed include failure mode analysis (FMEA), reliability prediction (MIL-HDBK-217F), and expert judgment analysis (Meyer & Booker, 2001). The text emphasizes the necessity of thorough data collection and analysis to foster reliability improvement

across systems.

**Poisson Distribution** The **Poisson distribution** models the probability of a given number of events occurring within a fixed interval of time or space, assuming these events occur with a constant mean rate and independently of the time since the last event.

**Key Formula** If  $X$  is a random variable following a Poisson distribution, then:  $P(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$  where:  $\lambda$  = the average number of events (mean of the distribution).  $k$  = the number of events of interest (must be a non-negative integer).

**Key Properties** 1. Mean:  $E[X] = \lambda$  2. Variance:  $\text{Var}(X) = \lambda$

**Cumulative Probability** To calculate  $P(X < k)$ , sum the probabilities for all values of  $k$  less than the threshold:  $P(X < k) = \sum_{i=0}^{k-1} \frac{\lambda^i \cdot e^{-\lambda}}{i!}$

**Steps to Solve Poisson Problems** 1. Identify Parameters: Determine  $\lambda$  (mean number of events). 2. Plug into the Formula: - For exact probabilities ( $P(X = k)$ ): Use the formula directly. - For cumulative probabilities ( $P(X \leq k)$  or  $P(X < k)$ ): Sum the probabilities for the desired range. 3. Use Tables or Calculators: To save time, use cumulative probability tables or statistical calculators for larger values of  $k$ .

**Reliability Apportionment for Series Components** **Key Formula:** For a series system:  $R_{\text{total}} = R_1 \cdot R_2 \cdot R_3 \cdot R_4 \cdot R_5$  Where  $R_{\text{total}}$  is the total system reliability and  $R_i$  are individual component reliabilities.

**Steps to Solve:** 1. Compute  $R_{\text{remaining}} = R_{\text{total}} / (R_1 \cdot R_2 \cdot R_3 \cdot R_4)$ .

**Normal Distribution and  $B_{10}$  Life** **Key Concept:** The  $B_{10}$  life represents the time at which 10% of items have failed.

**Key Formula:**  $B_{10} = \mu + Z_{0.1} \cdot \sigma$  Where:  $\mu$  = mean,  $\sigma$  = standard deviation,  $Z_{0.1}$  = standard score for the 10th percentile. **Steps to Solve:** 1. Calculate  $\sigma = \sqrt{\text{variance}}$ . 2. Use python to compute  $Z_{0.1}$  (for 10th percentile in standard normal distribution). 3. Plug values into the formula to find  $B_{10}$ .

**Reliability Engineering:** - Focuses on the performance and consistency of a product over time. - Concerned with how well the product performs throughout its life cycle, especially under varying conditions. Measures: - Failure rate over time: Probability of failure occurring at any point during the product's operational life. - Product design: Involves designing products for durability and minimal failure under intended use.

## Consequences of Attributing Failure to Human Error Human Error Assumption in Failure Analysis:

When management assumes that human error is the main cause of a failure, it often leads to surface-level solutions that don't address underlying systemic issues. This approach typically focuses on individual blame, without considering potential flaws in systems, processes, training, or design.

**Consequences of Assuming Human Error as the Main Root Cause:**

- Repeat of the Same Failure:** Root cause not addressed: If human error is seen as the main cause, the systemic or procedural factors that contribute to the error may go unaddressed. This can result in the same failure occurring again.
- Reactive solutions:** Focusing on individual mistakes may lead to ineffective solutions, which don't prevent the recurrence of the issue.
- Disciplinary Action for the Employee:** Blaming the individual: Management may respond to human error by punishing or reprimanding employees, rather than investigating broader organizational issues (e.g., inadequate training, unclear procedures).
- Decreased morale:** This approach can lead to low employee morale and a culture of fear rather than one focused on continuous improvement.
- Less Productivity:** Fear of blame: When employees fear being blamed for errors, they may be less likely to take initiative or report issues, leading to reduced productivity.
- Stress and disengagement:** A focus on punishment for errors can lead to stress and disengagement among employees, further decreasing overall productivity.
- Cultural Impact:** Lack of continuous improvement: Focusing on individual blame rather than addressing process or system flaws reduces the opportunity for learning and improvement.
- Blame culture:** Encourages a culture where employees are afraid to speak up, which stifles innovation and continuous improvement.

## **\*\*Exponential Distribution\*\***

The **\*\*Exponential distribution\*\*** models the time between events in a process where events occur continuously and independently at a constant average rate. It is often used to model the time between failures or arrival times in queuing systems.

### **\*\*Key Formula\*\***

If  $(X)$  is a random variable following an Exponential distribution, then:  $P(X = x) = \lambda e^{-\lambda x}$  where:  $(\lambda)$  = the rate parameter (inverse of the mean, i.e.,  $\lambda = \frac{1}{\mu}$ ).  $(x)$  = the time between events (must be a non-negative value,  $x \geq 0$ ).

### **\*\*Key Properties\*\***

- \*\*Mean\*\*:**  $E[X] = \frac{1}{\lambda}$
- \*\*Variance\*\*:**  $\text{Var}(X) = \frac{1}{\lambda^2}$
- \*\*Memoryless Property\*\*:** The probability of an event occurring in the next  $(t)$  time units is independent of how much time has already passed.  $P(X > t + s \mid X > t) = P(X > s)$

This means the distribution "forgets" the past and has no

memory of previous events. **### \*\*Cumulative Distribution Function (CDF)\*\*** To calculate the probability that the event occurs before a certain time  $x$ , use the CDF:  $P(X \leq x) = 1 - e^{-\lambda x}$ . This represents the probability that the time between events is less than or equal to  $x$ . **### \*\*Survival Function\*\*** The survival function represents the probability that the event occurs after time  $x$ :  $P(X > x) = e^{-\lambda x}$ . **### \*\*Steps to Solve Exponential Distribution Problems\*\***

- 1. \*\*Identify Parameters\*\*:** Determine  $\lambda$ , the rate parameter (often given or derived from the mean  $\mu$ ).  $\lambda = \frac{1}{\mu}$ , where  $\mu$  is the mean time between events.
- 2. \*\*Calculate Probabilities\*\*:**
  - For exact probabilities  $P(X = x)$ : Use the probability density function (PDF) formula.
  - For cumulative probabilities  $P(X \leq x)$ : Use the CDF formula.
  - For survival probabilities  $P(X > x)$ : Use the survival function formula.
- 3. \*\*Use Tables or Calculators\*\*:** For quick calculations, use statistical tables or calculators that provide the cumulative distribution function or survival function values.

**Taguchi DOE Approach** The **Taguchi Design of Experiments (DOE)** approach includes:

- 1. Loss function concept.**
- 2. Variability reduction to meet target values.**
- 3. Continuous loss functions (not step functions).**

**Confidence Interval for Population Mean** **\*\*Key Formula:\*\*** For a 95% confidence interval (CI):  $\text{CI} = \bar{x} \pm Z_{0.025} \cdot \frac{s}{\sqrt{n}}$ . **Where:**  $\bar{x}$  = sample mean,  $s$  = sample standard deviation,  $n$  = sample size.

**\*\*Steps to Solve:\*\***

- 1.** Find  $Z_{0.025}$ . Using python for example
- 2.** Compute  $\frac{s}{\sqrt{n}}$ .
- 3.** Add/subtract from  $\bar{x}$ .

## Conditional Probabilities

**### \*\*Understanding the Context\*\*** In diagnostic problems (e.g., medical tests, quality control), we often deal with terms like:

- **\*\*True Positive (TP):\*\*** The test correctly identifies the presence of a condition (e.g., crack detected when a crack exists).
- **\*\*False Positive (FP):\*\*** The test incorrectly signals the presence of a condition (e.g., crack detected when there's no crack).
- **\*\*True Negative (TN):\*\*** The test correctly identifies the absence of a condition (e.g., no crack detected when there's no crack).
- **\*\*False Negative (FN):\*\*** The test fails to identify the presence of a condition (e.g., no crack detected when a crack exists).

In such scenarios:

- 1. \*\*Sensitivity (True Positive Rate):\*\***  $P(B|A)$ , the probability the test signals a crack when a crack is present.
- 2. \*\*False Positive Rate:\*\***  $P(B|\neg A)$ , the probability

the test signals a crack when no crack is present. 3. **\*\*Prevalence:\*\***  $P(A)$ , the proportion of cases where the condition (crack) exists in the population. --- **### \*\*When to Use Each Method\*\* ##### \*\*1. Conditional Probability\*\*** Use this to calculate the probability of one event occurring given that another event has already occurred. For instance, you might calculate:  $P(\text{crack present} | \text{test signals a crack}) = \frac{P(\text{test signals a crack AND crack present})}{P(\text{test signals a crack})}$  This is useful for understanding how reliable a test is when it gives a positive result. --- **##### \*\*2. Total Probability Rule\*\*** Use this when you want to find the overall probability of an event ( $P(B)$ ), accounting for all possible causes. In the problem:  $P(\text{test signals a crack}) = P(\text{test signals a crack} | \text{crack present}) \cdot P(\text{crack present}) + P(\text{test signals a crack} | \text{no crack}) \cdot P(\text{no crack})$  This helps aggregate the probabilities of a signal coming from both true and false scenarios. --- **##### \*\*3. Bayes' Theorem\*\*** Use this to update the probability of an event based on new evidence. For example, if a part signals a crack, Bayes' theorem helps calculate:  $P(\text{crack present} | \text{test signals a crack}) = \frac{P(\text{test signals a crack} | \text{crack present}) \cdot P(\text{crack present})}{P(\text{test signals a crack})}$  This is especially important when the prior probability of the condition ( $P(A)$ ) is low, as it adjusts for the rarity of the event and the accuracy of the test. --- **### \*\*Application in the Problem\*\*** 1. Use the **\*\*Total Probability Rule\*\*** to find  $P(B)$ , the probability that the test signals a crack (true or false). 2. Use **\*\*Bayes' Theorem\*\*** to find  $P(A|B)$ , the probability that the part actually has a crack given a positive test result. This process helps determine how reliable the test result is when it indicates a problem, balancing the high sensitivity against the false positive rate. --- **### Final Insight** This framework is applicable in fields like medical diagnostics, fraud detection, and quality control, where tests or algorithms produce signals, and we need to evaluate their real-world implications. Use these methods whenever you encounter probabilities of detection, errors, or prior information that influence a decision.

## **## \*\*Confidence Levels and Reliability\*\***

**### \*\*1. Key Definitions\*\*** - **\*\*Reliability ( $R$ )\*\*** Probability that a system performs its function without failure over a specified time or conditions. - **\*\*Confidence Level ( $CL$ )\*\*** Probability that the true reliability ( $R$ ) of a system is at least the specified value, based on test results. **### \*\*2. Binomial Testing Basics\*\*** Reliability and confidence levels are often assessed using a binomial distribution in pass/fail tests: -  $n$ : Total number of tests performed. -  $x$ : Number of successful tests (no failures). --- **### \*\*3. Reliability Test Formula\*\*** For a given reliability ( $R$ ) and confidence level ( $CL$ ),



the inequality to check is:  $\lceil CL \rceil \geq 1 - (1 - R)^n$  - Rearrange to compute  $\lceil n \rceil$ , the minimum number of tests required for a given  $\lceil R \rceil$  and  $\lceil CL \rceil$ :  $\lceil n \rceil \geq \frac{\log(1 - CL)}{\log(1 - R)}$

Alternatively, rearrange to check whether the result meets the customer's requirements:  $\lceil CL \rceil = 1 - (1 - R)^n$

### ### **\*\*Burn-In Testing\*\***

### **\*\*1. What is Burn-In Testing?\*\*** - **\*\*Definition:\*\*** A process where electronic assemblies or systems are operated under specified conditions (e.g., stress, elevated temperature) for a predetermined time before being placed in service. - **\*\*Purpose:\*\*** To detect and eliminate early-life failures (also known as "infant mortality failures") that occur due to manufacturing defects or material flaws. --- ### **\*\*2. Key Objectives of Burn-In Testing\*\*** - **\*\*Minimize Early Life Failures:\*\*** - Identify and remove weak units that would fail soon after deployment. - Ensures higher reliability for units that pass the test. - **\*\*Do Not Extend Product Life:\*\*** - Burn-in testing does not change the intrinsic reliability of a properly functioning product. - **\*\*Establish System Failure Rate:\*\*** - Useful for estimating the reliability parameters for a population of systems. - **\*\*Prevent Catastrophic Failures in the Field:\*\*** - Focused on ensuring quality for high-stakes applications (e.g., aerospace, medical devices). --- ### **\*\*3. When to Use Burn-In Testing?\*\*** - **\*\*Products with High Cost of Failure:\*\*** Electronics, critical systems, or high-reliability requirements. - **\*\*New Product Lines:\*\*** To ensure manufacturing processes are reliable. - **\*\*High Failure Variability:\*\*** If early-life failures are a significant proportion of total failures. --- ### **\*\*4. Key Benefits of Burn-In Testing\*\*** - Improves reliability and customer satisfaction by removing early failures. - Reduces warranty and repair costs. - Provides data to refine manufacturing processes.

### **Reliability tests:**

1. The **\*\*Arrhenius\*\*** model is typically used for thermal stress and describes the effect of temperature on failure rates through an exponential relationship. 2. The **\*\*Coffin-Manson\*\*** model is applied for fatigue failure under mechanical or thermal cycling, focusing on strain or stress cycles. 3. The **\*\*Inverse Power Law\*\*** model relates failure rates to a single stressor, like voltage or pressure, often in an accelerated life testing scenario. 4. The **\*\*Eyring\*\*** model, unlike the others, can incorporate multiple stress types (e.g., temperature and humidity) simultaneously, making it the exception.

**\*\*Weibull Distribution\*\*** The **\*\*Weibull distribution\*\*** is often used in reliability analysis to model the life data of products, particularly

### **when the failure rate is not constant over time.**

**\*\*Key Formula\*\*** The probability density function (PDF) for the Weibull distribution is:  $f(x; \lambda, k) = \frac{k}{\lambda} \left( \frac{x}{\lambda} \right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}$  where: -  $\lambda$  = scale parameter (characteristic life). -  $k$  = shape parameter (controls the failure rate behavior). **\*\*Key Properties\*\***

1. Mean:  $E[X] = \lambda \Gamma\left(1 + \frac{1}{k}\right)$ , where  $\Gamma$  is the Gamma function. 2. Variance:  $Var(X) = \lambda^2 \left( \Gamma\left(1 + \frac{2}{k}\right) - \left( \Gamma\left(1 + \frac{1}{k}\right) \right)^2 \right)$  **\*\*Common Use Cases\*\***

- **\*\* $(k = 1)$ \*\***: Exponential distribution (constant failure rate). - **\*\* $(k < 1)$ \*\***: Decreasing failure rate (early failures). - **\*\* $(k > 1)$ \*\***: Increasing failure rate (wear-out failures). For parametric estimation of the parameters, use Python to compute them. Use python to compute number of defaults after a given period of time and the CDF of the law.

### **\*\*Normal Distribution\*\* The \*\*Normal distribution\*\* is widely used for modeling symmetrical data, particularly for failure times when the data is not heavily skewed.**

**\*\*Key Formula\*\*** The probability density function (PDF) for the Normal distribution is:  $f(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  where: -  $\mu$  = mean (expected value). -  $\sigma^2$  = variance. **\*\*Key Properties\*\***

1. Mean:  $E[X] = \mu$  2. Variance:  $Var(X) = \sigma^2$  **\*\*Common Use Cases\*\*** - When failure times are symmetrically distributed around a central value (i.e., product life or performance).

### **\*\*Log-Normal Distribution\*\* The \*\*Log-Normal distribution\*\* is often used in reliability engineering when the logarithm of the variable follows a normal distribution. This is useful when modeling life data with a positive skew.**

**\*\*Key Formula\*\*** The probability density function (PDF) for the Log-Normal distribution is:  $f(x; \mu, \sigma^2) = \frac{1}{x \sigma \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$  where: -  $\mu$  = mean of the logarithmic values. -  $\sigma^2$  = variance of the logarithmic values. **\*\*Key Properties\*\*** 1. Mean:  $E[X] = e^{\mu + \frac{\sigma^2}{2}}$  2. Variance:  $Var(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

### **\*\*Failure Rate (Hazard Function)\*\* The \*\*failure rate\*\*, or \*\*hazard function\*\*, is crucial in reliability engineering, as it represents the instantaneous rate of failure at a given time.**

**\*\*Formula\*\*** The hazard function  $h(t)$  is defined as:  $h(t) =$

$\frac{f(t)}{S(t)}$  where: -  $f(t)$  is the probability density function (PDF). -  $S(t)$  is the survival function:  $S(t) = 1 - F(t)$ , where  $F(t)$  is the cumulative distribution function (CDF).  
**Interpretation** - **Constant hazard rate** implies exponential distribution. - **Increasing hazard rate** implies wear-out failure mode (e.g., Weibull with  $k > 1$ ).

**Mean Time Between Failures (MTBF)** **MTBF** is an important reliability metric representing the average time between two consecutive failures in a system.

**Formula**  $MTBF = \frac{\text{Total operating time}}{\text{Number of failures}}$

**Bath Tub Curve** The **Bath Tub Curve** is a graphical representation of the failure rate over time, showing three distinct phases: - **Infant mortality**: High failure rate at the start (early life failures). - **Normal life**: Constant failure rate (random failures). - **Wear-out**: Increasing failure rate at the end (late life failures).

This curve is often modeled using the **Weibull distribution** with  $k < 1$  for early failures and  $k > 1$  for wear-out failures.

**Population** The complete set of all possible observations or individuals of interest in a particular study.

**Sample** A subset of the population, selected to represent the population in a study.

**Parameter** A numerical characteristic or measure that describes a feature of a population.

**Statistic** A numerical characteristic or measure calculated from a sample. It is used to estimate the corresponding population parameter.

**Parametric Methods** Statistical methods that assume the data follows a specific distribution (e.g., normal distribution) and rely on known parameters.

**\*\*Nonparametric Methods\*\*** Statistical methods that do not assume any specific distribution for the data. These methods are more flexible and are used when the assumptions of parametric methods are not met.

**\*\*Mean\*\*** The average value of a data set. This can be calculated by summing up all of the individual values in the data set and dividing the total by the number of data values (n) in the set. The formula is:  $\mu = \frac{1}{n} \sum_{i=1}^n x_i$ , where  $x_i$  are the individual values.

**\*\*Median\*\*** The middle value in a sorted (i.e. low to high) data set. If there is an even number of values, then it is the average of the two middle values. If n is odd, the formula is:  $\text{Median} = x_{(n+1)/2}$ , where n is the number of data points.

**\*\*Range\*\*** The difference between the highest and lowest values of a data set.  $\text{Range} = x_{\text{max}} - x_{\text{min}}$ , where  $x_{\text{max}}$  is the highest value and  $x_{\text{min}}$  is the lowest value.

**\*\*Variation\*\*** A measure of how widely data values are spread out from the center of a data set. This can be calculated as the variance.

**\*\*Variance\*\*** A measure of how far the values in a data set are from the mean, on average. The formula for population variance is:  $\sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$ , and for sample variance:  $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ , where ( N ) is the population size, ( n ) is the sample size, (  $\mu$  ) is the population mean, and (  $\bar{x}$  ) is the sample mean.

**\*\*Standard Deviation\*\*** A measure of how far data values are spread around the mean of a data set. It is computed as the square root of the variance. The formula for population standard deviation is:  $\sigma =$

$\sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2}$ ), and for sample standard deviation:  $(s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2})$ .

**\*\*Samples\*\*** The sampling table gives the number of possible samples of size  $k$  out of a population of size  $n$ , under various assumptions about how the sample is collected. When sampling with replacement, each individual or item is returned to the population after being selected. The number of possible samples is given by the formula for combinations with replacement, which is represented as the binomial coefficient:  $\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$  where  $n$  is the population size, and  $k$  is the sample size. When sampling without replacement, once an individual or item is selected, it is not returned to the population. The number of possible samples is given by the binomial coefficient for combinations without replacement:  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  where  $n$  is the population size, and  $k$  is the sample size.

**\*\*Central Limit Theorem\*\*** The Central Limit Theorem (CLT) is one of the most important results in mathematics. Consider  $X_1, \dots, X_n$  as a sequence of independent and identically distributed random variables with mean  $\mu$  and standard deviation  $\sigma$ . Then,  $\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n (X_i - \mu)$  converges in distribution to  $N(0, 1)$  as  $n$  approaches infinity. This theorem asserts that the empirical mean of a large number of observations always converges to the Normal distribution, regardless of the nature of the phenomenon being studied.

**\*\*independence\*\*** Independent Events  $A$  and  $B$  are independent if knowing whether  $A$  occurred gives no information about whether  $B$  occurred. More formally,  $A$  and  $B$  (which have nonzero probability) are independent if and only if one of the following equivalent statements holds:  $P(A \cap B) = P(A)P(B)$ ,  $P(A|B) = P(A)$ ,  $P(B|A) = P(B)$ .

**\*\*conditional\_independence\*\***  $A$  and  $B$  are conditionally independent given  $C$  if:  $P(A \cap B|C) = P(A|C)P(B|C)$ . Conditional independence does not imply independence, and independence does not imply conditional independence.

**\*\*unions\_intersections\_complements\*\*** De Morgan's Laws: A useful identity

that can make calculating probabilities of unions easier by relating them to intersections, and vice versa.  $(A \cap B)^c = A^c \cup B^c$ ,  $(A \cup B)^c = A^c \cap B^c$ .

**\*\*joint\_marginal\_conditional\*\*** Joint Probability:  $P(A \cap B)$  or  $P(A, B)$  ? Probability of A and B. Marginal (Unconditional) Probability:  $P(A)$  ? Probability of A. Conditional Probability:  $P(A|B) = P(A, B)/P(B)$  ? Probability of A, given that B occurred.

**\*\*intersection\_union\_probabilities\*\*** Intersections via Conditioning:  $P(A, B) = P(A)P(B|A)$ ,  $P(A, B, C) = P(A)P(B|A)P(C|A, B)$ . Unions via Inclusion-Exclusion:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ,  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$ .

**\*\*law\_of\_total\_probability\*\*** Let  $B_1, B_2, B_3, \dots, B_n$  be a partition of the sample space (i.e., they are disjoint and their union is the entire sample space).  $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$ ,  $P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$ .

**\*\*bayes\_rule\*\*** Bayes' Rule:  $P(A|B) = P(B|A)P(A)/P(B)$ .  $P(A|B, C) = P(B|A, C)P(A|C)/P(B|C)$ . We can also write  $P(A|B, C) = P(A, B, C)/P(B, C) = P(B, C|A)P(A)/P(B, C)$ .

**\*\*ProbabilityMassFunction\*\*** Gives the probability that a discrete random variable takes on the value  $x$ .  $p_X(x) = P(X = x)$ . The PMF satisfies  $p_X(x) \geq 0$  and  $\sum p_X(x) = 1$ .

**\*\*CumulativeDistributionFunction\*\*** Gives the probability that a random variable is less than or equal to  $x$ .  $F_X(x) = P(X \leq x)$ .

**\*\*ProbabilityDensityFunction\*\*** The PDF is the derivative of the CDF.  $F'(x) = f(x)$ . If you integrate it over an interval  $[a, b]$ , it gives the probability that a random variable takes on a value in this interval  $[a, b]$ . A PDF is nonnegative and integrates to 1. To get from PDF to CDF,

integrate f to get F.

**\*\*reliability of system\*\*** To compute the reliability of a system with components in series, multiply the reliability of each component:  $R_{\text{system}} = R_1 \cdot R_2 \cdot \dots \cdot R_n$ . In a series configuration, failure of any component causes system failure. For components in parallel, calculate the complement of the product of their unreliabilities:  $R_{\text{system}} = 1 - [(1 - R_1) \cdot (1 - R_2) \cdot \dots \cdot (1 - R_n)]$ . In parallel systems, the system fails only if all components fail simultaneously.

**\*\*beta\*\*** The probability density function is  $f(t) = \frac{t^{\alpha-1} (1-t)^{\beta-1}}{B(\alpha, \beta)}$ . The distribution function is  $F(t) = \int_0^t \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du$ . The survivor function is  $R(t) = \int_t^1 \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du$ . The failure rate function is  $z(t) = \frac{\frac{t^{\alpha-1} (1-t)^{\beta-1}}{B(\alpha, \beta)}}{\int_t^1 \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du}$ . The mean time to failure is  $MTTF = \int_0^1 t \frac{t^{\alpha-1} (1-t)^{\beta-1}}{B(\alpha, \beta)} dt$ . The conditional survivor function is  $R(t | t_0) = \frac{\int_t^1 \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du}{\int_{t_0}^1 \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du}$ . The mean residual lifetime is  $MRL(t) = \frac{\int_t^1 u \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du}{\int_t^1 \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)} du}$ .

**\*\*Poisson\*\*** The probability density function is  $f(t) = \frac{\lambda^t e^{-\lambda}}{t!}$ . The distribution function is  $F(t) = \sum_{k=0}^t \frac{\lambda^k e^{-\lambda}}{k!}$ . The survivor function is  $R(t) = 1 - \sum_{k=0}^t \frac{\lambda^k e^{-\lambda}}{k!}$ . The failure rate function is  $z(t) = \frac{\frac{\lambda^t e^{-\lambda}}{t!}}{1 - \sum_{k=0}^t \frac{\lambda^k e^{-\lambda}}{k!}}$ . The mean time to failure is  $MTTF = \frac{1}{\lambda}$ . The conditional survivor function is  $R(t | t_0) = \frac{1 - \sum_{k=t+1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}}{1 - \sum_{k=t_0+1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}}$ . The mean residual lifetime is  $MRL(t) = \frac{1}{\lambda}$ .

**\*\*Sampling\*\*** To determine appropriate sample sizes or testing times, several statistical and reliability methods are used. Power analysis

helps identify sample sizes required to detect a specific effect, considering Type I (false positive) and Type II (false negative) errors, with common confidence levels of 95% or 99%. Formulas such as  $n = (Z_{\alpha/2} \cdot \sigma / E)^2$  for normal distribution and  $n = (Z_{\alpha/2} \cdot \sigma) / r$  for exponential distribution are used in reliability testing. Accelerated life testing (ALT) estimates testing times by accelerating failure rates. Sample size formulas for proportions and means are  $n = (Z_{\alpha/2} \cdot \sqrt{p(1-p)}) / E$  for proportions, and  $n = \sigma^2 / E^2$  for means with known variance. T-distribution, Chi-squared, and Normal distribution tables aid sample size determination. Practical considerations, such as balancing sample size with resources, expected failure rates, and available time, are essential for reliability testing.

**\*\*Sources of data\*\*** Quantitative system reliability analyses rely on four main types of input data. Technical data are needed to understand the functions and the functional requirements and to establish a system model. Technical data are usually supplied by the system vendors. Operational and environmental data are necessary to define the actual operating context for the system. Maintenance data, in the form of procedures, resources, quality, and durations, are necessary to establish the system model, and to be able to determine the system reliability. Failure data, that is, information about failure modes and failure causes, time-to-failure distributions, and various parameters. Operational, environmental, and maintenance data are system specific and can usually not be found in any databases. Reliability data can generally be obtained from the following sources: 1) Field (i.e., operational) failure event data from the company where the study object is to be used. The failure event data are usually available from the plant's computerized maintenance management system. To provide parameter estimates, the data has to be analyzed by methods as presented in Chapter 14. 2) Generic reliability databases where the items are classified in broad groups without information about manufacturer, make, and item specifications. OREDA (2015), for example, presents estimates for items such as "centrifugal pump; oil processing", "gas turbine; aeroderivative (3000-10000 kW)", and the like. 3) Sources providing information about failure modes and failure modes distributions, such as FMD (2016). 4) Expert judgment is sometimes the only option available to obtain input parameters. The procedure to obtain expert judgments can be more or less structured (e.g., see Meyer and Booker, 2001). 5) Data from manufacturers. These estimates may be based on (i) feedback to the manufacturer from practical use of the items, (ii) engineering analyses of the items, sometimes combined with some test results, (iii) warranty data, and obviously, a combination of all three types. 6) Reliability prediction models, usually combined with a base case component reliability database, such as MIL-HDBK-217F (1995). 7) Research reports



and papers sometimes present reliability studies of specific items, including the input reliability data. 8) Data from reliability testing. The testing may be part of the item's qualification process or be available from testing of similar items.

**\*\*FRACAS\*\*** A FRACAS is a system that provides a systematic way for reporting, classifying, analyzing failures and planning preventative or correct actions in response to those failures. A typical FRACAS system consists of the following steps; 1. Failure Reporting (FR). The failures related to a piece of equipment are reported through a standard form (such as failure information in a work order). 2. Analysis (A). Is using the data to identify the cause of failure. This may be using a Pareto Analysis to identify the most important issue to address and then using other techniques to dive into the issue and determine the cause. 3. Corrective Actions (CA). Once the cause has been identified, the corrective (or preventative) actions must be implemented to prevent the recurrence of the failure. Ideally, these are documented through a formal change management program to ensure the learnings are incorporated into new equipment designs.

**\*\*confidence interval\*\*** a is a percentage. A confidence interval with confidence level  $1-\alpha$  is such that  $1-\alpha$  of the time, the true value is contained in the confidence interval. For a population mean with known variance, the confidence interval (CI) is calculated as:  $CI = \bar{x} \pm Z \cdot \frac{\sigma}{\sqrt{n}}$ . When the population variance is unknown, the T-distribution is used:  $CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$ . For variance, the CI is computed using the Chi-squared distribution:  $CI = [(n-1)s^2]/\chi^2_{(1-\alpha/2)} \text{ to } [(n-1)s^2]/\chi^2_{(\alpha/2)}$ , where  $s^2$  is the sample variance and  $\chi^2$  is the Chi-squared critical value.

**\*\*Failure analysis\*\*** Here are the main tools used to analyze failures. Methods include the 'Five Why,' which asks 'why' repeatedly to find root causes; Ishikawa/Fishbone diagrams to categorize causes and effects; Cause and Effect Analysis/Causal Factor Tree to display causal dependencies; Failure Modes and Effects/Criticality Analysis to define failure modes and address critical issues; Fault or Logic Tree Analysis to trace failures to their roots; Barrier Analysis to examine pathways and barriers for hazards; Change Analysis/Kepner-Tregoe to compare problem and non-problem situations; Pareto Charts to prioritize problems based on frequency; and Data Analytics to transform and model data for insights.

**\*\*Statistical Process control\*\*** Statistical Process Control (SPC) and process capability studies are essential tools in quality control and reliability engineering. SPC uses control charts to monitor process stability and variability over time. Key control charts include the X-bar and R-chart for variable data and p-charts for attribute data. Process capability studies evaluate how well a process performs relative to its specification limits using indices such as  $C_p = (USL - LSL) / (6\sigma)$ , where USL and LSL are the upper and lower specification limits, and  $\sigma$  is the process standard deviation. The Cpk index, defined as  $C_{pk} = \min((USL - \bar{x}) / (3\sigma), (\bar{x} - LSL) / (3\sigma))$ , accounts for process centering. Reliable processes exhibit  $C_p$  and  $C_{pk}$  values above 1.33. By reducing variability, these methods improve product reliability and minimize defects.

**\*\*Censoring\*\*** When censoring occurs, we cannot always observe the true time-to-failure  $T$ ; instead, we observe the survival time, the time until failure or censoring. This involves two independent processes: a failure process and a censoring process. The observed time for item  $i$  is  $\min(T_i, C_i)$ , where  $T_i$  is the failure time and  $C_i$  is the censoring time. Each observation  $t_i$  has an indicator  $\delta_i$ , defined as  $\delta_i = 1$  if  $t_i$  ends with a failure ( $T_i < C_i$ ) and  $\delta_i = 0$  if it ends with censoring ( $T_i > C_i$ ). The dataset consists of  $n$  pairs  $(t_i, \delta_i)$ , representing survival time and whether the event ended with failure or censoring. Survival time is typically measured from when the item is new, but in practical cases, items may already have an initial age  $t(0)_i$  when observation begins.

**Poisson Distribution** The **\*\*Poisson distribution\*\*** models the probability of a given number of events occurring within a fixed interval of time or space, assuming these events occur with a constant mean rate and independently of the time since the last event.

**\*\*Key Formula\*\*** If  $X$  is a random variable following a Poisson distribution, then:  $P(X = k) = (\lambda^k * e^{-\lambda}) / k!$  where:  $\lambda$  = the average number of events (mean of the distribution).  $k$  = the number of events of interest (must be a non-negative integer). **\*\*Key**

**Properties\*\*** 1. Mean:  $E[X] = \lambda$  2. Variance:  $Var(X) = \lambda$

**\*\*Cumulative Probability\*\*** To calculate  $P(X < k)$ , sum the probabilities for all values of  $k$  less than the threshold:  $P(X < k) = \sum_{i=0}^{k-1} (\lambda^i * e^{-\lambda}) / i!$

**\*\*Steps to Solve Poisson Problems\*\*** 1. Identify Parameters: Determine  $\lambda$  (mean number of events). 2. Plug into the Formula: - For exact probabilities ( $P(X = k)$ ): Use the formula directly. - For cumulative probabilities ( $P(X \geq k)$  or  $P(X < k)$ ): Sum

the probabilities for the desired range. 3. Use Tables or Calculators:  
To save time, use cumulative probability tables or statistical  
calculators for larger values of k.

**Reliability Apportionment for Series Components** **\*\*Key Formula:\*\*** For a series system:  $R_{\text{total}} = R_1 \cdot R_2 \cdot R_3 \cdot R_4 \cdot R_5$   
Where  $R_{\text{total}}$  is the total system reliability and  $R_i$  are individual component reliabilities.

**\*\*Steps to Solve:\*\*** 1. Compute  $R_{\text{remaining}} = R_{\text{total}} / (R_1 \cdot R_2 \cdot R_3 \cdot R_4)$ .

**Normal Distribution and  $B_{10}$  Life** **\*\*Key Concept:\*\*** The  $B_{10}$  life represents the time at which 10% of items have failed.

**\*\*Key Formula:\*\***  $B_{10} = \mu + Z_{0.1} \cdot \sigma$  Where: -

$\mu$  = mean,  $\sigma$  = standard deviation,  $Z_{0.1}$  = standard score for the 10th percentile. **\*\*Steps to Solve:\*\*** 1.

Calculate  $\sigma = \sqrt{\text{variance}}$ . 2. Use python to compute  $Z_{0.1}$  (for 10th percentile in standard normal distribution). 3. Plug values into the formula to find  $B_{10}$ .

**Reliability Engineering: - Focuses on the performance and consistency of a product over time. - Concerned with how well the product performs throughout its life cycle, especially under varying conditions. Measures:**  
**- Failure rate over time: Probability of failure occurring at any point during the product's operational life. - Product design: Involves designing products for durability and minimal failure under intended use.**

**Consequences of Attributing Failure to Human Error Human Error Assumption in Failure Analysis:**

When management assumes that human error is the main cause of a failure, it often leads to surface-level solutions that don't address underlying systemic issues. This approach typically focuses on individual blame, without considering potential flaws in systems, processes, training, or design. Consequences of Assuming Human Error as the Main Root Cause:  
Repeat of the Same Failure: Root cause not addressed: If human error is seen as the main cause, the systemic or procedural factors that contribute to the error may go unaddressed. This can result in the same failure occurring again. Reactive solutions: Focusing on individual mistakes may lead to ineffective solutions, which don't prevent the recurrence of the issue. Disciplinary Action for the Employee: Blaming the individual: Management may respond to human error by punishing or reprimanding employees, rather than investigating broader organizational

issues (e.g., inadequate training, unclear procedures). Decreased morale: This approach can lead to low employee morale and a culture of fear rather than one focused on continuous improvement. Less Productivity: Fear of blame: When employees fear being blamed for errors, they may be less likely to take initiative or report issues, leading to reduced productivity. Stress and disengagement: A focus on punishment for errors can lead to stress and disengagement among employees, further decreasing overall productivity. Cultural Impact: Lack of continuous improvement: Focusing on individual blame rather than addressing process or system flaws reduces the opportunity for learning and improvement. Blame culture: Encourages a culture where employees are afraid to speak up, which stifles innovation and continuous improvement.

## **\*\*Exponential Distribution\*\***

The **\*\*Exponential distribution\*\*** models the time between events in a process where events occur continuously and independently at a constant average rate. It is often used to model the time between failures or arrival times in queuing systems. **### \*\*Key Formula\*\*** If  $(X)$  is a random variable following an Exponential distribution, then:  $P(X = x) = \lambda e^{-\lambda x}$  where:  $\lambda$  = the rate parameter (inverse of the mean, i.e.,  $\lambda = \frac{1}{\mu}$ ).  $x$  = the time between events (must be a non-negative value,  $x \geq 0$ ). **### \*\*Key Properties\*\*** 1. **\*\*Mean\*\***:  $E[X] = \frac{1}{\lambda}$  2. **\*\*Variance\*\***:  $\text{Var}(X) = \frac{1}{\lambda^2}$  3. **\*\*Memoryless Property\*\***: The probability of an event occurring in the next  $(t)$  time units is independent of how much time has already passed.  $P(X > t + s \mid X > t) = P(X > s)$  This means the distribution "forgets" the past and has no memory of previous events. **### \*\*Cumulative Distribution Function (CDF)\*\*** To calculate the probability that the event occurs before a certain time  $(x)$ , use the CDF:  $P(X \leq x) = 1 - e^{-\lambda x}$  This represents the probability that the time between events is less than or equal to  $(x)$ . **### \*\*Survival Function\*\*** The survival function represents the probability that the event occurs after time  $(x)$ :  $P(X > x) = e^{-\lambda x}$  **### \*\*Steps to Solve Exponential Distribution Problems\*\*** 1. **\*\*Identify Parameters\*\***: Determine  $\lambda$ , the rate parameter (often given or derived from the mean  $\mu$ ).  $\lambda = \frac{1}{\mu}$ , where  $\mu$  is the mean time between events. 2. **\*\*Calculate Probabilities\*\***: - For exact probabilities  $(P(X = x))$ : Use the probability density function (PDF) formula. - For cumulative probabilities  $(P(X \leq x))$ : Use the CDF formula. - For survival probabilities  $(P(X > x))$ : Use the survival function formula. 3. **\*\*Use Tables or Calculators\*\***: - For quick calculations, use statistical tables or calculators that provide the cumulative distribution function or survival function values.

**Taguchi DOE Approach The **\*\*Taguchi Design of Experiments (DOE)\*\*** approach**

includes: 1. Loss function concept. 2. Variability reduction to meet target values. 3. Continuous loss functions (not step functions).

**Confidence Interval for Population Mean** **\*\*Key Formula:\*\*** For a 95% confidence interval (CI): 
$$\text{CI} = \bar{x} \pm Z_{0.025} \cdot \frac{s}{\sqrt{n}}$$
 Where: -  $\bar{x}$  = sample mean,  $s$  = sample standard deviation,  $n$  = sample size.

**\*\*Steps to Solve:\*\*** 1. Find  $Z_{0.025}$ . Using python for example  
2. Compute  $\frac{s}{\sqrt{n}}$ . 3. Add/subtract from  $\bar{x}$   
).

## Conditional Probabilities

### **\*\*Understanding the Context\*\*** In diagnostic problems (e.g., medical tests, quality control), we often deal with terms like: - **\*\*True Positive (TP):\*\*** The test correctly identifies the presence of a condition (e.g., crack detected when a crack exists). - **\*\*False Positive (FP):\*\*** The test incorrectly signals the presence of a condition (e.g., crack detected when there's no crack). - **\*\*True Negative (TN):\*\*** The test correctly identifies the absence of a condition (e.g., no crack detected when there's no crack). - **\*\*False Negative (FN):\*\*** The test fails to identify the presence of a condition (e.g., no crack detected when a crack exists). In such scenarios: 1. **\*\*Sensitivity (True Positive Rate):\*\***  $P(B|A)$ , the probability the test signals a crack when a crack is present. 2. **\*\*False Positive Rate:\*\***  $P(B|\neg A)$ , the probability the test signals a crack when no crack is present. 3. **\*\*Prevalence:\*\***  $P(A)$ , the proportion of cases where the condition (crack) exists in the population. --- ### **\*\*When to Use Each Method\*\*** ##### **\*\*1. Conditional Probability\*\*** Use this to calculate the probability of one event occurring given that another event has already occurred. For instance, you might calculate: 
$$P(\text{crack present} | \text{test signals a crack}) = \frac{P(\text{test signals a crack AND crack present})}{P(\text{test signals a crack})}$$
 This is useful for understanding how reliable a test is when it gives a positive result. --- ##### **\*\*2. Total Probability Rule\*\*** Use this when you want to find the overall probability of an event ( $P(B)$ ), accounting for all possible causes. In the problem: 
$$P(\text{test signals a crack}) = P(\text{test signals a crack} | \text{crack present}) \cdot P(\text{crack present}) + P(\text{test signals a crack} | \text{no crack}) \cdot P(\text{no crack})$$
 This helps aggregate the probabilities of a signal coming from both true and false scenarios. --- ##### **\*\*3. Bayes' Theorem\*\*** Use this to update the probability of an event based on new evidence. For example, if a part signals a crack, Bayes' theorem helps calculate: 
$$P(\text{crack present} | \text{test signals a crack}) = \frac{P(\text{test signals a crack} | \text{crack present}) \cdot P(\text{crack present})}{P(\text{test signals a crack})}$$

present))} {P(\text{test signals a crack})} \] This is especially important when the prior probability of the condition ( $P(A)$ ) is low, as it adjusts for the rarity of the event and the accuracy of the test. --- ### **Application in the Problem** 1. Use the **Total Probability Rule** to find  $P(B)$ , the probability that the test signals a crack (true or false). 2. Use **Bayes' Theorem** to find  $P(A|B)$ , the probability that the part actually has a crack given a positive test result. This process helps determine how reliable the test result is when it indicates a problem, balancing the high sensitivity against the false positive rate. --- ### **Final Insight** This framework is applicable in fields like medical diagnostics, fraud detection, and quality control, where tests or algorithms produce signals, and we need to evaluate their real-world implications. Use these methods whenever you encounter probabilities of detection, errors, or prior information that influence a decision.

## ## **Confidence Levels and Reliability**

### **1. Key Definitions** - **Reliability ( $R$ )**: Probability that a system performs its function without failure over a specified time or conditions. - **Confidence Level ( $CL$ )**: Probability that the true reliability ( $R$ ) of a system is at least the specified value, based on test results. ### **2. Binomial Testing Basics** Reliability and confidence levels are often assessed using a binomial distribution in pass/fail tests: -  $n$ : Total number of tests performed. -  $x$ : Number of successful tests (no failures). --- ### **3. Reliability Test Formula** For a given reliability ( $R$ ) and confidence level ( $CL$ ), the inequality to check is:  $CL \geq 1 - (1 - R)^n$  - Rearrange to compute  $n$ , the minimum number of tests required for a given ( $R$ ) and ( $CL$ ):  $n \geq \frac{\log(1 - CL)}{\log(1 - R)}$  Alternatively, rearrange to check whether the result meets the customer's requirements:  $CL = 1 - (1 - R)^n$

## ### **Burn-In Testing**

### **1. What is Burn-In Testing?** - **Definition**: A process where electronic assemblies or systems are operated under specified conditions (e.g., stress, elevated temperature) for a predetermined time before being placed in service. - **Purpose**: To detect and eliminate early-life failures (also known as "infant mortality failures") that occur due to manufacturing defects or material flaws. --- ### **2. Key Objectives of Burn-In Testing** - **Minimize Early Life Failures**: - Identify and remove weak units that would fail soon after deployment. - Ensures higher reliability for units that pass the test. - **Do Not Extend Product Life**: - Burn-in testing does not change the intrinsic reliability of a properly functioning product. - **Establish System Failure Rate**: - Useful for estimating the reliability parameters for

a population of systems. - **Prevent Catastrophic Failures in the Field:** - Focused on ensuring quality for high-stakes applications (e.g., aerospace, medical devices). --- **3. When to Use Burn-In Testing?** - **Products with High Cost of Failure:** Electronics, critical systems, or high-reliability requirements. - **New Product Lines:** To ensure manufacturing processes are reliable. - **High Failure Variability:** If early-life failures are a significant proportion of total failures. --- **4. Key Benefits of Burn-In Testing** - Improves reliability and customer satisfaction by removing early failures. - Reduces warranty and repair costs. - Provides data to refine manufacturing processes.

## Reliability tests:

1. The **Arrhenius** model is typically used for thermal stress and describes the effect of temperature on failure rates through an exponential relationship. 2. The **Coffin-Manson** model is applied for fatigue failure under mechanical or thermal cycling, focusing on strain or stress cycles. 3. The **Inverse Power Law** model relates failure rates to a single stressor, like voltage or pressure, often in an accelerated life testing scenario. 4. The **Eyring** model, unlike the others, can incorporate multiple stress types (e.g., temperature and humidity) simultaneously, making it the exception.

**Weibull Distribution** The **Weibull distribution** is often used in reliability analysis to model the life data of products, particularly when the failure rate is not constant over time.

**Key Formula** The probability density function (PDF) for the Weibull distribution is:  $f(x; \lambda, k) = \frac{k}{\lambda} \left( \frac{x}{\lambda} \right)^{k-1} e^{-\left( \frac{x}{\lambda} \right)^k}$  where: -  $\lambda$  = scale parameter (characteristic life). -  $k$  = shape parameter (controls the failure rate behavior). **Key Properties**

1. Mean:  $E[X] = \lambda \Gamma\left(1 + \frac{1}{k}\right)$ , where  $\Gamma$  is the Gamma function. 2. Variance:  $Var(X) = \lambda^2 \left( \Gamma\left(1 + \frac{2}{k}\right) - \left( \Gamma\left(1 + \frac{1}{k}\right) \right)^2 \right)$  **Common Use Cases** -  $k = 1$ : Exponential distribution (constant failure rate). -  $k < 1$ : Decreasing failure rate (early failures). -  $k > 1$ : Increasing failure rate (wear-out failures). For parametric estimation of the parameters, use Python to compute them. Use python to compute number of defaults after a given period of time and the CDF of the law.

**Normal Distribution** The **Normal distribution** is widely used for modeling symmetrical data, particularly for failure times when the data

**is not heavily skewed.**

**\*\*Key Formula\*\*** The probability density function (PDF) for the Normal distribution is:  $f(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  where:

-  $\mu$  = mean (expected value). -  $\sigma^2$  = variance. **\*\*Key Properties\*\***

1. Mean:  $E[X] = \mu$  2. Variance:  $Var(X) = \sigma^2$

**\*\*Common Use Cases\*\*** - When failure times are symmetrically distributed around a central value (i.e., product life or performance).

**\*\*Log-Normal Distribution\*\*** The **\*\*Log-Normal distribution\*\*** is often used in reliability engineering when the logarithm of the variable follows a normal distribution. This is useful when modeling life data with a positive skew.

**\*\*Key Formula\*\*** The probability density function (PDF) for the Log-

Normal distribution is:  $f(x; \mu, \sigma^2) = \frac{1}{x \sigma \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$  where: -  $\mu$  = mean of the logarithmic values. -  $\sigma^2$  = variance of the logarithmic values. **\*\*Key Properties\*\*** 1. Mean:  $E[X] = e^{\mu + \frac{\sigma^2}{2}}$  2. Variance:  $Var(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$

**\*\*Failure Rate (Hazard Function)\*\*** The **\*\*failure rate\*\***, or **\*\*hazard function\*\***, is crucial in reliability engineering, as it represents the instantaneous rate of failure at a given time.

**\*\*Formula\*\*** The hazard function  $h(t)$  is defined as:  $h(t) = \frac{f(t)}{S(t)}$  where: -  $f(t)$  is the probability density function (PDF). -  $S(t)$  is the survival function:  $S(t) = 1 - F(t)$ , where  $F(t)$  is the cumulative distribution function (CDF).

**\*\*Interpretation\*\*** - **\*\*Constant hazard rate\*\*** implies exponential distribution. - **\*\*Increasing hazard rate\*\*** implies wear-out failure mode (e.g., Weibull with  $k > 1$ ).

**\*\*Mean Time Between Failures (MTBF)\*\*** **\*\*MTBF\*\*** is an important reliability metric representing the average time between two consecutive failures in a system.

**\*\*Formula\*\***  $MTBF = \frac{\text{Total operating time}}{\text{Number of failures}}$

**\*\*Bath Tub Curve\*\*** The **\*\*Bath Tub Curve\*\*** is a graphical representation of the failure rate over time, showing three distinct phases: - **\*\*Infant mortality\*\***: High failure rate at the start (early life failures). - **\*\*Normal life\*\***: Constant failure rate (random failures). - **\*\*Wear-out\*\***: Increasing failure rate at the end (late life failures).



This curve is often modeled using the **Weibull distribution** with  $k < 1$  for early failures and  $k > 1$  for wear-out failures.

**Risk Management Techniques:** Risk management involves the application of tools and processes such as FMEA, FTA, and Risk Matrices to identify, document, and track concerns in reliability engineering. These techniques prioritize risks based on safety, economic, performance, and customer satisfaction concerns.

**Types of Risk:** Types of risks include technical, scheduling, safety, and financial risks. Each type affects reliability differently and requires tailored management strategies.

**Failure Mode and Effects Analysis (FMEA):** Failure Mode and Effects Analysis (FMEA) is a systematic methodology used to identify potential failure modes, their causes, and their effects on system performance. It helps prioritize risks based on severity, occurrence, and detection. Formulas include: Risk Priority Number (RPN):  $RPN = S \times O \times D$ , Criticality Index (CI):  $CI = \frac{S}{O} \times \frac{O}{D}$ . Example: For Severity (S) = 8, Occurrence (O) = 5, and Detection (D) = 4,  $RPN = 8 \times 5 \times 4 = 160$ .

**Common Mode Failure (CMF):** Common Mode Failure (CMF) occurs when two or more components fail simultaneously due to a shared cause. It is critical in redundant systems where multiple failures can compromise reliability. Formulas include: Beta Factor Model:  $P_{CMF} = \beta \times P_{individual}$ , Miller's Multiplicity Factor:  $P_{CMF} = 1 - (1 - P_{individual})^n$ . Example: For two components (failure probability = 0.01) and a beta factor of 0.2,  $P_{CMF} = 0.2 \times 0.01 = 0.002$ .

**Hazard Analysis (HA):** Hazard Analysis (HA) is a structured process to identify potential hazards and system demands. Methods include Preliminary Hazard Analysis (PHA), Hazard and Operability Analysis (HAZOP), and FMECA. Outputs comprise a comprehensive hazard list, identification of causes, hazard frequency estimation, qualitative assessment, and non-Safety Instrumented System (SIS) protection layers.

In Reliability Block Diagrams (RBD), identifying single points of failure

**is crucial yet challenging in complex systems. Relevant components impact system functionality, while irrelevant ones do not, defined as follows for an irrelevant component  $(i)$ :**

```
python def irrelevant_component(phi_i, x): return phi_i(1, x) ==
phi_i(0, x)`` A coherent structure, composed of relevant components
and possessing a nondecreasing structure function, is characterized by:
 $(\phi(0) = 0)$  (all components failed) and  $(\phi(1) = 1)$  (all
functioning). Its performance bounds are given by: python product(x)
<= phi(x) <= sum(x)`` Additionally, redundancy properties are
expressed as: python phi(x ? y) >= phi(x) + phi(y) # System level
phi(x ? y) <= phi(x) * phi(y) # Component level`` Understanding
component relevancy is context-dependent, influencing specific system
functions and enhancing overall reliability.
```

**The minimal cut set  $(K_j)$  for a binary function and the overall structure function  $(\phi(x))$  for a parallel system of minimal cut structures can be defined using Python functions. The cut set is represented by:**

```
python def k_j(x): return all(x[i] for i in K_j) - prod(1 - x[i]
for i in K_j)`` The overall structure function is: python def
phi(x): return 1 - prod(1 - rho(j, x) for j in range(1, p + 1))``
where  $(\rho(j, x))$  evaluates minimal paths as: python def rho(j,
x): return prod(x[i] for i in minimal_path(j))`` In pivotal
decomposition, the reliability of focusing on component  $(i)$  is:
python def pivotal_decomposition(x, i): return x[i] * phi(1, x) +
(1 - x[i]) * phi(0, x)`` For a bridge structure, using pivotal
decomposition: python def phi_bridge(x): return x[3] * ((x[1] |
x[2]) * (x[4] | x[5])) + (1 - x[3]) * (x[1] * x[4] + x[2] * x[5])`` In
a 2-out-of-3 (2oo3) structure, both minimal path sets  $(P)$  and minimal
cut sets  $(K)$  are defined similarly. Each path set can be evaluated
with: python def rho_j(x): return prod(x[i] for i in P_j)``
The overall structure function for this is given by: python def
phi(x): return 1 - prod(1 - rho_j(x) for j in range(1, p + 1))``
These foundations illustrate the critical role of minimal paths and cuts
in reliability analysis.
```

**A coherent module is a subset of components within a system, organized to operate independently and influence the system exclusively through their interactions. For a system  $(C, \phi)$  with  $(C)$  as the set of components and  $(\phi)$  as the structure function, a subset  $(A \subseteq C)$  and its complement  $(A^c = C - A)$  define a state vector  $(x_A = (x_{i_1}, x_{i_2}, \dots, x_{i_{nu}}))$ . The coherent module condition is that  $(\phi(x))$  can be expressed as  $(\psi(\chi(x_A), x_{A^c}))$ . For example, if  $(C = \{1, 2, \dots, 10\})$  and  $(A = \{5, 6, 7\})$ , then  $(\chi(x_A) = (x_5 \text{ or } x_6)(x_5 \text{ or } x_7))$ . In**

**Python, this is represented as:**

```
```python def chi(x_A):    return (x_A[0] | x_A[1]) & (x_A[0] | x_A[2])
def phi(x):    return psi(chi(x[:3]), x[:3])``` Thus,  $\chi(A, \chi)$ 
characterizes a coherent module in the system.
```

**A Bayesian Network (BN) is a directed acyclic graph (DAG) used for system reliability analysis, where nodes represent states and directed edges indicate cause-effect relationships. In this structure, root nodes have no parents, and leaf nodes have no children; for example, if node A influences node B, it is represented as  $A \rightarrow B$ . Various system architectures can be modeled:**

1. **Series Structure**: The system  $S = 1$  if both  $A = 1$  and  $B = 1$ .  
```python def series\_structure(A, B): return A \* B # Logical AND```  
2. **Parallel Structure**: The system  $S = 1$  if either  $A = 1$  or  $B = 1$ .  
```python def parallel\_structure(A, B): return A | B # Logical OR```  
3. **2-out-of-3 (2oo3) Structure**: The system functions if at least two of  $A, B, C$  are 1.  
```python def two\_out\_of\_three(A, B, C): return (A + B + C) >= 2 # At least 2 must be operational```  
These formulations are essential for fault tree analysis and reliability assessments.

**To ensure optimal lubrication system functionality, maintain sufficient oil throughput and quality. For throughput, at least one cooler, one open filter, a working pump, and unobstructed pipelines are essential. Oil quality requires both coolers operational, clear filters, and a functional separator system. Construct fault trees for the top events: "Too low throughput of oil/lubricant" and "Too low quality of the oil/lubricant," using MOCUS to identify all minimal cut sets. Utilize these key equations:**

1. For parallel structures: ````python def phi_parallel(x, y): return phi(x) + phi(y) # x ? y````  
2. For series structures: ````python def phi_series(x, y): return phi(x) * phi(y) # x ? y````  
3. For dual structures: ````python def dual_phi(x): return 1 - phi(1 - x) # Dual representation````  
These relationships link minimal cut sets to minimal path sets within the system analysis while ensuring the sufficiency criteria for oil throughput and quality are met.

**This section explores statistical distributions relevant to time-to-failure analysis, including exponential, gamma, Weibull, normal, lognormal, and extreme value distributions, alongside discrete distributions like binomial and Poisson. The state variable  $X(t)$  denotes whether an item is functioning (1) or failed (0), while the time-**

to-failure, a random variable  $(T)$ , captures the duration from operation to failure, usually treated as continuous. Historical datasets, such as  $(\{t_1, t_2, \dots, t_n\})$  (with  $(n = 60)$ ), are essential for reliability predictions under independent conditions. Key metrics include the survivor function  $(R(t))$ , failure rate  $(z(t))$ , Mean Time-To-Failure (MTTF), conditional survivor function, and Mean Residual Lifetime (MRL). Python functions for state representation, time-to-failure calculation, relative frequency distribution, and the empirical survivor function provide tools for analysis and predictive modeling.

**The k-th moment of a random variable  $(T)$  is defined as:**

```

python mu_k = integrate(t**k * f(t), (t, 0, infinity)) # Equation (1)
''' It can also be expressed using the reliability function  $(R(t))$ :
python mu_k = k * integrate(t**(k-1) * R(t), (t, 0, infinity)) #
Equation (2) ''' For  $(k=1)$ , this represents the mean of  $(T)$ . The
percentile function is defined as: python t_p = F_inv(p) # for  $0 < p$ 
 $< 1$  ''' The median lifetime  $(t_m)$  is: python t_m = R_inv(0.50)
# where  $R(t_m) = 0.50$  ''' The mode  $(t_{\text{mode}})$  is located at:
python t_{\text{mode}} = argmax(f(t)) #  $f(t_{\text{mode}}) = \max f(t)$  ''' The
Mean Time To Failure (MTTF), median, and mode are key metrics in
evaluating lifetime distributions. An example of the survivor function is
given by  $(R(t) = \frac{1}{(0.2t + 1)^2})$  for  $(t \geq 0)$ .

```

The rotary pump has a constant failure rate  $(\lambda = 4.28 \times 10^{-4})$  hours $(^{-1})$ . The survival probability after one month (730 hours) is  $(R(730) = e^{-\lambda t} \approx 0.732)$ , and the Mean Time To Failure (MTTF) is approximately  $(\frac{1}{\lambda} \approx 2336)$  hours (3.2 months). After 1460 hours without failure, the failure probability in the next month is  $(P_{\text{fail}} = 1 - e^{-\lambda \cdot 730})$ . For two independent components with failure rates  $(\lambda_1)$  and  $(\lambda_2)$ , the probability that component 1 fails first is  $(P_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2})$ . The survival function for a mixture of items from two plants with rates  $(\lambda_1, \lambda_2)$  is  $(R_t = p e^{-\lambda_1 t} + (1 - p)e^{-\lambda_2 t})$ . Average failure rate  $(\lambda_t)$  can be computed based on operational and standby periods.

The lognormal distribution  $(T \sim \text{lognorm}(\nu, \tau))$  is commonly used to model repair times, indicating a rapid initial repair rate that eventually slows due to complications. For symmetric intervals  $(t_L, t_U)$ , the probabilities are computed as  $(\Pr(t_L < T \leq$

**$t_U = 1 - 2\alpha$  with bounds:**

$t_L = e^{-u_\alpha \tau}$ ,  $t_U = e^{u_\alpha \tau}$ , where  $u_\alpha$  is the upper  $\alpha$  percentile of the normal distribution. The MTTF is given by:  $MTTF = e^{\nu + \frac{\tau^2}{2}}$ ,  $t_m = e^\nu$ ,  $t_{\text{mode}} = e^{\nu - \frac{\tau^2}{2}}$ , and the variance is:  $\text{var}(T) = e^{2\nu + 2\tau^2} - e^{\tau^2}$ . The survivor function is  $R(t) = 1 - \Phi\left(\frac{\log t - \nu}{\tau}\right)$ , with the failure rate function defined as  $z(t) = \frac{\nu - \log t}{\tau} \cdot \frac{\phi\left(\frac{\nu - \log t}{\tau}\right)}{t}$ . In fatigue analysis, the relationship between cycles  $N$  and stress  $s$  is modeled as:  $N s^b = c \Rightarrow \log N = \log c - b \log s \Rightarrow Y = \alpha + \beta x + \text{error}$ , foundational for the Weibuller curve. Overall, the lognormal distribution is effective in reliability contexts, mapping real-world failure data to statistical models.

**Extreme value distributions are essential in reliability analysis and studies on metal corrosion, particularly in systems with identical components. For independent, identically distributed random variables  $(T_1, T_2, \dots, T_n)$  with a continuous distribution function  $F_T(t)$ , the minimum and maximum are defined as  $T_{(1)} = \min\{T_1, T_2, \dots, T_n\}$  and  $T_{(n)} = \max\{T_1, T_2, \dots, T_n\}$ . Their distribution functions are given by  $F_{U_n}(u) = 1 - [1 - F_T(u)]^n$  and  $F_{V_n}(v) = [F_T(v)]^n$ . As  $n \rightarrow \infty$ ,  $(Y_n = n F_T(U_n))$  and  $(Z_n = n [1 - F_T(V_n)])$  converge to exponential distributions:  $F_Y(y) = 1 - e^{-y}$ . The Gumbel distribution applies when  $f_T(t)$  decays exponentially, represented by  $F_{T(1)}(t) = 1 - e^{-e^{-(t-x)/\alpha}}$  and  $F_{T(n)}(t) = e^{-e^{-(t-x)/\alpha}}$ . These models are critical for understanding extreme behavior, such as pitting corrosion in steel pipes.**

**The binomial distribution,  $(X \sim \text{bin}(n, p))$ , models the number of successes in  $n$  independent Bernoulli trials, with probability mass function  $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ . Its mean and variance are  $E(X) = n \cdot p$  and  $\text{var}(X) = n \cdot p \cdot (1-p)$ . The geometric distribution,  $(Z \sim \text{geom}(p))$ , represents trials until the first success, with  $P(Z = z) = (1-p)^{z-1} p$ , mean  $E(Z) = \frac{1}{p}$ , and variance  $\text{var}(Z) = \frac{1-p}{p^2}$ . The negative binomial distribution counts trials until the  $r$ -th success:  $P(Z_r = z) = \binom{z-1}{r-1} p^r (1-p)^{z-r}$ . Lastly, the homogeneous Poisson process (HPP) models event occurrences over a time interval, with  $(N(t) \sim \text{Poisson}(\lambda t))$  and  $P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ .**

**In a homogeneous Poisson process (HPP) governed by rate parameter  $\lambda$ , the observation interval  $(t, t + \Delta t]$  is significant. The probability of observing one event  $E$  in a short interval  $(t, t + \Delta t]$  is approximately:**

$\Pr(N(\Delta t) = 1) \approx \lambda \Delta t$  The mean and variance of events in  $(0, t]$  are both given by:  $E[N(t)] = \text{Var}[N(t)] = \lambda t$  An unbiased estimator for  $\lambda$  is:  $\hat{\lambda} = \frac{N(t)}{t}$  The time until the first event  $T_1$  is exponentially distributed:  $F_{T_1}(t) = 1 - e^{-\lambda t}$  The cumulative distribution function for the  $k$ -th event time  $S_k$  is:  $F_{S_k}(t) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$  with the probability density function:  $f_{S_k}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}$   $(t \geq 0)$ . The inter-occurrence times  $(T_1, T_2, \dots)$  are independent and exponentially distributed with parameter  $\lambda$ , and as  $t \rightarrow \infty$ ,  $N(t)$  converges to a normal distribution.

**The time-to-failure  $T$  has a survivor function  $R(t)$ . If  $E(T^r) < \infty$ , then:**

$E(T^r) = \int_0^\infty t^r R(t) dt$  The failure rate  $z(t)$  leads to:  $\Pr(T > t_2 | T > t_1) = e^{-\int_{t_1}^{t_2} z(u) du}$  For  $n$  components with failure rates  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ , the probability of the  $i$ -th component failing first is:  $P(\text{Component } i \text{ fails first}) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$  The overall failure density is:  $f(t) = p f_1(t) + (1 - p) f_2(t)$  where  $p$  is the proportion of failures due to a specific cause. For independent Poisson variables  $(N_1)$  and  $(N_2)$ : 1.  $(N_1 + N_2) \sim \text{Poisson}(\lambda_1 + \lambda_2)$  2.  $(N_1 | (N_1 + N_2 = n)) \sim \text{Binomial}(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})$  Independent gamma variables  $(T_1)$  and  $(T_2)$  maintain a gamma distribution sum. The cumulative failure rate diverges as:  $\int_0^\infty z(t) dt \rightarrow \infty$  For a component's failure rate  $(z(t) = kt)$ , calculations yield MTTF, survival probabilities, and distributions tailored to time-varying failure rates. The expected value and variance of a negative binomial variable  $(Z_r)$  are:  $E(Z_r) = \frac{r}{p}$ ,  $\text{Var}(Z_r) = \frac{r(1-p)}{p^2}$

**This chapter expands upon deterministic models from Chapter 4 by exploring reliability block diagrams (RBDs), fault trees, and Bayesian**

**networks with integrative probabilities for failures and repairs. It addresses nonrepairable systems using exact formulas for reliability calculations, while for repairable systems, maintenance occurs post-failure. Key reliability measures include the independence of components. The reliability of series systems is computed as the product of individual component reliabilities:**

$[ R_{\text{series}} = R_1 \times R_2 \times \dots \times R_n ]$  In contrast, parallel system reliability is based on the probability that at least one component functions:  $[ R_{\text{parallel}} = 1 - (1 - R_1)(1 - R_2) \dots (1 - R_n) ]$  Moreover, the reliability of a k-out-of-n configuration is modeled by a binomial distribution, denoting the likelihood that 'k' out of 'n' components are operational. The chapter concludes with a focus on quantitative analysis in Bayesian networks, enhancing the reliability assessment framework.

**This chapter on system reliability employs binary state variables for components, denoted  $( X_i(t) )$ , with the probability of a functional component represented as  $( p_i(t) )$  and the system's overall reliability at time  $( t )$  as  $( p_S(t) )$ . Assuming independent failures, the reliability for series and parallel configurations is determined by:**

- Series:  $( p_S(t) = p_1(t) \cdot p_2(t) \cdot \dots \cdot p_n(t) )$  - Parallel:  $( p_S(t) = 1 - (1 - p_1(t))(1 - p_2(t)) \cdot \dots \cdot (1 - p_n(t)) )$  The expected value of a component's state is  $( E[X_i(t)] = p_i(t) )$ . For  $( k )$ -out-of- $( n )$  structures, reliability utilizes binomial distributions for identical component reliabilities, highlighting the critical role of design and maintenance strategies. Overall, complex system reliability can be evaluated through structure functions under the assumption of component independence.

**The survivor function  $( R_S(t) )$  for a k-out-of-n system varies by configuration: for a single component,  $( R_S(t) = e^{-\lambda t} )$ , while for a 2-out-of-3 (2oo3) structure, it is  $( R_S(t) = 3 e^{-2\lambda t} - 2 e^{-3\lambda t} )$ . The Mean Time-to-Failure (MTTF) for a single component is  $( MTTF = \frac{1}{\lambda} )$ , and for the 2oo3 system,  $( MTTF_{2oo3} = \frac{6}{5\lambda} )$ , reflecting a 16% reduction in lifespan compared to a single component, although it exhibits higher early survival probability. For general k-out-of-n systems, the survivor function is  $( R_S(t) = \sum_{x=k}^n \binom{n}{x} e^{-\lambda t x} (1 - e^{-\lambda t})^{n-x} )$ . This highlights that as k increases, MTTF often decreases, underscoring the balance between reliability and longevity with redundancy enhancing system performance.**

**The survivor function  $( R_S(t) )$  for standby systems quantifies the**

probability that a system remains operational over time  $t$ . For a system with  $n$  items, it is given by  $R_S(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ . In a cold standby system with two items, this simplifies to  $R_S(t) = e^{-\lambda_1 t} + (1-p) \lambda_1 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_1 - \lambda_2)\tau} d\tau$ , or  $R_S(t) = e^{-\lambda t} + (1-p) \lambda t e^{-\lambda t}$  if both items have the same failure rate  $\lambda$ . The mean time to failure (MTTFS) is calculated as  $MTTFS = \int_0^\infty R_S(t) dt$ . For example, with  $\lambda = 10^{-3}$  failures/hour and  $p = 0.015$ ,  $R_S(1000) \approx 0.7302$  yields  $MTTFS \approx 1985$  hours. As  $n$  increases, the distribution of total time-to-failure approaches normal distribution by the Central Limit Theorem.

In repairable systems, the Rate of Occurrence of Failures (ROCOF) is approximated by  $ROCOF \approx 1 / (\text{Mean Uptime} + \text{Mean Downtime})$ , where Mean Uptime (MUT) and Mean Downtime (MDT) are critical for understanding system reliability. In systems with constant failure rate  $\lambda$  and repair rate  $\mu$ , ROCOF stabilizes to  $w = \lambda / (\lambda + \mu)$ , while the Mean Time Between Failures (MTBF) is defined as  $MTBF = 1/\lambda + 1/\mu$ . Vesely's Failure Rate,  $V(t) = w(t) / A(t)$ , measures the probability of failure in small time intervals, with  $A(t)$  indicating item availability. For a system of  $n$  independent components, availability is given by  $A_S(t) = E[X(t)] = h[A(t)]$ . The system's average up-time (MUT<sub>S</sub>) and down-time (MDT<sub>S</sub>) reflect the intricate interactions among components, affecting overall system reliability and failure frequency,  $w_S$ .

Fault Tree Analysis (FTA) is a deductive method for identifying root causes of system failures using logic gates (AND, OR). The TOP event, representing system failure, is assessed through basic events  $B_i$ . Key probability terms include  $q_i(t)$ , the probability of event  $B_i$  occurring at time  $t$ , and  $Q_0(t)$ , the probability of the TOP event. Assumptions include binary basic events, independence, and instantaneous transitions. The probabilities for logical structures are calculated as follows: for a single AND-gate,  $Q_0(t) = q_1(t) \times q_2(t) \times \dots \times q_n(t)$ ; for a single OR-gate,  $Q_0(t) = 1 - \prod (1 - q_i(t))$ . Additional guidelines are available in publications like NUREG0492 and NASA resources. FTA involves defining the problem, constructing the fault tree, minimizing cut/path sets, and conducting both qualitative and quantitative analyses.

Bayesian Networks (BNs) model relationships among random variables, where



each node's state depends solely on its parent nodes, making it conditionally independent of non-descendant nodes when parent states are known. The joint distribution for nodes A, B, and C is expressed as  $\Pr(A, B, C) = \Pr(C | B) * \Pr(B | A) * \Pr(A)$ . Causal influence is established when a parent node A affects a child node B, confirmed through correlation, temporal precedence, and absence of hidden variables. Independence is defined by  $\Pr(A | B) = \Pr(A)$ , while conditional independence states that A and B are independent given C if  $\Pr(A, B | C) = \Pr(A | C) * \Pr(B | C)$ . Bayes' theorem updates probabilities with new evidence:  $\Pr(A=0 | B=0) = [\Pr(B=0 | A=0) * \Pr(A=0)] / \Pr(B=0)$ . Inference methodologies, including Monte Carlo simulations, assist in learning BN structures and probabilities, aligning with observed data and expert judgments.

This section analyzes the reliability of parallel systems with independent and identical components. For three components, each with a reliability of 98%, the system reliability is calculated as  $R_{\text{System}} = 1 - (1 - 0.98)^3 \approx 0.9992$ , indicating very high reliability. In a scenario requiring a five-component system to achieve 99% reliability, the individual component reliability must be computed. For a 2-out-of-3 (2oo3) structure with constant failure rate  $\lambda$ , the failure rate limit approaches  $\lambda_S(t) = 2\lambda$  as time tends to infinity, emphasizing sustained reliability. The system survivor function can be expressed as  $R(S) \geq h[e^{-t/\mu_1}, e^{-t/\mu_2}, \dots, e^{-t/\mu_n}]$  under increasing failure rates. Parameters from reliability block diagrams help calculate system availability when components are independent and non-repairable.

This chapter emphasizes the significance of reliability importance metrics for system components, highlighting that components in series are more critical for reliability than those in higher-order cut sets. It defines eight metrics, including Birnbaum's and Fussell-Vesely's, presented in both Reliability Block Diagram (RBD) and fault tree formats, under the assumption of simple systems without complex gates. These metrics, which also include criticality and risk achievement worth, prioritize components for improvements and maintenance, identify those likely to cause failures, and guide data collection for reliability analyses. Key assumptions include binary states of components and continuous failure time distributions, facilitating effective resource allocation in evaluations, particularly in nuclear safety and other fields, by pinpointing components that significantly influence overall reliability.

A basic event  $(E_i)$  is critical for a top event when its occurrence is essential given the states of other components, specifically if  $(\phi(1_i, x) = 1)$  (event happens) and  $(\phi(0_i, x) = 0)$  (event does not happen), leading to  $(\phi(1_i, x) - \phi(0_i, x) = 1)$ . The total number of configurations meeting this condition is denoted as  $(\eta_{\phi(i)})$ . Birnbaum's metric for structural importance is defined as:  $[I_{\phi}^B(i) = \frac{\eta_{\phi(i)}}{2^{n-1}}]$ . For reliability importance, the metric is given by:  $[I_B(i | t) = \text{Probability}(\phi[1_i, X(t)] - \phi[0_i, X(t)] = 1)]$ , with specific calculations for series and parallel systems:  $[I_B(i) = p_i \cdot \prod_{j \neq i} p_j \text{ (series)}]$ ,  $[I_B(i) = (1 - p_i) \cdot \prod_{j \neq i} (1 - p_j) \text{ (parallel)}]$ . Understanding these metrics is crucial for optimizing data collection for critical components in complex systems.

Criticality importance  $(I_{CR}(i | t))$  quantifies the probability that component  $(i)$  caused system failure at time  $(t)$ , calculated as  $(I_{CR}(i) = \frac{I_B(i) \cdot q_i}{Q_0(t)})$ , where  $(I_B(i))$  is Birnbaum's importance,  $(q_i)$  is the failure probability of component  $(i)$ , and  $(Q_0(t))$  is the overall system failure probability. Fussell-Vesely's metric evaluates component significance through minimal cut sets, formulated as  $(I_{FV}(i | t) = P(\text{At least one cut set fails} | \text{System failure}))$ , facilitating easier calculations in complex systems. Improvement potential  $(I_{IP}(i) = h(1_i, \mathbf{p}) - h(\mathbf{p}))$  indicates the reliability increase from replacing component  $(i)$  with a perfect version, linking metrics with  $(I_B(i) = \frac{I_{IP}(i)}{1 - p_i})$  to prioritize maintenance and assess resilience to reliability changes using the differential importance metric (DIM).

Risk metrics in safety systems express time-dependent factors using fault tree terminology, where risk is defined as the probability of a Top Event (TOP) occurring. Risk Achievement Worth (RAW) for a basic event  $(E_i)$  is the ratio of TOP probability with  $(E_i)$  present to that without  $(E_i)$ , showing  $(E_i)$ 's significance in risk management. Conversely, Risk Reduction Worth (RRW) measures the risk reduction achieved by replacing a safety feature with an ideal one, expressed as  $(RR(i) = Q_0 - Q_0(E_i^*))$ . Both RAW and RRW serve distinct roles: IP (Improvement Potential) aids design, while RRW supports safety feature decisions. Barlow-Proschan's metric further evaluates component importance through failure probabilities, and risk metrics may vary based on system

structure but provide insight into system reliability and the implications of safety feature variations in nuclear applications.

This section addresses the calculation of reliability importance metrics?Birnbaum, Fussell-Vesely, and Criticality?for components in a non-repairable structure comprising six independent components, with the reliability of component  $(i)$  at time  $(t)$  denoted as  $(p_i(t))$ . To analyze component 3, we compute the metrics using realistic reliability values, facilitating a comparison between criticality importance and Fussell-Vesely's metric. The Birnbaum importance is calculated as  $(B_k = B_j \cdot B_{\{k|j\}})$ , where  $(B_k)$  is the Birnbaum metric for component  $(k)$ ,  $(B_j)$  is the module's importance, and  $(B_{\{k|j\}})$  is the importance of component  $(k)$  within module  $(j)$ . We also verify that the relationship presented in equation 7.55 holds for other metrics, reaffirming the accuracy of importance metric calculations across various reliability assessments in this coherent structure.

This chapter discusses the limitations of assuming independent failures in reliability systems, focusing on dependent failures such as cascading and common-cause failures, with a detailed analysis of the latter. Events  $(E_1)$  and  $(E_2)$  are independent if the occurrence of one does not affect the probability of the other, expressed mathematically as  $(P(E_1 | E_2) = P(E_1))$  and  $(P(E_1 \cap E_2) = P(E_1) \cdot P(E_2))$ . In contrast, dependent events affect each other's probabilities, leading to positive or negative dependencies, indicated by  $(P(E_2 | E_1) > P(E_2))$  for positive dependence. The chapter emphasizes the significant impact of common-cause failures on system reliability and notes the complexity of analyzing cascading failures, often requiring simulation. Overall, it underscores the critical importance of understanding these dependencies in reliability assessments.

Two discrete random variables,  $(X_1)$  and  $(X_2)$ , are independent if their joint probability equals the product of their individual probabilities:

$$[ P(X_1 = x_1 \text{ and } X_2 = x_2) = P(X_1 = x_1) \cdot P(X_2 = x_2). ]$$

They are dependent if this condition fails for any pair. Events  $(E_1)$  and  $(E_2)$  are mutually exclusive if they cannot occur

simultaneously, indicated by  $[ P(E_1 \cap E_2) = 0, ]$  implying

mutual exclusivity is a form of dependence. The correlation between  $($

$X_1)$  and  $(X_2)$  is measured by covariance:  $[ \text{cov}(X_1, X_2)$

$= E[(X_1 - E[X_1])(X_2 - E[X_2])]$ , and expressed through Pearson's correlation coefficient  $\rho(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma_{X_1} \sigma_{X_2}}$ , which ranges from -1 (total negative correlation) to 1 (total positive correlation), emphasizing that correlation does not imply causation or independence.

**The Basic Parameter Model (BPM) by Fleming et al. (1983) evaluates fault multiplicity in reliability systems using k-out-of-n:F voting methods, where combinations of k failing components from n are equally probable, and removing j components does not affect the remaining failure probabilities. For n=3 identical components, the failure probabilities are defined as g<sub>1,3</sub> for a single fault, g<sub>2,3</sub> for double faults, and g<sub>3,3</sub> for total failure, which inform configurations like 1oo3, 2oo3, and 3oo3. Concurrently, the Beta-Factor Model (Fleming, 1975) streamlines the analysis of Common Cause Failures (CCFs) by partitioning the constant failure rate,  $\lambda$ , into independent failures ( $\lambda(i)$ ) and CCFs ( $\lambda(c)$ ), with  $\lambda = \lambda(c)/\beta$  quantifying the proportion of failures attributable to CCFs. These models rely on observed data for estimation but struggle with incomplete information on CCFs, producing critical outputs for failure probability assessments in component reliability.**

**This section discusses reliability models for systems with identical components, particularly focusing on common cause failures (CCFs) in a 2-out-of-3 Good (2oo3:G) system. The beta-factor model assesses failure rates using individual failures at  $\lambda(1 - \beta)\lambda t$  and CCFs at  $\beta\lambda t$ . Reliability for n identical components is expressed as  $R_S(t) = e^{-n(1 - \beta)\lambda t} \cdot e^{-\beta\lambda t}$ , with Mean Time to Failure (MTTF) defined as  $MTTF = \frac{1}{n\lambda(n - (n-1)\beta)}$ . Failure probabilities for three components are  $Q_1 = (1 - \beta)Q$ ,  $Q_2 = \beta(1 - \gamma)Q$ , and  $Q_3 = \beta\gamma Q$ . As  $\beta$  approaches 1, MTTF and reliability improve, but complexity increases for nonidentical components. Alternate models like the alpha-factor and multiple beta-factor models provide additional framework for reliability assessments, especially in aerospace and nuclear applications.**

**In a homogeneous Poisson process (HPP) with failure rate  $\lambda$ , individual failure consequences ( $V_1, V_2, \dots$ ) are independent and exponentially distributed with parameter  $\lambda$ . The total consequence from n failures follows a gamma distribution. The mean time to total item failure, defined as the time until the total consequence exceeds a threshold c, is given by:  $E(T_c) = \frac{1}{\lambda} (1 + \lambda c)$  This formula indicates that if**

consequences are exponentially distributed, the time to total failure adheres to an Increasing Failure Rate Average (IFRA) distribution, which applies to any consequence distribution. Furthermore, the relationship with renewal processes is critical in reliability analysis, aiding in the calculations of availability and failure predictions, ultimately optimizing spare part allocation. Even in cases where the initial consequence  $(V_0 = 0)$ , this average time until total failure remains consistent, underscoring the robustness of the IFRA characterization.

The renewal function  $(W(t))$  for a Weibull distribution simplifies to  $(W(t) = \frac{\lambda t}{\Gamma(2)})$  when  $(\alpha = 1)$ , representing an exponential distribution with parameter  $(\lambda)$ . The age  $(Z(t))$  is determined as  $(Z(t) = t)$  if  $(N(t) = 0)$  and  $(Z(t) = t - S_{\{N(t)\}})$  for  $(N(t) > 0)$ . The remaining lifetime  $(Y(t))$  is given by  $(Y(t) = S_{\{N(t)+1\}} - t)$  and can be linked to failure times via  $(\Pr(Y(t) > y) = \frac{\Pr(T > y+t)}{\Pr(T > t)})$ . The mean remaining lifetime is calculated as  $(E[Y(t)] = \frac{1}{\Pr(T > t)} \int_t^{\infty} \Pr(T > u) \, du)$ , which equals  $(\frac{1}{\lambda})$  for exponentially distributed  $(T)$ . As  $(t \rightarrow \infty)$ ,  $(Y(t))$  and  $(Z(t))$  converge to a limiting distribution, with insights from Barlow and Proschan informing about survival probabilities and bounds on renewal processes.

Brown and Proschan's imperfect repair model describes an item that can undergo perfect repairs, restoring it to like-new condition with probability  $(p)$ , or minimal repairs, which degrade it with probability  $(1 - p)$ . For instance, with  $(p = 0.02)$ , an item is replaced on average after 50 failures. This model combines renewal processes (where  $(p=1)$ ) and non-homogeneous Poisson processes (NHPP, where  $(p=0)$ ). Extensions by Block et al. (1985) introduced age-dependent repairs, using failure rate functions  $(z(t) = \frac{f(t)}{R(t)})$  and cumulative distributions  $(F(t) = 1 - e^{-\int_0^t z(x) \, dx})$ . Chan and Shaw (1993) examined reductions in the rate of occurrence of failures (ROCOF) post-repair, while Doyen and Gaudoin (2011) developed history-dependent failure rate reductions. Lastly, Malik approached virtual age reduction based on performance history.

A Markov process is a stochastic model with states  $\{0, 1, 2, \dots, r\}$ , characterized by stationary transition probabilities  $(P_{\{ij\}}(t))$ , where  $(P_{\{ij\}}(t))$  represents the probability of transitioning from

state  $(i)$  to state  $(j)$  by time  $(t)$ . The rows of the transition probability matrix sum to one, ensuring valid probabilities. The sojourn times  $(T_i)$  in each state are memoryless and follow an exponential distribution with rate  $(\alpha_i)$ , leading to a mean sojourn time of  $(E(T_i) = 1/\alpha_i)$ . The transition rate from state  $(i)$  to state  $(j)$  is defined as  $(a_{ij} = \alpha_i P_{ij})$ , with the transition rate matrix  $(A)$  having diagonal elements  $(a_{ii} = -\alpha_i)$ . This framework effectively models continuous-time Markov chains using discrete transitions and independent exponentially distributed sojourn times.

In a homogeneous Poisson process, the transition rate matrix  $(A)$  has diagonal elements  $(-\lambda)$  and off-diagonal elements  $(\lambda)$ . The Chapman-Kolmogorov equations describe transition probabilities as  $(P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s))$  or in matrix form,  $(P(t+s) = P(t) \cdot P(s))$ . The Kolmogorov differential equations provide the rates of change: the backward equation is  $(\frac{dP_{ij}(t)}{dt} = \sum_{k \neq i} a_{ik} P_{kj}(t) - \alpha_i P_{ij}(t))$  and the forward equation is  $(\frac{dP_{ij}(t)}{dt} = \sum_{k \neq j} a_{kj} P_{ik}(t) - \alpha_j P_{ij}(t))$ . In matrix form, these are  $(\dot{P}(t) = A \cdot P(t))$  and  $(\dot{P}(t) = P(t) \cdot A)$ . Steady-state probabilities are determined by  $(P \cdot A = 0)$  and  $(\sum_j P_j = 1)$ . For a two-state system, availability is given by  $(P_0(t) = \frac{\mu}{\lambda + \mu})$  and  $(P_1(t) = \frac{\lambda}{\lambda + \mu})$ .

In a series structure of two components, failure in one prevents operation of the other. The failure and repair rates for components 1 and 2 are denoted as  $\lambda_1, \lambda_2, \mu_1$ , and  $\mu_2$ . The average system availability  $(A_s)$  is given by  $(A_s = \frac{\mu_1 \mu_2}{\lambda_1 \mu_2 + \lambda_2 \mu_1 + \mu_1 \mu_2})$ , and the mean time to failure (MTTF) relates to failure rates, extending to  $(n)$  components as  $(A_s = \frac{1}{1 + \sum_{i=1}^n \frac{\lambda_i}{\mu_i}})$ . In parallel structures, system availability is calculated as  $(A_s = 1 - P_0 = 1 - q_1 q_2)$ , where  $(P_0)$  is the probability of both components failing, and failure frequency is given by  $(\omega_F = P_0(\mu_1 + \mu_2))$ . Both series and parallel structures exhibit absorbing states that prevent recovery post-failure, complicating the analysis of  $(n)$  component systems, generally yielding  $(A_s = \frac{1}{1 + \sum_{i=1}^n \frac{\lambda_i}{\mu_i}})$ .

To calculate the Mean Time to First Failure (MTTF) for a system, create a

transition rate matrix  $(A)$  based on states  $\{0, 1, \dots, r\}$  and define the initial distribution  $(P(0))$ . Identify absorbing failed states and form the reduced matrix  $(A_R)$  by removing these states. Compute the Laplace transform  $(P^*(s))$  excluding absorbing states to yield  $(P^*_R(s))$ , then establish  $(P^*_R(s) A_R = [s P^*(s) - P(0)]_R)$  and evaluate at  $(s=0)$ . MTTF is computed as  $(MTTF = \sum P_j(0))$  over non-absorbing states. In parallel structures of  $(n)$  identical components, MTTFs simplifies to  $(MTTFS = \sum_{i=1}^n \frac{1}{i\lambda})$ . For CCFs,  $(MTTF)$  adjusts to  $(\frac{1}{n\lambda(n-(n-1)\beta)})$ , and in load-sharing systems defined by failure rates  $(\lambda_h, \lambda_f)$  and repair rates  $(\mu_h, \mu_f)$ , MTTF can be derived based on transition matrices and Laplace transforms, focusing on reliability under various operational conditions.

Steady state probabilities  $(P)$  in parallel systems are derived using a reduced transition matrix with failure (state 0) treated as an absorbing state. Mean Time-to-Failure (MTTF) for non-repairable systems is calculated as  $MTTF = 1 / (\lambda_A + \lambda_B)$ . In cold standby systems, item B activates upon A's failure; thus, system failure occurs if B fails before A is repaired. Adjustments for imperfect switching and additional states can influence MTTF and reliability metrics, calculated from state transitions and failure rates. For repairable systems, employing Laplace transforms leads to an approximated MTTF of 43,421 hours. Steady-state probabilities are obtained by solving  $(P \cdot A = 0)$  with  $(\sum P_i = 1)$ . For systems with  $(r + 1)$  states, construct a transition rate matrix  $(A_R)$  omitting absorbing states, with MTTF expressed as  $MTTF = (\sum P_j(0))$  for non-absorbing states. The survivor function follows  $R(t) = P_1(t) + P_2(t)$ , along with its Laplace transform to capture system reliability.

This overview covers Semi-Markov Processes (SMP), Multiphase Markov Processes (MMP), and Piecewise Deterministic Markov Processes (PDMP), highlighting their distinct features and applications. SMPs accommodate general distributions for sojourn times in state  $(i)$ , with transition probabilities  $(P_{ij})$  and the mean sojourn time calculated as  $(\mathbb{E}[T_i] = \int_0^\infty t F_i(t) dt)$ . MMPs, on the other hand, implement parameter changes at specific times, affecting the transition matrix  $(A_i)$ , and utilize the probability distribution  $(\mathbf{P}(t) = \mathbf{P}(0) e^{A_i t})$  post-maintenance. PDMPs integrate both deterministic processes and stochastic jumps, where state transitions and continuous variables are modeled through differential equations and unique numerical methods. This diverse framework enhances modeling capabilities for various maintenance scenarios and system

reliability assessments when traditional Markov approaches prove insufficient.

This text discusses modeling reliability systems using Markov processes, emphasizing state transitions, availability calculations, steady-state probabilities, and failure analysis. A fail-safe valve with two failure modes (premature closure and fail to close) transitions through three states defined by specific failure rates and repair times, enabling calculations of valve availability and failure intervals. In a production system with components A, B, C, D, and E, the presence of components C and D alters dynamics, represented through transition matrices and state equations, highlighting capacity states (100%, 50%, 0%) influenced by failure rates. The analysis also covers systems like pumping and burner systems, deriving Mean Time to Failure (MTTF) and steady-state probabilities from state transition matrices. Overall, the insights gained enhance the understanding of system behavior, economic impacts of operational income versus repair costs, and reliability engineering practices.

This section outlines key concepts in maintenance strategy analysis, defining maintenance tasks within a task space  $\mathcal{T} = \{T_1, T_2, T_3, \dots\}$ . The selection of a task  $T_i$  is driven by a decision function,  $\delta(T_i) = T_j$ , which accounts for the operational context and relevant data ( $\mathcal{D}$ ). The overall strategy incorporates a cost function  $J(T_i, \mathcal{D}, \mathcal{C})$  that evaluates maintenance costs over a time horizon  $[0, T]$ . The average long-term cost  $\bar{C}$ , representing the mean cost per time unit, is derived from renewal cycles. Although closed forms for  $\bar{C}$  may be difficult to obtain, numerical methods can approximate results. The focus is on time-based preventive maintenance, particularly replacements, where timing serves as the sole decision variable, aligning decisions with cost and context to optimize maintenance efficiency.

In reliability strategies, block replacement with spare items utilizes a gamma distribution for time-to-failure, and its associated costs comprise a fixed cost plus expected failures and historical costs. The average cost for block replacement without spares is represented as:

$$C = C_F + C_R \int_0^\infty P(t) dt + C_O \int_0^\infty P(t) dt$$
  
Here, the failure distribution follows a Weibull function. As the replacement period extends, costs converge towards operating costs. The P-F interval strategy focuses on



detecting potential failures through inspections modeled as a Poisson process, optimizing intervals to reduce total costs that include preventive and corrective actions as well as inspection expenses. Perfect inspections facilitate deterministic interval identification, while imperfect inspections require adjustments, with delay-time models providing further differentiation between defect occurrences and actual failures.

**This section examines maintenance strategies for degrading systems subject to delayed maintenance and periodic inspections. Preventive maintenance (PM) is initiated when degradation reaches a level  $(m)$ , with a predetermined delay  $(\tau)$ . If failure occurs before PM starts, corrective maintenance (CM) takes over. The mean cost per unit time is computed as:  $[C(t) = c_m N_m(t) + c_l N_l(t) + \gamma d(t)]$  where  $(c_m)$  and  $(c_l)$  are renewal costs for successful preventive and corrective tasks, respectively, and  $(\gamma)$  denotes downtime costs impacted by delay. Degradation modeling often employs the Weibull distribution, reflecting increasing failure rates. Transition matrices in Markov processes model state probabilities, with repair rates  $(\mu_{k,k+1})$  allowing transitions back to functional states. System availability  $(A_S(t))$  incorporates item availabilities influenced by a constant failure rate  $(\lambda_c)$ , aiding in the optimization of PM thresholds and maintenance strategies, particularly in high-reliability systems like offshore energy facilities. Monte Carlo simulations further enhance the understanding of these strategies under varying degradation parameters.**

**This section addresses reliability concerns regarding item replacements and failure rates. For a machine operating 8 hours daily for 230 days a year, with a constant failure rate of 0.002 failures per hour and 5 hours of downtime, average availability is given by:**

$$[\text{Availability}] = \frac{\{\text{Operational Time}\}}{\{\text{Operational Time}\} + \{\text{Downtime}\}}]$$
 In an analysis with a failure rate of 0.0005 failures per hour and 6 hours of downtime, similar availability calculations apply. For wear-induced failures, the failure rate is modeled as  $(z_1(t) = \tau t)$  (where  $(\tau = 0.00000005 \text{ hours}^{-2})$ ). The survival probability  $(R(t))$  after 2000 hours is determined through integration of the failure rate. Adjusting for overhauls, the failure rate modifies to  $(\tau - \tau_k)$ , and the survival function before the  $(k+1)$ -th overhaul addresses these probabilities through conditional integration. Thus, reliability and availability metrics are critical in assessing machine performance over time.

**The Probability Safety Function (PSF) estimates the likelihood of failure**

on demand for proof-tested safety system items, like fire detection, where fires occur as a Poisson process with intensity ( $\lambda$ ). The critical situation rate, when a fire occurs during a detector failure, is  $\lambda \cdot \lambda_{ST}$  (probability of detector failure). The expected number of critical situations over time ( $t$ ) is given by  $E[NC(t)] = \lambda \cdot \lambda_{ST} \cdot t$ , with distribution  $Pr(NC(t) = n) = (\lambda \cdot \lambda_{ST} \cdot t)^n \cdot e^{-(\lambda \cdot \lambda_{ST} \cdot t)} / n!$  for  $n = 0, 1, \dots$ . Spurious trips (ST) impact reliability, calculated as  $A \cdot \lambda_{ST} \cdot \sum_{j=1}^m (ST(j) \cdot MDTST(j))$  for  $j = 1$  to  $m$  independent subsystems. In parallel sensor systems, the total spurious trip rate is  $(\sum_{i=1}^n \lambda_{ST,i})$ . Redundant designs can unintentionally increase spurious trips, as seen in configurations like 2oo3:G.

This analysis focuses on a 2oo3:G sensor system, comprising three independent sensors with a spurious trip rate of approximately  $3 \cdot 10^{-7}$  hours<sup>-1</sup>, calculated using a sensor failure rate ( $\lambda_{ST}$ ) of  $5 \cdot 10^{-7}$  failures/hour and a restoration time ( $t_r$ ) of 2 hours. The 2oo3:G structure enhances reliability and probability of failure on demand (PFD) by necessitating at least two sensors to trigger an alarm, thus reducing false alarms from single sensor failures. Diagnostic self-testing improves system availability ( $A_{DT}$ ) by detecting failures and minimizing downtime, with unavailability assessed through the sum of failure rates. The system achieves up to 70% diagnostic coverage for failure detection, and attention to common cause failures (CCFs) is critical, as they threaten redundancy and require explicit modeling for effective risk assessment.

A gas detector has a constant failure rate of  $\lambda = 1.8 \times 10^{-6}$  hours<sup>-1</sup>, resulting in a Mean Time To Failure (MTTF) of  $MTTF = 1 / \lambda = 555,556$  hours. The Probability of Failure on Demand (PFD) for a proof test every 6 months ( $\tau = 4,380$  hours) is calculated as  $PFD = \lambda \cdot \tau = 0.0079$ . In a 2-out-of-3 (2oo3) configuration, the probability of surviving without a critical failure over 12 months (8,760 hours) is  $P(\text{survive}) = (1 - PFD)^3$ . Considering common cause failures with a beta factor of 0.08 modifies the PFD, allowing for the estimation of undetected failures post-test. For fire detectors with a failure rate of  $0.21 \cdot 10^{-7}$  hours<sup>-1</sup>, testing over 3 months ( $\tau = 2,190$  hours) provides a PFD of  $0.00023$  (about 1 in 4350 undetected fires), illustrating reliability across different configurations.

Field data collection, exemplified by the OREDA project, records survival times of items between observation points ( $t_1$  and  $t_2$ ) and includes

variables such as item number, starting age, observation times, and failure/censoring status (1 for failed, 0 for censored). The "at-risk set" at any time  $t$  consists of items that have neither failed nor been censored, which is crucial for survival analysis. Exploratory Data Analysis (EDA) is essential for understanding the data structure and identifying anomalies through sample statistics (mean, median, standard deviation) and visualizations (histograms, Q-Q plots). A complete dataset for EDA, assumed to be sorted, comprises survival times  $t(1), t(2), \dots, t(n)$ . The mean, representing central tendency, is calculated as the sum of values divided by the count, illustrated by the R command `mean(survtime)`, which yields a result of 68.08. EDA's importance was highlighted by John W. Tukey.

The random variable  $N(t)$  follows a binomial distribution with success probability  $R(t)$ , where the mean is  $E[N(t)] = nR(t)$  and variance  $\text{var}[N(t)] = nR(t)(1 - R(t))$ . The survival probability for  $m$  successes is calculated as  $P(m \text{ successes}) = \binom{n}{m} R(t)^m (1 - R(t))^{n-m}$ . In R, the survival curve can be plotted using the 'survival' package. To assess normality, Q-Q plots can be created to compare dataset quantiles against a theoretical distribution  $F(t)$ . Use `qqnorm()` for normal distributions and `qqplot()` for general comparisons, such as with the exponential distribution. A linear Q-Q plot indicates validity of the normality assumption. As  $n$  increases, the variance of the estimator  $R(t)$  decreases, ensuring it remains unbiased.

An estimator, denoted as  $\hat{\theta}$ , is a statistic used to estimate a parameter  $\theta$  from observed data. Estimators can be point, providing specific values, or interval, offering confidence intervals that indicate the likelihood of containing the true parameter. An estimator is unbiased if  $E(\hat{\theta}) = \theta$ , and its mean squared error (MSE) is defined as  $\text{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = \text{bias}^2 + \text{variance}$ . Estimators are consistent if they converge to the true parameter as sample size  $n$  increases, meaning the probability of deviation from  $\theta$  approaches zero. Chebyshev's Inequality can provide bounds for estimation error. Common methods for estimation include the Method of Moments, Maximum Likelihood Estimation, and Bayesian estimation, each with unique approaches to derive  $\hat{\theta}$ .

Maximum Likelihood Estimation (MLE) estimates parameters like the rate

parameter  $\lambda$  for exponential and Weibull distributions from observed data. For  $n$  events over time interval  $\tau$ , the likelihood function for an exponential distribution is given by  $L(\lambda | n, \tau) = \frac{(\lambda \tau)^n e^{-\lambda \tau}}{n!}$ , leading to the log-likelihood  $\ell(\lambda | n, \tau) = n \log(\lambda) - \lambda \tau$ . Setting  $\frac{d\ell}{d\lambda} = 0$  yields MLE  $\hat{\lambda} = \frac{n}{\tau}$ . For example, with  $n = 10$  and  $\tau = 68,450$  hours,  $\hat{\lambda} \approx 1.461 \times 10^{-4}$  hours<sup>-1</sup>, though it's biased; an unbiased estimator is  $\lambda^* = \frac{(n-1)\hat{\lambda}}{n}$ . In Weibull distributions with shape  $\alpha$  and scale  $\theta$ , MLE can be implemented via R packages like WeibullR, applicable for both complete and censored data analysis, demonstrating MLE's flexibility in statistical modeling.

This section explores maximum likelihood estimation (MLE) for parameters in time-to-failure analysis using the Weibull distribution. The likelihood function  $L(\alpha, \theta | t)$  is given by  $L(\alpha, \theta | t) \propto \alpha^r \cdot \theta^{-\alpha r}$  multiplied by the product of exponential terms based on ordered failure and censoring times. MLE estimates for the shape parameter  $\alpha^*$  and scale parameter  $\lambda^*$  are derived from specific summations of time-to-failure data. Additionally, the Kaplan-Meier estimator offers a nonparametric method to estimate the survival function  $R(t) = P(T > t)$ , calculated as the product of sequential survival probabilities adjusted for the at-risk population. These models form the foundation for effective survival data analysis and further statistical exploration.

The Kaplan-Meier estimator, introduced by Kaplan and Meier in 1958, is a nonparametric method for calculating the survivor function  $R(t) = \Pr(T > t)$  and graphically representing survival data, particularly with censored data. For complete datasets,  $R(t)$  is estimated using:  $R(t(i)) = \prod_{j=1}^i R(t(j) | t(j-1))$ . In the presence of censored data, the estimator is adapted to account for the observed events, defined as:  $\hat{R}(t) = \prod_{j: t(j) < t, \delta_j = 1} \left(1 - \frac{d_j}{n_j}\right)$  where  $d_j$  is the number of failures and  $n_j$  is the at-risk group size just before  $t(j)$ . This estimator serves as a nonparametric maximum likelihood estimator with asymptotic normality, and its variance can be quantified using Greenwood's formula. The Kaplan-Meier estimator visually illustrates survival probabilities over time and captures the influence of censored data.

The scaled Total-Time-on-Test (TTT) plot, generated via the AdequacyModel

**package in R, aids in failure data analysis, particularly for optimizing age replacements. The average replacement cost, determined by age  $(t_0)$ , is given by:**

$$[\text{Cost}] = \frac{c + k \cdot F(t_0)}{t} \int_0^{t_0} [1 - F(t)] dt$$

To minimize costs, solve for the optimal value  $(v_0 = F(t_0))$  by setting:  $[\frac{dC(v_0)}{dv_0} = 0]$  The optimal replacement age  $(t_0)$  can be identified graphically by drawing a tangent from the point  $((-c/k, 0))$  to the TTT transform graph. If  $(v_0 = 1)$ , preventive replacements are unnecessary (i.e.,  $(t_0 = \infty)$ ). While TTT plotting effectively visualizes cumulative failure data, incorporating additional methods like Kaplan-Meier and Nelson-Aalen enhances understanding of the distribution function  $(F(t))$  throughout an item's lifecycle.

**The Proportional Hazards (PH) model analyzes the failure rate function defined as  $(z(t | s) = z_0(t) \cdot g(s))$ , with  $(z_0(t))$  as the baseline failure rate (time-dependent and independent of covariates) and  $(g(s))$  as a covariate function. The hazard ratio,  $(HR(s_1, s_0) = \frac{g(s_1)}{g(s_0)})$ , compares two covariate vectors. The cumulative failure rate is  $(Z(t | s) = Z_0(t) \cdot g(s))$ , and the survivor function is  $(R(t | s) = \exp[-Z(t | s)] = [R_0(t)]^{g(s)})$ . The Cox model, a specific PH model, simplifies to  $(z(t | s) = z_0(t) \cdot \exp(\beta \cdot s))$ , where parameters  $(\beta)$  are estimated through partial likelihood, allowing analysis without specifying  $(z_0(t))$ . Practical applications, such as the MIL-HDBK-217 prediction method, utilize factors like stress and temperature, and can be implemented in R using the `coxph` function.**

**This analysis examines datasets of compressor repair times from 90 failures (1968-1989) and braided cord strength, analyzing trends and distributions. Compressor repair times ranged from 1.25 to 135 hours. Key analyses included constructing a TTT plot to evaluate life distribution and assess correlations between repair times and compressor age, an empirical distribution function based on independent, identically distributed times, and a lognormality assessment via plotting paper. For material strength, involving 48 right-censored specimens, a Kaplan-Meier plot was used for survival analysis, discussing the implications of censoring on failure rates. Additionally, methods for estimating parameters of Weibull, Pareto, and uniform distributions were introduced, utilizing moment estimators and maximum likelihood estimators to enhance reliability analysis.**

**In Bayesian data analysis, analysts select a probability model for a**

random variable  $(X)$  using either a continuous density function  $(f(x | \theta))$  or a discrete probability mass function  $(P(X = x | \theta))$  based on prior knowledge of the unknown parameter  $(\theta)$ . Prior beliefs about  $(\theta)$  are represented by a prior density  $(\pi(\theta))$ , illustrating the analyst's initial uncertainty. Upon observing data  $(x)$ , Bayes' theorem updates the prior to the posterior distribution  $(\pi(\theta | x))$  using the likelihood function  $(L(\theta | \text{data}))$ , which assesses how well  $(\theta)$  fits the observed data. Analysts may categorize prior distributions as informative, weakly informative, or non-informative, reflecting subjective, objective, or empirical approaches, respectively. Common models include the binomial distribution for discrete data and the exponential distribution for continuous data, often utilizing the beta distribution as a prior for probabilities.

Bayesian estimation involves calculating the Bayesian estimator for parameter  $\theta$  as the average of the posterior distribution derived from the prior density  $\pi(\theta)$  and observed independent and identically distributed data  $(x_1, x_2, \dots, x_n)$ . The posterior density can be defined as:

$$\pi(\theta | X_1, \dots, X_n) \propto L(\theta | X_1, \dots, X_n) * \pi(\theta)$$
 Credible intervals, the Bayesian equivalent of confidence intervals, are defined by bounds  $(a(d), b(d))$  such that:  $\int_a^b \pi(\theta | d) d\theta = 1 - \alpha$  These can be symmetrical or based on the highest posterior density region (HPD), capturing the most probable values of  $\theta$ . Moreover, the marginal density of random variable  $X$ , considering  $\theta$  as a realization of random variable  $\theta$ , is expressed as: 
$$f_X(x) = \int f_{X|\theta}(x | \theta) * \pi(\theta) d\theta$$
 This integral represents the prior predictive distribution of  $X$ . In complex models with multiple parameters, multidimensional prior distributions may require computational methods for solution.

## **\*\*Bayesian Estimation in Reliability Analysis\*\***

This section explores Bayesian estimation in reliability analysis, focusing on probabilities associated with test results. The Bayesian estimator for a parameter  $(\theta)$  that minimizes mean absolute error is the median of the posterior distribution derived from data  $(?)$ . For a binomially distributed variable with a uniform prior, the posterior density for probability  $(\theta)$  after observing  $(X = x)$  leads to Bayesian estimates for parameters such as mean time to failure (MTTF). For instance, observing 19 failures from 7 cars with an exponential failure distribution and a gamma prior allows for MTTF estimation and a 90% credible interval on reliability. When data follows a normal distribution  $(N(\theta, \sigma^2))$ , the estimate combines the prior mean and maximum likelihood estimate. As sample size increases, the posterior variance decreases below the prior variance. The section also contrasts Bayesian

credible intervals with frequentist confidence intervals, highlighting the distinct challenges in reliability assessments.

The probability density function (pdf) for failure is defined as  $f(t) = -R'(t) = \frac{0.4}{(0.2t + 1)^3}$ , with the failure rate function  $z(t) = \frac{1}{t + 1}$ . The survival function, indicating the probability of survival by time  $t$ , is expressed as  $R(t) = \frac{1}{(0.2t + 1)^2}$ , and the Mean Time To Failure (MTTF) is calculated as  $MTTF = \int_0^5 R(t) dt = 5$  months. The mean residual lifetime (MRL) at age  $x$  is defined as  $MRL(x) = \frac{\int_x^\infty R(t) dt}{R(x)}$ . Furthermore,  $g(x) = \frac{MTTF}{MRL(x)}$  establishes a relationship between MTTF and MRL. Critical relationships include  $F(t) = 1 - R(t)$  and  $z(t) = \frac{f(t)}{R(t)}$ . The cumulative failure rate is modeled as  $Z(t) = \int_0^t z(u) du$ , leading to  $R(t) = e^{-Z(t)}$ , which aids in reliability assessment across various distributions in engineering contexts.

The average availability  $A_{avg}$  of a repairable item is computed using  $A_{avg} = \frac{MUT}{MUT + MDT}$ , where  $MUT$  is Mean Up-Time and  $MDT$  is Mean Downtime. For a machine with a Mean Time to Failure (MTTF) of 1,000 hours and an MDT of 5 hours,  $A_{avg} \approx 99.5\%$ , resulting in approximately 44 hours of downtime annually. Unavailability is defined as  $U_{avg} = \frac{MDT}{MUT + MDT}$ . Operational availability  $A_{OP}$  is given by  $A_{OP} = 1 - \left(\frac{\text{Mean Downtime}}{\text{Mission Period}}\right)$ . As time approaches infinity, limiting availability  $A$  converges to  $A_{avg}$ . Additional metrics, such as deliverability and on-stream availability, evaluate system performance against benchmarks, while failure rate influences the Rate of Occurrence of Failures (ROCOF).

The High-Performance Poisson Process (HPP) has independent interoccurrence times  $(T_1, T_2, \dots)$  following an exponential distribution with rate  $\lambda$ . Arrival times  $(S_n)$  are gamma distributed with parameters  $(n)$  and  $(\lambda)$ , and the counting process  $(N(t))$  satisfies  $(N(0) = 0)$  and has independent, stationary increments. The probability distribution for a given interval is given by  $(P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!})$ . The expected value is  $(E[N(t)] = \lambda t)$  and the variance is  $(\text{Var}(N(t)) = \lambda t)$ . As  $(t \rightarrow \infty)$ ,  $(N(t))$  approaches a normal distribution:  $(\frac{N(t) - \lambda t}{\sqrt{\lambda t}} \rightarrow N(0,1))$ . An unbiased estimator for  $(\lambda)$  is  $(\hat{\lambda} =$

$\frac{N(t)}{t}$ ), with variance  $\frac{\lambda}{t}$ ). Compound HPPs arise from summing rates of independent events.

In renewal theory, which models the timing of replacements in stochastic processes with independent identically distributed intervals  $(T_1, T_2, \dots)$ , the expected number of renewals by time  $t$ , denoted  $W(t)$ , satisfies the fundamental renewal equation:  $W(t) = F_T(t) + \int_0^t W(t-x) dF_T(x)$ . As  $t$  increases,  $W(t) \approx \frac{t}{\mu}$ , where  $\mu$  is the average renewal length. The renewal density  $w(t)$ , given by  $w(t) = \lambda e^{-\lambda t} \frac{1 - e^{-2\lambda t}}{2}$ , approaches  $\frac{1}{\mu}$  as  $t \rightarrow \infty$ . For Weibull periods with shape parameter  $\alpha$  and scale  $\lambda$ , the renewal function is expressed as:  $W(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k (\lambda t)^k}{\Gamma(k\alpha + 1)}$ , with  $A_k$  recursively determined. This series converges to  $\lambda t$  for  $\alpha=1$ . Additionally,  $w(t)$  can also be derived via differentiation and Laplace transforms, illustrating the connections among mean, variance, and renewal dynamics. Notably, Blackwell's theorem refines estimates for small intervals, approximating the mean number of renewals in  $(t, t+u]$  as  $\frac{u}{\mu}$ . The  $n$ th arrival time is described by  $S_n = T_1 + T_2 + \dots + T_n$ , supporting foundational statistical results.

The Weibull distribution is a versatile tool in reliability engineering, used to model time-to-failure data. Its probability density function is given by

$f(t) = \frac{\alpha}{\theta} \left(\frac{t}{\theta}\right)^{\alpha-1} e^{-\left(\frac{t}{\theta}\right)^\alpha}$  for  $t > 0$ , where  $\alpha$  is the shape parameter and  $\theta$  is the scale parameter.

The survivor function is  $R(t) =$

$e^{-\left(\frac{t}{\theta}\right)^\alpha}$ , and the failure rate function is  $z(t) = \frac{\alpha}{\theta} \left(\frac{t}{\theta}\right)^{\alpha-1}$ ,

which varies based on the value of  $\alpha$ . The Mean Time to Failure (MTTF) is  $\text{MTTF} = \theta \Gamma(1 + \frac{1}{\alpha})$ , and for  $n$  components in series, it modifies to  $R_s(t) = e^{-\sum_{i=1}^n \left(\frac{t}{\theta_i}\right)^\alpha}$ ,

$\text{MTTF}_s = \frac{\theta \Gamma(1 + \frac{1}{\alpha})}{n^{1/\alpha}}$ . In a three-parameter model with threshold  $\xi$ , the MTTF becomes  $\text{MTTF} = \xi + \theta \Gamma(1 + \frac{1}{\alpha})$ . The variance is  $\text{Var}(T) = \theta^2 [\Gamma(2 + \frac{1}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})]$ , demonstrating its flexibility for various scenarios in reliability analysis.



The failure rate function  $z(t)$  for a gamma distribution diverges as  $t \rightarrow 0$  when  $(0 < \alpha < 1)$  and approaches zero for  $(\alpha > 1)$ , showing a discontinuity at  $(\alpha = 1)$ . For independent gamma variables  $(T_1 \sim \text{Gamma}(\alpha_1, \lambda))$  and  $(T_2 \sim \text{Gamma}(\alpha_2, \lambda))$ , their sum  $(T_1 + T_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda))$ . The gamma PDF is given by  $f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)}$  with mean time to failure  $(\text{MTTF}) = \frac{k}{\lambda}$  and variance  $(\text{var}(T) = \frac{k}{\lambda^2})$ . Notably, special cases include the exponential distribution  $(\alpha = 1)$  and the chi-square distribution  $(\alpha = n/2, \lambda = 1/2)$ . Python implementations for gamma distribution include functions for the PDF, survival function, and failure rate calculation:

```
python def gamma_pdf(t, alpha, lambda_): return (lambda_ *
(lambda_ * t)**(alpha - 1) * np.exp(-lambda_ * t)) / gamma(alpha) def
survival_function(t, alpha, lambda_): return 1 - gamma.cdf(t,
a=alpha, scale=1/lambda_) def failure_rate_function(t, alpha, lambda_):
return gamma_pdf(t, alpha, lambda_) / survival_function(t, alpha,
lambda_)`` This framework allows for effective modeling of time-to-
failure, particularly as  $(\alpha)$  approaches 1.
```

The reliability of systems is significantly influenced by their structure, particularly in series vs. parallel configurations. For a series structure with component reliabilities  $(p_1, p_2, \dots, p_n)$ , system reliability is expressed as  $(h(p) = p_1 p_2 \dots p_n)$ , with component  $(i)$ 's importance determined by  $(I_B(i) = \prod_{j \neq i} p_j)$ . In contrast, for parallel structures, the system reliability is  $(h(p) = 1 - \prod_{j \neq i} (1 - p_j))$ , and the importance metric is  $(I_B(i) = \prod_{j \neq i} (1 - p_j))$ . Notably, when component reliabilities equal  $(\frac{1}{2})$ , the importance metrics align with structural importance. The improvement potential of replacing component  $(i)$  with a perfect one is quantified by  $(I_{IP}(i|t) = h[1_i, p(t)] - h[p(t)])$ . Recognizing weak or strong components aids in optimizing maintenance and reliability strategies.

This section discusses reliability models, particularly the beta-factor model, for assessing failure rates in parallel structures with both identical and nonidentical components, focusing on common cause failures (CCFs). Individual failures are represented as  $(3(1 - \beta) \lambda t)$  and CCFs as  $(\beta \lambda t)$ . The reliability function for  $(n)$  identical components is  $(R_S(t) = e^{-n(1 - \beta) \lambda t} e^{-\beta \lambda t})$ , and the Mean Time to Failure (MTTF) is  $(\text{MTTF} =$

$\frac{1}{n\lambda(n - (n-1)\beta)}$ . As  $(\beta)$  approaches 1, reliability increases, with maximum reliability observed in 2-out-of-3 Good (2003:G) systems. Failure probabilities for three components are  $(Q_1 = (1 - \beta)Q)$ ,  $(Q_2 = \beta(1 - \gamma)Q)$ , and  $(Q_3 = \beta\gamma Q)$ . While applicable to nonidentical components, complexity increases in defining  $(\beta)$ . Alternative models like the alpha-factor and multiple beta-factor models enhance reliability assessments, especially in aerospace and nuclear applications.

Non-Homogeneous Poisson Processes (NHPP) are valuable for modeling reliability in systems, particularly in failure rate analysis. The failure rate of an NHPP is defined as  $(w(t) = 6 - 2t)$  for  $(0 \leq t \leq 2)$ ,  $(w(t) = -18 + t)$  for  $(2 < t \leq 20)$ , and  $(w(t) = 0)$  for  $(t > 20)$ . The cumulative rate function is given by  $(W(t) = \int_0^t w(u) du)$ , which defines both the expected number of failures  $(E[N(t)])$  and the variance  $(\text{var}[N(t)])$  as  $(W(t))$ . The survival function for the time until the first failure is  $(R_1(t) = e^{-W(t)})$ . Availability is assessed using  $(A(t) = \frac{\text{MTTF}}{\text{MTTF} + \text{MDT}})$ . Importantly, the NHPP can transform into a homogeneous Poisson process when  $(W(t))$  is invertible, with failure probabilities modeled by  $(P(N = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!})$ .

In renewal theory, which addresses the timing of replacements in stochastic processes defined by independent and identically distributed intervals  $(T_1, T_2, \dots)$ , the expected number of renewals by time  $(t)$ , denoted  $(W(t))$ , satisfies the fundamental renewal equation:  $[W(t) = F_T(t) + \int W(t - x) dF_T(x).]$  As  $(t)$  increases,  $(W(t) \approx \frac{t}{\mu})$ , where  $(\mu)$  is the average renewal length, and the renewal density  $(w(t))$  approaches  $(\frac{1}{\mu})$ . Blackwell's theorem refines estimates for small intervals, suggesting the mean number of renewals in  $((t, t + u])$  approximates  $(\frac{u}{\mu})$ . The  $(n)$ -th arrival time is given by  $(S_n = T_1 + T_2 + \dots + T_n)$ , with  $(S_n \rightarrow \mu n)$  as  $(n \rightarrow \infty)$ . Specific distributions, like Weibull or gamma, feature unique renewal function forms, such as:  $[W(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\Gamma(k\alpha + 1)} A_k (\lambda t)^\alpha]$  for Weibull distributions, and  $(w(t) = \lambda e^{-\lambda t})$ . These dynamics underline the relationships among mean, variance, and renewal processes, impacting reliability metrics and availability.

In reliability management, optimal age and block replacement strategies

aim to minimize costs associated with equipment failure. The age replacement cost function  $C^*(x_0)$ , defined with  $x_0 = \frac{t_0}{\theta}$ , helps determine the optimal replacement age  $t_0$ . For time-to-failure  $(TF, i)$ , interval counts  $(N_i)$  form a renewal process with expected replacements  $(E(N_i))$  from a geometric distribution. Mean unavailability  $(A_{av}(t_0))$  relates to mean downtimes (MDTP and MDTF). The age replacement total expected costs are expressed as  $E[C(T_R)] = c + k F(t_0)$  and asymptotic costs as  $C_{\infty}(t_0) = \frac{E[C(T_R)]}{t_0}$ . The block replacement strategy features scheduled replacements with costs  $C_{\infty}(t_0; m) = \frac{c + k F_{(m+1)}(t_0) + k_u \int_0^{t_0} F_{(m+1)}(t) dt}{t_0}$ , optimizing maintenance efficiency against unavailability.

Degradation models are crucial for condition-based maintenance in reliability theory, representing item states through continuous or discrete degradation levels. The item condition is denoted by state  $(X(t))$ , while observable degradation is represented by  $(Y(t))$ , both treated as stochastic processes. The Remaining Useful Lifetime (RUL) at time  $(t_j)$  is expressed as  $(Pr(RUL(t_j) \leq t))$ . Total degradation increments  $(I(t_j, t_k))$  are modeled using exponential distributions, yielding expected increment  $(E[I(t_1, t_2)] = (t_2 - t_1)/\lambda)$  and variance  $(var[I(t_1, t_2)] = (t_2 - t_1)^2/\lambda^2)$ . Optimization techniques, including Wiener and gamma processes, improve RUL predictions, while trend models, like  $(Y(t_k) = c + a \cdot t_k + \epsilon(t_k))$ , facilitate the correlation of current degradation indicators with future reliability. Additionally, Monte Carlo methods help generate simulations and assess model accuracy through empirical cumulative distributions and probability density functions, enhancing maintenance strategies.

Maximum Likelihood Estimation (MLE), introduced by Ronald Aylmer Fisher in 1922, estimates parameters by maximizing the likelihood function. For an exponential distribution with  $(n)$  events over a time interval  $(\tau)$ , the likelihood is  $(L(\lambda | n, \tau) = \frac{\lambda^n}{n!} e^{-\lambda \tau})$  and the log-likelihood is  $(\ell(\lambda | n, \tau) = n \log(\lambda) - \lambda \tau)$ . Setting  $(\frac{d\ell}{d\lambda} = 0)$  yields the MLE  $(\hat{\lambda} = \frac{n}{\tau})$ . For example, with  $(n = 10)$  and  $(\tau = 68,450)$  hours,  $(\hat{\lambda} \approx 1.461 \times 10^{-4})$  hours<sup>-1</sup>; this estimate is biased, and an unbiased estimator is  $(\lambda^* = \frac{(n-1)}{\hat{\lambda}})$ . MLE is also applicable in the binomial and Weibull distributions, with tools like R's WeibullR package supporting analysis for complete and censored data. MLE is favored for

parameter estimation in various statistical models, providing properties like asymptotic unbiasedness and consistency.

The survivor function  $R(t)$  is defined with the continuous probability density  $f(t) = R'(t)$  where  $f(t) > 0$  for  $t > 0$ . The failure rate function is  $z(t) = \frac{f(t)}{R(t)}$ , leading to the cumulative failure rate  $Z(t) = \int_0^t z(u) du$  and the survival function represented as  $R(t) = e^{-Z(t)}$ . The Nelson-Aalen estimate for cumulative failure rate is given by  $\hat{Z}(t) = \sum_{t_j < t} \frac{d_j}{n_j}$ , where  $d_j$  is the number of failures at time  $t_j$  and  $n_j$  is the at-risk number, resulting in  $\hat{R}^*(t) = \exp(-\hat{Z}(t))$ . For specific distributions,  $Z(t) = \lambda t$  for exponential and  $Z(t) = \left(\frac{t}{\theta}\right)^\alpha$  for Weibull. TTT plots can illustrate failure rates, showing increasing or decreasing trends, and while Nelson-Aalen and Kaplan-Meier estimates agree well for shorter times, discrepancies arise at longer durations; both plots can be efficiently created using R, despite fewer specific packages for Nelson-Aalen visualizations.

The Total-Time-on-Test (TTT) transform quantifies the area under the survivor function  $R(t)$  for a life distribution  $F(t)$ , defined as:

$$H_{F^{-1}}(v) = \int_0^{F^{-1}(v)} [1 - F(u)] du \quad \text{for } 0 \leq v \leq 1.$$

For large samples, the ratios

$$\frac{\mathcal{T}(T_i)}{\mathcal{T}(T_n)} \text{ approximate } \frac{i}{n}.$$

If  $F(t)$  is exponential ( $F(t) = 1 - e^{-\lambda t}$ ), the transform simplifies to  $H_{F^{-1}}(v) = -\frac{1}{\lambda} \ln(1 - v)$ , yielding a linear function. For the Weibull distribution

( $F(t) = 1 - e^{-(\frac{t}{\theta})^\alpha}$ ), the inverse is

$$F^{-1}(v) = \theta [-\log(1 - v)]^{1/\alpha},$$

resulting in a TTT transform related to the incomplete gamma function. The mean time-to-failure is  $E(T) = H_{F^{-1}}(1) = \theta \Gamma(1/\alpha + 1)$ . This analysis provides insights into life distributions, failure rates, and empirical observations.

In Bayesian analysis of failure rates  $\lambda$  using a gamma prior  $\text{gamma}(\alpha, \beta)$ , the prior mean is  $E(\lambda) = \frac{\alpha}{\beta}$ . The likelihood for observed failures  $(n_1)$  over time  $(t)$  is given by  $L(\lambda | n_1, t) \propto \lambda^{n_1} e^{-\lambda t}$ , resulting in a posterior distribution that remains gamma with updated parameters  $(\alpha + n_1)$  and  $(\beta + t)$ .

Thus, the posterior mean is  $(E(\lambda | n_1, t) = \frac{\alpha + n_1}{\beta + t})$ . The iterative updating process refines our estimate of  $(\lambda)$  with  $(\alpha_n = \alpha_{n-1} + 1)$  and  $(\beta_n = \beta_{n-1} + t_n)$ . The marginal distribution of failures  $(N(t))$  follows a negative binomial distribution, and the use of noninformative priors allows for equal likelihoods, simplifying posterior estimation to predominantly reflect the likelihood.

In a 2-out-of-3 (2oo3) structure, minimal path sets and cut sets both play a crucial role in reliability analysis. The minimal path sets are defined as  $(P_1 = \{1, 2\}, P_2 = \{1, 3\}, P_3 = \{2, 3\})$ , and the minimal cut sets as  $(K_1 = \{1, 2\}, K_2 = \{1, 3\}, K_3 = \{2, 3\})$ . The path function for minimal sets is  $(\rho_j(x) = \prod_{i \in P_j} x[i])$ , leading to the overall structure function  $(\phi(x) = 1 - \prod_{j=1}^p (1 - \rho_j(x)))$ . For cut sets, the function is represented by  $(\kappa_j(x) = \prod_{i \in K_j} x[i])$ . Python functions can be employed for calculations, such as  $(\rho(j, x))$  for path evaluation and pivotal decomposition  $(\text{pivotal\_decomposition}(x, i) = x[i] \cdot \phi(1, x) + (1 - x[i]) \cdot \phi(0, x))$ . This framework is essential for evaluating structural reliability in applications like bridge assessments.

The time-to-failure  $(T)$  follows a Weibull distribution  $(\text{Weibull}(\alpha, \theta))$  characterized by the probability density function

$f(t) = \frac{\alpha}{\theta} \left(\frac{t}{\theta}\right)^{\alpha-1} e^{-\left(\frac{t}{\theta}\right)^\alpha}$ ,  $\text{quad } (t > 0)$  with survivor function  $R(t) = e^{-\left(\frac{t}{\theta}\right)^\alpha}$ ,  $\text{quad } (t > 0)$  and failure rate  $z(t) = \frac{\alpha}{\theta} \left(\frac{t}{\theta}\right)^{\alpha-1}$ ,  $\text{quad } (t > 0)$ . For independent components, the series survivor function is  $R_s(t) = e^{-\sum_{i=1}^n \left(\frac{t}{\theta_i}\right)^\alpha}$ , and for identical components, the equivalent scale parameter is  $\theta_s = \frac{\theta}{n^{1/\alpha}}$ , yielding  $MTTF_s = \frac{\theta}{\Gamma(1 + \frac{1}{\alpha})} n^{1/\alpha}$ . In a three-parameter model, with threshold  $(\xi)$ ,  $MTTF = \xi + \theta \Gamma(1 + \frac{1}{\alpha})$ , and variance is  $\text{Var}(T) = \theta^2 [\Gamma(1 + \frac{2}{\alpha}) - \Gamma(1 + \frac{1}{\alpha})^2]$ . The transformed variable  $(T^\alpha)$  is exponentially distributed, with  $(P(T^\alpha > t) = e^{-\frac{t}{\theta^\alpha}})$  and failure rate  $(\lambda = \frac{1}{\theta^\alpha})$ .

The survivor function  $(R_S(t))$  for standby systems quantifies the

probability that a system with  $(n)$  items remains operational over time  $(t)$ , defined by  $(R_S(t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t})$ . In a cold standby system with two items, this becomes  $(R_S(t) = e^{-\lambda_1 t} + (1-p) \lambda_1 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1 - \lambda_2) \tau} d\tau)$ , and reduces to  $(R_S(t) = e^{-\lambda t} + (1-p) \lambda t e^{-\lambda t})$  when both items share the same failure rate  $(\lambda)$ . The mean time to failure (MTTFS) is computed as  $(MTTFS = \int R_S(t) dt)$ . For instance, with  $(\lambda = 10^{-3})$  failures/hour and  $(p = 0.015)$ , we find  $(R_S(1000) \approx 0.7302)$  and  $(MTTFS \approx 1985)$  hours. The total time-to-failure  $(T_S)$  for  $(n)$  items tends toward a normal distribution as  $(n)$  increases, according to the Central Limit Theorem.

To calculate the Mean Time to First Failure (MTTF) for a system, construct a transition rate matrix  $(A)$  based on states  $\{0, 1, \dots, r\}$ , and define the initial state distribution  $(P(0))$ . Identify absorbing failed states and create a reduced matrix  $(A_R)$  by removing them. Compute the Laplace transform  $(P^*(s))$  for non-absorbing states, leading to  $(P^*_R(s) A_R = [s P^*(s) - P(0)]_R)$  and evaluate at  $(s=0)$ . MTTF is calculated as  $(\text{MTTF} = \sum P_j(0))$  over non-absorbing states. For  $(n)$  identical components, MTTF simplifies to  $(\text{MTTF} = \sum_{i=1}^n \frac{1}{i\lambda})$ , while for systems with common cause failures characterized by a beta-factor  $(\beta)$ , it is  $(\text{MTTF} = \frac{1}{n\lambda(n-(n-1)\beta)})$ . Reliability can be expressed with  $(R(t) = e^{-n(1-\beta)\lambda t} e^{-\beta\lambda t})$ . For load-sharing systems with varying failure and repair rates, similar analyses apply in evaluating MTTF under different operational stresses.

This section evaluates system reliability, focusing on the Probability of Failure on Demand (PFD) across configurations. For a 1oo2:G structure with fire detectors having failure rates  $\lambda_{DU,1}$  and  $\lambda_{DU,2}$ , the PFD can be approximated as  $PFD = (\lambda_{DU,1} + \lambda_{DU,2}) / 2$  for minimal  $\lambda_{DU,1}$  and  $\lambda_{DU,2}$ . The failure rate of fire detectors is about  $0.21 \times 10^{-6}$  failures per hour. Over 3 months (approximately 2190 hours) of testing, the PFD is about 0.00023, indicating that roughly 1 in 4350 fires may go undetected. In a 1-out-of-2 setup, the PFD is  $7.1 \times 10^{-6}$ , and for a 2-out-of-3 structure, it's about  $2.1 \times 10^{-6}$ . Setting  $\lambda$  at 4380 hours for gas detectors yields a PFD of approximately 0.0079. Furthermore, a beta-factor of 0.08 adjusts for common cause failures, shedding light on the implications of beta reduction and the similarity in PFDs of 1oo4:G and 2oo4:G structures under identical conditions, which aids in evaluating overall reliability.

**Maximum Likelihood Estimation (MLE)**, introduced by Ronald Aylmer Fisher in 1922, estimates parameters by maximizing the likelihood function across various statistical models. For an exponential distribution, the likelihood is  $L(\lambda | n, \tau) = \frac{(\lambda \tau)^n}{n!} e^{-\lambda \tau}$  with log-likelihood  $\ell(\lambda | n, \tau) = n \log(\lambda) - \lambda \tau$ . Setting  $\frac{d\ell}{d\lambda} = 0$  yields MLE  $\hat{\lambda} = \frac{n}{\tau}$ . For a binomial model,  $L(p | x) = \binom{n}{x} p^x (1-p)^{n-x}$ , leading to MLE  $\hat{p} = \frac{x}{n}$ . In a homogeneous Poisson process, the likelihood remains  $L(\lambda | n, \tau)$ ; similarly,  $\ell(\lambda | n, \tau) = n \log(\lambda \tau) - \lambda \tau$  leads to the same MLE. Additionally, for Weibull distributions with shape  $\alpha$  and scale  $\theta$ , MLE can be implemented via R packages, demonstrating MLE's flexibility in statistical modeling. MLE properties include asymptotic unbiasedness and consistency.

The probability density function for failure is defined as  $f(t) = -R'(t) = \frac{0.4}{(0.2t + 1)^3}$  with the survival function  $R(t) = \frac{1}{(0.2t + 1)^2}$  and failure rate  $z(t) = \frac{f(t)}{R(t)} = \frac{t}{t + 1}$ , which quantifies the conditional probability of failure given survival. The Mean Time to Failure (MTTF) is calculated as  $MTTF = \int_0^\infty t f(t) dt = 5$  months, while the Mean Residual Lifetime (MRL) at age  $x$  is given by  $MRL(x) = \frac{\int_x^\infty R(t) dt}{R(x)}$ . The cumulative failure rate is  $Z(t) = \int_0^t z(u) du$ , leading to  $R(t) = e^{-Z(t)}$ . Key relationships include  $F(t) = 1 - R(t)$  and  $g(x) = \frac{MTTF}{MRL(x)}$ , which are essential for assessing reliability in engineering contexts.