Quommentaries

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Contents

1 Leftoverture

This repository is dedicated to solve exercises and comment on Quantum Computing. Most of the discussion is based on Nielsen And Chuang's book "Quantum Computation and Quantum Information". In addition, Kaye, Laflamme and Mosca's "An Introduction to Quantum Computing" is used as a complementary book, as well as Yanofsky and Mannucci's "Quantum Computing for Computer Scientists" - recommended by Greati.

1.1 Objective

Although Nielsen and Chuang's book is very famous, some equations may be solved too quickly. This may discourage the reader to continue the studies if the basic concepts were not mastered. One of the objectives of this repository is to support those who are studying Quantum Computing and Quantum Information by explaining some of these equations step-by-step.

In addition, the exercises present in the book may not be trivial for beginners. Hence, this repository attempts to help the students by showing a detailed solution or, at least, a sketch.

1.2 Disclaimer

This repository is being constructed by an **undergaduate student**. Henceforth, the notes, commentaries and exercises are **suscetible to errors**. Please, **do not hesitate to give feedback** (gustavowl@lcc.ufrn.br).

2 Introduction

On August 19, 2018, the author was studying the *Section 2.5 - The Schmidt decomposition and purification* of Nielsen and Chuang's book. Up until this section, all exercises were fairly discussed in worked problem's website. Most of the answers are reasonably satisfactory, though some lack formalism and detailed explanation. However, this website only discusses exercises 2.1 to 2.76. Question 2.77 is discussed on StackExchange's website. Apparently, questions 2.78 onward are not commonly discussed. Therefore, this material will *initially* focus on these questions. Details on questions 2.1 to 2.76 will be added sporadically.

In addition, this material will contain details on some equations solved during each chapter. Most explanations will try to specify the steps using to jump from one equation to another. Also, some affirmations and equations may induce doubts in the author; who will try to state and clarify them in this document.

3 Nielsen and Chuang - Chapter 01

3.1 Section 1.2

3.1.1 Qubit representation in a Bloch Sphere

The Bloch Sphere's equations explanation is not given by the book.

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle . \tag{1}$$

However, Agnez came up with a simple explanation using spherical coordinates. Its details can be found in Bits2Qubits blog (the post is in Portuguese).

The details of this equation are not necessary until Chapter 4. At this point, however, it is sufficient for the reader to know that a Qubit can be represented as a point in a sphere. A detailed explanation of the Bloch Sphere equations can be found in Sections 8.1.1 and 8.1.2. However, it requires information that will be mentioned throughout Nielsen and Chuang's book's Chapter 2 or in Noson's book's Chapters 4 and 5.

After comprehending Sections 8.1.1 and 8.1.2, the following Equation is obtained.

$$|\psi\rangle = cos\theta|0\rangle + e^{i\phi}sin\theta|1\rangle , \tag{2}$$

where $\theta \in [0, \frac{\pi}{2}]$, and $\phi \in [0, 2\pi)$. Nielsen and Chuang's book uses $\theta \in [0, \pi]$. Thus obtaining Equation 1 .

The book's Equation (1.3) can be obtained by mutiplying Equation 1 by $e^{i\gamma}$. [1] Since it does not change the state, as explained in Section 7.1.1.

3.2 Section 1.4

3.2.1 $(-1)^{f(x)}|x\rangle(|0\rangle + |1\rangle)/\sqrt{2}$

TODO: ADD EXPLANATION OF $(-1)^{F(X)}|X\rangle(|0\rangle + |1\rangle)/\sqrt{2}$

3.2.2 Equation (1.43)

While explaining Deutsch's algorithm, state $|\psi_1\rangle$ is obtained.

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Then, the Unitary gate U_f is applied to state $|\psi\rangle$ and how the result obtained in state $|\psi_2\rangle$ may not be clear enough to the reader. First, recall that $f(x) : \{0, 1\} \rightarrow \{0, 1\}$. That is, the function maps the qubits in state $|0\rangle$ to either state $|0\rangle$ or $|1\rangle$. Analogously, qubits in state $|1\rangle$ are mapped to state $|0\rangle$ or $|1\rangle$.

Henceforth, there are for possible functions: two possibilities where $f(0) = f(1)$ and two possibilities where $f(0) \neq f(1)$.

- $f(0) = f(1)$
  - $f(0) = f(1) = 0$
  - $f(0) = f(1) = 1$
- $f(0) \neq f(1)$
  - $f(0) = 0, f(1) = 1$
  - $f(0) = 1, f(1) = 0$

[1] Actually, Equation 1 was originally obtained *from* Equation (1.3) by multiplying it by $e^{-i\gamma}$.

Note that U_f does not apply any operation to the first qubit (x), but applies $y \oplus f(x)$ to the second qubit (y). Note that, using the distributive property, the state $|\psi_1\rangle$ may be written as

$$|\psi_1\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Then, analysing what would happen if any of the four possibilities for U_f were applied:

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{3}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{4}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{5}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{6}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{7}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{8}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |11\rangle - |10\rangle}{2} \tag{9}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{10}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{11}$$

Henceforth, the first part of Nielsen and Chuangs's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) = f(1)$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 0, f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{12}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{13}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2} \tag{14}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{15}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 1, f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{16}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{17}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |10\rangle - |11\rangle}{2} \tag{18}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2} \tag{19}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{20}$$

Henceforth, the second part of Nielsen and Chuang's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) \neq f(1)$$

Also, note that something interesting happened. Even though the U_f was not supposed to alter the state of the first qubit ($|x\rangle$); it is, in fact, changed. As a result, measuring $|x\rangle$ is sufficient to determine the specified property of $f(x)$.

4 Nielsen and Chuang - Chapter 02

4.1 Section 2.1.4

4.1.1 Outer Product Representation of A

It is stated from Equation 2.25 that it is possible to "[...] see from this equation that A has matrix element $\langle w_j| A |v_i\rangle$". To see this, it is possible to compare the matrix and Dirac representations. Consider two systems V and W with dimensions n and m, respectively. In addition, suppose $|v\rangle \in V$ and $|w\rangle \in W$.

Using matrix representation:

$$|v\rangle\langle w| = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} w_1 & \cdots & w_j & \cdots & w_m \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} v_1w_1 & \cdots & v_1w_j & \cdots & v_1w_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_iw_1 & \cdots & v_iw_j & \cdots & v_iw_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_nw_1 & \cdots & v_nw_j & \cdots & v_nw_m \end{bmatrix} \tag{22}$$

Using Dirac notation and the last part of Equation 2.21:

$$|v\rangle\langle w| = \sum_{ij} v_i\,|i\rangle\, w_j\,\langle j| \tag{23}$$

$$= \sum_{ij} v_iw_j\,|i\rangle\langle j| \tag{24}$$

Then, comparing matrix and Dirac representation, it is easily verified that the matrix has elements $m_{ij} = v_iw_j$ for the i-th row and j-th column ($|i\rangle\langle j|$) with respect to the orthonormal basis $|i\rangle$ and $|j\rangle$ for systems V and W, respectively.

4.2 Section 2.1.5

4.2.1 Exercise 2.11

The eigenvectors, eigenvalues and diagonal representations of Y will be calculated. The process for X and Z is similar. All answers are summarised in the end of this section.

Recall that

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{25}$$

Therefore,

$$det(Y - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} \tag{26}$$

$$= (-\lambda)^2 - (-i^2) \tag{27}$$

$$= \lambda^2 - 1 = 0. \tag{28}$$

Thus, $\lambda = \pm 1$.

Calculate the corresponding eigenvectors, using row reduction. For $\lambda_0 = 1$,

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} {+i \cdot row_1} \sim \begin{bmatrix} -1 & -i \\ 0 & 0 \end{bmatrix}. \tag{29}$$

Therefore, since $-1a - ib = 0$, the eigenvectors of λ_0 are the span of

$$|\lambda_0\rangle = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}. \tag{30}$$

That is, if $z \in \mathbb{C}$, then $-1 \cdot 1 \cdot z - i \cdot i \cdot z = 0$. Finally, normalise $|\lambda_0\rangle$.

$$\frac{|\lambda_0\rangle}{\sqrt{\langle\lambda_0|\lambda_0\rangle}} = \frac{|\lambda_0\rangle}{\sqrt{1 \cdot 1 + (-i) \cdot -i}} \tag{31}$$
$$= \frac{1}{\sqrt{2}} |\lambda_0\rangle . \tag{32}$$

Henceforth, let $|\lambda_0\rangle$ denote the normalised eigenvector.

Calculate the eigenvector span for $\lambda_1 = -1$.

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} {-i \cdot row_1} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}. \tag{33}$$

Therefore, since $1a - ib = 0$, the eigenvectors of λ_1 are the span of

$$|\lambda_1\rangle = \left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}. \tag{34}$$

That is, if $z \in \mathbb{C}$, then $1 \cdot 1 \cdot z - i \cdot (-i) \cdot z = 0$. Finally, normalise $|\lambda_1\rangle$.

$$\frac{|\lambda_1\rangle}{\sqrt{\langle\lambda_1|\lambda_1\rangle}} = \frac{|\lambda_1\rangle}{\sqrt{1 \cdot 1 + i \cdot (-i)}} \tag{35}$$
$$= \frac{1}{\sqrt{2}} |\lambda_1\rangle . \tag{36}$$

Henceforth, let $|\lambda_1\rangle$ denote the normalised eigenvector.

Thus, the diagonal representation is $Y = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| = 1 |\lambda_0\rangle\langle\lambda_0| -1 |\lambda_1\rangle\langle\lambda_1|$.

Repeat the procedure for the remaining Pauli matrices. Table 1 summarises the exercise answers.

Pauli Matrix	λ_0	λ_1	$\|\lambda_0\rangle$	$\|\lambda_1\rangle$	Diagonal Representation
X	1	-1	$\|+\rangle$	$\|-\rangle$	$\|+\rangle\langle+\| - \|-\rangle\langle-\|$
Y	1	-1	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$	$\|\lambda_0\rangle\langle\lambda_0\| - \|\lambda_1\rangle\langle\lambda_1\|$
Z	1	-1	$\|0\rangle$	$\|1\rangle$	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$

Table 1: Answers of Exercise 2.11

4.3 Section 2.1.8

4.3.1 Exercise 2.35

In order to solve this exercise, it is necessary to find a spectral decomposition for $\vec{v}\vec{\sigma}$. Then, it is possible to apply the definition of Operator functions.

With the aid of Exercise 2.60, the required spectral decomposition is obtained:

$$\vec{v}\vec{\sigma} = +1P_+ - 1P_- \tag{37}$$
$$= \frac{I + \vec{v}\vec{\sigma}}{2} - \frac{I - \vec{v}\vec{\sigma}}{2} \tag{38}$$

Now, calculating the value of $exp(i\theta\vec{v}\cdot\vec{\sigma})$ and applying the definition of Operator functions:

$$\begin{aligned} exp(i\theta\vec{v}\cdot\vec{\sigma}) &= exp(i\theta P_+) + exp(-i\theta P_-) && (39)\\ &= exp(i\theta)P_+ + exp(-i\theta)P_- && (40)\\ &= e^{i\theta}P_+ + e^{-i\theta}P_- && (41)\end{aligned}$$

Then, applying Euler's Formula:

$$\begin{aligned} exp(i\theta\vec{v}\cdot\vec{\sigma}) &= cos(\theta)P_+ + isin(\theta)P_+ + cos(\theta)P_- - isin(\theta)P_- && (42)\\ &= cos(\theta)\frac{I+\vec{v}\vec{\sigma}}{2} + isin(\theta)\frac{I+\vec{v}\vec{\sigma}}{2} + cos(\theta)\frac{I-\vec{v}\vec{\sigma}}{2} - isin(\theta)\frac{I-\vec{v}\vec{\sigma}}{2} && (43)\\ &= cos(\theta)\left(\frac{I+\vec{v}\vec{\sigma}+I-\vec{v}\vec{\sigma}}{2}\right) + isin(\theta)\left(\frac{I+\vec{v}\vec{\sigma}-I+\vec{v}\vec{\sigma}}{2}\right) && (44)\\ &= cos(\theta)I + i\ sin(\theta)\vec{v}\vec{\sigma} && (45)\end{aligned}$$

Thus obtaining the required result.

4.3.2 Equation 2.60

It is known that $tr(UAU^\dagger) = tr(A)$. Therefore,

$$tr(A|\psi\rangle\langle\psi|) = tr(UA|\psi\rangle\langle\psi|U^\dagger)) \tag{46}$$

By Equation 2.22, $\sum_i |i\rangle\langle i| = I$. Since I is an Unitary Operator, it is possible to write

$$\begin{aligned} tr(UA|\psi\rangle\langle\psi|U^\dagger) &= tr(IA|\psi\rangle\langle\psi|I^\dagger)) && (47)\\ &= tr(IA|\psi\rangle\langle\psi|I)) && (48)\\ &= tr(\sum_{ij}|i\rangle\langle i|A|\psi\rangle\langle\psi||j\rangle\langle j|) && (49)\end{aligned}$$

Since $\langle i|A|\psi\rangle$ and $\langle\psi|j\rangle$ are scalars,

$$tr(\sum_{ij}|i\rangle\langle i|A|\psi\rangle\langle\psi||j\rangle\langle j|) = tr(\sum_{ij}\langle i|A|\psi\rangle\langle\psi|j\rangle|i\rangle\langle j|) \tag{50}$$

Similarly to Equation 2.25, $tr(\sum_{ij}\langle i|A|\psi\rangle\langle\psi|j\rangle|i\rangle\langle j|)$ is an Outer Product representation for $A|\psi\rangle\langle\psi|$ where element $m_{ij} = \langle i|A|\psi\rangle\langle\psi|j\rangle$ (check Section 4.1.1 for details). Then, by the definition of trace (Equation 2.59):

$$\begin{aligned} tr(\sum_{ij}\langle i|A|\psi\rangle\langle\psi|j\rangle|i\rangle\langle j|) &= m_{ii} && (51)\\ &= \sum_i\langle i|A|\psi\rangle\langle\psi|i\rangle && (52)\end{aligned}$$

4.3.3 $tr(|\psi\rangle\langle\varphi| = \langle\varphi|\psi\rangle)$

I decided to add this section because this equation is used throughout the book, e.g. Equation 2.208 to 2.209, and Exercise 2.82. I do not remember it being explicitly stated or explained, though.

Suppose two different states $|\psi\rangle$ and $|\varphi\rangle$. Then, following the same reasoning as Equation 2.60 and Equation 2.61:

$$\begin{aligned} tr(\langle\psi|\varphi\rangle) &= tr(I\langle\psi|\varphi\rangle) && (53)\\ &= \sum_i\langle i|I|\psi\rangle\langle\varphi|i\rangle && (54)\\ &= \sum_i\langle i|\psi\rangle\langle\varphi|i\rangle && (55)\end{aligned}$$

$$= \sum_i \langle\varphi| \, |i\rangle\langle i| \, |\psi\rangle \tag{56}$$

$$= \langle\varphi| \, I \, |\psi\rangle \tag{57}$$

$$= \langle\varphi|\psi\rangle \tag{58}$$

Note that although Equation 2.60 defines a orthonormal basis $|i\rangle$ containing $|\psi\rangle$, this is not necessary. The only restriction is that, if $|\psi\rangle \in V$ and $|\varphi\rangle \in W$, then $dim(V) = dim(W)$. In other words, that $|\psi\rangle\langle\varphi|$ is a square matrix.

4.4 Section 2.2.5

4.4.1 Equation (2.116)

This definition may be rather confusing since $\vec{v}$ is defined but the definition of $\vec{\sigma}$ is not recapitulated. More specifically, σ_1, σ_1, and σ_3 were defined in Table 2.2 of the book. However, since X, Y and Z are used more frequently to denote the Pauli Matrices, the reader may not remind of the equivalent σ notation.

In order to reduce the calculi on Exercise 2.60 (section 4.4.2), The matrix form of $\vec{v}\vec{\sigma}$ is computed in this section.

Recall that $\sigma_1 \equiv X$, $\sigma_2 \equiv Y$, and $\sigma_3 \equiv Z$, which have matrix form as defined in the book's Table 2.2. Since $\vec{v}$ is a vector with components $v_1, v_2, v_3 \in \mathbb{R}$, it is possible to interpret $\vec{\sigma}$ as being a vector of matrices, i.e. $\vec{\sigma} \in (\mathbb{R}^{2\times 2})^3$ Therefore:

$$\vec{v}\vec{\sigma} = v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} v_3 & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 \end{bmatrix} \tag{60}$$

4.4.2 Exercise 2.60

This section will only find the requested eigenvalues, and the Projector given by P_+. The projector given by P_- can be found by following the same steps as P_+'s solution.

<u>THE EIGENVALUES</u> of $\vec{v}\vec{\sigma}$ can be found by using basic Linear Algebra knowledge: $det(\vec{v}\vec{\sigma} - \lambda I) = 0$. Therefore, referring to Section 4.4.1, calculate

$$det(\vec{v}\vec{\sigma} - \lambda I) = 0 \tag{61}$$

$$\begin{vmatrix} v_3 - \lambda & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - \lambda \end{vmatrix} = 0 \tag{62}$$

$$\lambda^2 - v_3^2 - (v_1^2 + v_2^2) = 0 \tag{63}$$

$$\lambda^2 = v_1^2 + v_2^2 + v_3^2 \tag{64}$$

A bit of cleverness is required here. Recall that just before the definition of Equation 2.116, $\vec{v}$ is supposed to be a unit vector. This means that $\vec{v} \cdot \vec{v} = 1$. Since $\vec{v} \in \mathbb{R}^3$), $\vec{v} \cdot \vec{v} = v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3$. Therefore, $v_1^2 + v_2^2 + v_3^2 = 1$. Plugging this into the previous result to find the values of λ:

$$\lambda^2 = 1 \tag{65}$$

$$\lambda = \pm 1 \tag{66}$$

as requested.

<u>TO FIND THE PROJECTOR</u> P_+ it is necessary to calculate the eigenspace of the eigenvector of $+1$. In order to find the eigenspace, basic Linear Algebra knowledge may be used. Hence, for $\lambda = +1$ and applying row reducing:

$$\begin{bmatrix} v_3 - \lambda & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - \lambda \end{bmatrix} = \begin{bmatrix} v_3 - 1 & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - 1 \end{bmatrix} \tag{67}$$

$$= \begin{bmatrix} (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \\ (v_1 + v_2 i)(v_3 - 1) & 1 - v_3^2 \end{bmatrix} \tag{68}$$

Note that $1 - v_3^2 = v_1^2 + v_2^2$, since $v_1^2 + v_2^2 + v_3^2 = 1$. Therefore,

$$\begin{bmatrix} (v_3-1)(v_1+v_2i) & v_1^2+v_2^2 \\ (v_1+v_2i)(v_3-1) & 1-v_3^2 \end{bmatrix} = \begin{bmatrix} (v_3-1)(v_1+v_2i) & v_1^2+v_2^2 \\ (v_3-1)(v_1+v_2i) & v_1^2+v_2^2 \end{bmatrix} \tag{69}$$

$$= \begin{bmatrix} (v_3-1)(v_1+v_2i) & v_1^2+v_2^2 \\ 0 & 0 \end{bmatrix} \tag{70}$$

Therefore, if t is a scalar, the eigenspace can be given by:

$$t \begin{bmatrix} \frac{-(v_1-v_2i)}{v_3-1} \\ 1 \end{bmatrix} \tag{71}$$

because:

$$(v_3-1)(v_1+v_2i) \cdot \frac{-(v_1-v_2i)}{v_3-1} + (v_1^2+v_2^2) \cdot 1 = 0 \tag{72}$$

$$-(v_1^2+v_2^2) + (v_1^2+v_2^2) = 0 \tag{73}$$

However, it is not possible to use the definition and keep calculating with $P_m = |m\rangle\langle m|$, where $|m\rangle = \begin{bmatrix} \frac{-(v_1-v_2i)}{v_3-1} \\ 1 \end{bmatrix}$ because it is necessary that $|m\rangle$ is unitary ($\langle m|m\rangle = 1$). And, if $\langle m|m\rangle$ is calculated, the following result would be obtained:

$$\langle m|m\rangle = \begin{bmatrix} \frac{-(v_1+v_2i)}{v_3-1} & 1 \end{bmatrix} \begin{bmatrix} \frac{-(v_1-v_2i)}{v_3-1} \\ 1 \end{bmatrix} \tag{74}$$

$$= \begin{bmatrix} \frac{-v_1-v_2i}{v_3-1} & 1 \end{bmatrix} \begin{bmatrix} \frac{-v_1+v_2i}{v_3-1} \\ 1 \end{bmatrix} \tag{75}$$

$$= \frac{v_1^2+v_2^2}{(v_3-1)^2} + 1 \tag{76}$$

$$= \frac{1-v_3^2}{(v_3-1)^2} + \frac{(v_3-1)^2}{(v_3-1)^2} \tag{77}$$

$$= \frac{1-v_3^2+v_3^2-2v_3+1}{(v_3-1)^2} \tag{78}$$

$$= \frac{-2v_3+2}{(v_3-1)^2} \tag{79}$$

$$= \frac{-2(v_3-1)}{(v_3-1)^2} \tag{80}$$

$$= -\frac{2}{v_3-1} \tag{81}$$

Hence, it is necessary to normalize $|m\rangle$. Recall that the norm of a vector $|m\rangle$ is given by $\sqrt{\langle m|m\rangle}$. To normalize a vector, divide it by its norm.

$$|\psi\rangle = \frac{|m\rangle}{\sqrt{\langle m|m\rangle}} \tag{82}$$

$$= |m\rangle \Big/ \sqrt{-\frac{2}{v_3-1}} \tag{83}$$

$$= \sqrt{-\frac{v_3-1}{2}}\,|m\rangle \tag{84}$$

$$= \frac{i}{\sqrt{2}}\sqrt{v_3-1}\,|m\rangle \tag{85}$$

However, this would not be right because $\sqrt{\langle m|m\rangle} \geq 0$ and $\sqrt{\langle m|m\rangle} \in \mathbb{R}$. It is necessary to rearrange the value of $\langle m|m\rangle$:

$$|\psi\rangle = \sqrt{-\frac{v_3-1}{2}}\,|m\rangle \tag{86}$$

$$= \sqrt{\frac{1-v_3}{2}}\,|m\rangle \tag{87}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1-v_3}\begin{bmatrix}\frac{-(v_1-v_2i)}{v_3-1}\\ 1\end{bmatrix} \tag{88}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1-v_3}\begin{bmatrix}\frac{v_1-v_2i}{1-v_3}\\ 1\end{bmatrix} \tag{89}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix} \tag{90}$$

Now that the normalized vector was obtained, it is possible to calculate the respective Projector by:

$$|\psi\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1+v_2i}{\sqrt{1-v_3}} & \sqrt{1-v_3}\end{bmatrix} \tag{91}$$

$$= \frac{1}{2}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix}\begin{bmatrix}\frac{v_1+v_2i}{\sqrt{1-v_3}} & \sqrt{1-v_3}\end{bmatrix} \tag{92}$$

$$= \frac{1}{2}\begin{bmatrix}\frac{v_1^2+v_2^2}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} \tag{93}$$

$$= \frac{1}{2}\begin{bmatrix}\frac{1-v_3^2}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} \tag{94}$$

$$= \frac{1}{2}\begin{bmatrix}\frac{(1+v_3)(1-v_3)}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} \tag{95}$$

$$= \frac{1}{2}\begin{bmatrix}1+v_3 & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} \tag{96}$$

$$= (I+\vec{v}\vec{\sigma})/2 = P_+ \tag{97}$$

as requested. P_- can be easily obtained following the same steps, but with $\lambda = -1$.

4.4.3 Exercise 2.61

This Exercise can be done very easily by using Equations (2.103) and (2.104) alongside the value of P_+ obtained in Exercise 2.60.

The probability can be calculated by using Equation (2.103):

$$p(+1) = \langle 0|\,P_+\,|0\rangle \tag{98}$$

$$= \begin{bmatrix}1 & 0\end{bmatrix}(I+\vec{v}\vec{\sigma})/2\begin{bmatrix}1\\0\end{bmatrix} \tag{99}$$

$$= \frac{1}{2}\begin{bmatrix}1+v_3 & v_1-v_2i\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \tag{100}$$

$$= \frac{1+v_3}{2} \tag{101}$$

Then, using Equation (2.104) to obtain the state of the system after the measurement:

$$|\psi\rangle = \frac{P_+\,|0\rangle}{\sqrt{p(+1)}} \tag{102}$$

$$= \frac{1}{\sqrt{p(+1)}}\frac{1}{2}(I+\vec{v}+\vec{\sigma})\begin{bmatrix}1\\0\end{bmatrix} \tag{103}$$

$$= \frac{\sqrt{2}}{\sqrt{1+v_3}}\frac{1}{2}\begin{bmatrix}v_3+1\\ v_1+v_2i\end{bmatrix} \tag{104}$$

$$= \frac{(v_3+1)\,|0\rangle + (v_1+v_2i)\,|1\rangle}{\sqrt{2+2v_3}} \tag{105}$$

4.5 Section 2.2.8

4.5.1 Equation (2.123)

I would like to thank Rex (rexmedeiros@ect.ufrn.br) and LIB (leandro@ect.ufrn.br) for helping me to understand this equation. The present subsection mixes some doubts I had alongside with their explanation.

The definition of equation (2.122) will be needed for this section. In order to understand equation (2.123), it is necessary to recall the definition of inner product [2] between two states $|\psi\rangle$ and $|\varphi\rangle$:

$$(|\varphi\rangle, |\psi\rangle) = |\varphi\rangle^{\dagger} |\psi\rangle = \langle\varphi|\psi\rangle$$

However, the inner product on equation (2.123) is a composite system inner product. Since composite systems are described using tensor products, it is necessary to apply the definition of equation (2.49). Hence, it is possible to calculate

$$\begin{aligned} (U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= \left(\sum_m M_m|\varphi\rangle|m\rangle, \sum_{m'.} M_{m'}|\psi\rangle|m'\rangle\right) && (106)\\ &= \sum_{m,m'} (M_m|\varphi\rangle)^{\dagger} M_{m'.}|\psi\rangle\langle m|m'\rangle && (107) \end{aligned}$$

Then, from the definitions on section 2.1.6:

$$\sum_{m,m'} (M_m|\varphi\rangle)^{\dagger} M_{m'.}|\psi\rangle\langle m|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^{\dagger} M_{m'.}|\psi\rangle\langle m|m'\rangle \qquad (108)$$

The left side of equation (2.123) may be rather confusing, however. Because according to the definitions on section 2.16 $(U|\varphi 0\rangle)^{\dagger} = \langle\varphi 0| U^{\dagger}$. Also, accordingly to the properties on equation (2.53) $(U|\varphi\rangle|0\rangle)^{\dagger} = \langle\varphi|\langle 0| U^{\dagger}$. If this line of thought was followed, then equation

$$\langle\varphi|\langle 0| U^{\dagger} U|\psi\rangle|0\rangle = \sum_{m,m'} \langle\varphi|\langle m| M_m^{\dagger} M_{m'}|\psi\rangle|m'\rangle$$

would be obtained. Which would not match equation (2.49)'s definition.

It is a common practice in Physics, however, to write $(U|\varphi\rangle|0\rangle)^{\dagger} = \langle 0|\langle\varphi| U^{\dagger}$. In this case, the adjoint operators are read 'backwards'. So, for instance, U operates on $|\varphi\rangle$ (i.e. $U|\varphi\rangle$); while $U^{\dagger}$ operates on $\langle\varphi|$ (i.e. $\langle\varphi| U^{\dagger}$). Following this line of thought, $(U|\varphi\rangle|0\rangle)^{\dagger} = \langle\varphi|\langle 0| U^{\dagger}$ would not make sense because $U^{\dagger}$ should operate on $\langle\varphi|$, not on $\langle 0|$. Formally, imagine that an operator M operates on vector space V, $|v\rangle \in V$ and $|w\rangle \in W$, then $\langle v|\langle w| M^{\dagger}$ would not be valid because M only acts on vector space V, not W.

Hence, it is possible to rewrite equation (2.123) as:

$$\begin{aligned} (U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= (U|\varphi\rangle|0\rangle)^{\dagger} U|\psi\rangle|0\rangle && (109)\\ &= (\sum_m M_m|\varphi\rangle|m\rangle)^{\dagger} \sum_{m'} M_{m'}|\psi\rangle|m'\rangle && (110)\\ &= \sum_{m,m'} \langle m|\langle\varphi| M_m^{\dagger} M_{m'}|\psi\rangle|m'\rangle && (111) \end{aligned}$$

since $\langle\varphi| M_m^{\dagger} M_{m'}|\psi\rangle$ is a scalar:

$$\sum_{m,m'} \langle m|\langle\varphi| M_m^{\dagger} M_{m'}|\psi\rangle|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^{\dagger} M_{m'}|\psi\rangle\langle m|m'\rangle \qquad (112)$$

Which is another way to obtain equation (2.123).

[2] For more details, refer to Nielsen and Chuang's section 2.1.4

4.6 Section 2.5

4.6.1 Symmetry of $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$

This subsection is dedicated to calculate $tr((\rho^A)^2)$ for $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$. By Equation (2.138):

$$\rho^{AB} = \frac{(|00\rangle + |01\rangle + |11\rangle)}{\sqrt{3}} \frac{(\langle 00| + \langle 01| + \langle 11|)}{\sqrt{3}} \tag{113}$$

$$= \frac{(|00\rangle + |01\rangle + |11\rangle)(\langle 00| + \langle 01| + \langle 11|)}{3} \tag{114}$$

Before using Equations (2.177) and (2.178) it is necessary to apply the distributive property. However from Equation (2.178), the result will be similar to $\sum_{ijkl} |i\rangle \langle j| tr(|k\rangle \langle l|)$, where $i, j, k, l \in \{0, 1\}$. Since $tr(|a\rangle \langle b|) = \langle b|a\rangle$ (check Section 4.3.3):

$$\sum_{ijkl} |i\rangle \langle j| tr(|k\rangle \langle l|) = \sum_{ijkl} |i\rangle \langle j| \langle l|k\rangle \tag{115}$$

$$= \sum_{ijkl} |i\rangle \langle j| \delta_{lk} \tag{116}$$

Therefore, when applying the distributive property, it is not necessary to write $|i\rangle \langle j|$ if $l \neq k$. For instance, $tr_B(|00\rangle \langle 01|) = |0\rangle \langle 0| tr(|0\rangle \langle 1|) = |0\rangle \langle 0| \langle 0|1\rangle = 0 |0\rangle \langle 0|$. Also, since $\langle i|i\rangle = 1$:

$$\rho^A = \frac{|0\rangle \langle 0| + |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{3} \tag{117}$$

$$= \frac{2 |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{3} \tag{118}$$

Now, calculating $(\rho^A)^2$:

$$(\rho^A)^2 = \frac{(2 |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|)\ (2 |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|)}{3 \cdot 3} \tag{119}$$

$$= \frac{4 |0\rangle \langle 0| + 2 |0\rangle \langle 1| + |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{9} \tag{120}$$

$$= \frac{5 |0\rangle \langle 0| + 3 |0\rangle \langle 1| + |1\rangle \langle 0| + 2 |1\rangle \langle 1|}{9} \tag{121}$$

Now, calculate $tr((\rho^A)^2)$:

$$tr((\rho^A)^2) = \frac{1}{9}\ tr(5 |0\rangle \langle 0| + 3 |0\rangle \langle 1| + |1\rangle \langle 0| + 2 |1\rangle \langle 1|) \tag{122}$$

$$= \frac{1}{9}\ (5 \langle 0|0\rangle + 3 \langle 1|0\rangle + \langle 0|1\rangle + 2 \langle 1|1\rangle) \tag{123}$$

$$= \frac{1}{9}\ (5 \cdot 1 + 3 \cdot 0 + 0 + 2 \cdot 1) \tag{124}$$

$$= \frac{1}{9}\ (5 + 2) \tag{125}$$

$$= \frac{7}{9} \tag{126}$$

Using an analogous line of thought $tr((\rho^B)^2) = \frac{7}{9}$ is obtained.

4.6.2 Exercise 2.78

PRODUCT STATE IF AND ONLY IF SCHIMIDT NUMBER 1.

Suppose state $|\psi\rangle$ is a product state of systems A and B, i.e. $A \otimes B$. Then, there exist orthonormal states $|a\rangle$ and $|b\rangle$, respectively for systems A and B, such that $|\psi\rangle = |a\rangle |b\rangle$. Therefore, the only possible Schmidt decomposition for state $|\psi\rangle$ is $|\psi\rangle = 1\,|a\rangle\,|b\rangle + \sum_i 0\,|i_A\rangle\,|i_B\rangle$ where $|i_A\rangle$ and $|i_B\rangle$ are part of the orthonormal bases alongside $|a\rangle$ and $|b\rangle$; in other words: $\langle a|a\rangle = \langle i_A|i_A\rangle = 1$ and $\langle a|i_A\rangle = \langle i_A|a\rangle = 0$, analogously for $|b\rangle$ and $|i_B\rangle$. As a consequence, the Schmidt number is 1 (refer to equation $|\psi\rangle = 1\,|a\rangle\,|b\rangle + \sum_i 0\,|i_A\rangle\,|i_B\rangle$).

Suppose state $|\psi\rangle$ has Schmidt number 1. Then, from Theorem 2.7, $|\psi\rangle = \sum_i \lambda_i\,|i_A\rangle\,|i_B\rangle$. Since $|\psi\rangle$ has Schmidt number 1, there exist $\lambda_i = 1$ and $\lambda_j = 0$ such that $|\psi\rangle = 1\,|i_A\rangle\,|i_B\rangle + \sum_j 0\,|j_A\rangle\,|j_B\rangle = |i_A\rangle\,|i_B\rangle$. Therefore, $|\psi\rangle$ is a product state.

Quod erat demonstrandum.

PRODUCT STATE IF AND ONLY IF ρ^A ARE PURE STATES.

Suppose $|\psi\rangle$ is a product state of composite system $A \otimes B$, then $|\psi\rangle = |a\rangle\,|b\rangle$ where $|a\rangle$ and $|b\rangle$ are orthonormal states of systems A and B, respectively.

Then, by definition of density operator in Equation (2.138): $\rho = 1 \cdot |ab\rangle\langle ab|$, and ρ is pure if $tr(\rho^2) = 1$:

$$\begin{aligned} tr(\rho^2) &= tr(|ab\rangle\,\langle ab|ab\rangle\,\langle ab|) && (127)\\ &= tr(|ab\rangle\langle ab|) && (128)\\ &= tr(\rho) && (129) \end{aligned}$$

Them, by Theorem 2.5, $tr(\rho) = 1$.

If ρ is pure, consequently ρ^A and ρ^B are pure. Otherwise, $\langle a|a\rangle \neq 1 \neq \langle b|b\rangle$, which would be a contradiction with ρ being pure.

The converse can be proved naturally following these steps reversely. Suppose ρ^A is pure. Then ρ^B is pure. Then ρ is pure. Then $|\psi\rangle$ is a state product and can be written as $|\psi\rangle = |a\rangle\,|b\rangle$.

Quod erat demonstrandum.

4.6.3 Equations 2.208 and 2.209

When I firstly read these equations I thought there was a possibility that an extra explanation would be necessary. This thought raised, most likely, because I was unaccustomed to Tensor Product Properties and the Reduced Density Operator.

Using $|AR\rangle$ as defined in Equation 2.207:

$$\begin{aligned} |AR\rangle\langle AR| &= \left(\sum_i \sqrt{p_i}\,|i^A\rangle\,|i^R\rangle\right)\left(\sum_j \sqrt{p_j}\,\langle j^A|\,\langle j^R|\right) && (130)\\ &= \left(\sum_i \sqrt{p_i}\,|i^A\rangle \otimes |i^R\rangle\right)\left(\sum_j \sqrt{p_j}\,\langle j^A| \otimes \langle j^R|\right) && (131)\\ &= \sum_{ij} \sqrt{p_i p_j}(\ |i^A\rangle \otimes |i^R\rangle\ )(\ \langle j^A| \otimes \langle j^R|\ ) && (132) \end{aligned}$$

Then, by applying the properties as similarly defined in Equation 2.46:

$$\sum_{ij} \sqrt{p_i p_j}(\ |i^A\rangle \otimes |i^R\rangle\ )(\ \langle j^A| \otimes \langle j^R|\ ) = \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A| \otimes |i^R\rangle\langle j^R| \qquad (133)$$

Therefore, using the definition of the Reduced Density Operator (Equation 2.178):

$$\begin{aligned} tr_R(|AR\rangle\langle AR|) &= tr_R\left(\sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A| \otimes |i^R\rangle\langle j^R|\right) && (134)\\ &= \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A|\,tr(|i^R\rangle\langle j^R|) && (135) \end{aligned}$$

$$ \tag{136} $$

Thus obtaining Equation 2.208.

In order to obtain Equation 2.209 it is necessary to apply $tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle$. Also, recall that $|i^R\rangle$ are orthonormal states and that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Hence,

$$ \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A|\, tr(|i^R\rangle\langle j^R|) = \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A|\, \langle j^R|i^R\rangle \tag{137} $$

$$ = \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A|\, \delta_{ij} \tag{138} $$

as required.

4.6.4 Exercise 2.79

In order to solve this exercise, refer back to Theorem 2.7 and factor each state.

1. $$ \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle \tag{139} $$

   $$ = \sum_{i\in\{0,1\}} \lambda_i |i\rangle|i\rangle \tag{140} $$

   where $\lambda_0 = \lambda_1 = 1/\sqrt{2}$.

2. Since Schmidt's decomposition requires that $|i_A\rangle$ and $|i_B\rangle$ are orthonormal states and $|+\rangle$ and $|-\rangle$ are examples of such states:

   $$ \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,\frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{141} $$

   $$ = |+\rangle|+\rangle \tag{142} $$

   $$ = 1\cdot|+\rangle|+\rangle + 0\cdot|-\rangle|-\rangle \tag{143} $$

   $$ = \sum_{i\in\{+,-\}} \lambda_i |i\rangle|i\rangle \tag{144} $$

   where $\lambda_+ = 1$ and $\lambda_- = 0$.

3. Analysing the state [3] $|\psi\rangle = \frac{|00\rangle+|01\rangle+|10\rangle}{\sqrt{3}}$ for both systems separately:

   For the first qubit: $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$.

   For the second qubit: $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$ as well.

   It is easy to check that $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$ is orthonormal. However, following this line of thought may lead to erroneous solutions. A bit more of cleverness is required: it is possible to use the interesting results obtained for ρ^A, ρ^B, and their eigenvalues as stated in the paragraph that follows Theorem 2.7.

   Then, calculate ρ to obtain ρ^A and ρ^B afterwards. Since $|\psi\rangle$ is pure:

   $$ \rho = |\psi\rangle\langle\psi| \tag{145} $$

   $$ = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}\,\frac{\langle 00| + \langle 01| + \langle 10|}{\sqrt{3}} \tag{146} $$

   $$ = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{147} $$

[3] Note that similarly to state $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$ described in Section 4.6.1 , $|\psi\rangle$ has symmetry 7/9. Hence, obtaining its Schmidt Decomposition is not intuitive.

Calculating ρ^A using the definition in Equations (2.177) and (2.178), alongside $tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle$ (Section 4.3.3):

$$\rho^A = \frac{1}{3} tr_B(\ (|00\rangle + |01\rangle + |10\rangle)(\langle 00| + \langle 01| + \langle 10|)\ ) \tag{148}$$

$$= \frac{1}{3}(\ |0\rangle\langle 0|\, tr(|0\rangle\langle 0| + |1\rangle\langle 1|) + |0\rangle\langle 1|\, tr(|0\rangle\langle 0|) + |1\rangle\langle 0|\, tr(|0\rangle\langle 0|) + |1\rangle\langle 1|\, tr(|0\rangle\langle 0|)\ ) \tag{149}$$

$$= \frac{1}{3}(\ 2\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\ ) \tag{150}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{151}$$

Also, calculating ρ^B it is possible to verify that $\rho^B = \rho^A$. In order to write the Schmidt Decomposition, it is necessary to find the eigenvalues and eigenvectors of ρ^A:

$$\begin{vmatrix} 2/3 - v & 1 \\ 1 & 1/3 - v \end{vmatrix} = v^2 - v + \frac{1}{9} = 0 \tag{152}$$

Solving the polynomial, the eigenvalues found are $v = \frac{3\pm\sqrt{5}}{6}$. Calculate the corresponding eigenvectors $|v_1\rangle$ and $|v_2\rangle$.

For $v_1 = \frac{3+\sqrt{5}}{6}$: substitute and row reduce

$$\begin{bmatrix} \frac{4}{6} - \frac{3+\sqrt{5}}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{6} - \frac{3+\sqrt{5}}{6} \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1-\sqrt{5}}{6} \end{bmatrix} \tag{153}$$

$$\sim \begin{bmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{bmatrix} \tag{154}$$

$$\sim \begin{bmatrix} 1-\sqrt{5} & 2 \\ 0 & 0 \end{bmatrix} \tag{155}$$

Then, the eigenspace of v_1 is $\left\{\begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix}\right\}$. The orthonormal vector of the eigenspace is of interest. Therefore, normalise $|v_1\rangle$:

$$|v_1\rangle = \frac{1}{\sqrt{\langle v_1|v_1\rangle}}\,|v_1\rangle \tag{156}$$

$$= \sqrt{\frac{2}{5+\sqrt{5}}}\begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} \tag{157}$$

For $v_2 = \frac{3-\sqrt{5}}{6}$, the normalised vector $|v_2\rangle = \sqrt{\frac{2}{5-\sqrt{5}}}\begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix}$ is found. [4]

From the results that follow Theorem 2.7, it is known that $\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$ where λ_i^2 are the eigenvalues of ρ^A, i.e. $\lambda_i^2 = v_i$. Therefore, since rho^A and rho^B have the same eigenvalues, by Theorem 2.7, $|\psi\rangle$ has Schmidt Decomposition

$$|\psi\rangle = \sum_i \lambda_i\,|i_A\rangle\,|i_B\rangle = \sum_i \sqrt{v_i}\,|v_i\rangle\langle v_i| \tag{158}$$

$$\begin{aligned} &= \sqrt{\frac{3+\sqrt{5}}{6}}\left(\sqrt{\frac{2}{5+\sqrt{5}}}\begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix}\right) \otimes \left(\sqrt{\frac{2}{5+\sqrt{5}}}\begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix}\right) \\ &+ \sqrt{\frac{3-\sqrt{5}}{6}}\left(\sqrt{\frac{2}{5-\sqrt{5}}}\begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix}\right) \otimes \left(\sqrt{\frac{2}{5-\sqrt{5}}}\begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix}\right) \end{aligned} \tag{159}$$

[4] Verify that the values found match $\langle v_1|v_2\rangle = \langle v_2|v_1\rangle = 0$, $\langle v_1|v_1\rangle = \langle v_2|v_2\rangle = 1$, and $\rho^A = \rho^B = \sum_i v_i\,|v_i\rangle\langle v_i|$

$$= \sqrt{\frac{3+\sqrt{5}}{6}}\ \frac{2}{5+\sqrt{5}}\begin{bmatrix}\frac{4}{6-2\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ 1\end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{6}}\ \frac{2}{5-\sqrt{5}}\begin{bmatrix}\frac{4}{6+2\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ 1\end{bmatrix} \tag{160}$$

In order to keep calculating, it is necessary to rewrite $\sqrt{3+\sqrt{5}}$ with the help of quadratic polynomials:

$$\sqrt{3+\sqrt{5}} = \sqrt{\frac{1+\sqrt{5}^2+2\sqrt{5}}{2}} \tag{161}$$

$$= \sqrt{\frac{(1+\sqrt{5})^2}{2}} \tag{162}$$

$$= \frac{1+\sqrt{5}}{\sqrt{2}} \tag{163}$$

Similarly, $\sqrt{3-\sqrt{5}} = \frac{1-\sqrt{5}}{\sqrt{2}}$.

Therefore, substituting in Equation 160:

$$|\psi\rangle = \frac{1+\sqrt{5}}{\sqrt{2}\sqrt{6}}\ \frac{2}{5+\sqrt{5}}\begin{bmatrix}\frac{4}{6-2\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ 1\end{bmatrix} + \frac{1-\sqrt{5}}{\sqrt{2}\sqrt{6}}\ \frac{2}{5-\sqrt{5}}\begin{bmatrix}\frac{4}{6+2\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ 1\end{bmatrix} \tag{164}$$

$$= \frac{1+\sqrt{5}}{\sqrt{3}}\ \frac{1}{5+\sqrt{5}}\begin{bmatrix}\frac{2}{3-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ 1\end{bmatrix} + \frac{1-\sqrt{5}}{\sqrt{3}}\ \frac{1}{5-\sqrt{5}}\begin{bmatrix}\frac{2}{3+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ 1\end{bmatrix} \tag{165}$$

$$= \begin{bmatrix}\frac{2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(3-\sqrt{5})} + \frac{2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(3+\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(1-\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(1+\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(1-\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(1+\sqrt{5})}\\ \frac{1+\sqrt{5}}{\sqrt{3}(5+\sqrt{5})} + \frac{1-\sqrt{5}}{\sqrt{3}(5-\sqrt{5})}\end{bmatrix} = \begin{bmatrix}\frac{2(1+\sqrt{5})}{\sqrt{3}(10-2\sqrt{5})} + \frac{2(1-\sqrt{5})}{\sqrt{3}(10+2\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(-4\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(4\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(-4\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(4\sqrt{5})}\\ \frac{1+\sqrt{5}}{\sqrt{3}(5+\sqrt{5})} + \frac{1-\sqrt{5}}{\sqrt{3}(5-\sqrt{5})}\end{bmatrix} \tag{166}$$

$$= \begin{bmatrix}\frac{(1+\sqrt{5})(5+\sqrt{5})+(1-\sqrt{5})(5-\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(5-\sqrt{5})}\\ \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{(1+\sqrt{5})(5-\sqrt{5})+(1-\sqrt{5})(5+\sqrt{5})}{20\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{10+6\sqrt{5}+10-6\sqrt{5}}{20\sqrt{3}}\\ \frac{2\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{2\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{4\sqrt{5}-4\sqrt{5}}{20\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ 0\end{bmatrix} \tag{167}$$

$$= \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \tag{168}$$

as requested.

4.6.5 Exercise 2.80

If $|\psi\rangle$ and $|\varphi\rangle$ are pure states, then they can be written as $|\psi\rangle = \sum_i p_i |i_A\rangle |i_B\rangle$ where $p_i = 1$ for a specific i and $p_i = 0$ otherwise. Therefore, $|\psi\rangle = |i_A\rangle |i_B\rangle$, analogously $|\varphi\rangle = |j_A\rangle |j_B\rangle$ where $|i_A\rangle$, $|i_B\rangle$, $|j_A\rangle$, $|j_B\rangle$ are unitary.

Then, construct two unitary matrices U and V such that $|i_A\rangle = U|j_A\rangle$ and $|i_B\rangle = V|j_B\rangle$. Then, using Equation (2.46), $|\psi\rangle$ can be written as:

$$|\psi\rangle = |i_A\rangle |i_B\rangle \tag{169}$$

$$= |i_A\rangle \otimes |i_B\rangle \tag{170}$$

$$= U|j_A\rangle \otimes V|j_B\rangle \tag{171}$$

$$= (U \otimes V) |j_A\rangle \otimes |j_B\rangle \tag{172}$$
$$= (U \otimes V) |j_A\rangle |j_B\rangle \tag{173}$$
$$= (U \otimes V) |\varphi\rangle \tag{174}$$

4.6.6 Exercise 2.81

It is possible to solve this exercise by following the same logic as Exercise 2.80 (Section 4.6.5). Define an Unitary Operator $U_R \equiv \sum_i |v_i\rangle\langle w_i|$, where $|v_i\rangle$ are an orthonormal basis for R_1 and $|w_i\rangle$ for R_2. Therefore, $|v_i\rangle = U_R |w_i\rangle$. Then, by Equations (2.207) and (2.46):

$$|AR_1\rangle = \sum_i \sqrt{p_i} |i_A\rangle |v_i\rangle \tag{175}$$
$$= \sum_i \sqrt{p_i} |i_A\rangle \;\; U_R |w_i\rangle \tag{176}$$
$$= \sum_i I \sqrt{p_i} |i_A\rangle \otimes U_R |w_i\rangle \tag{177}$$
$$= (I \otimes U_R) \sum_i \sqrt{p_i} |i_A\rangle \otimes |w_i\rangle \tag{178}$$
$$= (I \otimes U_R) |AR_2\rangle \tag{179}$$

4.6.7 Exercise 2.82

1. From the definition of purification, it is necessary to prove that $\rho^A = tr_R(|AR\rangle \langle AR|)$.
   Suppose $\sum_i \sqrt{(p_i)} |\psi_i\rangle |i\rangle$ is a purification. From equation (2.138) of Nielsen and Chuang's book:

$$\rho^{AB} = \sum_{ij} (\sqrt{p_i} |\psi_i\rangle |i\rangle) \left(\sqrt{p_j} |\psi_j\rangle |j\rangle\right)^\dagger \tag{180}$$

   Since $p_j \in \mathbb{R}$, $p_j^\dagger = p_j$. And since it is a scalar:

$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j} (|\psi_i\rangle |i\rangle) (|\psi_j\rangle |j\rangle)^\dagger \tag{181}$$

   Recall that for any states $|\varphi\rangle$ and $|\gamma\rangle$ writing $|\varphi\rangle |\gamma\rangle$ is the same as $|\varphi\rangle \otimes |\gamma\rangle$. Then, applying equation (2.48):

$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j} (|\psi_i\rangle \otimes |i\rangle) (\langle\psi_j| \otimes \langle j|) \tag{182}$$
$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j} (|\psi_i\rangle \langle\psi_j|) \otimes (|i\rangle \langle j|) \tag{183}$$

   If $\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$ is a purification, then $\rho^A = tr_B(\;(|\psi_i\rangle |i\rangle)(\langle\psi_i| \langle i|)\;)$. The question gives $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, in other words $\rho = \rho^A$.
   Now, it is necessary to calculate $tr_B(\;(|\psi_i\rangle |i\rangle)(\langle\psi_i| \langle i|)\;)$. Then, using the definitions given in equations (2.177) and (2.178) of Nielsen and Chuang's book:

$$tr_B(\rho^{AB}) = tr_B\left(\sum_{ij} \sqrt{p_i p_j} (|\psi_i\rangle \langle\psi_j|) \otimes (|i\rangle \langle j|)\right) \tag{184}$$
$$= \sum_{ij} \sqrt{p_i p_j} (|\psi_i\rangle \langle\psi_j|) tr(|i\rangle \langle j|) \tag{185}$$

   Since $tr(|a\rangle \langle b|) = \langle b|a\rangle$ (refer back to Section 4.3.3):

$$tr_B(\rho^{AB}) = \sum_{ij} \sqrt{p_i p_j}\; (|\psi_i\rangle \langle\psi_j|) \langle j|i\rangle \tag{186}$$

$$= \sum_{ij} \sqrt{p_i p_j} \ |\psi_i\rangle \langle\psi_j| \ \delta_{ij} \tag{187}$$

$$= \sum_{i} \sqrt{p_i p_i} \ |\psi_i\rangle \langle\psi_i| \tag{188}$$

$$= \sum_{i} p_i \, |\psi_i\rangle \langle\psi_i| \tag{189}$$

Since $\rho^A = \sum_i p_i |\psi_i\rangle\langle\psi_i| = tr_B(\rho^{AB})$, it is possible to conclude that $\sum_i \sqrt{p_i}\,|\psi_i\rangle|i\rangle$ is a purification.

2. **TODO: ADD INTUITIVE SOLUTION/EXPLANATION**

   For this exercise, it is necessary to review Postulate 3. For the system $|AR\rangle$, there is a set of measurement operators $\{M_m\}$. Only system R is being measured, though. So, it is possible to define every measurement operator $M_m = I \otimes M_i$ where I is the identity operator acting on system A and M_i is the measurement operator acting on system R which corresponds to measuring the state $|i\rangle$.

   The probability of measuring $|i\rangle$ is requested, i.e. $p(i)$. Using $|\varphi_i\rangle = \sqrt{p_i}\,|\psi_i\rangle|i\rangle$ temporarily for simplicity and by Equation (2.92):

$$\begin{aligned}
p(i) &= \langle\varphi_i| M_m^\dagger M_m |\varphi_i\rangle && (190)\\
&= \langle\varphi_i| (I\otimes M_i)^\dagger (I\otimes M_i) |\varphi_i\rangle && (191)\\
&= \sqrt{p_i}\,\langle\psi_i|\langle i| (I\otimes M_i)^\dagger (I\otimes M_i)\sqrt{p_i}\,|\psi_i\rangle|i\rangle && (192)\\
&= p_i\,\langle\psi_i|\langle i| (I^\dagger\otimes M_i^\dagger)(I\otimes M_i)|\psi_i\rangle|i\rangle && (193)\\
&= p_i\,\langle\psi_i|\langle i| (I\otimes M_i^\dagger)(I\otimes M_i)|\psi_i\rangle|i\rangle && (194)
\end{aligned}$$

   Using equation (2.48):

$$\begin{aligned}
p(i) &= p_i(\langle\psi_i| I \otimes \langle i| M_i^\dagger)\ (I|\psi_i\rangle \otimes M_i|i\rangle) && (195)\\
&= p_i(\langle\psi_i| \otimes \langle i| M_i^\dagger)\ (|\psi_i\rangle \otimes M_i|i\rangle) && (196)
\end{aligned}$$

   Then, by the definition of inner product (equation (2.49)):

$$p(i) = p_i\,\langle\psi_i|\psi_i\rangle\,\langle i| M_i^\dagger M_i |i\rangle \tag{197}$$

   Recall that $|\psi_i\rangle$ and $|i\rangle$ are orthonormal. Also, since M_i is the measurement operator that corresponds to obtaining state $|i\rangle$, it is possible to consider $|i\rangle$ as a "measurement basis" defining $M_i = |i\rangle\langle i|$. A similar example can be seen in Nielsen and Chuang's book in a paragraph between Equations (2.95) and (2.96). Therefore,

$$\begin{aligned}
p(i) &= p_i \cdot 1 \cdot \langle i| (|i\rangle\langle i|)^\dagger (|i\rangle\langle i|)|i\rangle && (198)\\
&= p_i\,\langle i| (|i\rangle\langle i|)(|i\rangle\langle i|)|i\rangle && (199)\\
&= p_i\,\langle i|i\rangle\langle i|i\rangle\langle i|i\rangle && (200)\\
&= p_i \cdot 1 \cdot 1 \cdot 1 && (201)\\
&= p_i && (202)
\end{aligned}$$

   Hence, the probability of measuring state $|i\rangle$ is p_i. Now, it is requested to obtain the state of system A after the measurement M_m, which is described by Postulate 3 as:

$$\begin{aligned}
\frac{M_m|\varphi_i\rangle}{\sqrt{p_i}} &= \frac{(I\otimes M_i)\sqrt{p_i}\,|\psi_i\rangle|i\rangle}{\sqrt{p_i}} && (203)\\
&= (I|\psi_i\rangle)\otimes(M_i|i\rangle) && (204)\\
&= (I|\psi_i\rangle)\otimes(|i\rangle\langle i|i\rangle) && (205)\\
&= |\psi_i\rangle|i\rangle && (206)
\end{aligned}$$

   Therefore, the measurement of the system A will always be $|\psi_i\rangle$.

3. Goropikari attempted to solve this exercise as follows[5]:

[5]The original code can be found at SolutionForQuantumComputationAndQuantumInformation

Suppose $|AR\rangle$ is a purification of ρ such that $|AR\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle$. By exercise 2.81, the others purification is written as $(I \otimes U) |AR\rangle$.

$$\begin{aligned}(I \otimes U) |AR\rangle &= (I \otimes U) \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle U |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle\end{aligned}$$

where $U = \sum_i |i\rangle \langle r_i|$.

By (2), if we measure the system R w.r.t $|i\rangle$, post-measurement state for system A is $|\psi_i\rangle$ with probability p_i, which prove the assertion.

TODO: IS THE PREVIOUS SOLUTION PLAUSIBLE IN SOME WAY?

However, if system R is measured with respected to $|i\rangle$ (that is, the measurement operator $M_m = I \otimes |i\rangle \langle i|$ is applied to $\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$) the same result of Exercise 2.82(2) will be achieved.

A similar way to try to solve this problem is: define $|AR_i\rangle = |\psi_i\rangle |r_i\rangle$ and use the measurement operator $M_m = I \otimes |i\rangle \langle i|$. Then, the probability of measuring $|i\rangle$ is calculated according to Equation (2.92):

$$\begin{aligned}p(i) &= (\sqrt{p_i}\ (I \otimes |i\rangle\langle i|) |\psi_i\rangle |r_i\rangle)^\dagger\ \sqrt{p_i}\ (I \otimes |i\rangle\langle i|) |\psi_i\rangle |r_i\rangle && (207)\\ &= p_i \langle r_i| \langle \psi_i| (|i\rangle\langle i| \otimes I)\ (I \otimes |i\rangle\langle i|) |\psi_i\rangle |r_i\rangle && (208)\end{aligned}$$

Then, using Equations (2.48) and (2.49):

$$\begin{aligned}p(i) &= p_i(\langle r_i|\, |i\rangle\langle i|) \otimes (\langle \psi_i| I)\ (I\, |\psi_i\rangle) \otimes (|i\rangle\langle i|\, |r_i\rangle) && (209)\\ &= p_i(\langle r_i|i\rangle \langle i|) \otimes \langle \psi_i|\ \ |\psi_i\rangle \otimes (\langle i|r_i\rangle |i\rangle) && (210)\\ &= p_i \langle r_i|i\rangle \langle i|r_i\rangle\ (\langle i| \langle \psi_i| |\psi_i\rangle |i\rangle) && (211)\\ &= p_i\ ||\, \langle i|r_i\rangle\, ||^2 && (212)\end{aligned}$$

if $|| \langle i|r_i\rangle ||^2 = 1/p_i$, the desired result would obtained. However, $0 < p_i < 1$, then $1/p_i > 1$, which is not possible since both $|i\rangle$ and $|r_i\rangle$ are orthonormal vectors. Independently, if the post-measurement state is calculated as given by Equation (2.93):

$$\begin{aligned}\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}} &= \frac{(I \otimes |i\rangle\langle i|)\sqrt{p_i} |\psi_i\rangle |r_i\rangle}{\sqrt{p_i || \langle i|r_i\rangle ||^2}} && (213)\\ &= \frac{\sqrt{p_i} |\psi_i\rangle |i\rangle\langle i| |r_i\rangle}{\sqrt{p_i}\ ||\, |i\rangle\langle r_i|\, ||} && (214)\\ &= \frac{\langle i|r_i\rangle |\psi_i\rangle |i\rangle}{|| \langle i|r_i\rangle ||} && (215)\end{aligned}$$

Therefore, $\frac{\langle i|r_i\rangle}{\sqrt{\langle i|r_i\rangle \langle r_i|i\rangle}} = p_i$. **TODO: IN CONCLUSION ?????**

4.7 Chapter 2 Problems

4.7.1 Problem 2.1

This problem can be solved easily by combining the logic of Exercises 2.35 and 2.60.

It is known from exercise 2.60 that $\vec{n}\vec{\sigma}$ has spectral decomposition $+1P_+ - 1P_- = +1\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) - 1\left(\frac{I-\vec{n}\vec{\sigma}}{2}\right)$. Therefore, $\theta\vec{n}\vec{\sigma} = \theta\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) - \theta\left(\frac{I-\vec{n}\vec{\sigma}}{2}\right)$. Then, by applying the definition of function operators and the distributive property:

$$\begin{aligned}f(\theta\vec{n}\vec{\sigma}) &= f(\theta)\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) + f(-\theta)\left(\frac{I-\vec{n}\vec{\sigma}}{2}\right) && (216)\\ &= \frac{f(\theta)}{2} I + \frac{f(-\theta)}{2} I + \frac{f(\theta)}{2}\vec{n}\vec{\sigma} - \frac{f(-\theta)}{2}\vec{n}\vec{\sigma} && (217)\\ &= \frac{f(\theta)+f(-\theta)}{2} I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\vec{\sigma} && (218)\end{aligned}$$

4.7.2 Problem 2.2

1. Since $|\psi\rangle$ is pure, it follows from Theorem 2.7, that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$. In addition, $\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$. From the rank-nullity theorem, it is known that $rank(A)+nullity(A) = dim(A)$ where $dim(A)$ is the dimension of the matrix A. Also, $rank(A) = dim(Col(A))$ where $dim(Col(A))$ is the dimension of the columnspan of A. Notwithstanding, $row(A) \cup kernel(A)$ spans $\mathbb{C}^n$, and is linearly independent. Thus, forming a basis for $\mathbb{C}^n$.

   To prove that $rank(\rho^A) = Sch(|\psi\rangle)$, it is possible to use the rank-nullity theorem. Henceforth, find $kernel(\rho^A)$, that is, $\{ \ |x\rangle \in \mathbb{C}^n \}$ such that $\rho^A |x\rangle = \vec{0}$. Assume the possibility that $\exists k, \lambda_k = 0$. Also, since $|i_A\rangle$ spans $\mathbb{C}^n$, any vector $|x\rangle$ can written as a linear combination of $|i_A\rangle$. Then,

$$\left(\sum_i \lambda_i^2 |i_A\rangle\langle i_A|\right)|x\rangle = \left(\sum_i \lambda_i^2 |i_A\rangle\langle i_A|\right)\sum_j x_j |j_A\rangle = \sum_{ij} \lambda_i^2 x_j |i_A\rangle \langle i_A|j_A\rangle \tag{219}$$

$$= \sum_j \lambda_j^2 x_j |j_A\rangle = \sum_{j\neq k} \lambda_j^2 x_j |j_A\rangle + \sum_k \lambda_k^2 x_k |k_A\rangle = \sum_{j\neq k} \lambda_j^2 x_j |j_A\rangle + \sum_k 0 x_k |k_A\rangle \tag{220}$$

   Therefore, $kernel(\rho^A) = |k_A\rangle$ where $\forall k, \lambda_k = 0$. As such, $Sch(|\psi\rangle) = n - dim(kernel(\rho^A)) = n - nullity(\rho^A)$. Then, using the rank-nullity theorem conclude that $rank(\rho^A) = Sch(|\psi\rangle)$.

   ....

2. To prove that $j > Sch(\psi)$ note that if $j > 1$, then it is possible to obtain an orthonormal basis for $|\psi\rangle$ such that $|\psi\rangle$ is equal to the tensor product of exactly one element of each subsystem's basis ($|\psi\rangle = |a\rangle \otimes |b\rangle$).

   To prove that $j = Sch(\psi)$, suppose that $|\alpha_j\rangle$ and $|\beta_j\rangle$ are linearly independent (thus forming a basis for a $|\psi\rangle$). Then, obtain bases $|a_i\rangle$ and $|b_i\rangle$ through Gram-Schmidt. Therefore, $Sch(\psi) = i = j$.

   In addition, no restriction is given regarding the number of j elements. Since they are not normalised, it is possible that $|\alpha_j\rangle$ and $|\beta_j\rangle$ are not linearly independent. Thus, by obtaining an orthonormal basis $|a_i\rangle$ and $|b_i\rangle$ from $|\alpha_j\rangle$ and $|\beta_j\rangle$ via Gram-Schmidt, $j > i$. Also proving that $j > Sch(\psi)$.

   ....

3. The solution when $\alpha = 0$ or $\beta = 0$ is trivial, since $Sch(\psi) = max(Sch(\varphi), Sch(\gamma))$. Henceforth, assume that $\alpha \neq 0$ and $\beta \neq 0$. Also, assume that $|\varphi\rangle \neq \mu |\gamma\rangle$, $\mu \in \mathbb{C}$.

   $|\psi\rangle$ is a pure state of A and B, and it is a linear combination of $|\varphi\rangle$ and $|\gamma\rangle$. Also, note that $|\psi\rangle$, $|\varphi\rangle$, and $|\gamma\rangle$ must be written in the same orthonormal basis $|a_i\rangle$ for subsystem A and $|b_i\rangle$ analogously for B. Therefore,

$$|\psi\rangle = |\psi\rangle \tag{221}$$

$$\sum_i \lambda_i |a_i\rangle |b_i\rangle = \alpha |\varphi\rangle + \beta |\gamma\rangle \tag{222}$$

$$\sum_i \lambda_i |a_i\rangle |b_i\rangle = \alpha \sum_i \varphi_i |a_i\rangle |b_i\rangle + \beta \sum_i \gamma_i |a_i\rangle |b_i\rangle \tag{223}$$

$$\tag{224}$$

   Where $\varphi_i, \gamma_i \in \mathbb{C}$ respectively are the indexes of the linear combinations of $|\varphi\rangle$ and $|\gamma\rangle$, according to the basis $|a_i\rangle |b_i\rangle$. Then,

$$\sum_i \lambda_i |a_i\rangle |b_i\rangle = \sum_i (\alpha\varphi_i + \beta\gamma_i) |a_i\rangle |b_i\rangle \tag{225}$$

$$\lambda_i = \alpha\varphi_i + \beta\gamma_i \tag{226}$$

   Using set theory, define $\Psi \equiv \{ i \mid \lambda_i \neq 0 \}$, $\Phi \equiv \{ i \mid \varphi_i \neq 0 \}$, and $\Gamma \equiv \{ i \mid \gamma_i \neq 0 \}$. Note that it is possible that $\alpha\varphi_i + \beta\gamma_i = 0$, thus, it is useful to define another set $\Delta \equiv \{ i \mid \alpha\varphi_i + \beta\gamma_i = 0 \wedge \varphi_i \neq 0 \wedge \gamma_i \neq 0 \}$ (where $\wedge$ indicates logical conjunction). Then, $Sch(\psi) = |\Psi|$, $Sch(\varphi) = |\Phi|$, and $Sch(\gamma) = |\Gamma|$. Therefore,

$$\Psi = (\Phi \cup \Gamma) - \Delta \tag{227}$$

$$|\Psi| = |(\Phi \cup \Gamma)| - |\Delta| \tag{228}$$

   Note that whenever $(\forall\varphi_i \neq 0, \alpha\varphi_i + \beta\gamma_i = 0) \vee (\forall\gamma_i \neq 0, \alpha\varphi_i + \beta\gamma_i = 0)$, then $max(|\Delta|) = min(|\Phi|, |\Gamma|)$, where $\vee$ indicates logical disjunction. Thus, $Sch(\psi) = |Sch(\varphi) - Sch(\gamma)|$. Also, $min(|\Delta|) = 0$ whenever $\forall\varphi_i \neq 0 \wedge \gamma_i \neq 0,\ \alpha\varphi_i + \beta\gamma_i \neq 0$. Thus, $Sch(\psi) > |Sch(\varphi) - Sch(\gamma)|$. Therefore, $Sch(\psi) \geq |Sch(\varphi) - Sch(\gamma)|$, as required.

4.7.3 Problem 2.3

Q, R, S, T are unitary matrices. Referring back to Equation 2.116,

$$Q = \begin{bmatrix} q_3 & q_1 - q_2 i \\ q_1 + q_2 i & -q_3 \end{bmatrix} \tag{229}$$

Note that $Q = Q^\dagger$ and that

$$QQ^\dagger = \begin{bmatrix} q_3 & q_1 - q_2 i \\ q_1 + q_2 i & -q_3 \end{bmatrix} \begin{bmatrix} q_3 & q_1 - q_2 i \\ q_1 + q_2 i & -q_3 \end{bmatrix} \tag{230}$$

$$= \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 & q_3(q_1 - q_2 i) - q_3(q_1 - q_2 i) \\ q_3(q_1 - q_2 i) - q_3(q_1 - q_2 i) & q_1^2 + q_2^2 + q_3^2 \end{bmatrix} \tag{231}$$

Since $\vec{q} \in \mathbb{R}^3$ is a unit vector, $q_1^2 + q_2^2 + q_3^2 = 1$,

$$QQ^\dagger = I \tag{232}$$

Thus proving that Q is an unitary matrix (and analogously for R, S, and T).

Calculating $(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2$, and writing it as M^2

$$\begin{aligned} M^2 = &(Q \otimes S)(Q \otimes S) + (Q \otimes S)(R \otimes S) + (Q \otimes S)(R \otimes T) - (Q \otimes S)(Q \otimes T) + \\ &(R \otimes S)(Q \otimes S) + (R \otimes S)(R \otimes S) + (R \otimes S)(R \otimes T) - (R \otimes S)(Q \otimes T) + \\ &(R \otimes T)(Q \otimes S) + (R \otimes T)(R \otimes S) + (R \otimes T)(R \otimes T) - (R \otimes T)(Q \otimes T) - \\ &(Q \otimes T)(Q \otimes S) - (Q \otimes T)(R \otimes S) - (Q \otimes T)(R \otimes T) + (Q \otimes T)(Q \otimes T) \end{aligned} \tag{233}$$

Using Equation 2.48,

$$\begin{aligned} M^2 = &QQ \otimes SS + QR \otimes SS + QR \otimes ST - QQ \otimes ST + \\ &RQ \otimes SS + RR \otimes SS + RR \otimes ST - RQ \otimes ST + \\ &RQ \otimes TS + RR \otimes TS + RR \otimes TT - RQ \otimes TT - \\ &QQ \otimes TS - QR \otimes TS - QR \otimes TT + QQ \otimes TT \end{aligned} \tag{234}$$

Since $QQ^\dagger = I$ and $Q = Q^\dagger$, then $QQ = I$, analogously for R, S, T.

$$\begin{aligned} M^2 = &I \otimes I + QR \otimes I + QR \otimes ST - I \otimes ST + \\ &RQ \otimes I + I \otimes I + I \otimes ST - RQ \otimes ST + \\ &RQ \otimes TS + I \otimes TS + I \otimes I - RQ \otimes I - \\ &I \otimes TS - QR \otimes TS - QR \otimes I + I \otimes I \end{aligned} \tag{235}$$

Refactoring,

$$\begin{aligned} M^2 = &4I \otimes I + (QR + RQ - RQ - QR) \otimes I + I \otimes (-ST + ST + TS - TS) + \\ &QR \otimes ST - RQ \otimes ST + RQ \otimes TS - QR \otimes TS \end{aligned} \tag{236}$$

$$= 4I + QR \otimes ST - RQ \otimes ST + RQ \otimes TS - QR \otimes TS \tag{237}$$

$$= 4I + QR \otimes (ST - TS) - RQ \otimes (ST - TS) \tag{238}$$

$$= 4I + (QR - RQ) \otimes (ST - TS) \tag{239}$$

Therefore, by the definition of commutator (Equation 2.66),

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T] \tag{240}$$

Recall that $Var(x) = E(x^2) - E(x)^2$ (as described in the book's Appendix 1). Also, since $Var(x) \geq 0$,

$$E(x^2) - E(x)^2 \geq 0 \tag{241}$$

$$E(x)^2 \leq E(x^2) \tag{242}$$
$$E(x) \leq \sqrt{E(x^2)} \tag{243}$$

Then, calculate $E(x^2)$. Recall that $E(M) = \langle\psi| M |\psi\rangle \equiv \langle M\rangle$ (as described in Equations 2.110 to 2.115). Also, $E(M+N) = E(M) + E(N)$ and $E(MN) = E(M)E(N)$ Thus,

$$E(M^2) = \langle M^2\rangle \tag{244}$$
$$= \langle 4I + [Q,R] \otimes [S,T]\rangle \tag{245}$$
$$= \langle 4I\rangle + \langle [Q,R] \otimes [S,T]\rangle \tag{246}$$

Calculate the mean for $[Q,R]$. The mean of $[S,T]$ is analagous.

$$\langle [Q,R]\rangle = \langle QR - RQ\rangle \tag{247}$$
$$= \langle QR\rangle - \langle RQ\rangle \tag{248}$$
$$= \langle\psi| QR |\psi\rangle - \langle\psi| RQ |\psi\rangle \tag{249}$$
$$= \langle\psi| QR |\psi\rangle - \langle\psi| R^\dagger Q^\dagger |\psi\rangle \tag{250}$$
$$= \langle\psi| QR |\psi\rangle - \langle\psi| (QR)^\dagger |\psi\rangle \tag{251}$$
$$= \langle\psi| QR |\psi\rangle - (\langle\psi| QR |\psi\rangle)^\dagger \tag{252}$$

Since $\langle\psi| QR |\psi\rangle \in \mathbb{C}$, write the resulting complex number as $\langle\psi| QR |\psi\rangle = a + bi$, where $a, b \in \mathbb{R}$. Therefore,

$$\langle\psi| QR |\psi\rangle - (\langle\psi| QR |\psi\rangle)^\dagger = a + bi - (a+bi)* \tag{253}$$
$$= a + bi - (a - bi) \tag{254}$$
$$= 2bi \in [-2i, 2i] \tag{255}$$

Note that QR is unitary ($(QR)^\dagger QR = R^\dagger Q^\dagger QR = R^\dagger IR = I$). As a consequence, the inner product between vectors is preserved. Thus, from $\langle\psi|\psi\rangle$, it is possible to conclude that $(a+bi)(a+bi)* = a^2 + b^2 = 1$. As such, $-1 \leq b \leq 1$. Back to the mean calculus, and considering $\langle\langle\psi| ST |\psi\rangle = c + di\rangle$,

$$\langle 4I\rangle + \langle [Q,R] \otimes [S,T]\rangle = \langle\psi| 4I |\psi\rangle + 2bi \cdot 2di \tag{256}$$
$$= 4\langle\psi|\psi\rangle + 4bdi^2 \tag{257}$$
$$= 4 - 4bd \tag{258}$$

Where $-1 \leq bd \leq 1$. Therefore, the maximum value of $E(x^2)$ is 8 (whenever $bd = -1$). Since $E(x) \leq \sqrt{E(x^2)}$,

$$\langle Q \otimes S + R \otimes S + R \otimes T - Q \otimes T\rangle \leq \sqrt{4 - 4bd} \tag{259}$$
$$\leq 2\sqrt{2} \tag{260}$$

As requested.

5 Nielsen and Chuang - Chapter 04

5.1 Section 4.2

5.1.1 Bloch Vector

At this point, it is strongly recommended that the reader understands the contents of Section 3.1.1 and the sections mentioned therein. Also, in order to properly understand the Bloch Vector, refer back to Figure 1.3 in the book.

The given vector can be easily understood by interpreting a Bloch Sphere as two unit circles with an overlapping axis. For instance, the first circle is determined by the x and y axes (the "equatorial line circle"), while the second circle is determined by z and any appropriate axis in the "equatorial line" (this axis depends on the point's position).

The z position is the easiest one to understand. It is simply the projection of the point ($|\psi\rangle$) along the z axis ($\langle z|\psi\rangle$). Since it does not change depending on the azimuthal angle φ.

The positions described by both x and y axes depend on the projection of the point $|\psi\rangle$ on the "equatorial circle" (name it $|\psi_{xy}\rangle$). Consider the x axis, if $z = 0$, then the projection of $|\psi_{xy}\rangle$ along x would equal $cos(\varphi)$. Note that $x \neq cos(\varphi)$ otherwise. This happens because $|\psi_{xy}\rangle$ depends on $sin(\theta)$. Thus, $\langle x|\psi\rangle = cos(\varphi)sin(\theta)$. Analogously, $\langle y|\psi\rangle = sin(\varphi)sin(\theta)$.

In conclusion, the Bloch Vector is described by $(x, y, z) = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta)$.

5.1.2 Exercise 4.1

It is recommended to answer Exercise 2.11 beforehand. Additionally, the reader should attempt to understand the Bloch Sphere Equation. Throughout this exercise, the reader may constantly refer back to the Bloch Sphere Equation,

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi} sin\frac{\theta}{2}|1\rangle . \tag{261}$$

- The eigenvectors of X are $|+\rangle$ and $|-\rangle$;
  - $|+\rangle$;
    - $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Hence, $cos\frac{\theta}{2}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$. Thus, $\theta = \frac{\pi}{2}$;
    - $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Since $\theta = \frac{\pi}{2}$, $e^{i\varphi}sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}|1\rangle$. Thus, $e^{i\varphi} = 1$, and $\varphi = 0$;
    - Substituting the values of θ and φ in the Bloch Vector formula, $|+\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (1, 0, 0)$;
  - $|-\rangle$;
    - $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. Hence, $cos\frac{\theta}{2}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$. Thus, $\theta = \frac{\pi}{2}$;
    - $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. Since $\theta = \frac{\pi}{2}$, $e^{i\varphi}sin\frac{\pi}{4}|1\rangle = -\frac{1}{\sqrt{2}}|1\rangle$. Thus, $e^{i\varphi} = -1$, and $\varphi = \pi$;
    - Substituting the values of θ and φ in the Bloch Vector formula, $|-\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (-1, 0, 0)$;
- The eigenvectors of Y are $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-i|1\rangle}{\sqrt{2}}$;
  - $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$;
    - $cos\frac{\theta}{2}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$. Thus, $\theta = \frac{\pi}{2}$;
    - Since $\theta = \frac{\pi}{2}$, $e^{i\varphi}sin\frac{\pi}{4}|1\rangle = \frac{i}{\sqrt{2}}|1\rangle$. Thus, $e^{i\varphi} = i$, and $\varphi = \frac{\pi}{2}$;
    - Substituting the values of θ and φ in the Bloch Vector formula, $|+\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (0, 1, 0)$;
  - $\frac{|0\rangle-i|1\rangle}{\sqrt{2}}$;
    - $cos\frac{\theta}{2}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$. Thus, $\theta = \frac{\pi}{2}$;
    - Since $\theta = \frac{\pi}{2}$, $e^{i\varphi}sin\frac{\pi}{4}|1\rangle = -\frac{i}{\sqrt{2}}|1\rangle$. Thus, $e^{i\varphi} = -i$, and $\varphi = \frac{3\pi}{2}$;

  - ∗ Substituting the values of θ and φ in the Bloch Vector formula, $|+\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (0, -1, 0)$;

- The eigenvectors of Z are $|0\rangle$ and $|1\rangle$;
  - ∘ $|0\rangle$;
    - ∗ $cos\frac{\theta}{2}|0\rangle = 1|0\rangle$. Thus, $\theta = 0$;
    - ∗ Since $\theta = 0$, $sin(0) = 0$. And φ can assume any value in the $[0, 2\pi)$ range;
    - ∗ Substituting the values of θ and φ in the Bloch Vector formula, $|0\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (0, 0, 1)$
  - ∘ $|1\rangle$;
    - ∗ $cos\frac{\theta}{2}|0\rangle = 0|0\rangle$. Thus, $\theta = \pi$;
    - ∗ Thus, $|1\rangle = e^{i\varphi} sin\frac{\pi}{2}|1\rangle$. It is known that multiplying a state by any complex number does not change it. Hence, $e^{i\varphi} sin\frac{\pi}{2}|1\rangle = e^{-i\varphi}e^{i\varphi} sin\frac{\pi}{2}|1\rangle = sin\frac{\pi}{2}|1\rangle$. In conclusion, the value of φ is negligible, and can assume any value in the $[0, 2\pi)$ range;
    - ∗ Substituting the values of θ and φ in the Bloch Vector formula, $|1\rangle = (cos\varphi sin\theta, sin\varphi sin\theta, cos\theta) = (0, 0, -1)$.

5.1.3 Exercise 4.2

In Nielsen and Chuang's Section 2.1.8, operator functions are discussed. Thus, A has spectral decomposition, and

$$A^2 = \sum_{ab} a|a\rangle\langle a|\, b|b\rangle\langle b| \tag{262}$$
$$= \sum_{ab} ab|a\rangle\langle a|b\rangle\langle b| \tag{263}$$
$$= \sum_{ab} ab\langle a|b\rangle|a\rangle\langle b|\,. \tag{264}$$

Since $\{\ |a\rangle \mid \forall a\}$ form an orthonormal basis defined by the eigenspace - and $\{\ |b\rangle \mid \forall b\}$ describes the same basis - $\langle a|b\rangle = \delta_{ab}$ [6],

$$A^2 = \sum_{ab} ab\delta_{ab}|a\rangle\langle b| \tag{265}$$
$$= \sum_{a} a^2|a\rangle\langle a| \tag{266}$$
$$= I. \tag{267}$$

Due to the completeness relation $\sum_a |a\rangle\langle a| = I$, it is possible to conclude that $a = \pm 1$.

Compute the value of $exp(iAx)$ using the definition of operator functions Nielsen and Chuang's Section 2.1.8, and Euler's Formula.

$$exp(iAx) = \sum_a exp(iax)|a\rangle\langle a| \tag{268}$$
$$= \sum_a cos(ax)|a\rangle\langle a| + i\ sin(ax)|a\rangle\langle a|\,. \tag{269}$$

Recall that $a = \pm 1$. From trigonometry, it is known $cos(x) = cos(-x)$. Also, $sin(-x) = -sin(x)$. Thus,

$$exp(iAx) = \sum_a cos(x)|a\rangle\langle a| + i\ sin(x)a|a\rangle\langle a| \tag{270}$$
$$= cos(x)I + i\ sin(x)A. \tag{271}$$

Using this to verify Equations (4.4) to (4.6) is straightforward, since $X^2 = Y^2 = Z^2 = I$. For $R_x(\theta)$,

$$e^{-i\theta X/2} = cos(-\frac{\theta}{2})I + i\ sin(-\frac{\theta}{2})X \tag{272}$$

[6] δ_{ij} is defined in the paragraph that follows Nielsen and Chuang's Equation (2.16)

$$= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}X \tag{273}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} & 0 \\ 0 & cos\frac{\theta}{2} \end{bmatrix} - i \begin{bmatrix} 0 & sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & 0 \end{bmatrix} \tag{274}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} & -i\ sin\frac{\theta}{2} \\ -i\ sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}. \tag{275}$$

For $R_y(\theta)$,

$$e^{-i\theta Y/2} = cos(-\frac{\theta}{2})I + i\ sin(-\frac{\theta}{2})Y \tag{276}$$

$$= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}Y \tag{277}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} & 0 \\ 0 & cos\frac{\theta}{2} \end{bmatrix} - i \begin{bmatrix} 0 & -i\ sin\frac{\theta}{2} \\ i\ sin\frac{\theta}{2} & 0 \end{bmatrix} \tag{278}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} & -sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}. \tag{279}$$

For $R_z(\theta)$,

$$e^{-i\theta Z/2} = cos(-\frac{\theta}{2})I + i\ sin(-\frac{\theta}{2})Z \tag{280}$$

$$= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}Z \tag{281}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} & 0 \\ 0 & cos\frac{\theta}{2} \end{bmatrix} - i \begin{bmatrix} sin\frac{\theta}{2} & 0 \\ 0 & -sin\frac{\theta}{2} \end{bmatrix} \tag{282}$$

$$= \begin{bmatrix} cos\frac{\theta}{2} - i\ sin\frac{\theta}{2} & 0 \\ 0 & cos\frac{\theta}{2} + isin\frac{\theta}{2} \end{bmatrix} \tag{283}$$

$$= \begin{bmatrix} cos(-\frac{\theta}{2}) + i\ sin(-\frac{\theta}{2}) & 0 \\ 0 & cos\frac{\theta}{2} + isin\frac{\theta}{2} \end{bmatrix} \tag{284}$$

$$= \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \tag{285}$$

5.1.4 Exercise 4.3

Compute $\pi/4$,

$$R_z(\pi/4) = cos\frac{\pi}{8}I - sin\frac{\pi}{8}Z \tag{286}$$

$$= \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}. \tag{287}$$

Note that this almost matches T - refer back to Equation (4.3). Thus,

$$T = e^{i\pi/8}R_z(\pi/4). \tag{288}$$

5.1.5 Exercise 4.4

It is desired to write H in terms of a product of R_x, R_z, and $e^{i\varphi}$. It may be tempting to assign in the range $[0, \pi]$, but the requested solution would not be obtained.

First, find the corresponding rotation operator for H. This is possible since $H^2 = I$ (refer back to Exercise 4.2). Thus,

$$R_h(\theta) = e^{-i\theta H/2} \tag{289}$$

$$= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}H. \tag{290}$$

Note that

$$R_h(\pi) = -iH. \tag{291}$$

And, multiplying both sides by $e^{i\frac{\pi}{2}}$,

$$\begin{aligned} e^{i\frac{\pi}{2}} R_h(\pi) &= e^{i\frac{\pi}{2}} \cdot (-iH) && (292)\\ &= i(-iH) && (293)\\ &= H. && (294) \end{aligned}$$

Now, express R_h in terms of R_x, R_z, and $e^{i\psi}$, according to the equation $R_h(\theta) = R_x(\theta)R_z(\theta)e^{i\psi}$. Recall that $H = \frac{X+Z}{\sqrt{2}}$. Also, suppose $\psi = \theta\phi/2$ for some ϕ. Then,

$$\begin{aligned} R_h(\theta) &= R_x(\theta)R_z(\theta)e^{i\psi} && (295)\\ e^{-i\theta(\frac{X+Z}{\sqrt{2}})/2} &= e^{-i\theta X/2}\ e^{-i\theta(X+Z)/2}\ e^{i\theta\phi/2} && (296)\\ &= e^{-i\theta(X+Z-\phi)/2}. && (297) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{X+Z}{\sqrt{2}} &= X+Z-\phi && (298)\\ \phi &= X+Z-\frac{X+Z}{\sqrt{2}} && (299)\\ \phi &= \frac{\sqrt{2}-1}{\sqrt{2}}(X+Z). && (300) \end{aligned}$$

Henceforth, $\psi = \theta\frac{\sqrt{2}-1}{\sqrt{2}}(X+Z)/2$, and

$$R_h(\theta) = R_x(\theta)R_z(\theta)e^{i\theta\frac{\sqrt{2}-1}{\sqrt{2}}(X+Z)/2}. \tag{301}$$

Merging this result with Equations 292 to 294,

$$\begin{aligned} H &= R_h(\pi)e^{i\frac{\pi}{2}} && (302)\\ &= R_x(\pi)R_z(\pi)e^{i\pi\frac{\sqrt{2}-1}{\sqrt{2}}(X+Z)/2}e^{i\frac{\pi}{2}} && (303)\\ &= R_x(\pi)R_z(\pi)e^{i\pi\frac{\sqrt{2}-1}{\sqrt{2}}(X+Z)/2\ +\ i\frac{\pi}{2}}. && (304) \end{aligned}$$

Thus, the requested value for φ in $e^{i\varphi}$ is

$$\varphi = \pi\frac{\sqrt{2}-1}{\sqrt{2}}(X+Z)/2\ +\ \frac{\pi}{2}. \tag{305}$$

5.1.6 Exercise 4.5

A proof analogous to $(\hat{n}-\vec{\sigma})^2 = I$ can be found in the beginning of Problem 2.3's solution (until Equation 232). To prove $(\hat{n}-\vec{\sigma})^2 = I$, simply rename the variables.

Notwithstanding, proving Equation 4.8 is also straightforward. Just substitute $(\hat{n}\cdot\vec{\sigma})^2 = I$ for A in Equation 4.7, which was proved in Exercise 4.2. Thus, using Equation 2.116,

$$\begin{aligned} R_{\hat{n}} &\equiv exp(-i\theta\hat{n}\cdot\vec{\sigma}/2) && (306)\\ &= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}(\hat{n}\cdot\vec{\sigma}) && (307)\\ &= cos\frac{\theta}{2}I - i\ sin\frac{\theta}{2}(n_xX+n_yY+n_zZ). && (308) \end{aligned}$$

5.1.7 Exercise 4.6

The original idea was to use $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ to construct a basis to represent $R_{\hat{n}}(\theta)$. Similar to how $\{x,y,z\}$ is a basis for $\mathbb{R}^3$. However, I was neither able to handle the Math; nor to come up with a formal proof which would satisfy me. ***If the reader manages to solve it this way, please contact me!*** The solution hereby detailed is based on the idea proposed by Jalil Moqadam. Thus, I thank him.

The idea is to rotate a state $|\psi\rangle$ – with corresponding Bloch Vector $\vec{r} = [r_1, r_2, r_3]^\dagger$ – by α degrees about the $\hat{n}$ vector – $n = [n_1, n_2, n_3]^\dagger$. Thus obtaining another stade $|\varphi\rangle$ with corresponding Bloch Vector $\vec{r'} = [r'_1, r'_2, r'_3]^\dagger$. In other words,

$$R_{\hat{n}}(\alpha) |\psi\rangle = |\varphi\rangle \,. \tag{309}$$

However, the information of the Bloch Vectors is implicit in a state. On the other hand, if density operator notation is used, the information about the Bloch Vector is explicitly stated (refer to **TODO: EXERCISE 2.72**),

$$|\psi\rangle\langle\psi| = \rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}. \tag{310}$$

Thus, applying the rotation,

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = R_{\hat{n}}(\alpha) |\psi\rangle\langle\psi| R_{\hat{n}}(\alpha)^\dagger = |\varphi\rangle\langle\varphi| = \rho' = \frac{I + \vec{r'}\cdot\vec{\sigma}}{2}. \tag{311}$$

Hence, it is possible to compare the obtained $\vec{r'}$ with the result of

$$R\vec{r} = \vec{r'}, \tag{312}$$

where R is the rotation matrix from axis and angle.

By computing $R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger$,

$$\begin{aligned}
R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger &= \exp(-i\alpha\hat{n}\cdot\vec{\sigma})\frac{I + \vec{r}\cdot\vec{\sigma}}{2}\exp(-i\alpha\hat{n}\cdot\vec{\sigma})^\dagger && (313)\\
&= \exp(-i\alpha\hat{n}\cdot\vec{\sigma})\frac{I + \vec{r}\cdot\vec{\sigma}}{2}\exp(i\alpha\hat{n}\cdot\vec{\sigma}) && (314)\\
&= \frac{1}{2}\left(e^{-i\alpha\hat{n}\cdot\vec{\sigma}}Ie^{i\alpha\hat{n}\cdot\vec{\sigma}} + e^{-i\alpha\hat{n}\cdot\vec{\sigma}}\vec{r}\cdot\vec{\sigma}e^{i\alpha\hat{n}\cdot\vec{\sigma}}\right) && (315)\\
&= \frac{1}{2}\left(\ I + \left(\cos\left(\frac{\alpha}{2}\right)I - i\sin\left(\frac{\alpha}{2}\right)\hat{n}\cdot\vec{\sigma}\right)\vec{r}\cdot\vec{\sigma}\left(\cos\left(\frac{\alpha}{2}\right)I + i\sin\left(\frac{\alpha}{2}\right)\hat{n}\cdot\vec{\sigma}\right)\ \right) && (316)\\
&= \frac{1}{2}(\ I + \cos^2\frac{\alpha}{2}I\vec{r}\cdot\vec{\sigma}I + i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}I\vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma} - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}I -\\
&\quad i^2\sin^2\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma}\ ) && (317)\\
&= \frac{1}{2}(\ I + \cos^2\frac{\alpha}{2}\vec{r}\cdot\vec{\sigma} + i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma} - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma} +\\
&\quad \sin^2\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma}\ ) && (318)
\end{aligned}$$

is obtained.

It is desirable to isolate the common σ_i factor, i.e. to rewrite the previous equation such that

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} = \frac{I + \sum_{i=1}^{3} r_i\sigma_i}{2}. \tag{319}$$

Thus, for the sake of clarity, Equation 318 will be computed separately for each part as follows

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{I + a + b + c}{2}. \tag{320}$$

Where

$$a = \cos^2\frac{\alpha}{2}\vec{r}\cdot\vec{\sigma}, \tag{321}$$

$$b = i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma} - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}, \tag{322}$$

and

$$c = \sin^2\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma}. \tag{323}$$

a) Equation 321 is already in the desired format, since

$$a = \cos^2\left(\frac{\alpha}{2}\right)\vec{r}\cdot\vec{\sigma} = \cos^2\left(\frac{\alpha}{2}\right)\sum_{i=1}^{3} r_i\sigma_i. \tag{324}$$

b) Equation 322 can be rewritten as follows

$$\begin{aligned} b &= i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma} - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma} &(325)\\ &= i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} r_i\sigma_i\ n_j\sigma_j - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} n_j\sigma_j\ r_i\sigma_i &(326)\\ &= i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} r_i n_j\ \sigma_i\sigma_j - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} n_j r_i\ \sigma_j\sigma_i. &(327)\end{aligned}$$

Whenever $j = i$,

$$i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3} r_i n_i\ \sigma_i\sigma_i - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sum_{j=1}^{3} n_i r_i\ \sigma_i\sigma_i = 0. \tag{328}$$

Hence,

$$b = i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j\neq i} r_i n_j\ \sigma_i\sigma_j - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sum_{j=1}^{3}\sum_{j\neq i} n_j r_i\ \sigma_j\sigma_i. \tag{329}$$

Since $\sigma_j\sigma_i = -\sigma_i\sigma_j$ if $i \neq j$ – i.e., $YX = -XY$, $ZX = -XZ$, and $ZY = -YZ$ –,

$$\begin{aligned} b &= i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j\neq i} r_i n_j\ \sigma_i\sigma_j + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sum_{j=1}^{3}\sum_{j\neq i} n_j r_i\ \sigma_i\sigma_j &(330)\\ &= 2i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j\neq i} r_i n_j\ \sigma_i\sigma_j &(331)\\ &= 2i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}\sum_{j> i}(r_i n_j - r_j n_i)\sigma_i\sigma_j. &(332)\end{aligned}$$

Nevertheless, it is possible to write the product of two distinct Pauli Matrices in terms of another Pauli Matrix. Namely,

- $\sigma_1\sigma_2 = i\sigma_3$ and $\sigma_2\sigma_1 = -i\sigma_3$;
- $\sigma_1\sigma_3 = -i\sigma_2$ and $\sigma_3\sigma_1 = i\sigma_2$;
- $\sigma_2\sigma_3 = i\sigma_1$ and $\sigma_3\sigma_2 = -i\sigma_1$.

Thus, it is possible to rewrite b with the aid of the function

$$f(x) = (x \bmod 3) + 1 \tag{333}$$

as follows

$$b = -2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sum_{i=1}^{3}(r_{f(i)}n_{f(i+1)} - r_{f(i+1)}n_{f(i)})\sigma_i, \tag{334}$$

which is in the desired format.

c) Equation 323 can be rewritten as follows

$$\begin{aligned} c &= \sin^2\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\ \vec{r}\cdot\vec{\sigma}\ \hat{n}\cdot\vec{\sigma} &(335)\\ &= \sin^2\frac{\alpha}{2}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} n_i r_j n_k\ \sigma_j\sigma_k\sigma_k\right) &(336)\end{aligned}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} n_i \sigma_i \left(\sum_{j=1}^{3} \sum_{k=1}^{3} r_j n_k \ \sigma_j \sigma_k \right) \right) \tag{337}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} n_i \sigma_i \left(\sum_{j=1}^{3} r_j n_j I + \sum_{j=1}^{3} \sum_{k \neq j} r_j n_k \ \sigma_j \sigma_k \right) \right) \tag{338}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \sum_{j=1}^{3} n_i r_j n_j \sigma_i + \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k \neq j} n_i r_j n_k \sigma_i \sigma_j \sigma_k \right) \tag{339}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} n_i r_j n_j \sigma_i + \sum_{k \neq i} n_i r_i n_k \ I \sigma_k + \sum_{j \neq i} \sum_{k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k \right) \right) \tag{340}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} n_i r_j n_j \sigma_i + \sum_{j \neq i} n_i r_i n_j \ \sigma_j + \sum_{j \neq i} \sum_{k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k \right) \right) \tag{341}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} \left(n_i n_j (r_j \sigma_i + r_i \sigma_j) + \sum_{k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k \right) \right) \right) \tag{342}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} \left(n_i n_j (r_j \sigma_i + r_i \sigma_j) - n_i^2 r_j \sigma_j + \sum_{k \neq j, k \neq i} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k \right) \right) \right). \tag{343}$$

The last summation equals 0. To see this, note that,

$$\sum_{i=1}^{3} \sum_{j \neq i} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k = \sum_{j=1}^{3} \sum_{i \neq j} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k \tag{344}$$

$$= - \sum_{j=1}^{3} \sum_{i \neq j} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_i \sigma_k \sigma_j \tag{345}$$

$$= \sum_{j=1}^{3} \sum_{i \neq j} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_k \sigma_i \sigma_j \tag{346}$$

$$= - \sum_{j=1}^{3} \sum_{i \neq j} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_k \sigma_j \sigma_i . \tag{347}$$

Since for a fixed value of j there are only two possible permutations,

$$\sum_{j=1}^{3} \sum_{i \neq j} \sum_{k \neq i, k \neq j} n_i r_j n_k \ \sigma_i \sigma_j \sigma_k = \sum_{j=1}^{3} (n_i r_j n_k \ \sigma_i \sigma_j \sigma_k + n_k r_j n_i \ \sigma_k \sigma_j \sigma_i) \tag{348}$$

$$= \sum_{j=1}^{3} (n_i r_j n_k \ \sigma_i \sigma_j \sigma_k - n_k r_j n_i \ \sigma_i \sigma_j \sigma_k) \tag{349}$$

$$= \sum_{j=1}^{3} 0 \tag{350}$$

$$= 0. \tag{351}$$

Hence,

$$c = \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} \left(n_i n_j (r_j \sigma_i + r_i \sigma_j) - n_i^2 r_j \sigma_j \right) \right) \right) \tag{352}$$

$$= \sin^2 \frac{\alpha}{2} \left(\sum_{i=1}^{3} \left(n_i^2 r_i \sigma_i + \sum_{j \neq i} n_i n_j r_j \sigma_i \right) + \sum_{i=1}^{3} \sum_{j \neq i} \left(n_i n_j r_i \sigma_j - n_i^2 r_j \sigma_j \right) \right). \tag{353}$$

Since addition and multiplication are commutative, the previous equation can be rearranged as follows (basically, what is being done here is similar to rewriting the set of tuples $\{ (1,2), (1,3), (2,1), (2,3), (3,1), (3,2) \}$ as

$\{(2,1),(3,1),(1,2),(3,2),(1,3),(2,3)\}$, which is the same set),

$$c = \sin^2\frac{\alpha}{2}\left(\sum_{i=1}^{3}\left(n_i^2 r_i\sigma_i + \sum_{j\neq i} n_i n_j r_j \sigma_i\right) + \sum_{i=1}^{3}\sum_{j\neq i}\left(n_j n_i r_j \sigma_i - n_j^2 r_i \sigma_i\right)\right) \tag{354}$$

$$= \sin^2\frac{\alpha}{2}\left(\sum_{i=1}^{3}\left(n_i^2 r_i + \sum_{j\neq i}\left(n_i n_j r_j + n_j n_i r_j - n_j^2 r_i\right)\right)\sigma_i\right) \tag{355}$$

$$= \sin^2\frac{\alpha}{2}\left(\sum_{i=1}^{3}\left(n_i^2 r_i + \sum_{j\neq i}\left(2 n_j n_i r_j - n_j^2 r_i\right)\right)\sigma_i\right), \tag{356}$$

which is in the desired format.

Concatenating these results, Equation 318 can be rewritten as follows,

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{1}{2}\Bigg(I + \sum_{i=1}^{3}\Bigg(\Big(r_i\cos^2\frac{\alpha}{2} - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(r_{f(i)}n_{f(i+1)} - r_{f(i+1)}n_{f(i)}) + \sin^2\frac{\alpha}{2}\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big)\Big)\,\sigma_i\Bigg)\Bigg). \tag{357}$$

Using the trigonometric identities

- $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$;
- $\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$;
- $\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \frac{\sin\alpha}{2}$;

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{1}{2}\Bigg(I + \sum_{i=1}^{3}\Bigg(\Big(\frac{r_i}{2}(1+\cos\alpha) - \sin\alpha(r_{f(i)}n_{f(i+1)} - r_{f(i+1)}n_{f(i)}) + \frac{1}{2}(1-\cos\alpha)\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big)\Big)\,\sigma_i\Bigg)\Bigg). \tag{358}$$

Let

$$d = \sin\alpha(r_{f(i)}n_{f(i+1)} - r_{f(i+1)}n_{f(i)}); \tag{359}$$

Then,

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{1}{2}\left(I + \sum_{i=1}^{3}\left(\left(\frac{r_i}{2}(1+\cos\alpha) + \frac{1}{2}(1-\cos\alpha)\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big) - d\right)\sigma_i\right)\right) \tag{360}$$

$$= \frac{1}{2}\left(I + \sum_{i=1}^{3}\left(\frac{1}{2}\Big(r_i(1+\cos\alpha) + (1-\cos\alpha)\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big)\Big)\sigma_i - d\sigma_i\right)\right). \tag{361}$$

Let

$$\hat{d} = \frac{1}{2}\Big(r_i(1+\cos\alpha) + (1-\cos\alpha)\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big)\Big). \tag{362}$$

Then,

$$R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{1}{2}\left(I + \sum_{i=1}^{3}(\hat{d}\sigma_i - d\sigma_i)\right). \tag{363}$$

Computing $\hat{d}$,

$$\hat{d} = \frac{1}{2}\Big(r_i(1+\cos\alpha) + (1-\cos\alpha)\big(n_i^2 r_i + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2)\big)\Big) \tag{364}$$

$$= \frac{1}{2}\Big(n_i^2 r_i(1-\cos\alpha) + r_i + r_i\cos\alpha + \sum_{j\neq i}(2r_j n_i n_j - r_i n_j^2 - 2\cos(\alpha) r_j n_i n_j + \cos(\alpha) r_i n_j^2)\Big) \tag{365}$$

$$= \frac{1}{2}\Big(n_i^2 r_i(1-\cos\alpha) + r_i\left(1 - \sum_{j\neq i} n_j^2\right) + r_i\cos\alpha\left(1 + \sum_{j\neq i} n_j^2\right) + \sum_{j\neq i}(2r_j n_i n_j - 2\cos(\alpha) r_j n_i n_j)\Big). \tag{366}$$

Since $\hat{n}$ is a real unit vector, $1 - n_i^2 = \sum_{j\neq i} n_j^2$. Hence,

$$\hat{d} = \frac{1}{2}\Big(n_i^2 r_i(1-\cos\alpha) + r_i n_i^2 + r_i\cos\alpha\,(2-n_i^2) + \sum_{j\neq i} r_j n_i n_j(2-2\cos\alpha)\Big) \tag{367}$$

$$= \frac{1}{2}\Big(n_i^2 r_i(2-\cos\alpha) + 2r_i\cos\alpha - r_i n_i^2\cos\alpha + \sum_{j\neq i} r_j n_i n_j(2-2\cos\alpha)\Big) \tag{368}$$

$$= \frac{1}{2}\Big(n_i^2 r_i(2-2\cos\alpha) + 2r_i\cos\alpha + \sum_{j\neq i} r_j n_i n_j(2-2\cos\alpha)\Big) \tag{369}$$

$$= n_i^2 r_i(1-\cos\alpha) + r_i\cos\alpha + \sum_{j\neq i} r_j n_i n_j(1-\cos\alpha). \tag{370}$$

In summary,

$$\begin{aligned} R_{\hat{n}}(\alpha)\rho R_{\hat{n}}(\alpha)^\dagger = \frac{1}{2}\Bigg(I + \sum_{i=1}^{3}\bigg(\Big(n_i^2 r_i(1-\cos\alpha) + r_i\cos\alpha + \sum_{j\neq i} r_j n_i n_j\big(1-\cos(\alpha)\big) -\\ \sin\alpha(r_{f(i)}n_{f(i+1)} - r_{f(i+1)}n_{f(i)})\Big)\sigma_i\bigg)\Bigg) \end{aligned} \tag{371}$$

$$\begin{aligned} = \frac{1}{2}\Bigg(I + \sum_{i=1}^{3}\bigg(\Big(n_i^2 r_i(1-\cos\alpha) + r_i\cos\alpha + \sum_{j\neq i} r_j n_i n_j\big(1-\cos(\alpha)\big) -\\ \sin\alpha(r_{f(i+1)}n_{f(i)} - r_{f(i)}n_{f(i+1)})\Big)\sigma_i\bigg)\Bigg). \end{aligned} \tag{372}$$

In conclusion, the obtained Bloch Vector $r' = (r_1', r_2', r_2')^\dagger$ is described – for each r_i', $1 \leq i \leq 3$ – as follows,

$$r_i' = n_i^2 r_i(1-\cos\alpha) + r_i\cos\alpha + \sum_{j\neq i} r_j n_i n_j\big(1-\cos\alpha\big) + \sin\alpha(r_{f(i+1)}n_{f(i)} - r_{f(i)}n_{f(i+1)}). \tag{373}$$

To assert this result, compute Equation 312 and verify that the results match. That is, verify that each r_i' computed using Equation 373 matches the values r_1', r_2' and r_3' obtained through

$$\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} n_1^2(1-\cos\alpha)+\cos\alpha & n_1 n_2(1-\cos\alpha) - n_3\sin\alpha & n_1 n_3(1-\cos\alpha) + n_2\sin\alpha \\ n_1 n_2(1-\cos\alpha) + n_3\sin\alpha & n_2^2(1-\cos\alpha)+\cos\alpha & n_2 n_3(1-\cos\alpha) - n_1\sin\alpha \\ n_1 n_3(1-\cos\alpha) - n_2\sin\alpha & n_2 n_3(1-\cos\alpha) + n_1\sin\alpha & n_3^2(1-\cos\alpha)+\cos\alpha \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{374}$$

As the book states, the "mysterious factor 2" appears because of the representation of a state in the Bloch Sphere (refer to Section 3.1.1).

5.2 Exercise 4.7

It is fairly simple to show that $XYX = -Y$.

$$XYX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{375}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{376}$$

$$= \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{377}$$

$$= -Y. \tag{378}$$

Hence, computing $XR_y(\theta)X$ and using the results of Exercise 2.35 (Section 4.3.1) and the definition of $R_y(\theta)$,

$$XR_y(\theta)X = X\left(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y\right)X \tag{379}$$

$$= \cos\frac{\theta}{2}XIX - i\sin\frac{\theta}{2}XYX \tag{380}$$

$$= \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Y \tag{381}$$

$$= e^{i\theta Y/2} \tag{382}$$

$$= R_y(-\theta). \tag{383}$$

as requested.

6 Nielsen and Chuang - Chapter 06

6.1 Section 6.1

6.1.1 Exercise 6.1

This Exercise refers to Equation (6.5), not to Equation (6.3), which describes the oracle's action. Recall that the δ_{ij} notation means

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{384}$$

Thus, Equation (6.5) essentially means that if $x = 0$, then $|0\rangle \to |0\rangle$, else, $x \neq 0$ and $|x\rangle \to -|x\rangle$. This is exactly the behaviour of $2|0\rangle\langle 0| - I$, which is described in the following paragraphs.

First, the operator is easily verifiable to be unitary,

$$\begin{aligned} (2|0\rangle\langle 0| - I)^\dagger (2|0\rangle\langle 0| - I) &= (2|0\rangle\langle 0| - I)^2 && (385)\\ &= 4|0\rangle\langle 0|0\rangle\langle 0| - 2\cdot 2I|0\rangle\langle 0| + I^2 && (386)\\ &= 4|0\rangle\langle 0| - 4|0\rangle\langle 0| + I && (387)\\ &= I. && (388) \end{aligned}$$

Show that $(2|0\rangle\langle 0| - I)|0\rangle = |0\rangle$,

$$\begin{aligned} (2|0\rangle\langle 0| - I)|0\rangle &= 2|0\rangle\langle 0|0\rangle - I|0\rangle && (389)\\ &= 2|0\rangle - |0\rangle && (390)\\ &= |0\rangle. && (391) \end{aligned}$$

For any other state [7] $|x\rangle$, the phase is shifted,

$$\begin{aligned} (2|0\rangle\langle 0| - I)|x\rangle &= 2|0\rangle\langle 0|x\rangle - I|x\rangle && (392)\\ &= 0|0\rangle - |x\rangle && (393)\\ &= -|x\rangle. && (394) \end{aligned}$$

These actions can be easily inferred by analysing $(2|0\rangle\langle 0| - I)$'s matricial form,

$$\begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}. \tag{395}$$

6.1.2 Equation 6.6

In case it is not clear how this Equation was obtained, the step by step solution is as follows,

$$\begin{aligned} H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} &= H^{\otimes n}(2|0\rangle\langle 0| H^{\otimes n} - H^{\otimes n}) && (396)\\ &= 2H^{\otimes n}|0\rangle\langle 0| H^{\otimes n} - H^{\otimes n}H^{\otimes n} && (397)\\ &= 2|+\rangle^{\otimes n}\langle +|^{\otimes n} - I. && (398) \end{aligned}$$

Then, using $|\psi\rangle$ as defined in Equation 6.4,

$$2|+\rangle^{\otimes n}\langle +|^{\otimes n} - I = 2|\psi\rangle\langle\psi| - I. \tag{399}$$

Thus obtaining the desired result.

[7] Recall that Grover's Algorithm operates on the database's indices, and that they form an orthonormal basis, i.e. $\langle i|j\rangle = \delta_{ij}$.

6.1.3 Exercise 6.2

Applying the operation to the general state,

$$(2\left|\psi\right\rangle\left\langle\psi\right| - I)\sum_k \alpha_k \left|k\right\rangle = \sum_k 2\alpha_k \left|\psi\right\rangle\left\langle\psi|k\right\rangle - \alpha_k \left|k\right\rangle . \tag{400}$$

Using Equation (6.4), and the definition of δ_{ij},

$$\begin{aligned}\sum_k 2\alpha_k \left|\psi\right\rangle\left\langle\psi|k\right\rangle - \alpha_k\left|k\right\rangle &= \sum_k 2\alpha_k \frac{1}{N}\sum_{ij}\left|i\right\rangle\left\langle j|k\right\rangle - \alpha_k\left|k\right\rangle &(401)\\ &= \sum_k 2\frac{\alpha_k}{N}\sum_{ij}\left|i\right\rangle \delta_{jk} - \alpha_k\left|k\right\rangle &(402)\\ &= \sum_k 2\frac{\alpha_k}{N}\sum_i \left|i\right\rangle - \alpha_k\left|k\right\rangle . &(403)\end{aligned}$$

Note that $\sum_k \alpha_k/N$ is a constant (written as $\langle\alpha\rangle$). Thus,

$$\sum_k 2\frac{\alpha_k}{N}\sum_i\left|i\right\rangle - \alpha_k\left|k\right\rangle = 2\langle\alpha\rangle\sum_i\left|i\right\rangle - \sum_k\alpha_k\left|k\right\rangle . \tag{404}$$

Since $\left|k\right\rangle$ and $\left|i\right\rangle$ correspond to the same orthonormal basis, it is possible to rename and rearrange

$$\begin{aligned}2\langle\alpha\rangle\sum_i\left|i\right\rangle - \sum_k\alpha_k\left|k\right\rangle &= \sum_k 2\langle\alpha\rangle\left|k\right\rangle - \sum_k\alpha_k\left|k\right\rangle &(405)\\ &= \sum_k\left[2\langle\alpha\rangle - \alpha_k\right]\left|k\right\rangle . &(406)\end{aligned}$$

Thus obtaining the desired answer.

6.1.4 Notes About Grover's Algorithm

At a first glance, it may appear that Grover's Algorithm cannot find a value if its index is 0. This erroneous thought may come to one's mind due to the phase shift step; when $\left|0\right\rangle$ is left unchanged. To show that Grover's Algorithm works perfectly, two scenarios will be considered: the searched value is *not* in the "database"; the searched value has index 0 in the "database". It can be seen that, for both scenarios, the obtained result is different.

Value not found. To leave $\left|0\right\rangle$ unchanged is important in case the searched value is not in the "database". To illustrate this scenario, let the Grover iteration be denoted by

$$H^{\otimes n}P_h H^{\otimes n} O\left|\varphi\right\rangle , \tag{407}$$

where P_h is the phase shift, O is the oracle. Let $\left|\varphi\right\rangle = \left|\psi\right\rangle$, the state described in Equation 6.4 – immediately before the first Grover Iteration. Thus,

$$\begin{aligned}H^{\otimes n}P_h H^{\otimes n} O\left|\psi\right\rangle &= H^{\otimes n}P_h H^{\otimes n}\left|\psi\right\rangle &(408)\\ &= H^{\otimes n}P_h\left|0\right\rangle^{\otimes n} &(409)\\ &= H^{\otimes n}\left|0\right\rangle^{\otimes n} &(410)\\ &= \left|\psi\right\rangle . &(411)\end{aligned}$$

Therefore, it is clear that after $\sqrt{n}$ Grover iterations, the initial state $\left|\psi\right\rangle$ will remain unchanged if the desired value is not in the "database".

Nevertheless, note that the obtained result is the same if the Grover iteration was denoted with the aid of Equation 6.6,

$$\begin{aligned}(2\left|\psi\right\rangle\left\langle\psi\right| - I)O\left|\psi\right\rangle &= (2\left|\psi\right\rangle\left\langle\psi\right| - I)\left|\psi\right\rangle &(412)\\ &= 2\left|\psi\right\rangle\left\langle\psi|\psi\right\rangle - \left|\psi\right\rangle &(413)\\ &= \left|\psi\right\rangle &(414)\end{aligned}$$

Value in index 0. If the desired value is in the "database", and in the first index, i.e. 0, the Algorithm will also work perfectly. In this scenario, the first Grover iteration will perform the following action,

$$H^{\otimes n} P_h H^{\otimes n} O |\psi\rangle = H^{\otimes n} P_h H^{\otimes n} \left(-\frac{2|0\rangle^{\otimes n}}{\sqrt{2}^n} + |\psi\rangle \right) \tag{415}$$

$$= H^{\otimes n} P_h \left(-\frac{2|\psi\rangle}{\sqrt{2}^n} + |0\rangle^{\otimes n} \right) \tag{416}$$

$$= H^{\otimes n} \left(\frac{2}{\sqrt{2}^n} \left(|\psi\rangle - \frac{2}{\sqrt{2}^n} |0\rangle^{\otimes n} \right) + |0\rangle^{\otimes n} \right) \tag{417}$$

$$= H^{\otimes n} \left(\frac{2}{\sqrt{2}^n} |\psi\rangle - \frac{4}{2^n} |0\rangle^{\otimes n} + |0\rangle^{\otimes n} \right) \tag{418}$$

$$= H^{\otimes n} \left(\frac{1}{\sqrt{2}^{n-2}} |\psi\rangle + \frac{2^{n-2}-1}{2^{n-2}} |0\rangle^{\otimes n} \right) \tag{419}$$

$$= \frac{1}{\sqrt{2}^{n-2}} |0\rangle^{\otimes n} + \frac{2^{n-2}-1}{2^{n-2}} |\psi\rangle . \tag{420}$$

The same result would be obtained if Equation 6.6 was used to denote part of the Grover iteration,

$$(2|\psi\rangle\langle\psi| - I) O |\psi\rangle = (2|\psi\rangle\langle\psi| - I) \left(-\frac{2|0\rangle^{\otimes n}}{\sqrt{2}^n} + |\psi\rangle \right) \tag{421}$$

$$= -\frac{4|\psi\rangle\langle\psi|0\rangle^{\otimes n}}{\sqrt{2}^n} + 2|\psi\rangle + \frac{2|0\rangle^{\otimes n}}{\sqrt{2}^n} - |\psi\rangle \tag{422}$$

$$= -\frac{4}{2^n} |\psi\rangle + |\psi\rangle + \frac{2|0\rangle^{\otimes n}}{\sqrt{2}^n} \tag{423}$$

$$= -\frac{1}{2^{n-2}} |\psi\rangle + |\psi\rangle + \frac{1}{\sqrt{2}^{n-2}} |0\rangle^{\otimes n} \tag{424}$$

$$= \frac{2^{n-2}-1}{2^{n-2}} |\psi\rangle + \frac{1}{\sqrt{2}^{n-2}} |0\rangle^{\otimes n} . \tag{425}$$

Conclusion. Note that the obtained result for both scenarios is different. In the first, the original state is left unchanged. In the second, the original state is changed; though the obtained result may seem odd. In order to properly comprehend the result in the latter scenario, it may be useful to understand the geometric interpretation of the Grover iteration.

TODO: NOTES ABOUT THE GEOMETRIC INTERPRETATION

7 Noson - Chapter 04

7.1 Section 4.1

7.1.1 Equation (4.5)

I would like to thank Victor Santos de Souza for reviewing this section and helping me to keep on the rails of the mathematical and physical formalisms.

The value $|c_i|^2$ is divided by $|\,|\psi\rangle\,|^2$ because there is no assumption that $|\psi\rangle$ is unitary. This division guarantees that the probabilities sum up to 1 ($\sum_i |c_i|^2 = 1$).

More interesting, from this equation it is possible to deduce that by multiplying a state $|\psi\rangle = \sum_i c_i |x_i\rangle$ by any complex number $z \neq 0$ does not change it because the measurement probabilities proportion is maintained.

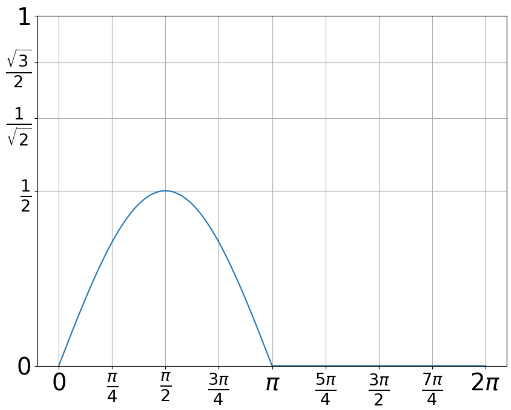
To conclude this, assume that $|\psi\rangle = \sum_i c_i |x_i\rangle$, and $|\varphi\rangle = z|\psi\rangle = \sum_i d_i |x_i\rangle = \sum_i c_i |y_i\rangle$, where $\forall i(c_i, d_i, z \in \mathbb{C} \wedge d_i = zc_i \wedge |y_i\rangle = z|x_i\rangle)$ - $\wedge$ denotes the logical and. Then prove that $\forall i, p(y_i) = p(x_i)$,

$$p(y_i) = \frac{|d_i|^2}{|\,|\varphi\rangle\,|^2} = \frac{|d_i|^2}{\sum_j |d_j|^2} = \frac{|zc_i|^2}{\sum_j |zc_j|^2} \tag{426}$$

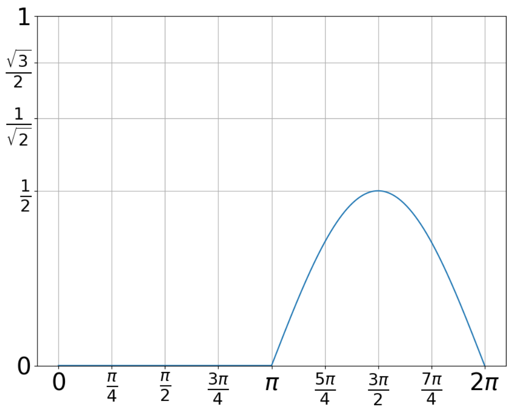
$$= \frac{|z|^2|c_i|^2}{\sum_j |z|^2|c_j|^2} = \frac{|z|^2|c_i|^2}{|z|^2\sum_j |c_j|^2} \tag{427}$$

$$= \frac{|c_i|^2}{\sum_j |c_j|^2} = p(x_i). \tag{428}$$

Due to the particle-wave duality, it may be interesting to visualise this result in a graph. Let a state be interpreted as a wave until the end of this section. Therefore, there exists a corresponding graph to the wave. Suppose $|\psi\rangle = \sum_i c_i |x_i\rangle$, where $c_i \in \mathbb{C}$ and $|x_i\rangle \in \mathbb{C}^n$ form an orthonnormal basis. It is known that a linear product can be defined as a function from two vectors to a complex number ($\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$). Therefore, it is possible to define $\psi(x)$ as a function that calculates $\langle x|\psi\rangle$ (the projection of $|\psi\rangle$ along $|x\rangle$) where $|x\rangle \in \{\ |\,x_i\rangle\,\}$). Analogously, for $|\varphi\rangle = \sum_i c_i |x_i\rangle$, define $\varphi(x) = \langle x|\varphi\rangle$. Similarly, it is also admissible to image $\psi(x)$ and $\varphi(x)$ as a functions that map the $|x\rangle$ domain to complex numbers ($\mathbb{C}^n \to \mathbb{C}$). It is interesting to interpret the linear product as a function because it is possible to calculate probabilites by using integration, similarly to what is done in statistics ($\langle\psi|\varphi\rangle = \int dx\, \langle\psi|x\rangle\,\langle x|\varphi\rangle$).



(a) Wave function graph of example state $|\psi\rangle$



(b) Wave function graph of example state $|\varphi\rangle$

Figure 1: Illustrative examples of wave functions graphs.

As an example, consider the given graphs. [8] Assume that $\psi(x)$ can also be written as

$$\psi(x) = \begin{cases} \frac{sin(x)}{2} & , x \in [0, \pi] \\ 0 & , \text{otherwise} \end{cases}. \tag{429}$$

[8]The graphs are *merely illustrative*.

The graph of $\psi(x)$ is illustrated in Figure 1a. Figure 1b illustrates the function $\varphi(x)$ assumed as

$$\varphi(x) = \begin{cases} \frac{sin(x-\pi)}{2} & , x \in (\pi, 2\pi] \\ 0 & , \text{otherwise} \end{cases}. \tag{430}$$

Additionally, let $\psi^*(x)$ denote the complex conjugate of $\psi(x)$, that is, $\psi^*(x) = \langle\psi|x\rangle$. In the given example, $\psi^*(x) = \psi(x)$ an $\varphi^*(x) = \varphi(x)$. Note that $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal.

$$\langle\psi|\varphi\rangle = \int dx \, \langle\psi|x\rangle \, \langle x|\varphi\rangle \tag{431}$$

$$= \int \psi^*(x)\varphi(x) \; dx \tag{432}$$

$$= \int \psi(x)\varphi(x) \; dx \tag{433}$$

$$= \int 0 \; dx \tag{434}$$

$$= 0 \tag{435}$$

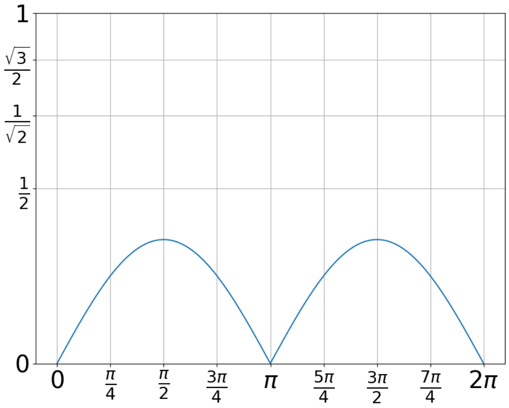


Figure 2: Wave function graph of example state $|\mu\rangle$

Therefore, it is possible to write another state $|\mu\rangle$ as a linear compination of $|\psi\rangle$ and $|\varphi\rangle$. The camel hump [9] graph in Figure 2 represents the superposition state $|\mu\rangle = \frac{1}{\sqrt{2}}|\psi\rangle + \frac{1}{\sqrt{2}}|\varphi\rangle$.

Analogously to Equation (4.5), the probability of obtaining state $|\psi\rangle$ after measuring $|\mu\rangle$ can be calculated by

$$p(\psi) = \frac{|c_i|^2}{\sum_j |c_j|^2} \tag{436}$$

$$= \frac{|\langle\psi|\mu\rangle|^2}{\langle\mu|\mu\rangle} \tag{437}$$

$$= \frac{|\int dx \, \langle\psi|x\rangle \, \langle x|\mu\rangle|^2}{\int dx \, \langle\mu|x\rangle \, \langle x|\mu\rangle} \tag{438}$$

$$= \frac{|\int \psi(x)\mu(x) \; dx|^2}{\int \mu(x)^2 \; dx} \tag{439}$$

$$= \frac{|\int_0^\pi \psi(x)\mu(x) \; dx + \int_\pi^{2\pi} \psi(x)\mu(x) \; dx|^2}{1^2} \tag{440}$$

$$= \left|\frac{1}{\sqrt{2}} + 0\right|^2 \tag{441}$$

$$= \frac{1}{2}. \tag{442}$$

Which matches $|\frac{1}{\sqrt{2}}|^2$, as defined by the Quantum Mechanics postulates.

[9] There is a good reason to mention camels.

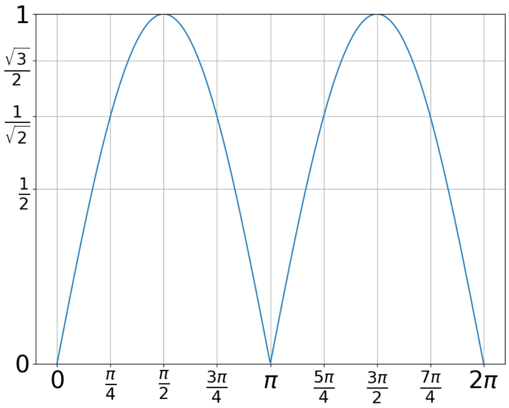


Figure 3: Graph of state $|\mu\rangle$ multiplied by $2\sqrt{2}$. The amplitudes of $|\mu\rangle$ are increased, but the proportion is maintained.

If $|\mu\rangle$ is multiplied by a complex number, $2\sqrt{2}$ for example, then the graph Figure 3 is obtained. The "new" $|\mu'\rangle$ is now described by $|\mu'\rangle = 2\sqrt{2}\,|\mu\rangle = 2\,|\psi\rangle + 2\,|\varphi\rangle$. However, by calculating $p(\psi)$, the previous result is also obtained $(\frac{1}{2})$,

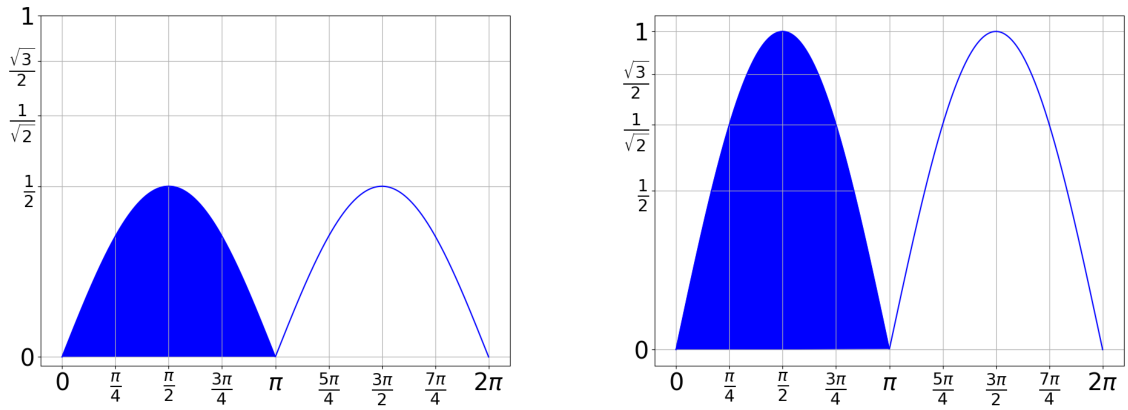
$$p(\psi) = = \frac{|\langle\psi|\mu'\rangle|^2}{\langle\mu'|\mu'\rangle} \tag{443}$$

$$= \frac{|2\sqrt{2}\,\langle\psi|\mu\rangle\,|^2}{((2\sqrt{2})^*\,\langle\mu|)\,(2\sqrt{2}\,|\mu\rangle)} \tag{444}$$

$$= \frac{|2\sqrt{2}|^2|\,\langle\psi|\mu\rangle\,|^2}{|2\sqrt{2}|^2\,\langle\mu|\mu\rangle} \tag{445}$$

$$= \frac{|\int dx\,\langle\psi|x\rangle\,\langle x|\mu\rangle\,|^2}{\int dx\,\langle\mu|x\rangle\,\langle x|\mu\rangle} \tag{446}$$

$$= \frac{1}{2}. \tag{447}$$



(a) The probability of $|\mu\rangle$ collapsing to $|\psi\rangle$ is highlighted in the blue area.

(b) The probability of $2\sqrt{2}\,|\mu\rangle$ collapsing to $2\sqrt{2}\,|\psi\rangle$ is highlighted in the blue area.

Figure 4: Both images highlight the part of $|\mu\rangle$ that corresponds to $|\psi\rangle$. Note that the proportion between the highlighted and non-highlighted areas is preserved, even if $|\mu\rangle$ is multiplied by any complex constant. Figures 4a and 4b illustrate the wave function of $|\mu\rangle$ and $2\sqrt{2}\,|\mu\rangle$, respectively.

By comparing these equations, it is possible to conclude that the probabilities are *proportional* to graph area. This result can be visualised in Figure 4 It is more accurate, however, to assert that the probability of measuring $|i\rangle$ is *proportional* to its projection on $|\psi\rangle = \sum_i c_i |i\rangle$, i.e. $P(i) \propto |\langle i|\psi\rangle|^2$. Which corresponds exactly to Postulate III of Quantum Mechanics as stated by Shankar's Principles of Quantum Mechanics.

8 Noson - Chapter 05

8.1 Section 5.4

8.1.1 Bloch Sphere - Equations (5.80) to (5.88)

Equations (5.80) to (5.88) explain how the Bloch Sphere representation of a Qubit is derived from the standard representation ($|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and $\beta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 = 1$). Since the logical steps of the derivation are flawlessly explained, only complementary comments are necessary (which are easily deductible if a solid background was built to this point).

- *Equations (5.81) and (5.82)*: Recall that any complex number can be written in the polar form. That is, if $z \in \mathbb{C}$ ($z = a + bi$, where $a, b \in \mathbb{R}$), then $z = r(cos\theta + i\ sin\theta)$, where $\theta \in [0, 2\pi)$, and $r \in \mathbb{R}$ and it is the norm (length) of z ($|z| = r = \sqrt{a^2 + b^2}$). To obtain Equations (5.81) and (5.82), apply Euler's formula ($e^{ix} = cos(x) + i\ sin(x)$). The polar form is useful because it simplifies complex numbers multiplications and exponentiation. Additionally, it gives a geometrical interpretation for complex numbers: they can be represented as a point in a plane;
- *Equation 5.84*: Multiplying any state $|\psi\rangle$ by a scalar $z \in \mathbb{C}, z \neq 0$ does not change the state $|\psi\rangle$. For a detailed explanation, refer back to Section 7.1.1;
- *Equations (5.86) and (5.87)*: It is possible to rename the r_0 and r_1 due to the Pythagorean trigonometric identity ($\sin^2\theta + cos^2\theta = 1$);
- *Equations (5.87) and (5.88)*: Substitute the values of Equation(5.87) in the final result of Equation(5.84);
- *Equation (5.88) - range of θ and ϕ*: The range for unique spherical coordinates is $0 \leq \theta \leq \pi$ for the polar angle (elevation) and $0 \leq \phi < 2\pi$ for the azimuthal angle. Noson states that the ranges are $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$. In this case, however, there would exist an unrepresentable point in the Bloch Sphere ($\theta = \frac{\pi}{2}$). Given the description of Exercise 5.4.4, it is possible to conclude that this was a typo. Therefore, the correct range for the polar angle θ is $0 \leq \theta \leq \frac{\pi}{2}$. The proof that $\theta \in [0, \frac{\pi}{2}]$ instead of $\theta \in [0, \pi]$ can be found in Exercise 5.4.4.

8.1.2 Exercise 5.4.4

A Qubit is represented in the Bloch Sphere by the formula

$$|\psi\rangle = cos\theta\,|0\rangle + e^{i\phi}sin\theta\,|1\rangle\,. \tag{448}$$

In spherical coordinates, the polar angle $\theta \in [0, \pi]$. From trigonometry, it is known that $cos(\theta) = -cos(\theta + \pi) = -cos(\pi - \theta)$, and $sin(\theta) = sin(\pi - \theta)$. Substituting these identities in $|\psi\rangle$,

$$|\psi\rangle = -cos(\pi - \theta)\,|0\rangle + e^{i\phi}sin(\pi - \theta). \tag{449}$$

From Equation (4.5), it is known that multiplying a Qubit $|\psi\rangle$ by $z \in \mathbb{C}$ does not change $|\psi\rangle$'s state. Therefore, using Euler's formula and multiplying $|\psi\rangle$ by $e^{i\pi}$,

$$\begin{aligned}
|\psi\rangle &= e^{i\pi}(-cos(\pi - \theta)\,|0\rangle + e^{i\phi}sin(\pi - \theta)) && (450)\\
&= (cos(\pi) + i\ sin(\pi))(-cos(\pi - \theta)\,|0\rangle) + e^{i(\phi+\pi)}sin(\pi - \theta) && (451)\\
&= (-1) - cos(\pi - \theta)\,|0\rangle) + e^{i(\phi+\pi)}sin(\pi - \theta) && (452)\\
&= cos(\pi - \theta)\,|0\rangle) + e^{i(\phi+\pi)}sin(\pi - \theta). && (453)
\end{aligned}$$

Hence, if $\theta > \pi/2$, it is possible to rewrite it in terms of $\theta \in [0, \frac{\pi}{2}]$ by simply adding π degrees to the azimuthal angle ϕ, and subtracting the polar angle θ from π.

By a similar line of thought, it is possible to prove that even if the ranges were inverted ($\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$), then they could be mapped back to the $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$ ranges.