3 Nielsen and Chuang - Chapter 01

3.1 Section 1.2

3.1.1 Qubit representation in a Bloch Sphere

The explanation to the following formula is not given by the book.

$$|\psi\rangle = e^{i\gamma}\left(cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle\right)$$

However, Agnez came up with a simple explanation using spherical coordinates. Its details can be found at TODO: add link to Computer Society

3.2 Section 1.4

3.2.1 Deutsch's Algorithm

While explaining Deutsch's algorithm, state $|\psi_1\rangle$ is obtained.

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Then, the Unitary gate U_f is applied to state $|\psi\rangle$ and how the result obtained in state $|\psi_2\rangle$ may not be clear enough to the reader. First, recall that $f(x) : \{0,1\} \rightarrow \{0,1\}$. That is, the function maps the qubits in state $|0\rangle$ to either state $|0\rangle$ or $|1\rangle$. Analogously, qubits in state $|1\rangle$ are mapped to state $|0\rangle$ or $|1\rangle$.

Henceforth, there are for possible functions: two possibilities where $f(0) = f(1)$ and two possibilities where $f(0) \neq f(1)$.

- $f(0) = f(1)$
  - $f(0) = f(1) = 0$
  - $f(0) = f(1) = 1$
- $f(0) \neq f(1)$
  - $f(0) = 0, f(1) = 1$
  - $f(0) = 1, f(1) = 0$

Note that U_f does not apply any operation to the first qubit (x), but applies $y \oplus f(x)$ to the second qubit (y). Note that, using the distributive property, the state $|\psi_1\rangle$ may be written as

$$|\psi_1\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Then, analyzing what would happen if any of the four possibilities for U_f were applied:

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{1}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{2}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{3}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{4}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{5}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{6}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |11\rangle - |10\rangle}{2} \tag{7}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{8}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{9}$$

Henceforth, the first part of Nielsen and Chuangs's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) = f(1)$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 0, f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{10}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{11}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2} \tag{12}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{13}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 1, f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{14}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{15}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |10\rangle - |11\rangle}{2} \tag{16}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2} \tag{17}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{18}$$

Henceforth, the second part of Nielsen and Chuangs's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) \neq f(1)$$

Also, note that something interesting happened. Even though the U_f was not supposed to alter the state of the first qubit ($|x\rangle$); it is, in fact, changed. That is why $|x\rangle$ is used to determine the property of $f(x)$.