4 Nielsen and Chuang - Chapter 02

4.1 Section 2.2.8

4.1.1 Equation (2.123)

I would like to thank Rex (rexmedeiros@ect.ufrn.br) and LIB (leandro@ect.ufrn.br) for helping me to understand this equation. The present subsection mixes some doubts I had alongside with their explanation.

The definition of equation (2.122) will be needed for this section. In order to understand equation (2.123), it is necessary to recall the definition of inner product [1] between two states $|\psi\rangle$ and $|\varphi\rangle$:

$$(|\varphi\rangle, |\psi\rangle) = |\varphi\rangle^\dagger |\psi\rangle = \langle\varphi|\psi\rangle$$

However, the inner product on equation (2.123) is a composite system inner product. Since composite systems are described using tensor products, it is necessary to apply the definition of equation (2.49). Hence, it is possible to calculate

$$\begin{aligned}(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= \left(\sum_m M_m|\varphi\rangle|m\rangle, \sum_{m'.} M_{m'}|\psi\rangle|m'\rangle\right) &(19)\\ &= \sum_{m,m'} (M_m|\varphi\rangle)^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle &(20)\end{aligned}$$

Then, from the definitions on section 2.1.6:

$$\sum_{m,m'} (M_m|\varphi\rangle)^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle \tag{21}$$

The left side of equation (2.123) may be rather confusing, however. Because according to the definitions on section 2.16 $(U|\varphi 0\rangle)^\dagger = \langle\varphi 0|U^\dagger$. Also, accordingly to the properties on equation (2.53) $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0|U^\dagger$. If this line of thought was followed, then equation

$$\langle\varphi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_{m,m'} \langle\varphi|\langle m|M_m^\dagger M_{m'}|\psi\rangle|m'\rangle$$

would be obtained. Which would not match equation (2.49)'s definition.

It is a common practice in Physics, however, to write $(U|\varphi\rangle|0\rangle)^\dagger = \langle 0|\langle\varphi|U^\dagger$. In this case, the adjoint operators are read 'backwards'. So, for instance, U operates on $|\varphi\rangle$ (i.e. $U|\varphi\rangle$); while $U^\dagger$ operates on $\langle\varphi|$ (i.e. $\langle\varphi|U^\dagger$). Following this line of thought, $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0|U^\dagger$ would not make sense because $U^\dagger$ should operate on $\langle\varphi|$, not on $\langle 0|$. Formally, imagine that an operator M operates

[1] For more details, refer to Nielsen and Chuang's section 2.1.4

on vector space V, $|v\rangle \in V$ and $|w\rangle \in W$, then $\langle v| \langle w| M^\dagger$ would not be valid because M only acts on vector space V, not W.

Hence, it is possible to rewrite equation (2.123) as:

$$
\begin{aligned}
(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= (U|\varphi\rangle|0\rangle)^\dagger U|\psi\rangle|0\rangle && (22)\\
&= (\sum_m M_m|\varphi\rangle|m\rangle)^\dagger \sum_{m'} M_{m'}|\psi\rangle|m'\rangle && (23)\\
&= \sum_{m,m'} \langle m|\langle\varphi| M_m^\dagger M_{m'}|\psi\rangle|m'\rangle && (24)
\end{aligned}
$$

since $\langle\varphi| M_m^\dagger M_{m'}|\psi\rangle$ is a scalar:

$$
\sum_{m,m'} \langle m|\langle\varphi| M_m^\dagger M_{m'}|\psi\rangle|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'}|\psi\rangle\langle m|m'\rangle \tag{25}
$$

Which is another way to obtain equation (2.123).