Quommentaries

Gustavo Bezerra
gustavowl@lcc.ufrn.br

August 23, 2018

Contents

1 Leftoverture

This repository is dedicated to solve exercises and comment on Quantum Computing. Most of the discussion is based on Nielsen And Chuang's book "Quantum Computation and Quantum Information". In addition, Kaye, Laflamme and Mosca's "An Introduction to Quantum Computing" is used as a complementary book, as well as Yanofsky and Mannucci's "Quantum Computing for Computer Scientists" - recommended by Greati.

1.1 Objective

Although Nielsen and Chuang's book is very famous, some equations may be solved too quickly. This may discourage the reader to continue the studies if the basic concepts were not mastered. One of the objectives of this repository is to support those who are studying Quantum Computing and Quantum Information by explaining some of these equations step-by-step.

In addition, the exercises present in the book may not be trivial for beginners. Hence, this repository attempts to help the students by showing a detailed solution or, at least, a sketch.

1.2 Disclaimer

This repository is being constructed by an **undergaduate student**. Henceforth, the notes, commentaries and exercises are **suscetible to errors**. Please, **do not hesitate to give feedback** (gustavowl@lcc.ufrn.br).

2 Introduction

On August 19, 2018, the author was studying the *Section 2.5 - The Schmidt decomposition and purification* of Nielsen and Chuang's book. Up until this section, all exercises were fairly discussed in worked problem's website. Most of the answers are reasonably satisfactory, though some lack formalism and detailed explanation. However, this website only discusses exercises 2.1 to 2.76. Question 2.77 is discussed on StackExchange's website. Apparently, question 2.78 onward are not commonly discussed. Henceforth, this material will *initially* focus on these questions. Details on questions 2.1 to 2.76 will be added sporadically.

In addition, this material will contain details on the equations solved during each chapter. Most explanations will try to specify the steps using to jump from one equation to another. Also, some affirmations and equations may induce doubts in the author; who will try to state and clarify them in this document.

3 Nielsen and Chuang - Chapter 01

3.1 Section 1.2

3.1.1 Qubit representation in a Bloch Sphere

The explanation to the following formula is not given by the book.

$$|\psi\rangle = e^{i\gamma}\left(cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle\right)$$

However, Agnez came up with a simple explanation using spherical coordinates. Its details can be found at TODO: add link to Computer Society

3.2 Section 1.4

3.2.1 Deutsch's Algorithm

While explaining Deutsch's algorithm, state $|\psi_1\rangle$ is obtained.

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Then, the Unitary gate U_f is applied to state $|\psi\rangle$ and how the result obtained in state $|\psi_2\rangle$ may not be clear enough to the reader. First, recall that $f(x) : \{0, 1\} \rightarrow \{0, 1\}$. That is, the function maps the qubits in state $|0\rangle$ to either state $|0\rangle$ or $|1\rangle$. Analogously, qubits in state $|1\rangle$ are mapped to state $|0\rangle$ or $|1\rangle$.

Henceforth, there are for possible functions: two possibilities where $f(0) = f(1)$ and two possibilities where $f(0) \neq f(1)$.

- $f(0) = f(1)$
  - $f(0) = f(1) = 0$
  - $f(0) = f(1) = 1$
- $f(0) \neq f(1)$
  - $f(0) = 0, f(1) = 1$
  - $f(0) = 1, f(1) = 0$

Note that U_f does not apply any operation to the first qubit (x), but applies $y \oplus f(x)$ to the second qubit (y). Note that, using the distributive property, the state $|\psi_1\rangle$ may be written as

$$|\psi_1\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Then, analysing what would happen if any of the four possibilities for U_f were applied:

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{1}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{2}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{3}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{4}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{5}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{6}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |11\rangle - |10\rangle}{2} \tag{7}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{8}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{9}$$

Henceforth, the first part of Nielsen and Chuangs's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) = f(1)$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 0, f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{10}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{11}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2} \tag{12}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{13}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 1, f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{14}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{15}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |10\rangle - |11\rangle}{2} \tag{16}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2} \tag{17}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{18}$$

Henceforth, the second part of Nielsen and Chuang's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) \neq f(1)$$

Also, note that something interesting happened. Even though the U_f was not supposed to alter the state of the first qubit ($|x\rangle$); it is, in fact, changed. As a result, measuring $|x\rangle$ is sufficient to determine the specified property of $f(x)$.

4 Nielsen and Chuang - Chapter 02

4.1 Section 2.2.8

4.1.1 Equation (2.123)

I would like to thank Rex (rexmedeiros@ect.ufrn.br) and LIB (leandro@ect.ufrn.br) for helping me to understand this equation. The present subsection mixes some doubts I had alongside with their explanation.

The definition of equation (2.122) will be needed for this section. In order to understand equation (2.123), it is necessary to recall the definition of inner product [1] between two states $|\psi\rangle$ and $|\varphi\rangle$:

$$(|\varphi\rangle, |\psi\rangle) = |\varphi\rangle^\dagger |\psi\rangle = \langle\varphi|\psi\rangle$$

However, the inner product on equation (2.123) is a composite system inner product. Since composite systems are described using tensor products, it is necessary to apply the definition of equation (2.49). Hence, it is possible to calculate

$$(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) = \left(\sum_m M_m|\varphi\rangle|m\rangle, \sum_{m'.} M_{m'}|\psi\rangle|m'\rangle\right) \tag{19}$$

$$= \sum_{m,m'} (M_m|\varphi\rangle)^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle \tag{20}$$

Then, from the definitions on section 2.1.6:

$$\sum_{m,m'} (M_m|\varphi\rangle)^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle \tag{21}$$

The left side of equation (2.123) may be rather confusing, however. Because according to the definitions on section 2.16 $(U|\varphi 0\rangle)^\dagger = \langle\varphi 0|U^\dagger$. Also, accordingly to the properties on equation (2.53) $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0|U^\dagger$. If this line of thought was followed, then equation

$$\langle\varphi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_{m,m'} \langle\varphi|\langle m|M_m^\dagger M_{m'}|\psi\rangle|m'\rangle$$

would be obtained. Which would not match equation (2.49)'s definition.

It is a common practice in Physics, however, to write $(U|\varphi\rangle|0\rangle)^\dagger = \langle 0|\langle\varphi|U^\dagger$. In this case, the adjoint operators are read 'backwards'. So, for instance, U operates on $|\varphi\rangle$ (i.e. $U|\varphi\rangle$); while $U^\dagger$ operates on $\langle\varphi|$ (i.e. $\langle\varphi|U^\dagger$). Following this line of thought, $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0|U^\dagger$ would not make sense because $U^\dagger$ should operate on $\langle\varphi|$, not on $\langle 0|$. Formally, imagine that an operator M operates

[1] For more details, refer to Nielsen and Chuang's section 2.1.4

on vector space V, $|v\rangle \in V$ and $|w\rangle \in W$, then $\langle v| \langle w| M^\dagger$ would not be valid because M only acts on vector space V, not W.

Hence, it is possible to rewrite equation (2.123) as:

$$\begin{aligned}
(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= (U|\varphi\rangle|0\rangle)^\dagger U|\psi\rangle|0\rangle && (22)\\
&= (\sum_m M_m|\varphi\rangle|m\rangle)^\dagger \sum_{m'} M_{m'}|\psi\rangle|m'\rangle && (23)\\
&= \sum_{m,m'} \langle m| \langle\varphi| M_m^\dagger M_{m'} |\psi\rangle |m'\rangle && (24)
\end{aligned}$$

since $\langle\varphi| M_m^\dagger M_{m'} |\psi\rangle$ is a scalar:

$$\sum_{m,m'} \langle m| \langle\varphi| M_m^\dagger M_{m'} |\psi\rangle |m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'} |\psi\rangle \langle m|m'\rangle \tag{25}$$

Which is another way to obtain equation (2.123).