Quommentaries

Gustavo Bezerra (Koruja)
gustavowl@lcc.ufrn.br

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Contents

1 Leftoverture

This repository is dedicated to solve exercises and comment on Quantum Computing. Most of the discussion is based on Nielsen And Chuang's book "Quantum Computation and Quantum Information". In addition, Kaye, Laflamme and Mosca's "An Introduction to Quantum Computing" is used as a complementary book, as well as Yanofsky and Mannucci's "Quantum Computing for Computer Scientists" - recommended by Greati.

1.1 Objective

Although Nielsen and Chuang's book is very famous, some equations may be solved too quickly. This may discourage the reader to continue the studies if the basic concepts were not mastered. One of the objectives of this repository is to support those who are studying Quantum Computing and Quantum Information by explaining some of these equations step-by-step.

In addition, the exercises present in the book may not be trivial for beginners. Hence, this repository attempts to help the students by showing a detailed solution or, at least, a sketch.

1.2 Disclaimer

This repository is being constructed by an **undergaduate student**. Henceforth, the notes, commentaries and exercises are **suscetible to errors**. Please, **do not hesitate to give feedback** (gustavowl@lcc.ufrn.br).

2 Introduction

On August 19, 2018, the author was studying the *Section 2.5 - The Schmidt decomposition and purification* of Nielsen and Chuang's book. Up until this section, all exercises were fairly discussed in worked problem's website. Most of the answers are reasonably satisfactory, though some lack formalism and detailed explanation. However, this website only discusses exercises 2.1 to 2.76. Question 2.77 is discussed on StackExchange's website. Apparently, questions 2.78 onward are not commonly discussed. Therefore, this material will *initially* focus on these questions. Details on questions 2.1 to 2.76 will be added sporadically.

In addition, this material will contain details on some equations solved during each chapter. Most explanations will try to specify the steps using to jump from one equation to another. Also, some affirmations and equations may induce doubts in the author; who will try to state and clarify them in this document.

3 Nielsen and Chuang - Chapter 01

3.1 Section 1.2

3.1.1 Qubit representation in a Bloch Sphere

The explanation to the following formula is not given by the book.

$$|\psi\rangle = e^{i\gamma} \left(cos\frac{\theta}{2} |0\rangle + e^{i\varphi} sin\frac{\theta}{2} |1\rangle \right)$$

However, Agnez came up with a simple explanation using spherical coordinates. Its details can be found at **TODO: ADD LINK TO COMPUTER SOCIETY**

3.2 Section 1.4

3.2.1 $(-1)^{f(x)} |x\rangle (|0\rangle + |1\rangle)/\sqrt{2}$

TODO: ADD EXPLANATION OF $(-1)^{F(X)} |X\rangle (|0\rangle + |1\rangle)/\sqrt{2}$

3.2.2 Equation (1.43)

While explaining Deutsch's algorithm, state $|\psi_1\rangle$ is obtained.

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Then, the Unitary gate U_f is applied to state $|\psi\rangle$ and how the result obtained in state $|\psi_2\rangle$ may not be clear enough to the reader. First, recall that $f(x) : \{0, 1\} \rightarrow \{0, 1\}$. That is, the function maps the qubits in state $|0\rangle$ to either state $|0\rangle$ or $|1\rangle$. Analogously, qubits in state $|1\rangle$ are mapped to state $|0\rangle$ or $|1\rangle$.

Henceforth, there are for possible functions: two possibilities where $f(0) = f(1)$ and two possibilities where $f(0) \neq f(1)$.

- $f(0) = f(1)$
  - $f(0) = f(1) = 0$
  - $f(0) = f(1) = 1$
- $f(0) \neq f(1)$
  - $f(0) = 0, f(1) = 1$
  - $f(0) = 1, f(1) = 0$

Note that U_f does not apply any operation to the first qubit (x), but applies $y \oplus f(x)$ to the second qubit (y). Note that, using the distributive property, the state $|\psi_1\rangle$ may be written as

$$|\psi_1\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Then, analysing what would happen if any of the four possibilities for U_f were applied:

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{1}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{2}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{3}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{4}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{5}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{6}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |11\rangle - |10\rangle}{2} \tag{7}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \tag{8}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{9}$$

Henceforth, the first part of Nielsen and Chuangs's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad if\ f(0) = f(1)$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 0, f(1) = 1$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{10}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 0\ )\rangle - |0\ (1 \oplus 0\ )\rangle + |1\ (0 \oplus 1\ )\rangle - |1\ (1 \oplus 1\ )\rangle}{2} \tag{11}$$

$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2} \tag{12}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{13}$$

- Apply U_f to $|\psi_1\rangle$ when $f(0) = 1, f(1) = 0$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus f(0)\ )\rangle - |0\ (1 \oplus f(0)\ )\rangle + |1\ (0 \oplus f(1)\ )\rangle - |1\ (1 \oplus f(1)\ )\rangle}{2} \tag{14}$$

$$|\psi_2\rangle = \frac{|0\ (0 \oplus 1\ )\rangle - |0\ (1 \oplus 1\ )\rangle + |1\ (0 \oplus 0\ )\rangle - |1\ (1 \oplus 0\ )\rangle}{2} \tag{15}$$

$$|\psi_2\rangle = \frac{|01\rangle - |00\rangle + |10\rangle - |11\rangle}{2} \tag{16}$$

$$|\psi_2\rangle = -\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2} \tag{17}$$

Then, inversely applying the distributive property:

$$|\psi_2\rangle = -\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \tag{18}$$

Henceforth, the second part of Nielsen and Chuang's *equation 1.43* was obtained:

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\ if\ f(0) \neq f(1)$$

Also, note that something interesting happened. Even though the U_f was not supposed to alter the state of the first qubit ($|x\rangle$); it is, in fact, changed. As a result, measuring $|x\rangle$ is sufficient to determine the specified property of $f(x)$.

4 Nielsen and Chuang - Chapter 02

4.1 Section 2.1.4

4.1.1 Outer Product Representation of A

It is stated from Equation 2.25 that it is possible to "[...] see from this equation that A has matrix element $\langle w_j| A |v_i\rangle$". To see this, it is possible to compare the matrix and Dirac representations. Consider two systems V and W with dimensions n and m, respectively. In addition, suppose $|v\rangle \in V$ and $|w\rangle \in W$.

Using matrix representation:

$$\langle v|w\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} w_1 & \cdots & w_j & \cdots & w_m \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} v_1 w_1 & \cdots & v_1 w_j & \cdots & v_1 w_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_i w_1 & \cdots & v_i w_j & \cdots & v_i w_m \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ v_n w_1 & \cdots & v_n w_j & \cdots & v_n w_m \end{bmatrix} \tag{20}$$

Using Dirac notation and the last part of Equation 2.21:

$$|v\rangle\langle w| = \sum_{ij} v_i |i\rangle w_j \langle j| \tag{21}$$

$$= \sum_{ij} v_i w_j |i\rangle\langle j| \tag{22}$$

Then, comparing matrix and Dirac representation, it is easily verified that the matrix has elements $m_{ij} = v_i w_j$ for the i-th row and j-th column ($\langle i|j\rangle$) with respect to the orthonormal basis $|i\rangle$ and $|j\rangle$ for systems V and W, respectively.

4.2 Section 2.1.8

4.2.1 Exercise 2.35

In order to solve this exercise, it is necessary to find a spectral decomposition for $\vec{v}\vec{\sigma}$. Then, it is possible to apply the definition of Operator functions.

With the aid of Exercise 2.60, the required spectral decomposition is obtained:

$$\vec{v}\vec{\sigma} = +1P_+ - 1P_- \tag{23}$$

$$= \frac{I + \vec{v}\vec{\sigma}}{2} - \frac{I - \vec{v}\vec{\sigma}}{2} \tag{24}$$

Now, calculating the value of $exp(i\theta\vec{v}\cdot\vec{\sigma})$ and applying the definition of Operator functions:

$$\begin{aligned} exp(i\theta\vec{v}\cdot\vec{\sigma}) &= exp(i\theta P_+) + exp(-i\theta P_-) && (25)\\ &= exp(i\theta)P_+ + exp(-i\theta)P_- && (26)\\ &= e^{i\theta}P_+ + e^{-i\theta}P_- && (27)\end{aligned}$$

Then, applying Euler's Formula:

$$\begin{aligned} exp(i\theta\vec{v}\cdot\vec{\sigma}) &= cos(\theta)P_+ + isin(\theta)P_+ + cos(\theta)P_- - isin(\theta)P_- && (28)\\ &= cos(\theta)\frac{I+\vec{v}\vec{\sigma}}{2} + isin(\theta)\frac{I+\vec{v}\vec{\sigma}}{2} + cos(\theta)\frac{I-\vec{v}\vec{\sigma}}{2} - isin(\theta)\frac{I-\vec{v}\vec{\sigma}}{2} && (29)\\ &= cos(\theta)\left(\frac{I+\vec{v}\vec{\sigma}+I-\vec{v}\vec{\sigma}}{2}\right) + isin(\theta)\left(\frac{I+\vec{v}\vec{\sigma}-I+\vec{v}\vec{\sigma}}{2}\right) && (30)\\ &= cos(\theta)I + i\ sin(\theta)\vec{v}\vec{\sigma} && (31)\end{aligned}$$

Thus obtaining the required result.

4.2.2 Equation 2.60

It is known that $tr(UAU^\dagger) = tr(A)$. Therefore,

$$tr(A\left|\psi\right\rangle\left\langle\psi\right|) = tr(UA\left|\psi\right\rangle\left\langle\psi\right|U^\dagger)) \tag{32}$$

By Equation 2.22, $\sum_i \left|i\right\rangle\left\langle i\right| = I$. Since I is an Unitary Operator, it is possible to write

$$\begin{aligned} tr(UA\left|\psi\right\rangle\left\langle\psi\right|U^\dagger) &= tr(IA\left|\psi\right\rangle\left\langle\psi\right|I^\dagger)) && (33)\\ &= tr(IA\left|\psi\right\rangle\left\langle\psi\right|I)) && (34)\\ &= tr(\sum_{ij}\left|i\right\rangle\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi\right|\left|j\right\rangle\left\langle j\right|) && (35)\end{aligned}$$

Since $\left\langle i\right|A\left|\psi\right\rangle$ and $\left\langle\psi|j\right\rangle$ are scalars,

$$tr(\sum_{ij}\left|i\right\rangle\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi\right|\left|j\right\rangle\left\langle j\right|) = tr(\sum_{ij}\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi|j\right\rangle\left|i\right\rangle\left\langle j\right|) \tag{36}$$

Similarly to Equation 2.25, $tr(\sum_{ij}\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi|j\right\rangle\left|i\right\rangle\left\langle j\right|)$ is an Outer Product representation for $A\left|\psi\right\rangle\left\langle\psi\right|$ where element $m_{ij} = \left\langle i\right|A\left|\psi\right\rangle\left\langle\psi|j\right\rangle$ (check Section 4.1.1 for details). Then, by the definition of trace (Equation 2.59):

$$\begin{aligned} tr(\sum_{ij}\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi|j\right\rangle\left|i\right\rangle\left\langle j\right|) &= m_{ii} && (37)\\ &= \sum_i\left\langle i\right|A\left|\psi\right\rangle\left\langle\psi|i\right\rangle && (38)\end{aligned}$$

4.2.3 $tr(|\psi\rangle\langle\varphi| = \langle\varphi|\psi\rangle)$

I decided to add this section because this equation is used throughout the book, e.g. Equation 2.208 to 2.209, and Exercise 2.82. I do not remember it being explicitly stated or explained, though.

Suppose two different states $|\psi\rangle$ and $|\varphi\rangle$. Then, following the same reasoning as Equation 2.60 and Equation 2.61:

$$
\begin{aligned}
tr(\langle\psi|\varphi\rangle) &= tr(I\,\langle\psi|\varphi\rangle) && (39)\\
&= \sum_i \langle i|\, I\, |\psi\rangle\, \langle\varphi|i\rangle && (40)\\
&= \sum_i \langle i|\psi\rangle\, \langle\varphi|i\rangle && (41)\\
&= \sum_i \langle\varphi|\, |i\rangle\langle i|\, |\psi\rangle && (42)\\
&= \langle\varphi|\, I\, |\psi\rangle && (43)\\
&= \langle\varphi|\psi\rangle && (44)
\end{aligned}
$$

Note that although Equation 2.60 defines a orthonormal basis $|i\rangle$ containing $|\psi\rangle$, this is not necessary. The only restriction is that, if $|\psi\rangle \in V$ and $|\varphi\rangle \in W$, then $dim(V) = dim(W)$. In other words, that $|\psi\rangle\langle\varphi|$ is a square matrix.

4.3 Section 2.2.5

4.3.1 Equation (2.116)

This definition may be rather confusing since $\vec{v}$ is defined but the definition of $\vec{\sigma}$ is not recapitulated. More specifically, σ_1, σ_1, and σ_3 were defined in Table 2.2 of the book. However, since X, Y and Z are used more frequently to denote the Pauli Matrices, the reader may not remind of the equivalent σ notation.

In order to reduce the calculi on Exercise 2.60 (section 4.3.2), The matrix form of $\vec{v}\vec{\sigma}$ is computed in this section.

Recall that $\sigma_1 \equiv X$, $\sigma_2 \equiv Y$, and $\sigma_3 \equiv Z$, which have matrix form as defined in the book's Table 2.2. Since $\vec{v}$ is a vector with components $v_1, v_2, v_3 \in \mathbb{R}$, it is possible to interpret $\vec{\sigma}$ as being a vector of matrices, i.e. $\vec{\sigma} \in (\mathbb{R}^{2\times 2})^3$ Therefore:

$$
\begin{aligned}
\vec{v}\vec{\sigma} &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} && (45)\\
&= \begin{bmatrix} v_3 & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 \end{bmatrix} && (46)
\end{aligned}
$$

4.3.2 Exercise 2.60

This section will only find the requested eigenvalues, and the Projector given by P_+. The projector given by P_- can be found by following the same steps as P_+'s solution.

<u>THE EIGENVALUES</u> of $\vec{v}\vec{\sigma}$ can be found by using basic Linear Algebra knowledge: $det(\vec{v}\vec{\sigma}-\lambda I) = 0$. Therefore, referring to Section 4.3.1, calculate

$$det(\vec{v}\vec{\sigma} - \lambda I) = 0 \tag{47}$$

$$\begin{vmatrix} v_3 - \lambda & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - \lambda \end{vmatrix} = 0 \tag{48}$$

$$\lambda^2 - v_3^2 - (v_1^2 + v_2^2) = 0 \tag{49}$$

$$\lambda^2 = v_1^2 + v_2^2 + v_3^2 \tag{50}$$

A bit of cleverness is required here. Recall that just before the definition of Equation 2.116, $\vec{v}$ is supposed to be a unit vector. This means that $\vec{v} \cdot \vec{v} = 1$. Since $\vec{v} \in \mathbb{R}^3$), $\vec{v} \cdot \vec{v} = v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3$. Therefore, $v_1^2 + v_2^2 + v_3^2 = 1$. Plugging this into the previous result to find the values of λ:

$$\lambda^2 = 1 \tag{51}$$

$$\lambda = \pm 1 \tag{52}$$

as requested.

<u>TO FIND THE PROJECTOR</u> P_+ it is necessary to calculate the eigenspace of the eigenvector of $+1$. In order to find the eigenspace, basic Linear Algebra knowledge may be used. Hence, for $\lambda = +1$ and applying row reducing:

$$\begin{bmatrix} v_3 - \lambda & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - \lambda \end{bmatrix} = \begin{bmatrix} v_3 - 1 & v_1 - v_2 i \\ v_1 + v_2 i & -v_3 - 1 \end{bmatrix} \tag{53}$$

$$= \begin{bmatrix} (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \\ (v_1 + v_2 i)(v_3 - 1) & 1 - v_3^2 \end{bmatrix} \tag{54}$$

Note that $1 - v_3^2 = v_1^2 + v_2^2$, since $v_1^2 + v_2^2 + v_3^2 = 1$. Therefore,

$$\begin{bmatrix} (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \\ (v_1 + v_2 i)(v_3 - 1) & 1 - v_3^2 \end{bmatrix} = \begin{bmatrix} (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \\ (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \end{bmatrix} \tag{55}$$

$$= \begin{bmatrix} (v_3 - 1)(v_1 + v_2 i) & v_1^2 + v_2^2 \\ 0 & 0 \end{bmatrix} \tag{56}$$

Therefore, if t is a scalar, the eigenspace can be given by:

$$t\begin{bmatrix}\frac{-(v_1-v_2i)}{v_3-1}\\1\end{bmatrix} \tag{57}$$

because:

$$(v_3-1)(v_1+v_2i)\cdot\frac{-(v_1-v_2i)}{v_3-1}+(v_1^2+v_2^2)\cdot 1=0 \tag{58}$$

$$-(v_1^2+v_2^2)+(v_1^2+v_2^2)=0 \tag{59}$$

However, it is not possible to use the definition and keep calculating with $P_m = |m\rangle\langle m|$, where $|m\rangle = \begin{bmatrix}\frac{-(v_1-v_2i)}{v_3-1}\\1\end{bmatrix}$ because it is necessary that $|m\rangle$ is unitary ($\langle m|m\rangle = 1$). And, if $\langle m|m\rangle$ is calculated, the following result would be obtained:

$$\langle m|m\rangle = \begin{bmatrix}\frac{-(v_1+v_2i)}{v_3-1} & 1\end{bmatrix}\begin{bmatrix}\frac{-(v_1-v_2i)}{v_3-1}\\1\end{bmatrix} \tag{60}$$

$$= \begin{bmatrix}\frac{-v_1-v_2i}{v_3-1} & 1\end{bmatrix}\begin{bmatrix}\frac{-v_1+v_2i}{v_3-1}\\1\end{bmatrix} \tag{61}$$

$$= \frac{v_1^2+v_2^2}{(v_3-1)^2}+1 \tag{62}$$

$$= \frac{1-v_3^2}{(v_3-1)^2}+\frac{(v_3-1)^2}{(v_3-1)^2} \tag{63}$$

$$= \frac{1-v_3^2+v_3^2-2v_3+1}{(v_3-1)^2} \tag{64}$$

$$= \frac{-2v_3+2}{(v_3-1)^2} \tag{65}$$

$$= \frac{-2(v_3-1)}{(v_3-1)^2} \tag{66}$$

$$= -\frac{2}{v_3-1} \tag{67}$$

Hence, it is necessary to normalize $|m\rangle$. Recall that the norm of a vector $|m\rangle$ is given by $\sqrt{\langle m|m\rangle}$. To normalize a vector, divide it by its norm.

$$|\psi\rangle = \frac{|m\rangle}{\sqrt{\langle m|m\rangle}} \tag{68}$$

$$= |m\rangle \Big/ \sqrt{-\frac{2}{v_3-1}} \tag{69}$$

$$= \sqrt{-\frac{v_3-1}{2}}\,|m\rangle \tag{70}$$

$$= \frac{i}{\sqrt{2}}\sqrt{v_3-1}\,|m\rangle \tag{71}$$

However, this would not be right because $\sqrt{\langle m|m\rangle} \geq 0$ and $\sqrt{\langle m|m\rangle} \in \mathbb{R}$. It is necessary to rearrange the value of $\langle m|m\rangle$:

$$\begin{aligned}
|\psi\rangle &= \sqrt{-\frac{v_3-1}{2}}\,|m\rangle && (72)\\
&= \sqrt{\frac{1-v_3}{2}}\,|m\rangle && (73)\\
&= \frac{1}{\sqrt{2}}\sqrt{1-v_3}\begin{bmatrix}\frac{-(v_1-v_2i)}{v_3-1}\\ 1\end{bmatrix} && (74)\\
&= \frac{1}{\sqrt{2}}\sqrt{1-v_3}\begin{bmatrix}\frac{v_1-v_2i}{1-v_3}\\ 1\end{bmatrix} && (75)\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix} && (76)
\end{aligned}$$

Now that the normalized vector was obtained, it is possible to calculate the respective Projector by:

$$\begin{aligned}
|\psi\rangle\langle\psi| &= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{v_1+v_2i}{\sqrt{1-v_3}} & \sqrt{1-v_3}\end{bmatrix} && (77)\\
&= \frac{1}{2}\begin{bmatrix}\frac{v_1-v_2i}{\sqrt{1-v_3}}\\ \sqrt{1-v_3}\end{bmatrix}\begin{bmatrix}\frac{v_1+v_2i}{\sqrt{1-v_3}} & \sqrt{1-v_3}\end{bmatrix} && (78)\\
&= \frac{1}{2}\begin{bmatrix}\frac{v_1^2+v_2^2}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} && (79)\\
&= \frac{1}{2}\begin{bmatrix}\frac{1-v_3^2}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} && (80)\\
&= \frac{1}{2}\begin{bmatrix}\frac{(1+v_3)(1-v_3)}{1-v_3} & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} && (81)\\
&= \frac{1}{2}\begin{bmatrix}1+v_3 & v_1-v_2i\\ v_1+v_2i & 1-v_3\end{bmatrix} && (82)\\
&= (I+\vec{v}\vec{\sigma})/2 = P_+ && (83)
\end{aligned}$$

as requested. P_- can be easily obtained following the same steps, but with $\lambda = -1$.

4.3.3 Exercise 2.61

This Exercise can be done very easily by using Equations (2.103) and (2.104) alongside the value of P_+ obtained in Exercise 2.60.

The probability can be calculated by using Equation (2.103):

$$
\begin{aligned}
p(+1) &= \langle 0| P_+ |0\rangle && (84)\\
&= \begin{bmatrix}1 & 0\end{bmatrix} (I + \vec{v}\vec{\sigma})/2 \begin{bmatrix}1\\0\end{bmatrix} && (85)\\
&= \frac{1}{2}\begin{bmatrix}1+v_3 & v_1 - v_2 i\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} && (86)\\
&= \frac{1+v_3}{2} && (87)
\end{aligned}
$$

Then, using Equation (2.104) to obtain the state of the system after the measurement:

$$
\begin{aligned}
|\psi\rangle &= \frac{P_+ |0\rangle}{\sqrt{p(+1)}} && (88)\\
&= \frac{1}{\sqrt{p(+1)}}\frac{1}{2}(I + \vec{v} + \vec{\sigma})\begin{bmatrix}1\\0\end{bmatrix} && (89)\\
&= \frac{\sqrt{2}}{\sqrt{1+v_3}}\frac{1}{2}\begin{bmatrix}v_3+1\\v_1+v_2 i\end{bmatrix} && (90)\\
&= \frac{(v_3+1)|0\rangle + (v_1+v_2 i)|1\rangle}{\sqrt{2+2v_3}} && (91)
\end{aligned}
$$

4.4 Section 2.2.8

4.4.1 Equation (2.123)

I would like to thank Rex (rexmedeiros@ect.ufrn.br) and LIB (leandro@ect.ufrn.br) for helping me to understand this equation. The present subsection mixes some doubts I had alongside with their explanation.

The definition of equation (2.122) will be needed for this section. In order to understand equation (2.123), it is necessary to recall the definition of inner product [1] between two states $|\psi\rangle$ and $|\varphi\rangle$:

$$(|\varphi\rangle, |\psi\rangle) = |\varphi\rangle^\dagger |\psi\rangle = \langle\varphi|\psi\rangle$$

However, the inner product on equation (2.123) is a composite system inner product. Since composite systems are described using tensor products, it is necessary to apply the definition of equation (2.49). Hence, it is possible to calculate

$$
\begin{aligned}
(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= \left(\sum_m M_m|\varphi\rangle|m\rangle, \sum_{m'.} M_{m'}|\psi\rangle|m'\rangle\right) && (92)\\
&= \sum_{m,m'} (M_m|\varphi\rangle)^\dagger M_{m'.}|\psi\rangle\langle m|m'\rangle && (93)
\end{aligned}
$$

[1] For more details, refer to Nielsen and Chuang's section 2.1.4

Then, from the definitions on section 2.1.6:

$$\sum_{m,m'} (M_m |\varphi\rangle)^\dagger M_{m'}. |\psi\rangle \langle m|m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'}. |\psi\rangle \langle m|m'\rangle \tag{94}$$

The left side of equation (2.123) may be rather confusing, however. Because according to the definitions on section 2.16 $(U|\varphi 0\rangle)^\dagger = \langle\varphi 0| U^\dagger$. Also, accordingly to the properties on equation (2.53) $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0| U^\dagger$. If this line of thought was followed, then equation

$$\langle\varphi|\langle 0| U^\dagger U |\psi\rangle |0\rangle = \sum_{m,m'} \langle\varphi|\langle m| M_m^\dagger M_{m'} |\psi\rangle |m'\rangle$$

would be obtained. Which would not match equation (2.49)'s definition.

It is a common practice in Physics, however, to write $(U|\varphi\rangle|0\rangle)^\dagger = \langle 0|\langle\varphi| U^\dagger$. In this case, the adjoint operators are read 'backwards'. So, for instance, U operates on $|\varphi\rangle$ (i.e. $U|\varphi\rangle$); while $U^\dagger$ operates on $\langle\varphi|$ (i.e. $\langle\varphi| U^\dagger$). Following this line of thought, $(U|\varphi\rangle|0\rangle)^\dagger = \langle\varphi|\langle 0| U^\dagger$ would not make sense because $U^\dagger$ should operate on $\langle\varphi|$, not on $\langle 0|$. Formally, imagine that an operator M operates on vector space V, $|v\rangle \in V$ and $|w\rangle \in W$, then $\langle v|\langle w| M^\dagger$ would not be valid because M only acts on vector space V, not W.

Hence, it is possible to rewrite equation (2.123) as:

$$\begin{aligned}
(U|\varphi\rangle|0\rangle, U|\psi\rangle|0\rangle) &= (U|\varphi\rangle|0\rangle)^\dagger U|\psi\rangle|0\rangle && (95)\\
&= (\sum_m M_m |\varphi\rangle |m\rangle)^\dagger \sum_{m'} M_{m'} |\psi\rangle |m'\rangle && (96)\\
&= \sum_{m,m'} \langle m|\langle\varphi| M_m^\dagger M_{m'} |\psi\rangle |m'\rangle && (97)
\end{aligned}$$

since $\langle\varphi| M_m^\dagger M_{m'} |\psi\rangle$ is a scalar:

$$\sum_{m,m'} \langle m|\langle\varphi| M_m^\dagger M_{m'} |\psi\rangle |m'\rangle = \sum_{m,m'} \langle\varphi| M_m^\dagger M_{m'} |\psi\rangle \langle m|m'\rangle \tag{98}$$

Which is another way to obtain equation (2.123).

4.5 Section 2.5

4.5.1 Symmetry of $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$

This subsection is dedicated to calculate $tr((\rho^A)^2)$ for $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$. By Equation (2.138):

$$\rho^{AB} = \frac{(|00\rangle + |01\rangle + |11\rangle)}{\sqrt{3}} \frac{(\langle 00| + \langle 01| + \langle 11|)}{\sqrt{3}} \tag{99}$$

$$= \frac{(|00\rangle + |01\rangle + |11\rangle)(\langle 00| + \langle 01| + \langle 11|)}{3} \tag{100}$$

Before using Equations (2.177) and (2.178) it is necessary to apply the distributive property. However from Equation (2.178), the result will be similar to $\sum_{ijkl} |i\rangle \langle j| \, tr(|k\rangle \langle l|)$, where $i, j, k, l \in \{0, 1\}$. Since $tr(|a\rangle \langle b|) = \langle b|a\rangle$ (check Section 4.2.3):

$$\sum_{ijkl} |i\rangle \langle j| \, tr(|k\rangle \langle l|) = \sum_{ijkl} |i\rangle \langle j| \langle l|k\rangle \tag{101}$$

$$= \sum_{ijkl} |i\rangle \langle j| \, \delta_{lk} \tag{102}$$

Therefore, when applying the distributive property, it is not necessary to write $|i\rangle \langle j|$ if $l \neq k$. For instance, $tr_B(|00\rangle \langle 01|) = |0\rangle \langle 0| \, tr(|0\rangle \langle 1|) = |0\rangle \langle 0| \langle 0|1\rangle = 0 \, |0\rangle \langle 0|$. Also, since $\langle i|i\rangle = 1$:

$$\rho^A = \frac{|0\rangle \langle 0| + |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{3} \tag{103}$$

$$= \frac{2 \, |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{3} \tag{104}$$

Now, calculating $(\rho^A)^2$:

$$(\rho^A)^2 = \frac{(2 \, |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|) \, (2 \, |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|)}{3 \cdot 3} \tag{105}$$

$$= \frac{4 \, |0\rangle \langle 0| + 2 \, |0\rangle \langle 1| + |0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{9} \tag{106}$$

$$= \frac{5 \, |0\rangle \langle 0| + 3 \, |0\rangle \langle 1| + |1\rangle \langle 0| + 2 \, |1\rangle \langle 1|}{9} \tag{107}$$

Now, calculate $tr((\rho^A)^2)$:

$$tr((\rho^A)^2) = \frac{1}{9} \, tr(5 \, |0\rangle \langle 0| + 3 \, |0\rangle \langle 1| + |1\rangle \langle 0| + 2 \, |1\rangle \langle 1|) \tag{108}$$

$$= \frac{1}{9} \, (5 \, \langle 0|0\rangle + 3 \, \langle 1|0\rangle + \langle 0|1\rangle + 2 \, \langle 1|1\rangle) \tag{109}$$

$$= \frac{1}{9} \, (5 \cdot 1 + 3 \cdot 0 + 0 + 2 \cdot 1) \tag{110}$$

$$= \frac{1}{9} \, (5 + 2) \tag{111}$$

$$= \frac{7}{9} \tag{112}$$

Using an analogous line of thought $tr((\rho^B)^2) = \frac{7}{9}$ is obtained.

4.5.2 Exercise 2.78

PRODUCT STATE IF AND ONLY IF SCHIMIDT NUMBER 1.

Suppose state $|\psi\rangle$ is a product state of systems A and B, i.e. $A \otimes B$. Then, there exist orthonormal states $|a\rangle$ and $|b\rangle$, respectively for systems A and B, such that $|\psi\rangle = |a\rangle |b\rangle$. Therefore, the only possible Schmidt decomposition for state $|\psi\rangle$ is $|\psi\rangle = 1 |a\rangle |b\rangle + \sum_i 0 |i_A\rangle |i_B\rangle$ where $|i_A\rangle$ and $|i_B\rangle$ are part of the orthonormal bases alongside $|a\rangle$ and $|b\rangle$; in other words: $\langle a|a\rangle = \langle i_A|i_A\rangle = 1$ and $\langle a|i_A\rangle = \langle i_A|a\rangle = 0$, analogously for $|b\rangle$ and $|i_B\rangle$. As a consequence, the Schmidt number is 1 (refer to equation $|\psi\rangle = 1 |a\rangle |b\rangle + \sum_i 0 |i_A\rangle |i_B\rangle$).

Suppose state $|\psi\rangle$ has Schmidt number 1. Then, from Theorem 2.7, $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$. Since $|\psi\rangle$ has Schmidt number 1, there exist $\lambda_i = 1$ and $\lambda_j = 0$ such that $|\psi\rangle = 1 |i_A\rangle |i_B\rangle + \sum_j 0 |j_A\rangle |j_B\rangle = |i_A\rangle |i_B\rangle$. Therefore, $|\psi\rangle$ is a product state.

Quod erat demonstrandum.

PRODUCT STATE IF AND ONLY IF ρ^A ARE PURE STATES.

Suppose $|\psi\rangle$ is a product state of composite system $A \otimes B$, then $|\psi\rangle = |a\rangle |b\rangle$ where $|a\rangle$ and $|b\rangle$ are orthonormal states of systems A and B, respectively.

Then, by definition of density operator in Equation (2.138): $\rho = 1 \cdot |ab\rangle\langle ab|$, and ρ is pure if $tr(\rho^2) = 1$:

$$tr(\rho^2) = tr(|ab\rangle \langle ab|ab\rangle \langle ab|) \tag{113}$$
$$= tr(|ab\rangle\langle ab|) \tag{114}$$
$$= tr(\rho) \tag{115}$$

Them, by Theorem 2.5, $tr(\rho) = 1$.

If ρ is pure, consequently ρ^A and ρ^B are pure. Otherwise, $\langle a|a\rangle \neq 1 \neq \langle b|b\rangle$, which would be a contradiction with ρ being pure.

The converse can be proved naturally following these steps reversely. Suppose ρ^A is pure. Then ρ^B is pure. Then ρ is pure. Then $|\psi\rangle$ is a state product and can be written as $|\psi\rangle = |a\rangle |b\rangle$.

Quod erat demonstrandum.

4.5.3 Equations 2.208 and 2.209

When I firstly read these equations I thought there was a possibility that an extra explanation would be necessary. This thought raised, most likely, because I was unaccustomed to Tensor Product Properties and the Reduced Density Operator.

Using $|AR\rangle$ as defined in Equation 2.207:

$$|AR\rangle\langle AR| = \left(\sum_i \sqrt{p_i}\,|i^A\rangle\,|i^R\rangle\right)\left(\sum_j \sqrt{p_j}\,\langle j^A|\,\langle j^R|\right) \tag{116}$$

$$= \left(\sum_i \sqrt{p_i}\,|i^A\rangle \otimes |i^R\rangle\right)\left(\sum_j \sqrt{p_j}\,\langle j^A| \otimes \langle j^R|\right) \tag{117}$$

$$= \sum_{ij} \sqrt{p_i p_j}(\ |i^A\rangle \otimes |i^R\rangle\ )(\ \langle j^A| \otimes \langle j^R|\ ) \tag{118}$$

Then, by applying the properties as similarly defined in Equation 2.46:

$$\sum_{ij} \sqrt{p_i p_j}(\ |i^A\rangle \otimes |i^R\rangle\ )(\ \langle j^A| \otimes \langle j^R|\ ) = \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A| \otimes |i^R\rangle\langle j^R| \tag{119}$$

Therefore, using the definition of the Reduced Density Operator (Equation 2.178):

$$tr_R(|AR\rangle\langle AR|) = tr_R\left(\sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A| \otimes |i^R\rangle\langle j^R|\right) \tag{120}$$

$$= \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A|\,tr(|i^R\rangle\langle j^R|) \tag{121}$$

$$\tag{122}$$

Thus obtaining Equation 2.208.

In order to obtain Equation 2.209 it is necessary to apply $tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle$. Also, recall that $|i^R\rangle$ are orthonormal states and that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Hence,

$$\sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A|\,tr(|i^R\rangle\langle j^R|) = \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A|\,\langle j^R|i^R\rangle \tag{123}$$

$$= \sum_{ij} \sqrt{p_i p_j}\,|i^A\rangle\langle j^A|\,\delta_{ij} \tag{124}$$

as required.

4.5.4 Exercise 2.79

In order to solve this exercise, refer back to Theorem 2.7 and factor each state.

1.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\,|0\rangle\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,|1\rangle \tag{125}$$

$$= \sum_{i\in\{0,1\}} \lambda_i\,|i\rangle\,|i\rangle \tag{126}$$

where $\lambda_0 = \lambda_1 = 1/\sqrt{2}$.

2. Since Schmidt's decomposition requires that $|i_A\rangle$ and $|i_B\rangle$ are orthonormal states and $|+\rangle$ and $|-\rangle$ are examples of such states:

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{127}$$
$$= |+\rangle |+\rangle \tag{128}$$
$$= 1 \cdot |+\rangle |+\rangle + 0 \cdot |-\rangle |-\rangle \tag{129}$$
$$= \sum_{i \in \{+,-\}} \lambda_i |i\rangle |i\rangle \tag{130}$$

where $\lambda_+ = 1$ and $\lambda_- = 0$.

3. Analysing the state [2] $|\psi\rangle = \frac{|00\rangle+|01\rangle+|10\rangle}{\sqrt{3}}$ for both systems separately:

For the first qubit: $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$.

For the second qubit: $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$ as well.

It is easy to check that $\frac{\sqrt{2}|0\rangle+|1\rangle}{\sqrt{3}}$ is orthonormal. However, following this line of thought may lead to erroneous solutions. A bit more of cleverness is required: it is possible to use the interesting results obtained for ρ^A, ρ^B, and their eigenvalues as stated in the paragraph that follows Theorem 2.7.

Then, calculate ρ to obtain ρ^A and ρ^B afterwards. Since $|\psi\rangle$ is pure:

$$\rho = |\psi\rangle\langle\psi| \tag{131}$$
$$= \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \frac{\langle 00| + \langle 01| + \langle 10|}{\sqrt{3}} \tag{132}$$
$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{133}$$

Calculating ρ^A using the definition in Equations (2.177) and (2.178), alongside $tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle$ (Section 4.2.3):

$$\rho^A = \frac{1}{3} tr_B(\ (|00\rangle + |01\rangle + |10\rangle)(\langle 00| + \langle 01| + \langle 10|) \ ) \tag{134}$$
$$= \frac{1}{3}(\ |0\rangle\langle 0|\, tr(|0\rangle\langle 0| + |1\rangle\langle 1|) + |0\rangle\langle 1|\, tr(|0\rangle\langle 0|) + |1\rangle\langle 0|\, tr(|0\rangle\langle 0|) + |1\rangle\langle 1|\, tr(|0\rangle\langle 0|) \ ) \tag{135}$$
$$= \frac{1}{3}(\ 2\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| \ ) \tag{136}$$
$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{137}$$

[2] Note that similarly to state $(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$ described in Section 4.5.1 , $|\psi\rangle$ has symmetry 7/9. Hence, obtaining its Schmidt Decomposition is not intuitive.

Also, calculating ρ^B it is possible to verify that $\rho^B = \rho^A$. In order to write the Schmidt Decomposition, it is necessary to find the eigenvalues and eigenvectors of ρ^A:

$$\begin{vmatrix} 2/3 - v & 1 \\ 1 & 1/3 - v \end{vmatrix} = v^2 - v + \frac{1}{9} = 0 \tag{138}$$

Solving the polynomial, the eigenvalues found are $v = \frac{3\pm\sqrt{5}}{6}$. Calculate the corresponding eigenvectors $|v_1\rangle$ and $|v_2\rangle$.

For $v_1 = \frac{3+\sqrt{5}}{6}$: substitute and row reduce

$$\begin{bmatrix} \frac{4}{6} - \frac{3+\sqrt{5}}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{6} - \frac{3+\sqrt{5}}{6} \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1-\sqrt{5}}{6} \end{bmatrix} \tag{139}$$

$$\sim \begin{bmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{bmatrix} \qquad \sim \begin{bmatrix} 1-\sqrt{5} & 2 \\ 0 & 0 \end{bmatrix} \tag{140}$$

Then, the eigenspace of v_1 is $\left\{ \begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} \right\}$. The orthonormal vector of the eigenspace is of interest. Therefore, normalise $|v_1\rangle$:

$$|v_1\rangle = \frac{1}{\sqrt{\langle v_1|v_1\rangle}} |v_1\rangle \tag{141}$$

$$= \sqrt{\frac{2}{5+\sqrt{5}}} \begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} \tag{142}$$

For $v_2 = \frac{3-\sqrt{5}}{6}$, the normalised vector $|v_2\rangle = \sqrt{\frac{2}{5-\sqrt{5}}} \begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix}$ is found. [3]

From the results that follow Theorem 2.7, it is known that $\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$ where λ_i^2 are the eigenvalues of ρ^A, i.e. $\lambda_i^2 = v_i$. Therefore, since rho^A and rho^B have the same eigenvalues, by Theorem 2.7, $|\psi\rangle$ has Schmidt Decomposition

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle = \sum_i \sqrt{v_i} |v_i\rangle\langle v_i| \tag{143}$$

$$\begin{aligned} &= \sqrt{\frac{3+\sqrt{5}}{6}} \left(\sqrt{\frac{2}{5+\sqrt{5}}} \begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} \right) \otimes \left(\sqrt{\frac{2}{5+\sqrt{5}}} \begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} \right) \\ &+ \sqrt{\frac{3-\sqrt{5}}{6}} \left(\sqrt{\frac{2}{5-\sqrt{5}}} \begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix} \right) \otimes \left(\sqrt{\frac{2}{5-\sqrt{5}}} \begin{bmatrix} \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix} \right) \end{aligned} \tag{144}$$

$$= \sqrt{\frac{3+\sqrt{5}}{6}} \frac{2}{5+\sqrt{5}} \begin{bmatrix} \frac{4}{6-2\sqrt{5}} \\ \frac{-2}{1-\sqrt{5}} \\ \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{6}} \frac{2}{5-\sqrt{5}} \begin{bmatrix} \frac{4}{6+2\sqrt{5}} \\ \frac{-2}{1+\sqrt{5}} \\ \frac{-2}{1+\sqrt{5}} \\ 1 \end{bmatrix} \tag{145}$$

[3] Verify that the values found match $\langle v_1|v_2\rangle = \langle v_2|v_1\rangle = 0$, $\langle v_1|v_1\rangle = \langle v_2|v_2\rangle = 1$, and $\rho^A = \rho^B = \sum_i v_i |v_i\rangle\langle v_i|$

In order to keep calculating, it is necessary to rewrite $\sqrt{3+\sqrt{5}}$ with the help of quadratic polynomials:

$$\begin{aligned}
\sqrt{3+\sqrt{5}} &= \sqrt{\frac{1+\sqrt{5}^2+2\sqrt{5}}{2}} && (146)\\
&= \sqrt{\frac{(1+\sqrt{5})^2}{2}} && (147)\\
&= \frac{1+\sqrt{5}}{\sqrt{2}} && (148)
\end{aligned}$$

Similarly, $\sqrt{3-\sqrt{5}} = \frac{1-\sqrt{5}}{\sqrt{2}}$.

Therefore, substituting in Equation 145:

$$|\psi\rangle = \frac{1+\sqrt{5}}{\sqrt{2}\sqrt{6}}\,\frac{2}{5+\sqrt{5}}\begin{bmatrix}\frac{4}{6-2\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ 1\end{bmatrix} + \frac{1-\sqrt{5}}{\sqrt{2}\sqrt{6}}\,\frac{2}{5-\sqrt{5}}\begin{bmatrix}\frac{4}{6+2\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ 1\end{bmatrix} \tag{149}$$

$$= \frac{1+\sqrt{5}}{\sqrt{3}}\,\frac{1}{5+\sqrt{5}}\begin{bmatrix}\frac{2}{3-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ \frac{-2}{1-\sqrt{5}}\\ 1\end{bmatrix} + \frac{1-\sqrt{5}}{\sqrt{3}}\,\frac{1}{5-\sqrt{5}}\begin{bmatrix}\frac{2}{3+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ \frac{-2}{1+\sqrt{5}}\\ 1\end{bmatrix} \tag{150}$$

$$= \begin{bmatrix}\frac{2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(3-\sqrt{5})} + \frac{2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(3+\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(1-\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(1+\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(1-\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(5-\sqrt{5})(1+\sqrt{5})}\\ \frac{1+\sqrt{5}}{\sqrt{3}(5+\sqrt{5})} + \frac{1-\sqrt{5}}{\sqrt{3}(5-\sqrt{5})}\end{bmatrix} = \begin{bmatrix}\frac{2(1+\sqrt{5})}{\sqrt{3}(10-2\sqrt{5})} + \frac{2(1-\sqrt{5})}{\sqrt{3}(10+2\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(-4\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(4\sqrt{5})}\\ \frac{-2(1+\sqrt{5})}{\sqrt{3}(-4\sqrt{5})} + \frac{-2(1-\sqrt{5})}{\sqrt{3}(4\sqrt{5})}\\ \frac{1+\sqrt{5}}{\sqrt{3}(5+\sqrt{5})} + \frac{1-\sqrt{5}}{\sqrt{3}(5-\sqrt{5})}\end{bmatrix} \tag{151}$$

$$= \begin{bmatrix}\frac{(1+\sqrt{5})(5+\sqrt{5})+(1-\sqrt{5})(5-\sqrt{5})}{\sqrt{3}(5+\sqrt{5})(5-\sqrt{5})}\\ \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{(1+\sqrt{5})(5-\sqrt{5})+(1-\sqrt{5})(5+\sqrt{5})}{20\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{10+6\sqrt{5}+10-6\sqrt{5}}{20\sqrt{3}}\\ \frac{2\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{2\sqrt{5}}{2\sqrt{5}\sqrt{3}}\\ \frac{4\sqrt{5}-4\sqrt{5}}{20\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ 0\end{bmatrix} \tag{152}$$

$$= \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \tag{153}$$

as requested.

4.5.5 Exercise 2.80

If $|\psi\rangle$ and $|\varphi\rangle$ are pure states, then they can be written as $|\psi\rangle = \sum_i p_i |i_A\rangle |i_B\rangle$ where $p_i = 1$ for a specific i and $p_i = 0$ otherwise. Therefore, $|\psi\rangle = |i_A\rangle |i_B\rangle$, analogously $|\varphi\rangle = |j_A\rangle |j_B\rangle$ where $|i_A\rangle$, $|i_B\rangle$, $|j_A\rangle$, $|j_B\rangle$ are unitary.

Then, construct two unitary matrices U and V such that $|i_A\rangle = U|j_A\rangle$ and $|i_B\rangle = V|j_B\rangle$. Then, using Equation (2.46), $|\psi\rangle$ can be written as:

$$|\psi\rangle = |i_A\rangle|i_B\rangle \tag{154}$$
$$= |i_A\rangle \otimes |i_B\rangle \tag{155}$$
$$= U|j_A\rangle \otimes V|j_B\rangle \tag{156}$$
$$= (U \otimes V)|j_A\rangle \otimes |j_B\rangle \tag{157}$$
$$= (U \otimes V)|j_A\rangle|j_B\rangle \tag{158}$$
$$= (U \otimes V)|\varphi\rangle \tag{159}$$

4.5.6 Exercise 2.81

It is possible to solve this exercise by following the same logic as Exercise 2.80 (Section 4.5.5). Define an Unitary Operator $U_R \equiv \sum_i |v_i\rangle\langle w_i|$, where $|v_i\rangle$ are an orthonormal basis for R_1 and $|w_i\rangle$ for R_2. Therefore, $|v_i\rangle = U_R|w_i\rangle$. Then, by Equations (2.207) and (2.46):

$$|AR_1\rangle = \sum_i \sqrt{p_i}|i_A\rangle|v_i\rangle \tag{160}$$
$$= \sum_i \sqrt{p_i}|i_A\rangle\ U_R|w_i\rangle \tag{161}$$
$$= \sum_i I\sqrt{p_i}|i_A\rangle \otimes U_R|w_i\rangle \tag{162}$$
$$= (I \otimes U_R)\sum_i \sqrt{p_i}|i_A\rangle \otimes |w_i\rangle \tag{163}$$
$$= (I \otimes U_R)|AR_2\rangle \tag{164}$$

4.5.7 Exercise 2.82

1. From the definition of purification, it is necessary to prove that $\rho^A = tr_R(|AR\rangle\langle AR|)$. Suppose $\sum_i \sqrt{(p_i)}|\psi_i\rangle|i\rangle$ is a purification. From equation (2.138) of Nielsen and Chuang's book:

$$\rho^{AB} = \sum_{ij} \left(\sqrt{p_i}|\psi_i\rangle|i\rangle\right)\left(\sqrt{p_j}|\psi_j\rangle|j\rangle\right)^\dagger \tag{165}$$

Since $p_j \in \mathbb{R}$, $p_j^\dagger = p_j$. And since it is a scalar:

$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j}\left(|\psi_i\rangle|i\rangle\right)\left(|\psi_j\rangle|j\rangle\right)^\dagger \tag{166}$$

Recall that for any states $|\varphi\rangle$ and $|\gamma\rangle$ writing $|\varphi\rangle|\gamma\rangle$ is the same as $|\varphi\rangle \otimes |\gamma\rangle$. Then, applying equation (2.48):

$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j}\left(|\psi_i\rangle \otimes |i\rangle\right)\left(\langle\psi_j| \otimes \langle j|\right) \tag{167}$$
$$\rho^{AB} = \sum_{ij} \sqrt{p_i p_j}(|\psi_i\rangle\langle\psi_j|) \otimes (|i\rangle\langle j|) \tag{168}$$

If $\sum_i \sqrt{p_i} \left|\psi_i\right\rangle \left|i\right\rangle$ is a purification, then $\rho^A = tr_B(\ (\left|\psi_i\right\rangle \left|i\right\rangle)(\left\langle\psi_i\right| \left\langle i\right|)\ )$. The question gives $\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$, in other words $\rho = \rho^A$.

Now, it is necessary to calculate $tr_B(\ (\left|\psi_i\right\rangle \left|i\right\rangle)(\left\langle\psi_i\right| \left\langle i\right|)\ )$. Then, using the definitions given in equations (2.177) and (2.178) of Nielsen and Chuang's book:

$$tr_B(\rho^{AB}) = tr_B\left(\sum_{ij} \sqrt{p_i p_j}(\left|\psi_i\right\rangle \left\langle\psi_j\right|) \otimes (\left|i\right\rangle \left\langle j\right|)\right) \tag{169}$$

$$= \sum_{ij} \sqrt{p_i p_j}(\left|\psi_i\right\rangle \left\langle\psi_j\right|) tr(\left|i\right\rangle \left\langle j\right|) \tag{170}$$

Since $tr(\left|a\right\rangle \left\langle b\right|) = \left\langle b|a\right\rangle$ (refer back to Section 4.2.3):

$$tr_B(\rho^{AB}) = \sum_{ij} \sqrt{p_i p_j}\ (\left|\psi_i\right\rangle \left\langle\psi_j\right|) \left\langle j|i\right\rangle \tag{171}$$

$$= \sum_{ij} \sqrt{p_i p_j}\ \left|\psi_i\right\rangle \left\langle\psi_j\right|\ \delta_{ij} \tag{172}$$

$$= \sum_{i} \sqrt{p_i p_i}\ \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{173}$$

$$= \sum_{i} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{174}$$

Since $\rho^A = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| = tr_B(\rho^{AB})$, it is possible to conclude that $\sum_i \sqrt{p_i} \left|\psi_i\right\rangle \left|i\right\rangle$ is a purification.

2. **TODO: ADD INTUITIVE SOLUTION/EXPLANATION**

For this exercise, it is necessary to review Postulate 3. For the system $\left|AR\right\rangle$, there is a set of measurement operators $\{M_m\}$. Only system R is being measured, though. So, it is possible to define every measurement operator $M_m = I \otimes M_i$ where I is the identity operator acting on system A and M_i is the measurement operator acting on system R which corresponds to measuring the state $\left|i\right\rangle$.

The probability of measuring $\left|i\right\rangle$ is requested, i.e. $p(i)$. Using $\left|\varphi_i\right\rangle = \sqrt{p_i} \left|\psi_i\right\rangle \left|i\right\rangle$ temporarily for simplicity and by Equation (2.92):

$$p(i) = \left\langle\varphi_i\right| M_m^\dagger M_m \left|\varphi_i\right\rangle \tag{175}$$

$$= \left\langle\varphi_i\right| (I \otimes M_i)^\dagger (I \otimes M_i) \left|\varphi_i\right\rangle \tag{176}$$

$$= \sqrt{p_i} \left\langle\psi_i\right| \left\langle i\right| (I \otimes M_i)^\dagger (I \otimes M_i) \sqrt{p_i} \left|\psi_i\right\rangle \left|i\right\rangle \tag{177}$$

$$= p_i \left\langle\psi_i\right| \left\langle i\right| (I^\dagger \otimes M_i^\dagger)(I \otimes M_i) \left|\psi_i\right\rangle \left|i\right\rangle \tag{178}$$

$$= p_i \left\langle\psi_i\right| \left\langle i\right| (I \otimes M_i^\dagger)(I \otimes M_i) \left|\psi_i\right\rangle \left|i\right\rangle \tag{179}$$

Using equation (2.48):

$$p(i) = p_i(\left\langle\psi_i\right| I \otimes \left\langle i\right| M_i^\dagger)\ (I \left|\psi_i\right\rangle \otimes M_i \left|i\right\rangle) \tag{180}$$

$$= p_i(\left\langle\psi_i\right| \otimes \left\langle i\right| M_i^\dagger)\ (\left|\psi_i\right\rangle \otimes M_i \left|i\right\rangle) \tag{181}$$

Then, by the definition of inner product (equation (2.49)):

$$p(i) = p_i \langle\psi_i|\psi_i\rangle \langle i| M_i^\dagger M_i |i\rangle \tag{182}$$

Recall that $|\psi_i\rangle$ and $|i\rangle$ are orthonormal. Also, since M_i is the measurement operator that corresponds to obtaining state $|i\rangle$, it is possible to consider $|i\rangle$ as a "measurement basis" defining $M_i = |i\rangle\langle i|$. A similar example can be seen in Nielsen and Chuang's book in a paragraph between Equations (2.95) and (2.96). Therefore,

$$\begin{aligned} p(i) &= p_i \cdot 1 \cdot \langle i| (|i\rangle\langle i|)^\dagger (|i\rangle\langle i|) |i\rangle &(183)\\ &= p_i \langle i| (|i\rangle\langle i|)(|i\rangle\langle i|) |i\rangle &(184)\\ &= p_i \langle i|i\rangle \langle i|i\rangle \langle i|i\rangle &(185)\\ &= p_i \cdot 1 \cdot 1 \cdot 1 &(186)\\ &= p_i &(187) \end{aligned}$$

Hence, the probability of measuring state $|i\rangle$ is p_i. Now, it is requested to obtain the state of system A after the measurement M_m, which is described by Postulate 3 as:

$$\begin{aligned} \frac{M_m |\varphi_i\rangle}{\sqrt{p_i}} &= \frac{(I \otimes M_i)\sqrt{p_i} |\psi_i\rangle |i\rangle}{\sqrt{p_i}} &(188)\\ &= (I |\psi_i\rangle) \otimes (M_i |i\rangle) &(189)\\ &= (I |\psi_i\rangle) \otimes (|i\rangle \langle i|i\rangle) &(190)\\ &= |\psi_i\rangle |i\rangle &(191) \end{aligned}$$

Therefore, the measurement of the system A will always be $|\psi_i\rangle$.

3. Goropikari attempted to solve this exercise as follows[4]:

> Suppose $|AR\rangle$ is a purification of ρ such that $|AR\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle$. By exercise 2.81, the others purification is written as $(I \otimes U) |AR\rangle$.
>
> $$\begin{aligned} (I \otimes U) |AR\rangle &= (I \otimes U) \sum_i \sqrt{p_i} |\psi_i\rangle |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle U |r_i\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle \end{aligned}$$
>
> where $U = \sum_i |i\rangle \langle r_i|$.
>
> By (2), if we measure the system R w.r.t $|i\rangle$, post-measurement state for system A is $|\psi_i\rangle$ with probability p_i, which prove the assertion.

TODO: IS THE PREVIOUS SOLUTION PLAUSIBLE IN SOME WAY?

However, if system R is measured with respected to $|i\rangle$ (that is, the measurement operator $M_m = I \otimes |i\rangle\langle i|$ is applied to $\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$) the same result of Exercise 2.82(2) will be achieved.

[4] The original code can be found at SolutionForQuantumComputationAndQuantumInformation

A similar way to try to solve this problem is: define $|AR_i\rangle = |\psi_i\rangle |r_i\rangle$ and use the measurement operator $M_m = I \otimes |i\rangle \langle i|$. Then, the probability of measuring $|i\rangle$ is calculated according to Equation (2.92):

$$p(i) = (\sqrt{p_i}\ (I \otimes |i\rangle\langle i|)\,|\psi_i\rangle\,|r_i\rangle)^\dagger\ \sqrt{p_i}\ (I \otimes |i\rangle\langle i|)\,|\psi_i\rangle\,|r_i\rangle \tag{192}$$
$$= p_i\,\langle r_i|\,\langle\psi_i|\,(|i\rangle\langle i| \otimes I)\ (I \otimes |i\rangle\langle i|)\,|\psi_i\rangle\,|r_i\rangle \tag{193}$$

Then, using Equations (2.48) and (2.49):

$$p(i) = p_i(\langle r_i|\,|i\rangle\langle i|) \otimes (\langle\psi_i|\,I)\ (I\,|\psi_i\rangle) \otimes (|i\rangle\langle i|\,|r_i\rangle) \tag{194}$$
$$= p_i(\langle r_i|i\rangle\,\langle i|) \otimes \langle\psi_i|\ \ |\psi_i\rangle \otimes (\langle i|r_i\rangle\,|i\rangle) \tag{195}$$
$$= p_i\,\langle r_i|i\rangle\,\langle i|r_i\rangle\ (\langle i|\,\langle\psi_i|\,|\psi_i\rangle\,|i\rangle) \tag{196}$$
$$= p_i\ ||\,\langle i|r_i\rangle\,||^2 \tag{197}$$

if $||\,\langle i|r_i\rangle\,||^2 = 1/p_i$, the desired result would obtained. However, $0 < p_i < 1$, then $1/p_i > 1$, which is not possible since both $|i\rangle$ and $|r_i\rangle$ are orthonormal vectors. Independently, if the post-measurement state is calculated as given by Equation (2.93):

$$\frac{M_m\,|\psi\rangle}{\sqrt{\langle\psi|\,M_m^\dagger M_m\,|\psi\rangle}} = \frac{(I \otimes |i\rangle\langle i|)\sqrt{p_i}\,|\psi_i\rangle\,|r_i\rangle}{\sqrt{p_i||\,\langle i|r_i\rangle\,||^2}} \tag{198}$$
$$= \frac{\sqrt{p_i}\,|\psi_i\rangle\,|i\rangle\langle i|\,|r_i\rangle}{\sqrt{p_i}\ ||\,|i\rangle\langle r_i|\,||} \tag{199}$$
$$= \frac{\langle i|r_i\rangle\,|\psi_i\rangle\,|i\rangle}{||\,\langle i|r_i\rangle\,||} \tag{200}$$

Therefore, $\frac{\langle i|r_i\rangle}{\sqrt{\langle i|r_i\rangle\,\langle r_i|i\rangle}} = p_i$. **TODO: IN CONCLUSION ?????**

4.6 Chapter 2 Problems

4.6.1 Problem 2.1

This problem can be solved easily by combining the logic of Exercises 2.35 and 2.60.

It is known from exercise 2.60 that $\vec{n}\vec{\sigma}$ has spectral decomposition $+1P_+ - 1P_- = +1\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) - 1\left(\frac{I-\vec{v}\vec{\sigma}}{2}\right)$. Therefore, $\theta\vec{n}\vec{\sigma} = \theta\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) - \theta\left(\frac{I-\vec{v}\vec{\sigma}}{2}\right)$. Then, by applying the definition of function operators and the distributive property:

$$f(\theta\vec{n}\vec{\sigma}) = f(\theta)\left(\frac{I+\vec{n}\vec{\sigma}}{2}\right) + f(-\theta)\left(\frac{I-\vec{v}\vec{\sigma}}{2}\right) \tag{201}$$
$$= \frac{f(\theta)}{2}I + \frac{f(-\theta)}{2}I + \frac{f(\theta)}{2}\vec{n}\vec{\sigma} - \frac{f(-\theta)}{2}\vec{n}\vec{\sigma} \tag{202}$$
$$= \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\vec{\sigma} \tag{203}$$

4.6.2 Problem 2.2

4.6.3 Problem 2.3