Ve460 Control Systems Analysis and Design
Chapter 7 Time Domain Analysis

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7-1 Introduction

Time Response: state and output responses with respect to time
$\rightarrow$ evaluate the performance of the system.

$$\text{Time response} \begin{cases} \text{Transient response} \\ \text{Steady-state response} \end{cases}$$

$y(t) =$	$y_t(t)$	$+$	$y_{ss}(t)$
	Transient		Steady-state
	goes to 0 as $t \to \infty$, $\lim_{t\to\infty} y_t(t) = 0$		what remains after transient is gone
	significant part of the dynamic behavior		not necessarily constant, *e.g.*, sinusoidal, ramp
			steady-state error, $\lim_{t\to\infty}(y(t) - r(t))$

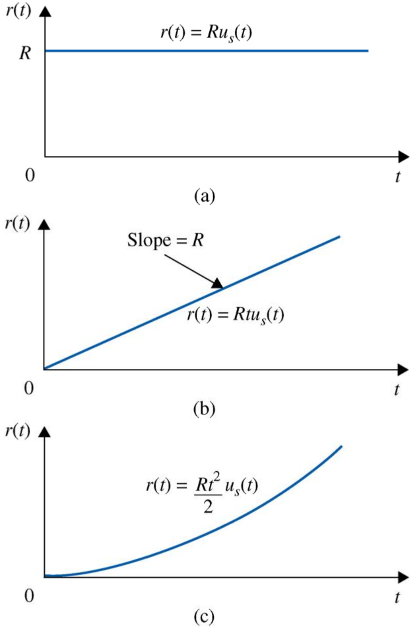
7-2 Typical test signals

Actual inputs of a control system are not exactly known ahead of time

- It is then difficult to design a control system performing well to all possible inputs;
- We then use some typical test signals so that the performance can be evaluated.

Benefits:

- Performance criteria can be specified with respect to these test signals;
- Particularly useful for linear systems.



1. Step function input

$$r(t) = \begin{cases} R, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or $r(t) = Ru_s(t)$.

2. Ramp function input

$$r(t) = Rtu_s(t).$$

3. Parabolic function input

$$r(t) = \frac{R}{2} t^2 u_s(t).$$

4. Jerk function input
$r(t) \propto t^3 u_s(t)$.

7-4 Steady-State Error

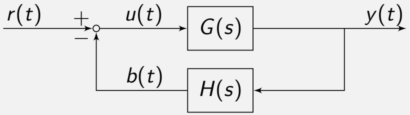
Control system: want output response to follow a reference, at least, in the steady state.

In real system, it is hard to completely eliminate steady state error. Then,

- Keep it small;
- Accuracy depends on control objective, *e.g.*, elevator v.s. space telescope.

7-4-1 Steady-State Error of Linear Continuous Control System

For linear continuous system, it depends on the type of reference signal and system.



$$e(t) = \underbrace{\text{reference signal}}_{\substack{\text{the signal that}\\ y(t)\text{ is to track}}} - y(t)$$

When $H(s) = 1$, *i.e.*, unity feedback, $r(t)$ is the reference.

$$\Rightarrow \quad e(t) = r(t) - y(t)$$
$$e_{ss} = \lim_{t\to\infty} e(t)$$

When $H(s) \neq 1$, actuating signal $u(t)$ may or may not be the error.

Example

Given $G(s) = \dfrac{1}{s^2(s+12)}$, $H(s) = 1$,

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{1}{s^2(s+12)}}{1+\dfrac{1}{s^2(s+12)}} = \frac{1}{s^3+12s^2+1}.$$

Unstable!

If $H(s) = \dfrac{5(s+1)}{s+5}$,

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{1}{s^2(s+12)}}{1+\dfrac{1}{s^2(s+12)}\cdot\dfrac{5(s+1)}{s+5}}$$

$$= \frac{s+5}{s^4+17s^3+60s^2+5s+5}.$$

s^4	1	60	5
s^3	17	5	0
s^2	$\frac{1015}{17}$	5	
s^1	$\frac{5075-1445}{17} = \frac{3630}{17}$		
s^0	5		

$\Rightarrow$ stable.
Since $H(0) = 1$, error can still be defined as

$$e(t) = r(t) - y(t).$$

Now consider a velocity control.

Input: step function;

Output: has a ramp in steady state.

$$R(s) = \frac{1}{s}, \quad G(s) = \frac{1}{s^2(s+12)}, \quad H(s) = K_t s,$$

where $H(s)$ is the transfer function for a tachometer and K_t is the tachometer constant.

The Closed-Loop TF is:

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{s^2(s+12)}}{1+\frac{1}{s^2(s+12)}K_t s} = \frac{1}{s(s^2+12s+K_t)}.$$

Due to the integrator, $r(t)$ and $y(t)$ are not of the same dimension. The reference is thus the desired velocity, not $r(t)$.

If $K_t = 10$ volts/(rad/sec), for a unit step input of 1 volt, the output is

$$
\begin{aligned}
& Y(s) = \frac{1}{s} \cdot M(s) \\
\Rightarrow \quad & y(t) = 0.1t - 0.12 - 0.000769e^{-11.1t} + 0.1208e^{-0.901t}.
\end{aligned}
$$

The desired velocity in the steady state is 0.1 rad/sec, as

$$K_t \cdot 0.1 \text{ rad/sec} = 1 \text{ volt} \quad \Rightarrow \quad \text{zero actuating signal}$$

The reference signal is thus $0.1t$. When $t \to \infty$,
$y(t) \to 0.1t - 0.12$.

$$\therefore \quad e_{ss} = \lim_{t\to\infty} (0.1t - y(t)) = 0.12.$$

Classify three types of systems

1. Unity feedback $H(s) = 1$;
2. Non-unity feedback but $H(0) = K_H = \text{const}$;
3. Non-unity feedback and $H(s)$ has zeros at $s = 0$ of order N.

Case 1. Unity Feedback System

If $sE(s)$ has no poles in the RHP or on the $j\omega$-axis

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+G(s)} \cdot R(s).$$

e_{ss} depends on the number of poles $G(s)$ has at $s = 0$. Let

$$G(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots(1+T_{m_1} s + T_{m_2} s^2)}{s^j(1+T_a s)(1+T_b s)\cdots(1+T_{n_1} s + T_{n_2} s^2)} e^{-T_d s}.$$

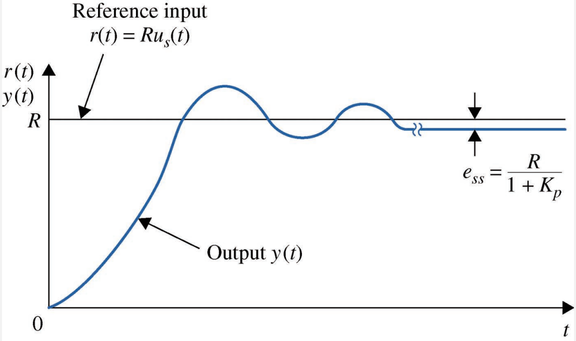
This TF is of system type j.

1. Step function input $r(t) = Ru_s(t)$, $R(s) = R/s$

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{sR(s)}{1+G(s)} \\
&= \lim_{s\to 0} \frac{R}{1+G(s)} \\
&= \frac{R}{1+\lim_{s\to 0} G(s)}.
\end{aligned}
$$

Define $K_p = \lim_{s\to 0} G(s)$ as the steady-state error constant, then

$$
e_{ss} = \frac{R}{1+K_p}.
$$



- Type 0: $j = 0$, $K_p = K$,

$$e_{ss} = \frac{R}{1+K} = \text{const};$$

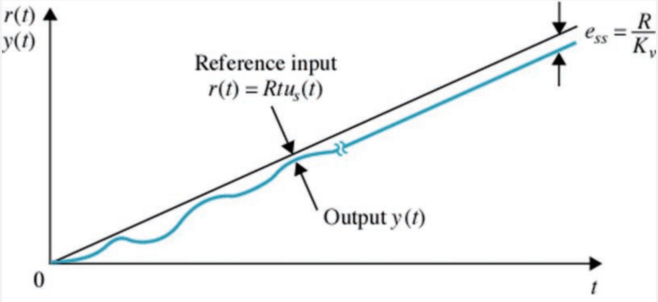
- Type 1 or higher: $j \geq 1$, $K_p = \infty$, $e_{ss} = 0$.

2. Ramp function input $r(t) = Rtu_s(t)$, $R(s) = R/s^2$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot R/s^2}{1+G(s)}$$
$$= \lim_{s\to 0} \frac{R}{s+sG(s)} = \lim_{s\to 0} \frac{R}{sG(s)}.$$

Define ramp error constant $K_\nu = \lim_{s\to 0} sG(s)$, then

$$K_\nu = \lim_{s\to 0} \frac{K}{s^{j-1}} \text{ and } e_{ss} = \frac{R}{K_\nu}.$$



- Type 0: $K_\nu = 0$, $e_{ss} = \infty$;
- Type 1: $K_\nu = K$,

$$e_{ss} = \frac{R}{K_\nu} = \text{const};$$

- Type ≥ 2: $K_\nu = \infty$, $e_{ss} = 0$.

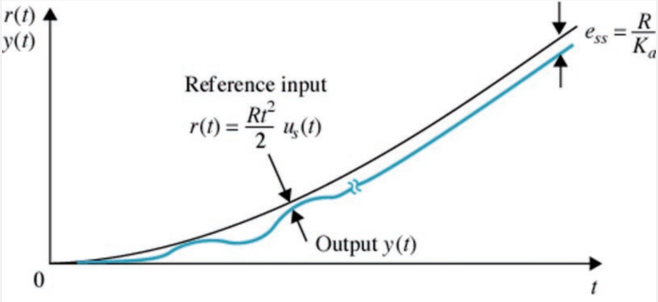
3. Parabolic function input

For $r(t) = \frac{R}{2}t^2 u_s(t)$, $R(s) = \frac{R}{s^3}$, we have

$$e_{ss} = \lim_{s\to 0} \frac{s \cdot R/s^3}{1+G(s)} = \lim_{s\to 0} \frac{R}{s^2 G(s)}.$$

Define parabolic error constant $K_a = \lim_{s\to 0} s^2 G(s)$, then

$$e_{ss} = \frac{R}{K_a} \text{ and } K_a = \lim_{s\to 0} \frac{K}{s^{j-2}}.$$



- Type 0: $K_a = 0$, $e_{ss} = \infty$;
- Type 1: $K_a = 0$, $e_{ss} = \infty$;
- Type 2: $K_a = K$, $e_{ss} = \dfrac{R}{K}$;
- Type ≥ 3: $K_a = \infty$, $e_{ss} = 0$.

Table 1: Summary of the Steady-State Errors due to Step-, Ramp- and Parabolic-Function Inputs for Unity-Feedback Systems.

Type j	Error Constants K_p	K_ν	K_a	Steady-State Error e_{ss}: Step $\frac{R}{1+K_p}$	Ramp $\frac{R}{K_\nu}$	Parabolic $\frac{R}{K_a}$
0	K	0	0	$\frac{R}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{R}{K}$	∞
2	∞	∞	K	0	0	$\frac{R}{K}$
3	∞	∞	∞	0	0	0

Case 2. Non-unity Feedback System

We have studied

Steady State Error vs $G(s)$ (Forward-path TF)

Now we want to study

Steady State Error vs Closed-loop TF

What is $G(0)$?

DC gain

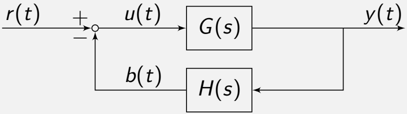
$$\longrightarrow \boxed{G(s)} \longrightarrow$$

$$G(0) = \lim_{s \to 0} G(s) = \lim_{s \to 0} s \cdot \left(G(s) \cdot \frac{1}{s} \right).$$

Suppose $\lim_{s\to 0} H(s) = H(0) = K_H = \text{const}$, $K_H \neq 0$ and finite, then $H(s)$ has no poles or zeros at 0.

As we want

$$r(\infty) = K_H \cdot y(\infty),$$
$$\text{(to make } u = 0\text{)}$$



we can define error signal as

$$e(t) = \frac{r(t)}{K_H} - y(t)$$

or

$$E(s) = \frac{1}{K_H}R(s) - Y(s) = \frac{1}{K_H}\left[1 - K_H \underbrace{\frac{Y(s)}{R(s)}}_{\text{CLTF}}\right]R(s).$$

Remember that control design is to arrange the locations of poles and zeros. Let's say we can achieve the following CLTF:

$$M(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.$$

$M(s)$ has no poles at $s = 0$, $n > m$. Now

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) = \frac{1}{K_H} \lim_{s\to 0} (1 - K_H \cdot M(s)) \cdot sR(s) \\ &= \frac{1}{K_H} \lim_{s\to 0} \frac{s^n + \cdots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \cdot sR(s) \end{aligned}$$

How to eliminate e_{ss}?

1. Step function input $R(s) = R/s$

$$e_{ss} = \frac{1}{K_H} \cdot \frac{a_0 - K_H b_0}{a_0} \cdot R$$

so if $a_0 - K_H b_0 = 0$, or $M(0) = \dfrac{b_0}{a_0} = \dfrac{1}{K_H}$, then $e_{ss} = 0$.

2. Ramp function input $R(s) = R/s^2$

$$e_{ss} = \frac{R}{K_H} \lim_{s \to 0} \frac{s^n + \cdots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)}$$

1. if $a_0 - b_0 K_H = 0$ and $a_1 - b_1 K_H = 0$, then $e_{ss} = 0$;
2. if $a_0 - b_0 K_H = 0$ and $a_1 - b_1 K_H \neq 0$, then

$$e_{ss} = \frac{R}{K_H} \cdot \frac{a_1 - b_1 K_H}{a_0} = \text{const};$$

3. if $a_0 - b_0 K_H \neq 0$, then $e_{ss} = \infty$.

3. Parabolic function input $R(s) = R/s^3$

$$e_{ss} = \frac{R}{K_H} \lim_{s\to 0} \frac{s^n + \cdots + (a_1 - b_1K_H)s + (a_0 - b_0K_H)}{s^2(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)}$$

1. if $a_k - b_kK_H = 0$ for $k = 0, 1, 2$, then $e_{ss} = 0$;
2. if $a_k - b_kK_H = 0$ for $k = 0, 1$, then $e_{ss} = \frac{R}{K_H} \cdot \frac{a_2 - b_2K_H}{a_0}$;
3. if either $a_0 - b_0K_H \neq 0$ or $a_1 - b_1K_H \neq 0$, then $e_{ss} = \infty$.

Case 3. Non-unity feedback and $H(s)$ has N-th order zero at $s = 0$

We compare $R(s)$ with N-th order derivative of output. Let

$$E(s) = \frac{1}{K_H s^N} R(s) - Y(s),$$

where $K_H = \lim_{s\to 0} H(s)/s^N$.

Example

Consider the case $N = 1$. We have

$$E(s) = \frac{1}{K_H}\left[\frac{1}{s} - K_H\frac{Y(s)}{R(s)}\right]R(s),$$

and then

$$e_{ss} = \frac{1}{K_H}\lim_{s\to 0}\frac{s^n + \cdots + (a_2 - b_1K_H)s^2 + (a_1 - b_0K_H)s + a_0}{s(s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0)}sR(s).$$

For step input R/s, we require that $a_0 = 0$ to completely eliminate the steady state error. Therefore,

$$\begin{aligned} e_{ss} &= \frac{1}{K_H}\lim_{s\to 0}\frac{s^n + \cdots + (a_2 - b_1K_H)s^2 + (a_1 - b_0K_H)s}{s(s^n + a_{n-1}s^{n-1} + \cdots + a_1s)}R \\ &= \frac{1}{K_H}\lim_{s\to 0}\frac{s^{n-1} + \cdots + (a_2 - b_1K_H)s + (a_1 - b_0K_H)}{s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1)}R. \end{aligned}$$

From

$$e_{ss} = \frac{1}{K_H} \lim_{s \to 0} \frac{s^{n-1} + \cdots + (a_2 - b_1 K_H)s + (a_1 - b_0 K_H)}{s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1)} R,$$

we have

1. if $a_1 - b_0 K_H \neq 0$, then $e_{ss} = \infty$;
2. if $a_1 - b_0 K_H = 0$ and $a_2 - b_1 K_H \neq 0$, then

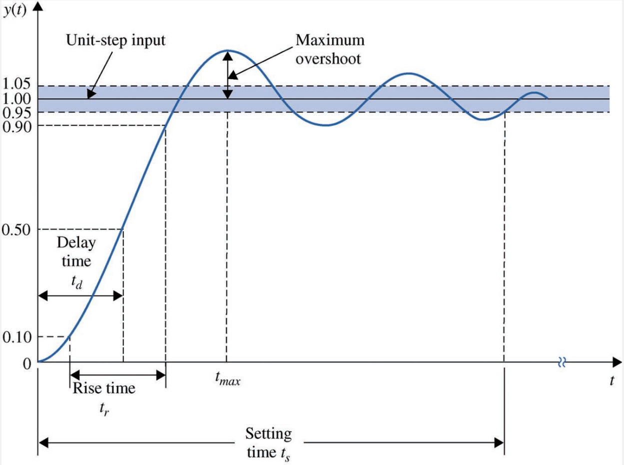
$$e_{ss} = \frac{R}{K_H} \frac{a_2 - b_1 K_H}{a_1} (a_1 \neq 0 \text{ to ensure stability });$$

3. if $a_1 - b_0 K_H = 0$ and $a_2 - b_1 K_H = 0$, then $e_{ss} = 0$.

7-3 Unit-step Response and Time-domain Specifications

Mainly study the transient response, since both amplitude and time duration must stay within tolerable or prescribed limits.

For linear control system, often studied is unit-step response.



Time-domain specifications

1. Maximum overshoot. Define

$$y_{\max} = \max_t y(t), \quad y_{ss} = \lim_{t\to\infty} y(t).$$

max overshoot: $y_{\max} - y_{ss}$

percent max overshoot: $\dfrac{y_{\max} - y_{ss}}{y_{ss}} \times 100\%$

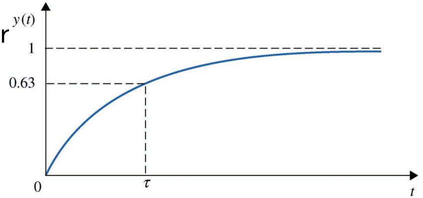
$\rightarrow$ used to measure the relative stability;

2. Delay time T_d: time required to reach $y_{ss}/2$;
3. Rise time t_r: time required from $0.1y_{ss}$ to $0.9y_{ss}$;
4. Settling time t_s: time required to stay within $\pm 5\%$ of y_{ss};
5. Steady state error $e_{ss} := \lim_{t\to\infty} (y(t) - r(t))$.

7-5 Time Response of 1st Order System

A simplified model of DC motor can be given as

$$G(s) = \frac{K}{s+a}, \quad a > 0$$



For unit step input,

$$Y(s) = G(s) \cdot \frac{1}{s} = \frac{K}{s(s+a)}$$
$$= \frac{K}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$
$$\Rightarrow \quad y(t) = \frac{K}{a}(1 - e^{-at}), \ t \geq 0$$

Observations

- No overshoot;
- Pole: $s = -a \Rightarrow$ stable;
- Steady state K/a;
- Rate of change depends on a.

7-6 Transient Response of 2nd Order System

Second order systems help understand, analyze, and design higher order systems.

Prototype 2nd-order system

Open-loop TF:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},$$

where ζ, $\omega_n \in \mathbb{R}$.

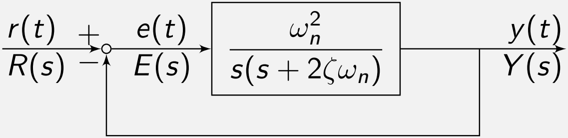


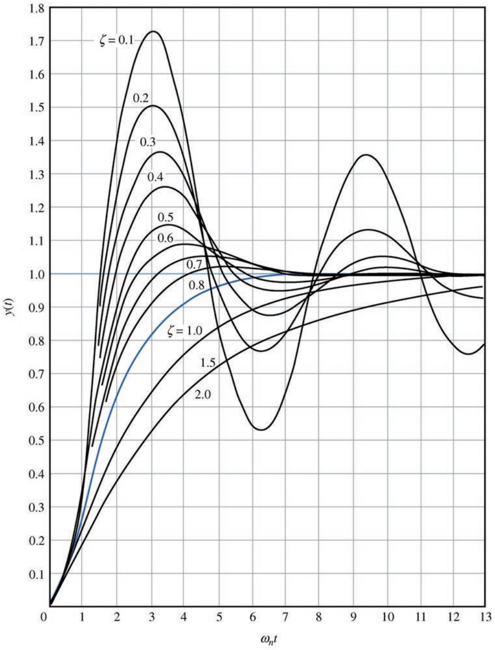
Figure 7.1: Prototype 2nd-order system

Then the closed-loop TF can be derived as:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

and the unit step response when $\zeta < 1$ is:

$$\begin{aligned} y(t) =& \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \\ =& 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \cos^{-1}\zeta\right). \end{aligned}$$



1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
y(t)
ωnt
ζ = 0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
ζ = 1.0
1.5
2.0

Matlab code

```
zeta=[0.1:0.1:1.0 1.5 2.0];

t=[0:0.1:13];
for k=1:length(zeta)
   sys1=tf(1, [1 2*zeta(k) 1]);
   y(k,:)=step(sys1,t);
end

plot(t, y)
xlabel('\omega_n t')
ylabel('y(t)')
axis([0 13 0 1.8])
```

7-6-1 Damping Ratio and Damping Factor

To study the effect of ζ and ω_n on step response, we consider the characteristic equation:

$$\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2 \\ = \ & s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2 \\ = \ & (s+\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2. \end{aligned}$$

For $\zeta \leq 1$,

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\alpha \pm j\omega, \tag{1}$$

where $\alpha = \zeta\omega_n$ and $\omega = \omega_n\sqrt{1-\zeta^2}$.

Now the unit step response can be written as

$$y(t) = 1 - \frac{e^{-\alpha t}}{\sqrt{1-\zeta^2}} \sin(\omega t + \cos^{-1}\zeta).$$

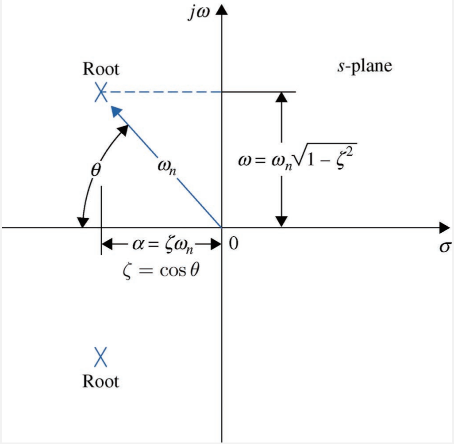
Therefore, α controls the rate of rise or decay, so α is called **damping factor**, or **damping constant**.

When two roots are real and equal, the system is **critically damped**. From Eq. (1), we then have $\zeta = 1$ and $\alpha = \omega_n$.

We can thus regard ζ as the **damping ratio**, *i.e.*,

$$\zeta = \text{damping ratio} = \frac{\alpha}{\omega_n} = \frac{\text{actual damping factor}}{\text{damping factor at critical damping}}.$$

7-6-2 Natural Undamped Frequency ω_n



When $\zeta = 0$,

$$y(t) = 1 - \cos\omega_n t,$$

where ω_n is the frequency of undamped sinusoidal response.

ω: **conditional frequency**, or **damped frequency**.

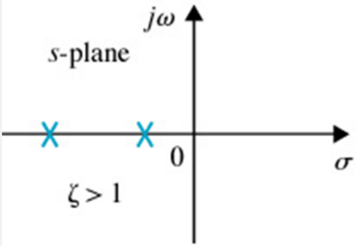
Step response comparison

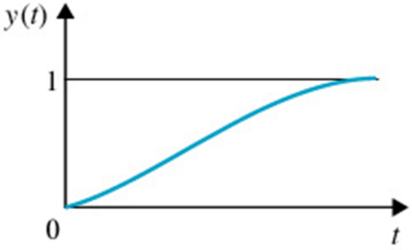
1. Overdamped ($\zeta > 1$)

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$M(s) = \frac{ab}{(s+a)(s+b)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{ab}{(s+a)(s+b)}\right\} = 1 - \frac{a}{a-b}e^{-bt} + \frac{b}{a-b}e^{-at}$$



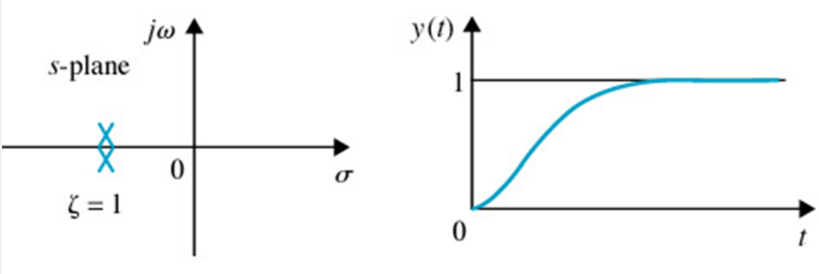


2. Critically damped ($\zeta = 1$)

$$s_{1,2} = -\omega_n$$

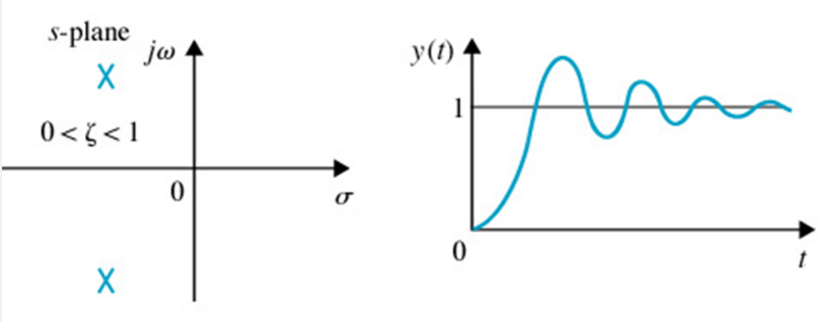
$$M(s) = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2}\right) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$



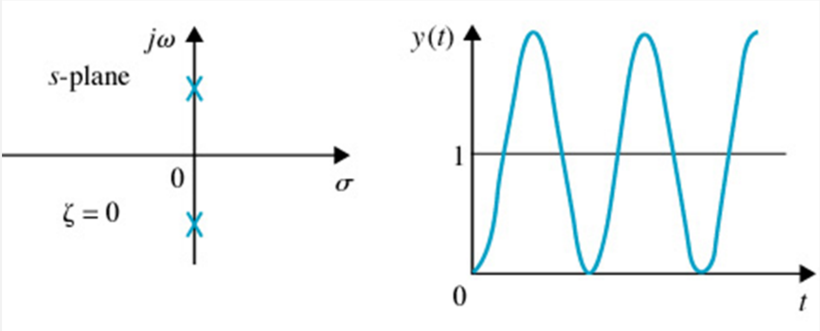
3. Underdamped ($0 < \zeta < 1$)

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$



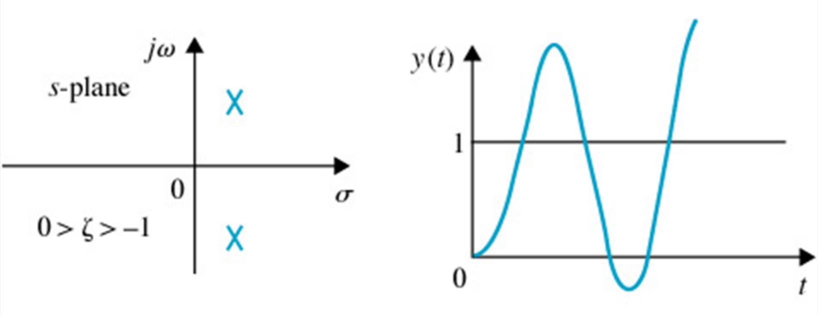
4. Undamped ($\zeta = 0$)

$$s_{1,2} = \pm j\omega_n$$
$$y(t) = 1 - \cos\omega_n t$$



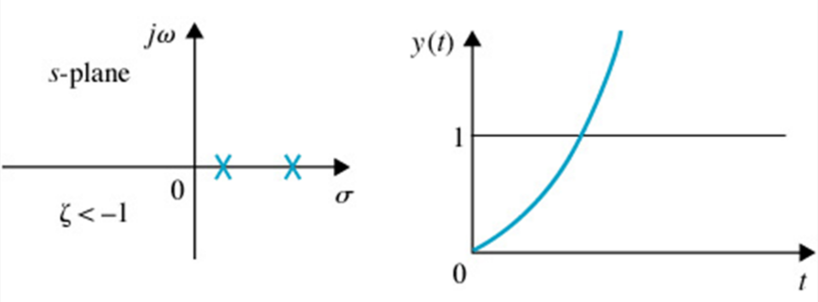
5. Negatively damped $(-1 < \zeta < 0)$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \omega_n > 0$$



6. Unstable ($\zeta < -1$)

$$s_{1,2} = a, b > 0$$



7-6-3 Maximum Overshoot

The analytic relation between the damping ratio and maximum overshoot can be obtained as follows.

Since

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2}t + \underbrace{\cos^{-1}\zeta}_{\theta}), \quad t \geq 0$$

we have

$$\begin{aligned} \frac{dy(t)}{dt} = & \zeta\omega_n \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2}t + \theta) \\ & - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_n \sqrt{1-\zeta^2}t + \theta) \cdot \omega_n \sqrt{1-\zeta^2} = 0. \end{aligned}$$

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Therefore,

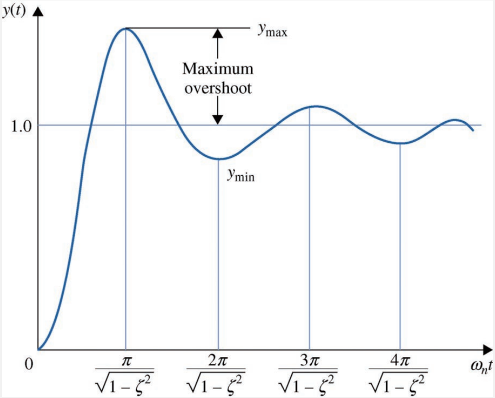
$$-\zeta \sin(\omega_n\sqrt{1-\zeta^2}t+\theta)+\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}t+\theta)=0.$$

Note that $\theta=\cos^{-1}\zeta$, hence $\zeta=\cos\theta$. Then

$$\sqrt{1-\zeta^2}=\sin\theta,$$

and

$$\begin{aligned} & \sin(\omega_n\sqrt{1-\zeta^2}t)=0 \\ \Rightarrow\quad & \omega_n\sqrt{1-\zeta^2}t=n\pi,\ \ n=0,1,2,\cdots \\ \Rightarrow\quad & t=\frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}. \end{aligned}$$



When $n = 1$,

$$t = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}.$$

Although the unit-step response for $\zeta \neq 0$ is not periodic, the overshoots and the undershoots do occur at periodic intervals.

Now

$$y(t)\Big|_{\max\,|\,\min} = 1 - \frac{e^{-n\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\sin(n\pi+\theta), \quad n = 1, 2, \cdots$$

Since

$$\sin(n\pi+\theta) = \sin n\pi \cos\theta + \sin\theta \cos n\pi = (-1)^n \sin\theta,$$

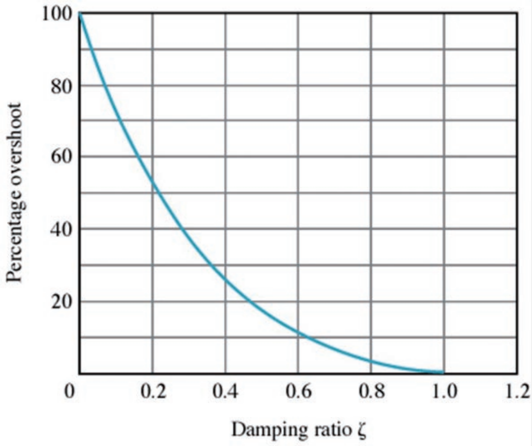
then

$$\begin{aligned} y(t)\Big|_{\max\,|\,\min} &= 1 - \frac{e^{-n\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}(-1)^n \sin\theta \\ &= 1 - (-1)^n e^{-\frac{n\pi\zeta}{\sqrt{1-\zeta^2}}}. \end{aligned}$$

Therefore, maximum overshoot is

$$y_{\max} - 1 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}},$$

which is a function of ζ only.



7-6-4 Delay Time and Rise Time

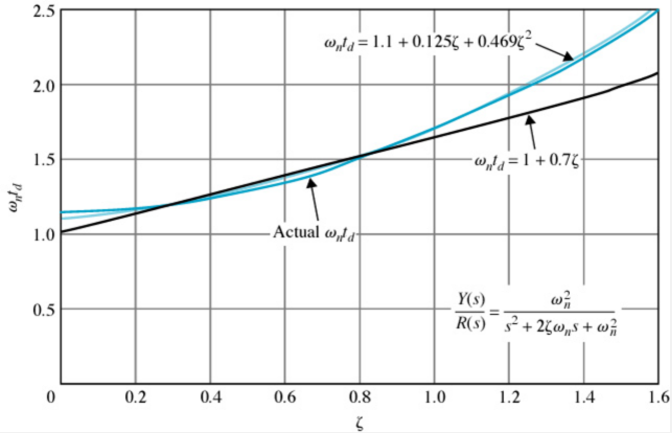
It is usually difficult to determine the analytical expressions of the delay time t_d, rise time t_r, and settling time t_s, even for prototype 2nd order system.

For the delay time t_d, we can set $y(t) = 0.5$:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t_d + \cos^{-1}\zeta) = \frac{1}{2}.$$

We can plot $\omega_n t_d$ versus ζ as in the following figure, and then approximate it by a straight line or a curve for $0 < \zeta < 1$:

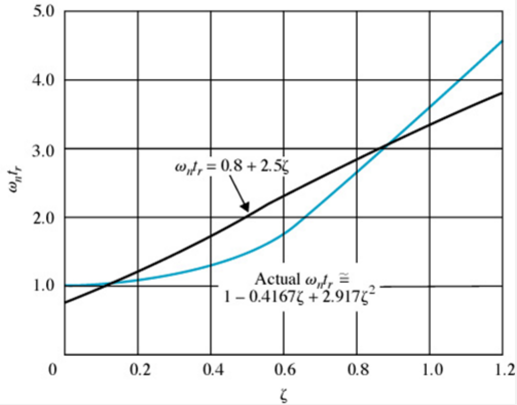
- 1st order approximation: $\omega_n t_d \approx 1 + 0.7\zeta$;
- 2nd order approximation: $\omega_n t_d \approx 1.1 + 0.125\zeta + 0.469\zeta^2$.



For the rise time t_r (the time from 10% to 90%), the exact value can be determined and the approximations are:

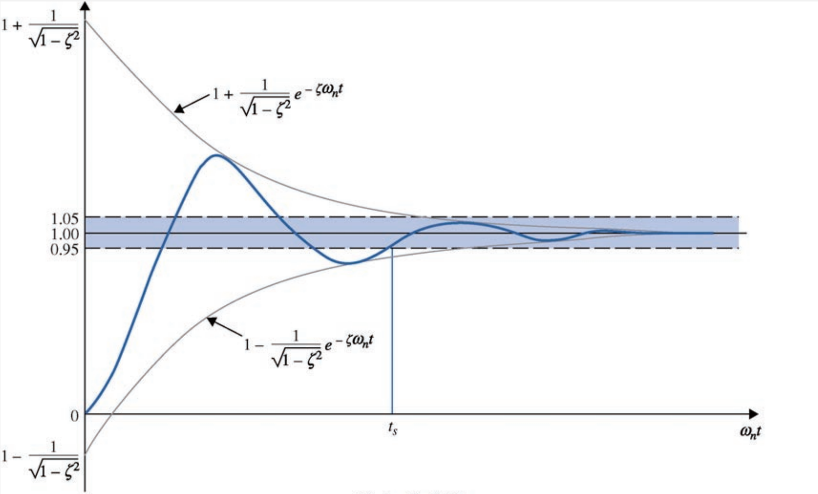
- 1st order approximation: $\omega_n t_r \approx 0.8 + 2.5\zeta$;
- 2nd order approximation: $\omega_n t_r \approx 1 - 0.4167\zeta + 2.917\zeta^2$.

t_d, t_r are proportional to ζ and inversely proportional to ω_n.



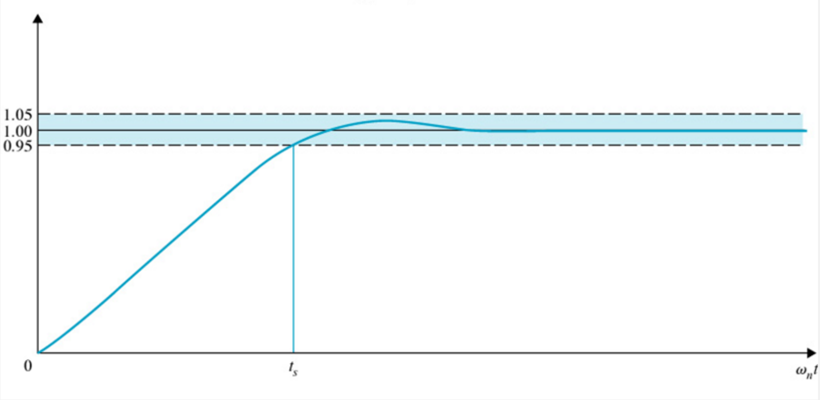
7-6-5 Settling Time

For $0 < \zeta < 0.69$, the unit-step response can enter the band from either top or bottom:



(a) $0 < \zeta < 0.69$

For $\zeta > 0.69$, the unit-step response can enter the band from bottom only:



Approximation: For $0 < \zeta < 0.69$, use intersection with envelope

$$1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_s} = 1.05$$

or

$$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_s} = 0.95.$$

In either case, we have

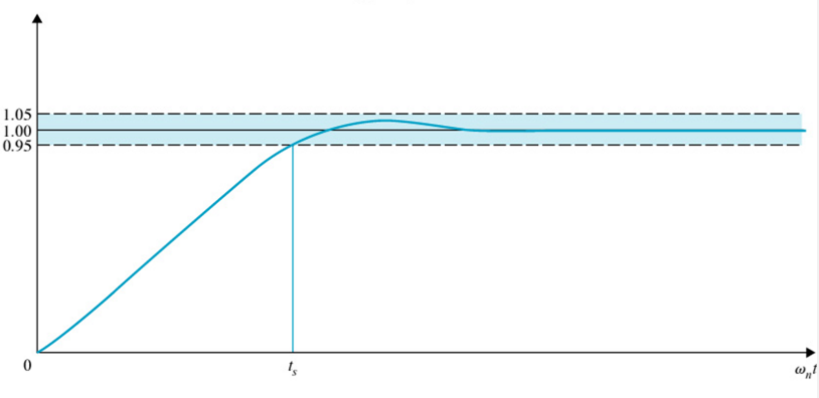
$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_s} = 0.05$$

$$e^{-\zeta\omega_n t_s} = 0.05\sqrt{1-\zeta^2}$$

$$\zeta\omega_n t_s = -\ln(0.05\sqrt{1-\zeta^2})$$

$$\omega_n t_s = -\frac{1}{\zeta}\ln(0.05\sqrt{1-\zeta^2}).$$

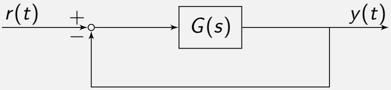
For $\zeta > 0.69$, $\omega_n t \propto \zeta$, $t_s = \dfrac{4.5\zeta}{\omega_n}$,



7-8 Adding poles and zeros to TF

7-8-1 Adding poles to OLTF

Consider



$$G_1(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},$$

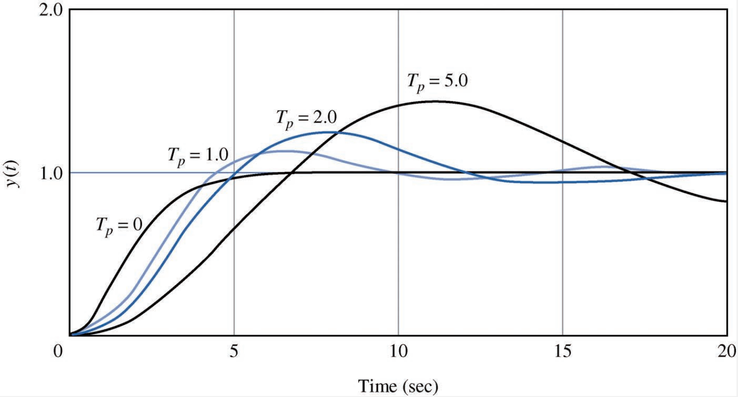
$$G_2(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)(1+T_p s)}.$$

Here $G_2(s)$ has an additional pole at $s = -1/T_p$.

$$M_1(s) = \frac{G_1(s)}{1+G_1(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},$$

$$M_2(s) = \frac{G_2(s)}{1+G_2(s)} = \frac{\omega_n^2}{T_p s^3+(1+2\zeta\omega_n T_p)s^2+2\zeta\omega_n s+\omega_n^2}.$$

1. Unit step response for $\zeta = 1$, $\omega_n = 1$, critically damped.

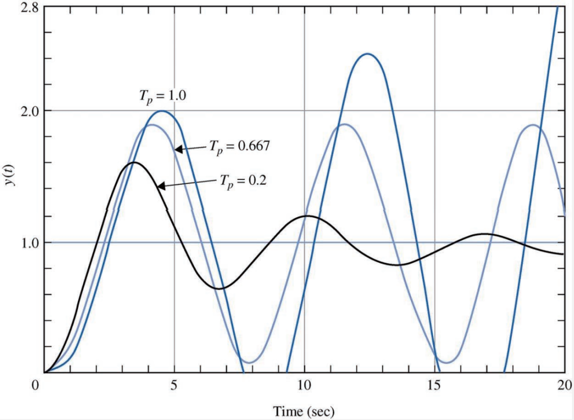


When $T_p \uparrow$, $-\dfrac{1}{T_p} \uparrow$, pole gets closer to the origin; and max overshoot $\uparrow$, rise time $\uparrow$.

Matlab code

```
t=[0:0.1:20];
zeta=1;
Tp=[0 1.0 2.0 5.0];
for k=1:length(Tp)
    sys1=tf(1, [Tp(k) 1+2*zeta*Tp(k) 2*zeta 1]);
    y(k,:)=step(sys1,t);
end
plot(t, y)
xlabel('Time (sec)')
ylabel('y(t)')
axis([0 20 0 2])
```

2. Unit step response for $\zeta = 0.25$, $\omega_n = 1$, underdamped. When $T_p \uparrow$, max overshoot $\uparrow$, rise time $\uparrow$.



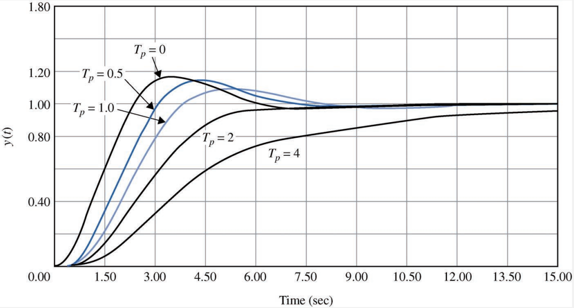
When $T_p > 0.667$, the system is unstable.

7-8-2 Adding a pole to CLTF

Consider

$$M_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad M_2(s) = M_1(s) \cdot \frac{1}{1 + T_p s}.$$

Unit step response for $\zeta = 0.5$, $\omega_n = 1$, underdamped.



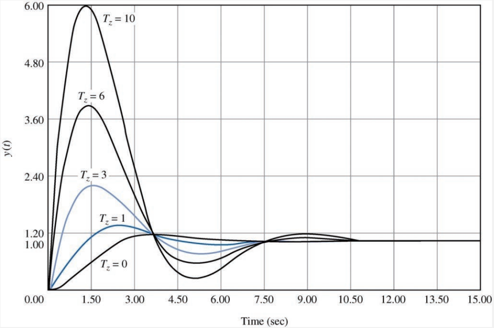
When $T_p \uparrow$, max overshoot $\downarrow$, rise time $\uparrow$.

7-8-3 Adding a zero to CLTF

Consider

$$M_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad M_2(s) = M_1(s) \cdot (1 + T_z s).$$

Unit step response for $\zeta = 0.5$, $\omega_n = 1$, underdamped:



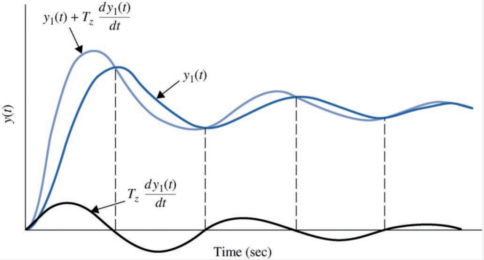
Rise time ↓, max overshoot ↑.

Since $M_2(s) = M_1(s) + T_z s \cdot M_1(s)$, if

$$y_1(t) = \mathcal{L}^{-1}\left\{M_1(s)\cdot\frac{1}{s}\right\}, \quad y_2(t) = \mathcal{L}^{-1}\left\{M_2(s)\cdot\frac{1}{s}\right\},$$

then

$$y_2(t) = y_1(t) + T_z\frac{dy_1(t)}{dt}$$



As $T_z \to \infty$, max overshoot$\to \infty$, but the system is still stable if overshoot is finite and $\zeta > 0$.

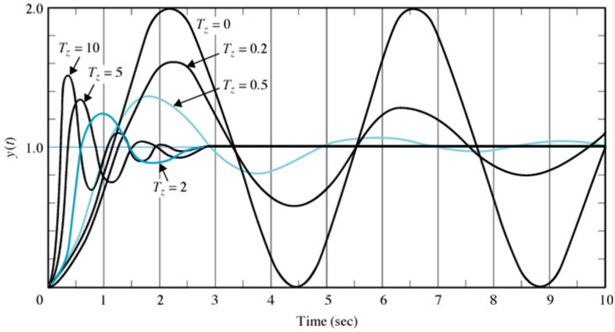
7-8-4 Adding a zero to OLTF

Consider

$$G_1(s) = \frac{6}{s(s+1)(s+2)}, \quad G_2(s) = G_1(s)(1+T_z s).$$

$$M_2(s) = \frac{6(1+T_z s)}{s^3+3s^2+(2+6T_z)s+6} \begin{array}{l} \longrightarrow \text{increase max overshoot} \\ \longrightarrow \text{reduce max overshoot} \end{array}$$

Affect both zeros and poles of CLTF.



- When $T_z = 0$, $s^3 + 3s^2 + 2s + 6 = (s^2 + 2)(s + 3)$, oscillating.
- When $T_z = 0.2, 0.5$, max overshoot $\downarrow$.
- When $T_z > 2$, max overshoot $\uparrow$.

Poles: affect relative damping and relative stability;

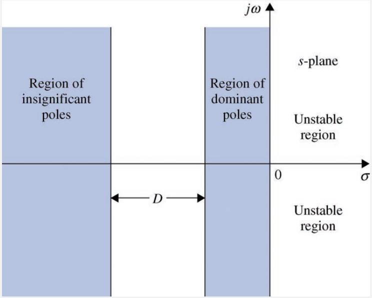
Zeros: affect transient response.

7-9 Dominant Poles of TF

Consider an example, $s_1 = -20$, $s_2 = -2$,

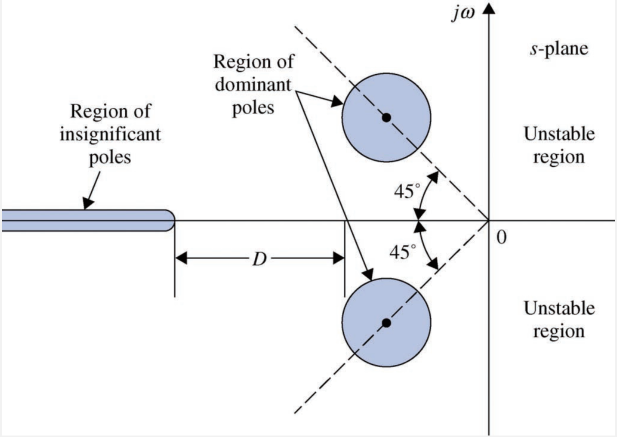
$$y(t) = 1 - e^{-20t} - e^{-2t},$$

e^{-20t} approaches 0 much faster than e^{-2t}, hence



- $s = -20$, insignificant pole;
- $s = -2$, dominant pole.

In design, we usually want poles in the shaded area.



7-9-1 Relative Damping Ratio

For system higher than 2nd-order, we don't have damping ratio ζ and natural undamped frequency ω_n defined.

However, if we have a pair of complex-conjugate dominant poles, we can still use ζ and ω_n.

ζ: relative damping ratio.

For example,

$$M(s) = \frac{20}{(s+10)(s^2+2s+2)},$$

then

$$s_1 = -10, \quad s_{2,3} = -1 \pm j, \quad \zeta = 1/\sqrt{2}.$$

How to neglect insignificant poles?

Need to maintain DC gain.

$$
\begin{aligned}
M(s) &= \frac{20}{(s+10)(s^2+2s+2)} \\
&= \frac{20}{10\left(1+\frac{s}{10}\right)(s^2+2s+2)} \\
&\approx \frac{2}{s^2+2s+2}
\end{aligned}
$$

when $\left|\frac{s}{10}\right| \ll 1$.