Ve460 Control Systems Analysis and Design
Chapter 9 Frequency Domain Analysis

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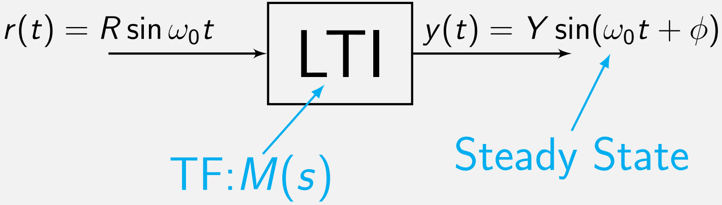
9-1 Introduction

Consider an LTI system



We have

$$Y(s) = M(s)R(s).$$

Replacing s by $j\omega$:

$$Y(j\omega) = M(j\omega)R(j\omega).$$

Writting $Y(j\omega)$ as $Y(j\omega) = |Y(j\omega)|\angle Y(j\omega)$, we have

$$|Y(j\omega)| = |M(j\omega)| \cdot |R(j\omega)|$$
$$\angle Y(j\omega) = \angle M(j\omega) + \angle R(j\omega)$$

Now $r(t) = R\sin\omega_0 t$,

$$Y = |Y(j\omega_0)| = |M(j\omega_0)| \cdot R$$
$$\phi = \angle Y(j\omega_0) = \angle M(j\omega_0)$$

$$\therefore M(s) \rightarrow \begin{matrix} |M(j\omega)| \\ \angle M(j\omega) \end{matrix} \rightarrow \begin{matrix} \text{completely describe steady-state} \\ \text{response for sinusoidal input} \end{matrix}$$

9-1-1 Frequency response of closed-loop systems

The closed-loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

Under sinusoidal steady state, $s = j\omega$,

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}.$$

Therefore,

$$|M(j\omega)| = \frac{|G(j\omega)|}{|1 + G(j\omega)H(j\omega)|}$$

$$\angle M(j\omega) = \angle G(j\omega) - \angle(1 + G(j\omega)H(j\omega))$$

9-1-2 Frequency-Domain Specifications

Define a set of frequency domain specifications to quantify the system performance.

For example,

- Resonant Peak:
$$M_r = \max |M(j\omega)|$$
  Large $M_r \Longleftrightarrow$ large max overshoot, desirable value: 1.1–1.5, $\longrightarrow$ also indicate relative stability
- Resonant Frequency ω_r: the frequency at which the peak resonance occurs.

- Bandwidth (BW): the frequency at which $|M(j\omega)|$ drops to 70.7% of $|M(0)|$.

$$20 \cdot \log_{10} \frac{1}{\sqrt{2}} = -10 \log_{10} 2 = -10 \cdot 0.3010 = -3dB$$

BW indicates the transient response properties:

$$\begin{array}{lcl} \text{Large BW} & \longrightarrow & \text{high freq signal pass through} \\ & \longrightarrow & \text{faster rise time} \\ \text{Small BW} & \longrightarrow & \text{only low freq signal pass through} \\ & \longrightarrow & \text{slower rise time} \end{array}$$

- Cutoff Rate: slope of $|M(j\omega)|$ at high frequency.

9-2 M_r, ω_r, and BW of the prototype 2nd-order system

9-2-1 Resonant Peak and Resonant Frequency

The closed-loop transfer function is given by

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Let $s = j\omega$:

$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}.$$

Now let $u = \dfrac{\omega}{\omega_n}$:

$$M(ju) = \frac{1}{1 - u^2 + j2\zeta u}.$$

Therefore,

$$|M(ju)| = \frac{1}{\sqrt{(1-u^2)^2 + 4\zeta^2 u^2}}, \quad \angle M(ju) = -\tan^{-1}\frac{2\zeta u}{1-u^2}$$

To find ω_r, $\max|M(ju)| \iff \min\,(1-u^2)^2 + 4\zeta^2 u^2$:

$$\begin{aligned}
\therefore \quad & \frac{d}{du}\left((1-u^2)^2 + 4\zeta^2 u^2\right) = 2(1-u^2)\cdot(-2u) + 8\zeta^2 u = 0 \\
\therefore \quad & u^2 = 1 - 2\zeta^2 \quad \text{or} \quad u = 0 \\
\therefore \quad & u = \sqrt{1-2\zeta^2} \quad \text{or} \quad u = 0
\end{aligned}$$

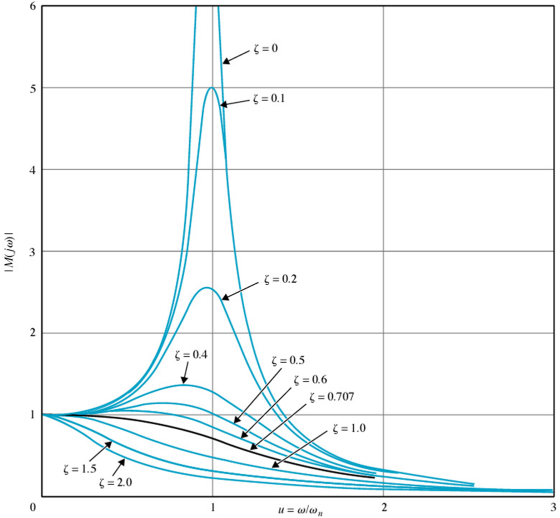
$u = 0$ implies that slope of $|M(ju)|_{u=0} = 0$. Therefore,

$$u = \sqrt{1-2\zeta^2}, \quad \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

$$\therefore \quad 2\zeta^2 \le 1, \quad \zeta \le \frac{\sqrt{2}}{2}.$$

For $\zeta \geq \frac{\sqrt{2}}{2}$, $M_r = 1$, $\omega_r = 0$;

For $\zeta \leq \frac{\sqrt{2}}{2}$, $M_r = \dfrac{1}{\sqrt{4\zeta^4 + 4\zeta^2(1-2\zeta^2)}} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$.



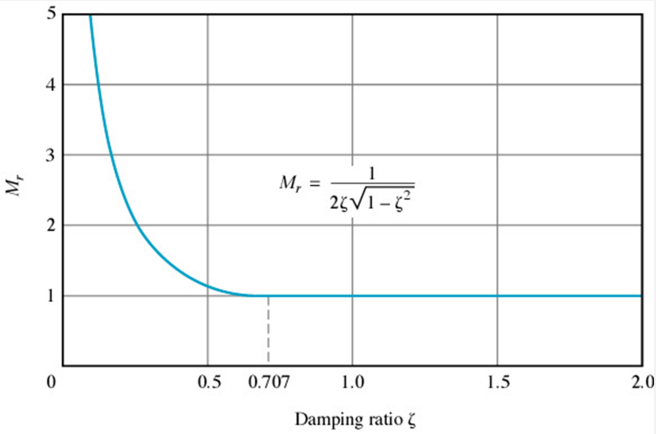


Figure 9.1: M_r versus damping ratio ζ

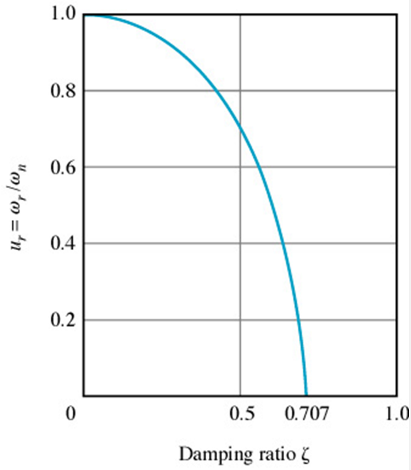


Figure 9.2: Normalized resonant frequency u_r versus damping ratio ζ: $u_r = \sqrt{1-2\zeta^2}$.

Since

- For $\zeta \geq \frac{\sqrt{2}}{2}$

$$M_r = 1, \omega_r = 0;$$

- For $\zeta \leq \frac{\sqrt{2}}{2}$,

$$\begin{aligned} M_r =& \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2(1-2\zeta^2)}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \\ \omega_r =& \omega_n\sqrt{1-2\zeta^2}. \end{aligned}$$

we have that

- M_r is a function of ζ only;
- ω_r is a function of both ζ and ω_n;
- Analytic method not recommended for higher order system $\rightarrow$ graphical & computer methods.

9-2-2 Bandwidth

$$\because |M(ju)| = \frac{1}{\sqrt{(1-u^2)^2 + 4\zeta^2 u^2}}, \quad \therefore \quad |M(0)| = 1$$

thus we need to find u such that $|M(ju)| = \frac{1}{\sqrt{2}}$. It yields that

$$\begin{aligned} & (1-u^2)^2 + 4\zeta^2 u^2 = 2 \\ \Rightarrow \quad & u^4 - (2-4\zeta^2)u^2 - 1 = 0 \\ \therefore \quad & u^2 = 1 - 2\zeta^2 \pm \sqrt{(1-2\zeta^2)^2 + 1} \end{aligned}$$

We should choose the plus sign, and thus

$$\mathrm{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{(1-2\zeta^2)^2 + 1}}$$

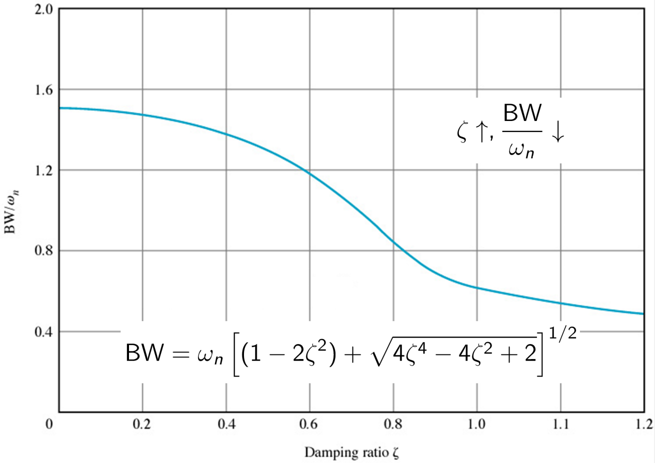


Figure 9.3: BW/ω_n versus damping ratio ζ.

Further discussion

1. M_r depends on ζ only:

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad M_r \downarrow \text{ as } \zeta \uparrow .$$

- When $\zeta = 0$, $M_r = \infty$.

  In this case, $\omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n$ and $M(s) = \dfrac{\omega_n^2}{s^2+\omega_n^2}$.

  Consider a sinusoidal input $R(s) = \dfrac{\omega_n}{s^2+\omega_n^2}$. The response is

  $$Y(s) = M(s)\cdot R(s) = \frac{\omega_n^2}{s^2+\omega_n^2}\cdot\frac{\omega_n}{s^2+\omega_n^2} = \frac{\omega_n^3}{(s^2+\omega_n^2)^2},$$

  $$y(t) = \frac{1}{2}\sin\omega_n t - \frac{1}{2}\omega_n t\cos\omega_n t \quad \longrightarrow \quad \text{unstable}$$

- When $\zeta < 0$, negatively damped $\longrightarrow$ unstable

- When $\zeta \geq \frac{\sqrt{2}}{2} \rightarrow M_r = 1$, $\omega_r = 0$.

  Recall that for unit step response,

$$\text{max overshoot} = \begin{cases} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}, & 0 \leq \zeta \leq 1 \\ 0, & \zeta \geq 1. \end{cases}$$

  So the max overshoot also depends only on ζ, but it is zero when $\zeta \geq 1$.

2. BW $\propto \omega_n$:

$$\mathrm{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{(1 - 2\zeta^2)^2 + 1}}$$

i.e., BW $\uparrow$ as $\omega_n \uparrow$.
For unit step response, rise time $\downarrow$ as $\omega_n \uparrow$ (See Chap 7-6-4).

So

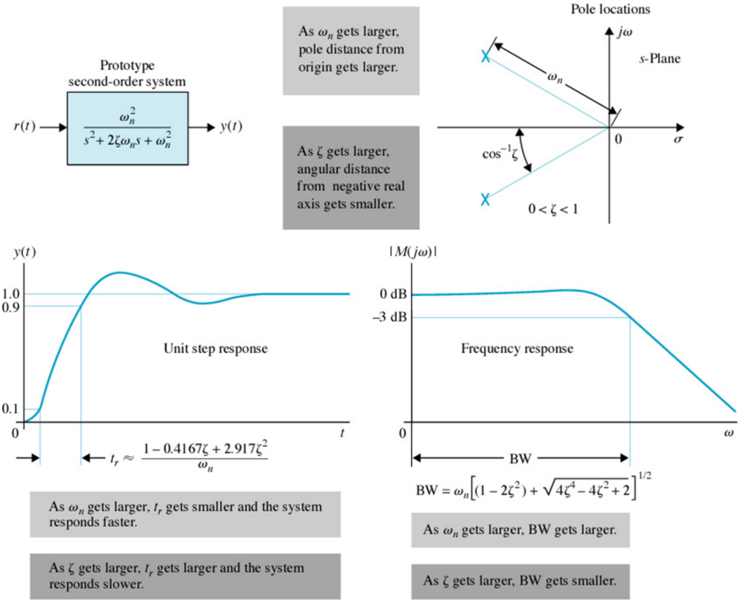
- $\omega_n \uparrow \Rightarrow$ BW $\uparrow$ and rise time $\downarrow$;
- $\zeta \uparrow \Rightarrow$ BW $\downarrow$ and rise time $\uparrow$.

Therefore, BW and rise time are inversely proportional.
$\Rightarrow$ the larger the BW is, the faster the system will respond.

3. For $0 \leq \zeta \leq \frac{\sqrt{2}}{2}$, we have

- $\zeta \uparrow \Rightarrow M_r \downarrow$ and $\zeta \uparrow \Rightarrow$ BW $\downarrow$.

Therefore, BW & M_r are proportional to each other in this range.


Prototype second-order system
$r(t)$
$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y(t)$
As ω_n gets larger, pole distance from origin gets larger.
As ζ gets larger, angular distance from negative real axis gets smaller.
Pole locations
$j\omega$
s-Plane
ω_n
$\cos^{-1}\zeta$
0
σ
$0 < \zeta < 1$
$y(t)$
1.0
0.9
0.1
0
Unit step response
t
$t_r \approx \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n}$
$|M(j\omega)|$
0 dB
–3 dB
Frequency response
0
ω
BW
$\mathrm{BW} = \omega_n\left[(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{1/2}$
As ω_n gets larger, t_r gets smaller and the system responds faster.
As ζ gets larger, t_r gets larger and the system responds slower.
As ω_n gets larger, BW gets larger.
As ζ gets larger, BW gets smaller.

9-3 Effect of adding a zero to the forward path TF

Study the effect on frequency domain response when poles & zeros are added to the forward path TF.

Consider the 2nd-order prototype OLTF:

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}.$$

Add a zero at $s = -\dfrac{1}{T}$ $\quad\Rightarrow\quad G(s) = \dfrac{\omega_n^2(1+Ts)}{s(s+2\zeta\omega_n)}.$

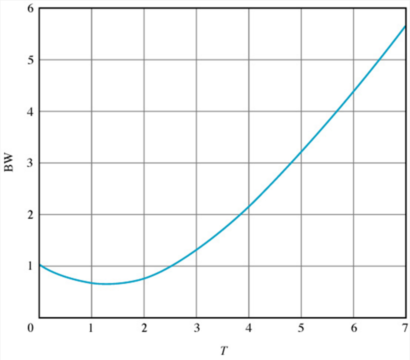
Therefore the CLTF is given by:

$$M(s) = \frac{\omega_n^2(1+Ts)}{s^2+(2\zeta\omega_n+T\omega_n^2)s+\omega_n^2}$$

$$\Rightarrow \quad \mathrm{BW} = \left(-b+\frac{1}{2}\sqrt{b^2+4\omega_n^4}\right)^{\frac{1}{2}},$$

$$b = 4\zeta^2\omega_n^2 + 4\zeta\omega_n^3T - 2\omega_n^2 - \omega_n^4T^2.$$

Let $\zeta = \frac{1}{\sqrt{2}}$ and $\omega_n = 1$. The following figure shows BW v.s. T:



When T is small, $T \uparrow \Rightarrow \text{BW} \downarrow$; but generally, $T \uparrow \Rightarrow \text{BW} \uparrow$.

Therefore, adding a zero $\Rightarrow$ increase BW.

Generally, $T \uparrow \Rightarrow \text{BW} \uparrow$ except when T small.

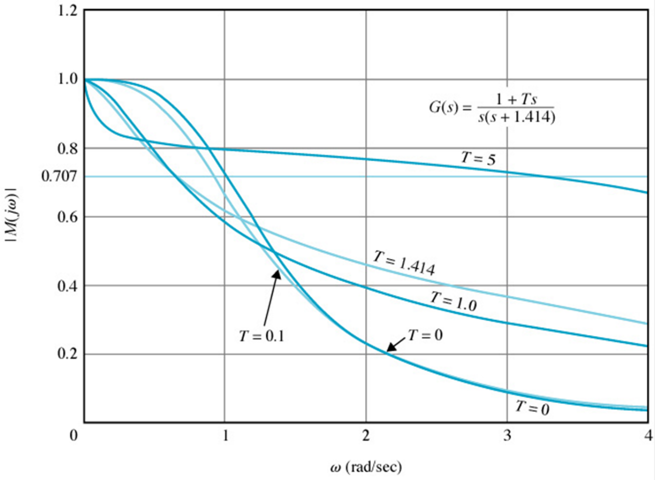


Figure 9.4: Magnification curves with the OLTF $G(s)$: $\zeta = \frac{1}{\sqrt{2}}$, $\omega_n = 1$.

Generally, $T\uparrow \Rightarrow$ BW $\uparrow$ except when T small.

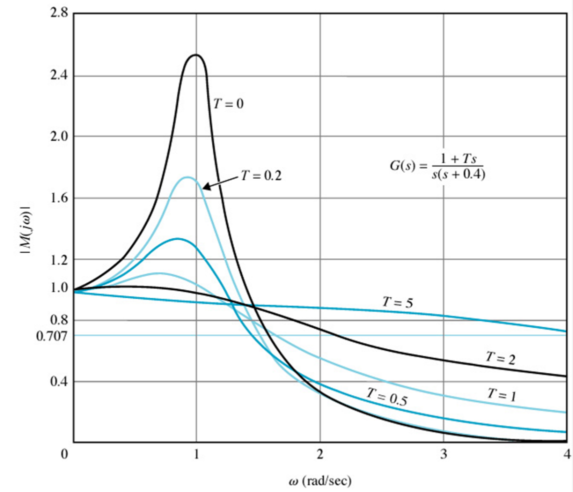


Figure 9.5: Magnification curves with the OLTF $G(s)$: $\zeta = 0.2$, $\omega_n = 1$.

Higher BW $\Rightarrow$ faster rise time $\Rightarrow$ longer settling time

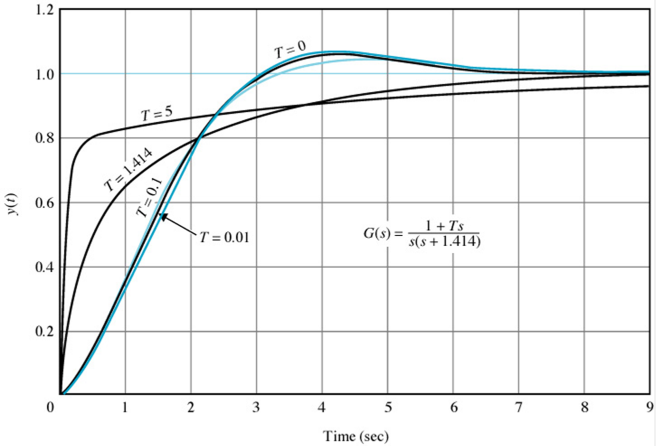


Figure 9.6: Unit-step responses of a 2nd-order system with a forward-path TF $G(s)$: $\omega_n = 1$, $\zeta = \frac{1}{\sqrt{2}}$.

Higher BW $\Rightarrow$ faster rise time $\Rightarrow$ longer settling time

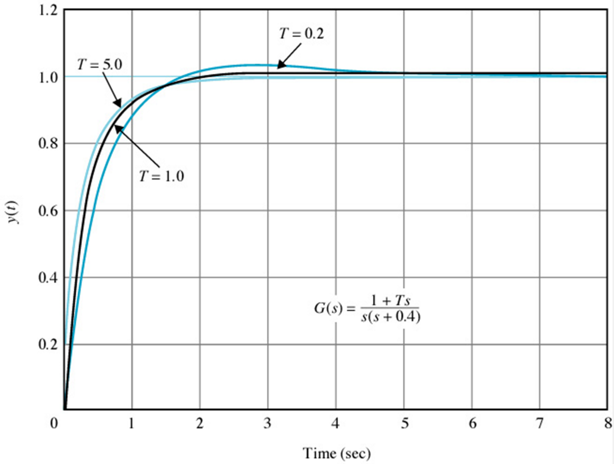
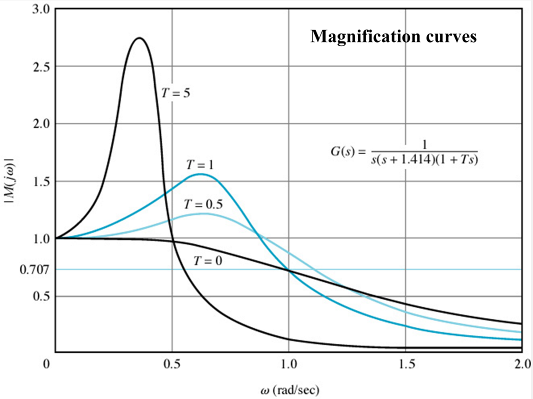


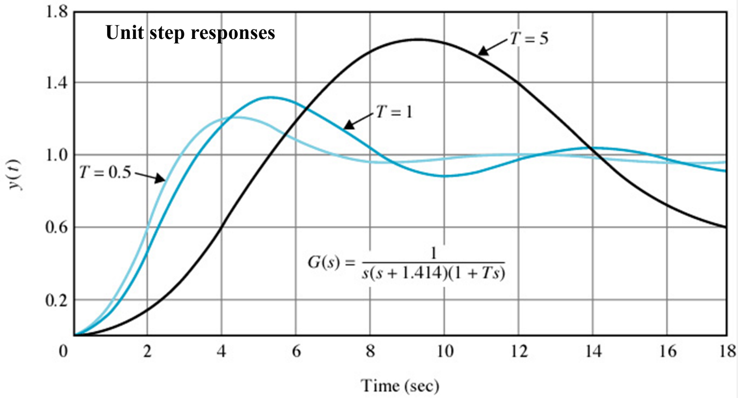
Figure 9.7: Unit-step responses of a 2nd-order system with a forward-path TF $G(s)$: $\omega_n = 1$, $\zeta = 0.2$.

9-4 Effects of adding a pole to the forward path TF

Consider a 2nd-order prototype TF $G(s) = \dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$. Add a pole $s = -\frac{1}{T}$, and set $\omega_n = 1$ and $\zeta = \frac{1}{\sqrt{2}}$:



For small T, BW $\uparrow$ and $M_r \uparrow$; for large T, BW $\downarrow$ and $M_r \uparrow$.



Slower response $\Rightarrow$ larger rise time

Generally, make the CL system less stable while decreasing BW.

9-5 Nyquist stability criterion: fundamentals

- So far, two methods to investigate stability:
  Routh-Hurwitz criterion & root locus
- Nyquist plot of the loop transfer function $KL(s)$ is a plot of $\mathrm{Im}[KL(j\omega)]$ vs $\mathrm{Re}[KL(j\omega)]$ as $\omega : 0 \to \infty$.
- Use $L(s)$ to study CL system.

Advantages

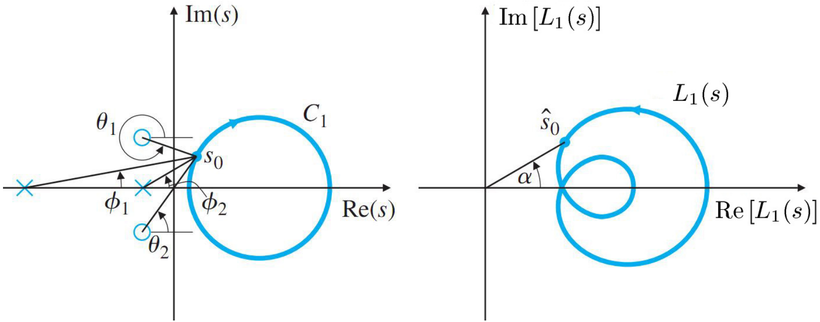
1. Give info on relative stability;
2. Easy to obtain;
3. Give info on frequency domain characteristic such as M_r, ω_r, and BW;
4. Can treat pure time delay.

9-5-4 Argument Principle

Consider a loop TF $L_1(s)$: zeros & poles are indicated as below. We want to evaluate $L_1(s)$ for values of s on clockwise path C_1. Take a point s_0 on C_1, $L_1(s_0) = re^{i\alpha}$, where

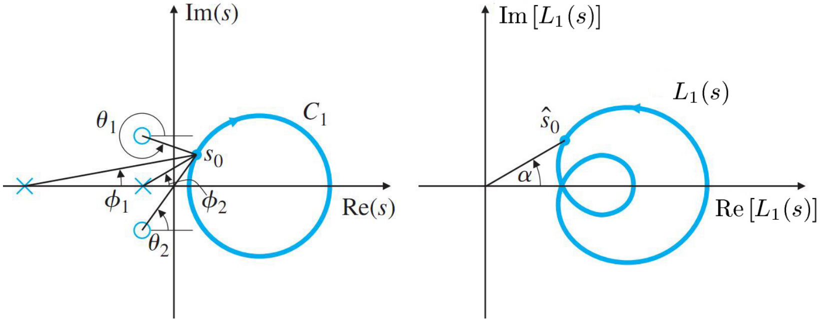
$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2.$$

As s travels along C_1, α will change as well.



Note that if C_1 contains no zeros or poles of $L_1(s)$, α will not undergo a net change of 2π.

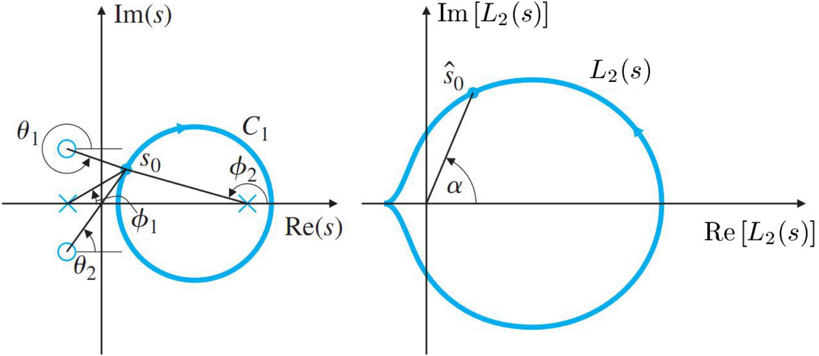
This is because none of θ_1, θ_2, ϕ_1, ϕ_2 goes through a net revolution, and then $L_1(C_1)$ will not encircle the origin.



Consider another function $L_2(s)$ as below:

- θ_1, θ_2, ϕ_1 change, but they return to original values;
- ϕ_2 undergoes a net change of -2π;

Therefore, α undergoes a net change of 2π.
$\Rightarrow L_2(C_1)$ encircles the origin in the counter clockwise direction.



Argument Principle

A complex function $L(C_1)$ will encircle the origin N times, where $N = Z - P$, and

- Z is the number of zeros of $L(s)$ inside C_1; and
- P is the number of poles of $L(s)$ inside C_1.

Discussions

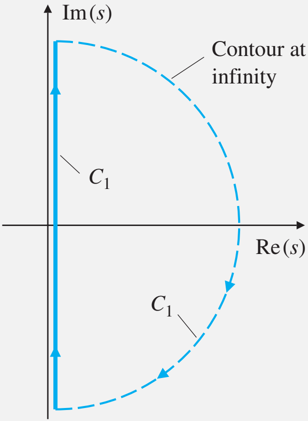
Take clockwise (CW) C_1:

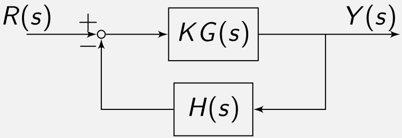
- $Z > P \Rightarrow N > 0$ and $L(C_1)$ CW
- $Z = P \Rightarrow N = 0$ and $L(C_1)$ does not encircle the origin
- $Z < P \Rightarrow N < 0$ and $L(C_1)$ counter clockwise (CCW)

In the previous example, we have $Z = 0$ and $P = 1$. Therefore, $L_2(C_1)$ encircles the origin -1 times, where the negative sign indicates the CCW direction.

Application to control design

Let C_1 encircle the entire RHP.





$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Letting $L(s) = G(s)H(s)$, we will use the Nyquist plot of $KL(s)$ to determine the closed-loop stability.

If $L(s)$ can be written as $L(s) = \frac{b(s)}{a(s)}$, we have

$$1 + KL(s) = 1 + K\frac{b(s)}{a(s)} = \frac{a(s) + Kb(s)}{a(s)}$$

- poles of $1 + KL(s)$ are also poles of $L(s)$;
  $\longrightarrow$ open-loop poles, easy to get.
- zeros of $1 + KL(s)$ are closed-loop poles.

Now from Argument Principle,

$$\begin{aligned}&\#\{(1 + KL(s)) \text{ encircling the origin}\}\\ =&\#\{\text{zeros of } (1 + KL(s)) \text{ in RHP}\} - \#\{\text{poles of } (1 + KL(s)) \text{ in RHP}\}.\end{aligned}$$

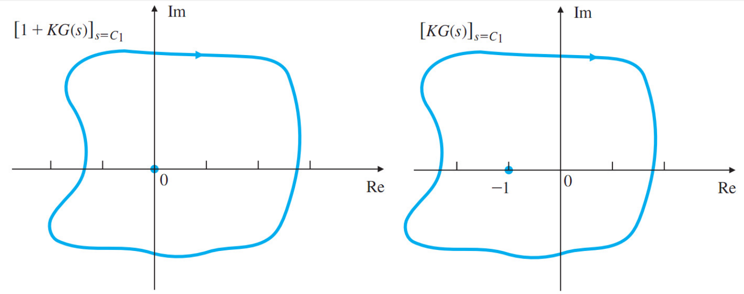
Hence,

$$\begin{aligned}\#\{\text{closed-loop poles in RHP}\} =&\#\{\text{open-loop poles in RHP}\}\\ &+\#\{(1 + KL(s)) \text{ encircling the origin}\}\end{aligned}$$

$$\text{If LHS} \neq 0 \quad \Rightarrow \quad \text{unstable (minimum phase)}$$

Further,

$$1 + KL(s) \text{ encircles the origin } \iff KL(s) \text{ encircles } (-1, 0)$$



$$\Rightarrow \quad \#\{\text{unstable closed-loop poles}\}$$
$$= \#\{\text{unstable open-loop poles}\} + \#\,\{KL(s) \text{ encircling } (-1, 0)\}$$

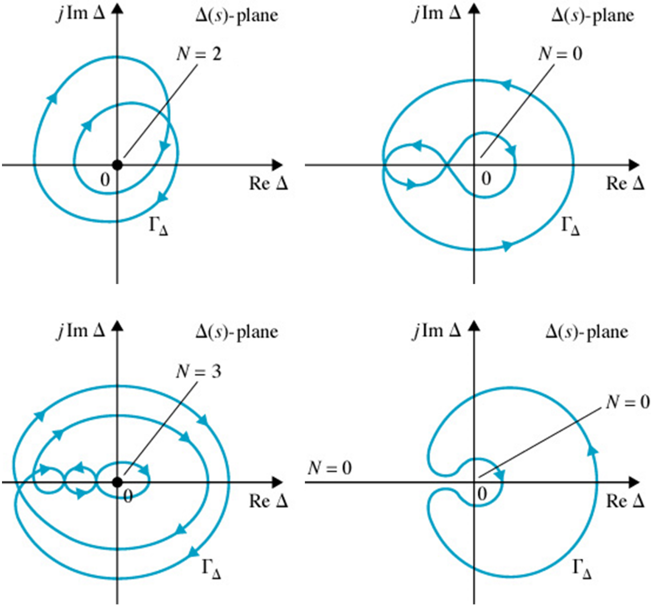
If $L(s)$ is stable, *i.e.*, it has no RHP poles, then an encirclement of $(-1, 0)$ by $KL(s) \Rightarrow$ an unstable closed-loop pole.

How to apply Nyquist stability criterion

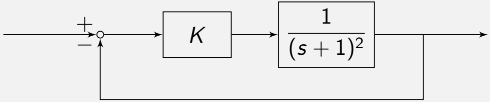
1. Plot $KL(s)$, $-j\infty \leq s \leq j\infty$
2. Get N: $\#\{KL(s)$ encircling $(-1, 0)$, CW:$+$, CCW:$-\}$
3. Get P: $\#\{\text{RHP poles of } L(s)\}$
4. $Z = N + P \rightarrow$ unstable closed-loop poles

We wish to have $Z = 0$.

How to determine N?



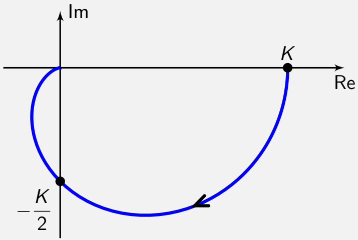
Example 1



CLTF: $\dfrac{K\frac{1}{(s+1)^2}}{1+K\frac{1}{(s+1)^2}}$

Then $KL(s) = \dfrac{K}{(s+1)^2}$. Letting $s = j\omega$, we have

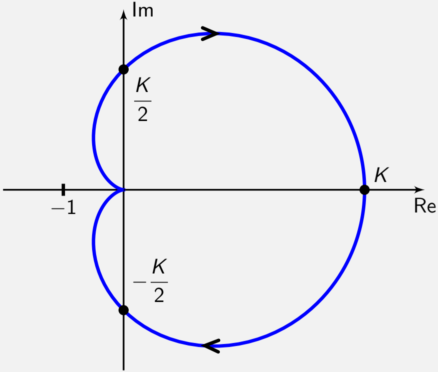
$$KL(j\omega) = K\frac{1}{(j\omega+1)^2} = \frac{K}{1-\omega^2+2j\omega}.$$



When $\omega = 0$, $KL(j\omega) = K$;
When $\omega = 1$, $KL(j\omega) = -\frac{K}{2}j$;
When $\omega = \infty$, $|KL(j\omega)| = 0$, and

$$\angle KL(j\omega) = \angle\frac{K}{(j\omega)^2} = \pi.$$

$$KL(s) = \frac{K}{(s+1)^2}.$$



$$P = 0, N = 0$$
$$\Rightarrow \quad Z = N + P = 0.$$

Therefore, it is stable no matter what K is.

Discussions

1. When $s = \infty$,

$$KL(s) = K\frac{b(s)}{a(s)} = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, n > m.$$

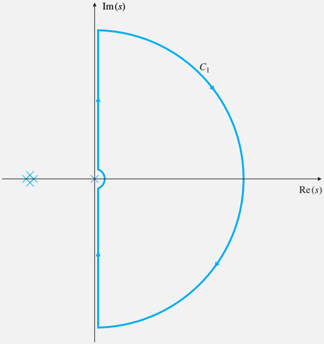
Let $s = Re^{i\theta}$ where $R \to \infty$,

$$KL(s)\Big|_{Re^{i\theta}} = K\frac{\prod_j(Re^{i\theta}+z_j)}{\prod_k(Re^{i\theta}+p_k)} = \frac{K}{R^{n-m}}e^{-i(n-m)\theta}.$$

Therefore, $|KL(s)| \to 0$. Since $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$, we have

$$\angle KL(s) : -(n-m)\frac{\pi}{2} \to (n-m)\frac{\pi}{2}.$$

2. $s=0$. If $s=0$ is not a pole of $KL(s)$, $KL(s)\Big|_{s=0} = K\dfrac{z_1\cdots z_m}{p_1\cdots p_n}$.



If $s=0$ is a pole of $KL(s)$, *i.e.*,

$$KL(s) = \frac{K(s+z_1)\cdots(s+z_m)}{s^v(s+p_1)\cdots(s+p_n)}.$$

Around the origin, let $s=\epsilon e^{i\theta}$, where $\theta : -\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$, and $\epsilon \longrightarrow 0$ (counter clockwise).

Then,

$$KL(s)\Big|_{s=\epsilon e^{i\theta}} = \frac{K(\epsilon e^{i\theta}+z_1)\cdots(\epsilon e^{i\theta}+z_m)}{(\epsilon e^{i\theta})^v(\epsilon e^{i\theta}+p_1)\cdots(\epsilon e^{i\theta}+p_n)}$$

$|KL(s)| = \infty$, $\angle KL(s) : v\dfrac{\pi}{2} \rightarrow -v\dfrac{\pi}{2}$, clockwise.

Example 2

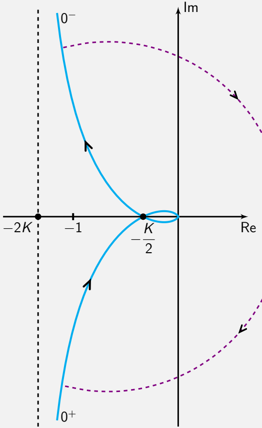
$$KL(s) = K\frac{1}{s(s+1)^2}.$$

Replace s by $j\omega$:

$$\begin{aligned} KL(j\omega) &= \frac{K}{j\omega(1+j\omega)^2} = \frac{K}{j\omega(1-\omega^2+2\omega j)} \\ &= \frac{K}{\omega(-2\omega+(1-\omega^2)j)} = \frac{K}{\omega}\cdot\frac{-2\omega-(1-\omega^2)j}{4\omega^2+(1-\omega^2)^2} \\ &= \frac{K}{\omega}\cdot\frac{-2\omega-(1-\omega^2)j}{(1+\omega^2)^2}. \end{aligned}$$

Therefore,

$$\begin{cases} x = \dfrac{-2K}{(1+\omega^2)^2}, \\ y = \dfrac{-K(1-\omega^2)}{\omega(1+\omega^2)^2}. \end{cases}$$



$$\begin{cases} x = \dfrac{-2K}{(1+\omega^2)^2}, \\ y = \dfrac{-K(1-\omega^2)}{\omega(1+\omega^2)^2}. \end{cases}$$

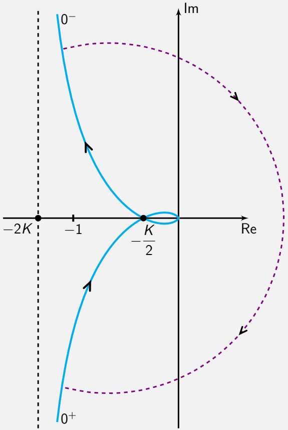
Then,

$$\text{when } \omega = 0, x = -2K, y = \infty;$$

$$0^- \to 0^+, \frac{\pi}{2} \to -\frac{\pi}{2}.$$

$$\text{when } \omega = \infty, x = y = 0.$$

$$y = 0 \quad \Rightarrow \quad \omega^2 = 1 \quad \Rightarrow \quad x = -\frac{K}{2}.$$



Because $P = 0$, we have $Z = N$.
We want $-\frac{K}{2} > -1$, $K < 2$.

$$
\begin{aligned}
& 1 + KL(s) \text{ encircles } 0 \\
\iff & KL(s) \text{ encircles } (-1, 0) \\
\iff & L(s) \text{ encircles } \left(-\frac{1}{K}, 0\right).
\end{aligned}
$$

Example 9-1

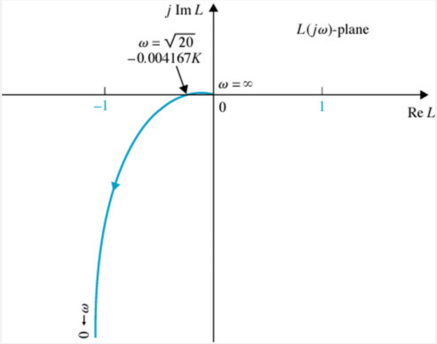
$$L(s) = \frac{1}{s(s+2)(s+10)}.$$

Replace s by $j\omega$:

$$\begin{aligned} L(j\omega) &= \frac{1}{j\omega(j\omega+2)(j\omega+10)} = \frac{1}{j\omega(20-\omega^2+12\omega j)} \\ &= \frac{1}{\omega(-12\omega+(20-\omega^2)j)} \\ &= \frac{1}{\omega}\cdot\frac{-12\omega-(20-\omega^2)j}{144\omega^2+(20-\omega^2)^2}. \end{aligned}$$

Therefore,

$$\begin{cases} x = \dfrac{-12}{144\omega^2+(20-\omega^2)^2}, \\ y = \dfrac{-(20-\omega^2)}{\omega(144\omega^2+(20-\omega^2)^2)}. \end{cases}$$



$$\begin{cases} x = \dfrac{-12}{144\omega^2 + (20-\omega^2)^2}, \\ y = \dfrac{-(20-\omega^2)}{\omega(144\omega^2 + (20-\omega^2)^2)}. \end{cases}$$

Then,

$$s: 0^- \to 0^+, \frac{\pi}{2} \to -\frac{\pi}{2}, \text{CW.} \quad \text{When } \omega = 0, x = -\frac{12}{400} = -\frac{3}{100}.$$

$$y = 0 \quad \Rightarrow \quad \omega^2 = 20, \quad x = \frac{-12}{144 \cdot 20} = -\frac{1}{240}$$

$$-\frac{1}{240} > -\frac{1}{K}, K < 240 \quad \Rightarrow \quad \text{stable}$$

Example: pure time delay

Consider a pure time delay TF:

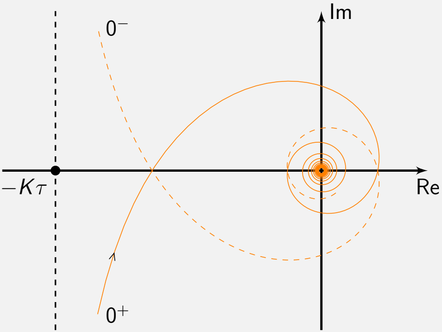
$$KL(s) = \frac{Ke^{-\tau s}}{s}.$$

Replace s by $j\omega$:

$$\begin{aligned} KL(j\omega) &= \frac{Ke^{-\tau j\omega}}{j\omega} = \frac{-jKe^{-j\omega\tau}}{\omega} = \frac{-K}{\omega} j(\cos\omega\tau - j\sin\omega\tau) \\ &= \frac{K}{\omega}(-\sin\omega\tau - j\cos\omega\tau). \end{aligned}$$

Therefore,

$$\begin{cases} x = -\dfrac{K\sin\omega\tau}{\omega}, \\ y = -\dfrac{K\cos\omega\tau}{\omega}. \end{cases} \quad \text{and also} \quad \begin{cases} |KL(j\omega)| = \dfrac{K}{\omega}, \\ \angle KL(j\omega) = -\omega\tau - \dfrac{\pi}{2}. \end{cases}$$



$$\begin{cases} x = -\dfrac{K\sin\omega\tau}{\omega}, \\ y = -\dfrac{K\cos\omega\tau}{\omega}. \end{cases}$$

$$\begin{cases} |KL(j\omega)| = \dfrac{K}{\omega}, \\ \angle KL(j\omega) = -\omega\tau - \dfrac{\pi}{2}. \end{cases}$$

$$\text{when } \omega = 0, \quad x = -\lim_{\omega\to 0}\frac{K\sin\omega\tau}{\omega} = -K\tau, \quad y = \infty;$$

$$\text{when } \omega \to \infty, \quad x \to 0, \quad y \to 0.$$

$$y = 0, \quad \cos\omega\tau = 0, \quad \omega\tau = n\pi + \frac{\pi}{2}, \quad x = (-1)^{n+1}\frac{K\tau}{n\pi + \frac{\pi}{2}}.$$

$$n = 0, \quad x = -\frac{2K\tau}{\pi} > -1, \quad K < \frac{\pi}{2\tau} \quad \Rightarrow \quad \text{stable}$$

Conditional Stability

Increasing gain may result in instability. Consider a TF

$$KL(s) = \frac{K(0.1s+1)^2}{(s+1)^3(0.01s+1)^2}.$$

Replace s by $j\omega$:

$$\begin{aligned} L(j\omega) &= \frac{(0.1j\omega+1)^2}{(j\omega+1)^3(0.01j\omega+1)^2} \\ &= \frac{1-0.01\omega^2+0.2\omega j}{(1-3\omega^2+3j\omega-j\omega^3)(1-0.0001\omega^2+0.02\omega j)} \\ &= \frac{1-0.01\omega^2+0.2\omega j}{\alpha+j\beta}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= 1-3.061\omega^2+0.0203\omega^4, \\ \beta &= 3.02\omega-1.0603\omega^3+0.0001\omega^5. \end{aligned}$$

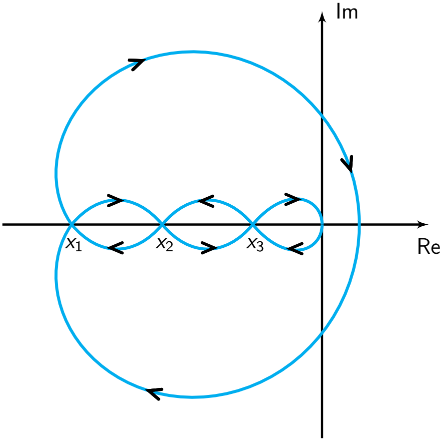
Therefore,

$$x = \frac{K}{\alpha^2+\beta^2}(1 - 2.4661\omega^2 - 0.161159\omega^4 - 0.000183\omega^6),$$
$$y = \frac{K}{\alpha^2+\beta^2}\omega(-2.82 + 0.478480\omega^2 - 0.006643\omega^4 + 10^{-6}\omega^6).$$

Hence $\omega = 0,\ L(0) = 1$. Further,

$$\begin{array}{cccccc} y = 0 & \Rightarrow & \omega = 0, & \pm 2.5446, & \pm 8.1415, & \pm 81.0570 \\ & & x = 1, & -5.2063e\text{-}2, & -2.9330e\text{-}3, & -7.5569e\text{-}5 \\ & & & \uparrow & \uparrow & \uparrow \\ & & & x_1 & x_2 & x_3 \end{array}$$

1. $-\frac{1}{K} < x_1 \quad \Rightarrow \quad K < 19.2$, stable;
2. $x_1 \leq -\frac{1}{K} \leq x_2 \quad \Rightarrow \quad 19.2 \leq K \leq 334.11$, unstable;
3. $x_2 < -\frac{1}{K} < x_3 \quad \Rightarrow \quad 334.11 < K < 13233$, stable;
4. $x_3 < -\frac{1}{K} \quad \Rightarrow \quad K \geq 13233$, unstable.

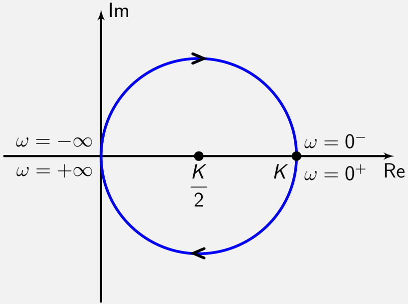


9-9 Adding zeros or poles to $L(s)$

Adding poles at $s = 0$.

Consider $L_0(s) = \dfrac{K}{1+T_1 s}$. Replace s by $j\omega$:

$$L_0(j\omega) = \frac{K}{1+j\omega T_1} = \frac{K(1-j\omega T_1)}{1+\omega^2 T_1^2}.$$



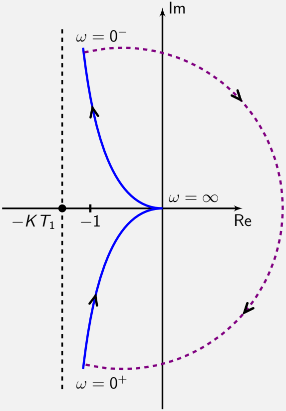
$$\begin{cases} x = K\dfrac{1}{1+\omega^2 T_1^2}, \\ y = -K\dfrac{\omega T_1}{1+\omega^2 T_1^2}. \end{cases}$$

$$\Downarrow$$

$$\left(x - \frac{K}{2}\right)^2 + y^2 = \left(\frac{K}{2}\right)^2.$$

Adding a pole at $s = 0$: $L(s) = \dfrac{K}{s(1+T_1 s)}$.

$$L(j\omega) = \frac{1}{j\omega} \cdot \frac{K}{1+T_1 j\omega} = \frac{1}{j\omega} \cdot \frac{K(1-T_1 j\omega)}{1+\omega^2 T_1^2}$$



Phase reduced by $\dfrac{\pi}{2}$,

$$\Big|L(j\omega)\Big|_0 = \infty.$$

$$x = \frac{-KT_1}{1+\omega^2 T_1^2}, \quad x\Big|_{\omega=0} = -KT_1.$$

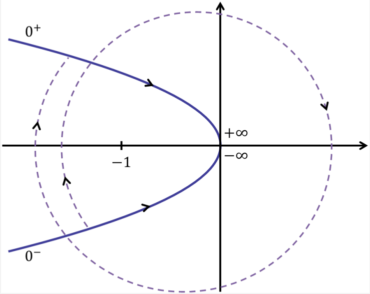
Generally, if adding a pole of multiplicity p at $s = 0$, we have

$$L(j\omega) = \frac{1}{(j\omega)^p} \cdot \frac{K}{1 + T_1 j\omega}$$

$$\begin{cases} \angle L(j\omega)\Big|_{\infty} = -(p+1)\dfrac{\pi}{2}, \\ \angle L(j\omega)\Big|_{0} = -p\dfrac{\pi}{2}; \end{cases} \qquad \begin{cases} \Big|L(j\omega)\Big|_{\infty} = 0, \\ \Big|L(j\omega)\Big|_{0} = \infty. \end{cases}$$

Now consider $L(s) = \dfrac{K}{s^2(1+T_1 s)}$. We have

$$L(j\omega) = \frac{-K}{\omega^2(1+j\omega T_1)} = \frac{-K(1-j\omega T_1)}{\omega^2(1+\omega^2 T_1^2)}.$$



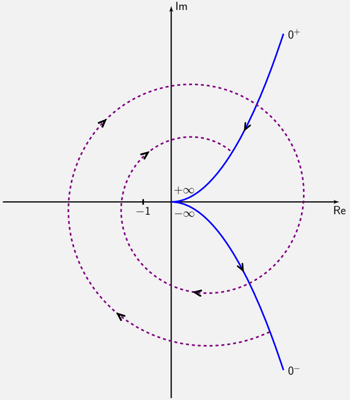
$$\begin{cases} x = \dfrac{-K}{\omega^2(1+\omega^2 T_1^2)}, \\ y = \dfrac{KT_1}{\omega(1+\omega^2 T_1^2)}. \end{cases}$$

$$0^- \to 0^+, \quad \epsilon e^{i\theta} : -\frac{\pi}{2} \to \frac{\pi}{2}$$

$N = 2 \;\Rightarrow\;$ 2 unstable poles.

Next consider $L(s) = \frac{K}{s^3(1+T_1 s)}$. We have

$$L(j\omega) = \frac{K}{-j\omega^3(1+j\omega T_1)} = \frac{K(\omega T_1 + j)}{\omega^3(1+\omega^2 T_1^2)}.$$



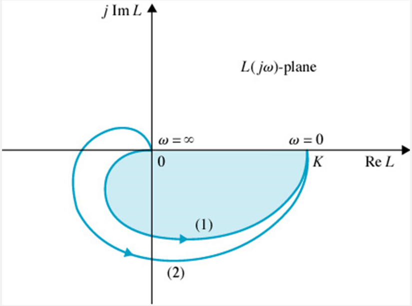
$$\begin{cases} x = \dfrac{KT_1}{\omega^2(1+\omega^2 T_1^2)}, \\ y = \dfrac{K}{\omega^3(1+\omega^2 T_1^2)}. \end{cases}$$

$$0^- \to 0^+, \quad \epsilon e^{i\theta} : -\frac{\pi}{2} \to \frac{\pi}{2}, \quad \frac{3\pi}{2} \to -\frac{3\pi}{2}$$

$N = 2 \;\Rightarrow\;$ 2 unstable poles.

$\boxed{\text{Conclusion: reduce stability.}}$

Adding nonzero poles



Nominal system:

$$L(s) = \frac{K}{(1 + T_2 s)(1 + T_1 s)}$$

$$L(j\omega)\Big|_{\infty} = \frac{-K}{\omega^2 T_1 T_2}\Big|_{\infty}$$

Add a nonzero pole: $L(s) = \dfrac{K}{(1 + T_3 s)(1 + T_2 s)(1 + T_1 s)}$.

At $\omega = 0$, $\Rightarrow$ same.
At $\omega = \infty$, phase shifted by $-\dfrac{\pi}{2}$.

$\longrightarrow$ reduce stability.

Adding zeros

Overall effect: reducing overshoot & increasing stability.

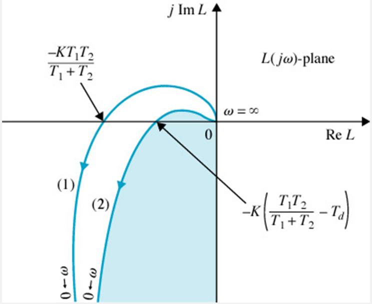
Consider a TF

$$L(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}.$$

Replace s by $j\omega$:

$$\begin{aligned} L(j\omega) &= \frac{K}{j\omega}\cdot\frac{1}{1+T_1 j\omega}\cdot\frac{1}{1+T_2 j\omega} \\ &= \frac{K}{\omega j}\cdot\frac{1}{1-\omega^2 T_1 T_2 + j\omega(T_1+T_2)} \\ &= \frac{K}{\omega}(-j)\frac{1-\omega^2 T_1 T_2 - j\omega(T_1+T_2)}{(1-\omega^2 T_1 T_2)^2+\omega^2(T_1+T_2)^2}. \end{aligned}$$

$$
\begin{cases}
x = \dfrac{-K(T_1 + T_2)}{(1-\omega^2 T_1 T_2)^2 + \omega^2 (T_1 + T_2)^2}, \\
y = -\dfrac{K}{\omega} \dfrac{1-\omega^2 T_1 T_2}{(1-\omega^2 T_1 T_2)^2 + \omega^2 (T_1 + T_2)^2}.
\end{cases}
$$



$$\omega = 0, \quad x = -K(T_1 + T_2),$$

$$\epsilon e^{i\theta} : -\frac{\pi}{2} \to \frac{\pi}{2}; r \to \infty, \frac{\pi}{2} \to -\frac{\pi}{2}.$$

$$y = 0 \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{T_1 T_2}},$$

$$x = \frac{-K}{\frac{1}{T_1 T_2}(T_1 + T_2)} = \frac{-K T_1 T_2}{T_1 + T_2}.$$

To ensure stability,

$$\frac{-K T_1 T_2}{T_1 + T_2} > -1, \quad K < \frac{T_1 + T_2}{T_1 T_2}.$$

Now add a zero, $L(s) = \dfrac{K(1+T_d s)}{s(1+T_1 s)(1+T_2 s)}$,

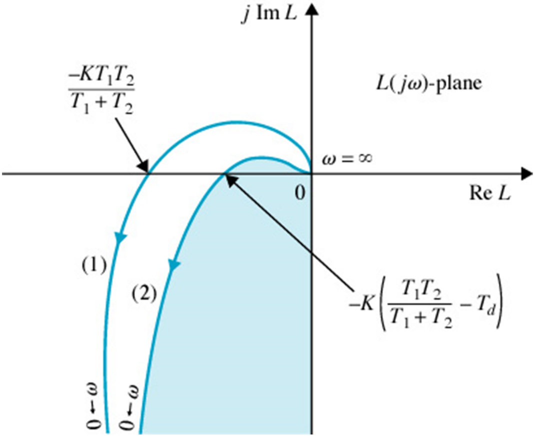
$$
\begin{aligned}
L(j\omega) &= \frac{K}{j\omega}\cdot\frac{1+j\omega T_d}{1-\omega^2 T_1 T_2 + j\omega(T_1+T_2)} \\
&= \frac{K(\omega T_d - j)(1-\omega^2 T_1 T_2 - j\omega(T_1+T_2))}{\omega[(1-\omega^2 T_1 T_2)^2+\omega^2(T_1+T_2)^2]} \\
&= \frac{K}{\omega}\cdot\frac{\omega T_d(1-\omega^2 T_1 T_2) - \omega(T_1+T_2) + (\omega^2 T_1 T_2 - 1 - \omega^2 T_d(T_1+T_2))j}{(1-\omega^2 T_1 T_2)^2+\omega^2(T_1+T_2)^2}
\end{aligned}
$$

$$
\begin{aligned}
\omega = 0 \quad &\Rightarrow \quad x = K(T_d - (T_1+T_2)) \\
y = 0 \quad &\Rightarrow \quad \omega^2(T_1 T_2 - T_d(T_1+T_2)) = 1 \\
&\Rightarrow \quad \omega^2 = \frac{1}{T_1 T_2 - T_d(T_1+T_2)} \\
&\Rightarrow \quad 1-\omega^2 T_1 T_2 = -\omega^2 T_d(T_1+T_2)
\end{aligned}
$$

$$
\begin{aligned}
x &= K \cdot \frac{T_d(1-\omega^2 T_1 T_2) - (T_1+T_2)}{(1-\omega^2 T_1 T_2)^2 + \omega^2 (T_1+T_2)^2} \\
&= K \cdot \frac{T_d(-\omega^2 T_d(T_1+T_2)) - (T_1+T_2)}{(-\omega^2 T_d(T_1+T_2))^2 + \omega^2 (T_1+T_2)^2} \\
&= -K \frac{(T_d^2\omega^2+1)(T_1+T_2)}{(\omega^2 T_d^2+1)\omega^2(T_1+T_2)^2} \\
&= \frac{-K}{T_1+T_2} \cdot \frac{T_d^2\omega^2+1}{\omega^2(1+T_d^2\omega^2)} = \frac{-K}{T_1+T_2} \cdot \frac{1}{\omega^2} \\
&= -K \cdot \frac{T_1 T_2 - T_d(T_1+T_2)}{T_1+T_2} = -K\left(\frac{T_1 T_2}{T_1+T_2} - T_d\right)
\end{aligned}
$$

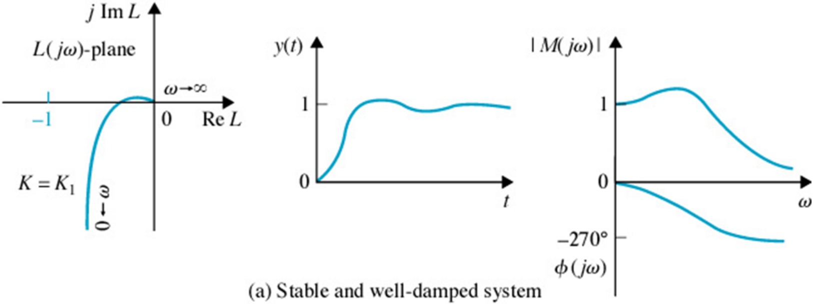
To ensure stability, $-K\left(\frac{T_1T_2}{T_1+T_2}-T_d\right)>-1$,

$$\therefore 0<K<\frac{T_1+T_2}{T_1T_2-T_d(T_1+T_2)}$$

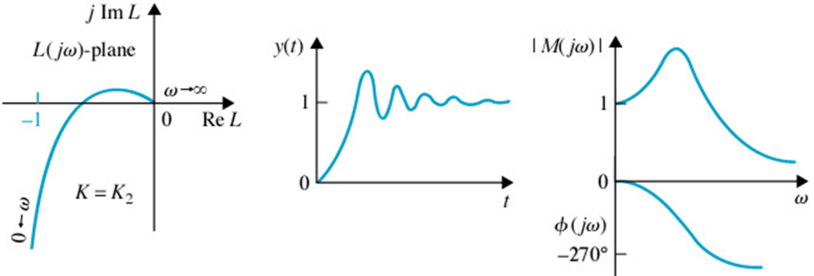


9-10 Relative Stability: GM & PM

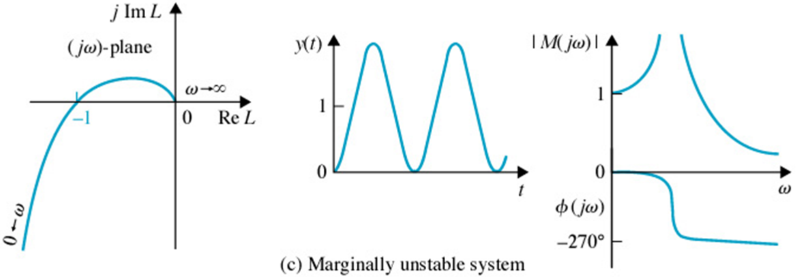
How stable a system is



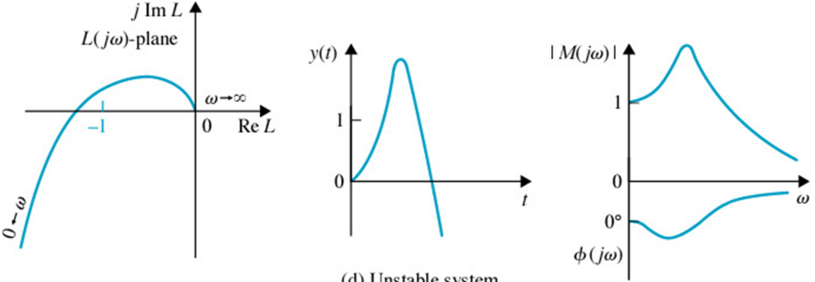
(a) Stable and well-damped system



(b) Stable but oscillatory system

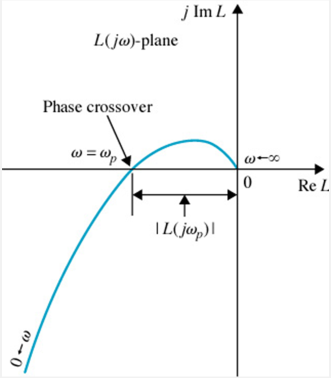


(c) Marginally unstable system



(d) Unstable system

9-10-1 Gain Margin (GM)



Phase crossover frequency ω_p

$$\angle L(j\omega_p) = \pi$$

Gain Margin (GM)

$$= 20\log_{10}\frac{1}{|L(j\omega_p)|}$$

$$= -20\log_{10}|L(j\omega_p)| \text{ dB}$$

In communication, it is standard to measure power gain in decibel (dB or db)

$$|G|_{\text{dB}} = 10 \log_{10} \frac{P_2}{P_1}$$

where P_1 is the input power, P_2 the output power.

As power $\propto$ (voltage)^2

$$\Rightarrow \quad |G|_{\text{dB}} = 20 \log_{10} \frac{V_2}{V_1}$$

- If $L(j\omega)$ does not intersect negative x-axis,

$$|L(j\omega_p)| = 0, \mathrm{GM} = \infty;$$

- If $L(j\omega)$ intersects negative x-axis between $(0, 1)$

$$0 < |L(j\omega_p)| < 1, \mathrm{GM} > 0;$$

- If $L(j\omega)$ passes through $(-1, j0)$;

$$|L(j\omega_p)| = 1, \mathrm{GM} = 0;$$

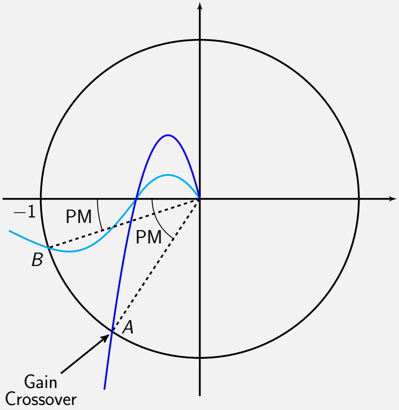
- If $L(j\omega)$ encloses $(-1, j0)$,

$$|L(j\omega_p)| > 1, \mathrm{GM} < 0.$$

9-10-2 Phase Margin (PM)

Sometimes, GM alone is not enough to determine relative stability.

For example, same GM but different stability margin:



- Gain crossover: point A
- Gain Crossover Frequency: $|L(j\omega_g)| = 1$
- PM: $\angle L(j\omega_g) - \pi$

Conclusions

Advantage of Nyquist Plot

- Can be used for non-minimum phase system
- Easy to check stability

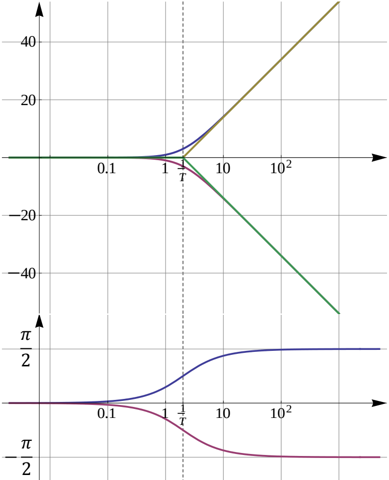
Disadvantage

- Not easy for design

9-11 Bode Plot

Bode plot of $G(s)$ (Loop Transfer Function): two plots.

1. $20\log_{10}|G(j\omega)|$ vs $\log_{10}\omega$
2. $\angle G(j\omega)$ vs $\log_{10}\omega$



Advantages of Bode plot

1. System in series $\Rightarrow$ addition in Bode plots;

$$G(s) = G_1(s)G_2(s)$$

a)

$$\begin{aligned} |G(j\omega)|_{\text{dB}} &= 20\log_{10}|G(j\omega)| = 20\log_{10}|G_1(j\omega)G_2(j\omega)| \\ &= 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)| \\ &= |G_1(j\omega)|_{\text{dB}} + |G_2(j\omega)|_{\text{dB}} \end{aligned}$$

b)

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

c) When $G_2(s) = K$,

$$|G(j\omega)|_{\text{dB}} = 20\log K + |G_1(j\omega)|_{\text{dB}},$$

shift up or down $|G_1(j\omega)|_{\text{dB}}$.

2. Logarithm scale permits a much wider range of frequencies to be displayed;

3. Bode plot can be determined experimentally;

4. Control design can be based entirely on Bode plots.

Consider a loop gain function

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{s^j(s+p_1)(s+p_2)\cdots(s+p_n)} \cdot e^{-T_d s}$$

Zeros: $-z_k$; Poles: $-p_k$; Time delay: T_d

$\longrightarrow$ preferred form for root locus.

For Bode plot, we want to write it as

$$G(s) = \frac{Kz_1 \cdots z_m \left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_m}\right)}{p_1 \cdots p_n \cdot s^j \left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_n}\right)} \cdot e^{-T_d s}.$$

Now let

$$K_1 = \frac{Kz_1 \cdots z_m}{p_1 \cdots p_n},$$
$$T_1 = \frac{1}{z_1}, \cdots, T_m = \frac{1}{z_m},$$
$$\tau_1 = \frac{1}{p_1}, \cdots, \tau_n = \frac{1}{p_n},$$

we have

$$G(s) = K_1 \frac{(1+T_1 s)(1+T_2 s)\cdots(1+T_m s)}{s^j(1+\tau_1 s)(1+\tau_2 s)\cdots(1+\tau_n s)} e^{-T_d s}.$$

For example, consider

$$G(s) = K\frac{(1+T_1 s)(1+T_2 s)}{s(1+\tau_1 s)(1+2\zeta/\omega_n s + s^2/\omega_n^2)}e^{-T_d s}, \quad 0<\zeta<1,$$

$$\Rightarrow \quad G(j\omega) = K\frac{(1+T_1 j\omega)(1+T_2 j\omega)}{j\omega(1+\tau_1 j\omega)(1+2\zeta/\omega_n j\omega - \omega^2/\omega_n^2)}e^{-T_d j\omega}.$$

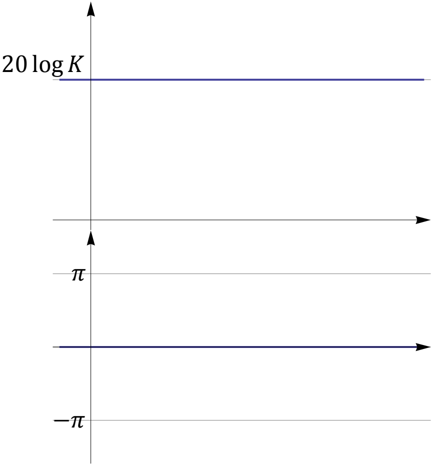
Now

$$\begin{aligned}
|G(j\omega)|_{\text{dB}} =& 20\log_{10}|G(j\omega)| \\
=& 20\log|K| + 20\log|1+j\omega T_1| + 20\log|1+j\omega T_2| \\
& - 20\log|j\omega| - 20\log|1+j\omega\tau_1| \\
& - 20\log\left|1+2\zeta/\omega_n j\omega - \omega^2/\omega_n^2\right| \\
\angle G(j\omega) =& \angle(1+T_1 j\omega) + \angle(1+T_2 j\omega) - \angle j\omega \\
& - \angle(1+\tau_1 j\omega) - \angle(1+2\zeta/\omega_n j\omega - \omega^2/\omega_n^2) - \omega T_d
\end{aligned}$$

Five types of Transfer Functions

In general, $G(j\omega)$ has five different types.

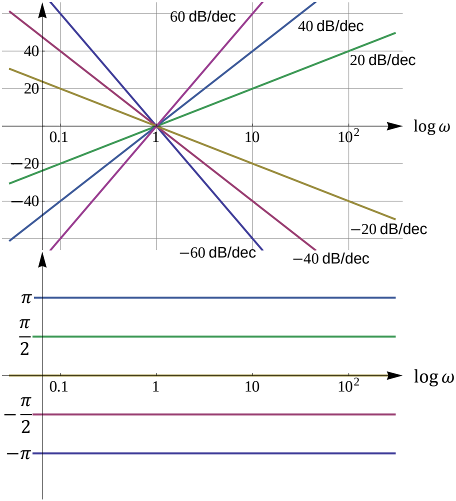
1. Constant K;
2. Poles or zeros at the origin of order p,
$$s^{\pm p},\ (j\omega)^{\pm p};$$
3. Poles or zeros at $s = -\dfrac{1}{T}$ of order q,
$$(1+sT)^{\pm q},\ (1+j\omega T)^{\pm q};$$
4. Complex poles or zeros of order r,
$$(1+2\zeta/\omega_n s+s^2/\omega_n^2)^{\pm r},\ (1+2\zeta/\omega_n j\omega-\omega^2/\omega_n^2)^{\pm r}$$
5. Pure time delay,
$$e^{-T_d s},\ e^{-j\omega T_d}.$$



1. $G(s) = K$, $G(j\omega) = K$,

$$|G(j\omega)|_{\mathrm{dB}} = 20\log|K|$$

$$\angle G(j\omega) = \begin{cases} 0, & K > 0 \\ \pi, & K < 0 \end{cases}$$



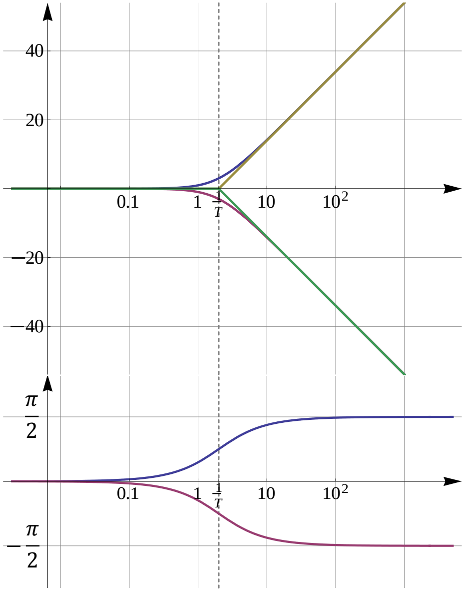
❷ $G(s) = s^{\pm p}$,
$G(j\omega) = (j\omega)^{\pm p}$,

$$|G(j\omega)|_{\mathrm{dB}} = 20\log|(j\omega)^{\pm p}| = \pm 20p\log\omega$$

slope: $\pm 20p$ dB/decade

$$\begin{aligned} \omega &= 1 & \to\quad & 0 \\ \omega &= 10 & \to\quad & \pm 20p \\ \omega &= 10^2 & \to\quad & \pm 40p \end{aligned}$$

$$\angle (j\omega)^{\pm p} = \pm p \cdot \frac{\pi}{2}$$



3. $G(s) = 1 + sT$, $G(j\omega) = 1 + j\omega T$,

$\angle G(j\omega) = \tan^{-1}\omega T$,
$|G(j\omega)|_{\mathrm{dB}} = 20\log|G(j\omega)| = 20\log\sqrt{1+\omega^2T^2}$

a) At low frequency, $\omega T \ll 1$, $|G(j\omega)|_{\mathrm{dB}} = 20\log 1 = 0$, $\angle G(j\omega) = 0$.

b) At high frequency, $\omega T \gg 1$, $|G(j\omega)|_{\mathrm{dB}} = 20\log\omega T = 20\log T + 20\log\omega$. $\rightarrow$ a line with slope 20 dB/dec. $\angle G(j\omega) = \frac{\pi}{2}$

When $\omega = \dfrac{1}{T}$ (corner frequency or break point), these two asymptotes intersect.

a) $\omega = \dfrac{1}{T}$, $G(j\omega) = 1 + j$,

$$\begin{aligned}\text{Error} &= |G(j\omega)|_{\text{dB}} = 20\log|1+j| = 20\log\sqrt{2}\\ &= 10\log 2 = 3\text{dB}\end{aligned}$$

$$\angle(j\omega) = \frac{\pi}{4}$$

b) $\omega = \dfrac{2}{T}$,

$$\begin{aligned}\text{Error} &= 20\log|1+2j| - 20\log 2 = 10\log 5 - 20\log 2\\ &= 10(0.6990) - 20(0.3010) = 6.990 - 6.020\\ &= 0.970 \sim 1\text{dB}\end{aligned}$$

Procedure:

i) Locate $\omega = \frac{1}{T}$;

ii) Draw 20 dB/dec line & horizontal line at 0 dB, intersecting at $\omega = \frac{1}{T}$,

iii) Add in error: $\omega = \frac{1}{T}$, 3 dB; $\omega = \frac{2}{T}$, 1 dB.

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

$$\begin{aligned} |G(j\omega)|_{\text{dB}} &= 20 \log \frac{1}{|1 + j\omega T|} \\ &= -20 \log |1 + j\omega T| = -|1 + j\omega T|_{\text{dB}} \end{aligned}$$

$$\angle G(j\omega) = -\tan^{-1} \omega T = -\angle(1 + j\omega T)$$

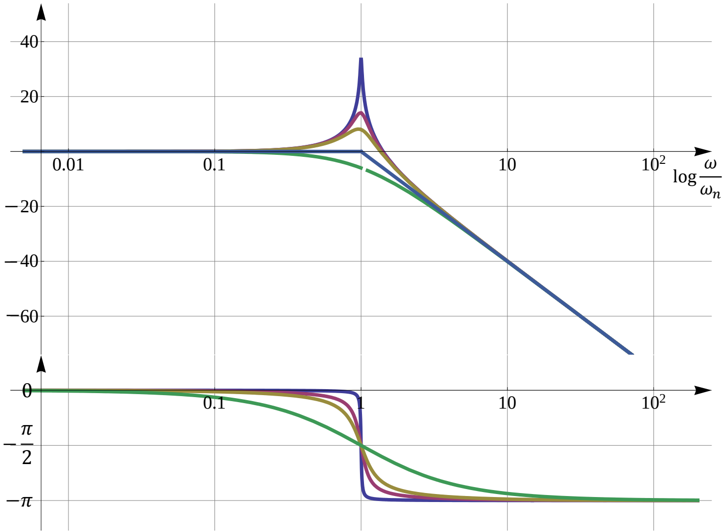
4. Complex poles or zeros ($0 < \zeta < 1$)

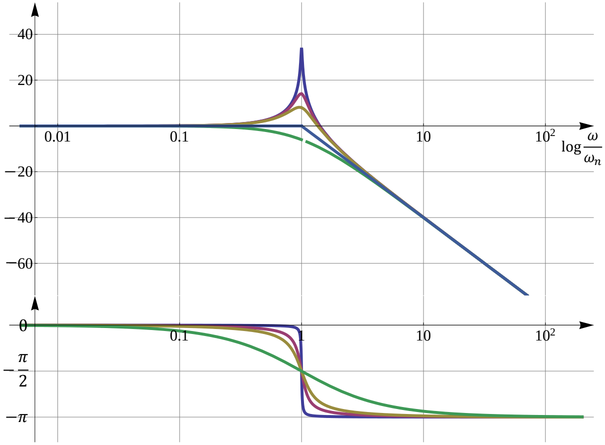
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{1 + (2\zeta/\omega_n)s + s^2/\omega_n^2},$$

$$G(j\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2\zeta\frac{\omega}{\omega_n}j},$$

$$|G(j\omega)|_{\mathrm{dB}} = 20\log|G(j\omega)| = -10\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}\right],$$

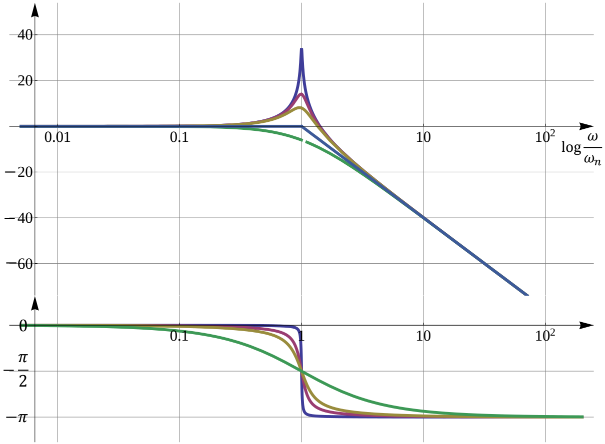
$$\angle G(j\omega) = -\tan^{-1}\frac{2\zeta\dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}}.$$


40
20
−20
−40
−60
0.01
0.1
10
10^2
$\log \frac{\omega}{\omega_n}$
0
$-\frac{\pi}{2}$
$-\pi$
0.1
1
10
10^2



At very low frequency, $\omega/\omega_n \ll 1$

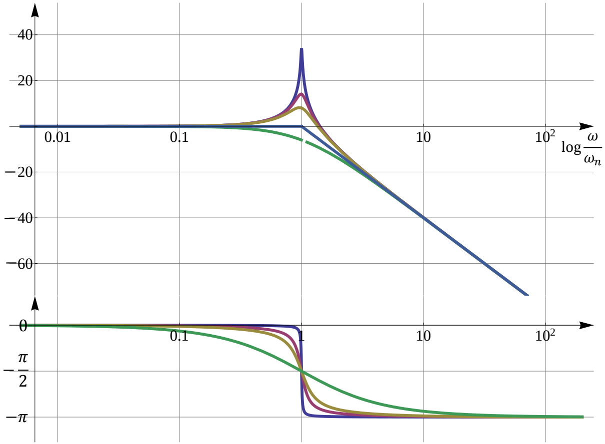
$$|G(j\omega)|_{\mathrm{dB}} = -10\log 1 = 0, \angle G(j\omega) = 0.$$



At very high frequency, $\omega/\omega_n \gg 1$,

$$|G(j\omega)|_{\mathrm{dB}} = -10\log\left(\omega^2/\omega_n^2\right)^2 = -40\log\frac{\omega}{\omega_n},$$

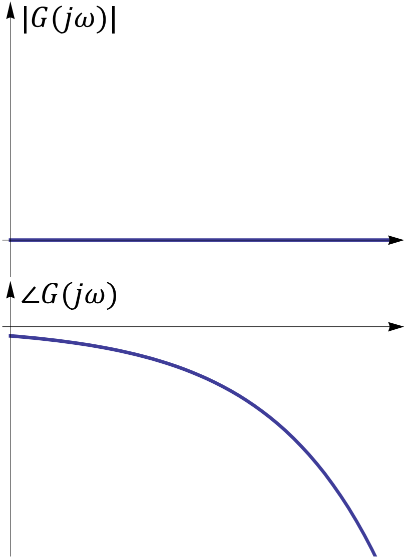
$$\angle G(j\omega) = \angle - \frac{1}{(\omega/\omega_n)^2} = -\pi.$$



Intersect $\omega = \omega_n$: differs a lot from the asymptote curve,

$$|G(j\omega)|_{\text{dB}} = -10\log 4\zeta^2 = -20\log 2\zeta \uparrow \text{ as } \zeta \downarrow;$$

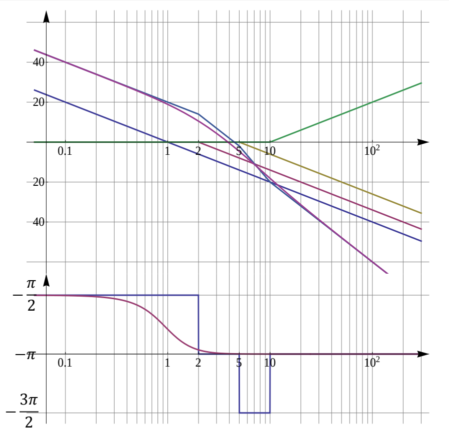
$$\text{when } \zeta = \frac{1}{2}, |G(j\omega)|_{\text{dB}} = 0; \quad \angle G(j\omega) = \angle\frac{1}{2\zeta j} = -\frac{\pi}{2}.$$



5. Pure time delay

$$
\begin{aligned}
G(s) &= e^{-T_d s}, \\
G(j\omega) &= e^{-T_d j\omega}, \\
|G(j\omega)|_{\text{dB}} &= 20\log\left|e^{-j\omega T_d}\right| \\
&= 0 \\
\angle G(j\omega) &= -\omega T_d.
\end{aligned}
$$

Example



$$G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$

$$= \frac{100\left(1+\frac{s}{10}\right)}{10s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)}$$

$$= 10\frac{1+\frac{s}{10}}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)},$$

Let $s = j\omega$, $G(j\omega) = 10\dfrac{1+\frac{j\omega}{10}}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{5}\right)}$.

Corner frequency: $\omega = 2, 5, 10$ rad/sec.

1) $K = 10$, mag=20dB, phase=0.

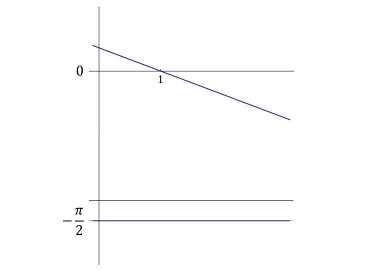


Figure 9.8: 2) $\frac{1}{j\omega}$

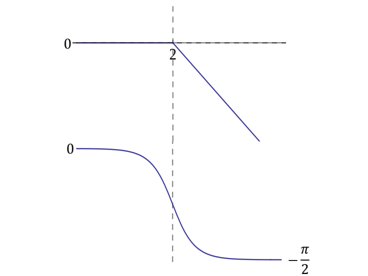


Figure 9.9: 3) $\frac{1}{1+\frac{j\omega}{2}}$

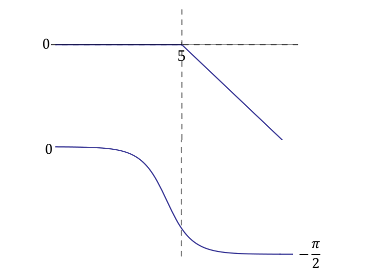


Figure 9.10: 4) $\dfrac{1}{1+\dfrac{j\omega}{5}}$

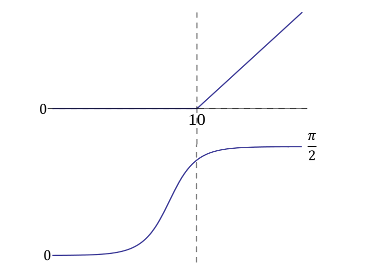


Figure 9.11: 5) $1+\dfrac{j\omega}{10}$

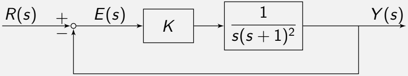
Discussions

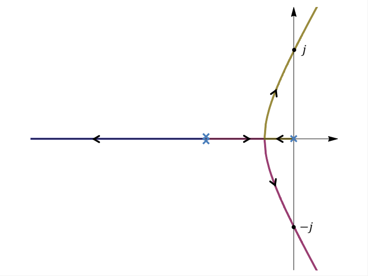
$$(0,2]: -20\ \mathrm{dB/dec},\quad [2,5]: -40\ \mathrm{dB/dec},$$
$$[5,10]: -60\ \mathrm{dB/dec},\quad [10,\infty): -40\ \mathrm{dB/dec}.$$

Bode Plots & Stability

Example

Consider a system





$$1 + K\frac{1}{s(s+1)^2} = 0$$

$$\Rightarrow \quad s^3 + 2s^2 + s + K = 0$$

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & K \\ s^1 & \frac{2-K}{2} & \\ s^0 & K & \end{array} \quad \Rightarrow \quad \begin{array}{c} 2-K \geq 0, \\ K > 0, \\ K = 2 \end{array} \quad \Rightarrow \quad \begin{array}{c} s^3 + 2s^2 + s + 2 = 0, \\ (s+2)(s^2+1) = 0, \\ \therefore s = \pm j. \end{array}$$

Root Locus & Bode Plots

All points on root locus satisfy

$$|KG(s)| = 1 \ \& \ \angle G(s) = \pi \quad (s = \sigma + j\omega).$$

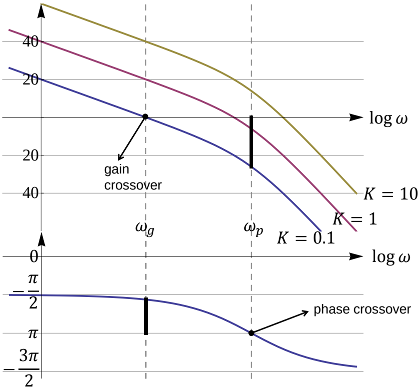
In particular, at the point of marginally stable $\sigma = 0$ and thus

$$|KG(j\omega)| = 1 \ \& \ \angle G(j\omega) = \pi.$$

Then on Bode plot, marginally stable system satisfies

$$|KG(j\omega)|_{\text{dB}} = 0 \ \& \ \angle G(j\omega) = \pi.$$

Stability Margins



Gain Margin: define phase crossover frequency ω_p as

$$\angle G(j\omega_p) = -\pi.$$

$$\begin{aligned} \text{GM: } 0 - |G(j\omega_p)|_{\text{dB}} &= -20\log|G(j\omega_p)| \\ &= 20\log\frac{1}{|G(j\omega_p)|} \end{aligned}$$

Phase Margin: define gain crossover frequency ω_g as

$$|G(j\omega_g)| = 1 \quad \Rightarrow \quad |G(j\omega_g)|_{\text{dB}} = 0$$

$$\text{PM: } \angle G(j\omega) + \pi.$$

Nonminimum Phase System: has zero in the RHP

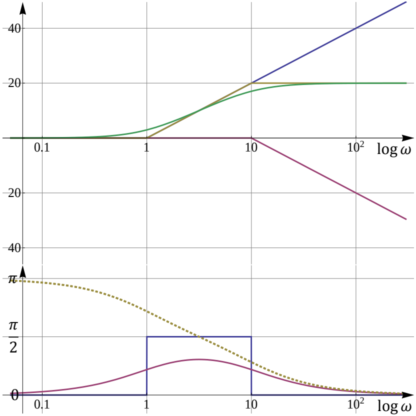
Example

Consider

$$G_1(s) = 10\frac{s+1}{s+10}, \qquad G_2(s) = 10\frac{s-1}{s+10},$$

$$G_1(j\omega) = \frac{1+j\omega}{1+j\omega/10}, \qquad G_2(j\omega) = \frac{-1+j\omega}{1+j\omega/10},$$

$$|G_1(j\omega)| = \sqrt{\frac{1+\omega^2}{1+(\omega/10)^2}}, \qquad |G_2(j\omega)| = \sqrt{\frac{1+\omega^2}{1+(\omega/10)^2}}.$$

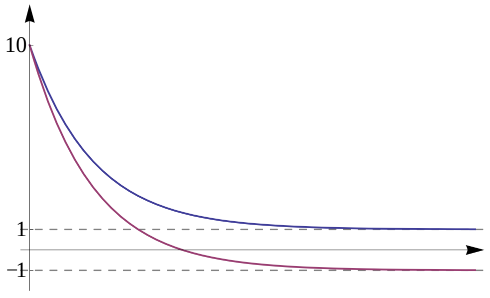


$$G_2(j\omega) = \frac{-1 + j\omega}{1 + j\omega/10}.$$

Nonminimum phase $\angle G_2(j\omega)$:
When $\omega \ll 1$, $\angle G_2(j\omega) = \pi$,
When $\omega \gg 1$, $\angle G_2(j\omega) = 0$,
When $\omega = 1$, $\angle G_2(j\omega) \approx \frac{3\pi}{4}$.

With a given mag curve, minimum phase system will produce smallest net change of phase as $\omega : 0 \to \infty$.

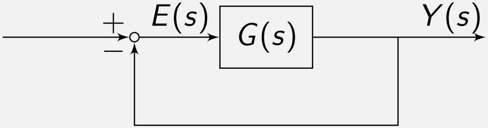
Unit step response



$$Y_1(s) = G_1(s) \cdot \frac{1}{s}, \quad y_1(t) = 9e^{-10t} + 1.$$

$$Y_2(s) = G_2(s) \cdot \frac{1}{s}, \quad y_2(t) = 11e^{-10t} - 1.$$

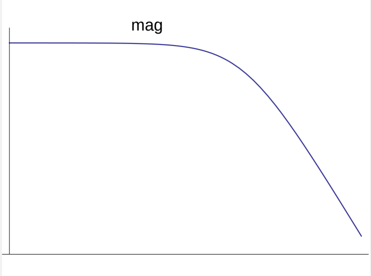
Steady state error



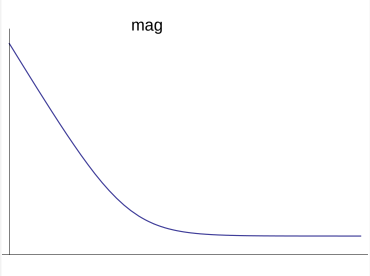
$$E(s) = \frac{1}{1+G(s)} \cdot R(s).$$

For unit step input, then

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+G(s)} \cdot \frac{1}{s} = \frac{1}{1+G(0)}.$$



(a) nonzero e_{ss}



(b) zero e_{ss}

$G(0) \approx |G(j\omega)|$ at very low frequency.

$\Rightarrow e_{ss} \downarrow$ when $|G(j\omega)| \uparrow$, ω: low frequency.

$\therefore$ larger low frequency magnitude $\Rightarrow$ smaller e_{ss}.