Ve460 Control Systems Analysis and Design
Chapter 3 Block Diagram and Signal Flow Graphs

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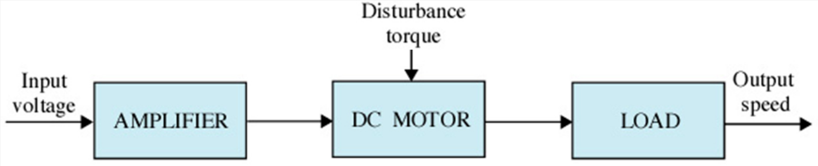
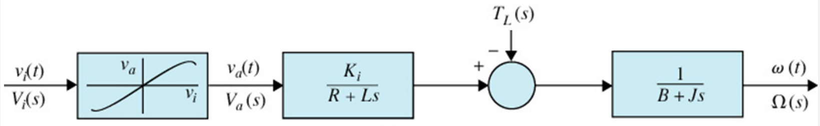
3-1 Block diagrams

- Used to model all types of system: plant and controller;
- Describe cause-and-effect relationship;
- Describe composition and interconnection.



(a) Block diagram of a DC-motor control system.



(b) Block diagram with TFs and amplifier characteristics.

3-1-1 Block Diagrams of Control Systems

Sensing devices: addition & subtraction

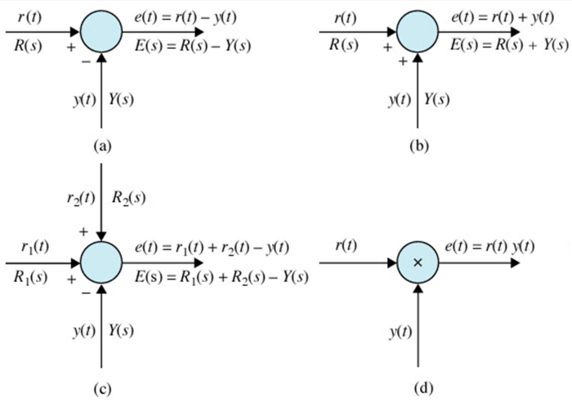
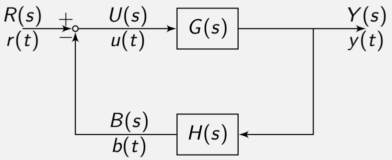


Figure 3.1: Block diagram element of typical sensing devices of control systems. (a) Subtraction; (b) Addition; (c) Addition and Subtraction; (d) Multiplication.

A feedback control system (an *LTI* system!)



r: reference input

y: output (controlled variable)

b: feedback signal

u: actuating signal

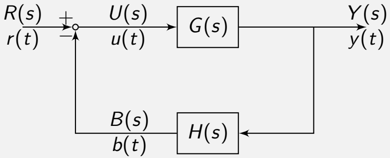
$H(s)$: feedback TF

$G(s)H(s) = L(s)$: Loop TF

$G(s)$: forward-path TF

$M(s) = Y(s)/R(s)$: closed-loop TF or system TF

How to calculate $M(s)$?



$$\begin{cases} Y(s) = G(s)U(s) \\ B(s) = H(s)Y(s) \\ U(s) = R(s) - B(s) \end{cases}$$

$$\begin{aligned} Y(s) &= G(s)(R(s) - B(s)) \\ &= G(s)R(s) - G(s)H(s)Y(s) \end{aligned}$$

$$\Rightarrow (1 + G(s)H(s))Y(s) = G(s)R(s)$$

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

3-1-2 Multivariable System

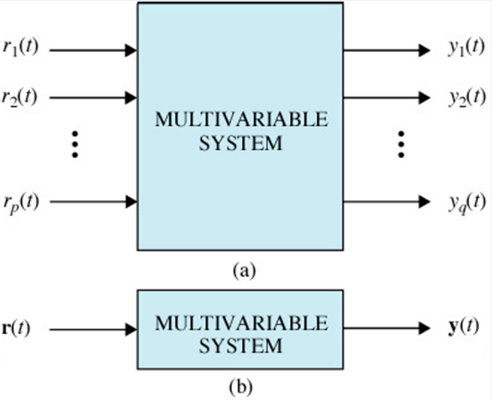
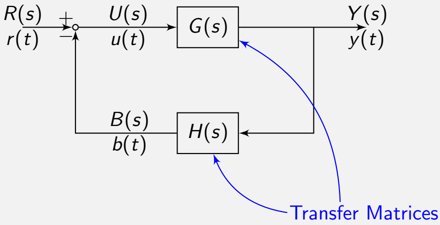


Figure 3.2: Block diagram representations of a multivariable system

A Multivariable Feedback Control System



$$\begin{cases} Y(s) = G(s)U(s) \\ U(s) = R(s) - B(s) \\ B(s) = H(s)Y(s) \end{cases}$$

Therefore

$$\begin{aligned} Y(s) &= G(s)R(s) - G(s)B(s) \\ &= G(s)R(s) - G(s)H(s)Y(s), \end{aligned}$$

and

$$(I + G(s)H(s))Y(s) = G(s)R(s)$$

$$M(s) = (I + G(s)H(s))^{-1}G(s).$$

Example 1

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{1}{s+2} \end{bmatrix}, \qquad H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$I + G(s)H(s) = \begin{bmatrix} \dfrac{s+2}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{s+3}{s+2} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

then

$$(I + G(s)H(s))^{-1} = \frac{1}{\Delta} \begin{bmatrix} \dfrac{s+3}{s+2} & \dfrac{1}{s} \\ -2 & \dfrac{s+2}{s+1} \end{bmatrix},$$

where

$$\Delta = \frac{s+2}{s+1}\cdot\frac{s+3}{s+2}+\frac{2}{s}=\frac{s^2+5s+2}{s(s+1)}.$$

Therefore

$$\begin{aligned}
M(s) &= (I+G(s)H(s))^{-1}G(s)\\
&= \frac{s(s+1)}{s^2+5s+2}\begin{bmatrix}\frac{s+3}{s+2} & \frac{1}{s}\\ -2 & \frac{s+2}{s+1}\end{bmatrix}\begin{bmatrix}\frac{1}{s+1} & \frac{1}{s}\\ 2 & \frac{2}{s}\end{bmatrix}\\
&= \frac{s(s+1)}{s^2+5s+2}\begin{bmatrix}\frac{3s^2+9s+4}{s(s+2)(s+1)} & -\frac{1}{s}\\ 2 & \frac{3s+2}{s(s+1)}\end{bmatrix}.
\end{aligned}$$

Matlab Control System Toolbox

We first use `tf` to create transfer function models.

tf

R2020a

Transfer function model

expand all in page

Description

Use `tf` to create real-valued or complex-valued transfer function models, or to convert dynamic system models to transfer function form.

Transfer functions are a frequency-domain representation of linear time-invariant systems. For instance, consider a continuous-time SISO dynamic system represented by the transfer function `sys(s) = N(s)/D(s)`, where `s = jw` and `N(s)` and `D(s)` are called the numerator and denominator polynomials, respectively. The `tf` model object can represent SISO or MIMO transfer functions in continuous time or discrete time.

You can create a transfer function model object either by specifying its coefficients directly, or by converting a model of another type (such as a state-space model `ss`) to transfer-function form. For more information, see Transfer Functions.

You can also use `tf` to create generalized state-space (`genss`) models or uncertain state-space (`uss`) models.

Creation

Syntax

```
sys = tf(numerator,denominator)
sys = tf(numerator,denominator,ts)
sys = tf(numerator,denominator,ltiSys)
```

Example

Create a Matlab transfer function model for:

$$G(s) = \frac{s}{s^2 + s + 1}.$$

```
>> num=[1 0];
>> den=[1 1 1];
>> Gs=tf(num, den)
Gs =
       s
  -----------
  s^2 + s + 1
Continuous-time transfer function.
```

Alternatively, we can use rational expression

```
>> s = tf('s');
>> Gs1=s/(s^2+s+1)
Gs1 =
       s
  -----------
  s^2 + s + 1
Continuous-time transfer function.
```

Consider Example 1 again:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{1}{s+2} \end{bmatrix}, \qquad H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

```
>> Gnum={1 -1; 2 1}; Gden={[1 1], [1 0]; 1 [1 2]};
>> Gs2=tf(Gnum, Gden)
Gs2 =
  From input 1 to output...
         1
   1:  -----
       s + 1
   2:  2

  From input 2 to output...
       -1
   1:  --
       s
         1
   2:  -----
       s + 2
Continuous-time transfer function.
```

We now use `feedback` in Matlab Control System Toolbox.

feedback

R2020a

Feedback connection of multiple models

collapse all in page

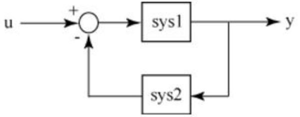
Syntax

```
sys = feedback(sys1,sys2)
sys = feedback(sys1,sys2,feedin,feedout)
sys = feedback(sys1,sys2,'name')

sys = feedback(___,sign)
```

Description

`sys = feedback(sys1,sys2)` returns a model object `sys` for the negative feedback interconnection of model objects `sys1,sys2`.

example



From the figure, the closed-loop model `sys` has u as input vector and y as output vector. Both models, `sys1` and `sys2`, must either be continuous or discrete with identical sample times.

```
>> sys=simplify( feedback(Gs2, eye(2)) )

sys =
  From input 1 to output...
             3 s^2 + 9 s + 4
   1:  ----------------------
       s^3 + 7 s^2 + 12 s + 4

           2 s^3 + 6 s^2 + 4 s
   2:  ----------------------
       s^3 + 7 s^2 + 12 s + 4

  From input 2 to output...
              -s^2 - 3 s - 2
   1:  ----------------------
       s^3 + 7 s^2 + 12 s + 4

             3 s^2 + 8 s + 4
   2:  ----------------------
       s^3 + 7 s^2 + 12 s + 4
Continuous-time transfer function.
```

From earlier computation:

$$\frac{s(s+1)}{s^2+5s+2} \cdot \begin{bmatrix} \frac{3s^2+9s+4}{s(s+2)(s+1)} & -\frac{1}{s} \\ 2 & \frac{3s+2}{s(s+1)} \end{bmatrix}$$

Using Matlab Symbolic Toolbox

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In Matlab, we can calculate it as

```
>> syms a b c d
>> inv([a b; c d])
ans =
[  d/(a*d - b*c), -b/(a*d - b*c)]
[ -c/(a*d - b*c),  a/(a*d - b*c)]

>> pretty(ans)
/      d              b      \
|  ---------,  - ---------   |
|  a d - b c     a d - b c   |
|                            |
|       c            a       |
| - ---------,  ---------    |
\   a d - b c   a d - b c    /
```

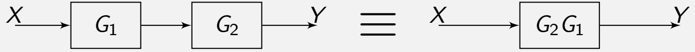
$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{1}{s+2} \end{bmatrix}, \qquad H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
>> syms s;
>> Gs=[1/(s+1) -1/s; 2 1/(s+2)];
>> Hs=eye(2);
>> simplify( inv(eye(2)+Gs*Hs)*Gs )
ans =
[ (3*s^2 + 9*s + 4)/(s^3 + 7*s^2 + 12*s + 4),  -(s + 1)/(s^2 + 5*s + 2)]
[              (2*s*(s + 1))/(s^2 + 5*s + 2), (3*s + 2)/(s^2 + 5*s + 2)]

>> pretty(ans)
/       2                                 \
|    3 s  + 9 s + 4            s + 1      |
| --------------------, - ------------    |
|  3      2                   2           |
| s  + 7 s  + 12 s + 4    s  + 5 s + 2    |
|                                         |
|      2 s (s + 1)          3 s + 2       |
|     ------------,      ------------     |
|      2                  2               |
\     s  + 5 s + 2       s  + 5 s + 2     /
```

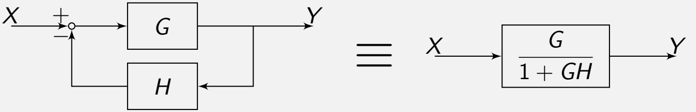
Simplification of Block Diagram

Simple cases

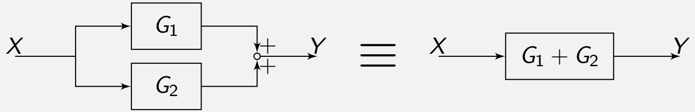
a) Combining blocks in series (Matlab command `series`)



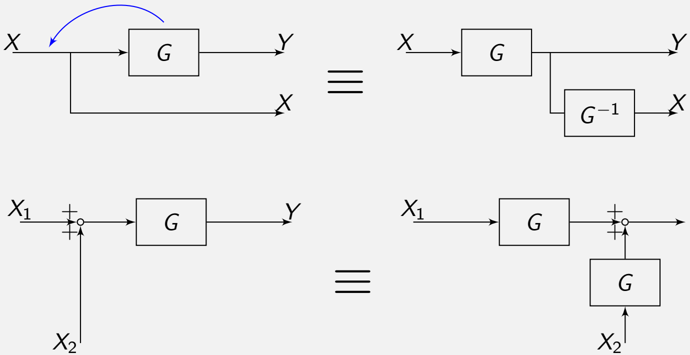
b) Elimination of feedback loop (Matlab command `feedback`)



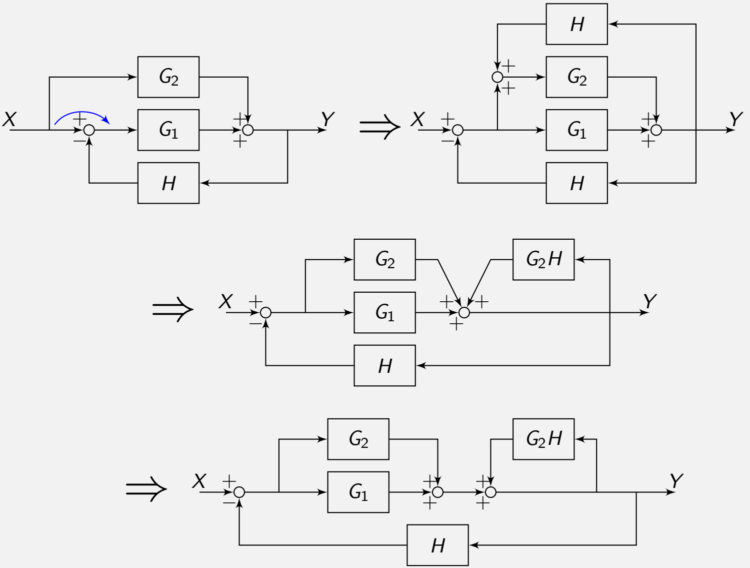
c) Combining blocks in parallel (Matlab command `parallel`)



d) Moving a block



Example

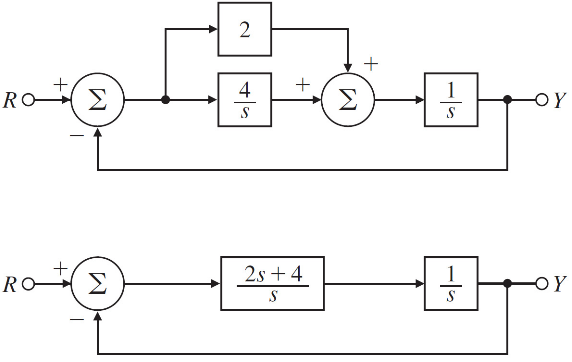


Therefore, we have

$$
\begin{aligned}
M(s) &= \frac{(G_1 + G_2) \cdot \dfrac{1}{1 - G_2 H}}{1 + (G_1 + G_2) \cdot \dfrac{1}{1 - G_2 H} \cdot H} \\
&= \frac{G_1 + G_2}{1 - G_2 H + (G_1 + G_2) H} \\
&= \frac{G_1 + G_2}{1 + G_1 H}.
\end{aligned}
$$

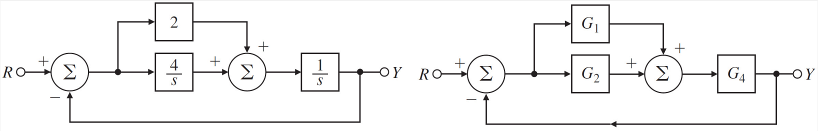
Block diagram simplification using Matlab

Consider the following example:



$$G(s) = \frac{\frac{2s+4}{s}\frac{1}{s}}{1+\frac{2s+4}{s}\frac{1}{s}} = \frac{2s+4}{s^2+2s+4}.$$

To use Matlab, label the blocks as follows:



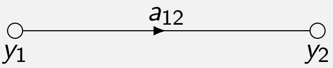

```
s=tf('s');                      % specify a transfer function using
                                  a rational function in the
                                  Laplace variable s
sysG1=2;                        % define subsystem G1
sysG2=4/s;                      % define subsystem G2
sysG3=parallel(sysG1,sysG2);    % parallel combination of G1 and G2
sysG4=1/s;                      % define subsystem G4
sysG5=series(sysG3,sysG4);      % series combination of G3 and G4
sysCL=feedback(sysG5,1,-1);     % feedback combination of G5 and G6

sysCL =
     2 s + 4
  -------------
  s^2 + 2 s + 4
Continuous-time transfer function.
```

3-2 Signal Flow Graphs (SFGs)

Simplified version of block diagram

Cause-and-effect relation:

$$y_2 = a_{12} y_1$$



SFG basic properties

- Applies only to linear systems;
- Cause-and-effect algebraic equations.

- Node: variable;
- Signal flows along the direction of arrow;
- $y_k \longrightarrow y_j$: arrow represents dependence;
- a_{kj}: gain.

SFG algebra

1. Value of node $= \sum$ all signals entering the node:

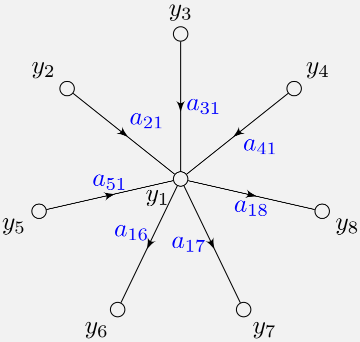
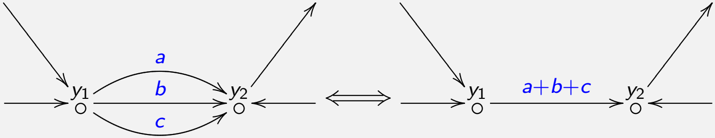


Figure 3.3: $y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$.

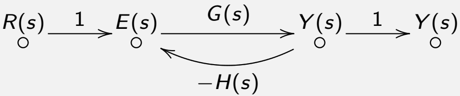
2. Value of node is transmitted through all branches leaving the node, *e.g.*, $y_6 = a_{16}y_1$, $y_7 = a_{17}y_1$, $y_8 = a_{18}y_1$.

3. Parallel branches in the same direction.



4. Series connection of unidirectional branches

Example: Feedback Control System



$$Y(s) = G(s) \cdot E(s) = G(s)(R(s) - H(s) \cdot Y(s))$$
$$\therefore (1 + G(s)H(s))Y(s) = G(s) \cdot R(s)$$
$$\therefore \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Example: Constructing SFG from algebraic equations

$$\begin{cases} y_2 = a_{12}y_1 + a_{32}y_3 \\ y_3 = a_{23}y_2 + a_{43}y_4 \\ y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4 \\ y_5 = a_{25}y_2 + a_{45}y_4 \end{cases}$$

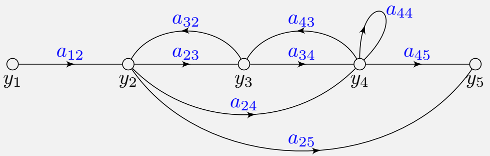
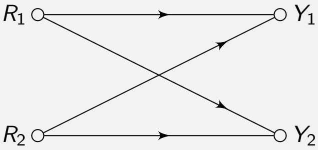


Figure 3.4: Signal Flow Graph example

Multivariable systems



$$Y_1(s) = G_{11}(s)R_1(s) + G_{21}(s)R_2(s)$$
$$Y_2(s) = G_{12}(s)R_1(s) + G_{22}(s)R_2(s)$$

Matrix representation:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{21}(s) \\ G_{12}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

In general, we have

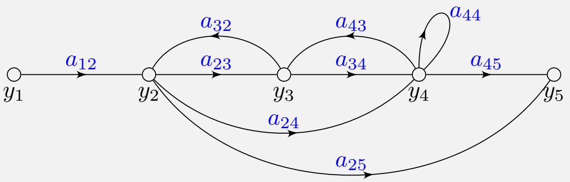
$$G_{ij}(s) = \frac{Y_j(s)}{R_i(s)}, \qquad Y(s) = G(s)R(s),$$

$$R(s) = \begin{bmatrix} R_1(s) \\ \vdots \\ R_p(s) \end{bmatrix}, \quad Y(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_q(s) \end{bmatrix},$$

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{21}(s) & \cdots & G_{p1}(s) \\ G_{12}(s) & G_{22}(s) & \cdots & G_{p2}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{1q}(s) & G_{2q}(s) & \cdots & G_{pq}(s) \end{bmatrix}.$$

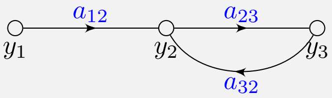
Note that the indices here are different from a usual matrix.

Definition of SFG Terms



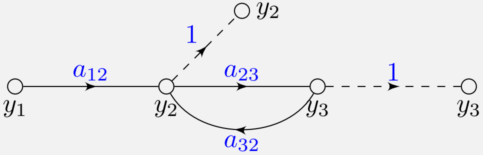
1. Input node: has only outgoing branch, y_1;
2. Output node: has only incoming branch, y_5.

Some SFG does not have output node, *e.g.*,



(a) Original signal flow graph

Can add a new node to make an output node:

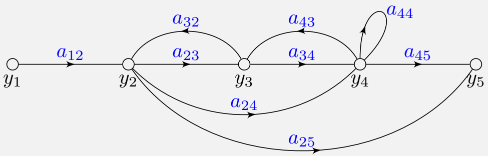


(b) Modified signal flow graph

More definitions of SFG Terms

3. Path: continuous succession of branches traversed in the same direction;
4. Path gain: product of branch gains along a path;
5. Forward path: a path from an input node to an output node, and no node traversed more than once;
6. Forward path gain: path gain of a forward path.

We have three forward paths between y_1 and y_5



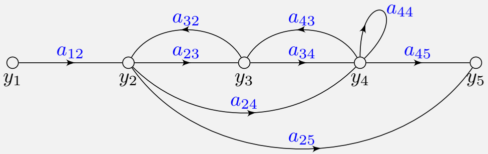
$$M_1 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow{a_{23}} y_3 \xrightarrow{a_{34}} y_4 \xrightarrow{a_{45}} y_5$$

$$M_2 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow[a_{25}]{} y_5$$

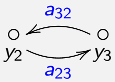
$$M_3 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow[a_{24}]{} y_4 \xrightarrow{a_{45}} y_5$$

$$M_1 = a_{45}a_{34}a_{23}a_{12}, \quad M_2 = a_{25}a_{12}, \quad M_3 = a_{45}a_{24}a_{12}.$$

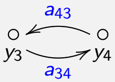
7. Loop: a path originates and terminates on the same node and no other node encountered more than once;
8. Loop gain: path gain of a loop



L_1 :

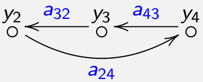


L_2 :



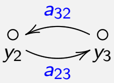
L_3 :

L_4 :



$$L_1 = a_{23}a_{32}, L_2 = a_{34}a_{43}, L_3 = a_{44}, L_4 = a_{24}a_{43}a_{32}.$$

9. Non-touching Loops: loops do not share a common node, *e.g.*,



and

3-2-6 Gain Formula for SFG (Mason's formula)

SFG with N forward paths and K loops:

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \frac{1}{\Delta}\sum_{k=1}^{N} M_k\Delta_k \text{ : gain between } y_{\text{in}} \text{ and } y_{\text{out}},$$

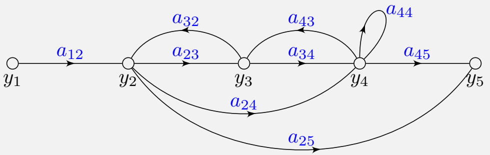
where

Δ : 1−(sum of the gains of all individual loops)
+(sum of products of gains of all possible combinations of TWO non-touching loops)
−(sum of products of gains of all possible combinations of THREE non-touching loops) $\cdots$

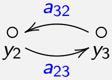
M_k : gain of the k-th forward path between y_{in} and y_{out},

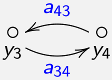
Δ_k : the Δ for that part of the SFG that is non-touching with kth forward path.

Example

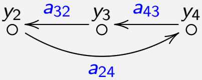


1. Loop gain L_k

L_1 :


L_2 :


L_3 :

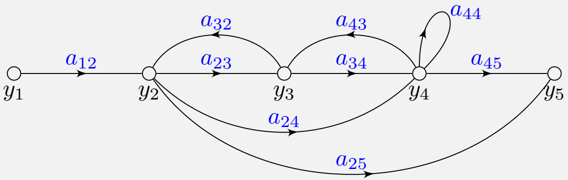

L_4 :


$$L_1 = a_{23}a_{32},\ L_2 = a_{34}a_{43},\ L_3 = a_{44},\ L_4 = a_{24}a_{43}a_{32}.$$

2. Two non-touching loops: L_1 and L_3, $(a_{23}a_{32})(a_{44})$.
3. No three non-touching loop.

$$\Delta = 1-(L_1+L_2+L_3+L_4)+(L_1L_3).$$

Forward path gain M_k and Δ_k

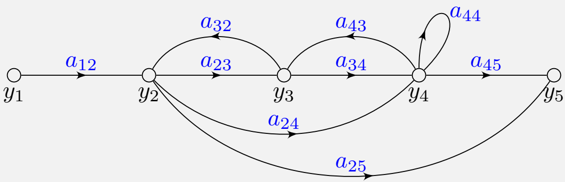


1. M_1 and Δ_1:

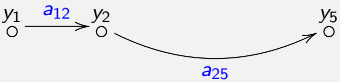
$$y_1 \xrightarrow{a_{12}} y_2 \xrightarrow{a_{23}} y_3 \xrightarrow{a_{34}} y_4 \xrightarrow{a_{45}} y_5$$

$M_1 = a_{12}a_{23}a_{34}a_{45}$;

Non-touching part: none $\Longrightarrow \Delta_1 = 1$.

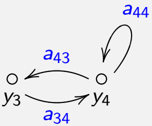


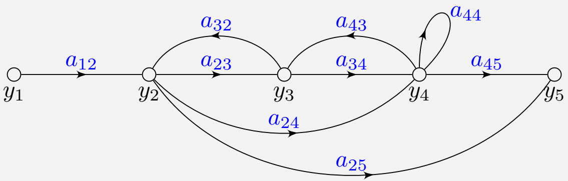
2. M_2 and Δ_2:



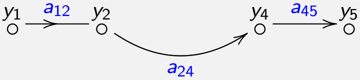
$M_2 = a_{12}a_{25}$;

Non-touching part: $\Longrightarrow \Delta_2 = 1 - (a_{43}a_{34} + a_{44})$.





3. M_3 and Δ_3:



$M_3 = a_{45}a_{24}a_{12}$;

Non-touching part: $\Longrightarrow \Delta_3 = 1$.

Putting everything together, we have

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2 + L_3 + L_4) + L_1 L_3 \\ &= 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{44} + a_{24}a_{43}a_{32}) + a_{23}a_{32}a_{44},\end{aligned}$$

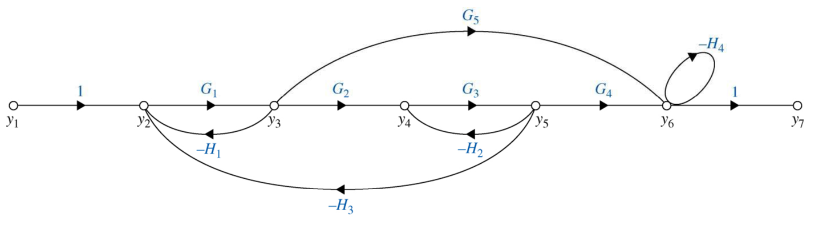
and

$$\begin{aligned} M_1 &= a_{12}a_{23}a_{34}a_{45}, \quad \Delta_1 = 1; \\ M_2 &= a_{12}a_{25}, \quad \Delta_2 = 1 - (a_{34}a_{43} + a_{44}); \\ M_3 &= a_{45}a_{24}a_{12}, \quad \Delta_3 = 1; \end{aligned}$$

Finally, we obtain

$$\begin{aligned}\therefore \frac{y_5}{y_1} &= \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta} \\ &= \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{25}(1 - a_{34}a_{43} - a_{44}) + a_{12}a_{24}a_{45}}{\Delta}.\end{aligned}$$

3-2-7 Gain formula between output nodes & non-input nodes



What if we want $\dfrac{y_7}{y_2}$, where y_2 is not an input?

$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\sum M_k\Delta_k|_{\text{from } y_1 \text{ to } y_7}/\Delta}{\sum M_k\Delta_k|_{\text{from } y_1 \text{ to } y_2}/\Delta},$$

Δ is independent of the inputs and outputs

$$\therefore \frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\sum M_k\Delta_k|_{\text{from } y_1 \text{ to } y_7}}{\sum M_k\Delta_k|_{\text{from } y_1 \text{ to } y_2}}$$

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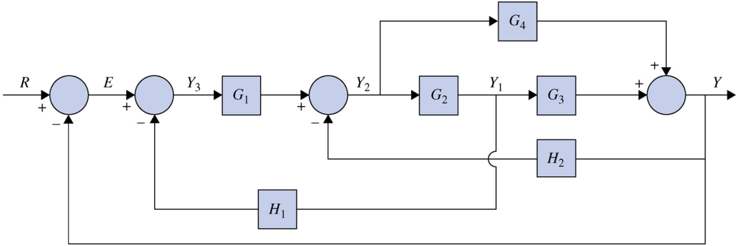
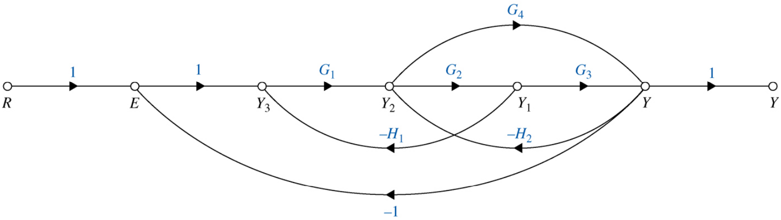
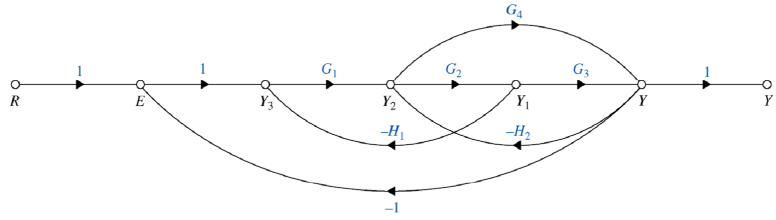
3-2-8 Gain Formula for Block Diagram



(a)

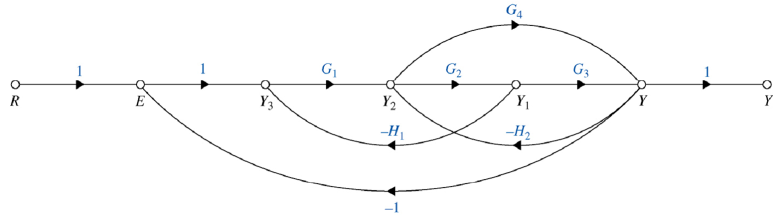




$$M_1: \quad R \to E \to Y_3 \to Y_2 \to Y_1 \to Y \qquad M_1 = G_1 G_2 G_3$$
$$M_2: \quad R \to E \to Y_3 \to Y_2 \to Y \qquad M_2 = G_1 G_4$$

Note that neither path has non-touching loops.

$$\begin{aligned}\Delta =& 1 - (\overbrace{G_1 G_2(-H_1)}^{Y_3\text{-}Y_2\text{-}Y_1\text{-}Y_3} + \overbrace{G_4(-H_2)}^{Y_2\text{-}Y\text{-}Y_2} + \overbrace{(G_2 G_3)(-H_2)}^{Y_2\text{-}Y_1\text{-}Y\text{-}Y_2} \\ &+ \underbrace{G_1 G_2 G_3(-1)}_{E\text{-}Y_3\text{-}Y_2\text{-}Y_1\text{-}Y\text{-}E} + \underbrace{1 \cdot G_1 \cdot G_4 \cdot (-1))}_{E\text{-}Y_3\text{-}Y_2\text{-}Y\text{-}E} \\ =& 1 + G_1 G_2 H_1 + G_4 H_2 + G_2 H_3 H_2 + G_1 G_2 G_3 + G_1 G_4\end{aligned}$$



$$\therefore \frac{Y(s)}{R(s)} = \frac{M_1 + M_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{\Delta}$$

$$\frac{E(s)}{R(s)} = \frac{1 - (\overbrace{G_1G_2(-H_1)}^{Y_3\text{-}Y_2\text{-}Y_1\text{-}Y_3} + \overbrace{G_4(-H_2)}^{Y_2\text{-}Y\text{-}Y_2} + \overbrace{(G_2G_3)(-H_2)}^{Y_2\text{-}Y_1\text{-}Y\text{-}Y_2})}{\Delta}$$

$$= \frac{1 + G_1G_2H_1 + G_4H_2 + G_2G_3H_2}{\Delta}$$

$$\therefore \frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_4H_2 + G_2G_3H_2}$$

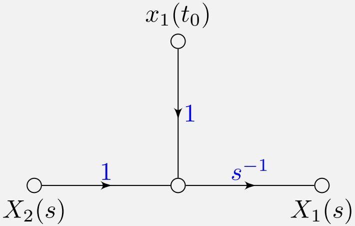
3-3 State Diagram

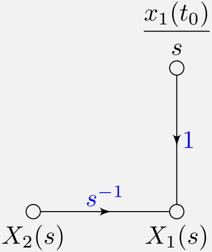
Use SFG to describe state equation and differential equation. Consider

$$\frac{dx_1(t)}{dt} = x_2(t),$$

$$sX_1(s) - x_1(t_0) = X_2(s),$$

$$\therefore X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}.$$





Now consider a differential equation

$$\frac{d^n y(t)}{dt^n} + a_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t)$$

$$\Rightarrow \quad \frac{d^n y(t)}{dt^n} = -a_n \frac{d^{n-1} y(t)}{dt^{n-1}} - \cdots - a_2 \frac{dy(t)}{dt} - a_1 y(t) + r(t)$$