

# Ve460 Control Systems Analysis and Design

## Chapter 2 Laplace Transform

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## 2-2 Laplace Transform

Laplace transform is used to **solve linear ordinary differential equations**. It has the following feature:

- **Converting the differential equation into an algebraic equation.**

Can perform simple algebraic rules to obtain the solution in the  $s$ -domain. The final solution is obtained by taking the inverse Laplace transform.

## 2-2-1 Definition of the Laplace Transform

Given the real function  $f(t)$  that satisfies the condition

$$\int_0^{\infty} |f(t)e^{-\sigma t}| dt < \infty$$

for some finite, real  $\sigma$ , the **Laplace transform of  $f(t)$**  is defined as

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt \quad (1)$$

or

$$F(s) = \text{Laplace transform of } f(t) = \mathcal{L}[f(t)]$$

The variable  $s$  is referred to as the **Laplace operator**, which is a complex variable; that is,  $s = \sigma + j\omega$ , where  $\sigma$  is the real component and  $\omega$  is the imaginary component.

The defining integration in Eq. (1) is also known as the **one-sided Laplace transform**, as the integration is evaluated from  $t = 0$  to  $\infty$ . This simply means that all information contained in  $f(t)$  prior to  $t = 0$  is ignored or considered to be zero.

### Example

Let  $u_s(t)$  be a unit-step function that is defined as

$$u_s(t) = \begin{cases} 1, & t \geq 0; \\ 0, & t < 0. \end{cases}$$

The Laplace transform of  $u_s(t)$  can be obtained as

$$F(s) = \mathcal{L}[u_s(t)] = \int_0^{\infty} u_s(t) e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}. \quad (2)$$

Eq. (2) is valid if

$$\int_0^{\infty} |u_s(t) e^{-\sigma t}| dt = \int_0^{\infty} |e^{-\sigma t}| dt < \infty,$$

which means that  $\sigma$ , the real part of  $s$ , must be greater than zero.

In practice, we simply refer to the Laplace transform of the unit-step function as  $1/s$ , and rarely do we have to be concerned with the region in the  $s$ -plane in which the transform integral converges absolutely.

### Example

Consider the exponential function

$$f(t) = e^{-\alpha t}, \quad t \geq 0,$$

where  $\alpha$  is a real constant. The Laplace transform of  $f(t)$  is written

$$F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right|_0^{\infty} = \frac{1}{s+\alpha}.$$

Replace  $\alpha$  by a complex number  $\alpha + i\beta$ , then

$$\mathcal{L} \left[ e^{-(\alpha+i\beta)t} \right] = \frac{1}{s + \alpha + i\beta}$$

$$\Rightarrow \mathcal{L} \left[ e^{-\alpha t} \cos \beta t - i e^{-\alpha t} \sin \beta t \right] = \frac{s + \alpha - i\beta}{(s + \alpha)^2 + \beta^2}$$

$$\Rightarrow \mathcal{L} \left[ e^{-\alpha t} \cos \beta t \right] = \frac{s + \alpha}{(s + \alpha)^2 + \beta^2}$$

$$\text{and } \mathcal{L} \left[ e^{-\alpha t} \sin \beta t \right] = \frac{\beta}{(s + \alpha)^2 + \beta^2}$$

In particular

$$\mathcal{L} [\cos \beta t] = \frac{s}{s^2 + \beta^2}$$

$$\mathcal{L} [\sin \beta t] = \frac{\beta}{s^2 + \beta^2}$$

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	$t$
4	$2!/s^3$	$t^2$
5	$3!/s^4$	$t^3$
6	$m!/s^{m+1}$	$t^m$
7	$\frac{1}{s+a}$	$e^{-at}$
8	$\frac{1}{(s+a)^2}$	$te^{-at}$
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1 - e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at - 1 + e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1 - e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt} - ae^{-at}$
17	$\frac{s}{s^2+a^2}$	$\sin at$
18	$\frac{s}{s^2+a^2}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at} \cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at} \sin bt$
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1 - e^{-at} \left( \cos bt + \frac{a}{b} \sin bt \right)$



We can use Matlab Symbolic Toolbox to calculate Laplace transform.

## laplace

**R2020a**

Laplace transform

[collapse all in page](#)

### Syntax

```
laplace(f)
laplace(f,transVar)
laplace(f,var,transVar)
```

### Description

`laplace(f)` returns the [Laplace Transform](#) of `f`. By default, the independent variable is `t` and the transformation variable is `s`.

[example](#)

`laplace(f,transVar)` uses the transformation variable `transVar` instead of `s`.

[example](#)

`laplace(f,var,transVar)` uses the independent variable `var` and the transformation variable `transVar` instead of `t` and `s`, respectively.

[example](#)

## Examples

```
>> syms a w t
```

```
>> laplace(t)
```

```
ans =
```

```
1/s^2
```

```
>> laplace(exp(-a*t))
```

```
ans =
```

```
1/(a + s)
```

```
>> laplace(cos(w*t))
```

```
ans =
```

```
s/(s^2 + w^2)
```

```
>> laplace(sin(w*t))
```

```
ans =
```

```
w/(s^2 + w^2)
```

```
>> laplace(exp(-a*t)*sin(w*t))
```

```
ans =
```

```
w/((a + s)^2 + w^2)
```

### 2-2-2 Inverse Laplace Transformation

Given the Laplace transform  $F(s)$ , the operation of obtaining  $f(t)$  is termed the **inverse Laplace transformation** and is denoted by

$$f(t) = \text{Inverse Laplace Transform of } F(s) = \mathcal{L}^{-1}[F(s)]$$

The inverse Laplace transform integral is given as

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$$

where  $c$  is a real constant that is greater than the real parts of all the singularities of  $F(s)$ .

## 2-2-3 Important theorems of Laplace Transform

Multiplication  
by a constant  $\mathcal{L}[kf(t)] = kF(s)$

Sum and difference  $\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$

Differentiation  $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$   
 $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \dots - f^{(n-1)}(0),$

where  $f^{(k)}(0) = \left. \frac{d^k f(t)}{dt^k} \right|_{t=0}$

Integration  $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$   
 $\mathcal{L}\left[\int_0^{t_n} \int_0^{t_{n-1}} \dots \int_0^{t_1} f(t)dt_1 dt_2 \dots dt_{n-1}\right]$   
 $= \frac{F(s)}{s^n}$

Shift in time	$\mathcal{L}[f(t - T)u_s(t - T)] = e^{-Ts}F(s)$
Initial Value Theorem	$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$
Final Value Theorem	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$ if $sF(s)$ does not have poles on or to the right of the imaginary axis in the $s$ -plane
Complex shifting	$\mathcal{L}[e^{\mp \alpha t} f(t)] = F(s \pm \alpha)$
Real convolution	$F_1(s)F_2(s) = \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t - \tau)d\tau\right]$ $= \mathcal{L}\left[\int_0^t f_2(\tau)f_1(t - \tau)d\tau\right]$ $= \mathcal{L}[f_1(t) * f_2(t)]$
Complex convolution	$\mathcal{L}[f_1(t)f_2(t)] = F_1(s) * F_2(s)$

## Differentiation

$$\mathcal{L} \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0).$$

## Proof

$$\begin{aligned} \mathcal{L} \left[ \frac{df(t)}{dt} \right] &= \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt = \int_0^{\infty} e^{-st} df(t) \\ &= f(t)e^{-st} \Big|_0^{\infty} - \int_0^{\infty} f(t) \frac{de^{-st}}{dt} dt \\ &= f(t)e^{-st} \Big|_0^{\infty} - \int_0^{\infty} f(t) (-s)e^{-st} dt \\ &= \underbrace{f(\infty)e^{-s \cdot \infty}}_{=0} - f(0) + s \cdot F(s) \\ &= sF(s) - f(0). \end{aligned}$$

$$\mathcal{L} \left[ \frac{d^n f(t)}{dt^n} \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - f^{(n-1)}(0),$$

where  $f^{(k)}(0) = \left. \frac{d^k f(t)}{dt^k} \right|_{t=0}$ .

### Proof

$$\begin{aligned} \mathcal{L} \left[ \frac{d^n f(t)}{dt^n} \right] &= \mathcal{L} \left[ \frac{d}{dt} \left( \frac{d^{n-1} f(t)}{dt^{n-1}} \right) \right] \\ &= s \cdot \mathcal{L} \left[ \frac{d^{n-1} f(t)}{dt^{n-1}} \right] - \left. \frac{d^{n-1} f(t)}{dt^{n-1}} \right|_{t=0} \\ &= s \left[ s \mathcal{L} \left[ \frac{d^{n-2} f(t)}{dt^{n-2}} \right] - \left. \frac{d^{n-2} f(t)}{dt^{n-2}} \right|_{t=0} \right] - f^{(n-1)}(0) \\ &= \dots \end{aligned}$$

## Integration

$$\mathcal{L} \left[ \int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s}.$$

## Proof

$$\begin{aligned} \text{LHS} &= \int_0^\infty \int_0^t f(\tau) d\tau e^{-st} dt \\ &= \int_0^\infty \int_0^t f(\tau) d\tau d \left( -\frac{1}{s} e^{-st} \right) \\ &= -\frac{1}{s} e^{-st} \cdot \int_0^t f(\tau) d\tau \Big|_{t=0}^\infty - \int_0^\infty \left( -\frac{1}{s} \right) e^{-st} f(t) dt \\ &= \frac{F(s)}{s}. \end{aligned}$$



## Shift in time

$$\mathcal{L}[f(t - T)u_s(t - T)] = e^{-Ts}F(s).$$

## Proof

$$\begin{aligned}\int_0^{\infty} f(t - T)u_s(t - T)e^{-st}dt &\stackrel{\tau=t-T}{=} \int_{-T}^{\infty} f(\tau)u_s(\tau)e^{-s(\tau+T)}d\tau \\ &= e^{-sT} \int_0^{\infty} f(\tau)e^{-s\tau}d\tau \\ &= e^{-Ts}F(s).\end{aligned}$$

## Final Value Theorem

If the Laplace transform of  $f(t)$  is  $F(s)$ , and if  $sF(s)$  **does not have poles on or to the right of the imaginary axis** in the  $s$ -plane

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

## Proof (not rigorous)

$$\mathcal{L} \left[ \frac{df(t)}{dt} \right] = \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0)$$

$$\lim_{s \rightarrow 0} \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt = \lim_{s \rightarrow 0} sF(s) - f(0)$$

$$\text{LHS} = \int_0^{\infty} \frac{df(t)}{dt} dt = \lim_{t \rightarrow \infty} f(t) - f(0)$$

$$\Rightarrow \lim_{s \rightarrow 0} sF(s) = \lim_{t \rightarrow \infty} f(t)$$

Initial Value Theorem is similar

$$\mathcal{L} \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0)$$

$$\begin{aligned} \lim_{s \rightarrow \infty} (sF(s) - f(0)) &= \lim_{s \rightarrow \infty} \mathcal{L} \left[ \frac{df(t)}{dt} \right] \\ &= \lim_{s \rightarrow \infty} \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt \\ &= \int_0^{\infty} \frac{df(t)}{dt} e^{-t\infty} dt = 0 \\ \Rightarrow \lim_{s \rightarrow \infty} sF(s) &= f(0) \end{aligned}$$

### Example 1 of Final Value Theorem

Consider the function

$$F(s) = \frac{5}{s(s^2 + s + 2)}.$$

Because  $sF(s)$  is analytic on the imaginary axis and in the Right Half Plane (**RHP**), the Final Value Theorem may be applied. Using Final Value Theorem, we have

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \frac{5}{s^2 + s + 2} = \frac{5}{2}.$$

## Example 2 of Final Value Theorem

Consider the function

$$F(s) = \frac{\omega}{s^2 + \omega^2},$$

which is the Laplace transform of  $f(t) = \sin \omega t$ .

Because the function  $sF(s)$  has two poles on the imaginary axis of the  $s$ -plane, the Final Value Theorem **cannot be applied** in this case. In other words, although the Final Value Theorem would yield a value of zero as the final value of  $f(t)$ , the result is **erroneous**.

## Rational Transfer Function

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t). \end{aligned}$$

With zero initial state, we have

$$\begin{aligned} a_n s^n Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) \\ = b_m s^m U(s) + \cdots + b_1 s U(s) + b_0 U(s) \\ \Rightarrow \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}. \end{aligned}$$

## Poles and Zeros

A rational transfer function can be described either as a ratio of two polynomials in  $s$ :

$$H(s) = \frac{b_ms^m + \cdots + b_1s + b_0}{a_ns^n + \cdots + a_1s + a_0} = \frac{N(s)}{D(s)},$$

or as a ratio in factored zero pole form:

$$H(s) = K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{j=1}^n (s - p_j)},$$

where  $K$  is called the transfer function gain.

## Zeros

The roots of the numerator  $z_1, z_2, \dots, z_m$  are called the **finite zeros** of the system. If  $s = z_i$ , then

$$H(s)|_{s=z_i} = 0.$$

The zeros have a significant effect on the transient properties of the system.

## Poles

The roots of the denominator  $p_1, p_2, \dots, p_n$  are called the **poles** of the system. If  $s = p_j$ , then

$$H(s)|_{s=p_j} = \infty.$$

The poles of the system determine its stability properties, as well as the natural or unforced behavior of the system, referred to as the **modes** of the system.



## 2-3 Inverse Laplace Transform by Partial Fraction Expansion

Consider the Inverse Laplace Transform of a rational function in  $s$ :

$$G(s) = \frac{Q(s)}{P(s)},$$

where  $P(s)$  and  $Q(s)$  are (coprime) polynomials of  $s$ .

It is assumed that the order of  $P(s)$  in  $s$  is greater than that of  $Q(s)$ . The polynomial may be written

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0,$$

where  $a_0, a_1, \cdots, a_{n-1}$  are real coefficients. The methods of partial fraction expansion will be given depending on the poles of  $G(s)$ :

- 1 Simple poles;
- 2 Multiple-order poles;
- 3 Complex conjugate poles.

### Case 1: $G(s)$ has simple poles

If all the poles of  $G(s)$  are simple and real, we have

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s + s_1)(s + s_2) \cdots (s + s_n)}, \quad (3)$$

where  $s_1 \neq s_2 \neq \cdots \neq s_n$ . Applying the partial fraction expansion, Eq. (3) can be written as

$$G(s) = \frac{K_{s_1}}{s + s_1} + \frac{K_{s_2}}{s + s_2} + \cdots + \frac{K_{s_n}}{s + s_n}. \quad (4)$$

The coefficient  $K_{s_i}$ ,  $i = 1, \cdots, n$ , can be determined by multiplying both sides of Eq. (4) by the factor  $(s + s_i)$  and then setting  $s$  equal to  $-s_i$ . For instance, to find  $K_{s_1}$ , we have

$$K_{s_1} = \left[ (s + s_1) \frac{Q(s)}{P(s)} \right] \Big|_{s=-s_1} = \frac{Q(-s_1)}{(s_2 - s_1)(s_3 - s_1) \cdots (s_n - s_1)}.$$

### Example

Consider the function

$$G(s) = \frac{5s + 3}{(s + 1)(s + 2)(s + 3)} = \frac{5s + 3}{s^3 + 6s^2 + 11s + 6},$$

which can be written in the partial fraction expansion form:

$$G(s) = \frac{K_{-1}}{s + 1} + \frac{K_{-2}}{s + 2} + \frac{K_{-3}}{s + 3}.$$

The coefficients  $K_{-1}$ ,  $K_{-2}$ , and  $K_{-3}$  are determined as follows:

$$\begin{aligned} K_{-1} &= [(s + 1)G(s)] \Big|_{s=-1} = \frac{5s + 3}{(s + 2)(s + 3)} \Big|_{s=-1} \\ &= \frac{5(-1) + 3}{(2 - 1)(3 - 1)} = -1, \end{aligned}$$

$$\begin{aligned}K_{-2} &= [(s+2)G(s)] \Big|_{s=-2} = \frac{5s+3}{(s+1)(s+3)} \Big|_{s=-2} \\&= \frac{5(-2)+3}{(1-2)(3-2)} = 7, \\K_{-3} &= [(s+3)G(s)] \Big|_{s=-3} = \frac{5s+3}{(s+1)(s+2)} \Big|_{s=-3} \\&= \frac{5(-3)+3}{(1-3)(2-3)} = -6.\end{aligned}$$

Thus, we have

$$G(s) = \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3},$$

and then

$$g(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}.$$

We can also use Matlab command `residue` to calculate partial fraction expansion.

## residue

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Partial fraction expansion (partial fraction decomposition)

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### Syntax

```
[r,p,k] = residue(b,a)  
[b,a] = residue(r,p,k)
```

### Description

`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a [Partial Fraction Expansion](#) of the ratio of two polynomials, where the expansion is of the form

[example](#)

The inputs to `residue` are vectors of coefficients of the polynomials  $b = [b_m \dots b_1 b_0]$  and  $a = [a_n \dots a_1 a_0]$ . The outputs are the residues  $r = [r_n \dots r_2 r_1]$ , the poles  $p = [p_n \dots p_2 p_1]$ , and the polynomial  $k$ . For most textbook problems,  $k$  is 0 or a constant.

If there are no multiple roots, then

$$\frac{b(s)}{a(s)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n} + k(s).$$

Consider our previous example

$$G(s) = \frac{5s + 3}{(s + 1)(s + 2)(s + 3)} = \frac{5s + 3}{s^3 + 6s^2 + 11s + 6},$$

we have

```
>> a=[1 6 11 6];
>> b=[5 3];
>> [r,p,k]=residue(b, a)
```

```
r =
-6.0000
 7.0000
-1.0000
```

```
p =
-3.0000
-2.0000
-1.0000
```

```
k =
```

```
[]
```

Hence,

$$\begin{aligned} & \frac{5s + 3}{s^3 + 6s^2 + 11s + 6} \\ &= \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \frac{r_n}{s - p_n} + k(s) \\ &= \frac{-6}{s + 3} + \frac{7}{s + 2} + \frac{-1}{s + 1} \end{aligned}$$

We can also use `ilaplace` in Matlab Symbolic Toolbox.

## ilaplace

**R2020a**

Inverse Laplace transform

[collapse all in page](#)

### Syntax

```
ilaplace(F)
ilaplace(F,transVar)
ilaplace(F,var,transVar)
```

### Description

`ilaplace(F)` returns the **Inverse Laplace Transform** of `F`. By default, the independent variable is `s` and the transformation variable is `t`. If `F` does not contain `s`, `ilaplace` uses the function `symvar`.

[example](#)

`ilaplace(F,transVar)` uses the transformation variable `transVar` instead of `t`.

[example](#)

`ilaplace(F,var,transVar)` uses the independent variable `var` and the transformation variable `transVar` instead of `s` and `t`, respectively.

[example](#)

## Examples

```
>> syms s t
>> ilaplace(-6/(s+3))
ans =
    -6*exp(-3*t)

>> ilaplace((5*s+3)/(s^3+6*s^2+11*s+6))
ans =
    7*exp(-2*t) - exp(-t) - 6*exp(-3*t)

>> ilaplace(1,s,t)
ans =
    dirac(t)
```



## Case 2: $G(s)$ has multiple-order poles

If  $r$  of the  $n$  poles of  $G(s)$  are identical, we say that the pole at  $s = -s_i$  is of multiplicity  $r$ . In this case,  $G(s)$  can be written as

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s + s_1)(s + s_2) \cdots (s + s_{n-r})(s + s_i)^r},$$

$i \neq 1, 2, \dots, n - r$ . Then  $G(s)$  can be expanded as

$$\begin{aligned} G(s) = & \frac{K_{s_1}}{s + s_1} + \frac{K_{s_2}}{s + s_2} + \cdots + \frac{K_{s_{n-r}}}{s + s_{n-r}} \\ & \left| \longleftarrow \quad n-r \text{ terms of simple poles} \quad \longrightarrow \right| \\ & + \frac{A_1}{s + s_i} + \frac{A_2}{(s + s_i)^2} + \cdots + \frac{A_r}{(s + s_i)^r} \\ & \left| \longleftarrow \quad r \text{ terms of repeated poles} \quad \longrightarrow \right| \end{aligned}$$

Then  $(n - r)$  coefficients,  $K_{s_1}, K_{s_2}, \dots, K_{s_{n-r}}$ , which correspond to simple poles, may be evaluated by the method described in Case 1. The determination of the coefficients that correspond to the multiple-order poles is described as follows:

$$A_r = [(s + s_i)^r G(s)] \Big|_{s=-s_i},$$

$$A_{r-1} = \frac{d}{ds} [(s + s_i)^r G(s)] \Big|_{s=-s_i},$$

$$A_{r-2} = \frac{1}{2!} \frac{d^2}{ds^2} [(s + s_i)^r G(s)] \Big|_{s=-s_i},$$

$$\vdots$$

$$A_1 = \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} [(s + s_i)^r G(s)] \Big|_{s=-s_i}.$$

## Example

Consider the function

$$G(s) = \frac{1}{s(s+1)^3(s+2)} = \frac{1}{s^5 + 5s^4 + 9s^3 + 7s^2 + 2s}.$$

We can write  $G(s)$  as

$$G(s) = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}.$$

The coefficients corresponding to the simple poles are

$$k_1 = [sG(s)] \Big|_{s=0} = \frac{1}{(s+1)^3(s+2)} \Big|_{s=0} = \frac{1}{2},$$

$$k_2 = [(s+2)G(s)] \Big|_{s=-2} = \frac{1}{s(s+1)^3} \Big|_{s=-2} = \frac{1}{2}.$$

The coefficients for the pole  $s = 1$  can be obtained:

$$a_3 = \left[ (s+1)^3 G(s) \right] \Big|_{s=-1} = \frac{1}{s(s+2)} \Big|_{s=-1} = -1,$$

$$\begin{aligned} a_2 &= \frac{d}{ds} \left[ (s+1)^3 G(s) \right] \Big|_{s=-1} = \frac{d}{ds} \left[ \frac{1}{s(s+2)} \right] \Big|_{s=-1} \\ &= \frac{-2s-2}{s^2(s+2)^2} \Big|_{s=-1} = 0, \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{d^2}{ds^2} \left[ (s+1)^3 G(s) \right] \Big|_{s=-1} = \frac{1}{2} \frac{d^2}{ds^2} \left[ \frac{1}{s(s+2)} \right] \Big|_{s=-1} \\ &= \frac{1}{2} \frac{6s^2 + 12s + 8}{s^3(s+2)^3} \Big|_{s=-1} = \frac{3(-1)^2 + 6(-1) + 4}{(-1)^3(-1+2)^3} = -1. \end{aligned}$$

The complete partial fraction expansion is

$$G(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3},$$

and then

$$g(t) = \frac{1}{2}u_s(t) + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}t^2e^{-t}.$$

Again, we can use Matlab `residue` command.

## residue

R2020a

Partial fraction expansion (partial fraction decomposition)

[collapse all in page](#)

### Syntax

```
[r,p,k] = residue(b,a)
[b,a] = residue(r,p,k)
```

### Description

`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a [Partial Fraction Expansion](#) of the ratio of two polynomials, where the expansion is of the form

[example](#)

The inputs to `residue` are vectors of coefficients of the polynomials  $b = [b_m \dots b_1 b_0]$  and  $a = [a_n \dots a_1 a_0]$ . The outputs are the residues  $r = [r_n \dots r_2 r_1]$ , the poles  $p = [p_n \dots p_2 p_1]$ , and the polynomial  $k$ . For most textbook problems,  $k$  is 0 or a constant.

If  $p(j) = \dots = p(j + m - 1)$  is a pole of multiplicity  $m$ , then the expansion includes terms of the form

$$\frac{r_j}{s - p_j} + \frac{r_{j+1}}{(s - p_j)^2} + \dots + \frac{r_{j+m-1}}{(s - p_j)^m}.$$

Consider our example

$$G(s) = \frac{1}{s(s+1)^3(s+2)} = \frac{1}{s^5 + 5s^4 + 9s^3 + 7s^2 + 2s}.$$

We have

```
>> [r,p,k]=residue(1, [1 5 9 7 2 0])
```

```
r =
```

```
0.5000
-1.0000
-0.0000
-1.0000
0.5000
```

This yields that

```
p =
```

```
-2.0000
-1.0000
-1.0000
-1.0000
0
```

```
k = []
```

$$\begin{aligned} & \frac{1}{s^5 + 5s^4 + 9s^3 + 7s^2 + 2s} \\ &= \frac{1/2}{s+2} + \frac{-1}{s+1} + \frac{-1}{(s+1)^3} + \frac{1/2}{s}. \end{aligned}$$

## Using Matlab Symbolic Toolbox

```
Gs=1/(s*(s+1)^3*(s+2));  
ilaplace(Gs)  
ans =  
exp(-2*t)/2 - exp(-t) - (t^2*exp(-t))/2 + 1/2
```

```
k1=simplify(s*Gs);          pretty(k1)  
ans =  
      1  
-----  
      3  
(s + 1) (s + 2)
```

```
k2=simplify((s+2)*Gs);      pretty(k2)  
ans =  
      1  
-----  
      3  
s (s + 1)
```



## Matlab command `diff`

```
a3=simplify((s+1)^3*Gs);    pretty(a3)
```

```
ans =
```

```
1
```

```
-----  
s (s + 2)
```

```
a2=simplify( diff(a3) );    pretty(a2)
```

```
ans =
```

```
2 (s + 1)
```

```
- -----
```

```
2      2
```

```
s (s + 2)
```

```
a1=simplify( 1/2*diff(a3,2) ); pretty(a1)
```

```
ans =
```

```
2
```

```
3 s  + 6 s + 4
```

```
-----
```

```
3
```

```
3
```

```
s (s + 2)
```



## Matlab command `eval`

```
s=0; eval(k1)
```

```
ans =  
    0.5000
```

```
s=-2; eval(k2)
```

```
ans =  
    0.5000
```

```
s=-1; [eval(a3) eval(a2) eval(a1)]
```

```
ans =  
    -1     0    -1
```

### Case 3: $G(s)$ has complex conjugate poles

The partial fraction expansion of  $G(s)$  is valid also for simple complex conjugate poles.

Because complex conjugate poles are more difficult to handle and are of special interest in control system studies, they deserve special treatment here.

Suppose that  $G(s)$  contains a pair of complex poles:

$$s = -\sigma + j\omega \quad \text{and} \quad s = -\sigma - j\omega.$$

The corresponding coefficients can be found by:

$$K_{-\sigma+j\omega} = (s + \sigma - j\omega)G(s) \Big|_{s=-\sigma+j\omega},$$

$$K_{-\sigma-j\omega} = (s + \sigma + j\omega)G(s) \Big|_{s=-\sigma-j\omega}.$$

## Example

Consider the **second-order prototype** function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Let us assume that  $|\zeta| < 1$  so that the poles of  $G(s)$  are complex. Then,  $G(s)$  is expanded as follows:

$$G(s) = \frac{K_{-\sigma+j\omega}}{s + \sigma - j\omega} + \frac{K_{-\sigma-j\omega}}{s + \sigma + j\omega},$$

where

$$\sigma = \zeta\omega_n,$$

and

$$\omega = \omega_n\sqrt{1 - \zeta^2}.$$

The coefficients are then determined as

$$\begin{aligned}K_{-\sigma+j\omega} &= (s + \sigma - j\omega)G(s) \Big|_{s=-\sigma+j\omega} \\&= \frac{\omega_n^2}{s + \sigma + j\omega} \Big|_{s=-\sigma+j\omega} = \frac{\omega_n^2}{2j\omega}, \\K_{-\sigma-j\omega} &= (s + \sigma + j\omega)G(s) \Big|_{s=-\sigma-j\omega} \\&= \frac{\omega_n^2}{s + \sigma - j\omega} \Big|_{s=-\sigma-j\omega} = -\frac{\omega_n^2}{2j\omega}.\end{aligned}$$

The complete partial fraction expansion is thus

$$G(s) = \frac{\omega_n^2}{2j\omega} \left[ \frac{1}{s + \sigma - j\omega} - \frac{1}{s + \sigma + j\omega} \right].$$

Taking the inverse Laplace transform yields

$$\begin{aligned} g(t) &= \frac{\omega_n^2}{2j\omega} e^{-\sigma t} (e^{j\omega t} - e^{-j\omega t}) = \frac{\omega_n^2}{\omega} e^{-\sigma t} \sin \omega t \\ &= \frac{\omega_n^2}{\omega_n \sqrt{1 - \zeta^2}} e^{-\sigma t} \sin \omega_n \sqrt{1 - \zeta^2} t, \end{aligned}$$

or,

$$g(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_n \sqrt{1 - \zeta^2} t, \quad t \geq 0.$$

Using Matlab command `ilaplace`, we get

```
>> syms zeta wn s;  
>> ilaplace(wn^2/(s^2+2*zeta*wn*s+wn^2))  
ans =  
(wn*exp(-t*wn*zeta)*sin(t*wn*(1 - zeta^2)^(1/2)))/  
(1 - zeta^2)^(1/2)
```