Ve460 Control Systems Analysis and Design
Chapter 6 Stability of Linear Control Systems

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6-1 Introduction

Design of linear control system:

> It's all about arranging the location of the zeros and poles such that the system will meet prescribed specifications

Stable: The most important requirement for control design.

This course only deals with LTI SISO:

- Absolute stability: stable or unstable
- Relative stability: how stable a system is, stability margin

Definition: BIBO (Bounded Input Bounded Output) stability

With zero initial conditions, a system is BIBO stable (or simply stable) if its output $y(t)$ is bounded to a bounded input $u(t)$, *i.e.*

$$|u(t)| \leq M \quad \Rightarrow \quad |y(t)| < \infty$$

Definition: Zero-input stability

An LTI system is zero-input stable if for any set of finite $y^{(k)}(t_0)$, there exists $M > 0$ such that

- $|y(t)| \leq M < \infty$, for all $t \geq t_0$;
- $\lim_{t\to\infty} |y(t)| = 0$.

If a system is BIBO stable, it must also be zero-input stable

Theorem

An LTI system is BIBO stable iff its impulse response $g(t)$ satisfies

$$\int_0^\infty |g(\tau)|d\tau \le Q < \infty.$$

Proof. $\Leftarrow$) Let input $u(t)$ be bounded, *i.e.*, $|u(t)| < M$. Hence,

$$\begin{aligned}
y(t) &= \int_0^\infty u(t-\tau)g(\tau)d\tau, \\
\Rightarrow \quad |y(t)| &= \left|\int_0^\infty u(t-\tau)g(\tau)d\tau\right| \\
&\le \int_0^\infty |u(t-\tau)g(\tau)|d\tau \\
&\le \int_0^\infty M\cdot|g(\tau)|d\tau \\
&\le M\cdot Q < \infty
\end{aligned}$$

$\Rightarrow$) Omitted.

6-2-1 Characteristic equation roots and stability

Let $G(s)$ be the transfer function,

$$\boxed{\text{BIBO stable} \Rightarrow \text{poles of } G(s) \text{ cannot be in RHP or on the } j\omega\text{-axis}}$$

Proof. We use contradiction. Suppose that $G(s)$ has a pole, say $s_1 = \sigma_1 + j\omega_1$, in RHP or on the $j\omega$-axis. That is,

$$G(s_1) = \infty \quad \text{and} \quad \sigma_1 \geq 0.$$

Because

$$G(s) = \mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt,$$

we have

$$|G(s)| = \left|\int_0^\infty g(t)e^{-st}dt\right| \leq \int_0^\infty |g(t)| \cdot |e^{-st}|dt.$$

Letting $s = s_1 = \sigma_1 + j\omega_1$, we have $|e^{-st}| = |e^{-\sigma_1 t}|$ and

$$|G(s_1)| \leq \int_0^\infty |g(t)| \cdot |e^{-\sigma_1 t}| dt.$$

Because $\sigma_1 \geq 0$, we get $|e^{-\sigma_1 t}| \leq 1$. Combining with $G(s_1) = \infty$, we obtain

$$\infty \leq \int_0^\infty |g(t)| \cdot |e^{-\sigma_1 t}| dt \leq \int_0^\infty |g(t)| dt < \infty.$$

The last inequality is from the assumption that the system is BIBO stable. Contradiction! $\square$

Stability Condition	Root Values
Stable	$\sigma_i < 0 \ \forall \ i$
Marginally stable	For any simple root, $\sigma_i = 0$ and no $\sigma_i > 0$
Unstable	$\exists$ a simple root in the RHP or $\exists$ a multiple-order root on the $j\omega$-axis

Example 1

Consider

$$G(s) = \frac{20}{(s+1)(s+2)(s+3)}.$$

$s = -1, -2, -3 \Rightarrow$ stable.

Example 2

$$G(s) = \frac{20(s+1)}{(s-1)(s^2+2s+2)}, \qquad s = 1 \quad \Rightarrow \quad \text{unstable.}$$

For example, the unit step response is

$$Y(s) = \frac{1}{s} \cdot G(s) = \frac{8}{s-1} - \frac{10}{s} + \frac{2s-4}{(s+1)^2+1}$$

$$\therefore y(t) = \underbrace{8e^t}_{\text{diverge}} - 10 + e^{-t}(2\cos t - 6\sin t)$$

Example 3

$$G(s) = \frac{20(s-1)}{(s+2)(s^2+4)}$$

$s = -2, \pm 2j \Rightarrow$ marginally stable.

Example 4: marginally stable

$$G(s) = \frac{1}{s^2+1}, \quad s = \pm j \quad \Rightarrow \quad \text{marginally stable}$$

The unit step response is

$$Y(s) = \frac{1}{s}G(s) = \frac{1}{s} - \frac{s}{s^2+1} \quad \Rightarrow \quad y(t) = 1 - \cos(t).$$

Sinusoidal input $\sin 2t$:

$$Y(s) = \frac{1}{s^2+4}G(s) = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$$

$$\therefore y(t) = \sin t/3 - \sin 2t/6.$$

Sinusoidal input $\sin t$:

$$Y(s) = \frac{1}{s^2+1}G(s) = \frac{1}{s^2+1} - \frac{s^2}{(s^2+1)^2}$$

$$\therefore y(t) = \sin t/2 - t\cos t/2.$$

Example 5

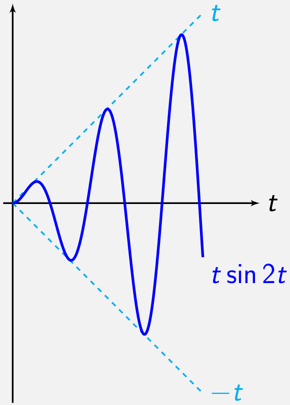
$$G(s) = \frac{16}{(s^2+4)^2}$$

$s = \pm 2j, \pm 2j \Rightarrow$ unstable. The unit step response is

$$\begin{aligned} Y(s) &= \frac{1}{s} \cdot G(s) \\ &= \frac{1}{s} - \frac{4s}{(s^2+4)^2} - \frac{s}{s^2+4} \end{aligned}$$

$$\Rightarrow$$

$$y(t) = 1 - t\sin 2t - \cos 2t$$



6-4 Methods of determining stability

- Routh-Hurwitz criterion — absolute stability
- Nyquist criterion
- Bode plot

Routh-Hurwitz criterion

Consider $G(s) = \dfrac{Q(s)}{P(s)}$, where

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0.$$

We need to determine whether the poles are in LHP.

Two necessary conditions

- All the coefficients have the same sign;
- None of the coefficients vanishes.

6-5 Routh-Hurwitz criterion

Routh-Hurwitz criterion is necessary and sufficient. We use an example to illustrate how to use it.

Example (Routh's array or Routh's tabulation)

Consider $a_6 s^6 + a_5 s^5 + \cdots + a_1 s + a_0 = 0$

$$
\begin{array}{c|c|cccc}
s^6 & a_6 & a_4 & a_2 & a_0 \\
s^5 & a_5 & a_3 & a_1 & 0 \\
s^4 & \dfrac{a_5 a_4 - a_3 a_6}{a_5} = A & \dfrac{a_5 a_2 - a_6 a_1}{a_5} = B & \dfrac{a_5 a_0}{a_5} = a_0 & 0 \\
s^3 & \dfrac{A a_3 - B a_5}{A} = C & \dfrac{A a_1 - a_0 a_5}{A} = D & 0 & 0 \\
s^2 & \dfrac{CB - AD}{C} = E & \dfrac{C a_0}{C} = a_0 & 0 & 0 \\
s^1 & \dfrac{ED - C a_0}{E} = F & 0 & 0 & 0 \\
s^0 & a_0 & & &
\end{array}
$$

Claims

1. 1st column elements are all of the same sign $\Rightarrow$ all the roots are in the LHP.
2. # of sign changes = # of roots in the RHP.

Example

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

$$\begin{array}{llll}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & \dfrac{1\cdot 3 - 2\cdot 5}{1} = -7 & \dfrac{1\cdot 10}{1} = 10 & 0 \\
s^1 & \dfrac{-7\cdot 5 - 10}{-7} = \dfrac{45}{7} & 0 & \\
s^0 & 10 & &
\end{array}$$

2 sign changes $\Rightarrow$ 2 roots in the RHP.

6-5-2 Special case: Routh's Tabulation terminates prematurely

Case 1: 1st element in one row is 0; others are not.
Strategy: replace 0 by a small, positive ε (may not be correct for pure imaginary roots).

Example

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

$$\begin{array}{cccc}
s^4 & 1 & 2 & 3 \\
s^3 & 1 & 2 & 0 \\
s^2 & 0(\varepsilon) & 3 & 0 \\
s^1 & \dfrac{2\varepsilon - 3}{\varepsilon} \approx -\dfrac{3}{\varepsilon} & 0 & 0 \\
s^0 & 3 & 0 & 0
\end{array}$$

2 sign changes $\Rightarrow$ 2 roots in the RHP.

Case 2: all zero row

Example

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

$$\begin{array}{llll}
s^5 & 1 & 8 & 7 \\
s^4 & 4 & 8 & 4 \\
s^3 & \frac{4\cdot 8-8}{4}=6 & \frac{4\cdot 7-4}{4}=6 & 0 \\
s^2 & \frac{6\cdot 8-4\cdot 6}{6}=4 & \frac{6\cdot 4}{6}=4 & 0 \\
s^1 & 0(8) & 0 & 0 \\
s^0 & 4 & 0 & 0
\end{array}$$

$\Rightarrow$ stable.

Strategy: Form auxiliary equation

$$A(s) = 4s^2 + 4 = 0$$

$$\frac{dA(s)}{ds} = 8s = 0$$

A simple design example

$$s^3 + 3Ks^2 + (K+2)s + 4 = 0$$

Find the range of K so that the system is stable.

$$\begin{array}{cccc}
s^3 & & 1 & K+2 \\
s^2 & & 3K & 4 \\
s^1 & & \dfrac{3K(K+2)-4}{3K} & 0 \\
s^0 & & 4 & 0
\end{array}$$

$$\begin{cases} 3K > 0 \\ 3K(K+2) - 4 > 0 \end{cases} \Rightarrow \begin{cases} K > 0 \\ 3K^2 + 6K - 4 > 0 \end{cases}$$

$$\therefore \quad K > \frac{\sqrt{36+48}-6}{6} = \frac{2\sqrt{21}-6}{6} = \frac{\sqrt{21}-3}{3}$$