Ve460 Control Systems Analysis and Design
Chapter 8 Root locus

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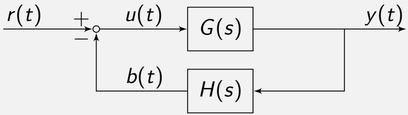
8-1 Introduction



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Characteristic equation:

$$1+G(s)H(s)=0$$

Suppose that

$$G(s)H(s) = K\frac{Q(s)}{P(s)},$$

where $Q(s)$ and $P(s)$ are polynomials:

$$Q(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0,$$
$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$

then

$$1+G(s)H(s) = 1 + K\frac{Q(s)}{P(s)} = 0.$$

Root Locus

The graph of all possible roots as K changes.

Let $L(s) = \dfrac{Q(s)}{P(s)}$, we have the following equivalent forms:

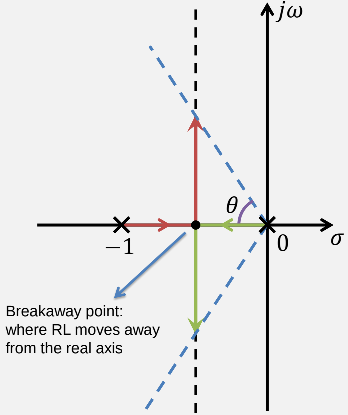
- $P(s) + KQ(s) = 0$;
- $1 + KL(s) = 0$;
- $L(s) = -\dfrac{1}{K}$.

Example (Motor Position Control: Design on Feedback Gain)

Consider $L(s) = \dfrac{1}{s(s+1)}$, $1 + KL(s) = 0$,

$$1 + \frac{K}{s(s+1)} = 0, \quad s^2 + s + K = 0$$

$$\Rightarrow \quad s_{1,2} = \frac{-1 \pm \sqrt{1-4K}}{2}.$$

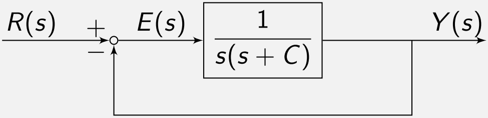


1. $K = 0$. $s_1 = 0$, $s_2 = -1$.
   $L(s) = -\dfrac{1}{K} = \infty$,
   $s \longrightarrow$ poles of $L(s)$;
2. $K = \infty$. $s_{1,2} = -\frac{1}{2} \pm j\infty$.
   $L(s) = -\dfrac{1}{K} = 0$,
   $s \longrightarrow$ zeros of $L(s)$;
3. $0 \leq K \leq \dfrac{1}{4}$, two real roots;
4. $K \geq \dfrac{1}{4}$,
   $s_{1,2} = -\dfrac{1}{2} \pm \dfrac{1}{2} j\sqrt{4K-1}$.

Now let's say we want to design K such that $\zeta = \dfrac{1}{2}$.

$$\cos\theta = \frac{1}{2},\ \ \theta = \frac{\pi}{3},\ \ \frac{1}{2}\sqrt{4K-1} = \frac{\sqrt{3}}{2} \ \Rightarrow\ K = 1.$$

Motor Position Control: Design on Pole Placement

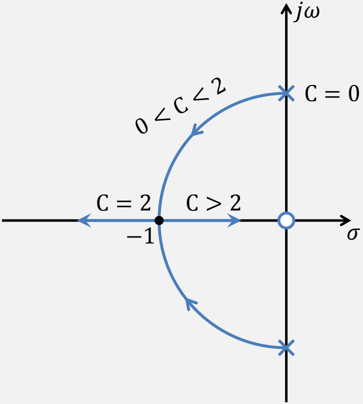


$$M(s) = \frac{\frac{1}{s(s+C)}}{1 + \frac{1}{s(s+C)}} = \frac{1}{s^2 + Cs + 1}.$$

$$s^2 + Cs + 1 = 0, \quad 1 + C \underbrace{\boxed{\frac{s}{s^2+1}}}_{L(s)} = 0,$$

$$s_{1,2} = \frac{-C \pm \sqrt{C^2 - 4}}{2}.$$

Example (Pole Placement Design)



1. $C = 0$, $s = \pm j$, poles of $L(s)$;
2. $C = 2$, $s_{1,2} = -1$;
3. $0 < C < 2$,

$$\begin{cases} x = -C/2 \\ y = \sqrt{4 - C^2}/2 \end{cases}$$

$$\Rightarrow \begin{cases} C^2 = 4x^2 \\ C^2 = 4 - 4y^2 \end{cases}$$

$$\Rightarrow x^2 + y^2 = 1$$

4. $C > 2$, two real roots.

When $C \gg 2$, $s_{1,2} = \dfrac{-C \pm C}{2} = -C, 0$.

8-2 Basic Properties of root locus

1. Magnitude condition. Consider $1 + KG_1(s)H_1(s) = 0$. We have that

$$G_1(s)H_1(s) = -\frac{1}{K} \Rightarrow |G_1(s)H_1(s)| = \frac{1}{|K|}$$

2. Angle condition.

$$\angle G_1(s)H_1(s) = \begin{cases} (2j+1)\pi, & K \geq 0; \\ 2j\pi, & K \leq 0; \end{cases} \quad j = 0, \pm 1, \pm 2, \cdots$$

Assume that

$$KG_1(s)H_1(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)},$$

then

$$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{k=1}^{n}|s+p_k|} = \frac{1}{|K|}.$$

For $K \geq 0$:

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{k=1}^{n}\angle(s+p_k) = (2j+1)\pi;$$

for $K \leq 0$:

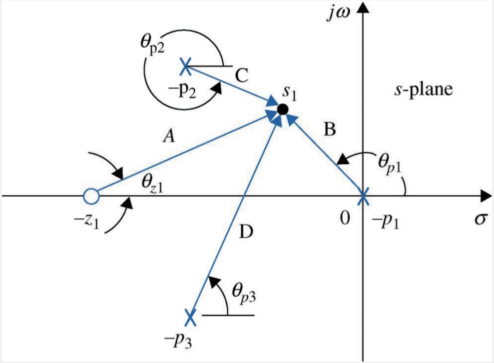
$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{k=1}^{n}\angle(s+p_k) = 2j\pi.$$

Example

Consider

$$G(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)}.$$

Select a trial point s_1. For $K \geq 0$, we have



$$\begin{aligned}&\angle(s_1+z_1)-\angle(s_1+p_2)-\angle(s_1+p_3)-\angle s_1\\=&\theta_{z_1}-\theta_{p_2}-\theta_{p_3}-\theta_{p_1}=(2j+1)\pi.\end{aligned}$$

$$\frac{|s_1+z_1|}{|s_1(s_1+p_2)(s_1+p_3)|}=\frac{1}{K} \quad \Rightarrow \quad \frac{A}{BCD}=\frac{1}{K}$$

3. Symmetry: root locus is symmetric w.r.t. real axis.
4. $K = 0$ and $K = \pm\infty$. From

$$G(s)H(s) = -\frac{1}{K},$$

when $K = 0$,

$$G(s)H(s) = -\infty,$$

s is a pole of $G(s)H(s)$;
when $K = \infty$,

$$G(s)H(s) = 0,$$

s is a zero of $G(s)H(s)$.
5. Consider $K \geq 0$. Root locus has n branches starting from poles, m branches ending at zeros:

$$G(s)H(s) = \frac{Q(s)}{P(s)}, \quad \begin{array}{c} m = \deg Q(s) \\ n = \deg P(s) \end{array}.$$

Consider an example $s(s+2)(s+3) + K(s+1) = 0$.

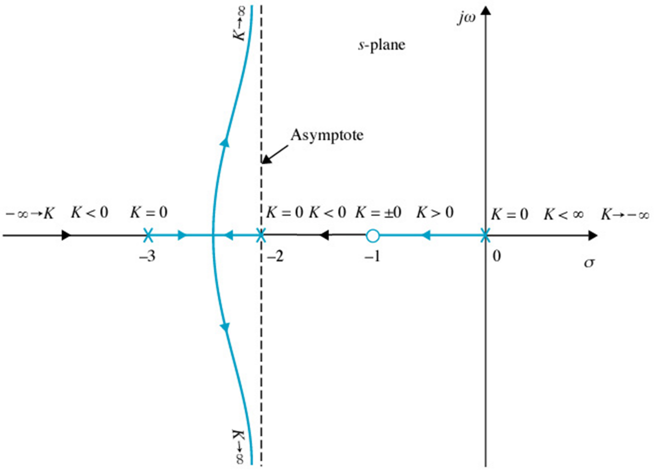


Figure 8.1: Root locus and asymptotes

6. RL on the real axis for $K \geq 0$ are found to the left of an odd number of real poles and real zeros.

For a trial point s_1 on the real axis, if s_1 is to the left of $-p$ or $-z$,

$$\angle(s_1 + p) = \pi, \quad \angle(s_1 + z) = \pi.$$

If s_1 is to the right of $-p$ or $-z$,

$$\angle(s_1 + p) = 0, \quad \angle(s_1 + z) = 0;$$

So they do not have contribution in

$$\sum_i \angle(s + z_i) - \sum_k \angle(s + p_k) = (2j + 1)\pi.$$

7. When $n \neq m$, some RL will go to infinity (called asymptotes of RL when $s \to \infty$). For $n \geq m$,

$$\theta_i = \frac{2i+1}{|n-m|}\pi, \quad i = 0, 1, 2, \cdots, |n-m|-1.$$

For our example in the previous figure,

$$G_1(s)H_1(s) = \frac{s+1}{s(s+2)(s+3)},$$

$$n = 3,\ m = 1,\ \theta_i = \frac{2i+1}{2}\pi, \text{ hence } \theta_i = \frac{\pi}{2}, \frac{3\pi}{2}.$$

8. Intersect of asymptotes

$$\sigma = \frac{\sum p_i - \sum z_i}{n-m} \quad \text{(center of gravity of RL)}$$

$$\Rightarrow \sigma = \frac{(-3)+(-2)-(-1)}{n-m} = \frac{-4}{2} = -2$$

9. Breakaway points on the RL correspond to multiple order roots, that is,

$$1 + G(s)H(s) = 1 + K\frac{Q(s)}{P(s)} = 0 \;\Rightarrow P(s) + KQ(s) = 0$$

has multiple order roots. Hence

$$P(s) + KQ(s) = (s - r_1)^q f(s), \;\; q \geq 2$$
$$\frac{dP(s)}{ds} + K\frac{dQ(s)}{ds} = q(s - r_1)^{q-1} f(s) + (s - r_1)^q \frac{df(s)}{ds}$$

Let $s = r_1$, then

$$\left(\frac{dP(s)}{ds} + K\frac{dQ(s)}{ds}\right)\bigg|_{s=r_1} = 0.$$

We also have $P(s) + KQ(s) = 0$, then $K = -\dfrac{P(s)}{Q(s)}$. Hence

$$\left.\left(\frac{dP(s)}{ds} - \frac{P(s)}{Q(s)}\frac{dQ(s)}{ds}\right)\right|_{s=r_1} = 0$$

$$\Rightarrow \quad Q(s)\frac{dP(s)}{ds} - P(s)\frac{dQ(s)}{ds} = 0$$

In our example,

$$Q(s) = s+1, \qquad P(s) = s(s+2)(s+3) = s^3 + 5s^2 + 6s,$$

$$\frac{dQ(s)}{ds} = 1, \qquad \frac{dP(s)}{ds} = 3s^2 + 10s + 6$$

$$\Rightarrow \quad Q(s)\frac{dP(s)}{ds} - P(s)\frac{dQ(s)}{ds}$$
$$= (s+1)(3s^2+10s+6) - (s^3+5s^2+6s)$$
$$= 3s^3+13s^2+16s+6-(s^3+5s^2+6s)$$
$$= 2s^3+8s^2+10s+6=0$$
$$\Rightarrow \quad s^3+4s^2+5s+3=0$$
$$\Rightarrow \quad s=-2.4656\ (\text{two complex roots} -0.7672 \pm 0.7926i)$$

10. Angles of departure and arrival of RL.
Angle of departure of RL at a pole: angle of the tangent to the RL near that pole can be determined by

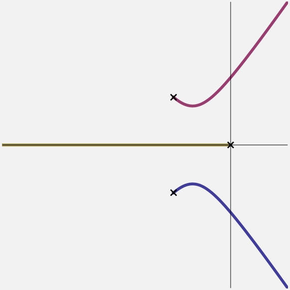
$$\angle G_1(s_1)H_1(s_1) = \sum_{k=1}^{m} \angle(s_1+z_k) - \sum_{j=1}^{n} \angle(s_1+p_j)$$
$$= (2j+1)\pi, \ \ K \geq 0.$$

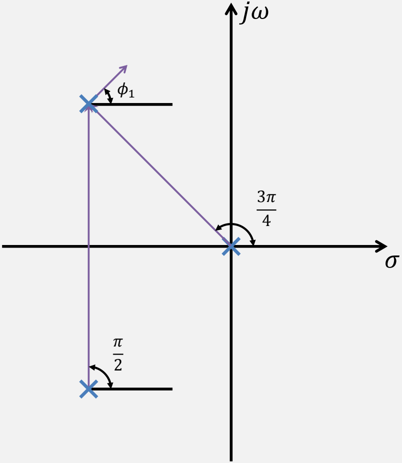
Consider $G_1(s)H_1(s)$ with no zeros and poles at 0, $\sigma \pm j\sigma$.

$$0 - \left(\phi_1 + \frac{\pi}{2} + \frac{3\pi}{4}\right) = \pm\pi, \pm 3\pi, \cdots$$

$$\phi_1 + \frac{5\pi}{4} = \pm\pi, \cdots$$

$$\Rightarrow \quad \phi_1 = -\frac{\pi}{4}$$

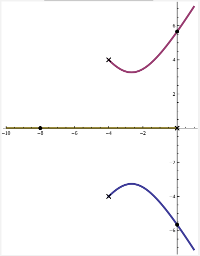




11 RL crosses the $j\omega$-axis $\rightarrow$ use Routh criterion.

$$1+\frac{K}{s((s+4)^2+16)}=0 \;\Rightarrow\; s^3+8s^2+32s+K=0.$$

s^3	1	32
s^2	8	K
s^1	$\frac{256-K}{8}$	0
s^0	K	



no pole in RHP $\Rightarrow$

$$\begin{cases} \dfrac{256-K}{8} > 0 \\ K > 0 \end{cases} \Rightarrow\; 0 < K < 256;$$

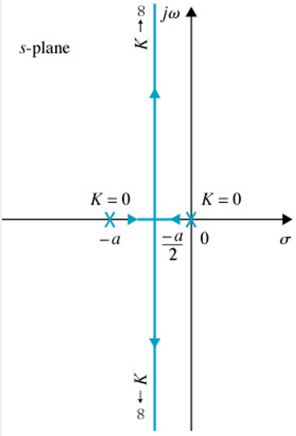
$K > 256$: unstable;
$K = 256$: on the $j\omega$-axis.

$$(j\omega)^3+8(j\omega)^2+32(j\omega)+256=0$$
$$\Rightarrow\; -\omega^3 j-8\omega^2+32\omega j+256=0$$
$$\Rightarrow\; -\omega^3+32\omega=0$$
$$\Rightarrow\; \omega=\pm\sqrt{32}$$

8-4 Design Aspects of the root loci

Addition of Poles to $G(s)H(s)$

$\rightarrow$ Pushing RL toward RHP.

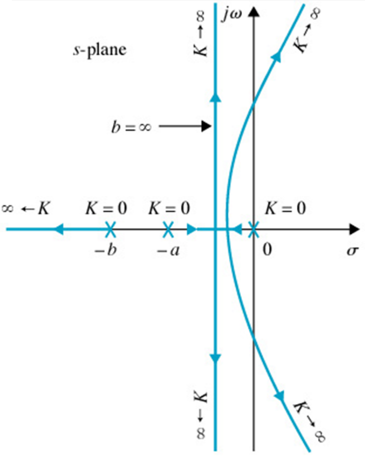


Consider

$$G(s)H(s) = \frac{K}{s(s+a)}, \quad a > 0.$$

- Poles: $s = 0$, $s = -a$;
- Breakaway point:
  $1 \cdot \frac{d}{ds}(s^2 + as) = 0 \Rightarrow s = -\frac{a}{2}$;
- Asymptote: $\theta_i = \frac{2i+1}{2}\pi$, $i = 0, \pm 1, \cdots$

$$\theta_i = \frac{\pi}{2}$$

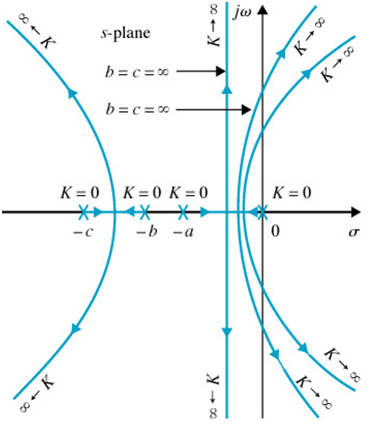


Now introduce a pole at $s=-b$ with $b>a$:

$$G(s)H(s)=\frac{K}{s(s+a)(s+b)}$$

- Asymptote angle: $\theta_i=\frac{2i+1}{3}\pi,\ i=0,\ \pm1,\ \pm2;$
- Asymptote intersection: $\frac{(-a)+(-b)}{3}=\frac{-(a+b)}{3};$
- Breakaway point: $\frac{d}{ds}(s^3+(a+b)s^2+abs)=0.$ Hence

$$3s^2+2(a+b)s+ab=0.$$



Add one more real pole:

$$\frac{K}{s(s+a)(s+b)(s+c)}$$

- Asymptote angle:

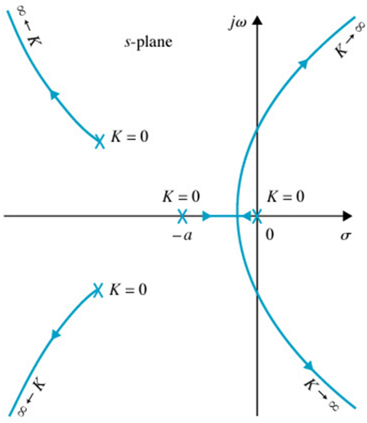
$$\theta_i = \frac{2i+1}{4}\pi,$$

for $i = 0, \pm 1, \cdots$. Then,

$$\theta_i = \pm\frac{\pi}{4}\pi, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}.$$

- Asymptote intersection:

$$-\frac{a+b+c}{4}.$$



Add a pair of complex conjugate poles

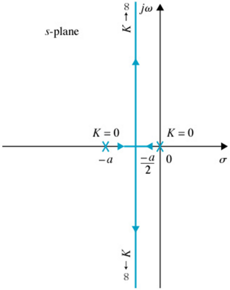
$$\frac{K}{s(s+a)(s^2+2\sigma_1 s+\sigma_1^2+\beta_1^2)}$$

- Asymptote angle: $\theta_i = \frac{2i+1}{4}\pi,\ i=0,\ \pm 1,\ \ldots,$

$$\theta_i = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}$$

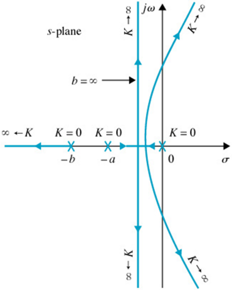
- Asymptote intersection:

$$-\frac{a+2\sigma_1}{4}$$



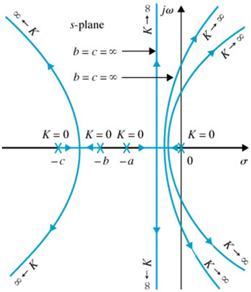
s-plane
jω
K → ∞
K = 0
K = 0
−a
−a/2
0
σ

(a)



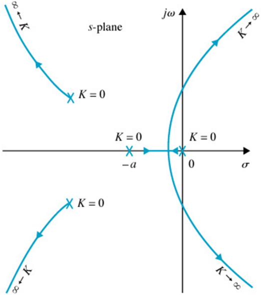
s-plane
jω
K → ∞
b = ∞
∞ ← K
K = 0
K = 0
K = 0
−b
−a
0
σ

(b)



s-plane
jω
K → ∞
∞ ← K
b = c = ∞
b = c = ∞
K = 0
K = 0
K = 0
K = 0
−c
−b
−a
0
σ

(c)



s-plane
jω
∞ ← K
K → ∞
K = 0
K = 0
K = 0
K = 0
−a
0
σ

(d)

Addition of Zeros to $G(s)H(s)$

Adding LHP zeros to $G(s)H(s)$ generally pushes RL toward LHP

Example

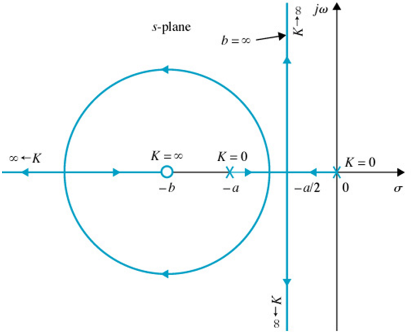


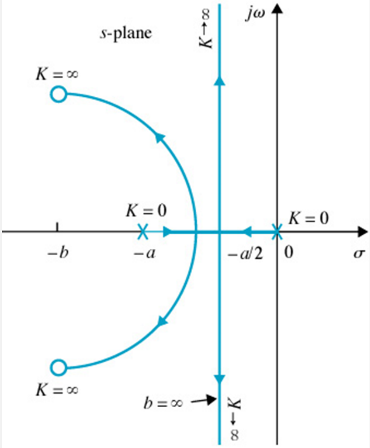
Figure 8.2: Adding a real zero

Consider

$$G(s)H(s)=\frac{K}{s(s+a)}, a>0.$$

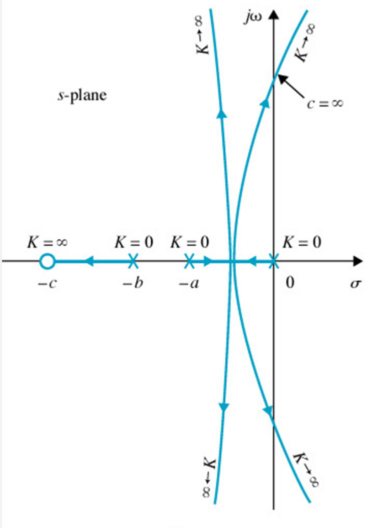
Adding a zero at $s=-b$ $(b>a)$, we have

$$G(s)H(s)=\frac{K(s+b)}{s(s+a)}, a>0.$$



Adding a pair of complex conjugate zeros.

Example



(c)

Now adding a zero $s = -c$ $(c > b)$ to

$$G(s)H(s) = \frac{K}{s(s+a)(s+b)},$$

we have

$$G(s)H(s) = \frac{K(s+c)}{s(s+a)(s+b)}.$$

Example

Consider $s^2(s+a)+K(s+b)=0$:

$$G(s)H(s)=\frac{K(s+b)}{s^2(s+a)}.$$

Breakaway point:

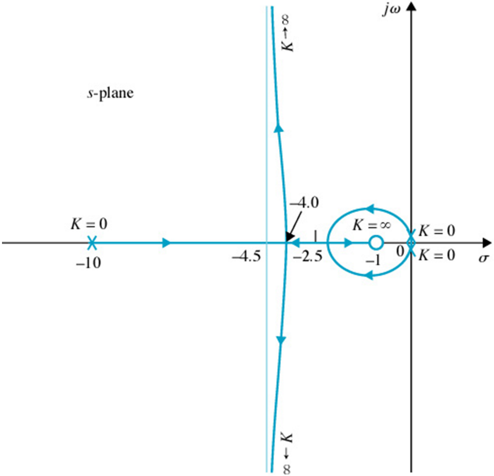
$$
\begin{aligned}
& (s+b)\frac{d}{ds}(s^3+as^2)-s^2(s+a)\frac{d}{ds}(s+b)=0 \\
\Rightarrow\quad & 2s^2+(a+3b)s+2ab=0 \\
\Rightarrow\quad & s_{1,2}=\frac{-(a+3b)\pm\sqrt{a^2-10ab+9b^2}}{4}
\end{aligned}
$$

Let $b=1$, then

$$s_{1,2}=\frac{-(a+3)\pm\sqrt{(a-9)(a-1)}}{4}$$

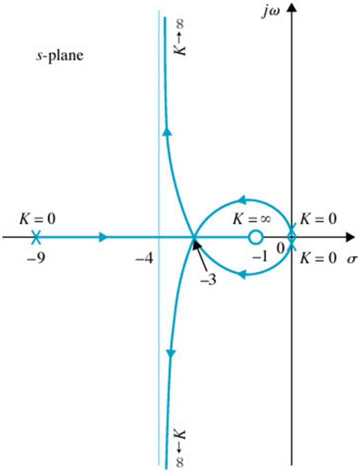
We want to study how the root locus will change as a changes.

Asymptote angle $= \pm\dfrac{\pi}{2}$; Asymptote intersection $= -\dfrac{a-1}{2}$;
When $a = 10$, breakaway points: -2.5, -4;



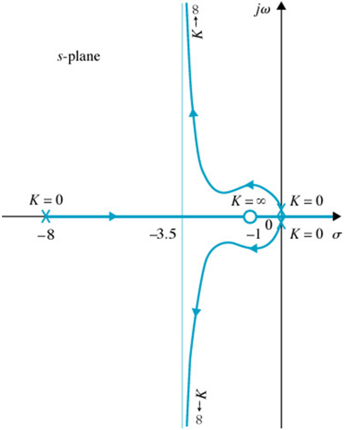
(a) $a = 10$

Asymptote angle $= \pm\dfrac{\pi}{2}$; Asymptote intersection $= -\dfrac{a-1}{2}$;
When $a = 9$, breakaway points: -3;



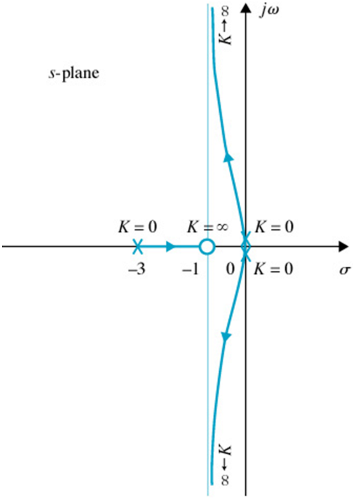
(b) $a = 9$

Asymptote angle $= \pm\dfrac{\pi}{2}$; Asymptote intersection $= -\dfrac{a-1}{2}$;
When $a = 8$, no breakaway points;



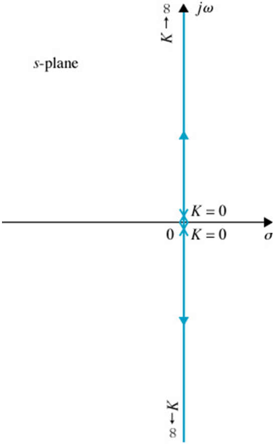
(c) $a = 8$

Asymptote angle $= \pm\frac{\pi}{2}$; Asymptote intersection $= -\frac{a-1}{2}$;
When $a = 3$, no breakaway points;

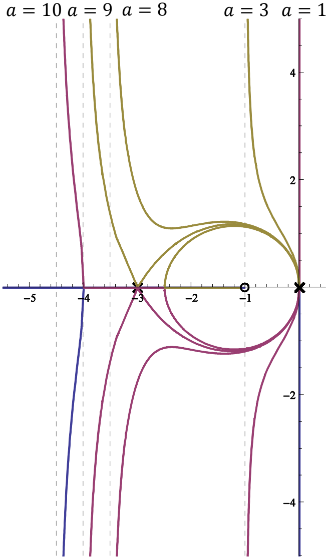


(d) $a = 3$

Asymptote angle $= \pm\frac{\pi}{2}$; Asymptote intersection $= -\frac{a-1}{2}$;
When $a = 1$, $G(s)H(s) = \frac{K}{s^2}$.



(e) $a = 1$



Asymptote angle $= \pm\frac{\pi}{2}$;

Asymptote intersection $= -\frac{a-1}{2}$;

When $a = 10$, breakaway points: -2.5, -4;

When $a = 9$, breakaway points: -3;

When $a = 8$, no breakaway points;

When $a = 3$, no breakaway points;

When $a = 1$, $G(s)H(s) = \frac{K}{s^2}$.