ECON W3213 Spring 2014 Jón Steinsson

Malthus Growth Model

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This set of recitation notes covers the **Malthus Growth Model** This is in no way a substitute for attending lectures, but just in case you dozed off or checked your boyfriend's Facebook page while Jón was working Calculus magic on the board, this set of notes may save you.

1 Introduction to Growth Models

In this set of notes and the next, you will be learning about growth models. Essentially, they answer the question

> What makes an economy grow?

And we're not talking about little movements (that economists call "cyclical"). Rather, we're talking about long run trajectories that last tens or even hundreds of years.

Do understand the characteristics of these models. What do I mean? Each model has its inherent assumptions on production and growth. Identify them:

- What is the key factor of production?
- What is the production dependent on?
- What happens in the long run?
- What happens during shocks both temporary and permanent?

This provides a good way for you to contrast the different models and to organize your thoughts. Otherwise, you'd be lost in a sea of information. Not even the CC swim test can help you with that.

One last thing – we are always talking about real income or real production in these models. We don't factor in inflation at all. In fact, you can think of production in terms of perhaps **Gatorade** (since Jón loves drinking it so much). We will include monetary effects much later on.

2 Malthus Model

The characteristics of the Malthus Model are

- Labor is the key factor of production
- Production is dependent on labor (or population), and so are wages
- Population always stays constant in the long run, with everyone earning subsistence wages
- Temporary shocks do nothing to the steady state. Permanent changes shift the steady state.

Let's see how we can arrive at these conclusions.

2.1 Production Function

The production function is assumed to be

$$Y_t = A_t D^\alpha L_t^{1-\alpha}$$

Where

- Y_t is output at time t
- A_t is productivity at t
- D is land, assumed to be constant over time
- L_t is labor at time t
- α is a constant less than 1. In fact, you can tell that this is a pretty standard Cobb-Douglas function.

Convince yourself that this function exhibits

1. Constant returns to scale
2. Diminishing returns to labor

Try it. It's really not that hard.

2.2 Labor Demand and Supply

The Malthus model focuses on labor. Let's derive the labor demand and supply then.

Remember that firms are always the ones demanding labor. They will always pay you up to your marginal product of labor. **Labor Demand** is hence

$$\text{MPL} = \frac{\partial Y_t}{\partial L} = (1-\alpha)A_t \left(\frac{D}{L_t}\right)^{\alpha} = w_t$$

We assume that labor supply is simply equal number of hours each person works times the population size. We also assume that number of hours worked for everyone is the same.

Labor Supply is

$$L_t = HN_t$$

Where

- H is the number of hours worked per person
- N_t is the population size

2.3 Malthus Theory for Population Growth

Malthus postulates that population growth is related to wage growth. Hence, we have the **population growth equation**:

$$\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)^{\gamma} \xi_t$$

Where

- w_s is subsistence wage
- γ is a constant where $0 < \gamma < 1$
- ξ_t represents exogenous shocks. In other words, if the Vesuvius eruption happens at $t = 79$, ξ_{79} will be very small. However, it reverts back to normal ($\xi_t = 1$) at $t = 80$.

2.4 Population Dynamics

Let's find the equilibrium of the labor market by equating labor supply and demand

$$w_t = (1-\alpha)A_t\left(\frac{D}{HN_t}\right)^{\alpha} = \phi A_t \frac{1}{N_t^{\alpha}}$$

Where $\phi = (1-\alpha)\left(\frac{D}{H}\right)^{\alpha}$ is simply a grouping of constants.

Now let's throw this w_t into the population growth equation

$$\begin{aligned}\frac{N_{t+1}}{N_t} &= \left(\frac{w_t}{w_s}\right)^{\gamma}\xi_t \\ &= \left(\frac{\phi A_t N_t^{-\alpha}}{w_s}\right)^{\gamma}\xi_t \\ N_{t+1} &= \left(\frac{\phi A_t}{w_s}\right)^{\gamma} N_t^{1-\alpha\gamma}\xi_t\end{aligned}$$

Now let's plot this little monster. It's not that hard. Just find your y and x axis variables.

$$\underbrace{N_{t+1}}_{\text{y}-\text{variable}} = \left(\frac{\phi A_t}{w_s}\right)^{\gamma} \underbrace{N_t^{1-\alpha\gamma}}_{\text{x}-\text{variable}} \xi_t$$

Since $1-\alpha\gamma < 1$ (remember what values α and γ took?), this is simply going to look like a $y = \sqrt{x}$ graph, except that the exponent may be a little different (but the shape still remains the shape).

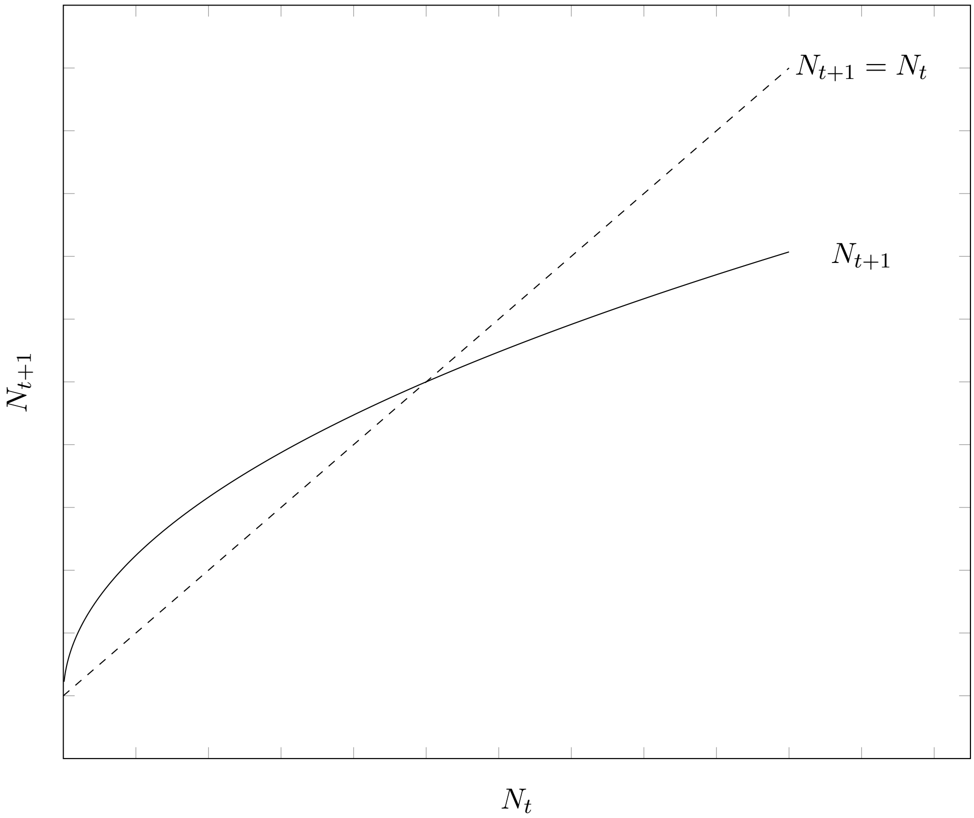

Figure 1: Plot of N_{t+1} against N_t

Now think intuitively. What does this graph tell us? This graph gives us the population at the next time period for a current level of population. A natural question we would then want to ask is: *is the population in the next period going to be higher than what we have right now?*

To answer that question, we draw the line $N_{t+1} = N_t$ to help us. Now any region above this dotted line is a situation where the population in the next period, N_{t+1}, is higher than what our current population N_t is.

Hence, let's say our current population at $t = 0$ is at the point indicated

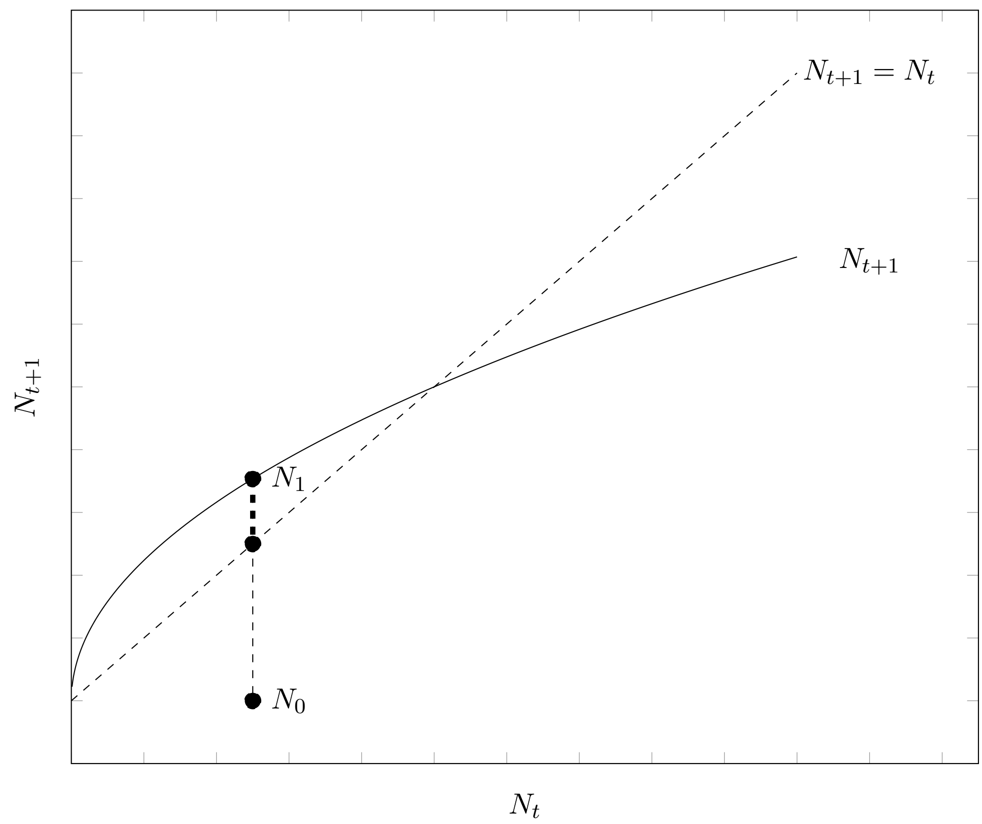

Figure 2: Plot of N_{t+1} against N_t

If population were to be the same as N_0 in the next time period, then we'd be on the dotted line. However, the population growth curve shows us that the population N_1 at the next period is higher than that (by precisely the height of the thick dotted line). Hence, there will be population growth from N_0 to N_1.

Since our population increases by $N_1 - N_0$, which is the vertical distance indicated by the thick dashed line, this is what happens in the next period.

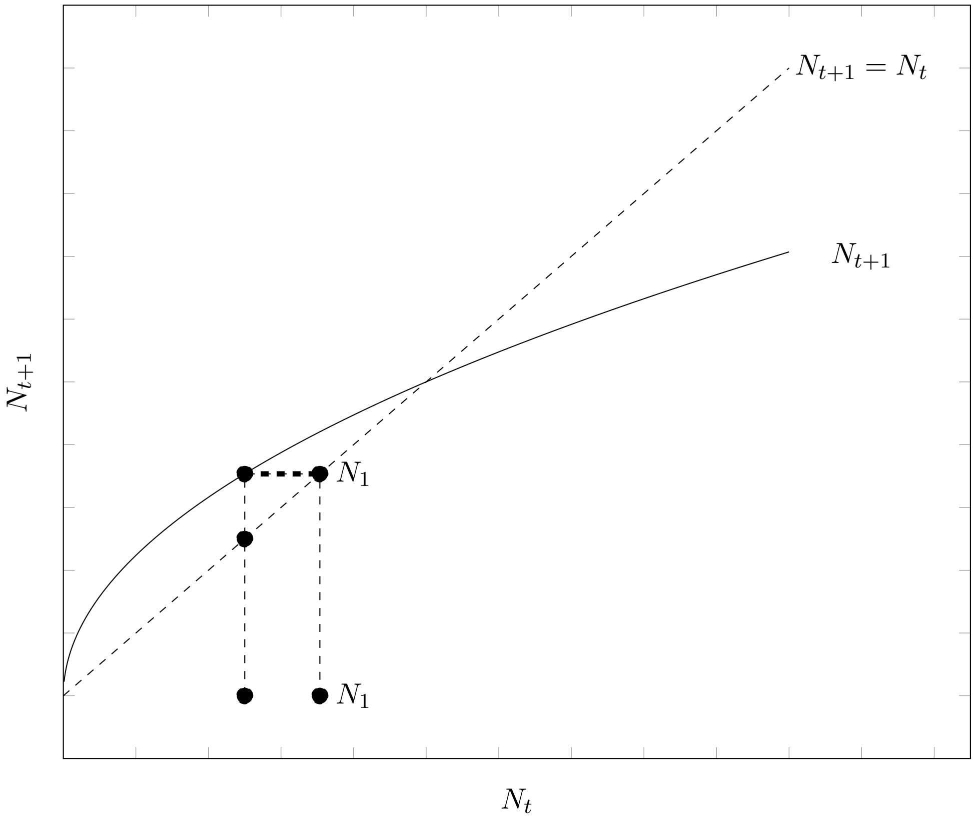

Figure 3: Plot of N_{t+1} against N_t

We arrive at N_1 on the x axis. The rightward shift indicated by the thick dashed line is exactly equal to the thick dashed line in figure 2 (prove to yourself that it's the same. After all, $N_{t+1} = N_t$ is a 45° line).

Again at N_1, we find that N_2 is going to be higher than N_1 and there is population growth.

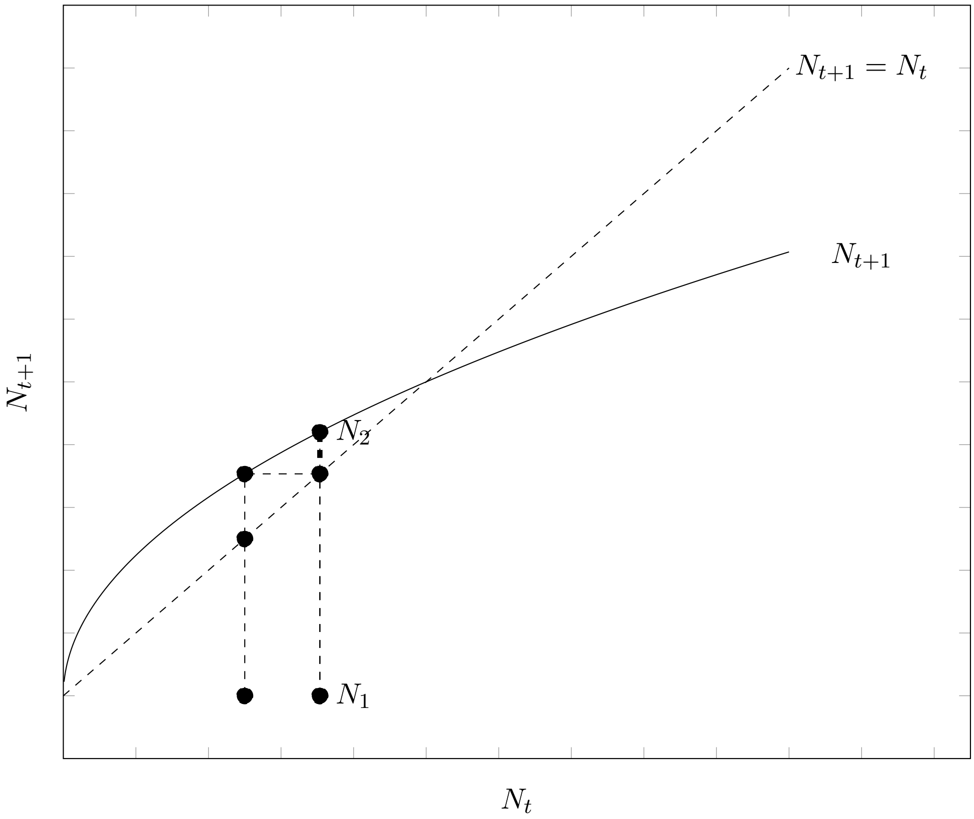

Figure 4: Plot of N_{t+1} against N_t

We can continue this stepwise movement upwards. The same cycle happens again. In fact, we can draw a ladder-ish movement until we reach the point where the two curves intersect.

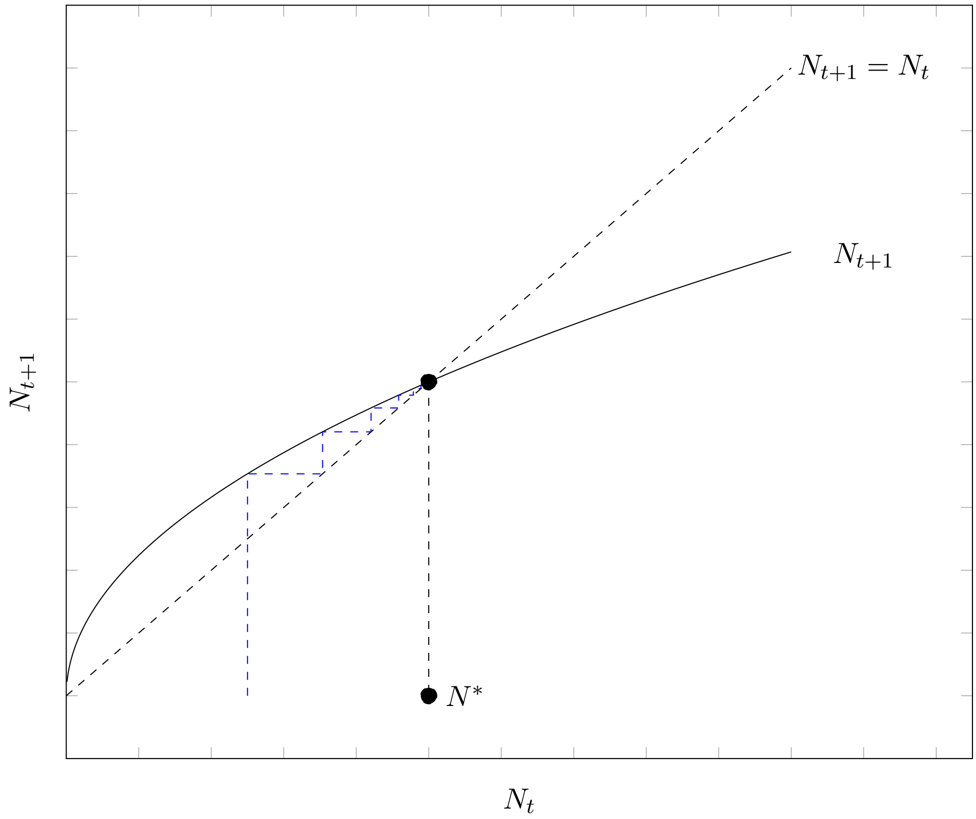

Figure 5: Plot of N_{t+1} against N_t

When $N_{t+1} = N_t$, in other words at N^*, population goes into what we call a "steady state", since it doesn't change anymore. This also works for any population above N^*.

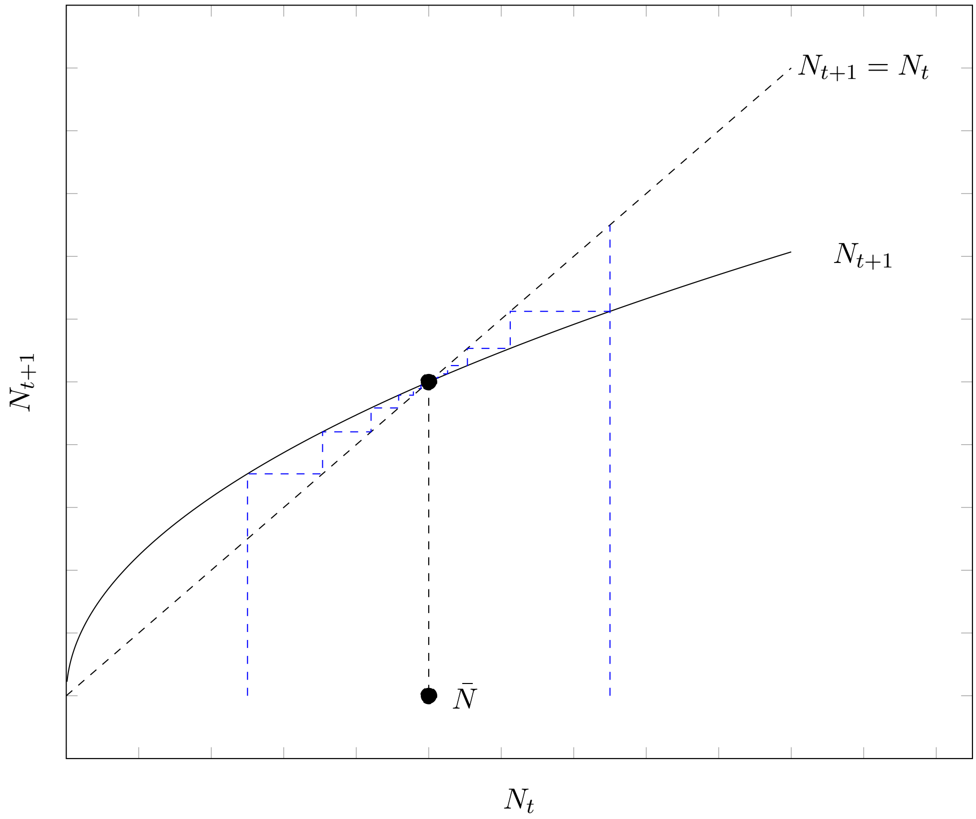

Figure 6: Plot of N_{t+1} against N_t

In other words, we conclude that in the long run, population will always be at the steady state $N*$ in the long run. We can solve for this $\bar{N}$.

$N*$ is when $N_{t+1} = N_t$

Hence, assuming $\xi_t = 1$

$$\bar{N} = \left(\frac{\phi A_t}{w_s}\right)^{\gamma} \bar{N}^{1-\alpha\gamma}$$
$$\bar{N} = \left(\frac{\phi A_t}{w_s}\right)^{\frac{1}{\alpha}}$$

2.5 Wage Dynamics

This has implications on wages. Since wages are

$$w_t = \phi A_t \frac{1}{N_t^\alpha}$$

Then at steady state $\bar{N}$

$$\bar{w} = \phi A_t \frac{1}{\bar{N}^\alpha}$$

Wait a minute. This looks a little familiar doesn't it? In fact, let's take the steady state population equation and switch the subject to w_s subsistence wages

$$\bar{N} = \left(\frac{\phi A_t}{w_s}\right)^{\frac{1}{\alpha}}$$
$$w_s = \phi A_t \frac{1}{\bar{N}^\alpha} = \bar{w}$$

This means that the only steady state wage is subsistence wage. This shouldn't come as a surprise at all. After all,

$$\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)^\gamma \xi_t$$

If $N_{t+1} = N_t = \bar{N}$, then $w_t = w_s = \bar{w}$. Now isn't this pretty sad? All we get to earn are subsistence wages.

2.6 Shocks

There are two different kinds of shocks.

- Temporary shocks (eg. disease)
- Permanent shock (eg. the discovery and synthesis of penicillin)

2.6.1 Temporary Shocks

In a temporary shock, say a disease, ξ decreases for that particular period, then goes back to 1 for all other periods.

In other words, the exogenous parameters, D, A did not shift. Hence, the N_{t+1} curve didn't shift at all and the steady state population is still the same.

The same happens if there was a sudden baby boom or influx of new immigrants. We'd just go back down to the original steady state.

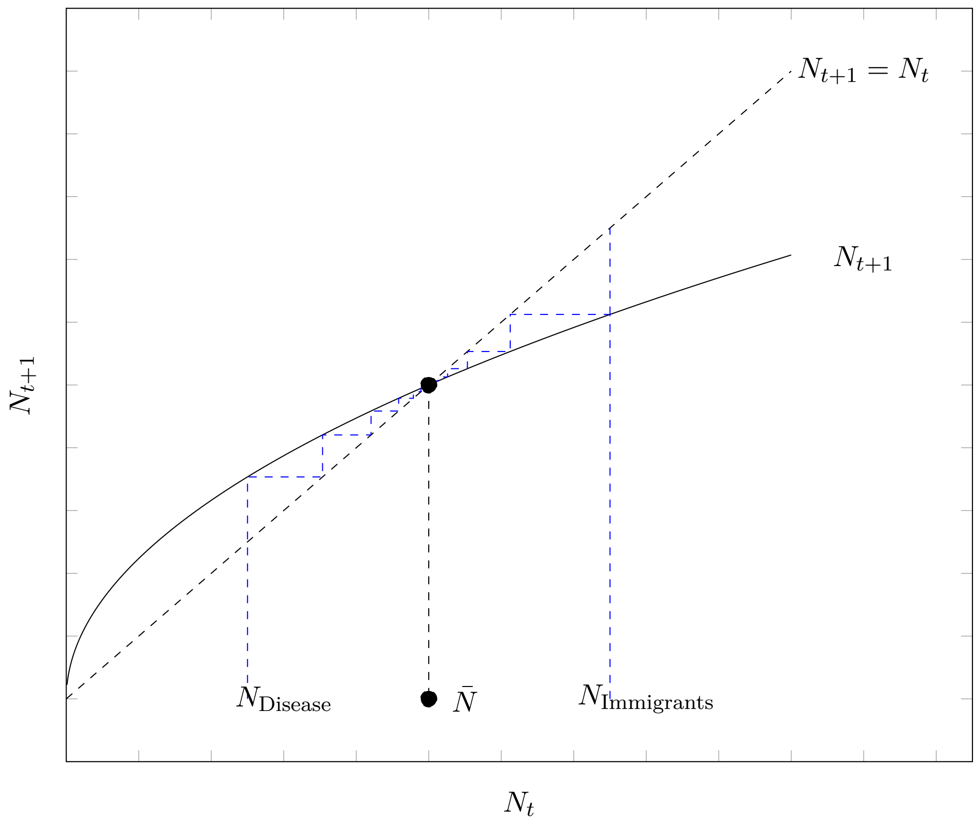

Figure 7: Plot of N_{t+1} against N_t

Steady state wages (ie. subsistence wages) don't change as well.

2.6.2 Permanent Shock

A permanent shock changes the exogenous variables. Examples include a new technology that increases A, discovery of new land that increases D and so on.

We then move up to an entirely different curve.

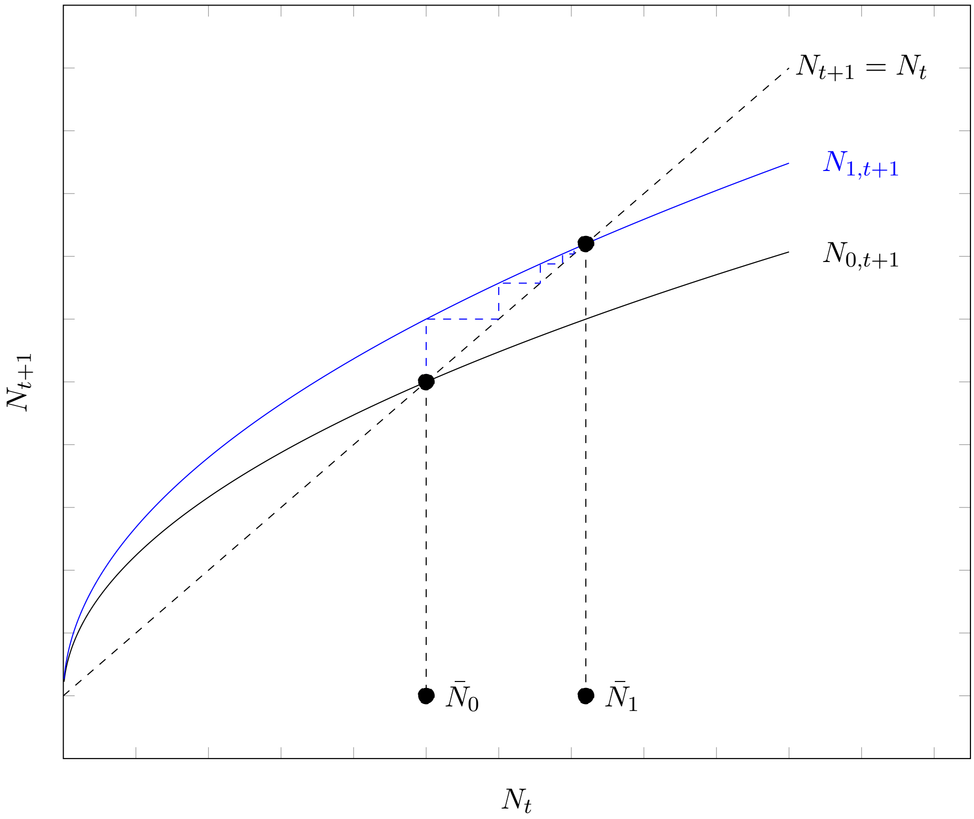

Figure 8: Plot of N_{t+1} against N_t

We will then move to a new steady state $\bar{N}_1$. Mathematically, we can solve for the new steady state using the equation for steady state population

$$\bar{N} = \left(\frac{\phi A_t}{w_s}\right)^{\frac{1}{\alpha}}$$

And the subsistence wages (or steady state wages) as

$$w_s = \phi A_t \frac{1}{\bar{N}^{\alpha}}$$

Paints a very sad picture doesn't it?