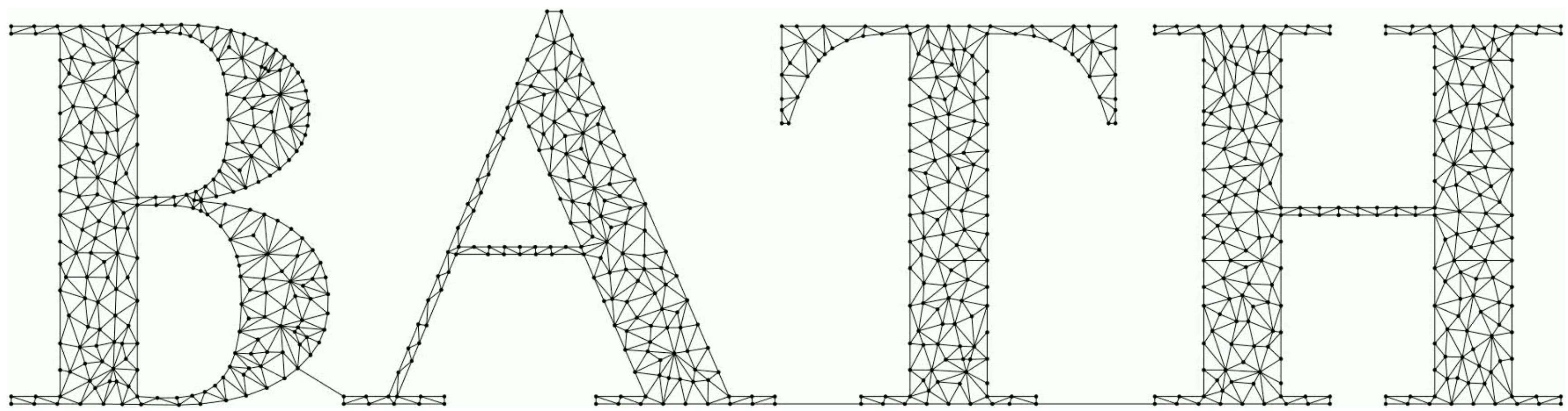
To contract/ or to delete– that is the question

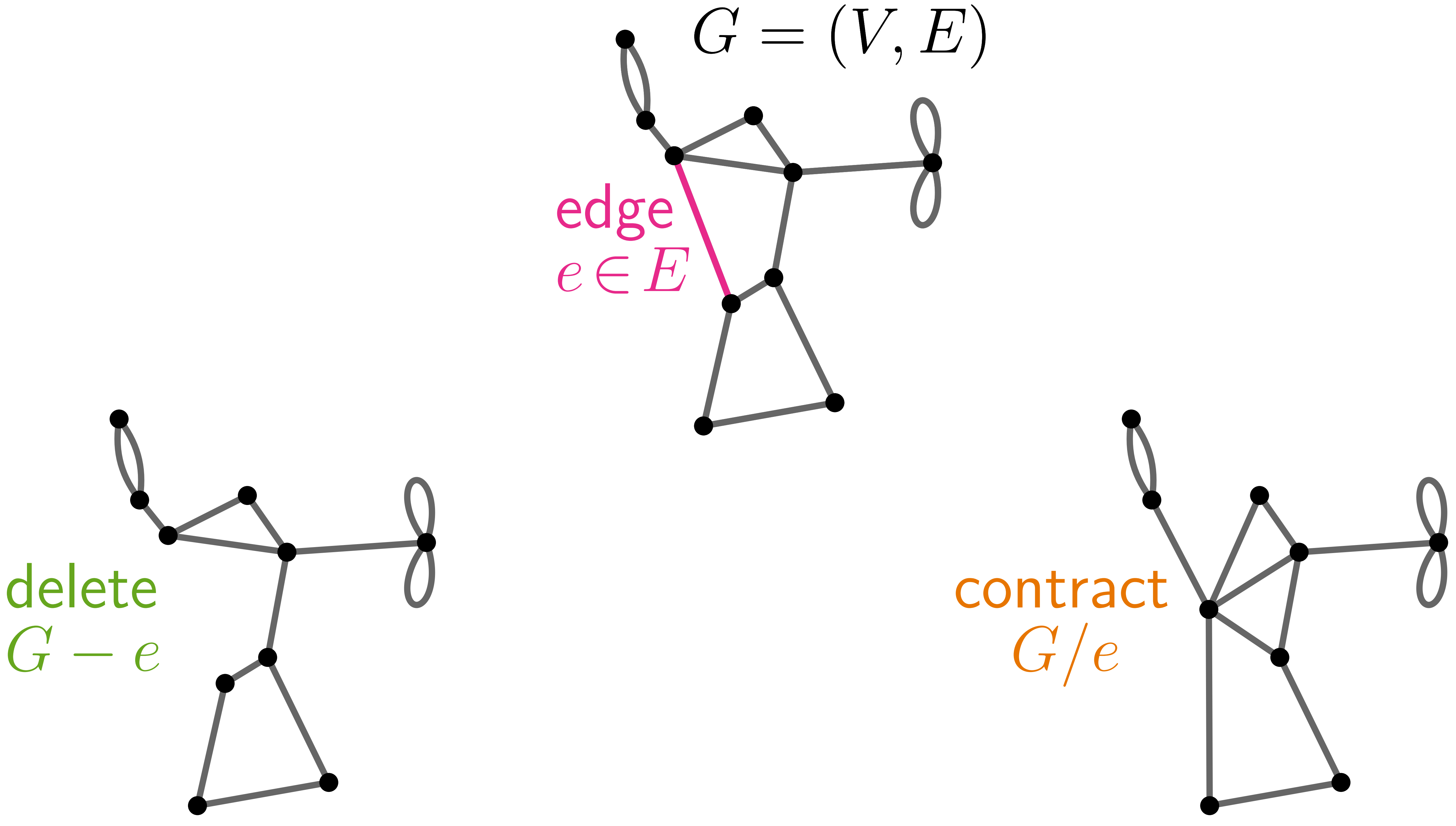


Gecia Bravo-Hermsdorff
joint work with **Lee M. Gunderson**

UNIVERSITY OF BATH

Thirteenth Meeting of the Southern Logic Seminar

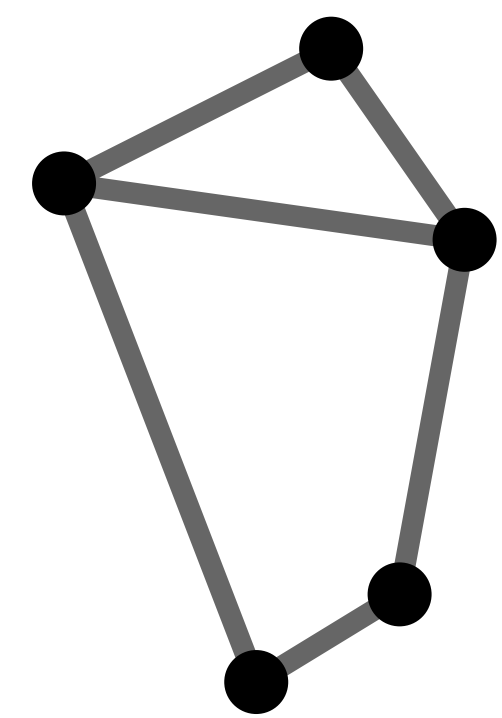
Deletion–Contraction/Example



Two Tantalizing Graph Invariants

Laplacian (spectrum)

$$0 \leq 1.38 \leq 2.38 \leq 3.62 \leq 4.62$$

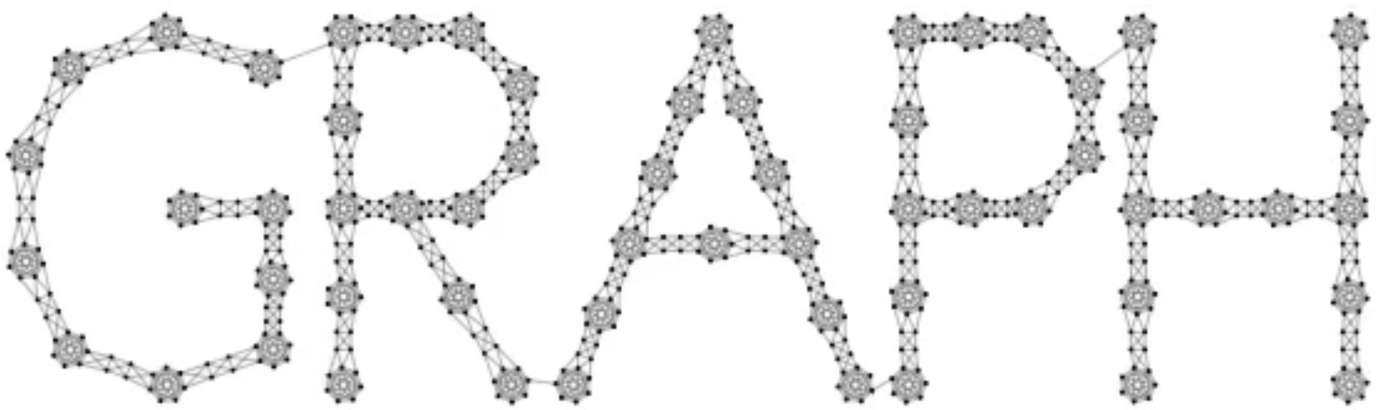


Tutte (polynomial)

$$x + y + 2x^2 + 2xy + y^2 + 2x^3 + x^2y + x^4$$

Two Tantalizing Graph Invariants

Laplacian (spectrum)



G. B-H & L. Gunderson. *NeurIPS*, 2019.
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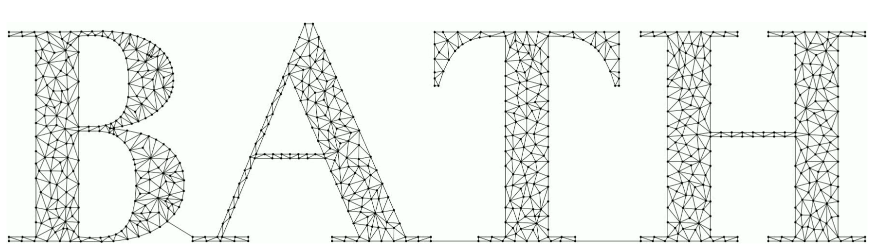

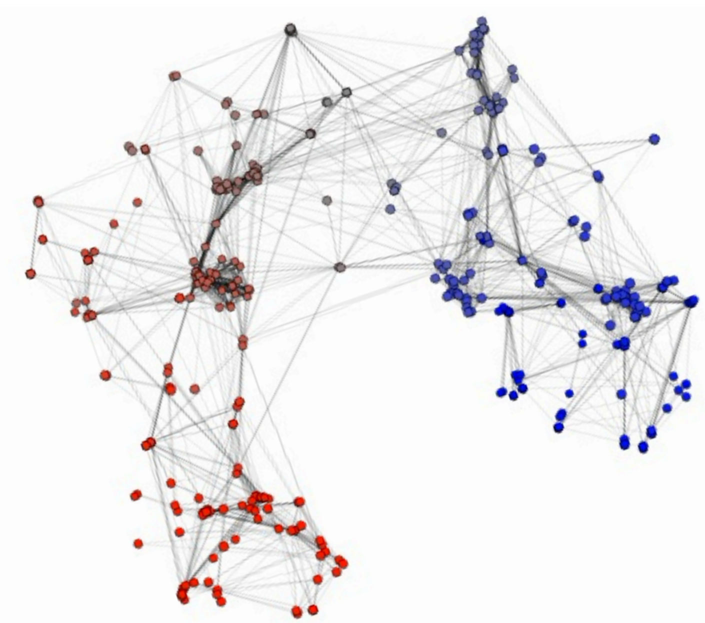

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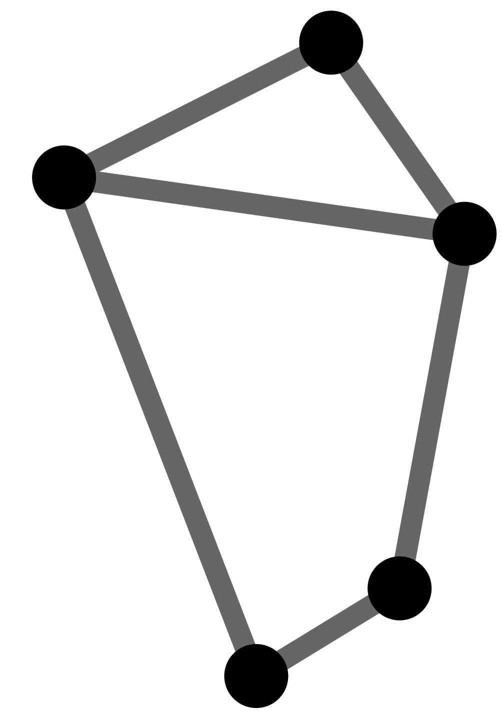


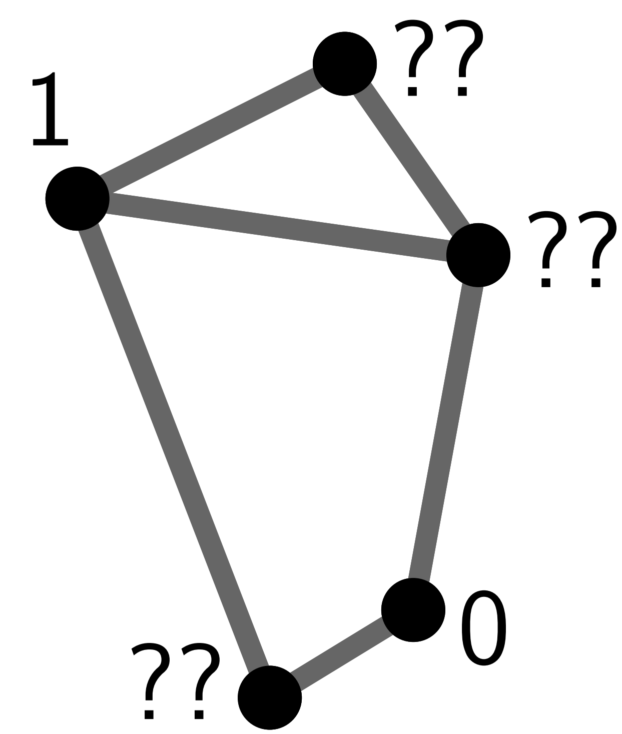
How to interpolate on graphs

Want: a function assigning a scalar to each vertex

$$x_i \in \mathbb{R} \quad \forall i \in V$$

Problem: only know $f$unction on some Fixed subset

$$x_i = f_i \quad \forall i \in F$$



How to interpolate on graphs

Solution: "be the average of your neighbors"

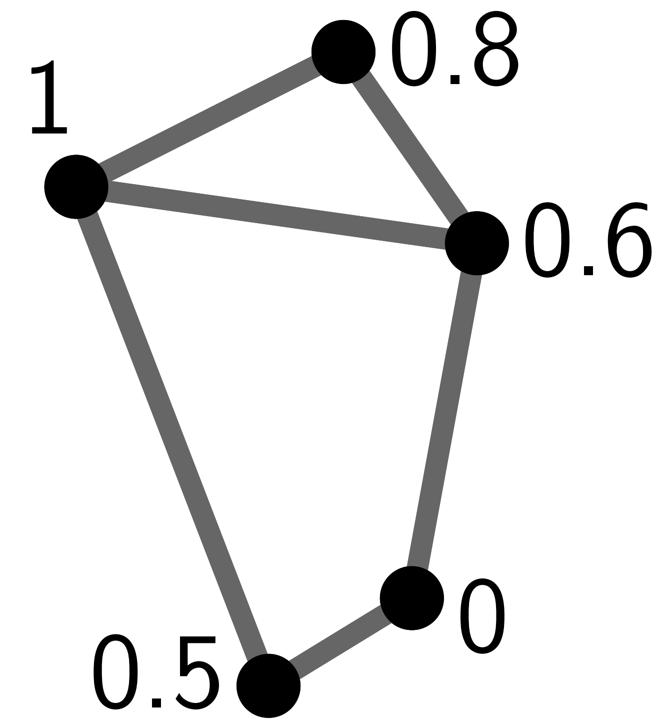
or

"minimize squared differences between neighbors"

$$x_i = \frac{1}{|N(i)|} \sum_{j \in N(i)} x_j$$

or

$$\text{argmin} \sum_{(i,j) \in E} \left(x_i - x_j\right)^2$$



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$$x_i = \frac{1}{|N(i)|} \sum_{j \in N(i)} x_j$$

$$\boldsymbol{L} \cdot \boldsymbol{x} = \boldsymbol{0}$$

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$$

or

$$\text{argmin} \sum_{(i,j) \in E} \left(x_i - x_j\right)^2$$

$$\text{argmin} \;\; \boldsymbol{x}^\top \boldsymbol{L} \boldsymbol{x}$$

$$\boldsymbol{L} = \boldsymbol{B}^\top \boldsymbol{B}$$

How to interpolate on graphs

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$$\boldsymbol{L} = \boldsymbol{B}^\top \boldsymbol{B}$$

$$\boldsymbol{L} = \sum_{k=1}^{n} \lambda_k \boldsymbol{v}_k \boldsymbol{v}_k^\top \qquad \lambda_1 = 0 < \lambda_2 \leq \ldots \leq \lambda_n$$

How to interpolate on graphs

Solution: "be the average of your neighbors"

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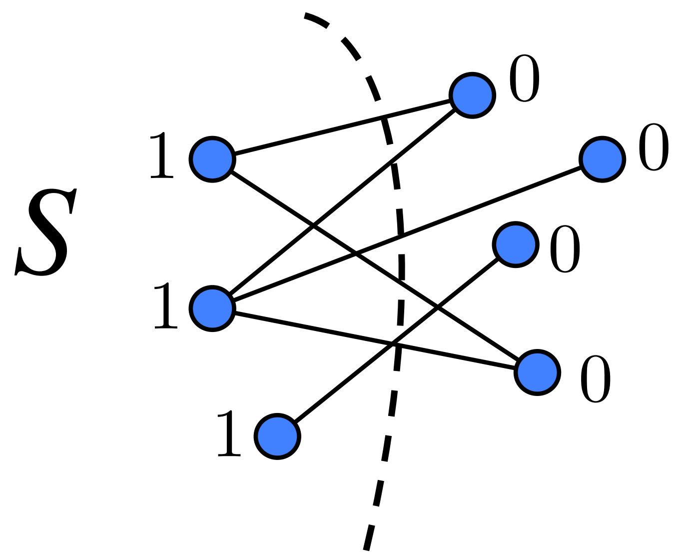

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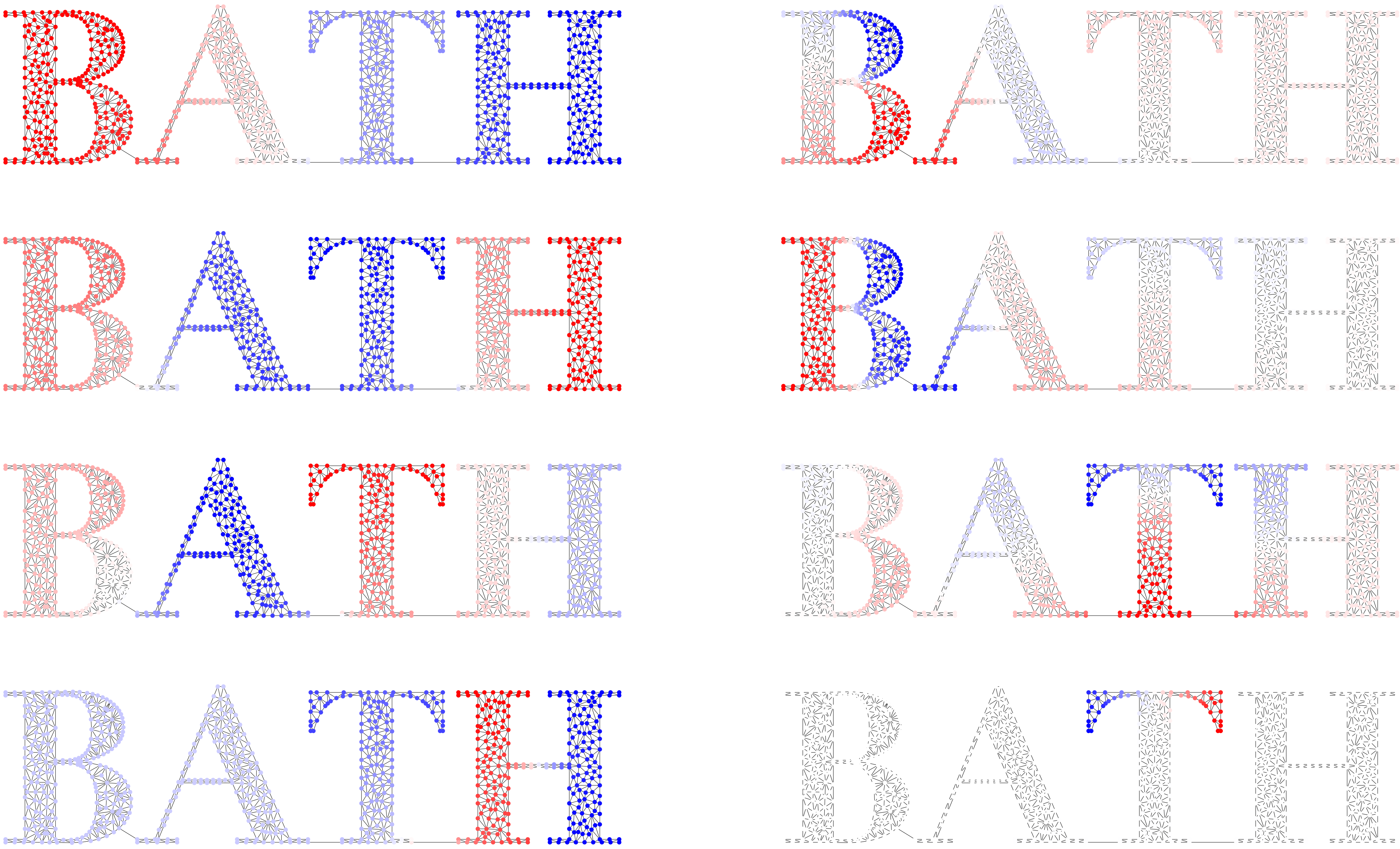
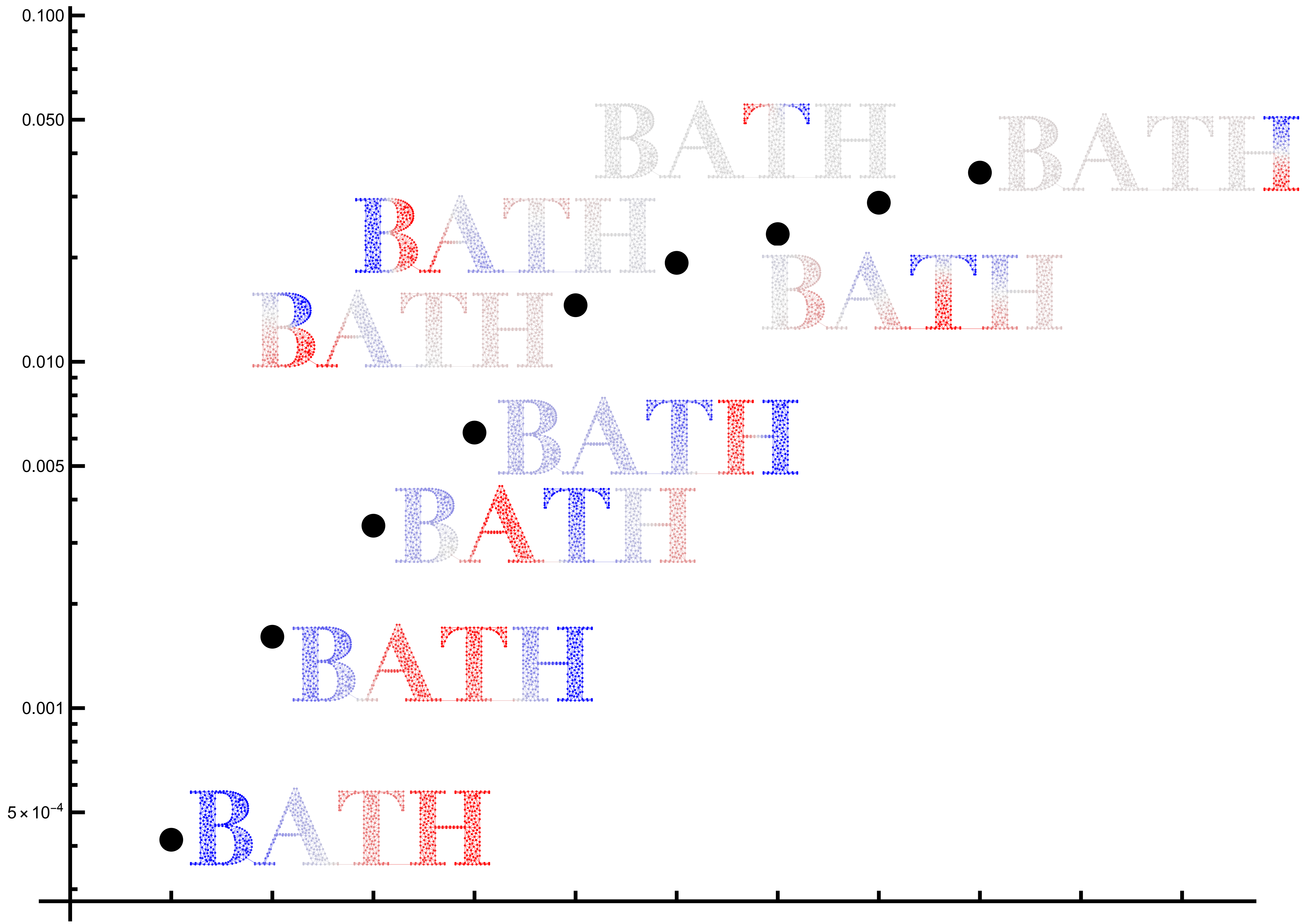
“Hearing” the shape of a (Bath) Graph



“Hearing” the shape of a (Bath) Graph



Kirchhoff's Matrix-Tree Theorem

$$\boxed{\prod_{i=2}^{|V|} \lambda_i \;=\; \Big(\#\text{ of spanning trees}\Big) \times \Big(\#\text{ of vertices}\Big)}$$

Effective resistance (distance) between nodes u and v is:

$$\Omega_{uv} := (\boldsymbol{\delta}_u - \boldsymbol{\delta}_v)^\top \boldsymbol{L}^\dagger (\boldsymbol{\delta}_u - \boldsymbol{\delta}_v)$$

where $\boldsymbol{\delta}$ is a unit indicator vector

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$$\Omega_{uv} := (\boldsymbol{\delta}_u - \boldsymbol{\delta}_v)^\top \boldsymbol{L}^\dagger (\boldsymbol{\delta}_u - \boldsymbol{\delta}_v)$$

where $\boldsymbol{\delta}$ is a unit indicator vector

The effective resistance has multiple interpretations:

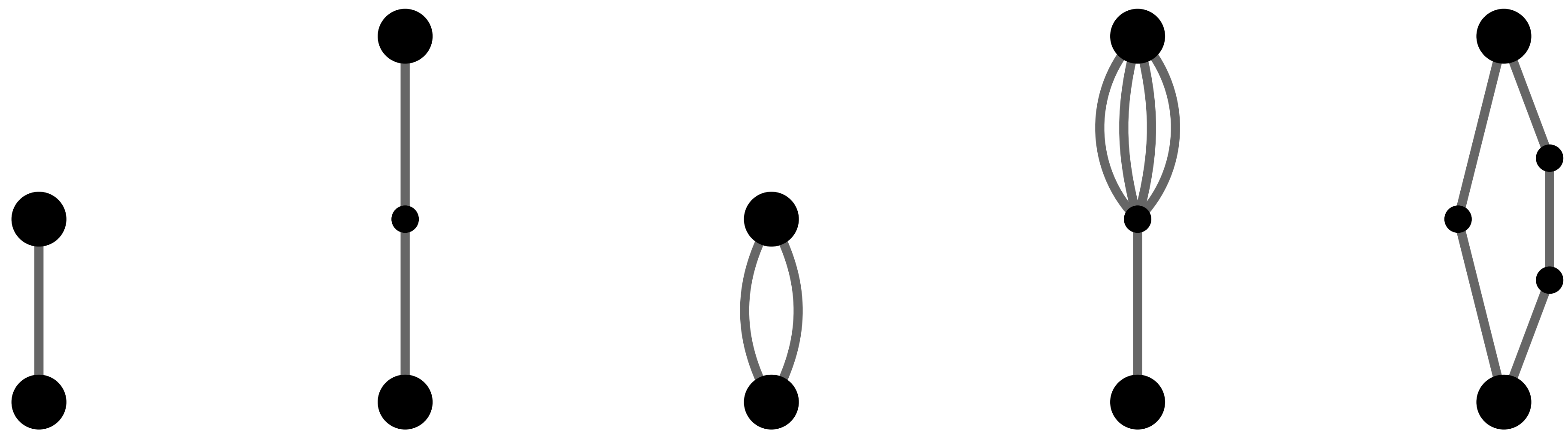
The fraction of spanning trees containing the edge (u, v).

As a network of resistors, the (reciprocal of the) current resulting from a unit of voltage between u and v.

The resulting difference in voltage between u and v when passing a unit of current through the network.

The expected number of steps for a random walker to get from u to v then back to u (divided by $2|E|$).

Effective resistance (distance)



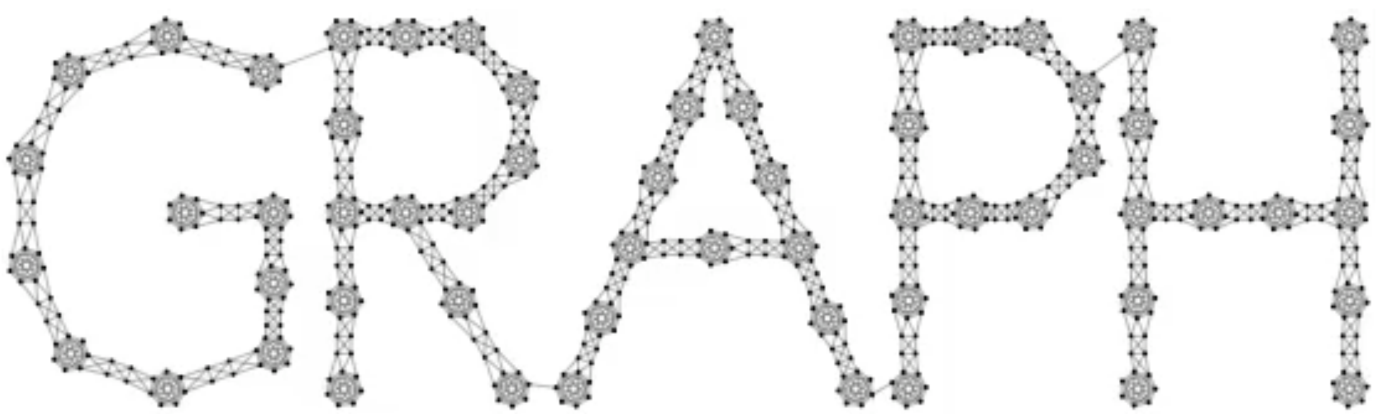
shortest path	1	2	1	2	2
effective resistance	1	2	0.5	1.25	1.2
				$1+\frac{1}{4}$	$\left(\frac{1}{2}+\frac{1}{3}\right)^{-1}$

series $\longrightarrow$ sum resistances

parallel $\longrightarrow$ sum reciprocals

Two Tantalizing Graph Invariants

Laplacian (spectrum)



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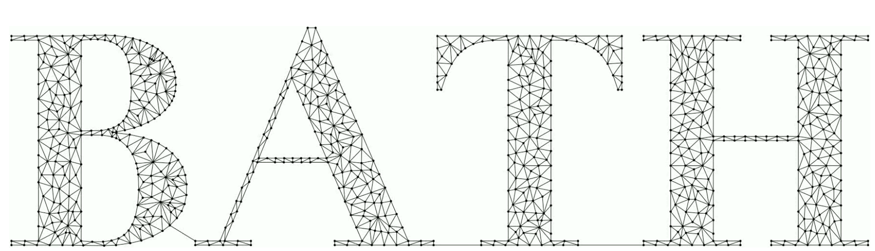
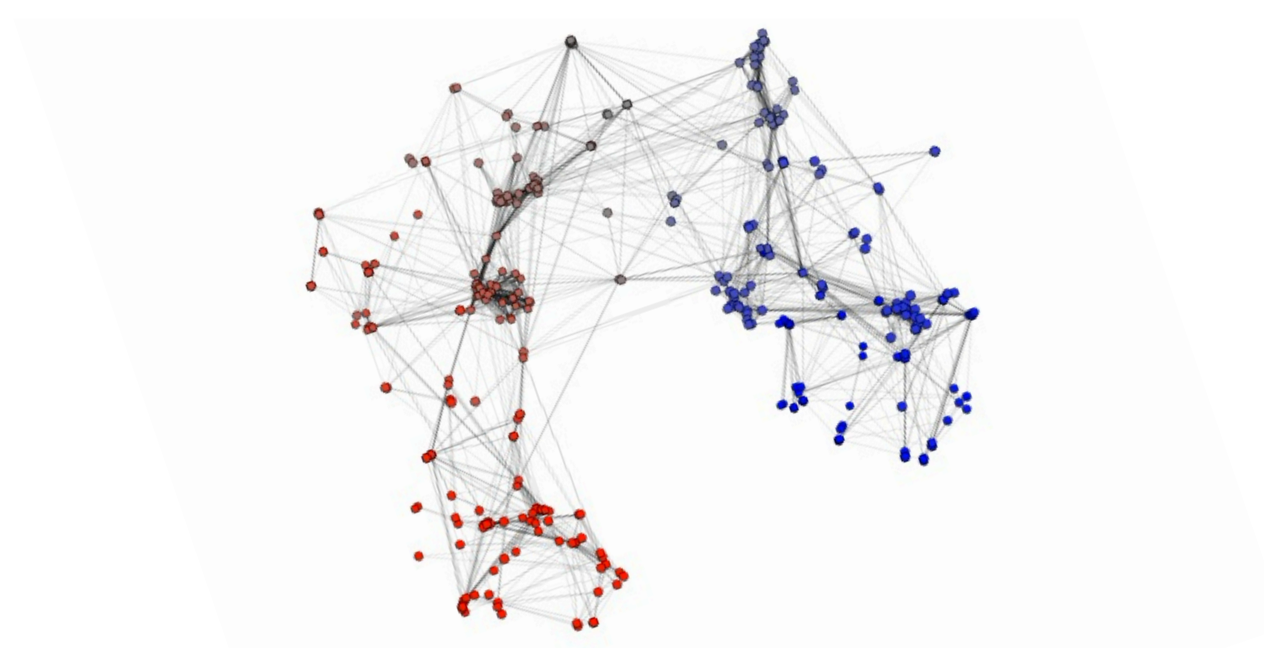




Algorithms for reducing a graph while preserving global structure

Spielman et. al ($\sim$ 2010)

edge deletion–

Laplacian quadratic form

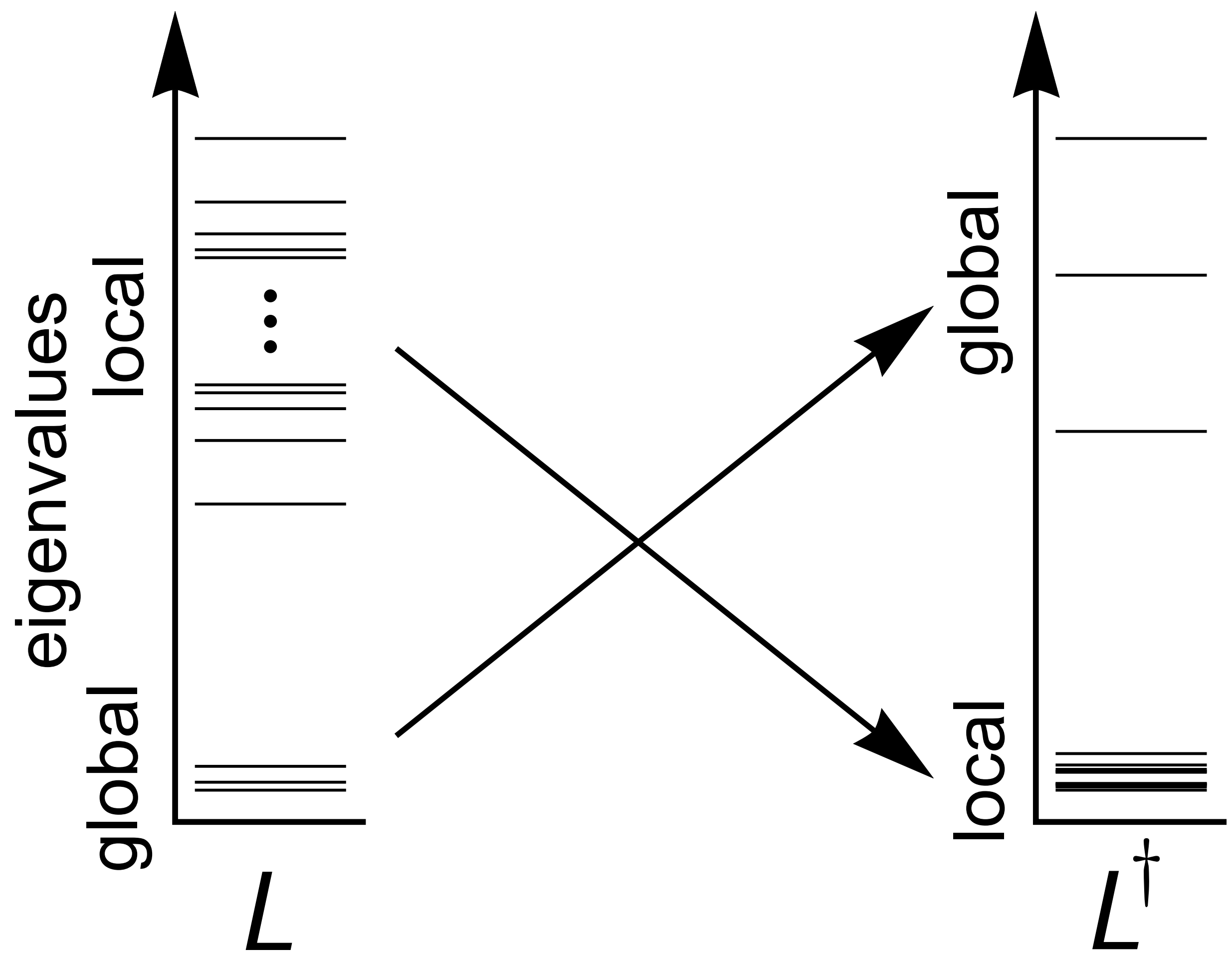
Our algorithm (2019)

edge deletion– and edge contraction/

Laplacian pseudoinverse

Probabilistically preserving the pseudoinverse

Want: $\mathbb{E}\left[\widetilde{\boldsymbol{L}^{\dagger}}\right] = \boldsymbol{L}^{\dagger}$



$$\boldsymbol{L}\boldsymbol{L}^{\dagger} = \boldsymbol{L}^{\dagger}\boldsymbol{L} = \boldsymbol{1} - \frac{1}{|V|}\boldsymbol{v}_1\boldsymbol{v}_1^{\top}$$

$$\boldsymbol{L}^{\dagger} = \sum_{k=2}^{n} \frac{1}{\lambda_k}\boldsymbol{v}_k\boldsymbol{v}_k^{\top}$$

Probabilistically preserving the pseudoinverse

$$\boxed{\textbf{Want:} \quad \mathbb{E}\left[\widetilde{\boldsymbol{L}}^{\dagger}\right] = \boldsymbol{L}^{\dagger}}$$

Change the weight of edge $e = (u, v)$ by Δw:

$$\widetilde{\boldsymbol{L}} = \boldsymbol{L} + \Delta w \, \boldsymbol{b}\boldsymbol{b}^{\top} \qquad (b_i)_{i \in V} = \begin{cases} +1 & \text{if } i = u \\ -1 & \text{if } i = v \\ 0 & \text{otherwise} \end{cases}$$

Probabilistically preserving the pseudoinverse

$$\boxed{\textbf{Want:} \quad \mathbb{E}\left[\widetilde{\boldsymbol{L}^\dagger}\right] = \boldsymbol{L}^\dagger}$$

Change the weight of edge $e = (u, v)$ by Δw:

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$$\widetilde{\boldsymbol{L}^\dagger} = \boldsymbol{L}^\dagger - \underbrace{\frac{\Delta w}{1 + \Delta w \, \boldsymbol{b}^\top \boldsymbol{L}^\dagger \boldsymbol{b}}}_{\text{scalar function of } \Delta w} \times \underbrace{\boldsymbol{L}^\dagger \boldsymbol{b}\boldsymbol{b}^\top \boldsymbol{L}^\dagger}_{\text{constant matrix}}$$

Probabilistically preserving the pseudoinverse

Change in $\boldsymbol{L}^{\dagger}$ $\quad\propto\quad$ $\underbrace{\dfrac{-\Delta w}{1 + \Delta w\, \Omega}}_{\text{scalar function of } \Delta w}$

$$\underbrace{\Omega = \boldsymbol{b}^{\top}\boldsymbol{L}^{\dagger}\boldsymbol{b}}_{\textbf{effective resistance}}$$

Deleting an edge: $\quad \Delta w \;\to\; -1$

$$\frac{-\Delta w}{1+\Delta w\,\Omega} \;\to\; \frac{1}{1-\Omega}$$

Contracting an edge: $\quad \Delta w \;\to\; +\infty$

$$\frac{-\Delta w}{1+\Delta w\,\Omega} \;\to\; -\frac{1}{\Omega}$$

To preserve $\mathbb{E}\left[\boldsymbol{L}^{\dagger}\right]$:

delete with probability $1 - \Omega$
contract with probability Ω

Algorithms for reducing a graph while preserving global structure

Spielman et. al ($\sim$ 2010)

edge deletion–

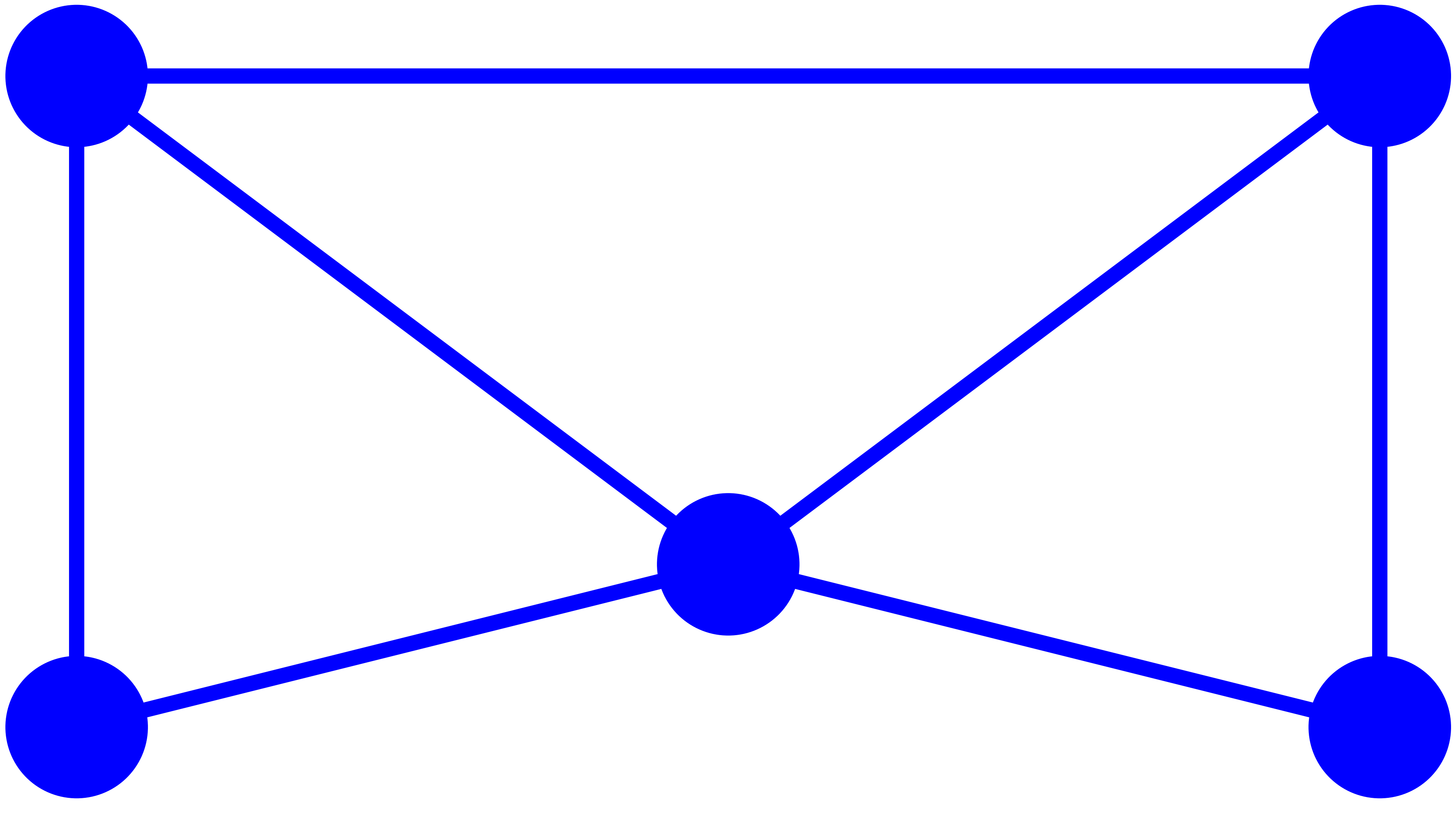
Laplacian quadratic form

Our algorithm (2019)

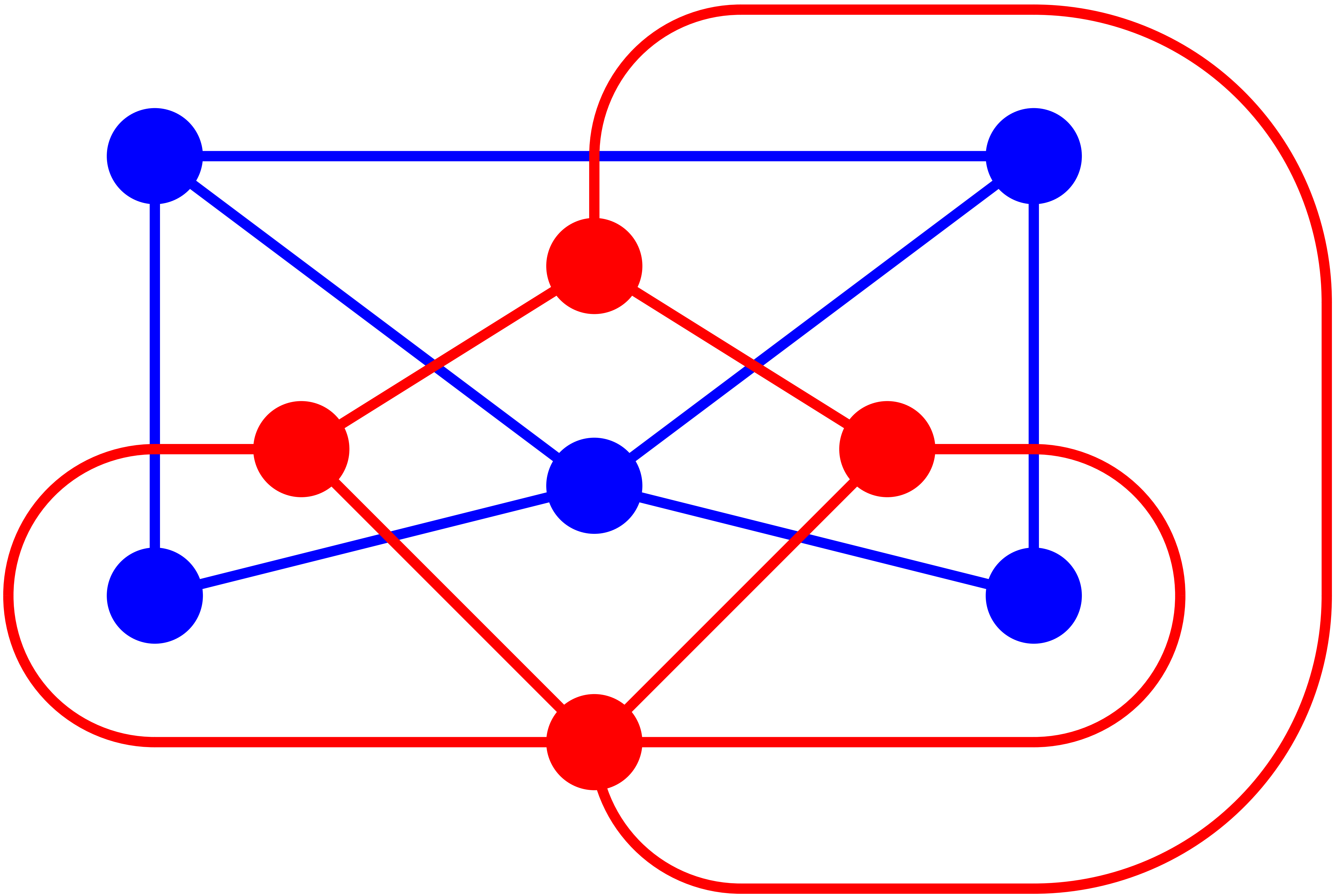
edge deletion– and edge contraction/

Laplacian pseudoinverse

Deletion–Contraction/Duality



Deletion–Contraction/Duality



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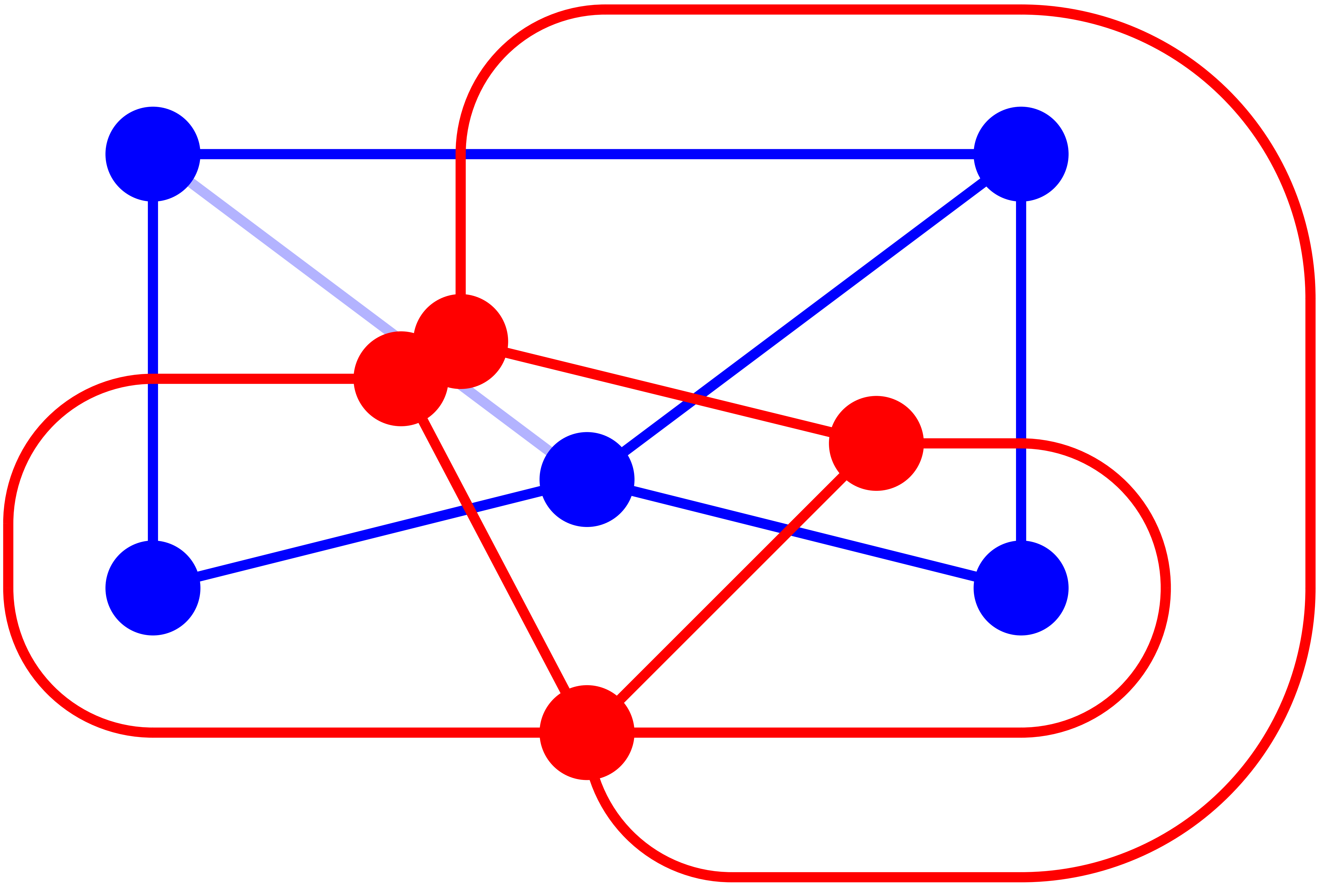

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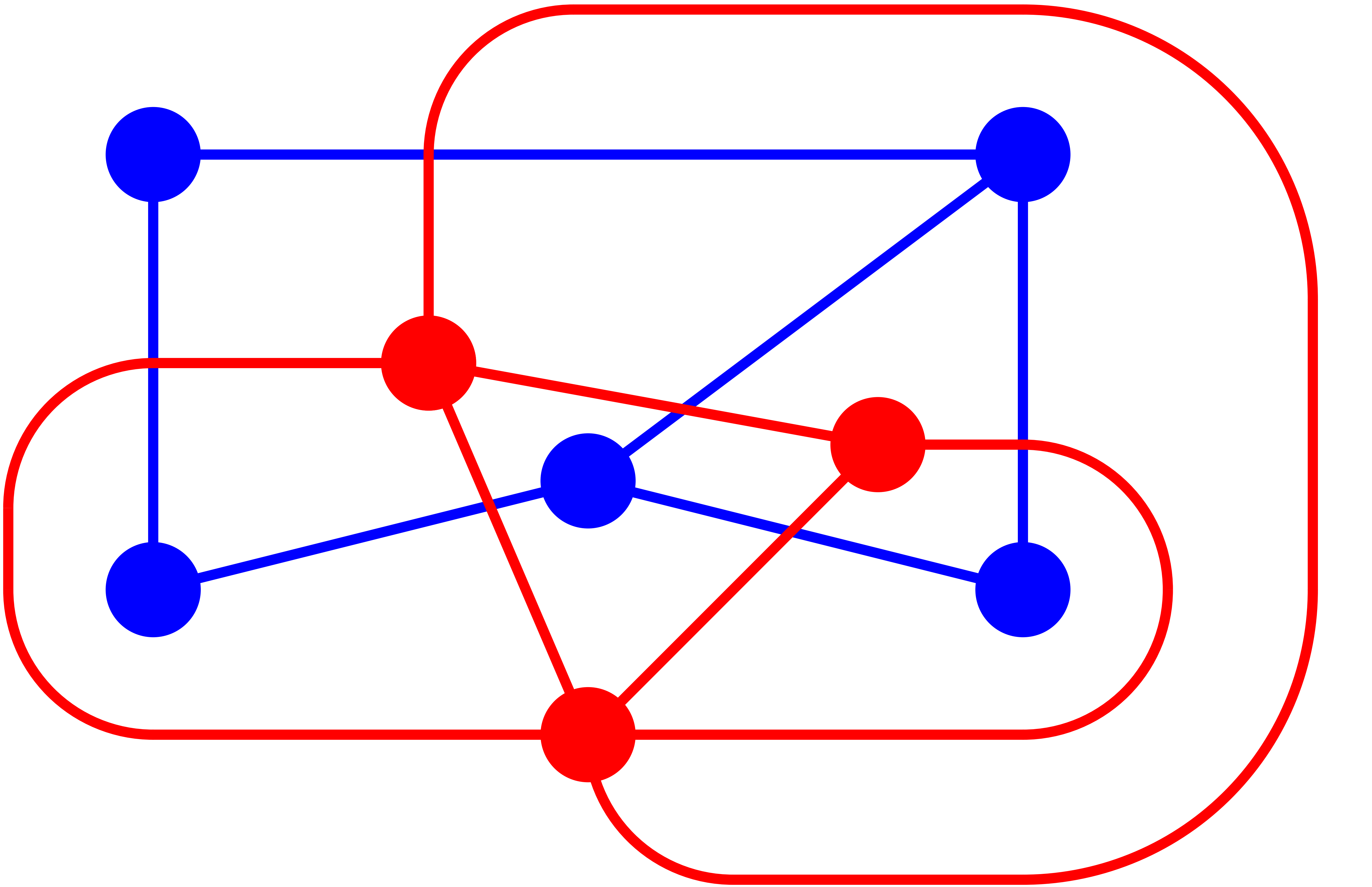
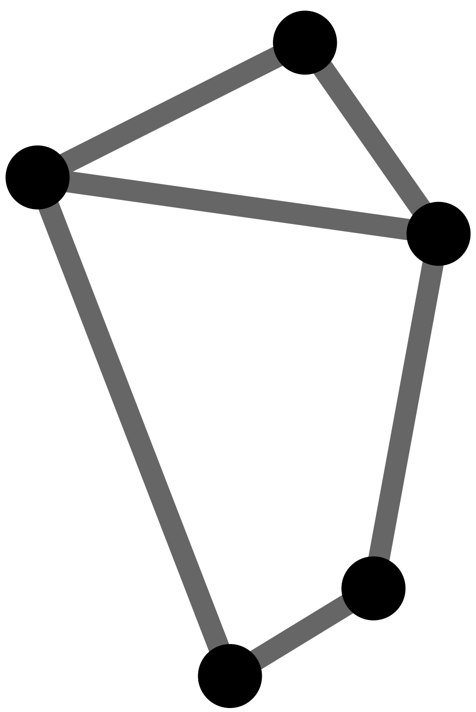
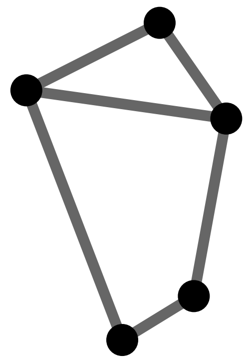




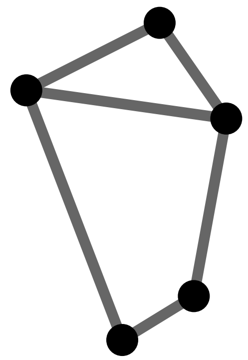
Tutte (polynomial)

$$x + y + 2x^2 + 2xy + y^2 + 2x^3 + x^2y + x^4$$



all edges

$$T(x, y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$



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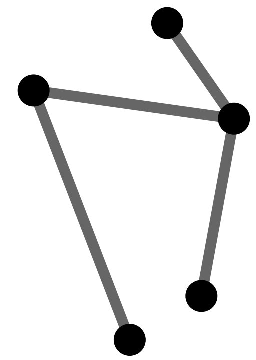
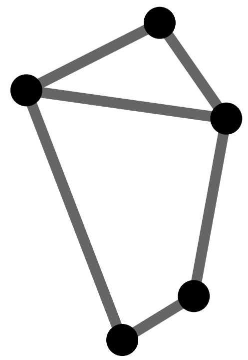


spanning tree

$$T(x,y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$

$T(1,1) = 11$ spanning trees



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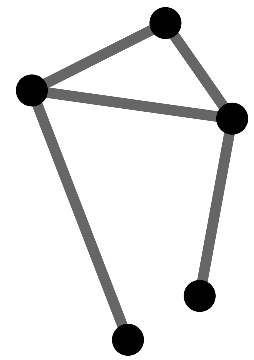
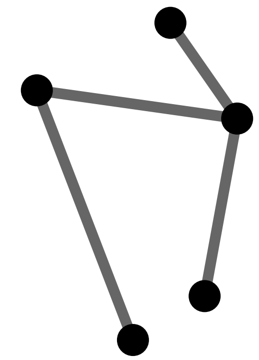


spanning connected subgraph

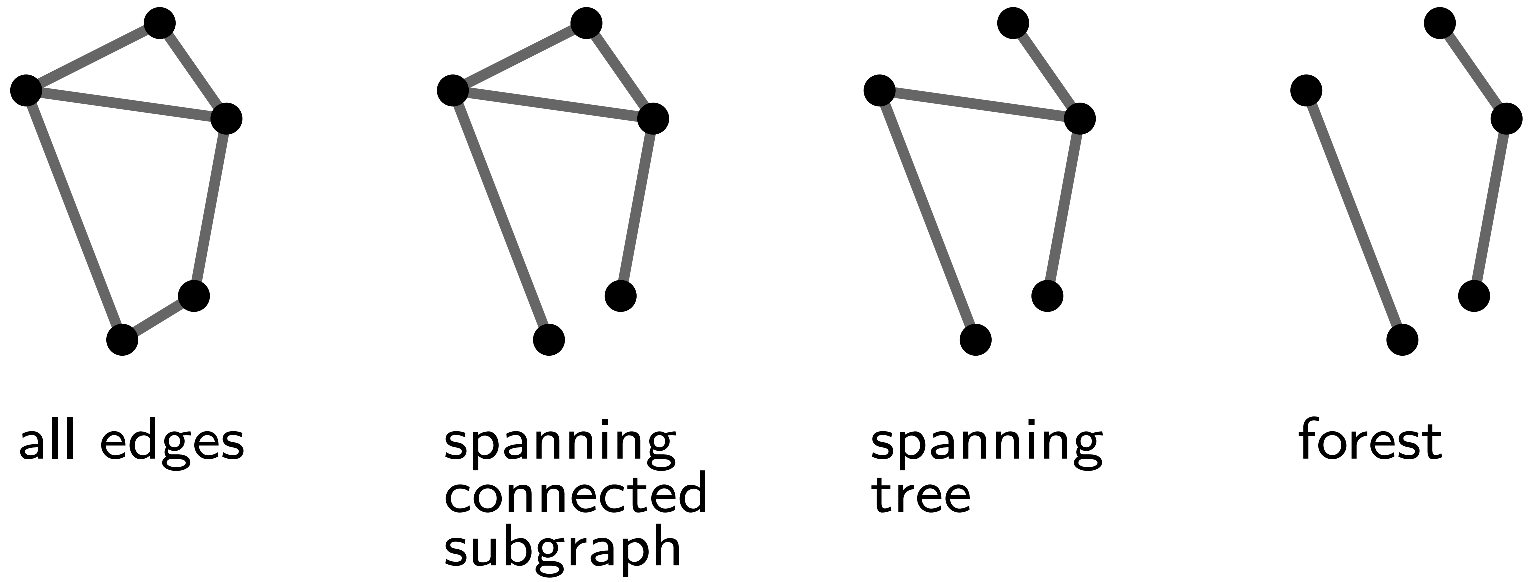


spanning tree

$$T(x,y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$

$T(1,1) = 11$ spanning trees

$T(1,2) = 18$ spanning connected subgraphs

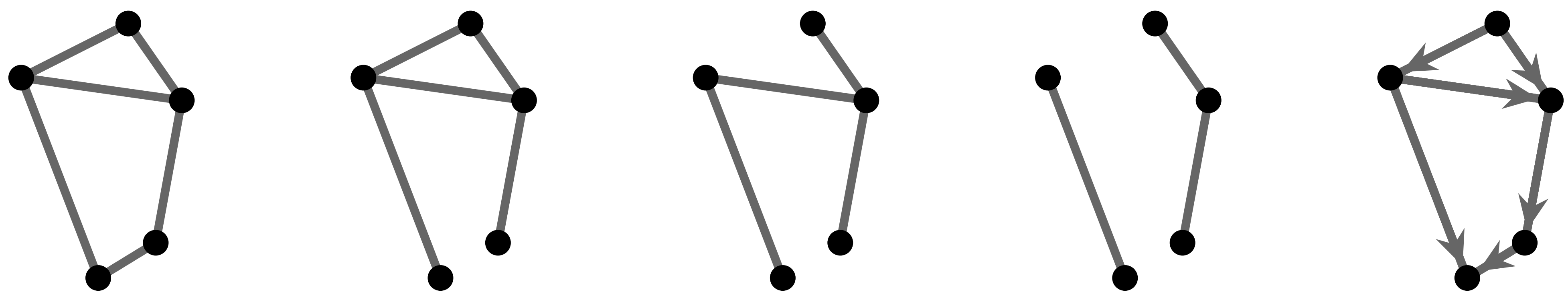


$$T(x,y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$

$T(1,1) = 11$ spanning trees

$T(1,2) = 18$ spanning connected subgraphs

$T(2,1) = 52$ forests



all edges — spanning connected subgraph — spanning tree — forest — acyclic orientation

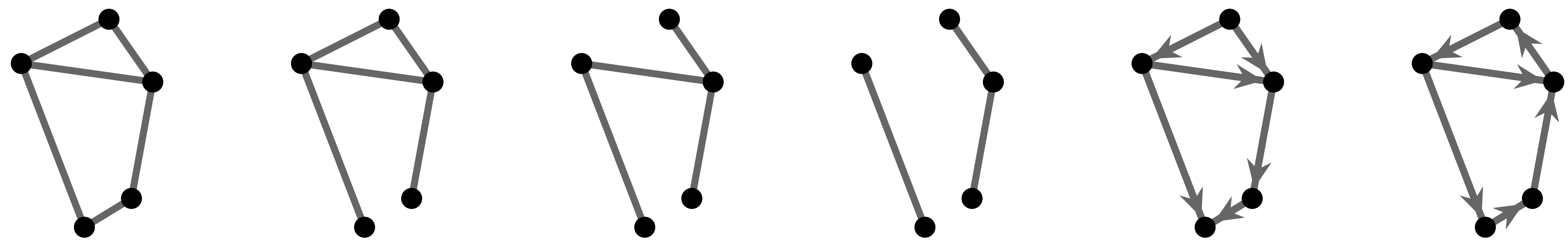
$$T(x,y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$

$T(1,1) = 11$ spanning trees

$T(1,2) = 18$ spanning connected subgraphs

$T(2,1) = 52$ forests

$T(2,0) = 42$ acyclic orientations



$$T(x, y) = x + 2x^2 + 2x^3 + x^4 + y + 2xy + x^2y + y^2$$

$T(1,1) = 11$ spanning trees

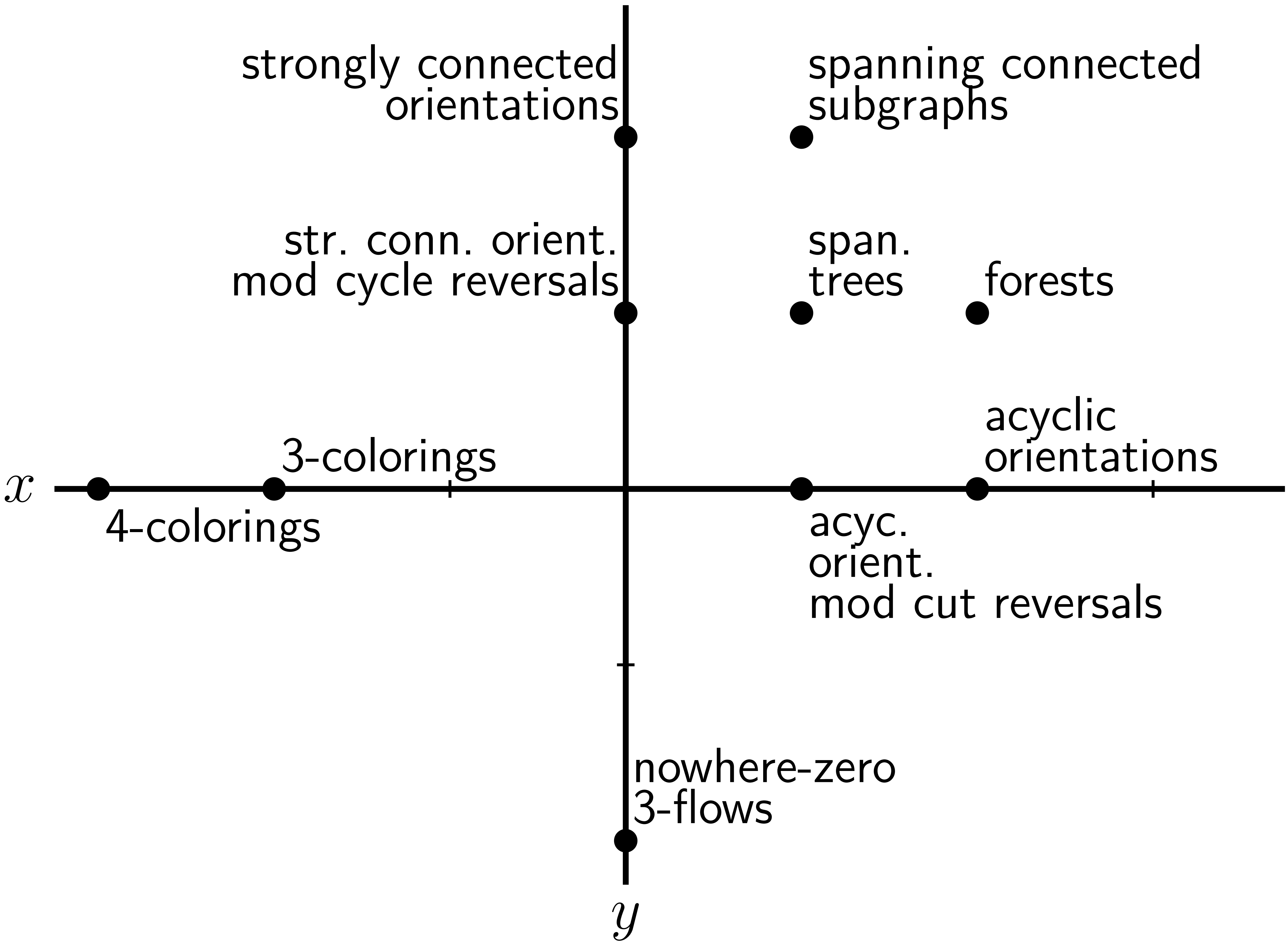
$T(1,2) = 18$ spanning connected subgraphs

$T(2,1) = 52$ forests

$T(2,0) = 42$ acyclic orientations

$T(0,2) = 6$ strongly connected orientations

Tutte–Grothendieck invariants



Specializations of the Tutte polynomial

polynomial	**curve of** $T[G](x,y)$
chromatic	x-axis
flow	y-axis
reliability	line $x = 1$
Jones	curve $xy = 1$
q-state Potts model	$(x-1)(y-1) = q$

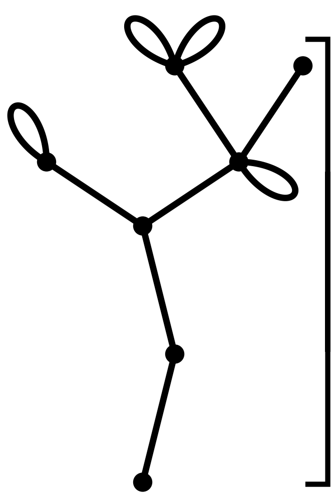
Def. the **Tutte polynomial** of a graph is:

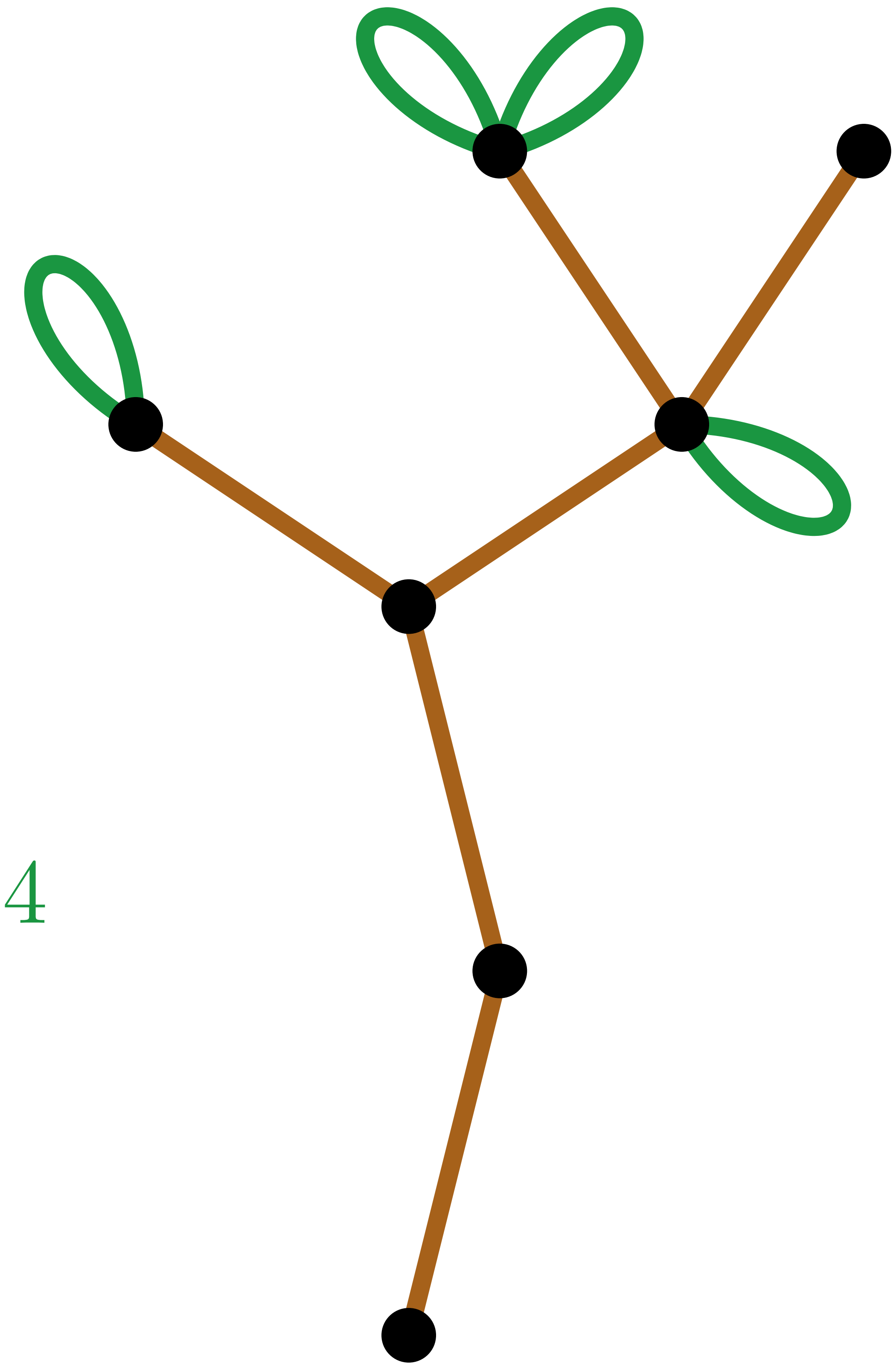
Tutte's deletion and contraction recurrence

Def. the **Tutte polynomial** of a graph is:

$$T[G](x,y) = \begin{cases} T[G-e] + T[G/e] & \text{if } e \text{ is an} \\ & \text{“ordinary” edge} \\ x^b y^l & \text{if } G \text{ consists of exactly} \\ & b \text{ bridges and } l \text{ self-loops} \end{cases}$$

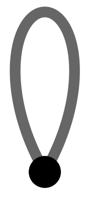
Monomials are “loopy trees”

$T\Bigg[$  $\Bigg] = x^6\, y^4$



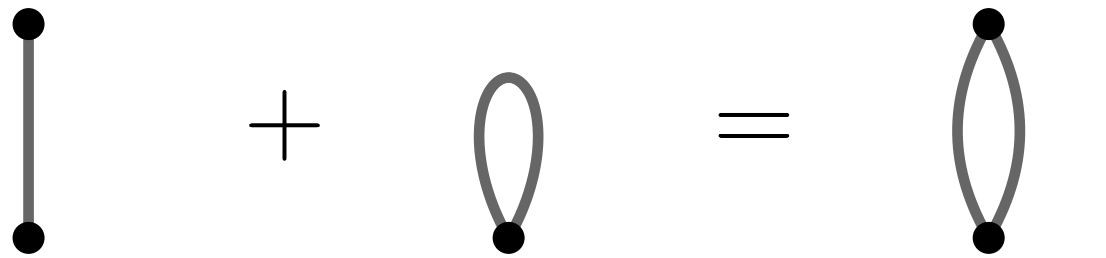
Hands-on Tutte evaluation

x

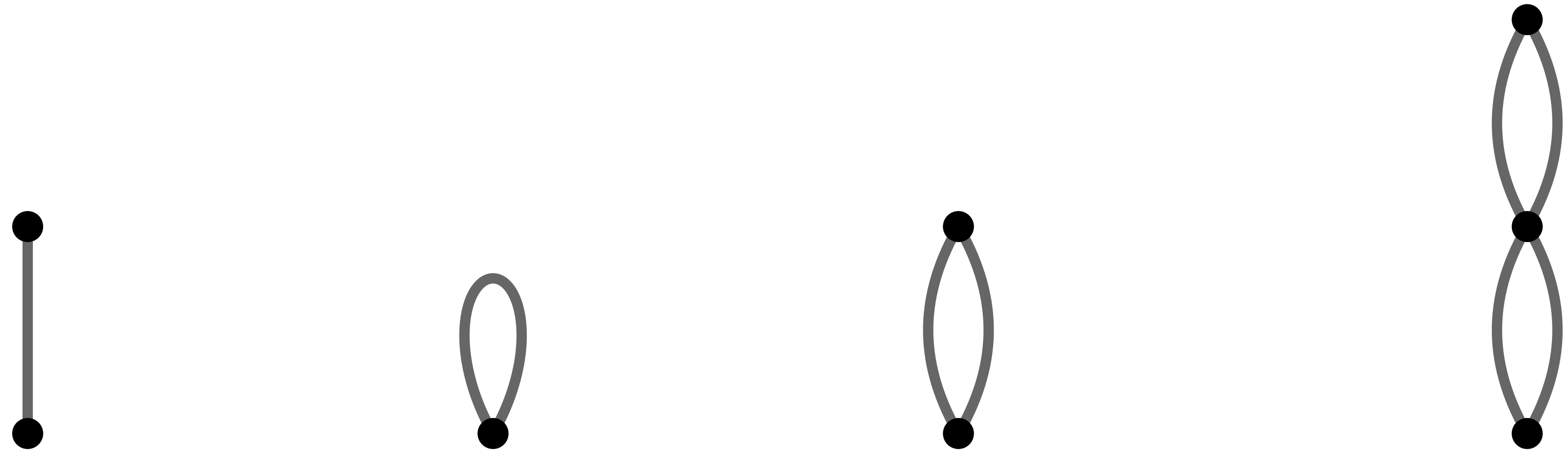


y

Hands-on Tutte evaluation

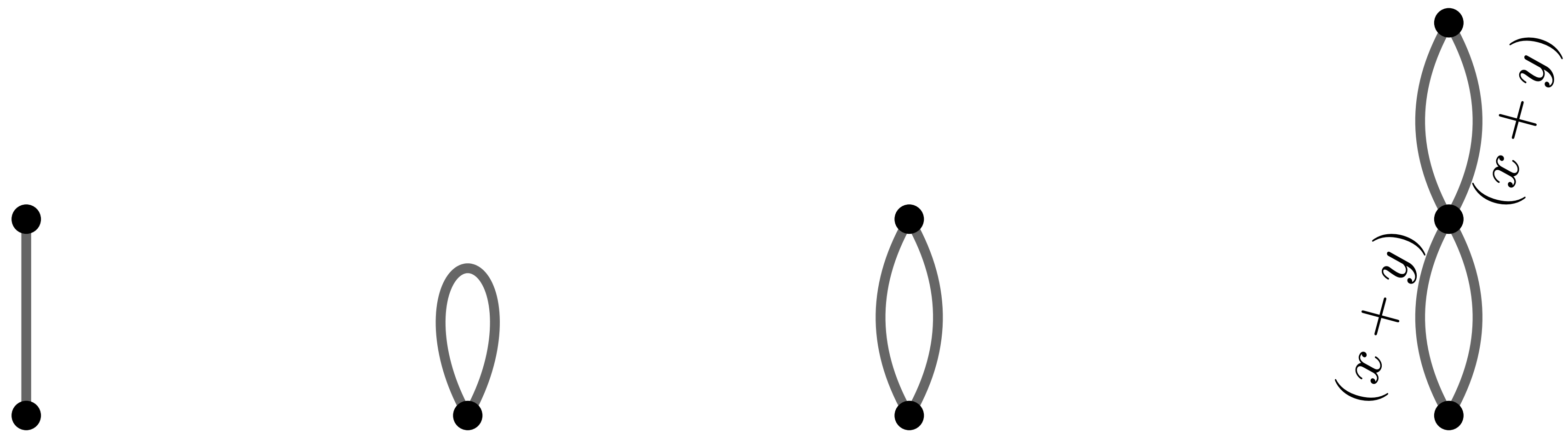


Hands-on Tutte evaluation



x $\quad$ y $\quad$ $x+y$ $\quad$ $x^2+2xy+y^2$

Hands-on Tutte evaluation



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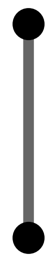
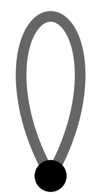
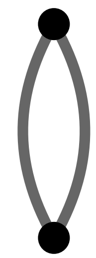


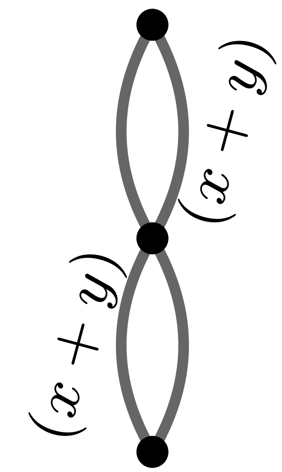
x



y



$x + y$



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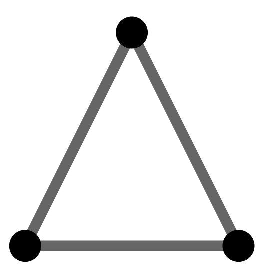
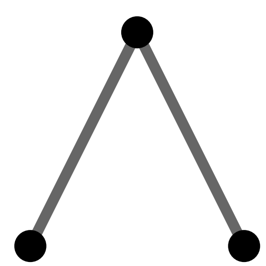
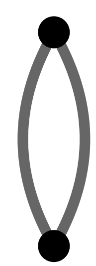


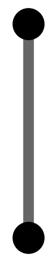
$=$



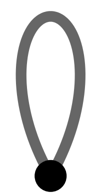
$+$



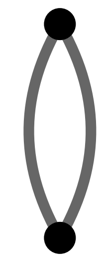
Hands-on Tutte evaluation



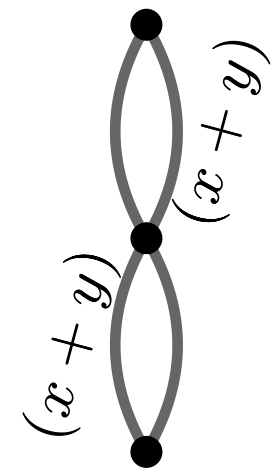
x



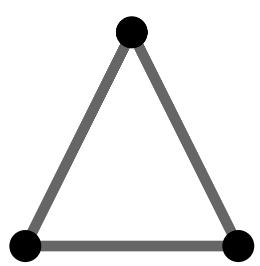
y



$x + y$

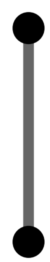


$x^2 + 2xy + y^2$



$x^2 + x + y$

Hands-on Tutte evaluation



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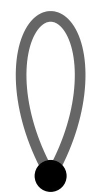

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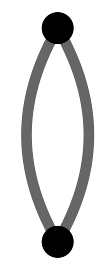
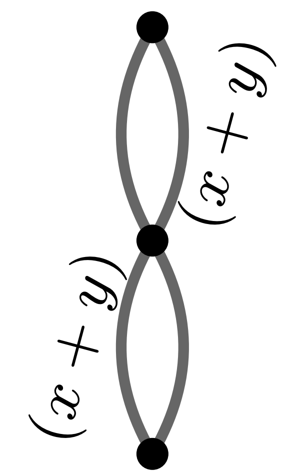
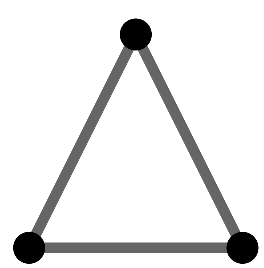


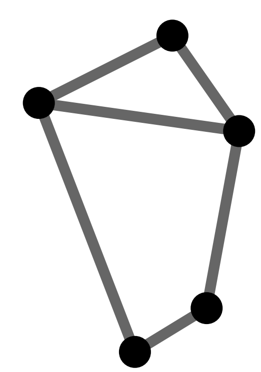
$x + y$



$x^2 + 2xy + y^2$



$x^2 + x + y$



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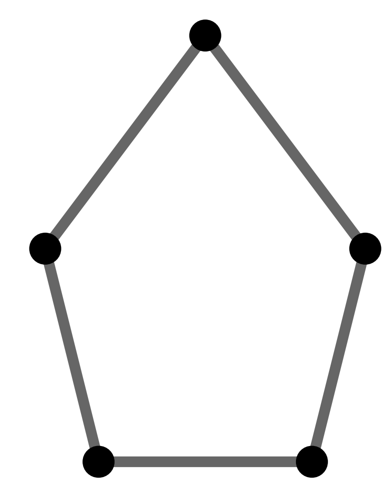

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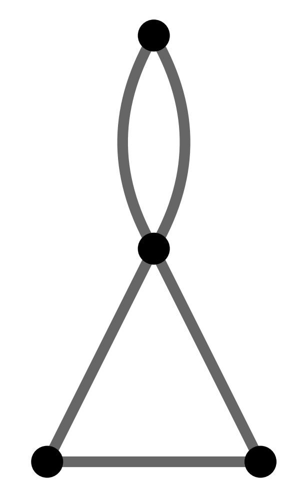
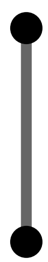


Hands-on Tutte evaluation



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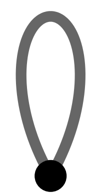

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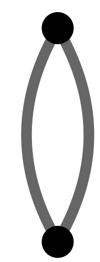
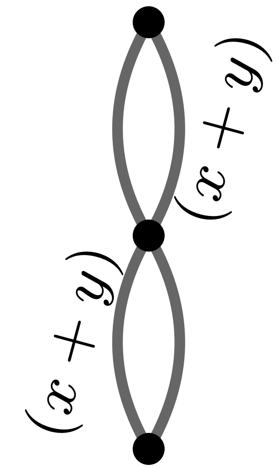
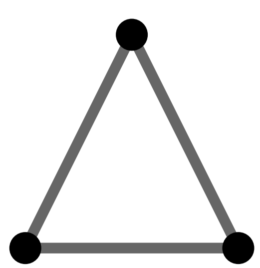


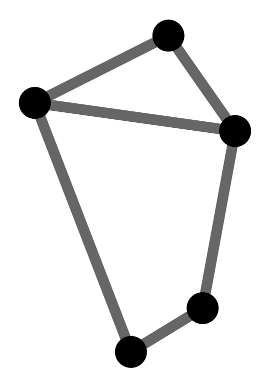
$x + y$



$x^2 + 2xy + y^2$



$x^2 + x + y$



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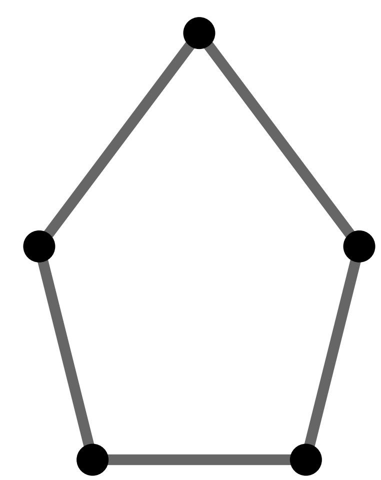
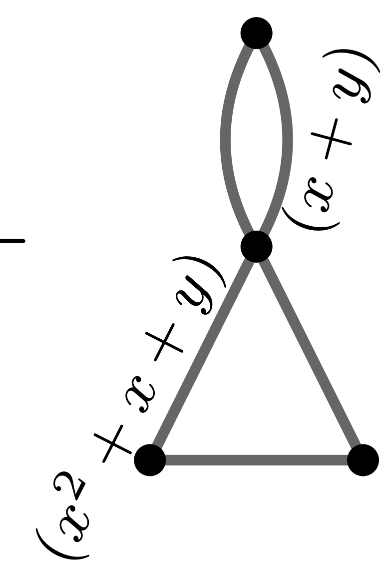
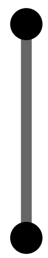


$x^4 + x^3 + x^2 + x + y$

$+$



Hands-on Tutte evaluation



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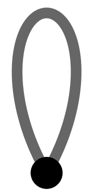

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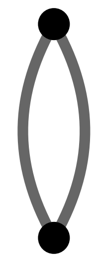
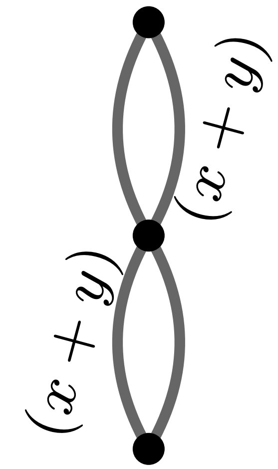
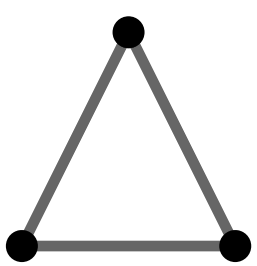


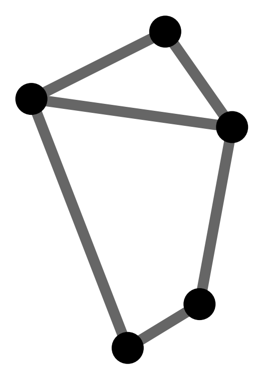
$x + y$



$x^2 + 2xy + y^2$



$x^2 + x + y$



$x^4 + x^3 + x^2 + x + y + (x^2 + x + y)(x + y)$

Def. For a connected graph $G = (V, E)$,
the **Tutte polynomial** is:

$$T[G](x, y) = \sum_{S \in E} = (x-1)^{\text{corank}(S)}(y-1)^{\text{nullity}(S)}$$

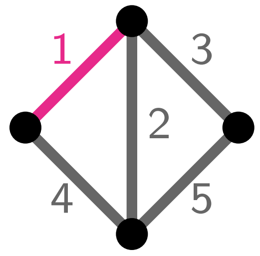
corank(S) = # **edges** needed to **add** to get a **connected** subgraph.

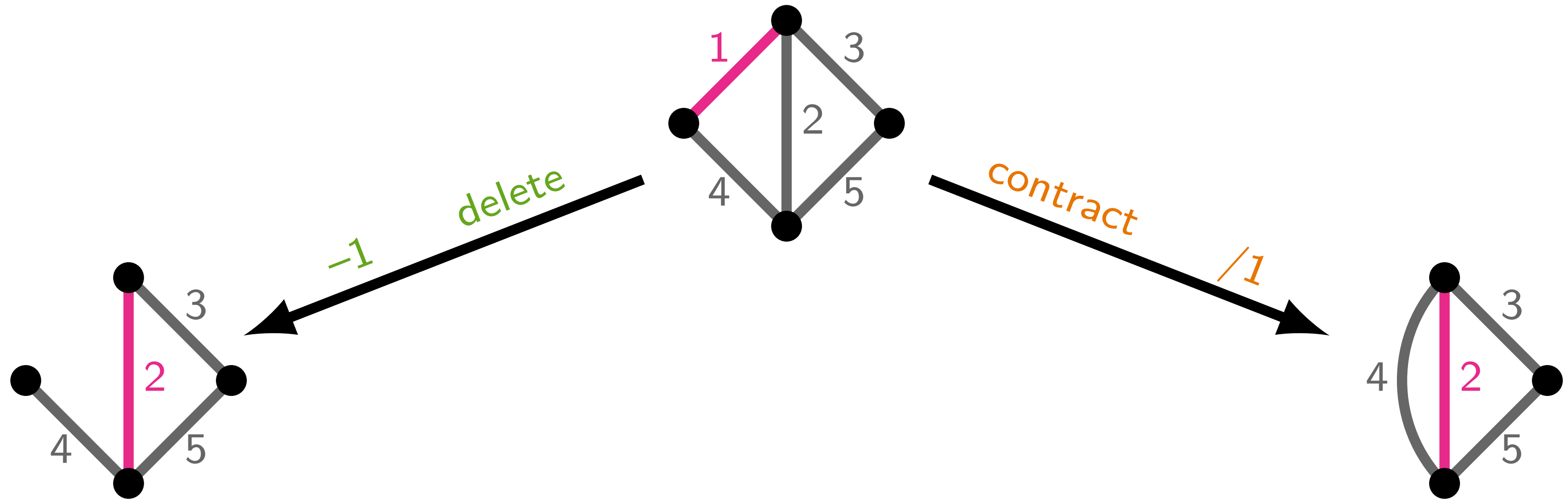
nullity(S) = # **edges** needed to **delete** to get an **acyclic** subgraph.

Tutte's deletion and contraction recurrence

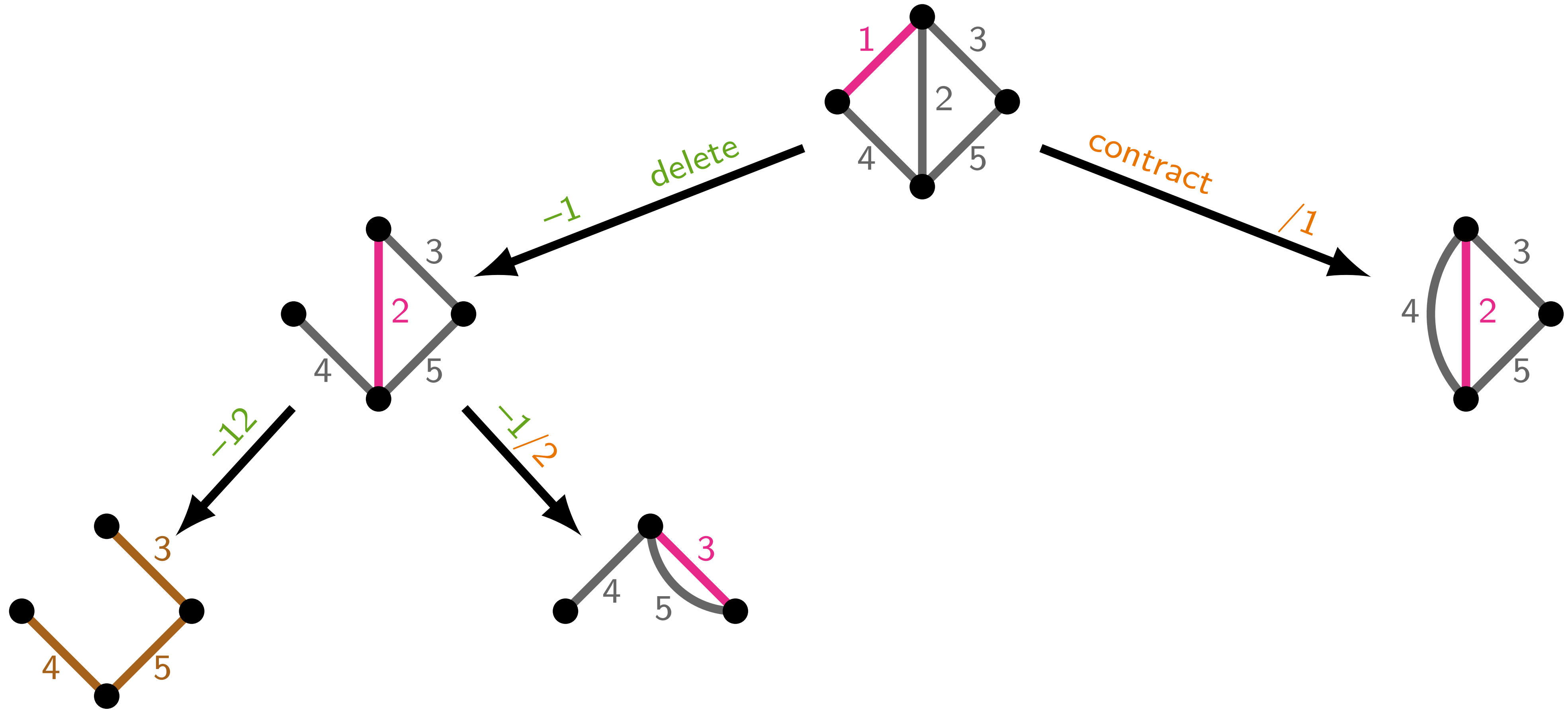
Def. the **Tutte polynomial** of a graph is:

$$T[G](x,y) = \begin{cases} T[G-e] + T[G/e] & \text{if } e \text{ is an} \\ & \text{“ordinary” edge} \\ x^b y^l & \text{if } G \text{ consists of exactly} \\ & b \text{ bridges and } l \text{ self-loops} \end{cases}$$

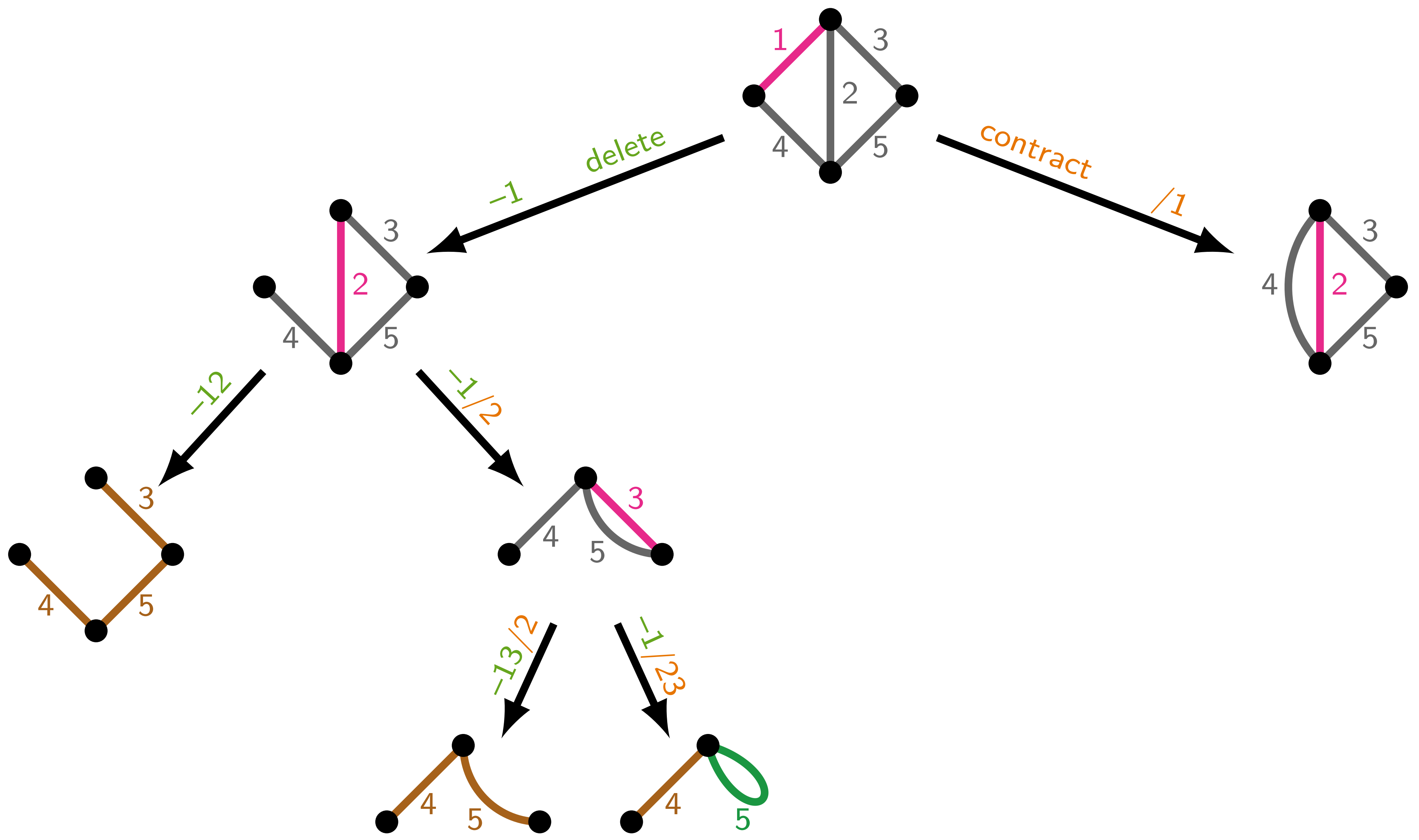

1
3
2
4
5



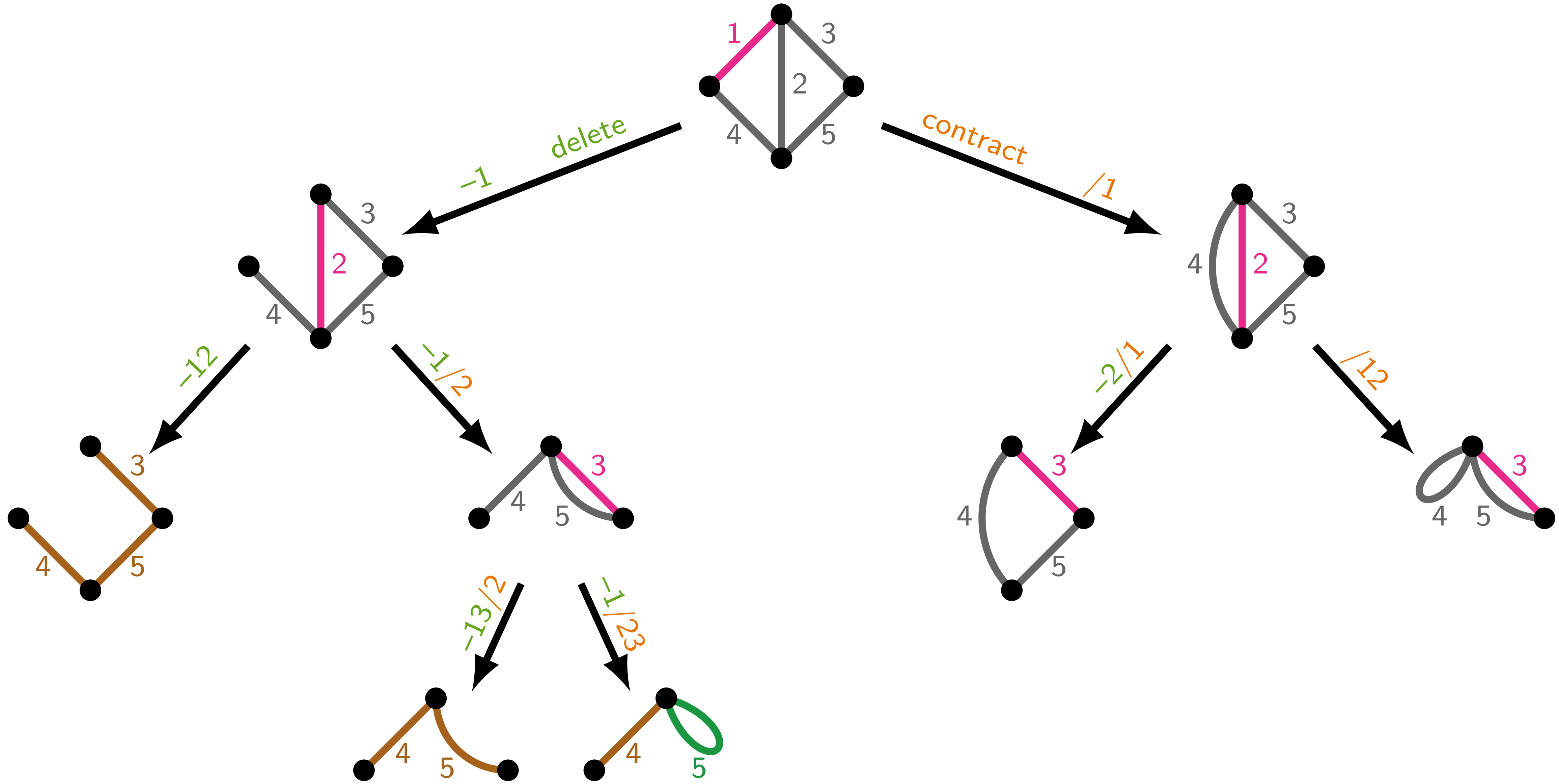
1
3
2
4
5
delete
−1
contract
/1
3
2
4
5
3
4
2
5



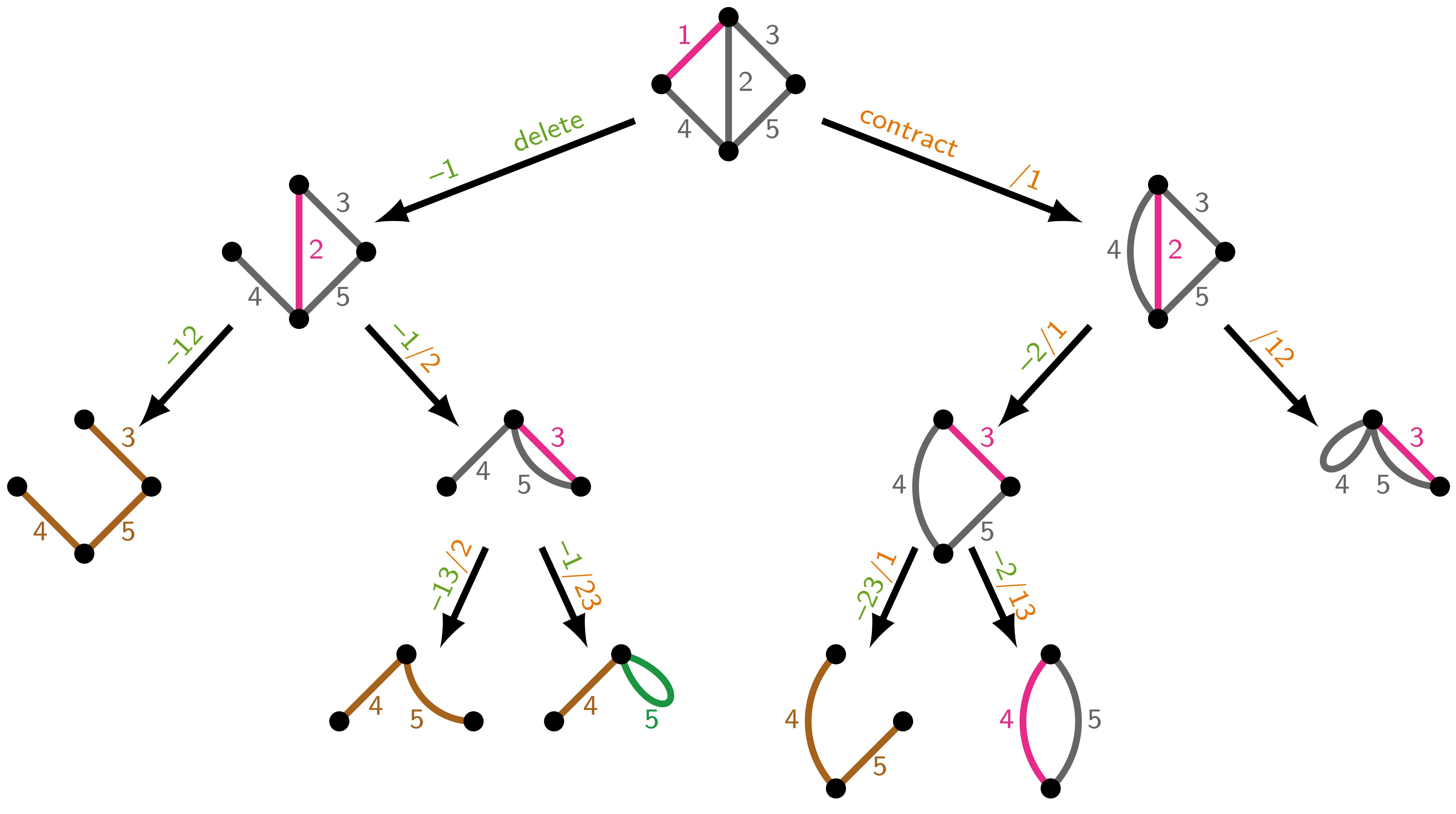
1
3
2
4
5
delete
−1
contract
/1
3
2
4
5
3
4
2
5
−12
−1/2
3
4
5
4
3
5



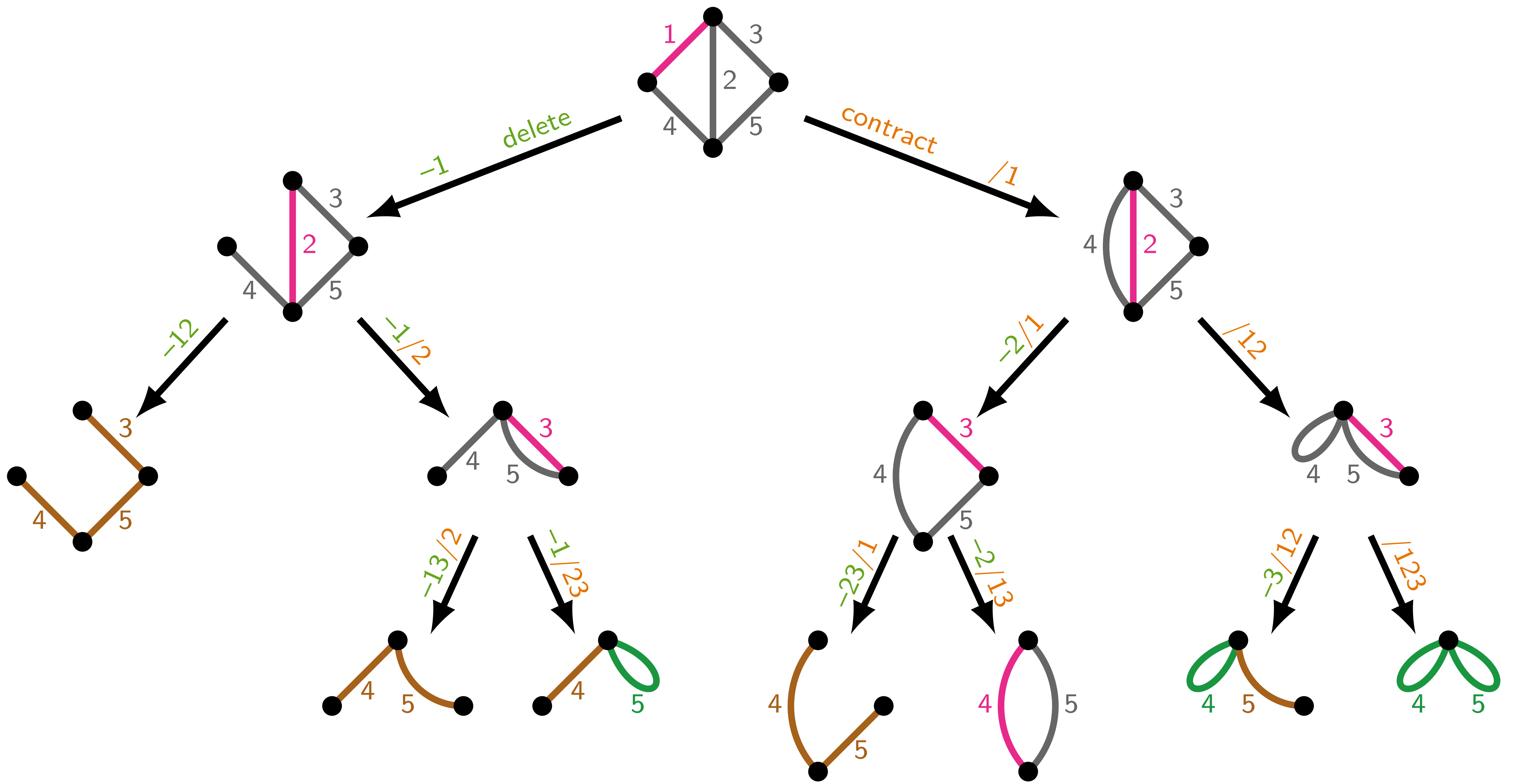
1
3
2
4
5
delete
−1
contract
/1
3
2
4
5
4
3
2
5
−12
−1/2
3
4
5
3
4
5
−13/2
−1/23
4
5
4
5



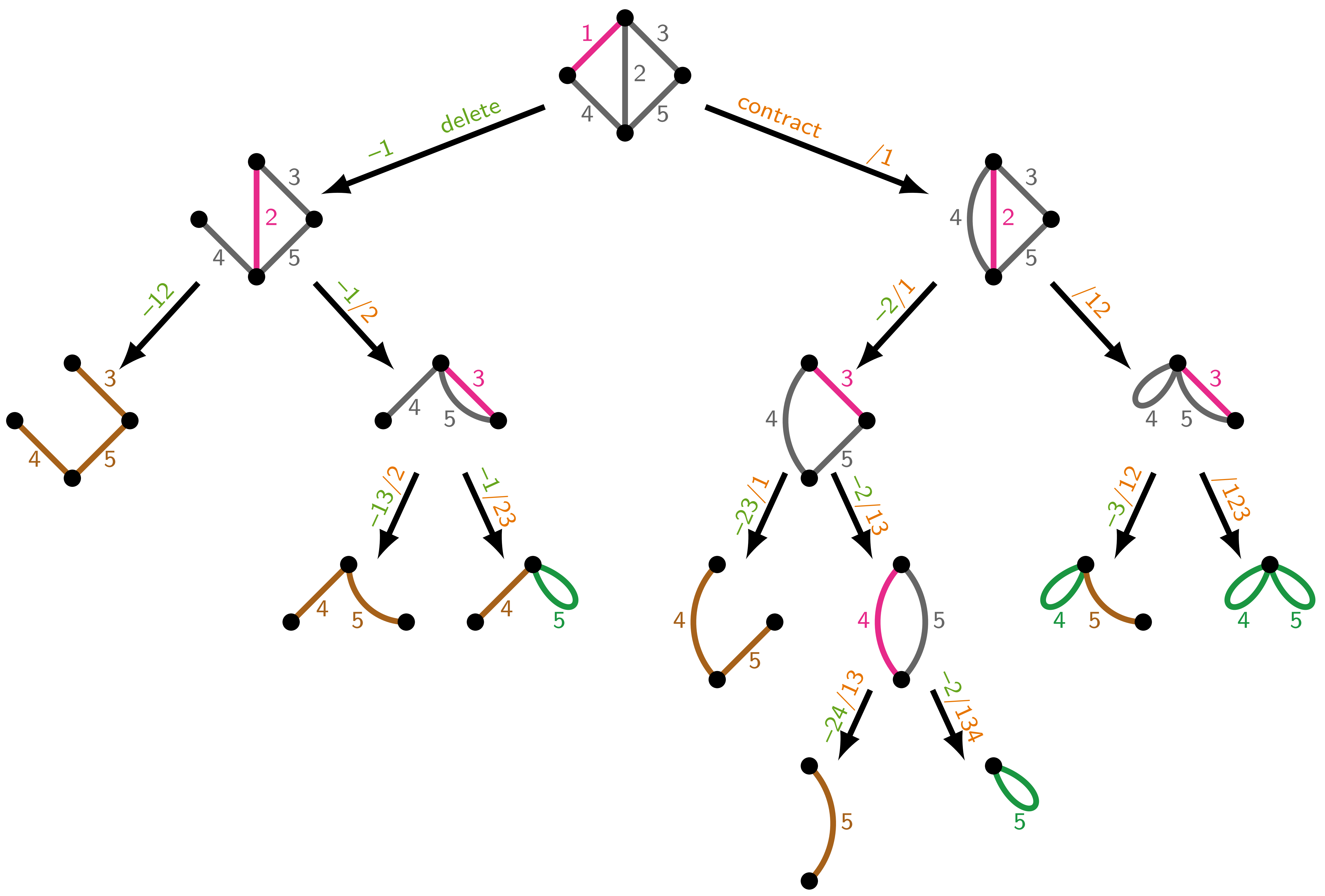
delete
contract
−1
/1
−12
−1/2
−2/1
/12
−13/2
−1/23
1
2
3
4
5



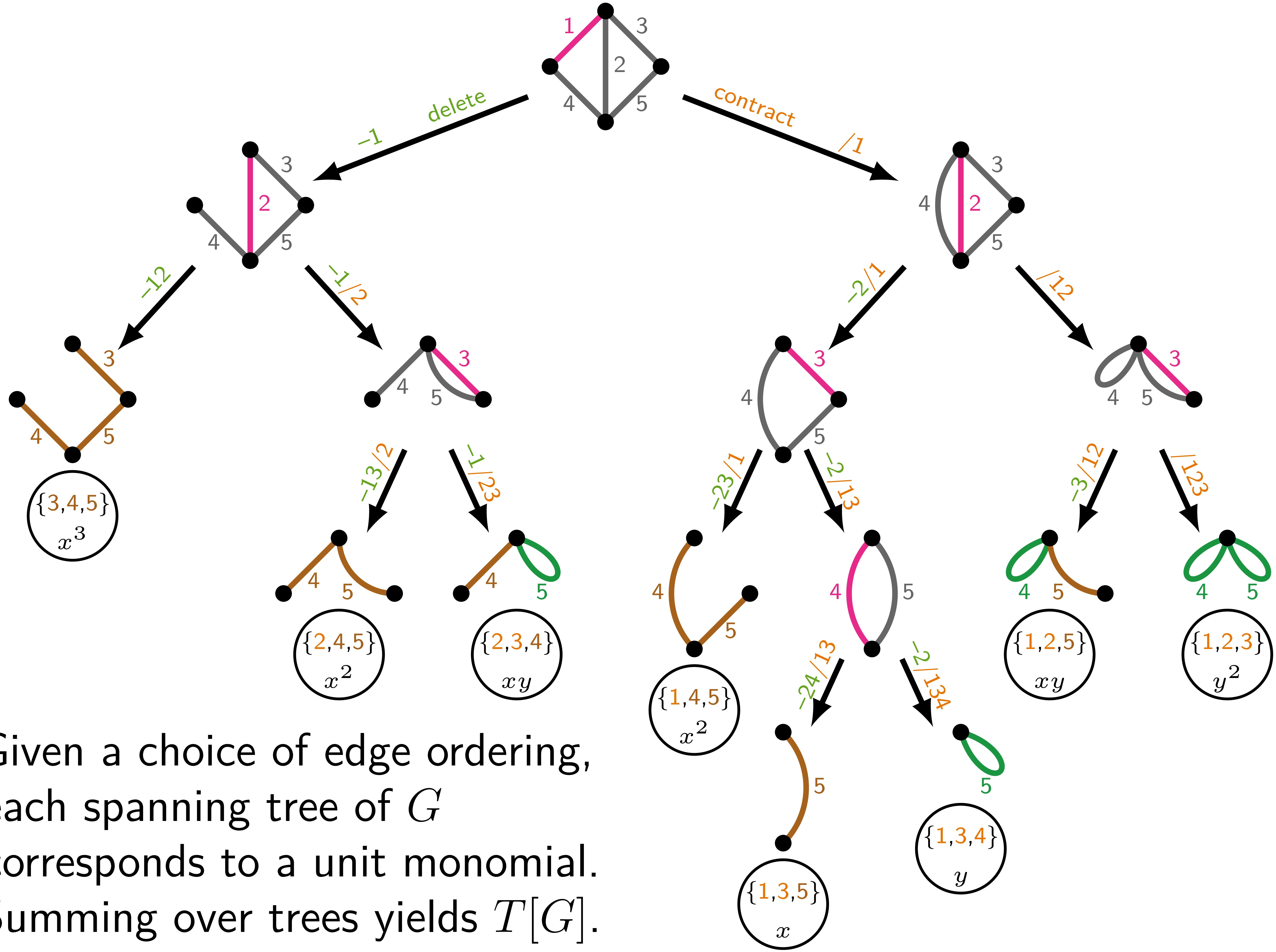
delete
−1
contract
/1
−12
−1/2
−2/1
/12
−13/2
−1/23
−23/1
−2/13
1
2
3
4
5



delete
contract
−1
/1
−12
−1/2
−2/1
/12
−13/2
−1/23
−23/1
−2/13
−3/12
/123

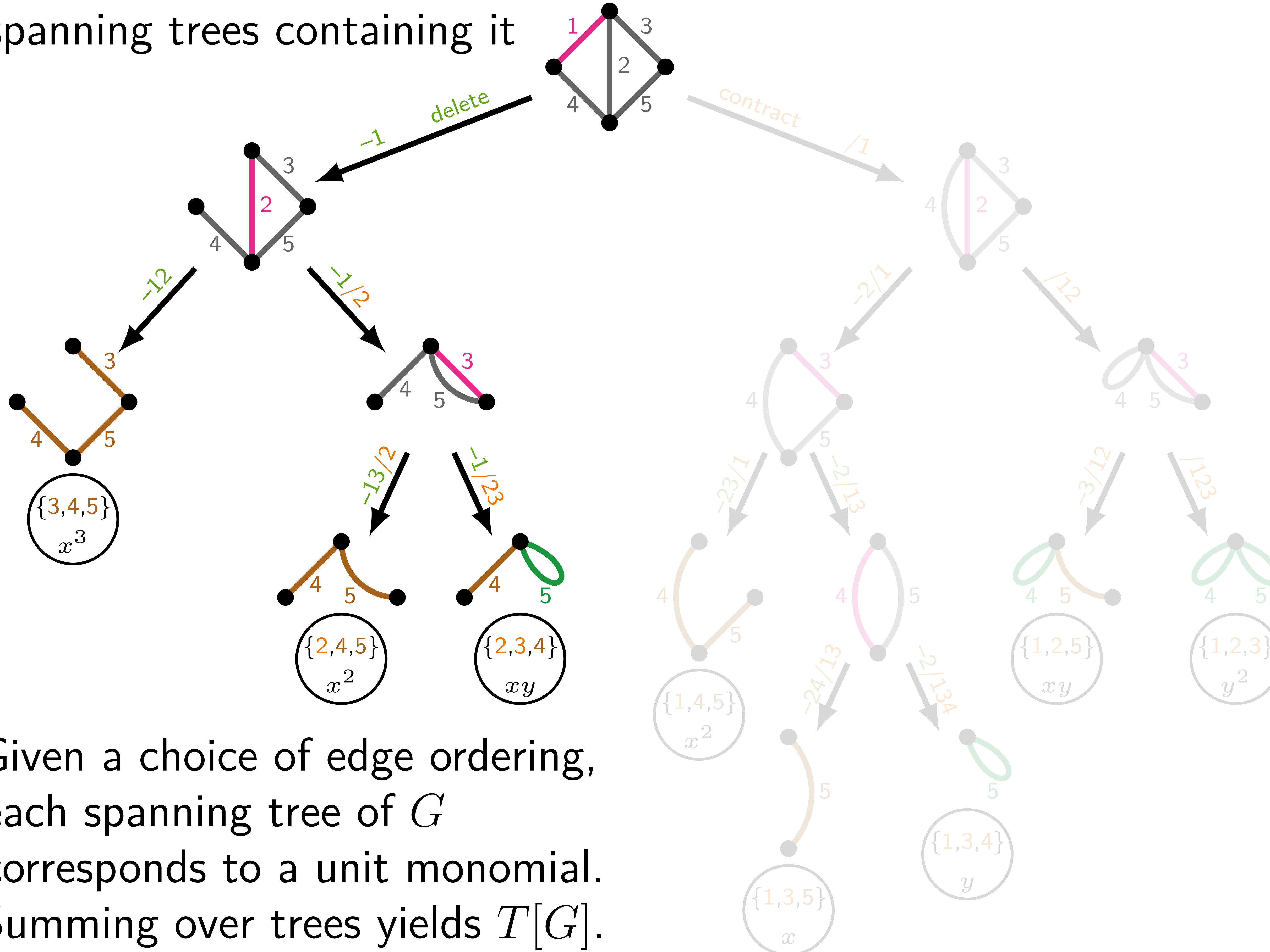


delete
contract
−1
/1
−12
−1/2
−2/1
/12
−13/2
−1/23
−23/1
−2/13
−3/12
/123
−24/13
−2/134



Given a choice of edge ordering, each spanning tree of G corresponds to a unit monomial. Summing over trees yields $T[G]$.

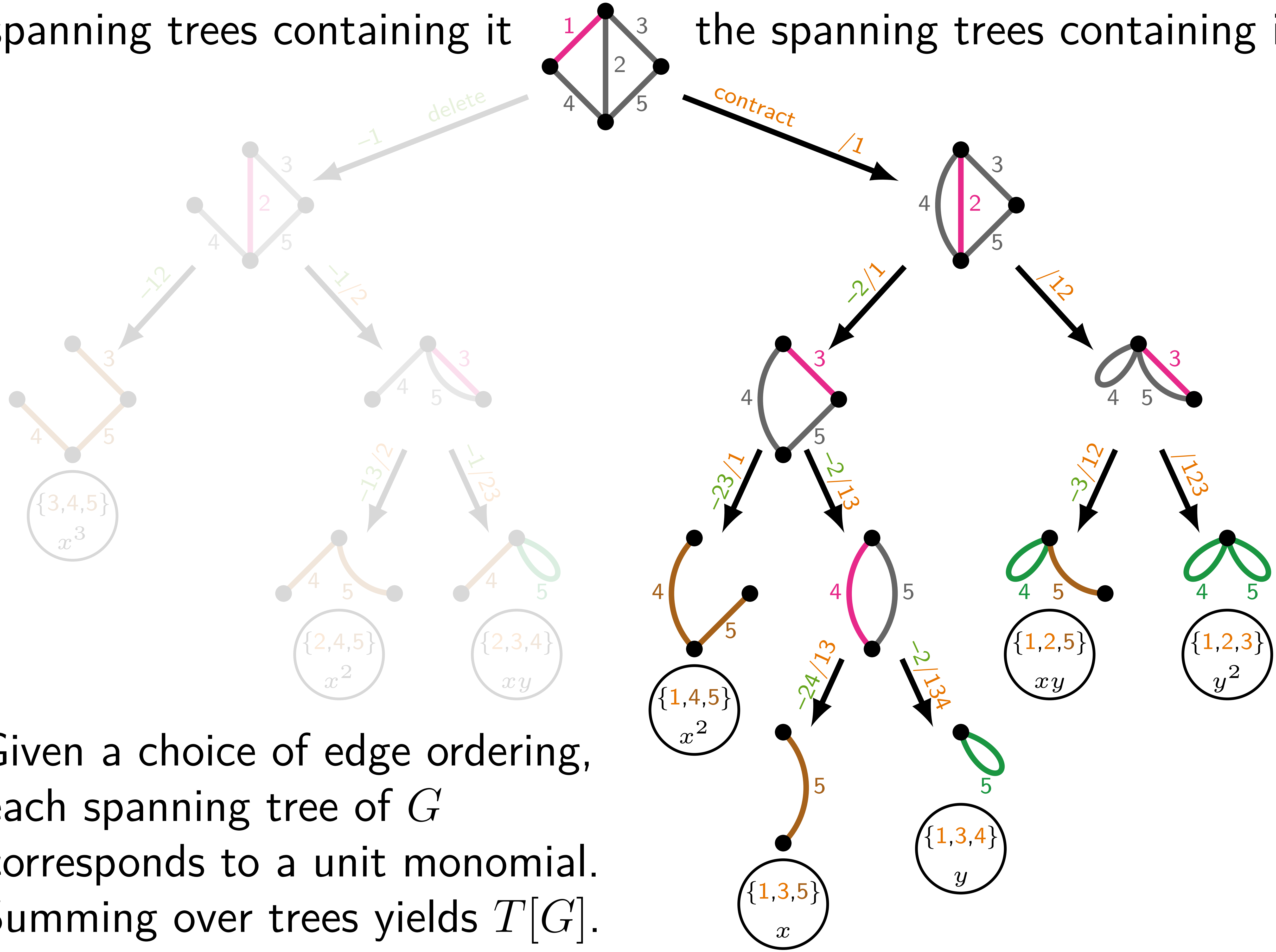
Deleting an edge **removes** the spanning trees containing it



Given a choice of edge ordering, each spanning tree of G corresponds to a unit monomial. Summing over trees yields $T[G]$.

Deleting an edge **removes** the spanning trees containing it

Contracting an edge keeps **only** the spanning trees containing it



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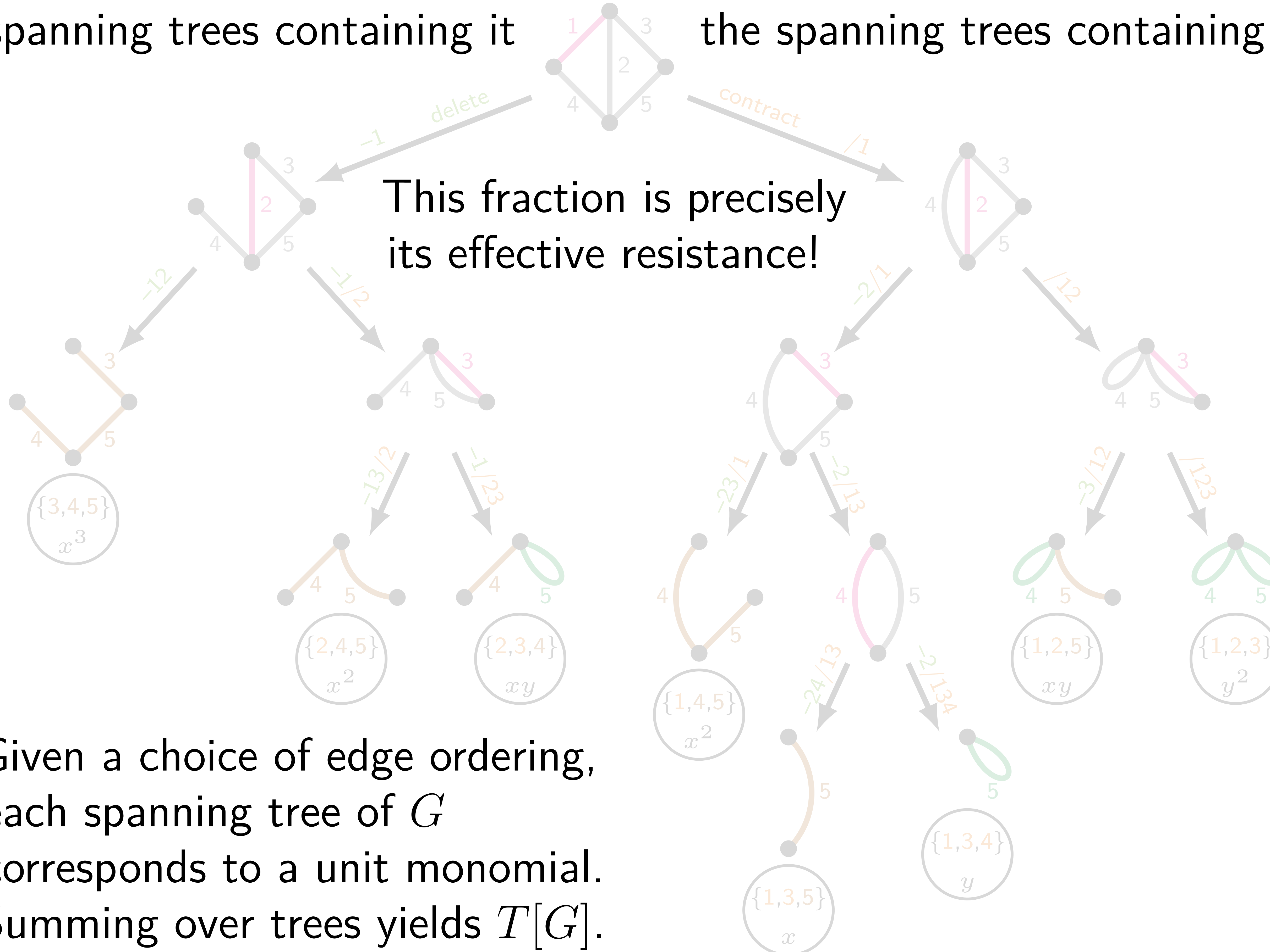
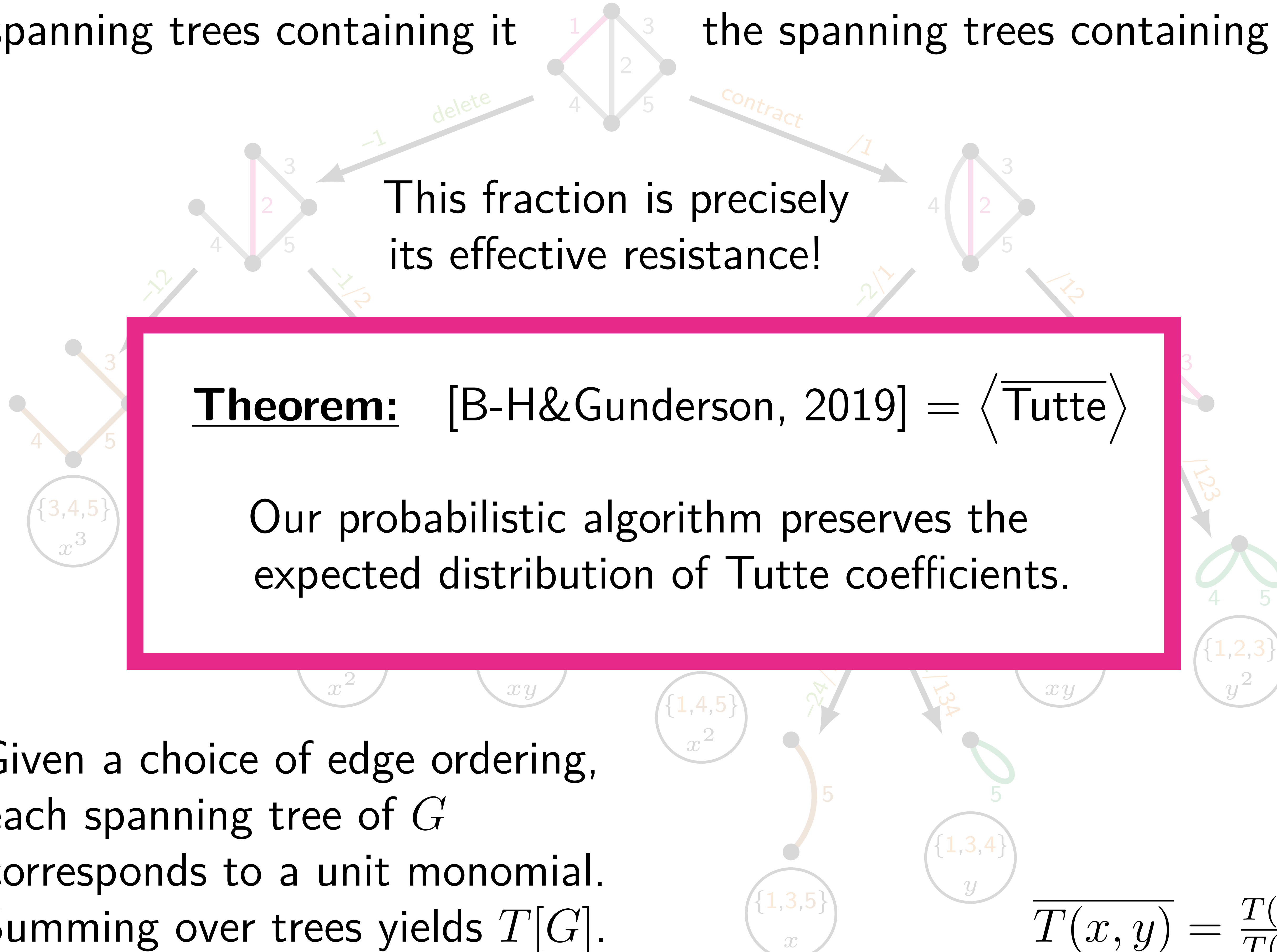


This fraction is precisely its effective resistance!

Given a choice of edge ordering, each spanning tree of G corresponds to a unit monomial. Summing over trees yields $T[G]$.

Deleting an edge **removes** the spanning trees containing it

Contracting an edge keeps **only** the spanning trees containing it



This fraction is precisely its effective resistance!

<u>Theorem:</u> [B-H&Gunderson, 2019] $= \left\langle \overline{\text{Tutte}} \right\rangle$

Our probabilistic algorithm preserves the expected distribution of Tutte coefficients.

Given a choice of edge ordering, each spanning tree of G corresponds to a unit monomial. Summing over trees yields $T[G]$.

$$\overline{T(x,y)} = \frac{T(x,y)}{T(1,1)}$$

Some Platonic Solids

Tetrahedron: $x^3 + 3x^2 + 4xy + 2x + y^3 + 3y^2 + 2y$

Cube: $x^7 + 5x^6 + 15x^5 + 6x^4y + 29x^4 + 24x^3y \cdots$

Octahedron: $y^7 + 5y^6 + 15y^5 + 6y^4x + 29y^4 + 24y^3x \cdots$

$$\boxed{T[G](x,y) = T[G^\star](y,x)}$$

Specializations of the Tutte polynomial

polynomial	**curve of** $T[G](x, y)$
chromatic	x-axis
flow	y-axis
reliability	line $x = 1$
Jones	curve $xy = 1$
q-state Potts model	$(x-1)(y-1) = q$

Flows

Assign direction to every edge in the graph G

A **nowhere-zero k-flow** assigns a flow value $f(e)$ from $\mathbb{Z}_k - \{0\}$ to every edge e of G such that, at every vertex, Kirchoff's current law holds, ie

flow in = flow out

- If G has a nowhere-zero k-flow,
  then G has no bridge edges.

- A bridgeless (ie, two-connected) planar graph G
  has a nowhere-zero k-flow
  iff its dual G^* is k-colorable.

Flow conjectures

Conjecture (Tutte, 1954)
There is a fixed number t such that every two-connected graph has a nowhere-zero t-flow.

Theorem (Jaeger, 1975)
Every two-connected graph has a nowhere-zero 8-flow.

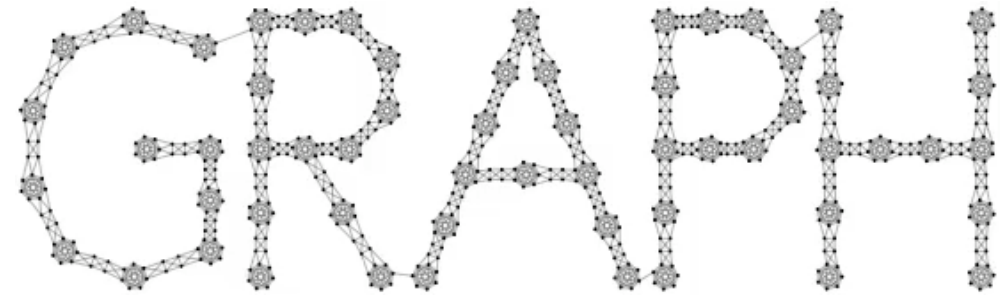
Flow conjectures

Conjecture (Tutte, 1954)
Every two-connected graph has a nowhere-zero 5-flow.

Tutte also conjectured that if, in addition, G doesn't have Petersen graph as minor, then G has a nowhere-zero 4-flow.

Theorem (Seymour, 1981)
Every two-connected graph has a nowhere-zero 6-flow.

Selected references



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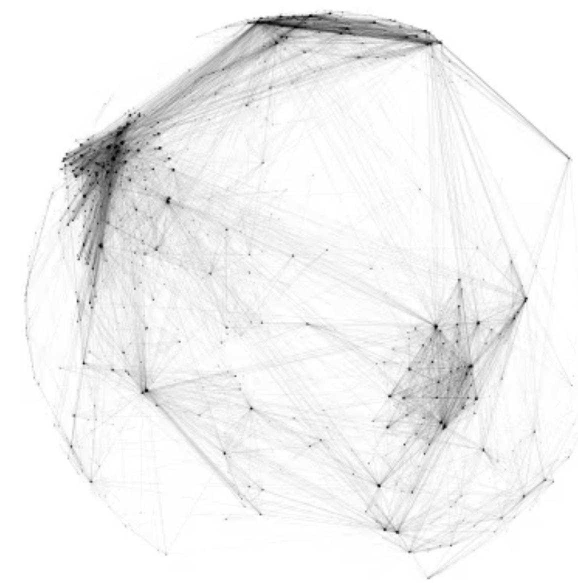
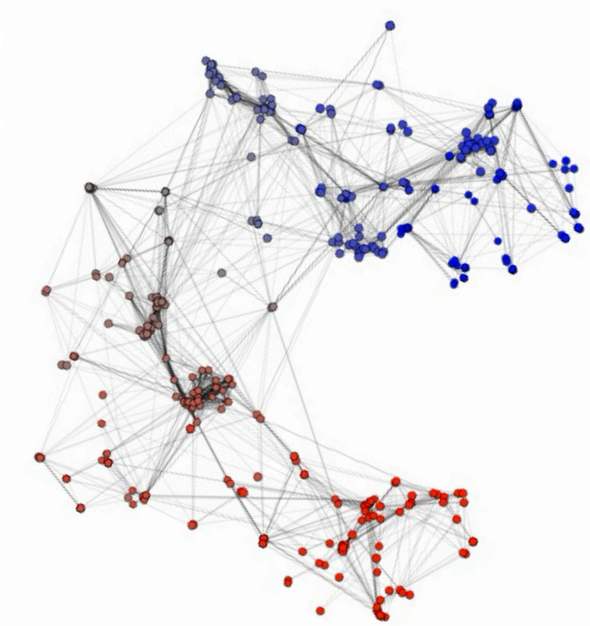




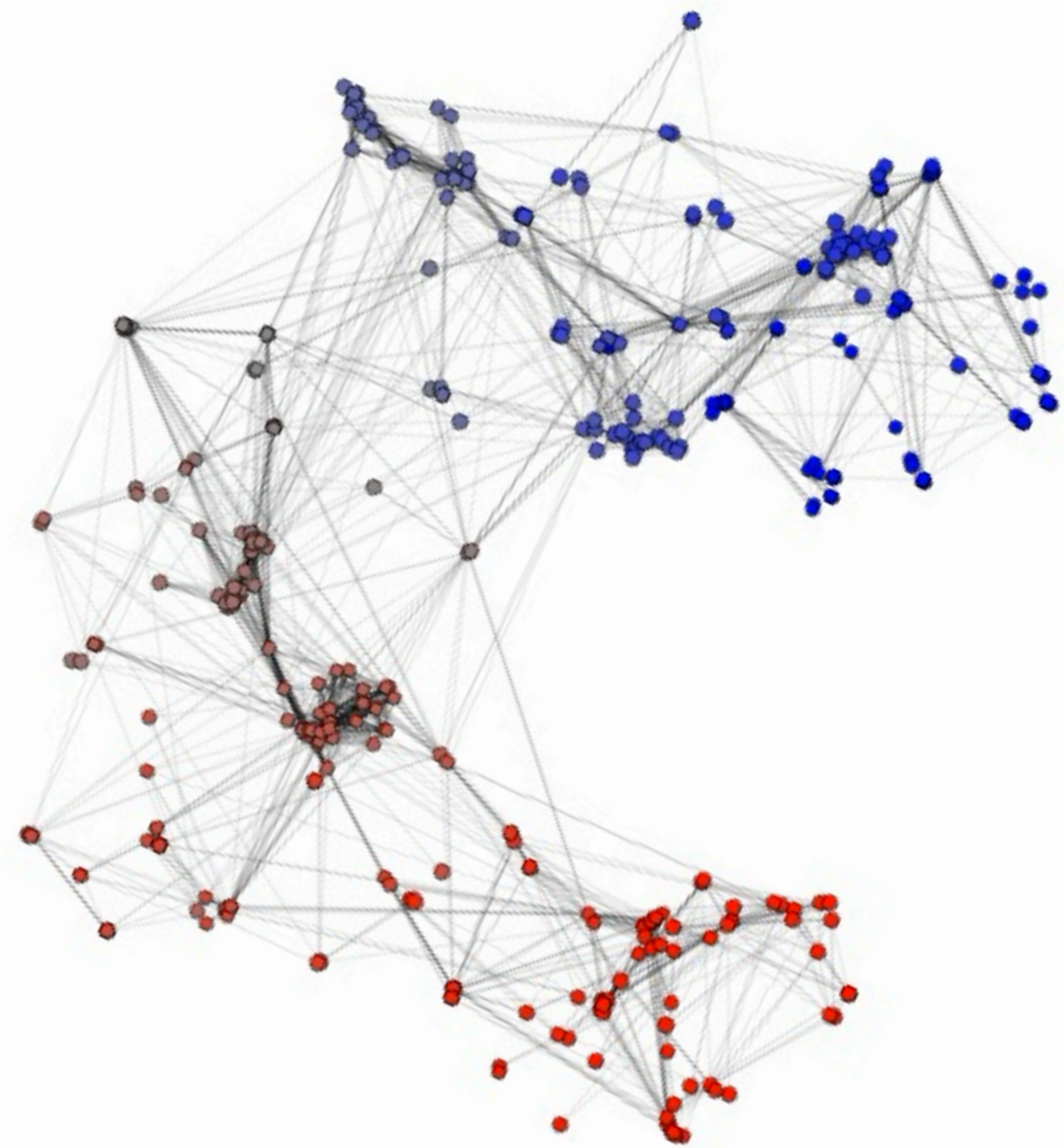
extra slides

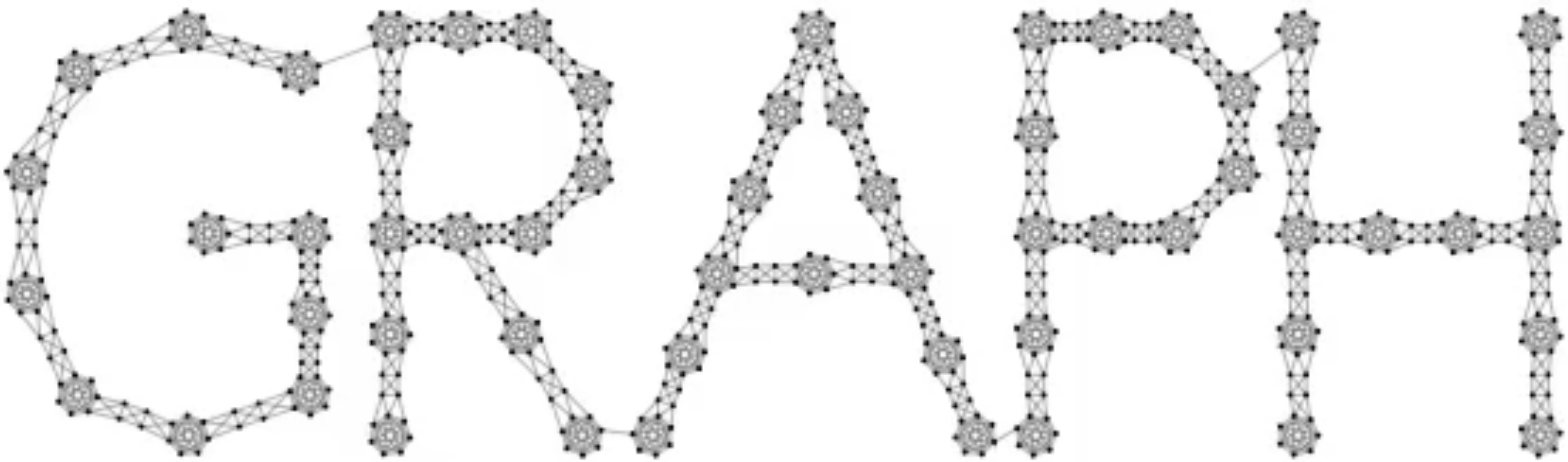
Deletion and contraction recurrence II

$$T[G](x,y) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ xT[G/e] & \text{if } e \text{ is a bridge} \\ yT[G-e] & \text{if } e \text{ is a self-loop} \\ T[G-e] + T[G/e] & \text{if } e \text{ is neither} \end{cases}$$

Merino-Welsh conjecture

$$\mathsf{max}\{T[G](2,0), T[G](0,2)\} \geq T[G](1,1)$$

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