# Quantum Computer Science: Assignment number 3

Macéo Ottavy

## 1 Question 1

Show that there exists orthogonal unit vectors $|v_0\rangle$, $|v_1\rangle$ such that starting with the state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and measuring each qubit in that basis, we obtain different results.

Let $|v_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|v_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.
Then $\|v_0\| = (\frac{1}{\sqrt{(2)}}.|1|)^2 + (\frac{1}{\sqrt{(2)}}.|i|)^2 = \frac{1}{2} + \frac{1}{2} = 1$ and $\|v_1\| = (\frac{1}{\sqrt{(2)}}.|1|)^2 + (\frac{1}{\sqrt{(2)}}.|-i|)^2 = \frac{1}{2} + \frac{1}{2} = 1$.
Moreover, $v_0$ and $v_1$ are linearly independent then we can also check that they create an orthogonal bases.
Let's calculate $|\Phi\rangle$ in our new base.
Notice that $|v_0 v_1\rangle = \frac{1}{2}\left(|00\rangle - i\,|01\rangle + i\,|10\rangle + |11\rangle\right)$ and $|v_1 v_0\rangle = \frac{1}{2}\left(|00\rangle + i\,|01\rangle - i\,|10\rangle + |11\rangle\right)$.
Therefore $\frac{1}{\sqrt{2}}\left(|v_0 v_1\rangle + |v_1 v_0\rangle\right) = \frac{1}{2\sqrt{2}}\left(2\,|00\rangle + 2\,|11\rangle\right) = |\Phi\rangle$

Then, Alice measure the system in base $\{|v_0\rangle, |v_1\rangle\}$.

- If outcome is $|v_0\rangle$ (with prob $\frac{1}{2}$) then the state become $|v_0 v_1\rangle$.
  Therefore, if Bob measure, he get $|v_1\rangle$ (with prob 1).
- If outcome is $|v_1\rangle$ (with prob $\frac{1}{2}$) then the state become $|v_1 v_0\rangle$.
  Therefore, if Bob measure, he get $|v_0\rangle$ (with prob 1).

Therefore, we obtain different measure.
So, there exist an orthogonal base that satisfy the conditions.

## 2 Question 2

Let $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Show that for any orthogonal unit vectors $|v0\rangle$, $|v1\rangle$, we have $\langle v0|\langle v0|\Psi\rangle = 0$ and $\langle v1|\langle v1|\Psi\rangle = 0$. This means that we always obtain different outcomes on the different qubits.

Let $\{|v_0\rangle, |v_1\rangle\}$ be an orthogonal base such that $v_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v_1 = \begin{bmatrix} c \\ d \end{bmatrix}$ with $a, b, c, d \in \mathbb{C}$.

$$\langle v_0|\langle v_0|\Psi\rangle = \langle v_0 \otimes v_0|\Psi\rangle = \begin{bmatrix} a.a \\ a.b \\ b.a \\ b.b \end{bmatrix}^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \overline{a^2} & \overline{a.b} & \overline{b.a} & \overline{b^2} \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(\overline{a.b} - \overline{b.a}) = 0$$

Similarly, $\langle v_1|\langle v_1|\Psi\rangle = 0$.
Then the property is proved.