Quantum Computer Science: Assignment number 4

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1 Question 1

Let $U = \begin{bmatrix} u_{0,0} & u_{0,1} \\ u_{1,0} & u_{1,1} \end{bmatrix}$, write the operators $U[1]$ and $U[2]$ for 2 qubits.

$$U[1] = U \otimes I_2 = \begin{bmatrix} u_{0,0} & 0 & u_{0,1} & 0 \\ 0 & u_{0,0} & 0 & u_{0,1} \\ u_{1,0} & 0 & u_{1,1} & 0 \\ 0 & u_{1,0} & 0 & u_{1,1} \end{bmatrix} \text{ and } U[2] = I_2 \otimes U = \begin{bmatrix} u_{0,0} & u_{0,1} & 0 & 0 \\ u_{1,0} & u_{1,1} & 0 & 0 \\ 0 & 0 & u_{0,0} & u_{0,1} \\ 0 & 0 & u_{1,0} & u_{1,1} \end{bmatrix}$$

Then, write operator $CNOT[3,1]$ for 3 qubits.

We denote by O_4 the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$CNOT[3,1] = CNOT \otimes I_2 \otimes CNOT = CNOT \otimes \begin{bmatrix} CNOT & O_4 \\ O_4 & CNOT \end{bmatrix}$$

$$= \begin{bmatrix} CNOT & O_4 & O_4 & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & CNOT & O_4 & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & CNOT & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & CNOT & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & O_4 & CNOT & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & O_4 & O_4 & CNOT \\ O_4 & O_4 & O_4 & O_4 & CNOT & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & CNOT & 0_4 & O_4 \end{bmatrix}$$

2 Question 2

Decompose Toffoli gate into 1 and 2 qubit gates.

Let $A = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}$ be a 1-qubit gate, then $A^{-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}$.

Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be an other 1-qubit gate, then $B^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$ be a 2-qubits gate.

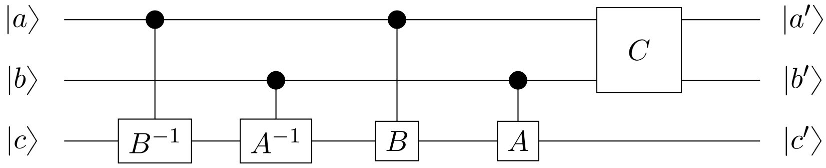

Figure 1: Quantum circuit to simulated Toffoli gate

If $|a\rangle = |0\rangle$ and $|b\rangle = |0\rangle$, then

- $|a'\rangle \otimes |b'\rangle = C. |ab\rangle = |ab\rangle$
- $|c'\rangle = |c\rangle$
- $|(a \wedge b) \oplus c\rangle = |c\rangle = |c'\rangle$

Then $Toffoli. |abc\rangle = |a'b'c'\rangle$

If $|a\rangle = |0\rangle$ and $|b\rangle = |1\rangle$, then

- $|a'\rangle \otimes |b'\rangle = C. |ab\rangle = |ab\rangle$
- $|c'\rangle = B.B^{-1}. |c\rangle = |c\rangle$
- $|(a \wedge b) \oplus c\rangle = |c\rangle = |c'\rangle$

Then $Toffoli. |abc\rangle = |a'b'c'\rangle$

If $|a\rangle = |1\rangle$ and $|b\rangle = |0\rangle$, then

- $|a'\rangle \otimes |b'\rangle = C. |ab\rangle = |ab\rangle$
- $|c'\rangle = A.A^{-1}. |c\rangle = |c\rangle$
- $|(a \wedge b) \oplus c\rangle = |c\rangle = |c'\rangle$

Then $Toffoli. |abc\rangle = |a'b'c'\rangle$

If $|a\rangle = |1\rangle$ and $|b\rangle = |1\rangle$, then
$A^{-1}.B^{-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and $A.B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$. Then $A.B.A^{-1}.B^{-1} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$.

- $|a'\rangle \otimes |b'\rangle = C. |ab\rangle = i. |ab\rangle$
- $|c'\rangle = A.B.A^{-1}.B^{-1}. |c\rangle = -i. |c\rangle$
- $|(a \wedge b) \oplus c\rangle = |1 \oplus c\rangle$

Then $|a'b'c'\rangle = i. |ab\rangle .(-i). |1 \oplus c\rangle = |ab(1 \oplus c)\rangle = Toffoli. |abc\rangle$

Then the circuit describe by the figure is a 1 and 2 qubits gate which simulate Toffoli gate.