MATHEMATICS

TEACHER'S GUIDE 12
GRADE

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FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

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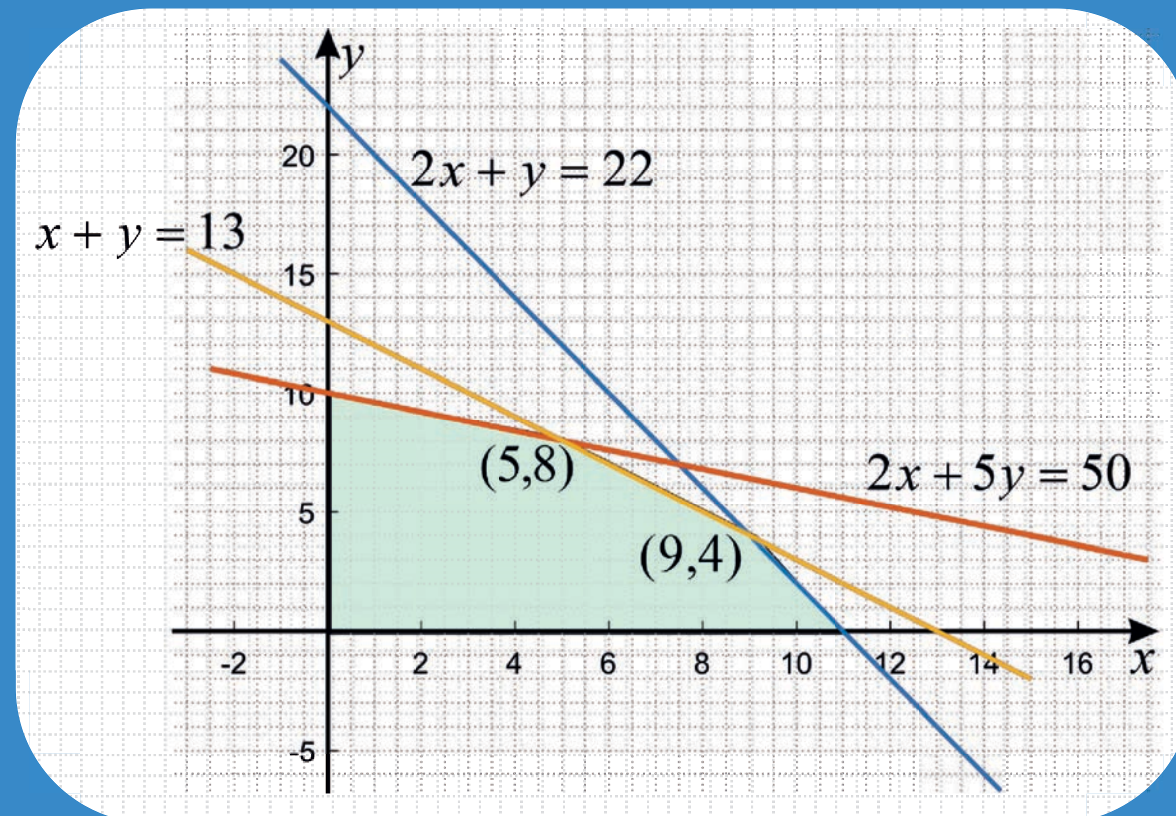

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MINISTRY OF EDUCATION

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TEACHER'S GUIDE GRADE 12

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Foreword

Education and development are closely related endeavors. This is the main reason why it is said that education is the key instrument in Ethiopia's development and social transformation. The fast and globalized world we now live in requires new knowledge, skill and attitude on the part of each individual. It is with this objective in view that the curriculum, which is not only the Blueprint but also a reflection of a country's education system, must be responsive to changing conditions.

It has been almost three decades since Ethiopia launched and implemented new *Education and Training Policy*. Since the 1994 *Education and Training Policy* our country has recorded remarkable progress in terms of access, equity and relevance. Vigorous efforts also have been made, and continue to be made, to improve the quality of education.

To continue this progress, the Ministry of Education has developed a new General Education Curriculum Framework in 2021. The Framework covers all pre-primary, primary, Middle level and secondary level grades and subjects. It aims to reinforce the basic tenets and principles outlined in the *Education and Training Policy*, and provides guidance on the preparation of all subsequent curriculum materials – including this Teacher Guide and the Student Textbook that come with it – to be based on active-learning methods and a competency-based approach.

In the development of this new curriculum, recommendations of the education Road Map studies conducted in 2018 are used as milestones. The new curriculum materials balance the content with students' age, incorporate indigenous knowledge where necessary, use technology for learning and teaching, integrate vocational contents, incorporate the moral education as a subject and incorporate career and technical education as a subject in order to accommodate the diverse needs of learners.

Publication of a new framework, textbooks and teacher guides are by no means the sole solution to improving the quality of education in any country. Continued improvement calls for the efforts of all stakeholders. The teacher's role must become more flexible ranging from lecturer to motivator, guider and facilitator. To assist this, teachers have been given, and will continue to receive, training on the strategies suggested in the Framework and in this teacher guide.

Teachers are urged to read this Guide carefully and to support their students by putting into action the strategies and activities suggested in it. For systemic reform and continuous improvement in the quality of curriculum materials, the Ministry of Education welcomes comments and suggestions which will enable us to undertake further review and refinement.

ADDIS ABABA, ETHIOPIA *FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA*

August 2023 *MINISTRY OF EDUCATION*

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GENERAL INTRODUCTION

According to the Educational and Training Policy of the Federal Democratic Republic of Ethiopia, the second cycle of the secondary education and training will enable your students to choose subjects or areas of training which will prepare them adequately for higher education and for the world of work. The study of mathematics at this cycle, Grades 11 and 12, should contribute to your students' growth into good, balanced and educated individuals and members of society. At this cycle, your students should acquire the necessary mathematical knowledge and develop skills and competencies needed in their further studies, working life, hobbies, and all-round personal development. Moreover, the study of mathematics at this level shall significantly contribute to your students' lifelong learning and self-development throughout their lives. These aims can be realized by closely linking mathematics learning with daily life, relating theorems with practice; paying attention to the practical application of mathematical concepts, methods and procedures by drawing examples from the fields of industry, agriculture and from sciences like physics, chemistry and engineering.

Mathematical study in Grade 12 should be understood as the unity of imparting knowledge, developing abilities and skills and forming convictions, attitudes and habits. Therefore, the didactic-methodical conception has to contribute to all these sides of the educational process and to consider the specifics of students' age, the function of the secondary school level in the present and prospective developmental state of the country, the pre-requisites of the respective secondary school and the guiding principles of the subject mathematics.

Besides learning to think effectively and efficiently, your students come to understand how mathematics deals with their daily and routine lives and with lives of the people at large. Your students are also expected to realize the changing power of mathematics and its national and international significance. To materialize the major goals stated above, encourage your students to apply high-level reasoning, and values to their daily life and to their understanding of the social, economic, and cultural realities of the surrounding context. This will in turn help the students to actively and effectively participate in the wider scope of the development activities of their nation. Your students are highly expected to gain solid knowledge of the fundamental theorems, rules and procedures of mathematics. It is also expected that your students should develop reliable skills for using this knowledge to solve problems independently and in groups.

To this end, the specific objectives of mathematics learning at this cycle are to enable the students to:

- gain solid knowledge on mathematical concepts, theorems, rules and methods.
- appreciate the changing power, dynamism, structure and elegance of mathematics.
- apply mathematics in their daily life.
- understand the essential contributions of mathematics to the fields of engineering, science, agriculture and economics at large.
- work with this knowledge more independently in the field of problem solving.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The traditional teaching-learning paradigm has been replaced by active, participatory and student-centered model. A student-centered classroom atmosphere and approach stimulates student's inquiry. Your role as a teacher in such student-oriented approach would be a mentor who guides the students construct their own knowledge and skills. A primary goal when you teach fundamental basics is for the students to discover the concept by themselves, particularly as you recognize threads and patterns in the data and theorems that they encounter under the teacher's guidance and supervision.

You are also encouraged to motivate your students to develop personal qualities that will help them in real life. For example, encourage students' self-confidence and their confidence in their knowledge, skills and general abilities. Motivate them to express their ideas and observations with courage and confidence. As the students develop personal confidence and feel comfortable on the subject, they would be motivated to address their material to groups and to express themselves and their ideas with strong conviction.

Support students and give them chance to stand before the class and present their opinion, observation and work. Similarly, help the students by creating favorable conditions for them to come together in groups and exchange views and ideas about what they have worked out, investigated and about the material they have read. In this process, the students are given opportunities to openly discuss the knowledge they have acquired and to talk about issues raised in the course of the discussion. Always remember that teamwork is one of the acceptable ways of approach in a student-centered classroom setting.

The students' textbook is "unitized," that is, divided into respective lessons with the principle of "one lesson, one topic" while each one lesson is basically composed of four components: Activity, Definition/Theorem/Note, Examples and Exercises. You are advised to follow the four components of each lesson and provide the required assistance to the students regularly. Here are some concrete suggestions on how to deal with the four components in the textbook:

Activity

This part of the lesson demands the students to revise what they have learnt or activate their background knowledge on the topic. The activity also introduces them to what they are going to learn in the lessons to follow. Use this part as an effective introduction to a new topic. However, you should refrain from spending too much time on this Activity part. Your introduction should be short and brief.

Definition/Theorem/Note

This part presents and explains new concepts to the students. Since definitions and theorems tend to be highly abstract, you should try to give concrete examples or explanations to the students to help them understand the concepts. Viewed from that viewpoint, Examples that immediately follow the definitions and theorems are very important to facilitate the students' understanding. Note, however, that every lesson may not begin with definition or theorem, especially when the lesson is a continuation of the previous one.

Example and Solution

Here, you will give the students specific examples to facilitate their understanding of the new content. Examples are always arranged from a simple, basic one to slightly advanced ones. To introduce a new definition or theorem, Example 1 is always the crucial one for the students to grasp the new concept. Therefore, you should follow the order of them. Since Example 1 and the first item of Exercise are mutually corresponding, it is effective and advised to give Exercise Item 1 immediately after Example 1. In your lessons, try to give plain and clear explanations on how to solve the examples. Always indicate the correct solution to the class. If your teaching time is not enough, you can only explain Example 1 and give other examples as homework.

Exercise

A few exercise problems are given at the end of each lesson. Among them, the first item is particularly important. It is designated as the "evaluation item" to check if the students have successfully understood the topic. Always give the first item of the Exercise after explaining Example 1 or at the end of one lesson. Give time for the students to solve the problem individually. While they are working, you may grasp the understanding of the students, and give support for those who have difficulty. At the end of the lesson, always provide the correct solutions in the whole class. This process is part of the formative assessment in the class. If you do not have enough time to cover all the exercise items, you can give other appropriate items as homework. But when you give some items as homework, you should provide their answers in the next lesson.

UNIT 1: SEQUENCE AND SERIES

Periods Allotted: 26 Periods

Introduction

In this unit, the first major outcome is to enable students understand the notion of sequences and series. The second one is to enable students to solve practical and real-life problems.

In sequences and series, students will learn how to make predictions, decisions and generalizations from given patterns. Some of the sub-topics of the unit begin with opening problems and activities related to the content of the topic.

In teaching sequences and series, select the methods which give more time for active students' participation and less time for the lecture format.

Unit Outcomes

After completing this unit, students will be able to:

- Revise the notions of sets and functions.
- Grasp the concept of sequence and series.
- Compute any terms of sequences from given rule.
- Find out possible rules (formula) from given terms.
- Identify the types of sequences and series.
- Compute partial and infinite sums of sequences.
- Apply the knowledge of sequence and series to solve practical and real life problems.

Suggested Teaching Aids

Students learn in a variety of different ways. Some are visually oriented and more inclined to acquire information from photographs or videos. Others do best when they hear instructions rather than read them. Teachers use teaching aids to provide these different ways of learning. Therefore, it is recommended that you may use models, charts, calculators, logarithmic tables and computers for this unit. You can use also other types of materials as long as they help the learners to get the skills required.

Teaching Notes

Under each sub-topic, a hint is given how to continue each sub-topic but your creativity is crucial. The purpose of the teaching notes is to provide the teacher information to use activities, opening problems and group-works to motivate and guide students rather than lecturing. This unit begins with an opening problem which may motivate students to follow the unit attentively. Therefore, before moving on to any subtopic of this unit insist that students discuss the opening problem.

1.1 Sequence

Answer to Activity 1.1

1. 5^{th} floor=76 people and 6^{th} floor=90 people

2. $a)$ 38, 35 $\quad$ $b)$ 42, 52

Answer to Exercise 1.1

1.

a) $a_n = 2n$:

$$a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10$$

b) $a_n = \left(\dfrac{1}{2}\right)^{n-1}$:

$$a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, a_4 = \frac{1}{8},\ a_5 = \frac{1}{16}$$

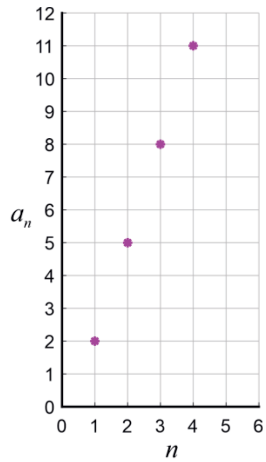
c) $a_n = \dfrac{n}{n+1}$:

$$a_2 = \frac{1}{2},\ a_2 = \frac{2}{3},\ a_3 = \frac{3}{4},\ a_4 = \frac{4}{5},\ a_5 = \frac{5}{6}$$

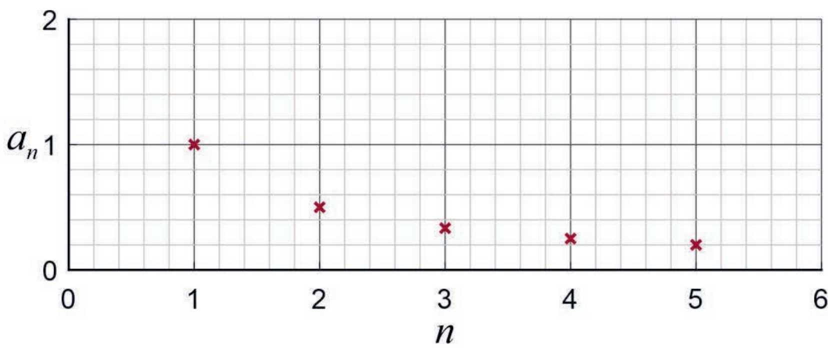
d) $a_n = n^3$

$$a_1 = 1 \quad a_2 = 8, \quad a_3 = 27, \quad a_4 = 64,\ a_5 = 125$$

2. *a)*



b)



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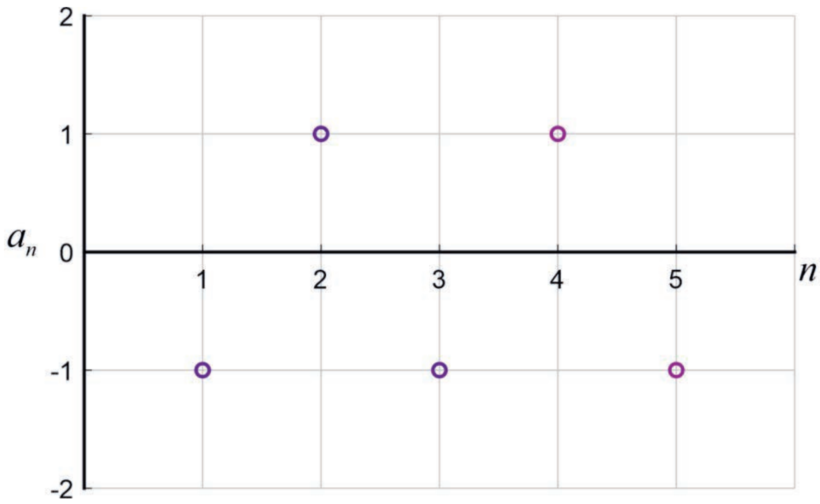


3. Grade four, five, six and seven contains 370, 450, 530 and 610 students respectively.

Answer to Activity 1.2

a) 104 *b*) 104

c) The area of the outer rectangle = the sum of the area of all inner squares

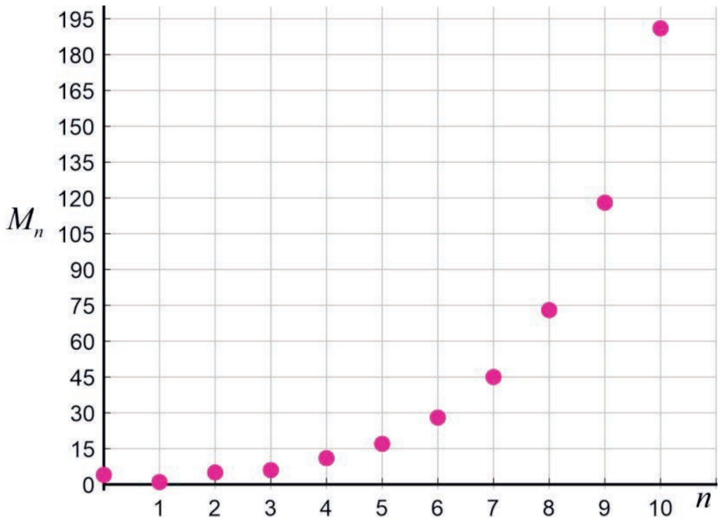
Answer to Exercise 1.2

1. Twelfth term = 55+89 = 144

2.

n	0	1	2	3	4	5	6	7	8	9	10
M_n	4	1	5	6	11	17	28	45	73	118	191

The graph of the Mulatu sequence is indicated as follows.



1.2 Arithmetic and Geometric Sequences

Answer to Activity 1.3

a. 4 b. -4 c. $\dfrac{1}{3}$

Answer to Exercise 1.3

1. a. $a_1 = 3$, $a_3 = 7$, $d = 2$, $a_6 = 13$
   b. $a_1 = 9$, $a_3 = 3$, $d = -3$, $a_6 = -6$

2. a. $6, 11, 16, 21, 26$ b. $\dfrac{3}{2}, 1, \dfrac{1}{2}, 0$

Answer to Exercise 1.4

1. a.

$$A_1 = 2, \quad d = 3.$$
$$A_n = A_1 + (n-1)d$$
$$A_n = 2 + 3(n-1)$$
$$A_n = 3n - 1$$

b.

$$A_1 = 10, \quad d = -5.$$
$$A_n = A_1 + (n-1)d$$
$$A_n = 10 + (-5)(n-1)$$
$$A_n = -5n + 15$$

2. $A_1 = 10, \quad d = -4$

$$A_{10} = A_1 + (10-1)(-4)$$
$$A_{10} = -26.$$

Answer to Exercise 1.5

1.

a. $A_4 = 15, \quad A_8 = 27.$

$$A_4 = A_1 + 3d, \qquad A_8 = A_1 + 7d$$
$$A_1 + 3d = 15$$
$$A_1 + 7d = 27$$
$$d = 3$$
$$A_1 = 6$$
$$\therefore A_n = 3n + 3$$

b. $A_5 = 20, \quad A_{10} = 0.$

$$A_5 = A_1 + 4d, \quad A_{10} = A_1 + 9d$$
$$A_1 + 4d = 20$$
$$A_1 + 9d = 0$$
$$d = -4$$
$$A_1 = 36$$
$$\therefore A_n = 40 - 4n$$

2.

$$A_1 + d = 3$$
$$A_1 + 4d = 24$$
$$d = 7 \;\; \& \;\; A_1 = -4$$
$$A_n = 7n - 11$$
$$A_{11} = 77 - 11 = 66$$

3. a. $a_n = 7_n - 3$

$$a_{n+1} - a_n = 7(n+1) - 3 - (7n-3) = 7$$

The difference between any two consecutive terms is constant. Hence, it is an arithmetic sequence.

b. $a_n = 3 - 5n$

$$a_{n+1} - a_n = 3 - 5(n+1) - (3 - 5n) = -5$$

Hence, it is an arithmetic sequence.

c. a_n is not an arithmetic sequence. d. a_n is not an arithmetic sequence.

Answer to Exercise 1.6

1.

$$x = \frac{3+7}{2} = 5$$

2.

$$\frac{1}{x} - \frac{1}{12} = \frac{1}{6} - \frac{1}{x}$$
$$x = 8$$

3. Arithmetic mean

$$= \frac{4+14}{2} = 9$$

4. When four arithmetic means are inserted between 4 and 14, then the common difference $d = \frac{14-4}{5} = 2$. Therefore, the arithmetic means are $6,\ 8,\ 10,\ 12$.

Answer to Activity 1.4

a. $r = 3,$ *b.* $r = -1$ *c.* $r = \frac{-1}{2}$

Answer to Exercise 1.7

1. 1^{st} term = 1,

   common ratio = 2.

   6^{th} term=32.

2. 3, 9, 27, 81, 243.

Answer to Exercise 1.8

1.

a) $G_1 = 2 \quad , \ r = 5$

$G_n = 2(5)^{n-1}$

b) $G_1 = 1 \quad , \ r = -3$

$G_n = (-3)^{n-1}$

c) $G_1 = 2 \quad , \ r = -2$

$G_n = 2(-2)^{n-1}$

d) $G_1 = -3 \quad , \ r = \frac{1}{2}$

$G_n = -3(\frac{1}{2})^{n-1}$

2.

a) $3, 6, 12, 24, \ldots$

$G_n = 3(2)^{n-1}$

b) $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$

$G_n = \frac{3}{2}(\frac{1}{2})^{n-1}$

c) $27, 9, 3, 1, \ldots$

$G_n = 27(\frac{1}{3})^{n-1}$

3. $G_1 = -3 \quad \& \ r = \frac{-1}{2}$

$$G_5 = -3\left(\frac{-1}{2}\right)^4 = \frac{-3}{16}$$

Answer to Exercise 1.9

1. $GM = \pm\sqrt{36} = \pm 6$

2.

$G_2 = 12 \quad \& \quad G_6 = 192$

$$\frac{G_6}{G_2} = \frac{192}{12}$$

$$\frac{r^5 G_1}{r G_1} = \frac{192}{12} \Rightarrow r^4 = 16$$

$r = 2$

$$\therefore G_{11} = (2)^{10}(6) = 1024(6) = 6144$$

3.

$$x, 4x+3, 7x+6$$
$$\frac{x}{4x+3} = \frac{4x+3}{7x+6}$$

$$x^2 + 2x + 1 = 0$$
$$x = -1$$

4. Let $\frac{a}{r}, a, ar$ be three numbers in geometric sequence.

$$\therefore \frac{a}{r} \times a \times ar = 1000$$
$$a^3 = 1000$$
$$a = 10$$
$$a\left[\frac{1}{r} + 1 + r\right] = 35$$
$$\frac{1}{r} + 1 + r = \frac{35}{a} = \frac{35}{10}$$
$$2(1 + r + r^2) = 7r$$
$$2r^2 - 5r + 2 = 0$$
$$(r-2)(2r-1) = 0$$
$$r = 2 \ or \ 2r = 1$$
$$if \ a = 10, \ r = 2$$
$$r = 2, r = \frac{1}{2}$$

for $r = 2, \frac{a}{r} = \frac{10}{2} = 5 . ar = 10 \times 2 = 20$

The numbers are 5, 10, 20

for $\quad r = \frac{1}{2}, \frac{a}{r} = \frac{10}{1/2} = 20, \ ar = 10 \times \frac{1}{2} = 5$

Thus, the three numbers are $20, 10, 5$

1.3 The Sigma Notation and Partial Sums

Answer to Exercise 1.10

1.

a) $S_5 = 1+3+5+7+9 = 25$

b) $S_{10} = 1+3+5+7+9+11+13+15+17+19 = 100$

2.

a) $a_n = 4n-3 \quad , S_5 = 45$

b) $a_n = 3-5n \quad , S_8 = -2-7-12-17-22-27-32-37 = -156$

c) $a_n = n^2+1 \quad , S_6$

$$S_n = \sum_{k=1}^{n}(k^2+1) \ ,$$

$$S_6 = 2+5+10+17+26+37 = 97$$

3. $S_n = \sum_{k=1}^{n}\frac{2}{k^2+5k+6}$

Answer to Exercise 1.11

1. In order to solve such types of problems, students should be able to recall the properties of sigma notation.

a) $\sum_{k=1}^{6} 2k = 2(1+2+3+4+5+6) = 42$

b) $\sum_{k=3}^{5} k^2 = 3^2+4^2+5^2 = 50$

c) $\sum_{k=1}^{n} 3^k = 3+3^2+3^3+\ldots+3^n$

d) $\sum_{k=3}^{5} k^3 = 3^3+4^3+5^3 = 216$

2.

$$2^2+4^2+6^2+8^2+10^2+12^2 = \sum_{k=1}^{6}(2k)^2$$

Answer to Exercise 1.12

1.

$$a)\ \sum_{k=1}^{4} 5k = 5(1+2+3+4) = 50$$

$$b)\ \sum_{k=1}^{5} (4k-1) = 3+7+11+15+19 = 55$$

$$c)\ \sum_{k=3}^{6} (k^2-4) = 5+12+21+32 = 70$$

$$d)\ \sum_{k=2}^{5} 3 = 3(4) = 12$$

$$e)\ \sum_{k=1}^{5} (k^3+2k^2-3k+5) = \sum_{k=1}^{5} k^3 + \sum_{k=1}^{5} 2k^2 - \sum_{k=1}^{5} 3k + \sum_{k=1}^{5} 5$$

$$\sum_{k=1}^{5} (k^3+2k^2-3k+5) = \sum_{k=1}^{5} k^3 + 2\sum_{k=1}^{5} k^2 - 3\sum_{k=1}^{5} k + \sum_{k=1}^{5} 5$$

$$\sum_{k=1}^{5} (k^3+2k^2-3k+5) = (1+8+27+64+125) + 2(1+4+9+16+25)$$

$$-3(1+2+3+4+5) + 5(5)$$

$$\sum_{k=1}^{5} (k^3+2k^2-3k+5) = 225+110-45+25 = 315$$

Answer to Activity 1.5

1. $S = 5+15+25+35+45+55+65+75+85+95 = 500$

Answer to Exercise 1.13

1.

$$a)\ S_{30} = 15(1+30) = 465$$

$$b)\ S_{99} = \frac{99}{2}(1+99) = 4950$$

$$c)\ S_{200} = 100(1+200) = 20100$$

2.

$$S_n = \frac{n}{2}(1+n)$$

$$3240 = \frac{n}{2}(1+n)$$

$$6480 = n^2 + n$$

$$n^2 + n - 6480 = 0$$

$$n = 80$$

Answer to Exercise 1.14

1.

(a) $S_{10} = \frac{10}{2}\left(2+21\right) = 115$

(b) $S_{26} = \frac{26}{2}\left(40+0\right) = 520$

2.

(a) $S_{10} = \frac{10}{2}\left(2(2)+(10-1)(3)\right) = 155$

(b) $S_{12} = \frac{12}{2}\left(60-55\right) = 30$

Answer to Exercise 1.15

1. $S_5 = 25$

2. $S_8 = 4\left(A_1 + A_8\right) \Rightarrow A_8 = 29$

3. Let S_1 denotes the sum of integers from 1 to 100 which are divisible by 2.

$$\therefore S_1 = 2+4+6+\ldots+100$$

$$\left(\text{an A.P with } a = 2, l = 100, n = \frac{100}{2} = 50\right)$$

$$= \frac{n}{2}(a+l) = \frac{50}{2}(2+100) = 25\times 102 = 2550$$

Let S_2 denote the sum of integers from 1 to 100 which are divisible by 5.

$$\therefore S_2 = 5 + 10 + 15 + \ldots + 100$$

$$\left(\text{an A.P with } a = 5, l = 100, n = \frac{100}{5} = 20\right)$$

$$= \frac{n}{2}(a + l) = \frac{20}{2}(5 + 100) = 10 \times 105 = 1050.$$

The lowest common multiple of 2 and 5 is 10. Multiples of 10 occurs in S_1 as well as S_2. Let S_3 denote the sum of integers divisible by both 2 and 5 i.e. by 10.

$$S_3 = 10 + 20 + 30 + \ldots + 100$$

$$\left(\text{an arthimetic progression with } a = 10, l = 100, n = \frac{100}{10} = 10\right)$$

$$= \frac{n}{2}(a + l) = \frac{10}{2}(10 + 100) = 5 \times 110 = 550$$

$$\therefore \text{Required sum} = S_1 + S_2 - S_3$$

$$= 2550 + 1050 - 550 = 3600 - 550 = 3050$$

4. The odd integers from 1 to 2001 are 1, 3, 5, …1999, 2001. This sequence forms an arithmetic progression. Here, first term, $a = 1$, common difference, $d = 2$

$$a + (n - 1)d = 2001$$

$$\Rightarrow 1 + (n - 1)(2) = 2001$$

$$\Rightarrow 2n - 2 = 2000$$

$$\Rightarrow n = 1001$$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\text{Hence, } S_n = \frac{1001}{2}[2 + 1(1000) \times 2] = \frac{1001}{2}[2 + 1000 \times 2]$$

$$= \frac{1001}{2} \times 2002 = 1001 \times 1001 = 1002001$$

Thus, the sum of odd numbers from 1 to 2001 is 1002001

Answer to Activity 1.6

1. (a) $S = 62$

   (b) $S = \dfrac{65}{27}$

2. $\dfrac{12197}{4887}$

Answer to Exercise 1.16

1.

$$(a)\ S_n = \frac{G_1(r^n - 1)}{r-1}$$

$$S_n = 3(2^n - 1)$$

$$b)\ S_n = -2\left(\left(\frac{1}{2}\right)^n - 1\right)$$

2.

$$S_n = \frac{(3^n - 1)}{2}$$

Answer to Exercise 1.17

1. $r = \sqrt[3]{-2}$, and $G_1 = \dfrac{9}{\sqrt[3]{4} + \sqrt[3]{-2} + 1}$

2. Let n be the number of terms needed. Given that, $a = 3$, $r = \dfrac{1}{2}$, $S_n = \dfrac{3069}{512}$

$$S_n = \frac{G_1(1 - r^n)}{1 - r}$$

$$\frac{3069}{512} = \frac{3\left[1 - \frac{1}{2^n}\right]}{1 - \frac{1}{2}} = 6\left[1 - \frac{1}{2^n}\right]$$

$$\frac{3069}{3072} = 1 - \frac{1}{2^n}$$

$$\frac{1}{2^n} = 1 - \frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$$

$2^n = 1024 = 2^{10}$, which gives $n = 10$.

3. a) The given geometric series is $0.15,\ 0.015,\ 0.00015,\ \ldots$

$$r = \frac{0.015}{0.15} = 0.1$$

$$S_n = G_1 \frac{(1-r^n)}{1-r}$$

$$\therefore S_{20} = \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1}$$

$$= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right]$$

$$= \frac{15}{90}\left[1-(0.1)^{20}\right]$$

$$= \frac{1}{6}\left[1-(0.1)^{20}\right]$$

b) The given geometric series is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Here, $G_1 = \sqrt{7}$

$$r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore S_n = \frac{\sqrt{7}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}}$$

$$\frac{\sqrt{7}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} \qquad \text{(by rationalizing)}$$

$$= \frac{\sqrt{7}(1+\sqrt{3})\left[1-(\sqrt{3})^n\right]}{1-3}$$

$$= \frac{-\sqrt{7}(1+\sqrt{3})}{2}\left[1-(3)^{\frac{n}{2}}\right]$$

$$=\frac{\sqrt{7}(1+\sqrt{3})}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

c) The given geometric series is

$1,-a,a^2,-a^3,\ldots$

Here, first term $G_1=1$, Common ratio $r=-a$

$$S_n=\frac{a_1(1-r^n)}{1-r^n}$$

$$\frac{1\left[1-(-a)^n\right]}{1-(-a)}=\frac{\left[1-(-a)^n\right]}{1+a}$$

d) The given G.P. is $x^3,x^5,x^7,\ldots$

Here $G_1=x^3$ and $r=x^2$

$$s_n=\frac{G_1(1-r^n)}{1-r}=\frac{x^3\left[1-(x^2)^n\right]}{1-x^2}=\frac{x^3(1-x^{2n})}{1-x^2}$$

1.4 Infinite Series

Periods allotted: 6 periods

Introduction

This sub-unit aims at introducing an infinite sum of terms of a series and to introduce informally the idea of the limit of a series just by using phrases such as, if n becomes "larger and larger" or as n tends to infinity. This idea is then used to describe convergence and divergence of the series.

Competencies

By the end of this sub-unit, students will be able to:

- define a series.
- identify divergent or convergent geometric series.
- show how infinite series can be divergent or convergent.
- show how recurring decimals converge.

Vocabulary: Infinite series, geometric series, convergent series, divergent series

Teaching Notes

In this section you need not introduce the concept of a limit by using phrases such as, if n becomes "larger and larger" or as n tends to infinity. At this moment, we may not use the symbol $\lim_{n\to\infty} S_n$. At this level, the students are expected to identify the partial sum S_n of the terms of the sequence a_n.

If $a_1, a_2, a_3, \ldots$ are terms of a sequence, then the partial sum is given by

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n.$$

Assessment

Activity 1.7 and opening problem at the beginning of the sub-topic can be used to assess the students' understanding. Suggest that debate on the activities and opening problems. You may set the activity and opening problem as a group work so that representatives may present their work for the whole class. But you have to ensure the participation of all students through questioning the group members while the representatives present. You should give homework, class-works and assess students by checking their exercise books. You can use Exercise 1.7 for assessment purposes.

Answer to Activity 1.7

Height of the tree will be 2 m in the long run.

Answer to Activity 1.8

1. i.

a) $1, 2, 3, 4, \ldots$

Since the sequence is an arithmetic sequence whose first term $A_1 = 1$ and the common difference $d = 1$, the formula for the n^{th} term is given by

$$A_n = A_1 + (n-1)d = 1 + (n-1)(1) = 1 + n - 1 = n.$$

b) To find whether the sequence converges, we have to evaluate the sum of the above infinite series

$$S_n = \frac{n}{2}[2A_1 + (n-1)d] = \frac{n}{2}[2(1)+(n-1)(1)] = \frac{n}{2}[2+n-1] = \frac{n}{2}(n+1).$$

As n becomes larger and larger, S_n gets "larger and larger". That is as n increases indefinitely, S_n also increase indefinitely. Or as n tends to infinity, S_n tends to infinity. Symbolically, as $n \to \infty$, $S_n \to \infty$.

ii.

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Since the above sequence is a geometric sequence whose first term $G_1 = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$, the formula for the n^{th} term is given by

$$G_n = G_1 r^{n-1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n.$$

b) To find whether the sequence converges, we have to evaluate the sum of the above infinite series

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^n.$$

As n becomes larger and larger, S_n gets "closer and closer to zero".

Symbolically, as $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$. Hence, the series converges

iii.

a) $-1, 1, -1, 1, \ldots$

Since the above sequence is a geometric sequence whose first term $G_1 = -1$ and the common ratio $r = -1$, the formula for the n^{th} term is given by

$$G_n = G_1 r^{n-1} = (-1)(-1)^{n-1} = (-1)^n.$$

b) To find whether the sequence converges, we have to evaluate the sum of the above infinite series

$$S_n = -1+1-1+1-1+\ldots+(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \text{-1} & \text{if } n \text{ is odd} \end{cases}$$

As $n \to \infty$, S_n is not a unique number. In such cases, the infinite sum doesn't exist.

iv.

a) $1, 1.5, 2.25, 3.375, \ldots$

Since the above sequence is a geometric sequence whose first term $G_1 = 1$ and the common ratio $r = 1.5$, the formula for the n^{th} term is given by

$$G_n = G_1 r^{n-1} = (1)(1.5)^{n-1} = (1.5)^{n-1}.$$

b) To find whether the sequence converges, we have to evaluate the sum of the above infinite series

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{(1)(1-(1.5)^n)}{1-1.5} = -2(1-(1.5)^n).$$

As n becomes larger and larger, S_n gets "larger and larger". That is as n increases indefinitely, S_n also increase indefinitely. Or as n tends to infinity, S_n tends to infinity. Symbolically, as $n \to \infty$, $S_n \to \infty$.

v.

a) $0.12,\ 0.0012,\ 0.000012,\ 0.00000012, \ldots$

Since the above sequence is a geometric sequence whose first term $G_1 = \frac{12}{100}$ and the common ratio $r = \frac{1}{100}$, the formula for the n^{th} term is given by

$$G_n = G_1 r^{n-1} = \frac{12}{100}\left(\frac{1}{100}\right)^{n-1}.$$

b) To find whether the sequence converges, we have to evaluate the sum of the above infinite series

$$S_n = 0.12 + 0.0012 + 0.000012 + \ldots$$
$$= \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \ldots$$

Since, $r = \frac{G_n}{G_{n-1}} = \frac{1}{100}$, $|r| < 1$ then $S_\infty = \frac{G_1}{1-r}$.

Answer to Exercise 1.18

a) Diverges b) Converges c) Converges

Answer to Exercise 1.19

a) It is a geometric series whose first term $G_1 = 3$ and common ratio $r = \frac{1}{3}$.

Therefore, the sum becomes

$$S_\infty = 3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots = \sum_{i=1}^{\infty} 3\left(\frac{1}{3}\right)^{n-1} = \frac{G_1}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{9}{2}.$$

b)

$$1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{81} + \ldots$$

It is a geometric series whose first term $G_1 = 1$ and common ratio $r = \frac{3}{4}$.

Therefore, the sum becomes

$$S_\infty = 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{81} + \ldots + \ldots = \sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^{n-1} = \frac{G_1}{1-r} = \frac{1}{1-\frac{3}{4}} = 4.$$

c) $\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots$

It is a geometric series whose first term $G_1 = \frac{1}{5}$ and common ratio $r = \frac{1}{2}$.

Therefore, the sum becomes

$$S_\infty = \frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots = \sum_{i=1}^{\infty}\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^{n-1} = \frac{G_1}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{2}} = \frac{2}{5}.$$

d)

$$\frac{1}{5} - \frac{1}{10} + \frac{1}{20} - \frac{1}{40} + \ldots$$

It is a geometric series whose first term $G_1 = \frac{1}{5}$ and common ratio $r = -\frac{1}{2}$. Therefore, the sum becomes

$$S_\infty = \frac{1}{5} - \frac{1}{10} + \frac{1}{20} - \frac{1}{40} + \ldots = \sum_{i=1}^{\infty}\left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)^{n-1} = \frac{G_1}{1-r} = \frac{\frac{1}{5}}{1+\frac{1}{2}} = \frac{2}{15}.$$

e)

$$7 + \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$$

It is a geometric series whose first term $G_1 = 7$ and common ratio $r = \frac{1}{10}$. Therefore, the sum becomes

$$S_\infty = 7 + \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots = \sum_{i=1}^{\infty}(7)\left(\frac{1}{10}\right)^{n-1} = \frac{G_1}{1-r} = \frac{7}{1-\frac{1}{10}} = \frac{70}{9}.$$

Answer to Exercise 1.20

$a)\ \sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k-1}$

It is a geometric series whose first term $G_1 = 5$ and common ratio $r = \frac{1}{3}$. Therefore, the sum becomes

$$S_\infty = \sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k-1} = 5(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...) = \frac{G_1}{1-r} = \frac{5}{1-\frac{1}{3}} = \frac{15}{2}.$$

$b)\ \sum_{k=1}^{\infty} 2^{1-k} = 2$

c) $\sum_{k=1}^{\infty} 5^{3-k}$

It is a geometric series whose first term $G_1 = 25$ and common ratio $r = \frac{1}{5}$. Therefore, the sum becomes

$$S_\infty = \sum_{k=1}^{\infty} 5^{3-k} = 5^2 + 5^1 + 5^0 + 5^{-1} + ... = \frac{G_1}{1-r} = \frac{25}{1-\frac{1}{5}} = \frac{125}{4}.$$

$d)\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-1}$

It is a geometric series whose first term $G_1 = \left(\frac{3}{4}\right)^4$ and common ratio $r = \frac{1}{2}$.

Therefore, the sum becomes

$$S_\infty = \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-1} = \left(\frac{3}{4}\right)^4 [1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}...] = \frac{G_1}{1-r} = \frac{\left(\frac{3}{4}\right)^4}{1-\left(\frac{1}{2}\right)} = \frac{243}{512}.$$

Answer to Exercise 1.21

a) $0.\dot{4}$

$$0.\dot{4} = 0.4 + 0.04 + 0.004 + ...$$
$$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ...$$
$$= \frac{4}{10}\left[1 + \frac{1}{10} + \frac{1}{100} + ...\right]$$

Therefore, the sum becomes

$$S_\infty = \frac{G_1}{1-r} = \frac{\frac{4}{10}}{1-\frac{1}{10}} = \frac{4}{9}.$$

b) $0.3\dot{7}$

$$\begin{aligned} 0.3\bar{7} &= 0.3 + 0.07 + 0.007 + \ldots \\ &= \frac{3}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots \\ &= \frac{3}{10} + \frac{7}{100}\left[1 + \frac{1}{10} + \frac{1}{100} + \ldots\right] \end{aligned}$$

Therefore, the sum becomes

$$S_\infty = \frac{3}{10} + \frac{G_1}{1-r} = \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{3}{10} + \frac{7}{90} = \frac{17}{45}.$$

c) $0.\dot{5}\dot{6} = \frac{56}{99}$

1.5 Applications Sequences and Series in Everyday Life

Periods allotted: 8 periods

Introduction

This sub-unit is devoted to the application of arithmetic and geometric progressions or geometric series that are associated with real life situations.

Competencies

At the end of this sub-unit, students will be able to:

- discuss the applications of arithmetic and geometric progressions and series in science and technology and daily life.

Vocabulary: Cross cutting issues

Teaching Notes

In this section, you need to illustrate real life problems with examples. You can use examples given in the students' textbook.

Assessment

You should give homework, class-works and assess students by checking their exercise books. For this purpose, you can use Exercise 1.8 and make the students do in groups particularly questions 7 to 12 so that the more able students may help the less able students.

Answer to Activity 1.9

1. a) The man gets a fixed increment of 320 ETB each month. Therefore, this forms an arithmetic progression whose first term, $A_1 = 5200$ and common difference $d = 320$.

a. Salary of 10^{th} month will be given by A_n, where $n = 10$. Therefore, Salary of 10th month A_{10}

$$\begin{aligned} A_{10} &= A_1 + (n-1)d = 5200 + (10-1)\times 320 \\ &= 5200 + 9\times 320 = 5200 + 2880 = 8080. \end{aligned}$$

b) Total earnings during the first year are equal to the sum of 12 terms of the arithmetic progression.

$$\begin{aligned} \text{Total earnings} &= S_{12} \\ &= \frac{n}{2}[2A_1 + (n-1)d] \\ &= \frac{12}{2}[2(5200) + (12-1)\times 320] \\ &= 6[104002 + 11\times 320] \\ &= 6(10400 + 3520) = 83,520. \end{aligned}$$

So, the total earnings of the man during the first year are 83,520.

2. $S = 192,\ A_1 = 5,\ d = 2.$

Now,

$$S_n = 192,$$

$$\Rightarrow \frac{n}{2}[2A_2 + (n-1)d] = 192$$

$$\Rightarrow \frac{n}{2}[2(5) + (n-1)\times 2] = 192$$

$$\Rightarrow \frac{n}{2}(10 + 2n - 2) = 192$$

$$\Rightarrow n(n+4) = 192$$

$$\Rightarrow n^2 + 4n - 192 = 0$$

$$\Rightarrow (n-12)(n+16) = 0$$

$$\Rightarrow (n-12) = 0 \quad \text{or} \quad (n+16) = 0$$

$$\Rightarrow n = 12 \quad \text{or} \quad n = -16$$

Since, n cannot be negative $n = 12.$

So, the carpenter takes 12 days to finish the job.

Answer to Exercise 1.22

1. Given:

$A_1 = 25,250$ ETB, $d = 250$ ETB, $n = 3\times 2 = 6.$

Required: A_7

The n^{th} term of an arithmetic sequence is calculated as $A_n = A_1 + (n-1)d \Rightarrow A_7 = (25,250 + (7-1)(250)) = 26750$ ETB.

Therefore, his/her annual salary is 26570 ETB at the end of the third year.

2. Given: $A_1 = 1,000$ ETB, $d = 200$ ETB, $n = 8.$ Required: $A_8.$

The n^{th} term of an arithmetic sequence is calculated as $A_8 = A_1 + (n-1)d \Rightarrow A_8 = (1,000 + (8-1)(200)) = 2400$ ETB.

Therefore, she saves 2400 ETB during the eighth year.

Answer to Exercise 1.23

1. The value at the end of the n^{th} year (in ETB) is given by

$$G_n = 28,000\left(1-\frac{1}{10}\right)^n = 28,000(0.9)^n$$
$$\Rightarrow G_4 = 28,000(0.9)^4 = 18,370.80.$$

2. At the end of the n^{th} year, the value of the boat (in ETB) is

$$G_n = 34,000\left(1-\frac{12}{100}\right)^n = 34,000(0.88)^n$$
$$\Rightarrow G_5 = 28,000(0.88)^5 = 17942.88517.$$

3. The population of the town at the end of the n^{th} year is

$$G_n = 100,000\left(1+\frac{2.5}{100}\right)^n = 100,000(1.025)^n.$$
$$\Rightarrow G_{10} = 100,000(1.025)^{10} = 128,008.45.$$

4. The money at the end of the n^{th} year (in ETB) is $G_n = 3500(1.06)^n$

a. $G_1 = 3,710.00$ b. $G_2 = 3,932.00$
c. $G_3 = 4,168.556$ d. $G_4 = 4,418.66936$
e. $G_n = 3,500(1.06)^n$ f. yes, $r = 1.06$

Answer to Exercise 1.24

1.

a. $A_n = A_1 + (n-1)d \Rightarrow A_{11} = 31,100 + (11-1)(1200) = 43,100.$

b. $B_n = B_1 + (n-1)d \Rightarrow B_{11} = 35,100 + (11-1)(900) = 44,100.$

c. $S_n = \frac{n}{2}(2A_1 + (n-1)d) \Rightarrow S_n = \frac{11}{2}(2(31,000) + (11-1)(1200)) = 408,100.$

d. $S_n = \frac{n}{2}(2A_1 + (n-1)d) \Rightarrow S_n = \frac{11}{2}(2(35,100) + (11-1)(900)) = 435,600.$

e. $\sum_{i=1}^{11} B_i - \sum_{i=1}^{11} A_i = 435,600 - 408,100 = 27,500.$

2. Given:

$$A_1 = 10{,}000, \quad d = -500$$

Required: $A_8, \quad S_8$

$$A_n = A_1 + (n-1)d = 10{,}000 + (n-1)(-500) \Rightarrow A_{18} = 10{,}000 + (18-1)(-500) = 1500.$$

$$S_n = \frac{n}{2}(2A_1 + (n-1)d) \Rightarrow S_{18} = \frac{18}{2}(2(10{,}000) + (18-1)(-500)) = 103{,}500.$$

3. Given:

$$S_{10} = 13{,}250, \quad d = 250, \quad n = 10$$

Required: A_1

$$A_n = A_1 - (n-1)d \Rightarrow A_{10} = A_1 - (10-1)(250) = A_1 - 9 \times 250$$

$$S_n = \frac{n}{2}(A_1 + A_n) \Rightarrow S_{10} = \frac{10}{2}(A_1 + A_1 - 9 \times 250)$$

$$\Rightarrow 13{,}250 = 5(2A_1 - 9 \times 250)$$

$$\Rightarrow A_1 = 2{,}450.$$

Answer to Exercise 1.25

1.

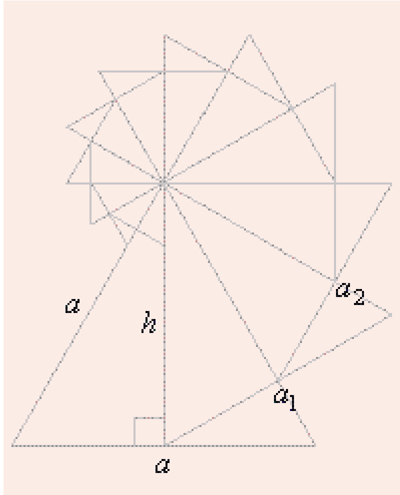
$$S_\infty = 2[rh + r^2h + ...] + h$$

$$= 2\left[\frac{G_1}{1-r}\right] + h = \frac{2h}{1-r} + h = \frac{2hr + h - rh}{1-r} = h\left(\frac{1+r}{1-r}\right)\text{m.}$$

2. $a_1 = h, \ a_2 = h_1, \ a_3 = h_2,$ and so on. Thus,

$$a_1 = h = \frac{a\sqrt{3}}{2}, a_2 = h_1 = \frac{a_1\sqrt{3}}{2} = \frac{\frac{a\sqrt{3}}{2}.\sqrt{3}}{2} = \frac{3a.}{4}$$

$$a_3 = h_2 = \frac{a_2\sqrt{3}}{2} = \frac{\frac{3a}{4}.\sqrt{3}}{2} = \frac{3\sqrt{3}a.}{8}$$



...

$$S_n = \frac{a^2\sqrt{3}}{4} + \frac{a_1^2\sqrt{3}}{4} + \frac{a_2^2\sqrt{3}}{4} + \ldots$$

$$= \frac{a^2\sqrt{3}}{4} + \frac{3}{4}\frac{a^2\sqrt{3}}{4} + \frac{9}{16}\frac{a^2\sqrt{3}}{4} + \ldots$$

Since,

$$r = \frac{G_n}{G_{n-1}} = \frac{3}{4}, \quad |r| < 1,$$

then $S_\infty = \dfrac{G_1}{1-r} = \dfrac{\frac{a^2\sqrt{3}}{4}}{1-\frac{3}{4}} = \dfrac{\frac{a^2\sqrt{3}}{4}}{\frac{1}{4}} = \sqrt{3}a^2.$

Answer to Review Exercise

1.

(a) $a_n = \dfrac{n^n}{n!}$,

$a_1 = \dfrac{1^1}{1!} = 1, \quad a_2 = \dfrac{2^2}{2!} = 2, \quad a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2}, \quad a_4 = \dfrac{4^4}{4!} = \dfrac{32}{3}, \quad a_5 = \dfrac{5^5}{5!} = \dfrac{625}{24}$

(b) $a_n = (-1)^n + (-1)^n \sin(n\pi)$

$a_1 = -1 \quad , a_2 = 1 \quad , a_3 = -1 \quad , a_4 = 1 \quad , a_5 = -1$

(c) $a_n = Sgn(3-n)$

$a_1 = Sgn(3-1) = 1$, $a_2 = Sgn(3-2) = 1$,

$a_3 = Sgn(3-3) = 0$, $a_4 = Sgn(3-4) = -1$, $a_5 = Sgn(3-5) = -1$

2.

(a) $a_n = 3^{\frac{1}{3}}(-1)^n$:

$a_1 = -3$

$a_2 = \sqrt{3}$

$a_3 = \sqrt[3]{3}$

$a_4 = \sqrt[4]{3}$

$a_5 = -\sqrt[5]{3}$

(b) $a_n = ne^{-2n}$:

$a_1 = e^{-2}$

$a_2 = 2e^{-4}$

$a_3 = 3e^{-6}$

$a_4 = 4e^{-8}$

$a_5 = 5e^{-10}$

(c) $a_n = \dfrac{(-2)^n + 6}{(n-)!}$:

$a_1 = 4$

$a_2 = 10$

$a_3 = -1$

$a_4 = \dfrac{11}{3}$

$a_5 = \dfrac{-13}{12}$

(d) $a_n = (-1)^n - \frac{1}{n^2}$:

$a_1 = -2$

$a_2 = \frac{3}{4}$

$a_3 = \frac{-10}{9}$

$a_4 = \frac{15}{16}$

$a_5 = \frac{-26}{25}$

(e) $a_n = \cos\left(\frac{n\pi}{2}\right)$:

$a_1 = 0$

$a_2 = -1$

$a_3 = 0$

$a_4 = 1$

$a_5 = 0$

3. a) is an arithmetic sequence

   b) is not an arithmetic sequence.

4. Arithmetic mean=5

5. $G_{10} = 5^{10}$ and $G_n = 5^n$

6. Let $\frac{a}{r}, a, ar$ be three numbers in geometric series.

$$\frac{a}{r} + a + ar = \frac{13}{3}$$

$$\frac{a^2}{r^2} + a^2 + a^2r^2 = \frac{91}{9}$$

$$\left(\frac{a}{r} + a + ar\right)^2 = \left(\frac{13}{3}\right)^2$$

$$\frac{a^2}{r^2} + a^2 + a^2r^2 + \frac{2a^2}{r} + 2a^2 + 2a^2r = \frac{169}{9}$$

$$\left(\frac{a^2}{r^2} + a^2 + a^2r^2\right) + 2a\left(\frac{a}{r} + a + ar\right) = \frac{169}{9}$$

$$\frac{91}{9} + 2a\left(\frac{13}{3}\right) = \frac{169}{9}$$

$$\frac{26a}{3} = \frac{169}{9} - \frac{91}{9}$$

$$\frac{26a}{3}=\frac{26}{3}$$
$$\Rightarrow a=1$$
$$\frac{1}{r}+1+r=\frac{13}{3}$$
$$\frac{1+r+r^2}{r}=\frac{13}{3}$$
$$3+3r+3r^2=13r$$
$$3r^2-10r+3=0$$
$$(r-3)(3r-1)=0$$
$$r=3\ or\ r=\frac{1}{3}$$
$$\text{for } r=3, \frac{a}{r}=\frac{1}{3}, ar=1\times 3=3$$
The three numbers are $\frac{1}{3},1,3$

$$\text{for } r=\frac{1}{3}, \frac{a}{r}=\frac{1}{1/3}=3; ar=1\times\frac{1}{3}=\frac{1}{3}$$

Thus, the three numbers are $3,1,\frac{1}{3}$.

7. (a) $\sum_{k=1}^{5}(k^3+2k^2-3k+5)=\sum_{k=1}^{5}k^3+\sum_{k=1}^{5}2k^2-\sum_{k=1}^{5}3k+\sum_{k=1}^{5}5$

$$\sum_{k=1}^{5}(k^3+2k^2-3k+5)=\sum_{k=1}^{5}k^3+2\sum_{k=1}^{5}k^2-3\sum_{k=1}^{5}k+\sum_{k=1}^{5}5$$

$$\sum_{k=1}^{5}(k^3+2k^2-3k+5)=(1+8+27+64+125)+2(1+4+9+16+25)$$
$$-3(1+2+3+4+5)+5(5)$$

$$\sum_{k=1}^{5}(k^3+2k^2-3k+5)=225+110-45+25=315$$

(b) $\sum_{k=2}^{5}3=3(4)=12$

(c) $\sum_{k=1}^{5}\frac{1}{k^2+5k+6}=\sum_{k=1}^{5}\frac{1}{(k+2)(k+3)}$

$\sum_{k=1}^{5}\frac{1}{k^2+5k+6}=\sum_{i=1}^{5}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)\rightarrow$ by partial fraction

$\sum_{k=1}^{5}\frac{1}{k^2+5k+6}=\frac{5}{24}$

$(d)\ \sum_{m=1}^{5}\ell n(\frac{m}{m+1})=\sum_{i=1}^{5}\left(\ell n(m)-\ell n(m+1)\right)=-\ell n6=\ell n\left(\frac{1}{6}\right)$

8. $(a)\sum_{i=1}^{5}(2x_i-3y_i)=2\sum_{i=1}^{5}x_i-3\sum_{i=1}^{5}y_i=2(27)-3(127)=-327$

$(b)\sum_{i=1}^{5}(2x_i+3y_i)=2\sum_{i=1}^{5}x_i+3\sum_{i=1}^{5}y_i=2(27)+3(127)=435$

$$(c)\sum_{i=1}^{5}\left(2x_i-3y_i\right)^2=4\sum_{i=1}^{5}x_i^2-12\sum_{i=1}^{5}x_iy_i+9\sum_{i=1}^{5}y_i^2$$
$$=4(303)-12(105)+9(50)$$
$$=402$$

$(d)\ \left(\sum_{i=1}^{5}x_i\right)^2=(37)^2=1369$

$(e)\ \sum_{i=1}^{5}(2x_i-5y_i+3)=29$

9. The natural numbers lying between 100 and 1000, which are multiples of 5, are $105,\ 110,\ \ldots\ 995$. Here $a=105, d=5$

$$a+(n-1)d=995$$
$$\Rightarrow 105+(n-1)5=995$$
$$\Rightarrow (n-1)5=995-105=890$$
$$\Rightarrow n-1=178$$
$$\Rightarrow n=179$$
$$\therefore S=\frac{179}{2}\left[2(105)+(179-1)(5)\right]$$
$$=179\left[105+(89)5\right]$$
$$=(179)(105+445)$$
$$=(179)(550)$$
$$=98450$$

Thus, the sum of all natural numbers lying between 100 and 1000, which are multiples of 5, is 98450

10. First term = 2

Let d be the common difference of the arithmetic series.

Therefore, the arthimetic series is $2,\ 2+d,\ 2+2d,\ 2+3d,\ \ldots$

Sum of first five terms $= 10 + 10d$

Sum of next five terms $= 10 + 35d$

According to the given condition,

$$10+10d=\frac{1}{4}(10+35)d$$
$$\Rightarrow 40+40d=10+35$$
$$\Rightarrow 30=-5d$$
$$\Rightarrow d=-6$$
$$\therefore a_{20}=a+(20-1)d=2+(19)(-6)=2-114=-112$$

Thus, the 20^{th} term of the A.P. is -112.

11. Let the sum of n terms of the given arithmetic series -2.5.

It is known that, $S_n = \frac{n}{2}[2a+(n-1)d]$ where n = number of terms, a = first term, and d = common difference

Here, a = -6

$$d = -\frac{11}{2} + 6 = \frac{-11+12}{2} = \frac{1}{2}$$

Therefore, we obtain.

$$-25 = \frac{n}{2}\left[2\times(-6)+(n-1)\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow -50 = n\left[-12+\frac{n}{2}-\frac{1}{2}\right]$$

$$\Rightarrow -50 = n\left[-\frac{25}{2}+\frac{n}{2}\right]$$

$$\Rightarrow -100 = n(-25+n)$$

$$\Rightarrow n^2 - 25n + 100 = 0$$

$$\Rightarrow n^2 - 5n - 20n + 100 = 0$$

$$\Rightarrow n = 20 \quad \text{or} \quad 5$$

12. Let the sum of n terms of the given an arithmetic series be 116.

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

Here, A_1 = 25 and d = 22 - 25 = - 3

$$\therefore S_n = \frac{n}{2}[2\times 25 + (n-1)(-3)]$$

$$\Rightarrow 116 = \frac{n}{2}[50 - 3n + 3]$$

$$\Rightarrow 232 = n(53-3n) = 53n - 3n^2$$

$$\Rightarrow 3n^2 - 53n + 232 = 0$$

$$\Rightarrow 3n^2 - 24n - 29n + 23 = 0$$

$$\Rightarrow 3n(n=8) - 20(n-8) = 0$$

$$\Rightarrow (n-8)(3n-29) = 0$$

$$\Rightarrow n = 8 \,\text{or}\, n = \frac{29}{3}$$

However, n cannot be equal to $\frac{29}{3}$. Therefore, $n = 8$

$$\therefore A_8 = A_1 + (n-1)d = 25 + (8-1)\ (-3) = 25 + (7)\ (-3) = 25\ -21 = 4$$

Thus, the last term of the arithmetic series is 4.

13. According to the given condition

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = pn + qn^2$$

$$\Rightarrow \frac{n}{2}\left[2a + nd\right] = pn + qn^2$$

$$\Rightarrow na + n^2\frac{d}{2} - n.\frac{d}{2} = pn + qn^2$$

Comparing the coefficients of n^2 on both sides, we obtain

$$\frac{d}{2} = q$$

$$\therefore d = 2q$$

Thus, the common difference of the A.P. is 2q.

14. $A_n = 8n - 4$ and $A_{10} = 76$

15. $0.3\dot{1}\dot{7} = \frac{157}{495}$

16. a. $0.3\dot{7} = \frac{17}{45}$ b. $3.23\dot{5}\dot{4} = \frac{1493281}{111000}$

17. a. $x = 3$ b. $\text{sum} = 8$

18.

$$\sum_{x=1}^{12}(24 - 3x) = 21 + 18 + 15 + \ldots$$

This is an arithmetic series with $A_1 = 21$ and $d = -3$.

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(21) + (12-1)(3)]$$

$$= 6[42 - 33]$$

$$= 54$$

$$\therefore \sum_{x=1}^{12}(24 - 3x) = 54$$

Write out the series on the LHS:

$$\sum_{k=1}^{\infty} 27p^k = 27p + 27p^2 + 27p^3 + \ldots$$

This is a geometric series with $G_1 = 27p$ and $r = p$ ($-1 < p < 1$ for the series to converge).

$$S_\infty = \frac{G_1}{1-r} = \frac{27p}{1-p}$$

$$\Rightarrow 54 = \frac{27p}{1-p} \Rightarrow 27p = 54 - 54p \Rightarrow p = \frac{54}{81} = \frac{2}{3}.$$

19. $4 \times 4^{\frac{1}{2}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times \ldots \times 4^{\frac{1}{2^n}} \times \ldots = 4^{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots} = 4^{\frac{1}{1-\frac{1}{2}}} = 4^2 = 16.$

20.

$$\sum_{k=1}^{\infty} 5^{kt} = \frac{1}{24} \Rightarrow 5^t + 5^{2t} + 5^{3t} + 5^{4t} + \ldots = \frac{1}{24}$$

It is a geometric series in which the first term is $G_1 = 5^t$ and the common ratio $r = 5^t$.

$$S_\infty = \frac{G_1}{1-r} = \frac{5^t}{1-5^t} = \frac{1}{24}$$

$$\Rightarrow 24(5^r) = 1 - 5^r$$

$$\Rightarrow 5^t(25) = 1$$

$$\Rightarrow 5^t = 5^{-2}$$

$$\Rightarrow t = -2.$$

21.

$$5^k.5^{k^2}.5^{k^3}\ldots=5\Rightarrow 5^{k+k^2+k^3+\ldots}=5\Rightarrow k+k^2+k^3+\ldots=1.$$

It is a geometric series in which the first term is $G_1=k$ and the common ratio $r=k$.

$$S_\infty=\frac{G_1}{1-r}=\frac{k}{1-k}=1$$
$$\Rightarrow k=1-k$$
$$\Rightarrow 2k=1$$
$$\Rightarrow k=\frac{1}{2}.$$

22.

$$\sum_{k=1}^{11}(2+3^k)=\sum_{k=1}^{11}(2)+\sum_{k=1}^{11}3^k=22+\sum_{k=1}^{11}3^k \qquad (1)$$

$$\sum_{k=1}^{11}3^k=3^1+3^2+3^3+\ldots 3^{11}$$

The terms of this sequence $3,\ 3^2,\ 3^3,\ \ldots$ forms a geometric series.

$$S_n=\frac{G_1(r^n-1)}{r-1}$$
$$\Rightarrow S_{11}=\frac{3\left[(3)^{11}-1\right]}{3-1}$$
$$\Rightarrow S_{11}=\frac{3}{2}\left(3^{11}-1\right)$$
$$\therefore \sum_{k=1}^{11}\frac{3}{2}\left(3^{11}-1\right)$$

Substituting this value in equation (1), we obtain

$$\sum_{k=1}^{11}(2+3^k)=22+\frac{3}{2}(3^{11}-1)$$

23. Let $\frac{a}{r}$, a, ar be the first three terms of the geometric series.

$$\frac{a}{r}+a+ar=\frac{39}{10} \qquad ...(1)$$

$$\left(\frac{a}{r}\right)(a)(ar) \qquad ...(2)$$

From (2), we obtain

$$a^3=1$$

$$\Rightarrow a=1 \text{ (considering real roots only)}$$

Substituting $a = 1$ in equation (1), we obtain

$$\frac{1}{r}+1+r=\frac{39}{10}$$

$$\Rightarrow 1+r+r^2=\frac{39}{10}r$$

$$\Rightarrow 10+10r+10r^2-39=0$$

$$10r^2-29+10=0$$

$$10r^2-25r-4r+10=0$$

$$5r(2r-5)-2(2r-5)=0$$

$$(5r-2)(2r-5)=0$$

$$r=\frac{2}{5} \text{ or } \frac{5}{2}$$

Thus, the three terms of geometric series are $\frac{5}{2},1,\frac{2}{5}$

24. The given geometric series is $3,3^2,3^3\dots$

Let n terms of this geometric series be required to obtain the sum as 120.

$$s_n=\frac{a(r^n-1)}{r-1}$$

Here, $a = 3$ and $r = 3$

$$\therefore s_n = 120 = \frac{3(3^n - 1)}{3 - 1}$$

$$\Rightarrow 120 = \frac{3(3^n - 1)}{2}$$

$$\Rightarrow \frac{120 \times 2}{3} = 3^n - 1 = 80$$

$$\Rightarrow 3^n - 1 = 80$$

$$3^n = 81$$

$$3^n = 3^4$$

$$n = 4$$

Thus, four terms of the given geometric series are required to obtain the sum as 120

25. Let the geometric series be $a + ar + ar^2 \ldots$

According to the given condition,

$$a + ar + ar^2 = 16 \text{ and } ar^3 + ar^4 + ar^5 = 128$$

$$a(1 + r + r^2) = 16 \ldots \qquad (1)$$

$$ar^3(1 + r + r^2) = 128 \ldots \qquad (2)$$

Dividing equation (2) by (1), we obtain

$$\frac{ar^3(1 + r + r^2)}{a(1 + r + r^2)} = \frac{128}{16}$$

$$\Rightarrow r^3 = 8$$

$$\therefore r = 2$$

Substituting $r = 2$ in (1), we obtain

$$a(1 + 2 + 4) = 16$$

$$\Rightarrow a(7) = 16$$

$$\Rightarrow a = \frac{16}{7}$$

$$S_n = \frac{16}{7} \frac{(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

26. $G_1 = 729 \qquad G_7 = 64$

Let r be the common ratio of the geometric series.

It is known that, $G_n = G_1 r^{n-1}$

$$G_7 = G_1 r^{7-1} = (729) r^6$$
$$\Rightarrow r^6 = \frac{64}{729}$$
$$\Rightarrow r^6 = \left(\frac{2}{3}\right)^6$$
$$\Rightarrow r = \frac{2}{3}$$

Also it is known that, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$\therefore S_7 = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}}$$
$$= 3\times 729\left[1-\left(\frac{2}{3}\right)^7\right]$$
$$=(3)^7\left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$
$$=(3)^7-(2)^7$$
$$=2187-128$$
$$=2059$$

27. Let a be the first term and r be the common ratio of the geometric series. According to the given situation,

$$G_4 = G_1 r^3 = x \ldots \qquad (1)$$
$$G_{10} = G_1 r^9 = y \ldots \qquad (2)$$
$$G_{16} = G_1 r^{15} = z \ldots \qquad (3)$$

Dividing (2) by (1), we obtain

$$\frac{y}{x} = \frac{G_1 r^9}{G_1 r^3} \Rightarrow \frac{y}{x} = r^6$$

Dividing (3) by (2), we obtain

$$\frac{z}{y} = \frac{G_1 r^{15}}{G_1 r^9} \Rightarrow \frac{z}{y} = r^6$$

$$\therefore \frac{y}{x} = \frac{z}{y}$$

Thus, x, y, z are in geometric series.

28. The given sequence is $8, 88, 888, 8888\ldots$

This sequence is not a geometric series. However, it can be changed to geometric series by writing the terms as

$$S_n = 8 + 88 + 888 + 8888 + \text{..........to } n \text{ terms}$$

$$= \frac{8}{9}\left[9 + 99 + 999 + 9999 + \text{...........to } n \text{ terms}\right]$$

$$= \frac{8}{9}\left[(10-1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \text{......to } n \text{ terms}\right]$$

$$= \frac{8}{9}\left[\left(10 + 10^2 + \text{.......} n \text{ terms}\right) - (1 + 1 + 1 + \text{......} n \text{ terms}\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$$

$$= \frac{80}{81}\left[(10^n - 1) - \frac{8}{9}n\right]$$

29. It has to be proved that the sequence $aA, arAR, ar^2AR^2, \ldots ar^{n-1}AR^{n-1}$ forms a geometric series.

$$\frac{\text{second term}}{\text{first term}} = \frac{arAR}{aA} = rR$$
$$\frac{\text{third term}}{\text{second term}} = \frac{ar^2AR^2}{arAR} = rR$$

Thus, the above sequence forms a geometric series and the common ratio is rR.

30. Let A be the first term and R be the common ratio of the geometric series.

According to the given information,

$$AR^{p-1} = a$$
$$AR^{q-1} = b$$
$$AR^{r-1} = c$$
$$a^{q-r}b^{r-p}c^{p-q}$$
$$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}$$
$$= Aq^{-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$$
$$= A^0R^0$$
$$= 1$$

Thus, the given result is proved.

31. The first term of the geometric series is a and the last term is b.

Therefore, the geometric series is $a, ar, ar^2, ar^3, ...ar^{n-1}$ where r the common ratio is.

$b = ar^{n-1}$.... (1)

$p =$ product of n terms

$=(a)(ar)(ar^2)...(ar^{n-1})$

$=(a\times a\times ...\times a)(r\times r\times ...\times r^{n\text{-}1})$

$=a^n r^{1+2+...(n-1)}$... (2)

Here $1, 2, ...(n-1)$ is an $A.P.$

$$\therefore 1+2+.........+(n-1)=\frac{n-1}{2}\left[2+(n-1-1)x1\right]=\frac{n-1}{2}\left[2+n-2\right]=\frac{n(n-1)}{2}$$

$P = a^n r^{\frac{n(n-1)}{2}}$

$\therefore P^2 = a^{2n} r^{n(n-1)}$

$=\left[a^2 r^{(n-1)}\right]^n$

$=(ab)^n$ $\left[\text{Using (1)}\right]$

Thus, the given result is proved

32.

a, b, c, d are in geometric series. Therefore,

$bc = ad$ ... (1)

$b^2 = ac$ ... (2)

$c^2 = bd$ ... (3)

it has to be proved that,

$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

R.H.S

$=(ab+bc+cd)^2$

$=(ab+ad+cd)^2$ [using (1)]

$=[ab+d(a+c)]^2$

$=a^2b^2+2abd(a+c)+d^2(a+c)^2$

$=a^2b^2+2a^2bd+2acbd+d^2(a^2+2ac+c^2)$

$=a^2b^2+2a^2c^2+2b^2c^2+d^2a^2+2d^2b^2+d^2c^2$ $\left[\text{Using (1) and (2)}\right]$

$=a^2b^2+a^2c^2+a^2c^2+b^2c^2+b^2c^2+d^2a^2+d^2b^2+d^2b^2+d^2c^2$

$=a^2b^2+a^2c^2+a^2d^2+b^2xb^2+b^2c^2+b^2d^2+c^2b^2+c^2xc^2+c^2d^2$

Using (2) and (3) and rearranging terms

$$= a^2(b^2+c^2+d^2)+b^2(b^2+c^2+d^2)+c^2(b^2+c^2+d^2)$$
$$= (a^2+b^2+c^2)(b^2+c^2+d^2)$$

$= \text{L.H.S}$

$= \text{L.H.S} = \text{R.H.S}$

$$\therefore (a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$$

33. Let G_1 and G_2 be two numbers between 3 and 81 such that the series, forms a geometric series.

Let a be the first term and r be the common ratio of the geometric series.

$$\therefore 81 = (3)(r)^3$$
$$\Rightarrow r^3 = 27$$
$$\therefore r = 3 \ (\text{taking real roots only})$$
$$\text{For } r = 3$$
$$G_1 = ar = (3)(3) = 9$$
$$G_2 = ar^2 = (3)(3)^2 = 27$$

Thus, the required two numbers are 9 and 27.

34. Let the two numbers be a and b

$G.M. = \sqrt{ab}$

According to the given condition,

$a+b=6\sqrt{ab}$

$\Rightarrow (a+b)^2 = 36ab$

Also,

$(a-b)^2 = (a+b)^2 - 4ab = 36ab - 4ab = 32ab$

$\Rightarrow a-b = \sqrt{32}\sqrt{ab}$

$= 4\sqrt{2}\sqrt{ab}$

adding (1) and (2), we obtain

$2a = (6{+}4\sqrt{2})\sqrt{ab}$

$\Rightarrow a = (3+2\sqrt{2})\sqrt{ab}$

Substituting the value of n in (1),we obtain

$b = 6\sqrt{ab} - (3+2\sqrt{2})\sqrt{ab}$

$\Rightarrow b = (3-2\sqrt{2})\sqrt{ab}$

$$\frac{a}{b} = \frac{(3+2\sqrt{2})\sqrt{ab}}{(3-2\sqrt{2})\sqrt{ab}} = \frac{(3+2\sqrt{2})}{(3-2\sqrt{2})}$$

Thus, the required ratio is=$(3+2\sqrt{2}):(3-2\sqrt{2})$

35. It is given that A and G are arithmetic mean and geometric mean between two positive numbers. Let these two positive numbers be a and b.

$\therefore A = \dfrac{a+b}{2}$ ..... (1)

$G = \sqrt{ab}$ ..... (2)

From (1) and (2) we obtain

$a+b = 2A$ ..... (3)

$ab = G^2$ ..... (4)

Substituting the value of a and b from (3)and (4)in the identity $(a\text{-}b)^2 = (a+b)^2 - 4ab$

$(a-b)^2 = 4A^2 - 4G^2 = 4(A^2 - G^2)$

$(a-b)^2 = 4(A+G)(A-G)$

$(a-b) = \pm 2\sqrt{(A+G)(A-G)}$ ..... (5)

From (3) and (5) we obtain

$2a=2A \pm 2\sqrt{(A+G)(A-G)}$

$\Rightarrow a = A \pm \sqrt{(A+G)(A-G)}$

substituting the value of a in (3),we obtain

$b = 2A - A \pm \sqrt{(A+G)(A-G)}$

$b = A \pm \sqrt{(A+G)(A-G)}$

36. 150 workers were engaged to finish a job in a certain number of days say k days.

$\therefore$ Number of wortkers who would have worked for k days.

$=(150+150+150+....+k$ terms)

$=150k$ ...(i)

But workers present on first day are 150 (given), on second day $150-4=146$

($\because$ Workers dropped on second day), on third day $=146=4=142$ and so on.

Because of the dropping out of workers, it took 8 more days to finish the work (given)

$\therefore$ Number of days taken to finish the work

$k+8=n$ (says) ...(ii)

$\therefore 150+146+142+....$to n

Terms (days) $=150k$ [By (i)]

The left hand series is an arithmetic series with $a=150$,

$d = 146 - 150 = -4,$

$\frac{n}{2}\left[2a + (n-1)d\right] = 150k$

putting $k = n - 8$ from (i)

$\Rightarrow \frac{n}{2}\left[300 + (n-1)(-4)\right] = 150(n-8)$

$\Rightarrow \frac{n}{2}\left[300 - 4n + 4\right] = 150(n-8)$

$\Rightarrow n(304 - 4n) = 300(n-8)$

$304n - 4n^2 = 300n - 2400$

$\Rightarrow -4n^2 + 4n + 2400 = 0$

dividing by -4, $\quad n^2 - n - 600 = 0$

$\Rightarrow n^2 - 25n + 24n - 600 = 0$

$\Rightarrow n(n-25) + 24(n-25) = 0$

$\Rightarrow (n-25)(n+24) = 0$

$\Rightarrow$ Either $n - 25 = 0$ *or* $n + 24 = 0$

i.e., $\quad n = 25 \quad$ *or* $\quad n = -24$

But $n = -24$ is impossible because number of days can't be negative

$\therefore n = 25$

UNIT 2: INTRODUCTION TO CALCULUS

Introduction

In this unit, there are several outcomes which are expected to be accomplished. Among those, the following two are the most important. The first one is to help students to understand the concepts of differentiation and integration. The second one is to assist students to apply differential and integral calculus in solving problems in different fields of study.

Therefore, students should be able to exercise the different rules and methods of differentiation and basic integration techniques.

The activities are to be used to introduce unfamiliar terms and create an opportunity for students to explore rules and methods of differentiation as well as integration. The exercises are prepared on the assumptions that of students will have learning gaps. The exercises are designed to be useful in assessing individual learning difficulties by indicating the rule which needs to be explained and represented to various students.

Unit Outcomes

By the end of this unit, students will be able to:

- Deduce rate of change of different quantities;
- Calculate rate of change of different quantities;
- Understand gradient of functions at a point;
- Analyze the geometrical and mechanical meaning of derivative;
- Find the derivative of simple functions using gradient method;
- Find area of a region bounded by a function and x-axis on a given interval;
- Understand definite integral;
- Real-world applications of derivatives and integrals ;
- Apply the knowledge of integral calculus to solve real-life mathematical problems.

Suggested Teaching Aids

- Prepare charts containing the rules of differentiation,
- the sum, difference,
- product and quotient rules,
- the chain rule,
- Prepare charts containing the derivatives of the standard functions,
- Prepare charts showing tangents to the graphs of standard functions at a point,
- Prepare charts relating to derivative and integrals,
- Integration techniques.

2.1 Introduction to Derivatives

Period allowed: 10 periods

Introduction

The purpose of this topic is to develop the concept or understanding of the formal definition of the derivative of a function. The topic begins by reviewing rate of change and relating rate of change with the slope of a secant line and then showing the derivative of a function as the slope of the tangent line to the graph of the function. Some problems in physics that involve the rate of change of a function are used to demonstrate the interpretation of the derivatives as an instantaneous rate of change.

Competencies

- Explain average rate of change of one quantity with respect to another by giving different examples. By so doing, students:
- Show skills in solving rate of change problems.
- Find average rate of change for different linear and non- linear curves.
- Show the process to find gradient of different curves at a point.
- Compute rational algebraic expressions to find general formula for gradient.
- Develop gradient formula for cubic function using the approach of square function.

Key words

Average rate of change, gradient, secant line, tangent line, velocity, displacement, instantaneous rate of change, quotient, difference, differentiation, derivatives, critical number, extreme value, relative maximum value, relative minimum value, absolute maximum value, absolute minimum value, anti-derivative, definite integrals, integration.

Teaching method

Your approach of teaching the contents in this unit should be a **learner-centered approach** than a **teacher –centered approach.** You should involve students in

- Finding average rates of change of functions between different points.
- Finding the slope of a secant line and a tangent line to the graph of a function.
- Developing the concept of the derivative of a function at a point.
- Developing the concepts and exploring the rules of differentiation and integration.

2.1.1 Understanding Rate of Change

Teaching Note

Activity 2.1 is designed to help students to understand the concept of the derivative by starting from what they already know, that is from rate of change and slope of a tangent line to a curve at a point. You can begin this section by forming different groups and encouraging students to attempt activity 2.1. You can also give different additional activities that you think might help students to link the concept of a slope to the derivative of a function.

Assessment

When the groups have completed the activity, you can present the topic and do different examples given in the text so that students can understand the concept. You can also give Exercise 2.1 as class work and home work. When students have completed the exercise, you may give the solutions to all the questions so that they can check their own work. The exercise is designed to assess how well students understand the topics and whether or not students are ready to start the next topic.

Answer to Activity 2.1

1.

a. $\Delta p = 4(4\text{ cm} + 1\text{ cm}) - 4(4\text{ cm}) = 20\text{ cm} - 16\text{ cm} = 4\text{ cm}$

b. $\Delta p = 4(4\text{ cm} + 1\text{ cm}) - 4(4\text{ cm}) = 20\text{ cm} - 16\text{ cm} = 4\text{ cm}$ at 1 sec

$\Delta p = 4(4\text{ cm} + 2\text{ cm}) - 4(4\text{ cm}) = 24\text{ cm} - 16\text{ cm} = 8\text{ cm}$ at 2 sec

$\Delta p = 4(4\text{ cm} + 3\text{ cm}) - 4(4\text{ cm}) = 28\text{ cm} - 16\text{ cm} = 12\text{ cm}$ at 3 sec

c. $\dfrac{\Delta p}{\Delta t} = \dfrac{p_6 - p_4}{6 - 4} = 4\text{ cm/sec}.$

2. A rate of change is a rate that describes how one quantity changes in relation to another quantity.

For instance:

- ✓ Change in velocity of a moving object with respect to time.
- ✓ Change in price of materials in relation to demand.
- ✓ Change in position of an object due to the applied force.

Answer to Exercise 2.1

a. $\text{rate of change} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{9-6}{5-4} = \dfrac{3}{1} = 3.$

b. $\text{rate of change} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-7}{5-4} = \dfrac{-1}{1} = -1.$

c. $\text{rate of change} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-4-2}{3-1} = \dfrac{-6}{2} = -3.$

d. $\text{rate of change} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-6-4}{3-(-2)} = \dfrac{-10}{5} = -2.$

e. $\text{rate of change} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{9-9}{9-0} = \dfrac{0}{9} = 0.$

f. rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-0}{8-8} = \dfrac{8}{0}$ (undefined).

g. rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1-(-1)}{1-3} = \dfrac{0}{-2} = 0.$

Answer to Exercise 2.2

1.

a. Average *r*ate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{3(2) - 2 - (3\times 0 - 2)}{2-0} = \dfrac{6}{2} = 3.$

b. Average *r*ate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{216 - 54 - (1-9)}{6-0} = \dfrac{170}{6} = \dfrac{85}{3}.$

c. Average *r*ate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{3(5) - 2 - (3\times 1 - 2)}{5-1} = \dfrac{12}{4} = 3.$

d. Average *r*ate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{k^2 + 3k + 1 - (1)}{k-0} = \dfrac{k(k+3)12}{k}$

$$= k + 3,\ k \neq 0.$$

2.

Average rate of change $= \dfrac{2.41 - 2.84}{2} = -0.215$

3. Begin by finding two pairs. The candle begins at 10cm in length. So a time $t = 0$, the length is 10cm. The ordered pair representing this is $(0,10)$. Thirty minutes later, the candle is 7km, so the second pair is $(30,7)$. Now, we have two points

$(x_1, y_1) = (0,10)$ and $(x_2, y_2) = (30,7)$

Thus, slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-10}{30-0} = -\dfrac{1}{10}$

Therefore, the length of the candle changes with a rate of $-\dfrac{1}{10}$ km/min.

At the time a candle completely burn out, its length $y_2 = 0$. Now, using this value in the formula, we have

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-10}{x_2 - 0} = -\frac{1}{10} \Rightarrow x_2 = 100.$$

Hence, the candle takes 100 minutes to completely burn to nothing.

Answer to Activity 2.2

i.

a. $m_1 = \dfrac{f(1.1) - f(1)}{1.1 - 1} = \dfrac{1.331 - 1}{0.1} = 3.31$

b. $m_2 = \dfrac{f(1.001) - f(1)}{1.001 - 1} = 3 + 3(0.001) + (0.001)^2 = 3.003001$

c. $m_3 = \dfrac{f(1+\Delta x) - f(1)}{\Delta x} = \dfrac{(1+\Delta x)^2 - 1}{\Delta x} = 3 + 3\Delta x + (\Delta x)^2$

ii. When Δx approaches to zero, $m_1 \to 3$.

Answer to Exercise 2.3

1. The instantaneous rate of change of a function f at $x = x_0$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + h) - f(x_0)}{h}$,

For the given function $f(x) = 6x^2 - 3$ at $x = 1$, the instantaneous rate of change is

$$\begin{aligned}\text{instantaneous rate of change } &= \frac{f(1+h) - f(1)}{h} \\ &= \frac{6(1+h)^2 - 3 - (6-3)}{h} = \frac{6(h^2 + 2h + 1) - 6}{h} \\ &= \frac{6h^2 + 12h + 6 - 6}{h} = \frac{6h^2 + 12h}{h} \\ &= 6h + 12, \text{ as } h \text{ moves closer and closer to zero} \\ &= 12.\end{aligned}$$

Thus, the instantaneous rate of change is 12.

2. Recall that the instantaneous rate of change of a function $f(x)$ when $x = x_0$ is

instantineous rate of change $=\dfrac{f(x_0+h)-f(x_0)}{h}$

$$=\frac{2\sqrt{x_0+h}-2\sqrt{x_0}}{h}, \text{ rationalizing the numinator}$$

$$=\frac{2\sqrt{x_0+h}-2\sqrt{x_0}}{h}\left(\frac{\sqrt{x_0+h}+\sqrt{x_0}}{\sqrt{x_0+h}+\sqrt{x_0}}\right)$$

$$=\frac{2h}{h(\sqrt{x_0+h}+\sqrt{x_0})}=\frac{2}{\sqrt{x_0+h}+\sqrt{x_0}}, \text{ as } h \text{ moves closer and closer to zero}$$

$$=\frac{1}{\sqrt{x_0}}.$$

3. Recall that the instantaneous rate of change of a function $f(t)$ when $t=t_0$ is

Instantineous rate of change $=\dfrac{f(3+h)-f(3)}{h}$

$$=\frac{81(3+h)^3+90-(81\times 3^3+90)}{h}$$

$$=\frac{81(h^3+9h^2+17h+27)-81\times 27}{h}$$

$$=\frac{81h^3+729h^2+1447h}{h}$$

$$=81h^2+729h+1447, \text{ as } h \text{ closer and closer to zero}$$

$$=1447.$$

Since the rate of change is positive, it is equivalent to the rate of growth.

Hence, the rate of growth of the biomass of a bacterial culture when $t=3$ minutes is $1447\ \dfrac{\text{mg}}{\text{min}}$.

2.1.2 Gradient of Curves and Rate of Changes

Teaching Note

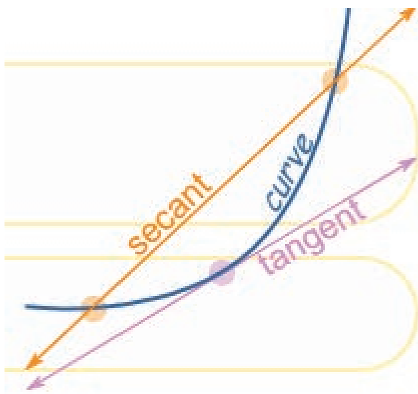
Start this section by encouraging students to discuss activity 2.2 to assess how they understand the meaning of secant line and tangent line. It is clear that, most of the students will give the definition of secant and tangent lines of a circle. After they discuss the problems in the activity and reflect on the problems, you can start the lesson by giving the definition for secant and tangent lines to a curve. Students should notice that a tangent line may cross a curve at some point other than a point of tangency. Then, briefly discuss the worked examples in the student book.

Assessment

After giving them the definition of a secant line and tangent line to a curve, briefly discuss the worked examples in the student's book, give some of the problems from Exercise 2.2 as class work and the remaining problems as home work. The aim of the exercise is to assess the performance of students in understanding the relationship between average rate of change, instantaneous rate of change, and slope of the secant and tangent line.

Answer to Activity 2.3

1. A **secant line** is a straight line joining two points on a curve or graph of a function. That is, a secant line is a line passing through two points of a curve. A **tangent line** is a straight line that touches a curve or a graph of a function at only one point.



As the two points are brought together (or, more precisely, as one is brought towards the other), the secant line tends to a tangent line.

2. **Gradient (slope)** is a number that describes steepness and direction of a line. To calculate the slope we need to find the ratio between vertical change and horizontal change, which is often described as 'rise over run' (rise is vertical and run is horizontal). Since on most graphs vertical is shown by the *y*-axis and horizontal by *x*-axis, we define slope as 'change in *y*' divided by 'change in *x*.

Answer to Exercise 2.4

1.

a. $$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{11+5}{3+3} = \frac{8}{3}.$$

b. $$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-7+9}{5-4} = 2.$$

2.

a. $$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{9+1-2}{3+1} = 2$$

Therefore, the slope a secant line to the graph of $f(x) = x^2 + 1$ with respect to x over the interval $[-1, 3]$ is 2.

b. $$\text{average rate of change} = \frac{g(x_2) - g(x_1)}{x_2 - x_1} = \frac{8-8+6}{2-1} = 6$$

Therefore, the slope a secant line to the graph of $f(x) = x^3 - 4x + 6$ with respect to x over the interval $[1, 2]$ is 6.

2.1.3 Gradient at a Point on a Curve

Teaching Note

You can begin the class lesson by asking students about the difference between the gradient of a line between two distinct points and gradient of a line at single point. Then encourage students to complete activity 2.4, which is designed to help them identify the difference between the gradient of a line at a point and at two points and how it is interpreted. When students complete the activity provide the answers to the questions in the activity so that they can check their answers.

Assessment

Ask as many questions as possible and give Exercise 2.3 as class-work and home-work. The aim of the exercise is to help students understand the concept of the derivative in a simple way.

Answer to Activity 2.4

1.

a) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x+h)^2-x^2}{h}=2x+h.$

b) $\dfrac{f(x)-f(x_0)}{x-x_0}=\dfrac{x^2-{x_0}^2}{x-x_0}=x+x_0$

2.

a. $m_1=\dfrac{f(1.1)-f(1)}{1.1-1}=\dfrac{1.331-1}{0.1}=3.31$

b. $m_2=\dfrac{f(1.001)-f(1)}{1.001-1}=3+3(0.001)+(0.001)^2=3.003001$

c. $m_3=\dfrac{f(1+h)-f(1)}{h}=\dfrac{h^3+3h^2+3h}{h}=h^2+3h+3$

i) When h approaches to zero, $m_3\to 3$.

Answer to Exercise 2.5

1. Gradient (or slope) = rate of change $=\dfrac{\text{vertical increase}}{\text{horizontal increase}}=\dfrac{f(x)-f(a)}{x-a}$

a. when $a=-1,$ the gradient (m) $=\dfrac{f(x)-f(-1)}{x+1}=\dfrac{x^2-1}{x+1}=x-1, x\neq -1$

As x get close to – 1, $m=-2$

when $a=4$, the gradient (m) $=\dfrac{f(x)-f(4)}{x-4}=\dfrac{x^2-16}{x-4}=x+4,\ x\neq 4$

Gradient (m) $=\dfrac{f(x)-f(4)}{x-4}=\dfrac{x^2-16}{x-4}=x+4, x\neq 4$

As x get close to 4, $m=8$.

Therefore, the gradient of the tangent line to the curve $y=f(x)=x^2$ at $x=4$ is 8.

b.

At $x=2$, gradient (or slope) =

$$\frac{f(x)-f(2)}{x-2}=\frac{3x^2-5x-2}{x-2}=\frac{(3x+1)(x-2)}{x-2}=3x+1, x\neq 2$$

As x gets close to 2, the expression $3x+1$ approaches 7. Therefore, the gradient of the tangent line to the curve $f(x)=3x^2-5x+4$ at $a=2$ is 7.

At $x=-\sqrt{2}$, gradient (or slope)

$$\frac{f(x)-f(-\sqrt{2})}{x+\sqrt{2}}=\frac{\left(3x^2-5x+4\right)-\left(10+5\sqrt{2}\right)}{x+\sqrt{2}}$$

$$=\frac{3x^2-5x-6-5\sqrt{2}}{x+\sqrt{2}}=\frac{\left(x+\sqrt{2}\right)\left(3x-3\sqrt{2}-5\right)}{x+\sqrt{2}}$$

$$=3x-3\sqrt{2}-5, x\neq -\sqrt{2}$$

As x get close to $-\sqrt{2}$, the gradient of the function is $-6\sqrt{2}-5$.

c.

At $x=4$,

The gradient (or slope) $=\dfrac{f(x)-f(4)}{x-2}=\dfrac{\sqrt{x}-2}{x-4}=\dfrac{x-4}{(x-4)\left(\sqrt{x}+2\right)}=\dfrac{1}{\sqrt{x}+2}, x\neq 4$

As x get close to 4, the gradient will be $\dfrac{1}{4}$.

d. At $(2,\frac{1}{2})$, Gradient $=\dfrac{f(x)-f(2)}{x-2}=\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}=\dfrac{-(x-2)}{2x(x-2)}=-\dfrac{1}{2x}, x\neq 2$

As x get close to 2, the gradient of the function equals $-\frac{1}{4}$.

2.

$$\text{gradient of } f(x)=\frac{f(x)-f(a)}{x-a}=\frac{x^3-a^3}{x-a}$$

$$=\frac{(x-a)\left(x^2+ax+a^2\right)}{x-a}=x^2+ax+a^2, x\neq a$$

As x get close to a, the gradient of the given function equals to $3a^2$.

2.1.4 Definition of Derivative

Period allowed: 10 periods

Competencies

By the end of this sub-unit, students will be able to:

- State the definition of the derivative the gradient function.
- Explain what will happen when the gradient is zero.
- Find the derivative of some functions at a point.
- Find derivatives of power functions, exponential functions, logarithmic functions and simple trigonometric functions.

2.1.4.1 Derivative of Function at a Point

Teaching Note

Before starting instruction formally, ask the students orally what they understand about the difference between the two problems in Activity 2.4. Then, give them the definition of the derivative of a function at a point by linking the slope of a tangent line to a curve at a point. After the students understand the definition of the derivative at a point, discuss the procedure one should follow to find the derivative of a function at a point using the worked examples in the on student' book.

Assessment

You can ask some problems related to the topic by developing yourself. Then give exercise 2.4 as home work.

Answer to Activity 2.5

Given $f(x)=\frac{1}{x^2}$.

a. $$\frac{f(x)-f(1)}{x-1}=\frac{\frac{1}{x^2}-1}{x-1}=-\left(\frac{x+1}{x^2}\right), x\neq 1.$$

b. $$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=-\left(\frac{2x-h}{x^2(x+h)^2}\right).$$

Answer to Exercise 2.6

a. $$\frac{f(2+h)-f(2)}{h}=\frac{18+9h-7-18+7}{h}=9$$

Thus, by definition $f'(2)=9$.

b. $$\frac{f(2+h)-f(2)}{h}=\frac{(2+h)^2-1-(-3)}{h}=\frac{h(h+4)}{h}=h+4$$

Thus, by definition $f'(2)=4$.

c. $$\frac{f(3+h)-f(3)}{h}=\frac{(3+h)^3+2(3+h)-33}{h}$$

$$=\frac{h^3+9h^2+29h}{h}=h^2+12h+29$$

Thus, by definition $f'(3)=29$.

d. $$\frac{f(3+h)-f(3)}{h}=\frac{\sqrt{3(h+3)-6}-\sqrt{3(3)-6}}{h}$$

$$=\frac{\sqrt{3h+3}-\sqrt{3}}{h}=\left(\frac{\sqrt{3h+3}-\sqrt{3}}{h}\right)\left(\frac{\sqrt{3h+3}+\sqrt{3}}{\sqrt{3h+3}+\sqrt{3}}\right)$$

$$=\left(\frac{3h+3-3}{h\left(\sqrt{3h+3}+\sqrt{3}\right)}\right)=\frac{3}{\sqrt{3h+3}+\sqrt{3}}$$

Thus, by definition $f'(3) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$.

2.1.4.2 The Derivative as a Function

Activity 2.6 is designed to define the derivative as a function at a point and to inform students that some functions are differentiable on the entire domain and that some are not differentiable at some points in their domain. Hence, you may start the lesson by discussing the activity with the class. Then discuss the worked examples using the different notations for derivatives.

Assessment

Exercise 2.7 is concerned with the derivative of a function at a point. Give the exercise as group work, your role is to help the groups as necessary. Finally, ask some students to demonstrate the solutions on the black board. The exercise is also designed to assess how well students have mastered finding the derivatives of a function.

Answer to Activity 2.6

a. $\mathbb{R}$ b. $\mathbb{R}\setminus\{0\}$ c. $\mathbb{R}\setminus\{0\}$

Answer to Exercise 2.7

a.

a. $\frac{f(x+h)-f(x)}{h} = \frac{x+h-x}{h} = 1.$ $\Rightarrow f'(x) = \frac{d(x)}{dx} = 1.$

b. $$\frac{f(x+h)-f(x)}{h} = \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= 3x^2+3xh+h^2 \quad as\ h \to 0$$
$$= 3x^2.$$

c. $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{-1}{x(x+h)}$ $\Rightarrow f'(x) = -\frac{1}{x^2}$

$\Rightarrow f'(x) = 3x^2.$

d.
$$\frac{f(x+h)-f(x)}{h}=\frac{2\sqrt{x+h}-2\sqrt{x}}{h}$$
$$=2\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$=\frac{2}{\sqrt{x+h}+\sqrt{x}}\Rightarrow f'(x)=\frac{1}{\sqrt{x}}.$$

2.1.4.3 Derivative of some functions

Teaching Note

Start the lesson by revising the concepts of constant, power, polynomial, trigonometric, exponential and logarithmic functions. After the revision from different groups and give Activity 2.7 as class-work and use the activity and the examples in the student book to discuss the differentiation of power function.

Assessment

Assess students by asking questions on the revision of power functions, polynomial functions, exponential functions and logarithmic functions. Give as many problems as possible until most of the students master the rules of differentiation of power functions.

Answer to Activity 2.7

1.

a. $$\frac{f(x+h)-f(x)}{h}=\frac{x+h-x}{h}=\frac{h}{h}=1 \qquad \Rightarrow f'(x)=1$$

b. $$\frac{f(x+h)-f(x)}{h}=\frac{x^2+2xh+h^2-x^2}{h}=\frac{2xh+h^2}{h}=2x+h$$
$$As\ h\to 0, 2x+h\to 2x \quad \Rightarrow f'(x)=2x$$

c.
$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^{-2}-x^{-2}}{h}=\frac{\frac{1}{x^2+2xh+h^2}-\frac{1}{x^2}}{h}$$

$$=\frac{-2xh-h^2}{hx^2\left(x^2+2xh+h^2\right)}=\frac{-2x-h}{x^2\left(x^2+2xh+h^2\right)}$$

$$As\ h\to 0,\frac{-2x-h}{x^2\left(x^2+2xh+h^2\right)}\to\frac{-2}{x^3}=-2x^{-3}\ \Rightarrow f'(x)=-2x^{-3}$$

2. $f'(x)=nx^{n-1}$.

Answer to Exercise 2.8

1.

a. $f'(x)=8x^7$

b. $f'(x)=27x^{26}$

c. $f(x)=2x^6\Rightarrow f'(x)=12x^5$

d. $f(x)=10x^{15}\Rightarrow f'(x)=150x^{14}$

e. $f(x)=4x^{-7}\Rightarrow f'(x)=-28x^{-8}$

f. $f(x)=3x^{-20}\Rightarrow f'(x)=-60x^{-21}$

g. $f(x)=4x^{\frac{-3}{2}}\Rightarrow f'(x)=-6x^{\frac{-5}{2}}$

h. $f(x)=-10x^{\frac{-2}{5}}\Rightarrow f'(x)=4x^{\frac{-7}{5}}$

2.

$f(x)=x^{-9}\Rightarrow f'(x)=-9x^{-10}$

a. $f'(1)=-9$

b. $f'(\frac{1}{3})=-9\left(\frac{1}{3}\right)^{-10}=-3^{12}$

c. $f'(k)=-9k^{-10}$

3.

Proof of corollary 2.1

Let $f(x)=x^{-n}$.The quotient difference

$$\frac{f(t)-f(x)}{t-x}=\frac{t^{-n}-x^{-n}}{t-x}$$
$$=\frac{x^n-t^n}{(t-x)t^nx^n}=-\frac{(t^n-x^n)}{(t-x)t^nx^n}$$
$$=-\frac{(t-x)(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+tx^{n-2}+x^{n-1})}{(t-x)t^nx^n}$$
$$=-[(t^{-n}x^{-n})(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+tx^{n-2}+x^{n-1})]$$
$$=-[x^{-n}t^{-1}+x^{-n+1}t^{-2}+x^{2-n}t^{-3}+\ldots+t^{1-n}x^{-2}+t^{-n}x^{-1}]$$

$$As\ t\to x, =-\underbrace{\left[x^{-n-1}+x^{-n-1}+x^{-n-1}+\ldots+x^{-n-1}\right]}_{n-terms}$$

1)
Thus, by the definition of a derivatives, $f'(x)=-nx^{-n-1}\quad or\ \frac{d(x^{-n})}{dx}=-nx^{-n-1}$

Proof of corollary 2.2

Let $f(x)=kx^n$, then quotient difference

$$\frac{f(t)-f(x)}{t-x}=\frac{kt^n-kx^n}{t-x}=\frac{k(t^n-x^n)}{t-x}$$
$$=\frac{k(t-x)(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+tx^{n-2}+x^{n-1})}{t-x}$$
$$=k(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+tx^{n-2}+x^{n-1})\ ;x\neq t$$

Thus, by the definition of a derivative
b.

$$f'(x)=\frac{d}{dx}(kx^n)=k\underbrace{\left[x^{-n-1}+x^{-n-1}+x^{-n-1}+\ldots+x^{-n-1}\right]}_{n-terms}=knx^{n-1}.$$

Derivatives of combinations of functions

Competencies

By the end of this sub-unit, students will be able to: apply the sum, difference, product, and quotient formulae of differentiation functions.

Teaching Note

Start this section by encouraging students to complete Activity 2.8 in pairs. After almost all the students complete the work ask the students what they learned from the second problem of the activity. Then encourage them to find formulae for the derivative of the sum, difference, product and quotient of two functions. Then discuss the sum, difference, product, and quotient rules using the example provided in the student book.

Assessment

To assess your students, you can use oral questions on combinations and compositions of functions; give the first problems in exercise 2.7 as class work and the remaining as home work.

Answer to Activity 2.8

a. $f'(x)+g'(x)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(x^3+2x\right)=2x+3x^2+2.$

b. $(f+g)'(x)=\frac{d}{dx}\left(x^2+x^3+2x\right)=2x+3x^2+2.$

c. $(f-g)'(x)=\frac{d}{dx}\left(x^2-x^3-2x\right)=2x-3x^2-2.$

d. $f'(x)-g'(x)=\frac{d}{dx}\left(x^2\right)-\frac{d}{dx}\left(x^3+2x\right)=2x-3x^2-2$

Answer to Exercise 2.9

1.

$$f(x)=x^4-3x^2+2\Rightarrow f'(x)=4x^3-6x$$

Thus, $f'(3)=4(3)^3-6(3)=90.$

2.

a. $f(x)=5x^4+\sqrt{x}\Rightarrow f'(x)=20x^3+\frac{1}{2\sqrt{x}}, x\neq 0.$

b. $f(x) = x + \frac{1}{x^2} \Rightarrow f'(x) = 1 - \frac{2}{x^3}.$

c. $f(x) = \frac{1}{6}x^6 - 2x - 4 \Rightarrow f'(x) = x^5 - 2.$

d. $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} = \frac{1}{2\sqrt{x}}\left(1 + \frac{1}{x}\right).$

e. $f(x) = x^{10} + \frac{1}{x} + 6x^{\frac{5}{3}} \Rightarrow f'(x) = 10x^9 - \frac{1}{x^2} + 10x^{\frac{2}{3}}.$

f. $f(x) = 6x^{\frac{4}{3}} - \frac{1}{3x^3} - \frac{1}{2}x^2 \Rightarrow f'(x) = 8x^{\frac{1}{3}} + \frac{1}{x^4} - x.$

3.

By the derivative of the sum rule, we have $(f+g)'(x) = f'(x) + g'(x).$

$$(f-g)'(x) = (f+(-g))'(x) = f'(x) + (-g)'(x) \quad \text{(the derivative of the sum rule)}$$
$$= f'(x) - g'(x).$$

Answer to Activity 2.9

$f(x) = x^2 \Rightarrow f'(x) = 2x$

a. $3f(x) = 3x^2 \Rightarrow (3f)' = (3x^2)' = 6x.$

b. $3f'(x) = 3(2x) = 6x.$

Answer to Exercise 2.10

1.

a. $f(x) = \frac{x^{10}}{10} \Rightarrow f'(x) = 10\left(\frac{x^{10-1}}{10}\right) = x^9$

b. $f(x) = 3x^{\frac{4}{3}} \Rightarrow f'(x) = 3\left(\frac{4x^{\frac{4}{3}-1}}{3}\right) = 4x^{\frac{1}{3}}$

c. $f(x)=\sqrt{3}x^3 \Rightarrow f'(x)=3\sqrt{3}x^2$

d. $f(x)=x^4-2x^3+3x-2 \Rightarrow f'(x)=4x^3-6x^2+3.$

2. a.

$$f(x)=2\sqrt{x} \Rightarrow f'(x)=\frac{1}{\sqrt{x}}.\text{Thus, } f'(1)=1 \text{ and } f'(4)=\frac{1}{2}.$$

b. $f(x)=\frac{9}{x^3} \Rightarrow f'(x)=-\frac{27}{x^4}.\text{Thus, } f'(-1)=-27 \text{ and } f'(3)=-\frac{1}{3}.$

c.

$$y=x^3+\frac{1}{2}x^2-\frac{2}{\sqrt{x}} \Rightarrow \frac{dy}{dx}=3x^2+x+\frac{1}{x\sqrt{x}}.\text{Thus, } \left(\frac{dy}{dx}\right)(1)=5 \text{ and } \left(\frac{dy}{dx}\right)(4)=\frac{417}{8}.$$

Answer to Activity 2.10

a. $(fg)(x)=x^2(x)=x^3 \Rightarrow (fg)'(x)=\frac{d}{dx}\left(x^3\right)=3x^2$

b. $f'(x)g(x)+f(x)g'(x)=\frac{d}{dx}(x^2)x+x^2\frac{d}{dx}(x)=2x^2+x^2=3x^2.$

Answer to Exercise 2.11

a.

$$f(x)=(2x+1)(x^2+x-5) \Rightarrow f'(x)=(x^2+x-5)\frac{d}{dx}(2x+1)+(2x+1)\frac{d}{dx}(x^2+x-5)$$
$$=2(x^2+x-5)+(2x+1)(2x+1)=6x^2+6x-9.$$

b.

$$f(x)=\left(x^2-2\right)(x^3+4x) \Rightarrow f'(x)=(x^3+4x)\frac{d}{dx}\left(x^2-2\right)+\left(x^2-2\right)\frac{d}{dx}(x^3+4x)$$
$$=2x(x^3+4x)+\left(x^2-2\right)\left(3x^2+4\right)=5x^4+6x^2-8.$$

c.

$$f(x)=x(x^2+\sqrt{x}) \Rightarrow f'(x)=(x^2+\sqrt{x})\frac{d}{dx}(x)+x\frac{d}{dx}(x^2+\sqrt{x})=x^2+\sqrt{x}+2x+\frac{1}{2\sqrt{x}}.$$

d.

$$f(x)=(x+2\sqrt{x})(x^2-2)$$

$$\Rightarrow f'(x)=(x^2-2)\frac{d}{dx}(x+2\sqrt{x})+(x+2\sqrt{x})\frac{d}{dx}(x^2-2)$$

$$=(x^2-2)\left(1+\frac{1}{\sqrt{x}}\right)+(x+2\sqrt{x})(2x)$$

Answer to Exercise 2.12

1.

a.

$$f(x)=\frac{1}{x^2+1}\Rightarrow f'(x)=\frac{\frac{d}{dx}(1)\left(x^2+1\right)-1\cdot\frac{d}{dx}\left(x^2+1\right)}{(x^2+1)^2}=-\frac{2x}{(x^2+1)^2}.$$

b.

$$f(x)=\frac{2x^3+3}{x-5}\Rightarrow f'(x)=\frac{\frac{d}{dx}(2x^3+3)(x-5)-\left(2x^3+3\right)\frac{d}{dx}(x-5)}{(x-5)^2}=\frac{4x^3-30x^2-3}{(x-5)^2}.$$

c.

$$f(x)=\frac{x^3+3x}{x^2}\Rightarrow f'(x)=\frac{\frac{d}{dx}(x^3+3x)\left(x^2\right)-\left(x^3+3x\right)\frac{d}{dx}\left(x^2\right)}{(x^2)^2}=1-\frac{3}{x^2}.$$

d.

$$f(x)=\frac{2x^4}{\sqrt{x}}\Rightarrow f'(x)=\frac{\frac{d}{dx}(2x^4)\left(\sqrt{x}\right)-\left(2x^4\right)\frac{d}{dx}\left(\sqrt{x}\right)}{(\sqrt{x})^2}=\frac{7x^4}{2x\sqrt{x}}.$$

2.1.4.4.5 The Chain Rule

Teaching Note

Before starting the lesson encourage students to do the first question of Activity 2.8 individually and observe the relation between the solutions for *c* and *d*. Then discuss some of the introductory example in the student textbook and order the students to do the second problem of Activity 2.8 in groups and allow them to explain how the derivatives of the powers could be determined without expanding the expressions.

Now, give them the precise chain rule formula to find the derivative of compositions of functions.

Assessment

Ask as many questions as possible so that students can understand the chain rule formula and give Exercise 2.8 as home-work. The exercise is designed to assess how well students have mastered the derivatives of composition of functions using the chain rule formula.

Answer to Activity 2.11

1.

a. $f'(x) = 2$

b. $f(g(x)) = f(x^2 - 1) = 2x^2 + 1$

c. $f'(g(x)) = 4x.$

d. $f'(g(x)) \cdot g'(x) = 2(2x) = 4x.$

2.

a. $f(x) = (3x^2 + 2)^2 = (3x^2 + 2)(3x^2 + 2)$

$\Rightarrow f'(x) = 6x(3x^2 + 2) + (3x^2 + 2)(6x) = 12x(3x^2 + 2)$

b. $$f(x) = \left(\sqrt{x^3 + 2x - 7}\right)^3 \Rightarrow f'(x) = 3\left(\sqrt{x^3 + 2x - 7}\right)^2 \cdot \frac{1}{2\sqrt{x^3 + 2x - 7}} \cdot \left(3x^2 + 2\right)$$

$$= \frac{3}{2}\left(3x^2 + 2\right)\left(\sqrt{x^3 + 2x - 7}\right).$$

Answer to Exercise 2.13

a. $f(x) = (2x+1)^4 \Rightarrow f'(x) = 8(2x+1)^3.$

b. $f'(x) = 25(x^3 + 2x)^{24}(3x^2 + 2)$

c. $f(x) = \sqrt{x^2 + 3x - 1} \Rightarrow f'(x) = \dfrac{2x+3}{2\sqrt{x^2 + 3x - 1}}.$

d. $f(x) = \dfrac{1}{(3x+2)^2} \Rightarrow f'(x) = -\dfrac{6}{(3x+2)^3}.$

e. $f'(x)=\frac{1}{2}\sqrt{\frac{x^2}{x^4+1}}\left(\frac{2x^5-2x}{x^4}\right)=\frac{x^3-1}{x^3}\sqrt{\frac{x^2}{x^4+1}}$

f. $f'(x)=6\left(\frac{3x-5}{2-7x}\right)^5\left[\frac{3(2-7x)+7(3x-5}{(2-7x)^2}\right]=-\frac{174(3x-5)^5}{(2-7x)^7}$

2.1.5 Maximum and Minimum Points

Introduction

The main objective of this section is to examine the use of differential calculus in finding extreme values, in writing the equation of a tangent line and the equation of a normal line to a curve.

Competences

By the end of this sub-unit, students will be able to:

- ✓ Find local maximum point and local minimum point of a given function on an interval.
- ✓ Write the equation of a tangent line and a normal line to a curve.

Teaching Note

You can begin this section with an opening problem which may motivate students to follow the lesson attentively and encourage students to do Activity 2.9 by forming groups, and letting them present their response to the whole class through a group representative before defining the technical terms which are vital in determining extreme values of a function. Activity 2.9 is designed to prepare students for the next discussion. For instance, the first question helps students to find critical numbers in the foregoing discussion. Give them feedback on the activity, provide the definitions of the technical terms required to determine extreme values of a function and discuss the procedure that which should be followed to find extreme value by using the worked examples in the student text book.

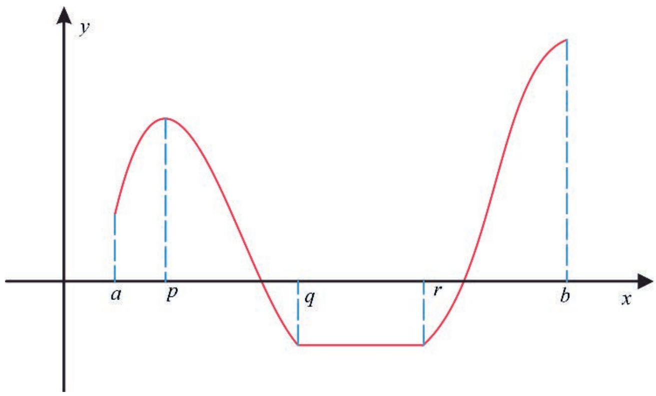
Assessment

To assess the extent your students have mastered the concepts of the lesson, you can ask as many questions as possible and give the first question in the Exercise 2.9 as class-work and the remaining question as home work.

Answer to Activity 2.12

1. a. $x = 2$ b. $x = -3$, $x = 4$ c. $x = 1$

2. Consider the graph of the following function.



The graph of the function f

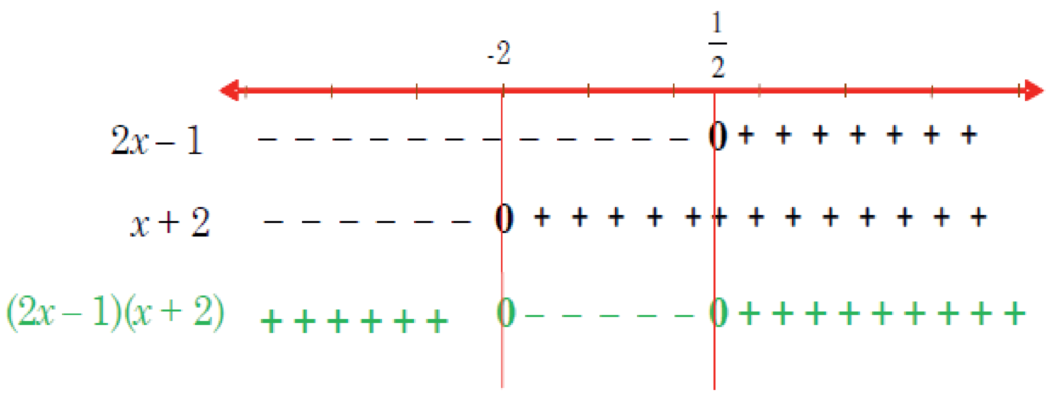
i. Increases on the interval $(a, p) \cup (r, b)$

ii. Falls on the interval (p, q)

3.

a. $2x^2 + 3x - 2 \geq 0 \Leftrightarrow (2x-1)(x+2) \geq 0$

The above results are shown in the sign chart given below:

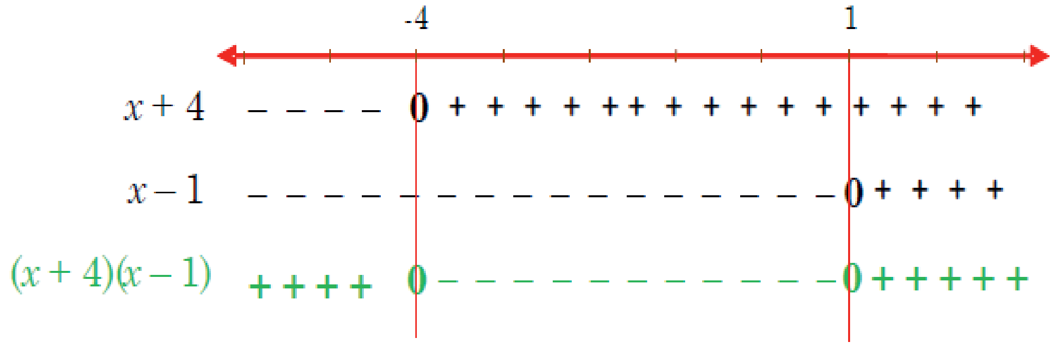


From the sign chart, you can conclude that the solution is

$$(-\infty,-2]\cup[\frac{1}{2},\infty).$$

b. $x^2+3x-4<0 \Leftrightarrow (x+4)(x-1)<0$

The results are shown in the sign chart given below:



From the sign chart, you can conclude that the solution is $(-4,1)$.

Answer to Exercise 2.14

1. a.

$$f(x)=x^3+6x^2 \Rightarrow f'(x)=3x^2+12x=3x(x+4)$$

$$\Rightarrow f'(x)=0 \text{ when } x=0 \text{ or } x=-4$$

Thus, the critical numbers are $x=-4,0$.

b. $f'(x)=\dfrac{4x(1-x^2)-4x^3}{1-x^2}=\dfrac{4x}{1-x^2}$

$\Rightarrow f'(x)=0$ if $x=0$ and $f'(x)$ is undefined at $x=-1$ and $x=1$.

Thus, the critical numbers of f are $x=0,\ x=-1$ and $x=1$.

c.

$$f(x)=2x^3+9x^2-24x \Rightarrow f'(x)=6(x^2+3x-4)=6(x+4)(x-1)$$

$$\Rightarrow f'(x)=0 \text{ when } x=-4 \text{ or } x=1.$$

Thus,the critical numbers are $x=-4,1$.

d. $f'(x)=8\sqrt{1-x^2}+8x\left(\dfrac{-2x}{2\sqrt{1-x^2}}\right)=\dfrac{8(1-2x^2)}{\sqrt{1-x^2}}$

$f'(x)=0 \Rightarrow 1-2x^2=0 \Rightarrow x=-\dfrac{\sqrt{2}}{2}\ or\ x=\dfrac{\sqrt{2}}{2}$

and $f'(x)$ is undefined at $x=-1\ or\ x=1$.

Thus, the critical numbers are $x=-\dfrac{\sqrt{2}}{2}$, $x=\dfrac{\sqrt{2}}{2}$, $x=-1$ and $x=1$.

2. a. $f(x)=x^3+6x^2$

$\Rightarrow f'(x)=3x^2+12x \Leftrightarrow f'(x)=3x(x+4)$

$f'(x)=0\ \ if\ \ x=0\ \ or\ x=-4$

	-4		0
$3x$	-----------	-----------------------------------	+++++++++++++++
$(x+4)$	-----------	++++++++++++++++++++++	+++++++++++++++
$f'(x)$	++++++++	-----------------------------------	+++++++++++++++

Hence, by FDT f is increasing on the interval $(-\infty,-4]\cup[0,\infty]$ and decreasing on the interval $[-4,0]$.

b. $f(x)=2x^2+6x-10$

$f'(x)=4x+6=2(2x+3)$

$f'(x)=0 \Rightarrow x=-\dfrac{3}{2}.$

	$\frac{-3}{2}$	
$2x+3$	-----------	++++++++++++++
$f'(x)=2(2x+3)$	-----------	++++++++++++++

Hence, by FDT f decreasing is decreasing on the interval $(-\infty,\dfrac{-3}{2}]$ and increasing on the interval $[\dfrac{-3}{2},\infty)$

c. $f(x)=6+12x+3x^2-2x^3$

$\Rightarrow f'(x)=12+6x-6x^2=-6(x^2-x-2)=-6(x-2)(x+1)$
$f'(x)=0\ \ if \Rightarrow x=2\ \ or\ x=-1$

	-1		2
$-6(x-2)$	+++++++	+++++++++++++++++++++	-----------------------------
$(x+1)$	-----------	++++++++++++++++++++	++++++++++++++++++
$f'(x)$	-----------	++++++++++++++++++++	-----------------------------

Hence, by FDT f is increasing on the interval $[-1,2]$ and decreasing on the interval $(-\infty,-1]\bigcup[2,\infty)$

d.

$$f(x)=x-6\sqrt{x-1}, dom f:(1,\infty)\Rightarrow f'(x)=\frac{\sqrt{x-1}-3}{\sqrt{x-1}}$$

$f'(x)=0\Rightarrow x=10$ and $f'(x)$ *is* undefined *at* $x=1$

	1	10
$\sqrt{x-1}-3$	-----------------------------------	++++++++++++++++++
$\sqrt{x-1}$	++++++++++++++++++++	++++++++++++++++++
$f'(x)=\dfrac{\sqrt{x-1}-3}{\sqrt{x-1}}$	-----------------------------------	++++++++++++++++++

Thus ,by FDT f is increasing on the interval $[10,\infty)$ and decreasing on the interval $(-\infty,1)$.

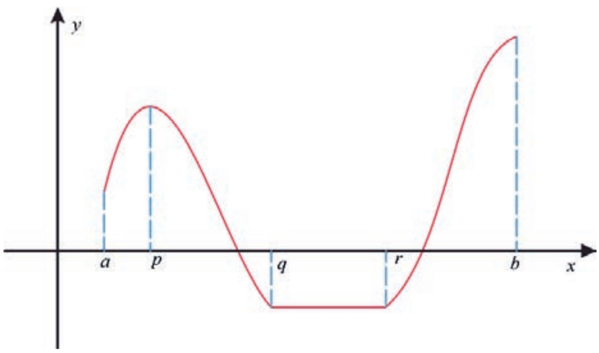
e.

$f(x)=(x+3)^2(x-1)^2$
$\Rightarrow f'(x)=(x+3)(x-1)(x+1)$
$f'(x)=0\Rightarrow x=-3,-1,1$

		-3		-1		1	
$x+3$	----	0	++++		++++		+++++
$x-1$	----		-----		-----	0	++++
$x+1$	----		-----	0	++++		++++
$f'(x)=(x+3)(x-1)(x+1)$	----	0	++++	0	-----	0	++++

Thus ,by FDT f is increasing on the interval $[-3,-1]\cup[1,\infty)$ and decreasing on the interval $(-\infty,-3]\cup[-1,1]$.

Answer to Activity 2.13



a. A function attains maximum value and minimum value at a point on the graph where the function has horizontal tangent line.

b. At both maximum point and minimum point the gradient of a function is zero.

Answer to Exercise 2.15

1.

a. $f'(x)=2x+2=2(x+1)$
$f'(x)=0\Rightarrow 2(x+1)=0\Rightarrow x=-1$

Therefore, the only critical numbers of f on the given interval is $x=-1$.

$f(-2)=(-2)^2+2(-2)+3=3$

$f(-1)=(-1)^2+2(-1)+3=2$

$f(2)=(2)^2+2(2)+3=11$

Hence, the absolute maximum value of f is 11 and the absolute minimum value of f is 2.

b. $f'(x) = 3x^2 + 6x - 9$

$f'(x) = 0 \Rightarrow 3x^2 + 6x - 9 = 0$

$\Rightarrow 3(x+3)(x-1) = 0$

$\Rightarrow x = -3 \; or x = 1.$

Therefore, the only critical numbers of f on the given interval is $x = 1$.

$f(-2) = 27, f(1) = 0 \; and \; f(2) = 7.$

Hence, the absolute maximum of f is 27 and its absolute minimum is 0.

c. $f'(x) = 12x^3 - 78x^2 + 120x = 6x(2x^2 - 13x + 20) = 6x(x-4)(2x-5)$

$f'(x) = 0 \; at \; x = 0, 4, \frac{5}{2}$. Thus, on the given interval the critical numbers are

$x = 4 \; and \; x = \frac{5}{2}$.

$f(1) = 26, f(\frac{5}{2}) = \frac{1199}{16} = 74.9, f(4) = 53 \; and \; f(5) = 114.$

Hence, the global maximum value of f on the interval is 114 and its minimum value is 26.

d. $f'(x) = \frac{2(x-1)(x-2) - (x^2 - 2x + 4)}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$.

$f'(x) = 0 \; if \; x = x \; or \; x = 4 \; and \; f'(x) \; is \; undefined \; if \; x = 2.$

Therefore, the only critical number of f on the given interval is $x = 0$.

$f(-3) = \frac{-19}{5}, f(0) = -2 and \; f(1) = -3.$

Hence, the maximum value of f is -2 and the minimum value is $\frac{-19}{5}$.

Answer to Exercise 2.16

1.

a. $f'(x)=6x^2-18x=6x(x-3)$
$f'(x)=0$ if $x=0$ or $x=3$

	$\quad 0$	$\quad 3$	
$6x$	------------	++++++++++++	++++++++++++++
x-3	--------------	-----------------	++++++++++++++
$f'(x)$	+++++++++	----------------	++++++++++++

Thus, $f(0)=-24$ is local minimum and $f(3)=-51$ is a local maximum value of the function.

b. $f'(x)=\frac{3}{4}x^2-3 \Rightarrow f'(x)=0 \Leftrightarrow 3x^2-12=3(x+2)(x-2)=0$

	$\quad -2$	$\quad 2$	
$3(x+2)$	-----------	++++++++++	++++++++++++
x-2	------------	--------------	+++++++++++++
$f'(x)$	++++++++++	-------------	++++++++++++

Therefore, f has local minimum at $x=2$ and local maximum at $x=-2$
$\Rightarrow f(2)=-4$ is local minimum value of f and $f(-2)=4$ is a local maximum value of f.

c. $f'(x)=\frac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2-x+1)^2}$

$$=\frac{2(1-x^2)}{\left(x^2-x+1\right)^2}$$

$$\Rightarrow f'(x)=0 \Leftrightarrow x=-1 \text{ or } x=1.$$

	-1		1
2(x+1)	------------	++++++++++	+++++++++++
x-1	+++++++++	++++++++++	--------------
f'(x)	--------------	+++++++++++	--------------

Therefore, $f(-1)=\frac{1}{3}$ is a local minimum value and $f(1)=3$ is a local maximum value of *f*.

2.

$$f(x)=x^3+ax+b \Rightarrow f'(x)=3x^2+a$$
f has minimum value at $x=2 \Rightarrow f'(2)=0$
$$\Rightarrow 3(2)^2+a=0 \Rightarrow a=-12$$
$$f(2)=4 \Rightarrow 8-24+b=4 \Leftrightarrow b=20$$

Equation of tangents and normal to curves

Teaching Note

Before starting the lesson motivate students by letting them complete Activity 2.10 in groups and asking opening problems. Activity 2.10 is designed to assess the background knowledge of students on the relationship between the slopes of two non- vertical perpendicular lines, and the slopes of their derivatives.

Assessment

Use Exercise 2.10 to assess how well your students understand the concept of the lesson. You should give the first two questions as class work and the remaining questions as homework and assess students by checking their exercise books.

Answer to Activity 2.14

1. Using two point equation a line : $y-y_0=\left(\frac{y_1-y_0}{x_1-x_0}\right)(x-x_0)$, we have

$$y-3=\left(\frac{3-3}{-2-1}\right)(x-1) \Leftrightarrow y-3=0 \Leftrightarrow y=3$$

2. Thus, the equation of the line containing $(1,3)$ and $(-2,3)$ is $y=3$.
3. The equation of a line perpendicular to the line in equation (1) containing the point (1,3) is $x=1$.
4. $y-5=3(x+2) \Leftrightarrow y=3x+11$
5. The product of the slope of two non-vertical perpendicular lines is -1. Thus the slope of the normal line is the negative reciprocal of the slope the tangent line.

The first derivative of a function at a point and the slope of a tangent line to a curve at that point is equal or the same.

Answer to Exercise 2.17

1.

a. $f'(x)=3x^2+1 \Rightarrow f'(0)=1.$

Therefore, the equation of the tangent line at $(0,2)$ is $y=(x-0)+2=x+2.$

And the equation of the normal line at $(0,2)$ is $y=-x+2$.

b. $f'(x)=-3x^2(\sqrt{x+2})+\dfrac{1}{2\sqrt{x+2}}(1-x^3)$

$$=\frac{-6x^2(x+2)+1-x^3}{2\sqrt{x+2}}=\frac{-7x^3-12x^2+1}{2\sqrt{x+2}}$$

$\Rightarrow f'(-1)=-2$

Therefore, the equation of the tangent line at $(-1,2)$ is

$$y=-2(x+1)+2=-2x.$$

And the equation of the normal line at $(-1,2)$ is

$$y=\frac{1}{2}(x+1)+2=\frac{1}{2}x+\frac{5}{2}.$$

2. The function will have a vertical tangent line at its first derivative is undefined in its domain.

a. $f(x)=x^3-3x^2+1 \Rightarrow f'(x)=3x^2-6x$

$\Rightarrow f$ has no verical tangent line.

$$f(x)=2\sqrt{x}\Rightarrow f'(x)=\frac{1}{\sqrt{x}}$$

b. $\Rightarrow f'(x)$ is undefined at $x=0$.

Thus, f has vertical tangent line at $x=0$.

3.

$f(x)=x^2+ax+b\Rightarrow f(1)=1+a+b=2$

$g(x)=x^3-c\Rightarrow g(1)=1-c=2=2$

$$\Rightarrow\begin{cases}a+b=1\\ c=-1\end{cases} \qquad (1)$$

The two functions $f(x)$ and $g(x)$ have the same tangent line at $(1,2)$. This implies that

$f'(1)=g'(1)\Rightarrow 2+a=3\Rightarrow a=1$ (2)

From equation (1) and equation (2), we obtain $b=0$.

Therefore, the values of a,b,c is respectively $1,0,-1$.

2.2 Applications of Derivatives

Period allowed: 10 periods

Competencies

By the end of this section students will be able to:

- Apply the concept of derivatives to solve problems on applications of derivative in different fields of study.

Teaching Note

You may start this section by summarizing the basic tools which are necessary to find the absolute maximum and absolute minimum and give Activity 2.11 as class work in groups. Then extend the notion to practical problems which may be encountered in our daily life. You may encourage students to think about the problems and be able to convert into mathematical language as illustrated by examples in the students' textbook. After having illustrated them thoroughly using the examples given in the student book, you can summarize how the students convert problems into mathematical language and solve them.

Assessment

Give various problems related to road traffic, tax education, customer protection, climate change and anti-doping as class work and home works followed by reflection. You can use Exercise 2.11 and Exercise 2.12 for assessment purpose.

Answer to Activity 2.15

1.
   a. $K = \{11, 14, 17, 20, 23\}$
   b. The smallest element of K is 11.
   c. The largest element of K is 23.

2.
   a. $K = (11, 23)$
   b. We cannot list all the elements of K.
   c. K has no smallest element.
   d. K has no largest element.

3.
   a. $K = [11, 23]$
   b. We cannot list all the elements of K.
   c. 11 is the smallest element of K.
   d. 23 is the largest element of K.

Answer to Exercise 2.18

1.

$$\text{Given } \frac{dr}{dt} = 0.3cm/s \text{ , } r = 15cm.$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=15cm} = 2\pi(15cm)(0.3cm/s) = 9\pi cm^2/s$$

2.

Given $\frac{dv}{dt} = 0.2\pi cm^3/s, r = 10cm$

$$V = \frac{4\pi r^3}{3} \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{0.2\pi}{4\pi r^2} = \frac{0.2}{4r^2} \Rightarrow \left.\frac{dr}{dt}\right|_{r=10cm} = 0.0005cm/\sec.$$

3.

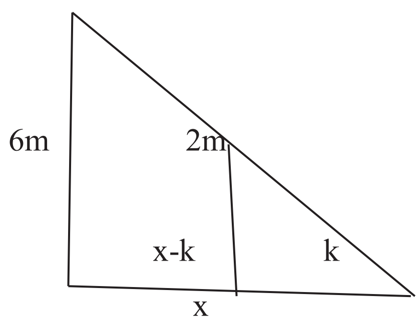
Given $\frac{ds}{dt} = 0.2cm/s, s = 20cm.$

$$A = s^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{ds}\frac{ds}{dt} = 2s\frac{ds}{dt}$$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{s=20cm} = 2x20cmx0.2cm/s = 8cm^2, s.$$

4.

Given $\frac{dx}{dt} = 8m/\min.$



$$\frac{x}{k} = \frac{6}{2} \Rightarrow k = \frac{1}{3}x$$

$$\therefore \frac{dk}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}(8m/\min) = \frac{8}{3}m/\min$$

5. $F = ma, \frac{dF}{dt} = 2N/\sec. \Rightarrow \frac{dF}{dt} = m\frac{da}{dt}$

$$\Rightarrow \frac{da}{dt} = \frac{1}{m}\frac{dF}{dt} = \frac{1}{m}(2N/\sec) = \frac{2}{m}N/\sec.$$

Answer to Activity 2.16

1. Derivatives have vast applications in economics, in road taxation, education, in finance and business, in physics, in different engineering fields etc.
2. yes it has vital application in these fields of study.

Answer to Exercise 2.19

Solution:

a. $P(x) = R(x) - C(x)$

$$= (x^3 - 12x^2 + 40x + 10) - (62x^2 + 27,500)$$
$$= x^3 - 74x^2 + 40x - 27,490$$

b. $C(x) = 62(2500) + 27,500 = 155,000 + 27,500 = 182,500$

$$R(50) = (50)^3 - 12(50)^2 + 40(50) + 10$$
$$= 125,000 - 30,000 + 2,000 + 10 = 97,010$$

$$P(50) = 125,000 - 74(2500) + 40(50) - 27,490 = 127,000 - 212,490 = -85,490$$

c. $MC = C'(x) = 124x$

i. $MC = C'(50) = 124 \times 50 = 6,200$

ii. $MR = R'(x) = 3x^2 - 24x + 40$

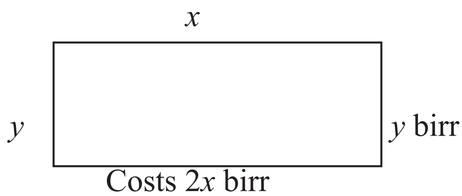
$$MR = R'(50) = 3(2500) - 24(50) + 40 = 7500 - 1200 + 40 = 6,340$$

iii. $MP = P'(x) = 3x^2 - 148x + 40$

$$MP = MR - MC = 6,340 - 6,200 = 140.$$

Answer to Exercise 2.20

1. Let x and y be the dimensions of the rectangular area as shown below.



The total area of the rectangle is given to be

$A = xy = 3200$

And the total cost of the material is

$C(x, y) = 4x + 2y$. But from $xy = 3200$, $y = \dfrac{3200}{x}$.

So, $C(x, y) = 4x + 2\left(\dfrac{3200}{x}\right) = 4x + \dfrac{6400}{x}$.

Now, find the critical number by the use of first derivative.

$C'(x, y) = 4 - \dfrac{6400}{x^2} \Rightarrow x = 40. Thus,\ y = 80.$

Therefore, the dimensions of the rectangle are 40ft for the sides costing 2 dollars and 80ft for the sides costing 1 dollar.

2. Let x be the side of the square and h be its height. Total surface are

$$S = x^2 + 4xh = 48. \Rightarrow h = \frac{48 - x^2}{4x}.$$

$$V = x^2 h = x^2\left(\frac{48 - x^2}{4x}\right) = \frac{48x - x^3}{4}$$

$$\frac{dV}{dx} = \frac{1}{4}\left(48 - 3x^2\right) \Rightarrow V'(x) = 0 \ if \ 3x^2 = 48$$

$$\Rightarrow x^2 = 16 \Leftrightarrow x = 4 \ and \ h = 2.$$

4

$x-4$	----------------	++++++++++++++++
$V'(x)$	+++++++++	-----------------------------

Therefore, $V(4)$ is maximum value. Hence, the dimension of the box that will give maximum volume is

$x=4$ and $h=2$ and the maximum volume is $V=16x2=32$cubic unit.

3. **Solution:** Let r be the base radius and h be the height of the cylindrical can.

$$Total\ area = 2\pi r^2 + 2\pi rh \ \ and\ V = \pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2}$$

Thus, $A(r) = 2\pi r^2 + \dfrac{256\pi}{r}$,

So, $A'(r) = 4\pi r - \dfrac{256\pi}{r^2} = 0 \Rightarrow r^3 - 64 = 0 \Rightarrow r = 4\ and\ h = \dfrac{128}{r^2} = 3.$

Therefore, the dimension of the can that will minimize the amount of material to be used is $r = 4\ and\ h = 3$.

2.3 Introduction to Integration

Answer to Activity 2.17

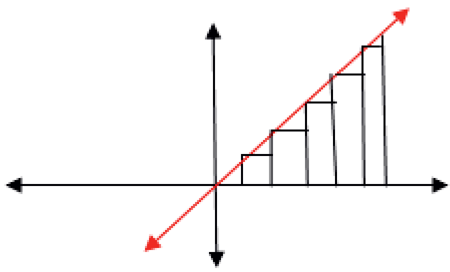
1. Yes
2. Yes
3.

   a) We see the region corresponding to the time interval $0 \le x \le 30$ is a triangle with base 30 and height 15 (since $v(30)=15$). Therefore, the area of this region is $A = \frac{1}{2}(30)(15) = 225\, ft/mi$. The object travelled $225\,ft$ during the first 30 mi.

   b) The region corresponding to the time interval $60 \le x \le 120$ is a trapezoid. It can be decomposed into a rectangle and a triangle. The rectangle has base 60 and height 30, and thus an area $A = (60)(30) = 1800\,ft$; the triangle has base 60 and height 30, for an area $A = \frac{1}{2}(60)(30) = 900\,ft$. Summing these, we get 2700.

Therefore, the object traveled 2700 ft between the one hour and the two hour.

4.



$$\Delta x = \frac{b-a}{n} = \frac{6-0}{6} = 1.$$

$$f(x_0) = f(0) = 2(0) = 0 \qquad f(x_4) = f(4) = 2(4) = 8$$
$$f(x_1) = f(1) = 2(1) = 2 \qquad f(x_5) = f(5) = 2(5) = 10$$
$$f(x_2) = f(2) = 2(2) = 4 \qquad f(x_6) = f(6) = 2(6) = 12$$
$$f(x_3) = f(3) = 2(3) = 6$$

$$\begin{aligned}\text{Area on } [0,6] &= \sum_{i=0}^{6} f(x_i)\Delta x \\ &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= 0(1) + (2)(1) + (4)(1) + (6)(1) + (8)(1) + (10)(1) \\ &= 30\end{aligned}$$

5.

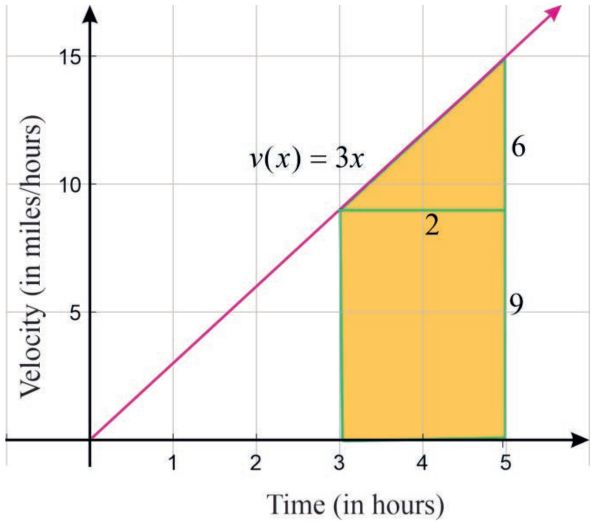
$$\begin{aligned}\text{Area on } [0,6] &= \sum_{i=0}^{6} f(x_i)\Delta x \\ &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= (2)(1) + (4)(1) + (6)(1) + (8)(1) + (10)(1) + (12)(1) \\ &= 42\end{aligned}$$

6. $$A = \int_0^6 2x\,dx = 2\left.\frac{x^2}{2}\right|_0^6 = (6)^2 - (0)^2 = 36 - 0 = 36.$$

7. $A(\text{lower sum}) = 30 \le A(\text{exact}) = 36 \le A(\text{upper sum}) = 42$

Answer to Exercise 2.21

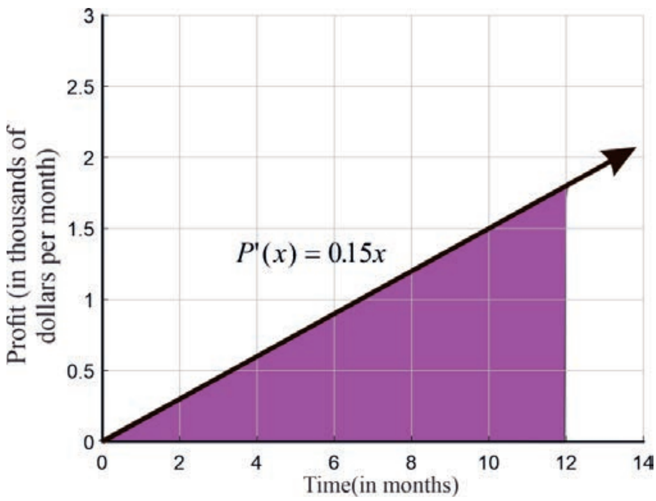
1.



Therefore, the area of this region is

$$A = A_{\square} + A_{\triangle}$$
$$= (2)(9) + \frac{1}{2}(2)(6)$$
$$= 24$$

2. The graph of P' is shown below:



For the 12-month period, the area is calculated by using the formula for a triangle:

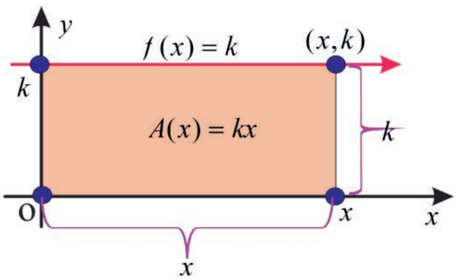
$$A=\frac{1}{2}(12)\left(1.8\frac{\text{Thousands of dollars}}{\text{month}}\right)=10.8 \text{ thousands of dollars.}$$

A company earned a total profit of \$10,800 in a year.

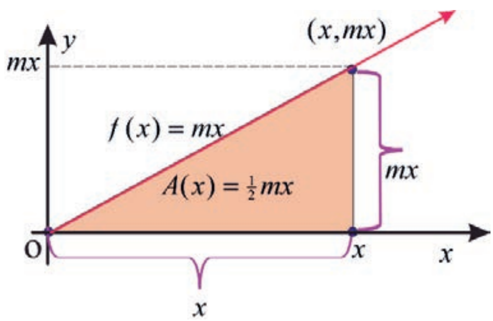
Answer to Activity 2.18

1.

✓ If $f(x)=k$, where k is a constant, its graph is a horizontal line of height k. The region under this graph over the interval $[0, x]$ is a rectangle, and its area is $A=k.x$ (height times base).



✓ If $f(x)=mx$, its graph is a line of slope m, passing through the origin. The region under this graph over an interval $[0, x]$ is a triangle, and its area is $A=\frac{1}{2}(x)(mx)=\frac{1}{2}mx^2$.



a. No b. No

c. We use either the concept of Riemann sum or Numerical techniques

2.

i. $5+10+15+20+25=\sum_{k=1}^{5}5k$

ii. $g(x_1)\Delta x+g(x_2)\Delta x+...+g(x_{19})\Delta x=\sum_{k=1}^{19}g(x_k)\Delta x$

Answer to Exercise 2.22

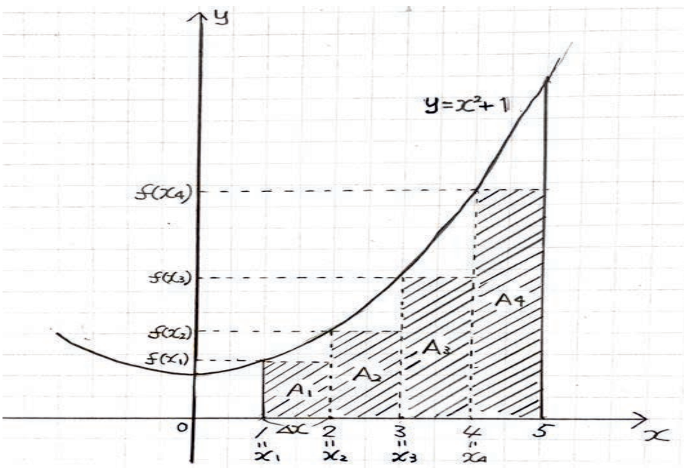
1 a. Use the following steps to find it.

Step 1: Interval $[1,\ 5]$ is divided into four subintervals, each having width $\Delta x=\frac{5-1}{4}=1$

Step2: Find the values of $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$, when $x_1=1$, $x_2=2$, $x_3=3$ and $x_4=4$

Step3: Since the area of the region is approximate the sum of the areas of four rectangles A_1, A_2, A_3 and A_4.

Calcúlate $A_1+A_2+A_3+A_4=\sum_{i=1}^{4}f(x_i)\Delta x$.



$A=A_1+A_2+A_3+A_4=2+5+10+17=34$ sq.unit

b. Find the approximate area of the same region in the same way as a, when the interval $[1,\ 5]$ is divided into eight subintervals, each having width $\Delta x=\frac{5-1}{8}=0.5$

2. $(1^2+1)+(2^2+2)+(3^2+3)+(4^2+4)+(5^2+5)+(6^2+6)$

Answer to Activity 2.19

1.

a. $$\Delta x = \frac{b-a}{n} = \frac{5-0}{5} = 1.$$

$$f(x_0) = f(0) = 3(0)^2 = 0$$
$$f(x_1) = f(1) = 3(1)^2 = 3$$
$$f(x_2) = f(2) = 3(2)^2 = 12$$
$$f(x_3) = f(3) = 3(3)^2 = 36$$
$$f(x_4) = f(4) = 3(4)^2 = 48$$
$$f(x_5) = f(5) = 3(5)^2 = 75$$

The Riemann sum of on $[0,5] = \sum_{i=1}^{5} f(x_i)\Delta x$

$$= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$
$$= 0(1) + (3)(1) + (12)(1) + (36)(1) + (48)(1) + (75)(1)$$
$$= 174$$

b.

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{10} = 0.5.$$

$$f(x_0) = f(0) = 3(0)^2 = 0$$
$$f(x_1) = f(0.5) = 3(0.5)^2 = 0.75$$
$$f(x_2) = f(1) = 3(1)^2 = 3$$
$$f(x_3) = f(1.5) = 3(1.5)^2 = 6.75$$
$$f(x_4) = f(2) = 3(2)^2 = 12$$
$$f(x_5) = f(2.5) = 3(2.5)^2 = 18.75$$
$$f(x_6) = f(3) = 3(3)^2 = 36$$
$$f(x_7) = f(3.5) = 3(3.5)^2 = 36.75$$
$$f(x_8) = f(4) = 3(4)^2 = 48$$
$$f(x_9) = f(4.5) = 3(4.5)^2 = 60.75$$
$$f(x_{10}) = f(5) = 3(5)^2 = 75$$

The Riemann sum of on $[0,5]=\sum_{i=1}^{5} f(x_i)\Delta x$

$$\begin{aligned}
&= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&\quad + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x + f(x_9)\Delta x + f(x_{10})\Delta x \\
&= 0(0.5)+(0.75)(0.5)+(3)(0.5)+(6.75)(0.5)+(12)(0.5)+(18.75)(0.5) \\
&\quad +(36)(0.5)+(36.75)(0.5)+(48)(0.5)+(60.75)(0.5)+(75)(0.5) \\
&= 148.875
\end{aligned}$$

Answer to Exercise 2.23

a. 6 b. 16 c. 12 d. 13.5

Answer to Activity 2.20

1. a) $f(x)=x^2$, b) $f(x)=x^2+5$ c) $f(x)=x^2-10$
2. All of the functions have same derivative $2x+3$.

Answer to Exercise 2.24

$a)\int x^5 dx=\frac{x^6}{6}+c$ $b)\int x^{-4}dx=-\frac{x^{-3}}{3}+c$ $c)\int x^{\frac{1}{3}}dx=\frac{3x^{\frac{4}{3}}}{4}+c$

$d)\int x^{\frac{-3}{2}}dx=-2x^{\frac{-1}{2}}+c$ $e)\int x^{10}dx=\frac{x^{11}}{11}+c$ $f)\int x^{-2}dx=-x^{-1}+c$

$g)\int x^{\frac{3}{2}}dx=\frac{2}{5}x^{\frac{5}{2}}+c$ $h)\int x^{-\frac{3}{5}}dx=\frac{5}{2}x^{\frac{2}{5}}+c$

Answer to Exercise 2.25

$a)\int 99dx=99x+c$ $b)\int 10x^4dx=2x^5+c$ $c)\int \frac{3}{x}dx=3\ln x+c$

$d)\int (x^2-x)dx=\frac{x^3}{3}-\frac{x^2}{2}+c$ $e)\int (3x^2-2x+5)dx=x^3-x^2+c$

$f)\int (-2x^2-x-6)dx=-\frac{2x^3}{3}-\frac{x^2}{2}-6x+c$

$g)\int (x^{-2}-4x^3+\sqrt{x})dx=-x^{-1}-x^4+\frac{1}{2\sqrt{x}}+c$

Answer to Exercise 2.26

1. **a)** $\int_1^3 (3x^2+2x)dx = 34$ b) $\int_{-1}^0 \left(x^3-3x+1\right)dx = \frac{9}{4}$

c) $\int_4^{15} (0.002x^4 - 0.3x^2 + 4x - 7)dx = \frac{783101}{2500}$

2. a) $A = 2\int_0^5 \left(x^2+1\right)dx = \frac{140}{3} = 46\frac{2}{3}$ b) $A = \int_{-2}^2 \left(x^2+1\right)dx = 12\text{sq.unit}$

$$A = \int_{-2}^2 \left(x^2+1\right)dx = 2\int_0^2 \left(x^2+1\right)dx = 12 \ sq.unit$$

Answer to Exercise 2.27

a) $\int_1^3 x^2 dx = \frac{26}{3}$ $b)$ $\int_{-2}^3 (-3x^2+4x-5)dx = -50$

$c)$ $\int_0^2 (x^2-2x+8)dx = \frac{44}{3}$ $d)$ $\int_1^4 \left(2x+\frac{1}{x^2}\right)dx = \frac{63}{4}$

Answer to Activity 2.21

a)

$$f(x) = k,$$

$$A = \int_a^b f(x)dx = \int_a^b kdx = kx\Big|_a^b = k(b-a).$$

b)

$$f(x) = x,$$

$$A = \int_a^b f(x)dx = \int_a^b xdx = \frac{x^2}{2}\Bigg|_a^b = \frac{1}{2}(b^2-a^2).$$

c)

$$f(x)=x^2,$$

$$A=\int_a^b f(x)dx=\int_a^b x^2dx=\left.\frac{x^3}{2}\right|_a^b=\frac{1}{2}(b^3-a^3).$$

Answer to Exercise 2.28

1. $a) A=\int_{-2}^{2}\left(x^2-4\right)dx=\frac{32}{3}$ sq.unit

b) $A=\int_{-2}^{2}\left(4-x^2\right)dx=\frac{32}{3}$ sq.unit

c) $A=\int_{0}^{2}\left(x^2-2x\right)dx=\frac{4}{3}$ sq.unit

d) $A=\int_{0}^{2}\left(-x^2-2x\right)dx=\frac{20}{3}$ sq.unit

2. **a)** $A=\int_{0}^{4}x^2dx=\frac{64}{3}$ sq.unit **b)** $A=\int_{-1}^{3}(x^2+3)dx=\frac{64}{3}$ sq.unit

3.

$$A=\int_{-2}^{1}\left(x(x+2)(x-1)\right)dx=\int_{-2}^{0}(x^3+x^2-2x)dx+\int_{0}^{1}(x^3+x^2-2x)dx=\frac{37}{12}\text{ sq.unit}$$

Answer to Activity 2.22

Integration has various applications in different fields of study such as business, physical sciences, economics, etc

Answer to Exercise 2.29

1. Marginal profit:

In calculus marginal profit is defined as the first order derivative of the profit function So, we can integrate the marginal profit function to get the profit function.

The below equation shows the relationship between the marginal profit and profit function.

$$P'(x) = \frac{d}{dx}P(x)$$

Here the marginal profit $P'(x)$, and the profit function $P(x)$.

Given:

- The marginal profit is: $P'(x) = \sqrt[5]{x}$.
- The depth is: $x = 250\,ft$.

Simplify the given marginal profit function

$$P'(x) = \sqrt[5]{x} \Rightarrow \frac{d}{dx}P(x) = \sqrt[5]{x} \Rightarrow dP(x) = (x)^{\frac{1}{5}}dx.$$

Integrating both side of the above function.

$$\int dP(x) = \int (x)^{\frac{1}{5}}dx \Rightarrow P(x) = \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} = \frac{x^{\frac{6}{5}}}{\frac{6}{5}} = \frac{5}{6}x^{\frac{6}{5}} + C$$

Where C is the integration constant.

Let us assume that there is no any fixed charges So, the profit only depends on the drilling. So, substitute $x=0$ and $P(0)=0$ in the above equation.

$$P(0) = \frac{5}{6}x^{\frac{6}{5}} + C \Rightarrow C = 0.$$

Substitute the value of constant in the profit function

$$P(x) = \frac{5}{6}x^{\frac{6}{5}} + 0 = \frac{5}{6}x^{\frac{6}{5}}.$$

Substitute $x = 250$ in the above function.

$$P(250) = \frac{5}{6}(250)^{\frac{6}{5}} = 628.56 \approx 629.$$

Therefore, the profit is 629 dollars.

2.

a. Given:

$$S'(t) = 20e^t, \qquad t - days$$

$$S(50 = \int_0^5 S'(t)dt = \int_0^5 20e^t dt = 20e^t \Big|_0^5 = 20(e^5 - 1) \text{ dollars}$$

b.

$$\text{Sales} = \int_2^5 S'(t)dt = \int_2^5 20e^t dt = 20e^t \Big|_2^5 = 20(e^5 - e^2) \text{ dollars}$$

Answer to Review Exercise

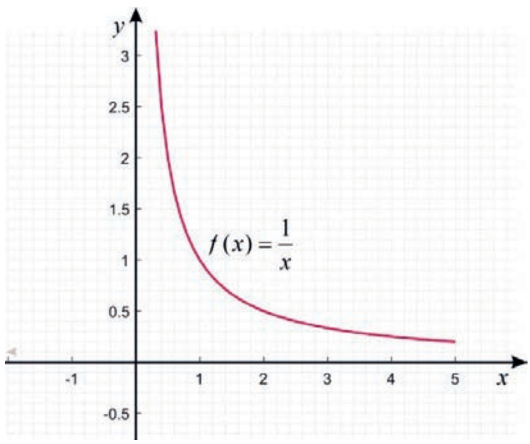
1. a) $f'(x) = \dfrac{x^4 - 3x^2}{x^4} = \dfrac{x^2 - 3}{x^2} \Rightarrow f'(2) = \dfrac{1}{4}$

b) $f'(x) = \dfrac{0 - 5(3x^2)}{\left(x^3 + 2\right)^2} = \dfrac{-15x^2}{\left(x^3 + 2\right)^2} \Rightarrow f'(-4) = \dfrac{-60}{961}$

c) $f'(x) = \dfrac{x^6 + 8x^3}{\left(x^3 + 2\right)^2} \Rightarrow f'(2) = \dfrac{32}{25}$

d)

$$f'(x) = \frac{-4x + 2}{\sqrt{x}\left(2x + 1\right)^2} \Rightarrow f'(4) = -\frac{7}{81}$$



2. **i** a) $\dfrac{-1}{9.3}$ **b)** $\dfrac{-1}{9.03}$ **c)** $\dfrac{-1}{9.003}$

ii. $slope = \dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$

iii. $Slope = 0$ as x approaches to zero.

3. Given $y = f(x) = 80 - 4.9t^2$

i.

a. $Average\ velocity = \dfrac{f(1.1) - f(1)}{1.1 - 1}$

$$= \frac{(80-4.9(1.1)^2)-(80-4.9)}{1.1-1}$$

$$= \frac{-5.929+4.9}{0.1} = -10.29$$

b. $\textit{Average velocity} = \dfrac{f(1.01)-f(1)}{1.01-1}$

$$= \frac{(80-4.9(1.01)^2)-(80-4.9)}{1.01-1}$$

$$= \frac{-4.998494+4.9}{0.01} = -9.849$$

c. $\textit{Average velocity} = \dfrac{f(1.001)-f(1)}{1.001-1}$

$$= \frac{(80-4.9(1.001)^2)-(80-4.9)}{1.001-1}$$

$$= \frac{-4.9098049+4.9}{0.001} = -9.8049$$

ii. $\textit{Average velocity} = \dfrac{f(1+\Delta t)-f(1)}{\Delta t}$

$$= \frac{(80-4.9(1+\Delta t)^2)-(80-4.9)}{\Delta t}$$

$$= \frac{-4.9(1+2\Delta t+\Delta t^2)+4.9}{\Delta t} = -9.8-4.9\Delta t$$

iii. As $\Delta t \to 0$, average velocity $= -9.8m/s$.

4.

i. $y = f(t) = t^2$

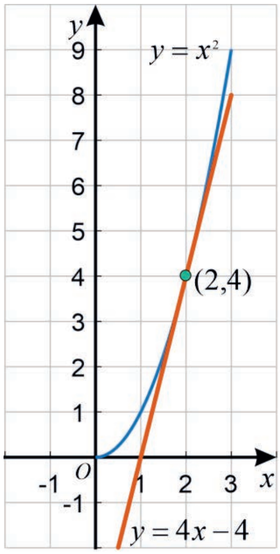
a. $\textit{Average velocity} = \dfrac{f(3)-f(2)}{3-2} = \dfrac{9-4}{1} = 5m/s$

b. $\textit{Average velocity} = \dfrac{f(2.1)-f(2)}{2.1-2} = \dfrac{4.41-4}{0.1} = 4.1m/s$

c. $\textit{Average velocity} = \dfrac{f(2.01)-f(2)}{2.01-2} = \dfrac{4.0401-4}{0.01} = 4.01m/s$

ii. $Average\ velocity = \frac{f(2+\Delta t)-f(2)}{\Delta t} = \frac{4+4\Delta t+\Delta t^2-4}{\Delta t} = (4+\Delta t)m/s$

iii. As $\Delta t \to 0$, average velocity $=4m/s$.



5.

a) Equation of tangent line: $y - f(-1) = f'(-1)(x+1) \Rightarrow y = -2x$

b) Equation of the normal line:

$$y - f(-1) = -\frac{1}{f'(-1)}(x+1) \Rightarrow y = \frac{1}{2}x$$

6.

a) $f(x) = x^2(2x+5) \Rightarrow f'(x) = 6x^2 + 10x$

b) $f(x) = \frac{x^2+x}{x^3-x+2} \Rightarrow f'(x) = \frac{-x^4-2x^3-x^2+4x+2}{\left(x^3-x+2\right)^2}$

c) $f(x) = \frac{x^2}{x+1} \Rightarrow f'(x) = \frac{x^2+2x}{\left(x+1\right)^2}$

d) $f(x) = x^3\left(1-\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)$

$$\Rightarrow f'(x) = 3x^2\left(1-\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right) + x\left(x^2-\frac{1}{x^2}\right) + x^3\left(1-\frac{1}{x}\right)\left(2x+\frac{2}{x^3}\right)$$

7. $f(x)=6x^5-50x^3-120 \Rightarrow f'(x)=30x^4-150x^2$

$=30x^4-150x^2=30x^2(x^2-5)$

$f'(x)=0 \ \ if \ \ x=0 \ and \ x=\pm\sqrt{5}$

$-\sqrt{5}$ 0 $\sqrt{5}$

$30x^2$	++++++	+++++++++++	+++++++++++	+++++++++++++++
$x+\sqrt{5}$	-----------	+++++++++++	++++++++++	++++++++++++++
$x-\sqrt{5}$	-----------	---------------	----------------	+++++++++++++++
$f'(x)$	+++++++	---------------	----------------	+++++++++++++++

Therefore,

a. f is increasing on $(-\infty,-\sqrt{5}]\cup[\sqrt{5},\infty)$.

b. f is decreasing on $[-\sqrt{5},0)\cup(0,\sqrt{5}]$.

c. f has relative maximum value at $x=-\sqrt{5}$.

d. F has relative minimum value at $x=\sqrt{5}$.

8.

$$2y=x^2+4 \Rightarrow y=\frac{1}{2}x^2+2 \Rightarrow \frac{dy}{dx}=x$$

Hence, $\left.\frac{dy}{dx}\right|_{x=4}=4$. when $x=4, y=10$. Thus the point is $(4,10)$.

9. $f(x)=x^2+px+q;\ p,q\in\mathbb{R}$

$\Rightarrow f'(x)=2x+p$. It is given that f ha extreme value at $(1,3)$.

This means $f'(1)=0 \ and \ f(1)=3.$

$\Rightarrow p=-2 \ and \ 1+p+q=3$

$\Rightarrow p=-2 \ and \ q=4.$

The value is minimum.

10. $h'(x) = 3ax^2 + 2bx + c.$

Given that a function h has extreme value at $(1,2)$ *and* $(2,3)$.

Thus $f'(1) = 3a + 2b + c = 0$ *and* $f'(2) = 12a + 4b + c = 0$ .....(*)

$f(1) = a + b + c + d = 2$ *and* $f(2) = 8a + 4b + 2c + d = 3$ .....(**)

From (*) we obtain $b = -\dfrac{9a}{2}$ and from (**) we get $7a + 3b + c = 1$

Now, combining these two we get $c = \dfrac{13a + 2}{2}$

Substituting these values of b and c in the first equation of (*), we have

$$3a + 2\left(\frac{-9a}{2}\right) + \frac{13a + 2}{2} = 0 \Rightarrow a = -2, b = 9, c = -12 \text{ and } d = 7.$$

11.

a. $f(x) = \sin x - \dfrac{1}{2}\cos 2x; [0, 2\pi]$

$$\Rightarrow f'(x) = \cos x + \sin 2x = \cos x + 2\sin x\cos x = \cos x(1 + 2\sin x)$$

$$Thus,\ f'(x) = 0 \Rightarrow \cos x = 0 \text{ or } \sin x = \frac{-1}{2}$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}.$$

$$f(0) = f(2\pi) = f\left(\frac{3\pi}{2}\right) = -\frac{1}{2},\ f\left(\frac{\pi}{2}\right) = \frac{3}{2},$$

$$f\left(\frac{7\pi}{6}\right) = f\left(\frac{11\pi}{6}\right) = -\frac{3}{4}.$$

Hence, the maximum value of f on the given interval is $f\left(\dfrac{\pi}{2}\right) = \dfrac{3}{2}$, while its minimum value is $-\dfrac{3}{4}$.

b. $f(x)=\begin{cases} x^3-\dfrac{x}{3}; 0\le x\le 1 \\ x^2+x-\dfrac{4}{3}; 1<x\le 2\end{cases}; on\,[0,2].$

$$\Rightarrow f'(x)=\begin{cases} 3x^2-\dfrac{1}{3}; 0\le x\le 1 \\ 2x+1; 1<x\le 2\end{cases}; on\,[0,2].$$

Now, $f'(x)=0\ if\ x=\pm\frac{1}{3}\ and\ x=-\frac{1}{2}$. Of these critical numbers only $x=\frac{1}{3}$ is found on $[0,2]$.

And $f(0)=0$, $f(\frac{1}{3})=-\frac{2}{27}$ $and\ f(2)=\frac{14}{3}$.

Thus, the absolute maximum of f on the given interval is $\frac{14}{3}$ and the absolute minimum is $-\frac{2}{27}$.

12.

Given $\dfrac{dy}{dt}=2\,ft/\sec,\qquad \dfrac{dx}{dt}=?$

$$y^2+x^2=25$$

$$\Rightarrow y=\sqrt{25-16}=3$$

$$2y\frac{dy}{dt}+2x\frac{dx}{dt}=0$$

$$\Rightarrow y\frac{dy}{dt}+x\frac{dx}{dt}=0$$

a) $\Rightarrow x\dfrac{dx}{dt}=-y\dfrac{dy}{dt}$

$$\Rightarrow \frac{dx}{dt}=\left(\frac{-y}{x}\right)\frac{dy}{dt}$$

$$\therefore \left.\frac{dx}{dt}\right|_{(4,3)}=\left(\frac{-3}{4}\right)(2\,ft/\sec)=\frac{-3}{2}\,ft/\sec.$$

$$b)\ A=\frac{1}{2}xy \Rightarrow \frac{dA}{dt}=\frac{1}{2}(y\frac{dy}{dt}+x\frac{dx}{dt})$$

$$\Rightarrow \left.\frac{dA}{dt}\right|_{(4,3)}=\frac{1}{2}\left(3\,ft(\frac{-3}{2}\,ft/\sec)+4\,ft(2\,ft/\sec)\right)$$

$$=\frac{1}{2}\left(-\frac{9}{2}+8\right)ft^2/\sec=\frac{7}{4}ft^2/\sec.$$

13.

$2x+y=240m$

$\Rightarrow y=240-2x$

$A(x)=x(240-2x)=240x-2x^2$

$\Rightarrow A'(x)=240-4x=4(60-x)$

$A'(x)=0 \Rightarrow x=60$

$\Rightarrow A'(x)$ Changes from positive to negative at $x=0$.

$\Rightarrow A(60)=240(60)-2(60)^2$

$=240(240-120)$

$=60(120)$

$=7200m^2$

is a maximum area.

$\therefore$ The dimensions of the rectangle that will give maximum area is $x=60$ and $y=120$.

14.

Given

$\frac{dr}{dt}=0.5cm/\sec.,\ r=4cm$

$A=4\pi r^2$

$\frac{dA}{dt}=8\pi r\frac{dr}{dt}$

$\Rightarrow \left.\frac{dA}{dt}\right|_{r=4cm}=8\pi(4cm)(0.5cm/\sec)=16\pi cm^2/\sec$

15.

a.

$$v_0 = 30km/hr, \quad v_f = 0, \quad t = 45\sec(0.0125hr)$$

$$\vec{a} = \frac{v_f - v_0}{t} = \frac{0-30}{0.0125} = -\frac{30}{0.0125}km/hr^2$$

$$a = \frac{dv}{dt} \Rightarrow dv = adt$$

$$\Rightarrow v(t) = \int adt = \int\left(-\frac{30}{0.0125}\right)dt = -\frac{30}{0.0125}t + c_1$$

$$Initial\ condition, t = 0, v(0) = 30 \Rightarrow c_1 = 30$$

$$\Rightarrow v(t) = -2400t + 30.$$

$$t = 20\sec = 0.0056hr, V(0.0056) = -2400 \times 0.0056 + 30 = 16.56km/hr$$

$$S(t) = \int(-2400t + 30)dt = -1200t^2 + 30t + c_2.$$

b.

$$Initial\ conditiont = 0, S(t) = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow S(t) = -1200t^2 + 30t$$

$$t = 45\sec = 0.0125hr, S(0.0125) = 0.1875km = 187.5m.$$

a.

$$\int_0^{16} 280t^{\frac{3}{2}}dt = 280\int_0^{16} t^{\frac{3}{2}}dt = 280[(t^{\frac{3}{2}})(\frac{2}{5}]\left[\left(t^{\frac{3}{2}}\right)\left(\frac{2}{5}\right)\right]_0^{16}$$

$$= 280\left(16^{\frac{5}{2}}\right)\left(\frac{2}{5}\right) = 280(4^5)\left(\frac{2}{5}\right)$$

$$= 280(409.6)$$

$$=114688 \text{ pollutants.}$$

$$N(t)=\left(280t^{\frac{5}{2}}\right)\left(\frac{2}{5}\right)\Rightarrow 50,000=\left(280t^{\frac{5}{2}}\right)\left(\frac{2}{5}\right)$$

b.

$$\Rightarrow 125,000=280t^{\frac{5}{2}}$$

$$\Rightarrow (446.43)^{\frac{2}{5}}=\left(t^{\frac{5}{2}}\right)^{\frac{2}{5}}\Rightarrow t=(446.43)^{\frac{2}{5}}=11.48.$$

After 12 month the lake will have exceeded 50,000 pound of pollutants.

UNIT 3: STATISTICS

Periods Allotted: 33 Periods

Introduction

In this unit, you first need to give a brief revision of the purpose of the field statistics in different sectors of social and economic situations. After this revision, it is important to point out that the unit mainly focuses on introducing students to absolute measure of dispersion and relative measures of dispersion. The unit also discusses the various forms of presenting data. Students should also be encouraged: to present a given set of data using graphs, and to compute different measures of central tendency and measures of variations. They are also expected to relate measures of central tendency and measures of dispersion in measuring consistency and symmetry. At the end of each main point, it is helpful to provide students with examples that are applicable to real life.

Unit Outcomes

By the end of this unit, students will be able to:

- know absolute and relative dispersion and their interpretation.
- deduce specific facts about measurement in statistical data.
- understand basic concepts about sampling techniques.
- appreciate the value of statistics in real life

Suggested instructional Materials for the unit

Since Statistics is one of the fields that are practically applied in our day-to-day experience, there might be a lot to apply depending on where the discussion is applied. Some of the teaching aids that are useful for teaching statistics and particularly this unit are: Colored chalks (white board markers), chalk board, different colored objects (marbles), and paper for drawing graphs, color pencils, and straight edged ruler for drawing. Different graphs, charts, different statistical data and figures of different organizations, text and other references and calculators are also useful. In addition, you can also use different software such as MS-EXCEL, SPSS, Fathom, etc.

Assessment

You can give several exercise problems on computing range of grouped and ungrouped data and in interpreting a given data.

Teaching Notes

You may begin this unit by encouraging students to complete Activity 3.1.

Answer to Activity 3.1

1. In statistics, the measure of dispersions helps to interpret the variability of data, that is, know how homogenous or heterogeneous the data is. In simple terms, it shows how squeezed or scattered the data is.
2. The measures of central tendency give us a bird's eye view of the entire data. They are called averages of the first order, and they serve to locate the center of the distribution but they do not reveal how the items are spread out on either side of the central value. But the measures of dispersion measure the extent to which the items vary from some central value.

3.1 Measure of Absolute Dispersion

3.1.1 Range and Inter-quartile Range

A. Range

Introduction

In previous grades, you studied about statistics and its basics, such as collection of data, presentation of data, measures of central tendency such as mean, median and mode, measures. In this sub-topic, they will discuss further on the range of given data.

Competencies

By the end of this sub-unit, students will be able to:

- compute the range of a given data.
- describe the relative significance of range as a measure for dispersion.

Vocabulary: range, upper class boundary of the highest class, lower class boundary of the lowest class.

Teaching Notes

You may begin this section by encouraging students to do Activity 3.2. With the help of examples of ungrouped data and grouped frequency distribution, give a brief revision of calculating the range.

Answer to Activity 3.2

a. The highest price is 15 and the lowest price is 5.
b. The difference between the highest and the lowest price of a cup of coffee is 10.

To help students revise these ideas, it is recommended that a discussion is conducted on the examples given in the student textbook. You can also use Exercise 3.1 for revision purpose and assessing student competencies.

Assessment

You can give several exercises on computing range of grouped and ungrouped data and in interpreting a given data.

Answer to Exercise 3.1

1.
   a. Range =25-16=9.
   b. Range =22-10=12.
2.
   a. Range=11-5=6.
   b. Range=100-0=100.

B. Inter-Quartile Range

Competencies

By the end of this sub-unit, students will be able to:

- calculate inter-quartile range of a given data set.
- describe the relative significance of inter-quartile range as a measure for dispersion.

Vocabulary: range, inter-quartile range, quartiles

Introduction

Students have previously discussed absolute measures of dispersion with a particular focus on range. In this sub-topic, they will discuss further the inter-quartile range of given data set.

Teaching Notes

You may begin this section by encouraging students to complete Activity 3.3. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the inter-quartile range.

Answer to Activity 3.3

1.

a.

For city A:

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right)+\left(\frac{n}{4}+1\right)}{2}\right)^{\text{th}} = \left(\frac{\left(\frac{10}{4}\right)+\left(\frac{10}{4}+1\right)}{2}\right)^{\text{th}} = 3^{\text{rd}}\text{item} = 25.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{\text{th}} = \left(\frac{\left(\frac{3\times10}{4}\right)+\left(\frac{3\times10}{4}+1\right)}{2}\right)^{\text{th}}\text{item} = 8^{\text{th}}\text{item} = 28.$$

For city B:

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right)+\left(\frac{n}{4}+1\right)}{2}\right)^{\text{th}} = \left(\frac{\left(\frac{10}{4}\right)+\left(\frac{10}{4}+1\right)}{2}\right)^{\text{th}} = 3^{\text{nd}}\text{item}=24.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times10}{4}\right)+\left(\frac{3\times10}{4}+1\right)}{2}\right)^{th} \text{item=8}^{\text{th}}\text{item} = 26.$$

b.

For city A: $Q_3 - Q_1 = 28 - 25 = 3.$

For city B: $Q_3 - Q_1 = 26 - 24 = 2.$

City A has a higher variation in temperature.

2. $Q_1 = L + \left(\frac{\frac{(n)}{4} - cf}{f}\right)w = 5 + \left(\frac{\frac{9}{4} - 1}{3}\right)5 = 7.08.$

$$Q_3 = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right)w = 10 + \left(\frac{\frac{3(9)}{4} - 4}{3}\right)5 = 14.58.$$

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student book. You can also use Exercise 3.2 for revision purpose and assessing student competencies.

Assessment

You can give several problems on determining inter-quartile range of grouped and ungrouped data and in interpreting a given data set.

Answer to Exercise 3.2

1.

a. $Q_1 = \left(\frac{\left(\frac{n}{4}\right)+\left(\frac{n}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{6}{4}\right)+\left(\frac{6}{4}+1\right)}{2}\right)^{th} = 2^{\text{nd}}\text{item} = 19$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times 6}{4}\right)+\left(\frac{3\times 6}{4}+1\right)}{2}\right)^{th} = 5^{\text{nd}}\text{item}$$

$$= 22$$

Therefore, IQR = 22-19 = 3.

b.

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right) + \left(\frac{n}{4} + 1\right)}{2}\right)^{\text{th}} = \left(\frac{\left(\frac{8}{4}\right) + \left(\frac{8}{4} + 1\right)}{2}\right)^{\text{th}} = 2.5^{\text{th}}\text{item}$$

$$= 14 + 0.5(16 - 14) = 15$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4} + 1\right)}{2}\right)^{\text{th}} = \left(\frac{\left(\frac{3 \times 8}{4}\right) + \left(\frac{3 \times 8}{4} + 1\right)}{2}\right)^{\text{th}}$$

$$= 6.5^{\text{th}}\text{item} = 0.5(19 - 18) = 18.5$$

Therefore, IQR = 18.5-15=3.5.

2.

a.

$$Q_1 = \frac{1(n+1)}{4} = \frac{59+1}{4} = \frac{60^{th}}{4}\text{item} = 15^{th} = 7.$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(59+1)}{4} = \frac{180^{th}}{4}\text{item} = 45^{th} = 9.$$

Therefore, IQR=9-7=2.

b.

$$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = 2^{nd}\text{item} = 23.$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = 6^{th}\text{item} = 24.$$

Therefore, IQR = 24-23=1.

c. $Q_1 = L + \left(\frac{\frac{(n)}{4} - cf}{f}\right)w = 0 + \left(\frac{\frac{11}{4} - 1}{3}\right)20 = 18.33$

$$Q_3 = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right)w = 60 + \left(\frac{\frac{3(11)}{4} - 4}{3}\right)20 = 71.25.$$

Therefore, IQR = 71.25-18.33=52.92.

d. $Q_1 = L + \left(\frac{\frac{(n)}{4} - cf}{f}\right)w = 0 + \left(\frac{\frac{9}{4} - 1}{3}\right)9 = 6.19$

$$Q_3 = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right)w = 60 + \left(\frac{\frac{3(9)}{4} - 4}{3}\right)9 = 26.75.$$

Therefore, IQR = 26.75-6.19 = 20.56.

3.1.2. Mean Deviation and Quartile Deviation

A. Mean Deviation

Introduction

Students have previously discussed absolute measure of dispersion with a particular focus on range and inter-quartile range. In this sub-topic, they will further discuss the mean deviation of a given data set.

Competencies

By the end of this sub-unit, students will be able to:

- compute mean deviation of a given data set.
- describe the relative significance of mean deviation as a measure for dispersion.

Vocabulary: mean, median, mode, mean deviation

Teaching Notes

You may begin this section by encouraging students to complete Activity 3.4. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the mean, median, mode and mean deviation.

Answer to Activity 3.4

1.

a. $\bar{x} = \frac{\sum_{i=1}^{n} f x_i}{N} = \frac{168}{15} = 11.2.$

x	f	fx	$f(x-\overline{x})$	$f\lvert x-\overline{x}\rvert$
8	1	8	-3.2	3.2
9	2	18	-4.4	4.4
10	5	50	-6	6
11	1	11	-0.2	0.2
12	2	24	1.6	1.6
13	1	13	1.8	1.8
14	1	14	2.8	2.8
15	2	30	7.6	7.6
Total	15	168	0	27.6

b. See the above table for the deviation of each value from the mean.

c. Mean of these deviations$=\frac{0}{15}=0$..

d. $\frac{\sum_{i=1}^{n} f|x_i - \overline{x}|}{n} = \frac{27.6}{15} = 1.84.$

e. The value in (d) is greater than the value in (c).

2. $\overline{x} = \frac{\sum_{i=1}^{n} fx_i}{n} = \frac{117}{10} = 11.7,$

$$m_d = \frac{\left(\frac{10}{2}\right)^{th} \text{item} + \left(\frac{10}{2}+1\right)^{th} \text{item}}{2} = \frac{12+12}{2} = 12$$

$m_0 = 13.$

3. $\overline{x} = \frac{\sum_{i=1}^{n} fm_i}{n} = \frac{1605}{10} = 160.5,$

$$m_d = L + \left(\frac{\frac{n}{2} - cf}{f}\right) w = 160 + \left(\frac{\frac{10}{2} - 4}{2}\right) 4 = 162.$$

$$m_o = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) w = 165 + \left(\frac{2}{2+4}\right) 4 = 166.3.$$

To help students revise these ideas, it is recommended that a discussion is conducted on the examples given in the student textbook. You can also use Exercise 3.3 for revision purpose and assessing student competencies.

Assessment

You can give several exercise problems on calculating mean deviation from (mean, median, mode) of grouped and ungrouped data.

Answer to Exercise 3.3

1. For (a)

$$\bar{x} = \frac{94}{7} = 13.4,$$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n}|x_i - \bar{x}|}{n} = \frac{27.4}{7} = 3.9$$

For (b)

$$\bar{x} = \frac{101}{10} = 10.1,$$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n}|x_i - \bar{x}|}{n} = \frac{26.1}{10} = 2.61$$

Answer to Exercise 3.4

For (a)

$m_d = 12$

$$MD(m_d) = \frac{\sum_{i=1}^{n}|x_i - m_d|}{n} = \frac{26}{7} = 3.7$$

For (b)

$m_d = 10$

$$MD(m_d) = \frac{\sum_{i=1}^{n}|x_i - m_d|}{n} = \frac{24}{10} = 2.4$$

Answer to Exercise 3.5

For (a)

$m_o = 11$

$$MD(m_o) = \frac{\sum_{i=1}^{n}|x_i - m_o|}{n} = \frac{18}{7} = 2.57.$$

For (b)

$m_o = 10$

$$MD(m_o) = \frac{\sum_{i=1}^{n} |x_i - m_o|}{n} = \frac{24}{10} = 2.4.$$

Answer to Exercise 3.6

x	f	fx	$f\|x - \bar{x}\|$
10	2	20	40
20	4	80	40
30	7	210	0
40	6	240	60
50	1	50	20
Total	20	600	160

$\bar{x} = \frac{600}{20} = 30.$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f|x_i - \bar{x}|}{n} = \frac{160}{20} = 8$$

Answer to Exercise 3.7

x	f	fx	$f\|x - m_d\|$
10	2	20	40
20	4	80	40
30	7	210	0
40	6	240	60
50	1	50	20
Total	20	600	160

$m_d = 30$

$$MD(m_d) = \frac{\sum_{i=1}^{n} f|x_i - m_d|}{n} = \frac{160}{20} = 8$$

Answer to Exercise 3.8

x	f	fx	$f\|x-m_o\|$
10	2	20	40
20	4	80	40
30	7	210	0
40	6	240	60
50	1	50	20
Total	20	600	160

$$m_o = 30$$

$$MD(m_0) = \frac{\sum_{i=1}^{n} f|x_i - m_o|}{n} = \frac{160}{20} = 8$$

Answer to Exercise 3.9

x	f	m	fm	$f\|m-\bar{x}\|$
0-9	4	4.5	18	76
10-19	3	14.5	43.5	27
20-29	6	24.5	147	6
30-39	5	34.5	172.5	55
40-49	2	44.5	89	42
Total	20		470	206

$$\bar{x} = \frac{470}{20} = 23.5$$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f|m - \bar{x}|}{n} = \frac{206}{20} = 10.3$$

Answer to Exercise 3.10

x	f	m	cf	$f\|m-m_d\|$
0-9	4	4.5	4	82
10-19	3	14.5	7	31.5
20-29	6	24.5	13	3
30-39	5	34.5	18	47.5
40-49	2	44.5	20	39
Total	20			203

$$m_d = L + \left(\frac{\frac{n}{2} - cf}{f}\right) w = 20 + \left(\frac{\frac{20}{2} - 7}{6}\right) 10 = 25$$

$$MD(m_d) = \frac{\sum_{i=1}^{n} f|m - m_d|}{n} = \frac{203}{20} = 10.15$$

Answer to Exercise 3.11

x	f	m	cf	$f\|m - m_o\|$
0-9	4	4.5	4	89
10-19	3	14.5	7	36.75
20-29	6	24.5	13	13.5
30-39	5	34.5	18	38.75
40-49	2	44.5	20	35.5
Total	20			213.5

$$m_o = 26.75$$

$$MD(m_d) = \frac{\sum_{i=1}^{n} f|m - m_o|}{n} = \frac{213.5}{20} = 10.675$$

Answer to Exercise 3.12

1. a) Mean deviation about the mean

$$\bar{x} = \frac{77}{7} = 11,$$

x	f	fx	$f\|x - \bar{x}\|$
7	1	7	4
9	1	9	2
10	1	10	1
12	3	36	3
15	1	15	4
Total	7	77	14

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f|x_i - \bar{x}|}{n} = \frac{14}{7} = 2$$

Mean deviation about the median

$m_d = 12$

x	f	fx	$f\lvert x - m_d \rvert$
7	1	7	5
9	1	9	3
10	1	10	2
12	3	36	0
15	1	15	3
Total	7	77	13

$$MD(m_d) = \frac{\sum_{i=1}^{n} f\lvert x_i - m_d \rvert}{n} = \frac{13}{7} = 1.86$$

Mean deviation about the mode

$m_o = 12$

x	f	fx	$f\lvert x - m_0 \rvert$
7	1	7	5
9	1	9	3
10	1	10	2
12	3	36	0
15	1	15	3
Total	7	77	13

$$MD(m_0) = \frac{\sum_{i=1}^{n} f\lvert x_i - m_0 \rvert}{n} = \frac{13}{7} = 1.86$$

2. a)

x	f	fx	$f\lvert x - \bar{x} \rvert$
12	10	120	15
13	12	156	6
14	9	126	4.5
15	11	165	16.5
Total	42	567	42

$\bar{x} = \frac{567}{42} = 13.5$, $m_d = 13$ and $m_o = 13$.

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} f|x_i - \bar{x}|}{n} = \frac{42}{42} = 1.0$$

$$MD(m_d) = \frac{\sum_{i=1}^{n} f|x_i - m_d|}{n} = \frac{41}{42} = 0.98$$

$$MD(m_o) = \frac{\sum_{i=1}^{n} f|x_i - m_o|}{n} = \frac{41}{42} = 0.98.$$

b)

x	f	m	fm	$f\lvert m - \bar{x}\rvert$
0-4	3	2	6	28.5
5-9	1	7	7	4.5
10-14	2	12	24	1
15-19	4	17	68	22
20-24	1	22	22	10.5
Total	11		127	66.5

$\bar{x} = \frac{127}{11} = 11.5$, $m_d = 13$ and $m_o = 16.6$.

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} = \frac{66.5}{11} = 6.04$$

$$MD(m_d) = \frac{\sum_{i=1}^{n} |x_i - m_d|}{n} = \frac{66}{11} = 6$$

$$MD(m_o) = \frac{\sum_{i=1}^{n} |x_i - m_o|}{n} = \frac{69.5}{11} = 9.32.$$

B. Quartile Deviation

Introduction

Students have previously discussed absolute measure of dispersion with a particular focus on range, inter-quartile range and mean deviation. In this sub-topic, they will further discuss the quartile deviation of a given data set.

Competencies

By the end of this sub-unit, students will be able to:

- calculate the quartile deviation of a given data set.
- describe the relative significance of quartile deviation as a measure for dispersion.

Vocabulary: range, inter-quartile range**,** quartile deviation

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.5. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the quartile deviation.

Answer to Activity 3.5

1. a.

R=L-S=600-250=350.

$$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = 2^{\text{nd}}\text{item} = 300.$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = 6^{\text{th}} \text{ item} = 540.$$

Therefore, IQR = 540-300 = 240.

2. The value of range is greater than the value of IQR.

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student textbook. You can also use Exercise 3.13 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating quartile deviation of grouped and ungrouped data.

Answer to Exercise 3.13

1.

a.

$$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = 2^{nd}\text{item=100.}$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = 6^{th}\text{item=109.}$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{109-100}{2} = 4.5$$

b.

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right) + \left(\frac{n}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{10}{4}\right) + \left(\frac{10}{4}+1\right)}{2}\right)^{th} = 3^{nd}\text{item} = 70$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times 10}{4}\right) + \left(\frac{3\times 10}{4}+1\right)}{2}\right)^{th}$$

$$= 8^{nd}\text{item} = 120$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{120-70}{2} = 25.$$

Answer to Exercise 3.14

a.

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right) + \left(\frac{n}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{8}{4}\right) + \left(\frac{8}{4}+1\right)}{2}\right)^{th} = 2.5^{nd}\text{item}$$

$$= 110 + 0.5(130-110) = 120.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4}+1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3\times 8}{4}\right) + \left(\frac{3\times 8}{4}+1\right)}{2}\right)^{th} = 6.5^{nd}\text{item}$$

$$= 140.$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{140 - 120}{2} = 10.$$

b.

$$Q_1 = L + \left(\frac{\frac{(n)}{4} - cf}{f}\right)w = 0 + \left(\frac{\frac{11}{4} - 0}{3}\right)4 = 3.67$$

$$Q_3 = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right)w = 15 + \left(\frac{\frac{3(11)}{4} - 6}{4}\right)4 = 17.25.$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{17.25 - 3.67}{2} = 6.79.$$

3.1.3 Variance and Standard Deviation

A. Variance

Introduction

Students have previously discussed absolute measure of dispersion with a particular focus on range, inter-quartile range, mean deviation and quartile deviation. In this sub-topic, they will further discuss the quartile deviation of given data.

Competencies

By the end of this sub-unit, students will be able to:

- find the variance of a given data set.
- describe the relative significance of variance as a measure of dispersion.

Vocabulary: mean, midpoint, variance

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.6. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the variance.

Answer to Activity 3.6

1.

R=L-S = 20 - 5=15.

$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = 2^{nd}$ item =10.

$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = 6^{th}$ item =17

$$IQR = Q_3 - Q_1 = 17 - 10 = 7.$$

$$\bar{x} = \frac{98}{7} = 14$$

$$MD(\bar{x}) \quad = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} = \frac{26}{7} = 3.71$$

2. R> $\bar{x} > IQR > MD(\bar{x})$.

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student book. You can also use Exercise 3.15 for revision purposes and assessing student competencies.

Assessment

You can give several exercise problems on calculating variance of ungrouped and grouped data.

B. Standard Deviation

Introduction

Students have previously discussed absolute measure of dispersion with a particular focus on range, inter-quartile range, mean deviation, quartile deviation and variance. In this sub-topic, they will further discuss the quartile deviation of given data.

Competencies

By the end of this sub-unit, students will be able to:

- calculate the standard deviation of a given data.
- describe the usefulness of standard deviation in interpreting the variability of a given data.

Vocabulary: mean, midpoint, variance

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.7. With the help of examples of ungrouped data distribution, give a brief revision of calculating the standard deviation.

Answer to Exercise 3.15

The mean ($\bar{x}$) is:

$$\bar{x} = \frac{5 + 10 + 4 + 7 + 9}{5} = 7$$

Calculate the deviation of each value from the mean and square it.

x	x-$\bar{x}$	$(x - \bar{x})^2$
5	-2	4
10	3	9
4	-3	9
7	0	0
9	2	4
Total		$\sum (x_i - \bar{x})^2 = 26$

Thus,

$$\sigma^2 = \frac{\sum(x - \bar{x})^2}{n} = \frac{26}{5} = 5.2.$$

Answer to Exercise 3.16

x	f	$x - \bar{x}$	$f(x - \bar{x})^2$
10	4	-16.5	1089
20	6	-6.5	253.5
30	5	3.5	61.25
40	3	13.5	546.75
50	2	23.5	1104.5
Total	$\sum f_i = 20$		$\sum f(x_i - \bar{x})^2 = 3055$

The mean

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{530}{20} = 26.5$$

Calculate the deviation of each value from the mean and square it. The required variance

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i} = \frac{3055}{20} = 152.75.$$

Answer to Exercise 3.17

$$\bar{x} = \frac{\sum f_i m_i}{\sum f_i} = \frac{125}{20} = 6.25$$

Calculate the deviation of each value from the mean and square it.

x	f	m_i	$f(m - \bar{x})^2$
0-10	2	5	3.125
10-20	8	15	612.5
20-30	5	25	1757.8125
30-40	4	35	3306.25
40-50	1	45	1501.5625
Total	$\sum f = 20$		$\sum f(m_i - \bar{x})^2 = 7181.25$

The required variance is, therefore,

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i (m_i - \bar{x})^2}{\sum_{i=1}^{n} f_i} = \frac{7181.25}{20} = 359.0625.$$

Answer to Activity 3.7

1.

$$\bar{x} = \frac{481}{10} = 48.1, \mathrm{R} = 80 - 25 = 55.$$

$$Q_1 = \left(\frac{\left(\frac{n}{4}\right) + \left(\frac{n}{4} + 1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{10}{4}\right) + \left(\frac{10}{4} + 1\right)}{2}\right)^{th} = 3^{nd}\text{item} = 35.$$

$$Q_3 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4} + 1\right)}{2}\right)^{th} = \left(\frac{\left(\frac{3 \times 10}{4}\right) + \left(\frac{3 \times 10}{4} + 1\right)}{2}\right)^{th} = 8^{nd}\text{item}$$

$$= 60.$$

$$IQR = 60 - 35 = 25.$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{60 - 35}{2} = 12.5.$$

2. $R > IQR$ and $\bar{x} > QD$.

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student textbook. You can also use Exercise 3.18 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating standard deviation of ungrouped and grouped data. In addition, you may give real-life problems whose data will be collected by the students themselves that can be organized into a frequency distribution. This can be used to calculate various absolute measures of disperstion. You can do this as a group assignment or project work.

Answer to Exercise 3.18

1. $\bar{x} = \frac{91}{9} = 10.1.$

$$\delta^2 = \frac{\sum_{i=1}^{9}(x_i - \bar{x})^2}{n} = \frac{124.08}{9} = 13.79$$

$$\delta = \sqrt{13.79} = 3.71.$$

2.

a. $\bar{x} = \frac{45}{12} = 3.75$

$$\delta^2 = \frac{\sum f(x_i - \bar{x})^2}{\sum f} = \frac{58.25}{12} = 4.85$$

$$\delta = \sqrt{4.85} = 2.20.$$

b. $\bar{x} = \frac{226}{8} = 28.25$

x	f	m	$f(m - \bar{x})^2$
10-19	2	14.5	378.125
20-29	3	24.5	42.1875
30-39	1	34.5	39.06
40-49	2	44.5	528.125
Total	8		987.5

$$\delta^2 = \frac{\sum f(m - \bar{x})^2}{\sum f} = \frac{987.5}{8} = 123.43$$

$$\delta = \sqrt{123.43} = 11.11.$$

3.2 Interpretation of relative Dispersion

Introduction

Students have previously discussed absolute measures of dispersion such as range, inter-quartile range, mean deviation, quartile deviation, variance and standard deviation. In this sub-topic, they will further discuss the relative dispersion and their interpretation of given data.

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.8.

Answer to Activity 3.8

1. We study relative dispersion to compare the variation of two or more distributions.
2.
   a. For team A mean is 53 and for team B it is also 53.
   b. Based on the means of the two teams, we cannot conclude about the performance of the two teams. That is why we study relative dispersion of the given data.

3.2.1 Coefficient of Range

Introduction

Students have previously discussed absolute measures of dispersion. In this sub-topic, they will further discuss relative measure of dispersion in particular on the coefficient range of a data set.

Competencies

By the end of this sub-unit, students will be able to:

- Calculate coefficient of range of a given data set.
- interpretation of two or more data by using their coefficient ranges.

Vocabulary: range, coefficient range

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.9. With the help of examples of ungrouped data and grouped data, give a brief revision of calculating the coefficient range.

Answer to Activity 3.9

1. For data A, R=10-2=8 and range for data B, R=10-2=8.
2. Since they have the same range we cannot compare interpret the two data sets based on range. This is why we study the coefficient range of the data.

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student book. You can also use Exercise 3.19 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating coefficient of range of ungrouped and grouped data.

Answer to Exercise 3.19

1. $L = 83$ and $S = 17$, then $CR = \frac{L-S}{L+S} = \frac{83-17}{83+17} = \frac{66}{100} = 0.66$.
2. $L = 20$ and $S = 10$, then $CR = \frac{L-S}{L+S} = \frac{20-10}{20+10} = \frac{10}{30} = 0.33.$
3.
   a. For company A, $CR = \frac{L-S}{L+S} = \frac{43200-1000}{43200+1000} = \frac{42200}{44200} = 0.95$

      For company B, $CR = \frac{L-S}{L+S} = \frac{36000-1200}{36000+1200} = \frac{34800}{37200} = 0.93.$

   b. Company A has more variable income.

3.2.2 Coefficient of Quartile Deviation (CQD)

Introduction

Students have previously discussed relative measures of dispersion with a particular focus on coefficient of range. In this sub-topic, they will further discuss the coefficient of quartile deviation of a given data set.

Competencies

By the end of this sub-unit, students will be able to:

- determine the coefficient of quartile deviation of a given data set.
- interpret two or more data sets by using their coefficient of quartile deviation.

Vocabulary: quartile, quartile deviation, coefficient quartile deviation,

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.10. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the coefficient of range.

Answer to Activity 3.10

1. For data A

$$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = \left(\frac{8}{4}\right)^{\text{th}} \text{item=}2^{\text{th}} item \;= 30 \quad \text{and}$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = \left(\frac{24}{4}\right)^{\text{th}} \text{item=}6^{\text{th}}\text{item} \;= 180$$

For data B

$$Q_1 = \frac{1(n+1)}{4} = \frac{7+1}{4} = \left(\frac{8}{4}\right)^{th} \text{item=}2^{th}\text{item} \;= 140 \quad \text{and}$$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(7+1)}{4} = \left(\frac{24}{4}\right)^{th} \text{item=}6^{th}\text{item} \;= 160$$

2. Range for data A, R = 190-20=170 and $IQR = Q_3 - Q_1 = 180 - 30 = 150.$

   Range for data B, R = 170-130 = 40 and $IQR = Q_3 - Q_1 = 160 - 140 = 20.$
3. Data A has higher variation.

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student textbook. You can also use Exercise 3.20 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating coefficient of quartile deviation of ungrouped and grouped data. Moreover, suggest that assist students watch the online video at https://www.embibe.com/study/coefficient-mean-deviation-in-statistics-concept. Give related problems so that the students may work in excel.

Answer to Exercise 3.20

1. $Q_1 = \frac{1(n+1)}{4} = \frac{9+1}{4} = \left(\frac{10}{4}\right)^{th} \text{item=}2.5^{th} item \;= 36 \quad \text{and}$

$$Q_3 = \frac{3(n+1)}{4} = \frac{3(9+1)}{4} = \left(\frac{30}{4}\right)^{th} \text{item}$$

$$=7.5^{th} item \quad = 45 + 0.5(47 - 45) = 46.$$

Therefore,

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{46-36}{46+36} = \frac{10}{82} = 0.122$$

2. After sorting the data in ascending order, we get

$Q_1 = 4$ and $Q_3 = 7$

Therefore,

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{7-4}{7+4} = \frac{3}{11} = 0.273.$$

Answer to Exercise 3.21

1.

Class boundaries	m	f	Cf
29.5-39.5	34.5	8	8
39.5-49.5	44.5	87	95
49.5-59.5	54.5	190	285
59.5-69.5	64.5	304	589
69.5-79.5	74.5	211	800
79.5-89.5	84.5	85	885
89.5-99.5	94.5	20	905
Total		905	

$$Q_1 = L + \left(\frac{\frac{(n)}{4} - cf}{f}\right) w = 49.5 + \left(\frac{\frac{905}{4} - 95}{285}\right) 10 = 54.1$$

$$Q_3 = L + \left(\frac{\frac{k(n)}{4} - cf}{f}\right) w = 69.5 + \left(\frac{\frac{3(905)}{4} - 589}{800}\right) 10 = 70.6.$$

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{70.6 - 54.1}{70.6 + 54.1} = \frac{16.5}{124.7} = 0.13.$$

2.

a. For company A, $CQD = \frac{Q_3-Q_1}{Q_3+Q_1} = \frac{54300-5000}{54300+5000} = \frac{49300}{59300} = 0.83$ and

for company B, $CQD = \frac{Q_3-Q_1}{Q_3+Q_1} = \frac{45320-1000}{45320+1000} = \frac{44320}{46320} = 0.96.$

b. Company A has more variable income.

3.2.3 Coefficient of Mean Deviation (CMD)

Introduction

Students have previously discussed relative measure of dispersion with a particular focus on coefficient of range and coefficient of quartile deviation. In this sub-topic, they will further discuss the coefficient of mean deviation of given data.

Competencies

By the end of this sub-unit, students will be able to:

- calculate the coefficient of mean deviation of a given data set.
- interpret two or more data sets by using their coefficient of mean deviation.

Vocabulary: mean, median, mode**,** mean deviation, coefficient mean deviation

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.11. With the help of examples of ungrouped data and grouped data, give a brief revision of calculating the coefficient of mean deviation.

Answer to Activity 3.11

1. $\bar{x} = 4.9,\ m_d = 5,\ m_o = 6$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n}|x_i - \bar{x}|}{n} = \frac{11.2}{10} = 1.12$$

$$MD(m_d) = \frac{\sum_{i=1}^{n}|x_i - m_d|}{n} = \frac{11}{10} = 1.1$$

$$MD(m_o) = \frac{\sum_{i=1}^{n}|x_i - m_o|}{n} = \frac{13}{10} = 1.3.$$

2. $MD(\bar{x}) < MD(m_d) < MD(m_o)$

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student book. You can also use Exercise 3.21 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating coefficient of mean deviation of ungrouped and grouped data.

Answer to Exercise 3.22

1. $CMD = \frac{MD}{\bar{x}} = \frac{4.22}{40} = 0.11.$

2.

x	m	f	$f\|m - m_0\|$	Cf
35-39	37	13	189.8	13
40-44	42	15	144	28
45-49	47	17	78.2	45
50-54	52	28	11.2	73
55-59	57	12	64.8	85
60-64	62	10	104	95
65-69	67	5	77	100
Total		100	669	

$$m_o = L + \left(\frac{\Delta_1}{\Delta_{1} + \Delta_2}\right)w = 50 + \left(\frac{11}{11 + 16}\right)4 = 51.63.$$

$$MD(m_o) = \frac{\sum_{i=1}^{n} |x_i - m_o|}{n} = \frac{669}{100} = 6.69.$$

$$CMD = \frac{MD}{m_o} = \frac{6.69}{51.6} = 0.13$$

3.

a. For company A, $CMD = \frac{MD}{m_d} = \frac{4430}{50000} = 0.089$ and

for company B, $CMD = \frac{MD}{m_d} = \frac{2200}{20000} = 0.1.$

b. Company A has the more consistent income.

3.2.4 Coefficient of Variation (CV)

Introduction

Students have previously discussed relative measure of dispersion with a particular focus on coefficient of range, coefficient of quartile deviation and coefficient of mean deviation. In this sub-topic, they will further discuss the coefficient of variation of given data.

Competencies

By the end of this sub-unit, students will be able to:

- Calculate coefficient of mean deviation of a given data.
- Interpretation of two or more data sets using their coefficient of mean deviation.

Vocabulary: mean, coefficient of variation

Teaching Notes

You may begin this section by encouraging students to attempt Activity 3.12. With the help of examples of ungrouped data distribution and grouped data, give a brief revision of calculating the coefficient of variation.

Answer to Activity 3.12

1. $\bar{x} = \frac{400}{5} = 80$

$$\delta^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} = \frac{6800}{5} = 1360$$

$$\delta = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}} = \sqrt{1360} = 36.88$$

To assist students to revise these ideas, it is recommended that a discussion is conducted on the examples given in the student book. You can also use Exercise 3.22 for revision purpose and assessing student competencies.

Assessment

You can give several problems on calculating the coefficient of variation of ungrouped and grouped data. In addition, you may give real-life problems whose data will be collected by the students themselves that can be used to calculate various relative measures of dispersion of the data sets and compare the data sets. You can do this through a group assignment or project work.

Answer to Exercise 3.23

1.

x	f	xf	$f(x-\bar{x})^2$
2	3	6	72.03
5	4	20	14.44
7	9	63	0.09
8	3	24	3.63
10	6	60	57.66
Total	25	173	147.85

$$\bar{x} = \frac{173}{25} = 6.9$$

$$\delta^2 = \frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n} = \frac{147.85}{25} = 5.9$$

$$\delta = \sqrt{\frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n}} = \sqrt{5.9} = 2.4$$

$$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{2.4}{6.9} \times 100 = 35.2.$$

2.

a. For farmer A

$$\bar{x} = \frac{530}{10} = 53$$

$$\delta^2 = \frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n} = \frac{4038}{10} = 403.8$$

$$\delta = \sqrt{\frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n}} = \sqrt{403.8} = 20.1$$

$$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{20.1}{53} \times 100 = 37.9$$

For farmer B

$$\bar{x} = \frac{490}{10} = 49$$

$$\delta^2 = \frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n} = \frac{13734}{10} = 1373.4$$

$$\delta = \sqrt{\frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{n}} = \sqrt{1373.4} = 37.1$$

$$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{37.1}{49} \times 100 = 75.6$$

b. Farmer A has more consistent prices.

3.

a. For company A

$$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{2400}{12000} \times 100 = 20$$

For company B

$$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{4400}{20000} \times 100 = 22$$

b. Company B has the more variable income.

3.3 Use of Frequency Curves

3.3.1 Relationship among Mean, Median and Mode

Introduction

So far, students have discussed the three measures of central tendency and skewness of the distribution of a data set, which may be symmetrical, skewed to the left or skewed to the right. They discussed skewness by observing histograms or frequency curves. Here, they will discuss how they can use measures of central tendency.

Competencies

By the end of this subunit, students will be able to:

- describe the relationship between mean, median and mode for grouped data using its frequency curve.
- use cumulative frequency graphs to determine the dispersion of values of data (in terms of mean, median and mode).
- determine the variability of data values in terms of quartiles by using a cumulative frequency graph.

Vocabulary: Symmetry, Skewness,

Teaching Notes

You may start the lesson by first asking the students to attempt Activity 3.13 and then discuss the result by comparing the mean, the mode and the median. Then, revise the three types of frequency curves (symmetrical, skewed to the left, and skewed to the right).

Answer to Activity 3.13

1.
   a. For data set A

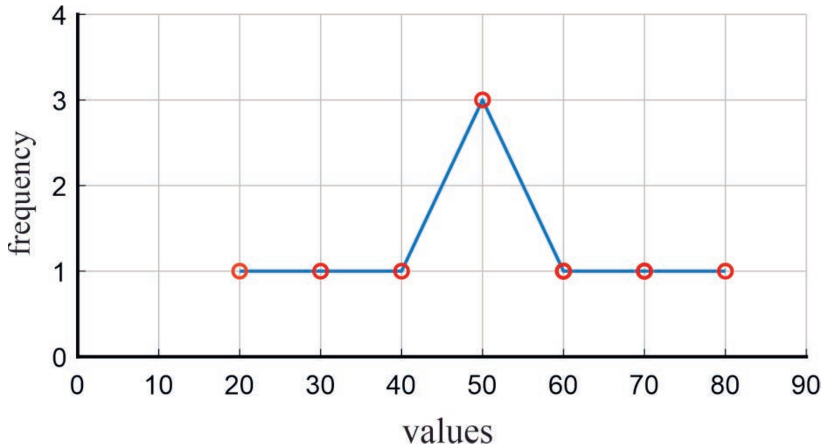
      mean=median=mode=50.

      For data set B

      mean=50, median=50 and Mode=80

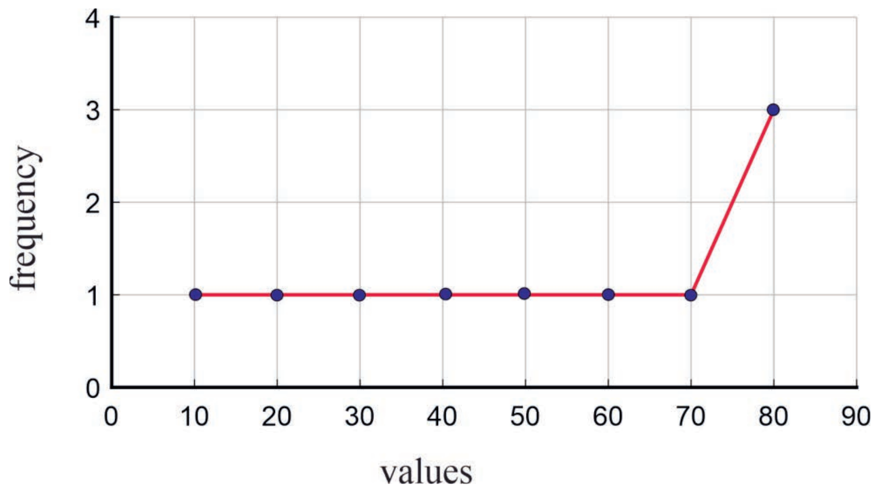
2. Constructing frequency curve for the following data:

   Data A: $20, 30, 40, 50, 50, 60, 50, 70, 80$



For data A: Mean = 50, Median = 50, Mode = 50.

Data B: 20, 30, 10, 40, 80, 50, 80, 60, 80



For data B: Mean = 50, Median = 50, Mode = 80.

mean of data A = mean of data B = 50,
median of data A = median of data A = 50,
mode of data A = 50 and mode of data B = 80.

b. Even though the two data have the same mean and median, they do have different Mode.

3.3.2 Relationship among Mean, Median and Standard Deviation

Introduction

So far, students have discussed the three measures of central tendency and skewness of the distribution of a data set, which may be symmetrical, skewed to the left or skewed to the right. They discussed skewness by observing histograms or frequency curves. Here, they will discuss how they can use measures of central tendency and measures of dispersion to determine the skewness of a distribution.

Competencies

By the end of this subunit, students will be able to:

- describe the relationship between mean, median and standard deviation for grouped data using its frequency curve.
- use cumulative frequency graphs to determine the dispersion of values of data (in terms of mean, median and standard deviation).
- determine the variability of data values in terms of quartiles by using cumulative frequency graph.

Vocabulary: Symmetry, skewness, Pearson's coefficient of skewness, Bowley's coefficient of skewness.

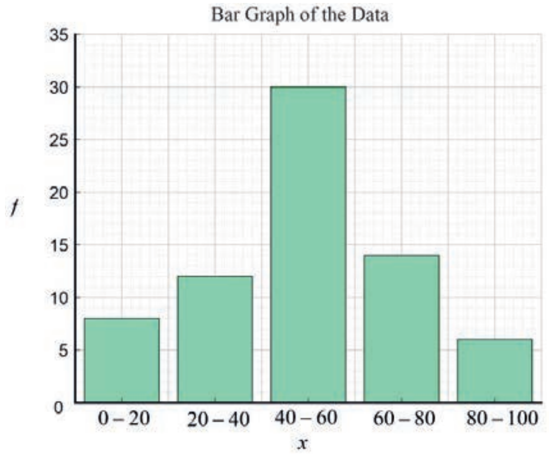
Teaching Notes

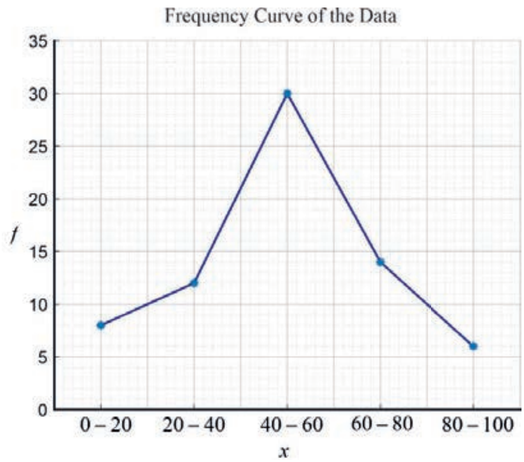
You may start the lesson by first asking the students to attempt Activity 3.14 and then discuss the result by comparing the mean, the mode and the standard deviation. Then, revise the three types of frequency curves (symmetrical, skewed to the left, or skewed to the right).

Answer to Exercise 3.24

x	0-20	20-40	40-60	60-80	80-100
f	8	12	30	14	6

a.





b. mean=49.3, median=46, mode=50.

c. mean< median<mode.

Therefore, the distribution is negatively skewed (Skewed to the left).

Answer to Activity 3.14

1. For data A

   mean=5=median and standard deviation=1.76.

   For data B

   mean=5=median and standard deviation=2.54

2. Mean and median of both data are equal but standard deviation of data B is greater than standard deviation of data A.

Answer to Exercise 3.25

x	10	20	30	40	50
f	4	6	5	3	2

Solution:

mean=26.5, median=25, standard deviation=12.36

$$\alpha = \frac{3(\text{mean} - \text{median})}{\text{Standard divation}} = \frac{3(26.5 - 25)}{12.36} = 0.364 > 0$$

Therefore, the distribution is positively skewed.

Answer to Exercise 3.26

1.

x	10	20	30	40	50
f	4	6	5	3	2

median=$Q_2 = 25, \quad Q_1 = 20, \quad Q_3 = 35$

$$\beta = \frac{Q_3 + Q_1 - 2(\text{median})}{Q_3 - Q_1} = \frac{1}{3} > 0$$

Therefore, the distribution is skewed positively (skewed to the right).

2. In both cases, the distribution is skewed positively (skewed to the right).

3.4 Sampling Techniques

Introduction

In this sub-topic, students are expected to identify the four sampling techniques, namely, simple random sampling, systematic sampling, stratified sampling and cluster sampling, and they will also determine the advantages and limitations of each technique. In order to teach these techniques, it is recommended to let the students say or write their understanding before they discuss each. Finally, it will be helpful to practice each in some practical problems.

Competencies

By the end of this sub-unit, students will be able to:

- describe the four methods/ techniques of sampling.
- explain the advantages and limitations of each sampling technique. of

Vocabulary: population, sample, random sampling, systematic sampling, stratified sampling, cluster sampling**,** sampling technique

Teaching Notes

It will be better to revise the purpose of field statistics in different sectors of social and economic situations. You can then proceed by refreshing students' memory with definitions of statistical terms such as population and sample; and by encouraging students to attempt Activity 3.15.

Answer to Activity 3.15

1. Collecting data is the basis for conducting statistical process. Without having an aggregate of facts or collected information, it is difficult to perform statistics. Hence, it is essential to collect data.
2. At this stage, we can think of different mechanisms of data collection such as using questionnaire, interview, observation, discussion, performing experiments, etc.
3. We can collect data either from a population or from a sample (which is represents a population). But here you are expected to explain to the students that trying to collect data from the entire population is difficult, time consuming and cumbersome. Thus, data would be collected from a sample, which is part of the population and is assumed to be representative of the population. The data in this case can be collected either primarily from the farmers themselves or from secondary sources such as the woreda bureau of agriculture and natural resources.

To assist students following the discussion above, encourage them to identify the concepts of a population and a sample. Right after the students have understood the concept of population, sample and the need for sampling, you can let them discuss some of the sectors in which statistics is useful. Some examples are given in the student book. Here you also need to let students understand the ideas of sample size, homogeneity and independence, equally likely, and representativeness of a sample to a population.

Answer to Exercise 3.27

   a. The size of the population=160
   b. Size of the sample=20

1. Not acceptable, because boys are not in the sample.
2. Not acceptable because sections B, C and D are not represented in the sample.

3. Not acceptable because 10 boys and 10 girls are not proportional to the population.
4. Not acceptable because sections are not represented proportionally in the sample.

Answer to Exercise 3.28

Roll Number	Random Number	Random Number in Ascending Order	Roll Number Selected
1	49	4	9
2	56	6	7
3	31	14	6
4	28	17	14
5	72	25	19
6	14	27	13
7	06	28	4
8	39	31	3
9	04	37	12
10	78	39	8
11	48	41	17
12	37	48	11
13	27	49	1
14	17	50	20
15	68	56	2
16	71	68	15
17	41	71	16
18	80	72	5
19	25	78	10
20	50	80	18

Which roll numbers did you select? 9, 7, 6, 14, and 19. Compare the results with others.

Answer to Exercise 3.29

a. Sampling interval= 60.

b. 13, 73, 133, 193, 253, 313, 373, 433, 493, 553, 613, 673, 733, 793, 853, 913, 973, 1033, 1093, and 1153.

Answer to Exercise 3.30

a. Stream (natural and social)

b. From social science 12 sections and from Natural Science 18 sections.

Answer to Exercise 3.31

a. Systematic sampling.

b. Simple random Sampling.

c. Stratified sampling.

d. Cluster sampling.

Answer to Exercise 3.32

a. 125

b. Total number of population=100+300+300+400+500+600+800+1000=4000.

1^{st} Woreda: $\frac{100}{4000}\times 1000 = 25.$

2^{nd} Woreda: $\frac{300}{4000}\times 1000 = 75.$

3^{rd} Woreda: $\frac{300}{4000}\times 1000 = 75.$

4^{th} Woreda: $\frac{400}{4000}\times 1000 = 100.$

5^{th} Woreda: $\frac{500}{4000}\times 1000 = 125.$

6^{th} Woreda: $\frac{600}{4000}\times 1000 = 150.$

7^{th} Woreda: $\frac{800}{4000}\times 1000 = 200.$

8^{th} Woreda: $\frac{1000}{4000}\times 1000 = 250.$

Therefore, the number of sample households to be selected from each Woreda are 25, 75, 75, 100, 125, 150, 200 and 250 respectively.

Answer to Review Exercise

1. $L = 35$ and $S = 15$, then R=35-15=20,

$Q_1 = 20.95$, $Q_3 = 27.33$, $IQR = Q_3 - Q_1 = 6.38$ $CQD = \dfrac{Q_3 - Q_1}{Q_3 + Q_1} = 0.132$ and

$CR = \frac{L-S}{L+S} = \frac{35-15}{35+15} = \frac{20}{20} = 0.4$.

2.

x_i	f	$x_i f_i$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
2.25	17	38.3	1.08	18.41
2.75	97	267	0.58	56.55
3.25	187	608	0.08	15.52
3.75	135	506	0.42	56.30
4.25	18	119	0.92	25.68
4.75	6	28.5	1.42	8.50
Total	470	1567	4.5	180.95

$$MD = \frac{\sum f_i \lvert x_i - \bar{x}\rvert}{\sum f_i} = \frac{180.95}{470} = 0.34$$

$$CMD = \frac{MD}{\bar{x}} \times 400 = \frac{0.34}{3.3} \times 100 = 11.8$$

3. $\sigma^2 = \frac{\sum(x-\mu)^2}{N} = 44.5$. Thus $\sigma = \sqrt{44.5} = 6.67$.

4.

a. $\sigma^2 = \frac{\sum(x-\mu)^2}{N} = 6$. Thus $\sigma = \sqrt{6}$.

b. $\sigma^2 = \frac{\sum(x+10-\mu)^2}{N} = 6$. Thus $\sigma = \sqrt{6}$.

c. $\sigma^2 = \frac{\sum(5x-\mu)^2}{N} = 150$ Thus $\sigma = 5\sqrt{6}$.

d. By adding the same number to the all values the standard deviation is the same but multiplying all values by some constant n then the new standard deviation is n times the old standard deviation.

5. $\sigma^2 = \frac{\sum(x-\mu)^2}{n} = 131.25$. $\bar{x} = 22.5$, $\sigma = 11.46$, $CV = \frac{\sigma}{\bar{x}} = \frac{11.46}{22.5} \times 100 = 50.93$.

6. Mean deviation about the mode=1.74,

$\bar{x} = 7.13$, $\sigma = 2.87$, $CV = \frac{\sigma}{\bar{x}} = \frac{2.87}{7.13} \times 100 = 40.25$.

7.

$$n = 15, \qquad \mu = 10, \qquad \delta = 5,$$

$$\sum x = n\mu = 15 \times 10 = 150$$

wrong observation value $= 8$

corrected observation value $= 23$

corrected total $= 150 - 8 + 23 = 165$

corrected $\mu = \frac{165}{15} = 11$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

incorrect $\sigma = 5 = \sqrt{\frac{\sum x^2}{n} - (10)^2}$

$$25 = \frac{\sum x^2}{n} - 100$$

$$\frac{\sum x^2}{n} = 125$$

i.e, $\frac{\sum x^2}{15} = 125$

$$\sum x^2 = 1875$$

corrected $\sum x^2 = 1875 - 8^2 + 23^2 = 23210$

corrected $\sigma = \sqrt{\frac{23210}{15} - (11)^2} = \sqrt{35} \approx 5.9$

8.

Technician A

$$CV = \frac{\sigma}{\mu} \times 100 = \frac{5}{40} \times 100 = 12.5$$

Technician B

$$CV = \frac{\sigma}{\mu} \times 100 = \frac{15}{160} \times 100 = 9.36$$

Employee B shows less variation.

9.

$R = n - 1,$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1+2+3+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$= \frac{1+2^2+\cdots+n^2}{n} - \left(\frac{1+2+\cdots+n}{n}\right)^2$$

$$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n(n+1)}{2n}\right)^2 = \frac{2n+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$= \frac{4n^2+6n+2-3n^2-6n-3}{12}$$

$$= \frac{n^2-1}{12}$$

Therefore, $\sigma = \sqrt{\frac{n^2-1}{12}}$

10. The number of strikes the bell make a

day=2(1+2+3+4+5+6+7+8+9+10+11+12)=$2 \times \frac{n(n+1)}{2} = 2 \times \frac{12\times13}{2} = 156.$

Standard deviation of the sum of the first n natural number = $\sqrt{\frac{n^2-1}{12}}$.

$\delta = 2 \times \sqrt{\frac{n^2-1}{12}} = 2 \times \sqrt{\frac{12^2-1}{12}}$=$2 \times \sqrt{\frac{143}{12}}$=6.90.

11. $\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} = \sqrt{\frac{33315}{1500} - \left(\frac{6750}{1500}\right)^2} = \sqrt{1.96}$=1.4.

12.

a.

X	$\bar{x}$	*SD*	*CV*
Death	21.20	3.83	$CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{3.83}{21.20} \times 100 = 18.07$
Heavy injury	47.20	12.85	$CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{12.85}{47.20} \times 100 = 27.22$
Light injury	88.40	12.70	$CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{12.70}{88.40} \times 100 = 14.37$
No. of accident	276.40	12.80	$CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{12.80}{276.40} \times 100 = 4.63$

b. Number of death is more variable than number of accident.

13. Let the two observations be x and y. $\bar{x} = 8$ and CV=50. Since $CV = \frac{\delta}{\bar{x}} \times 100$,

we get $\delta = \frac{CV \times \bar{x}}{100} = \frac{50 \times 8}{100} = 4$. Thus, variance, $\delta^2 = 16$.

Now, $\frac{\sum x}{n} = \frac{42+x+y}{7} = 8$, which implies $42+x+y = 56$.

Hence, $x+y=14$. .... (1)

$$\sum x = 42 + x + y$$

$$\sum x^2 = 460 + x^2 + y^2$$

$$\delta = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\delta^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{\sum x^2}{n} - \bar{x}^2$$

This implies $\frac{460+x^2+y^2}{7} - 64 = 16$. Thus, $x^2 + y^2 = 100$.

But $x^2 + y^2 = (x+y)^2 - 2xy$.

Therefore, $100=14^2-2xy$, which implies

$xy=48$..................... (2)

From equation (2), we get $y=\frac{48}{x}$ and substituting in (1), we obtain $x + \frac{48}{x} = 14$.

This implies

$$x^2 - 14x + 48 = (x-8)(x-6) = 0.$$

Hence, x=8 or x=6. If x=8, then y=6. If x=6, then y=8.

Thus, the two observations are 6 and 8.

14.

$\bar{\text{x}} = 60$ and CV=25. Thus, $\delta = \frac{CV \times \bar{\text{x}}}{100} = \frac{25 \times 60}{100} = 15$. Wrong values are 40 and 27 and correct values are 45 and 72. Wrong total $= \bar{\text{x}} \times n = 60 \times 100 = 6000$.

Corrected total=wrong total-wrong values +correct values=6000-40-27+45+72=6050.

Corrected $\bar{\text{x}} = \frac{\text{cottect total}}{100} = \frac{6050}{100} = 60.5$

Wrong $\delta = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} = \sqrt{\frac{\sum x^2}{100} - (60)^2} = 15.$

This implies wrong $\sum x^2$ =382500.

Thus, correct $\sum x^2 = 382500 - 40^2 - 27^2 + 45^2 + 72^2 = 387380.$

Correct $\delta = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} = \sqrt{\frac{\text{correct} \sum x^2}{n} - \left(\frac{\text{correct} \sum x_i}{n}\right)^2} =$

$\sqrt{\frac{387380}{100} - (60.5)^2}$=14.61.

Corrected = $CV = \frac{14.61}{60.5} \times 100 = 24.15.$

15. MD=180 and CMD=10. We know that $CMD = \frac{MD}{\bar{x}}$, which implies $\bar{x} = CMD = \frac{MD}{CMD} = \frac{180}{10} = 18$. Thus, $\bar{x} = 18 = \frac{\sum fm}{\sum f} = \frac{752+(f\times 20)}{44+f}$, which implies 792+18$f$=752+20$f$, solving for f we get f=20.

16. MD about the median= 440, CMD about the median=8 and total frequency=82.

Since $CMD = \frac{MD}{m_d}$, $m_d = \frac{MD}{CMD} = \frac{440}{8} = 55.$

x	*f*	*cf*
0-10	5	5
10-20	17	22
20-30	2	24
30-40	s	24+s
40-50	12	36+s
50-60	2	38+s
60-70	7	45+s
70-80	t	45+s+t
80-90	9	54+s+t
90-100	4	58+s+t

Since, median = 55 it lies in the class of 50-60. Thus L=50, w=10, cf=36+s, f=2 and

$m_d = L + \left(\frac{\frac{N}{2}-cf}{f}\right)w = 50 + \left(\frac{\frac{82}{2}-(36+s)}{2}\right)10 = 55$, which implies s=4. The fact that total frequency = 82=58+s+t, we get t=20.

Unit 4: INTRODUCTION TO LINEAR PROGRAMMING

Introduction

Dear teacher before starting the lesson of these chapter, please motivate students how the applications of mathematics in particular related to information technology changed the world in the last 100 years.

In earlier classes, you have discussed systems of linear equations and their applications in day to day problems. In grade 9, you have studied linear inequalities and systems of linear inequalities in two variables and their solutions by graphical method. Many applications in mathematics involve systems of inequalities/equations. In this chapter, you shall apply the systems of linear inequalities/equations to solve some real life problems.

Unit Outcomes

By the end of this unit, you will be able to:

- Deduce how to find regions of inequality in a plane.
- Solve systems of linear inequality.
- Formulate linear programing problems.
- Solve real life problems of linear programming problems graphically.
- Use spreadsheet to solve Linear Programming Problems.

Suggested Teaching Aids

You know that students learn in a variety of different ways. Some are visually oriented and more inclined to acquire information from photographs or videos. Others do best when they hear instructions rather than read them. Teachers use teaching aids to provide these different ways of learning. Therefore, it is recommended that you may use models of planes, charts, calculators, trigonometric tables and computers for this unit. You can use also other types of materials as long as they help the learners to get the skills required. To clarify the topic using ruler, calculators and color chalk is highly recommended.

Under each sub-topic, a hint is given how to continue each sub-topic but your creativity is very crucial. The purpose of the teaching notes is to provide the teacher information to use activities, opening problems and group-works to motivate and guide students make the teaching learning process student center. Now this unit begins with an introductory example which may motivate students to follow the unit attentively. Therefore, before passing on to any subtopic of this unit make students discuss the activities.

4.1 Graphical Solutions of System of Linear Inequalities

Minimum Learning Competencies

- Draw graph of linear inequality
- Draw of graph of systems of linear inequalities

Revise linear graphs and try to relate the concept of "linear motion and circular motion" from physics course. And also remind students of linear demand and supply curves from economics class. In a class please use teacher aids like, ruler, triangles, rectangles, and so on.

And also discuss with the students to recall the terms like, slope of a line, y-intercept and x-intercept of a graph of a line.

Answer to Activity 4.1

1. The graph of the equation $y=mx+b$ for different value of m is as shown below in Figure 4.1, $m = 2, m = -2$ and $m = -4$.

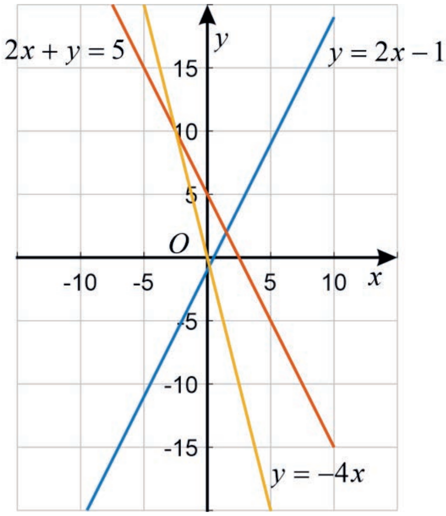


Figure 4.1

2. Find slope and y-intercept of the following

   a) $y = 3x + 2$. The equation is in the form of $y = mx + b$. So the slope $m = 3$ and y-intercept is at $(0, 2)$.

   b) $y = \frac{1}{3}x + 2$. The equation is in the form of $y = mx + b$. So the slope $m = \frac{1}{3}$ and y-intercept is at (0,2).

   c) $y = -4x + 5$. The equation is in the form of $y = mx + b$. So the slope $m = -4$ and y-intercept is at (0, 5).

   d) $x + 2y = 2$, write the equation in the form of $y = mx + b$. $y = -\frac{1}{2}x + 1$. So the slope is $m = -\frac{1}{2}$ and y-intercept is at (0,1).

3. The graph of the following linear equations given on the same coordinate axis in Figure 4.2

   a) $y = 2$, shown in red color.

   b) $x = -4$, shown in blue color.

   c) $y = -3$, shown in green color below the x-axis.

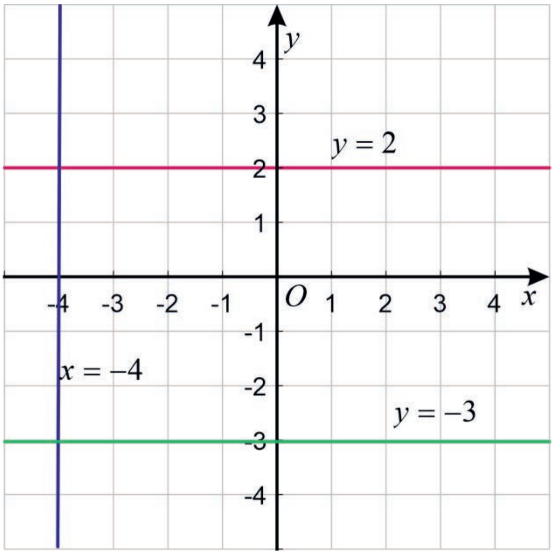


Figure 4.2

Start this section by asking questions from grade 9 and some other real life situation like height of the students in the class, the number of exercise books just now on the hand, the temperature of the place today and yesterday compare it.

4.1.1 Inequalities

Explain for the class almost all things are different in the world, to compare any two things in the universe they should have measured in the same unit.

AnsweEr to Exercise 4.1

1. Solve the following inequalities and show the graphs of their solutions on a number lines.

a. $5x + 1 < 2x + 7$

Collecting like terms,

$5x - 2x < 7 - 1,$

$3x < 6,$

$x < 2$ (Dividing both sides by 3). The solution become all real numbers less than 2.

b. $\begin{cases} 3x + 1 < 16 \\ -2x + 5 \leq 13 \end{cases}$

This equation have two inequalities.

From the first, $3x < 16 - 1 = 15$,

$x < 5$ (Dividing both sides by 3)

From the second $-2x \leq 13 - 5 = 8$

$x \geq -4$ (dividing by 2 and multiplying -1 will reverse the inequality).

Then, the answer is $-4 \leq x < 5$.

c. $-1 < -2x + 3 \leq 2$

This equation has two inequalities

$-1 < -2x + 3$ and $-2x + 3 \leq 2$

$-4 < -2x \wedge -2x \leq -1$,

$2x < 4 \wedge 2x \geq 1$ (Multiplying both sides by -1)

$x < 2 \wedge x \geq \frac{1}{2}$. The solution become $\frac{1}{2} \leq x < 2$ or all numbers in the interval$[\frac{1}{2}, 2)$.

d. $\frac{2x}{3} - 3 > \frac{16x}{21} - \frac{13}{3} - \frac{2x}{15}$

This inequality written as $\frac{2x+13}{3} - 3 > \frac{66x}{105}$, simplifying $35(2x + 4) > 66x$, this leads as $70x - 66x > -140$, finally $4x > -140$,

$x > -\frac{35}{2}$ (Dividing both sides by 4).

e. $\frac{x-4}{2} < \frac{7x}{2} - (3x + 2)$

Multiplying both sides by 2, we have

$x - 4 < 7x - 6x - 4, \Longrightarrow x - 4 < x - 4$. No number satisfies this condition. The answer is no solution.

f. $5(x - 4) + 17 - 14x > 13 - 4x$

Simplifying you obtain $-5x > 16$. $x < \frac{-16}{5}$ (Multiplying by -1 reverse the inequality and dividing by 5).

g. $12x - 1 \leq 3(4x - 3)$

Simplifying $12x - 12x \leq -8, 0 \leq -8$. There is no number that makes this inequality a true statement.

h. $11(2x - 15) < x + 3$

Multiplying $22x - 165 < x + 3$,

$21x < 168$,(collecting like terms)

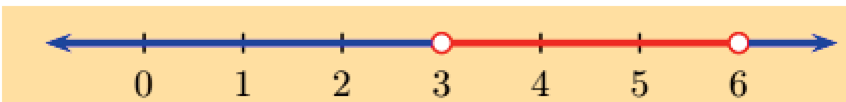
$x < 8$. (Dividing by 21 both sides). This is equivalent to the interval $(-\infty, 8)$.

i. $\frac{x}{x-3} < 2, x \neq 3$

$\frac{x}{x-3} < 2, x \neq 3$ First solve the equation $\frac{x}{x-3} < 2$, subtract 2 from both sides we obtain $\frac{x}{x-2} - 2 < 0$, equivalent to $\frac{-x+6}{x-3} < 0$ multiplying by -1 reverse the inequality and we obtain $f(x) = \frac{x-6}{x-3} > 0$. The ratio of two expression is positive either both of them positive or both negative. By using the following sign chart we obtain the solution

expression		$x = 3$	$3 < x < 6$	$x = 6$	
$x - 6$	-	-	-	0	+
$x - 3$	-	0	+	+	+
$f(x)$	+	undefined	-	0	+

From the table we see that $\boldsymbol{f(x)} > 0$ when $\boldsymbol{x} < 3$ or $\boldsymbol{x} > 6$ with $\boldsymbol{x} \neq 3$. The solution can be represented on the number line as follow:



j. $\frac{x+2}{x} - 1 \geq 0,\ x \neq 0.$

Solution: $\frac{x+2}{x} - 1 \geq 0,\ x \neq 0$ first simplify the equation $\frac{x+2}{x} - 1 \geq 0$,

$$\frac{x+2-x}{x} \geq 0,$$

$\frac{2}{x} \geq 0$. So $x > 0$. The solution is $x > 0$.

4.1.2 Word Problem of Linear inequality

In this section student will practice how word problems expressed mathematically in the form of inequality.

Answer to Exercise 4.2

1. **Solution**

   First express the problem mathematically. Let x be the mark obtained by Semira in fifth examination. $x \leq 100$.

   To score A, $\frac{87+82+81+80+x}{5} \geq 85$. Equivalently, $\frac{330+x}{5} \geq 85$ Solving for x we have $330 + x \geq 425, x \geq 95$. Semira has a chance to score A. If Semira will get a result between 95 and 100 in the fifth exam her average mark would be greater than or equal to 85.

2. **Solution**

   The pair of consecutive odd integers are by inspection (11, 13), (13, 15), (15, 17), and (17, 19). Algebraically, let x be positive integer a and b be two consecutive odd positive integers less than 20. $a = 2x - 1, b = 2x + 1$. From the two expression $x < \frac{19}{2} = 9.5$.

   $a + b > 21, 4x > 21, x > \frac{21}{4} = 5.25$. Therefore, x is any integer between 5.25 and 9.5. Those are 6,7,8,9. So $a = 11, 13, 15, 17$. Similarly, $b = 13, 15, 17, 19$.

3. **Solution:**

   a) Let the number of sessions be x. Then, $3 + \frac{3}{4}x \geq 12$, is an inequality.

   b) Let V is the volume of the prism, $V \leq 65\ cm^3$. Where, $V = l \times h \times w = 13\ cm \times 0.5 \times h \leq 65 cm^3$.

4.1.3 Linear Inequalities in Two variables and their Graphical Solutions

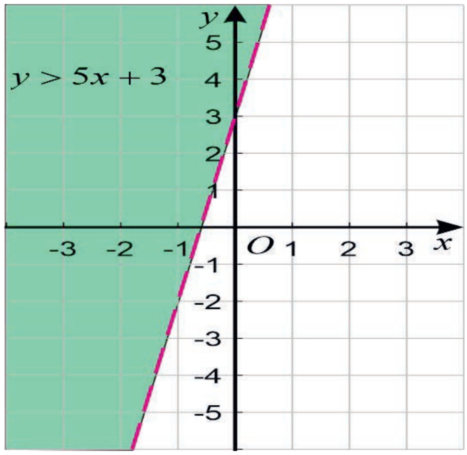
Answer to Activity 4.2

Choose any point(x, y). When $x < y$ the point is above the line $y = x$. When $x > y$ the point is below the line $y = x$.

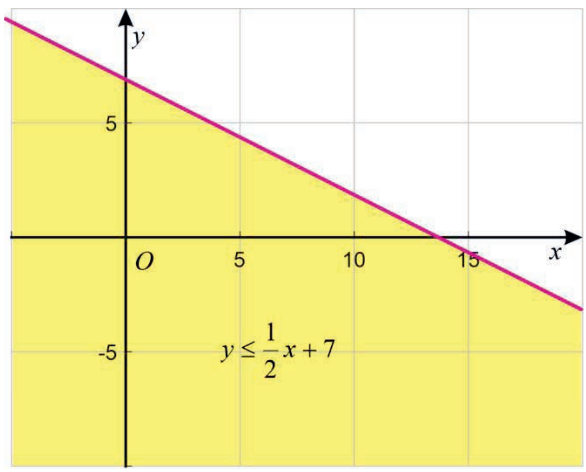
Answer to Exercise 4.3

1. Graph the solution set for the following inequality.

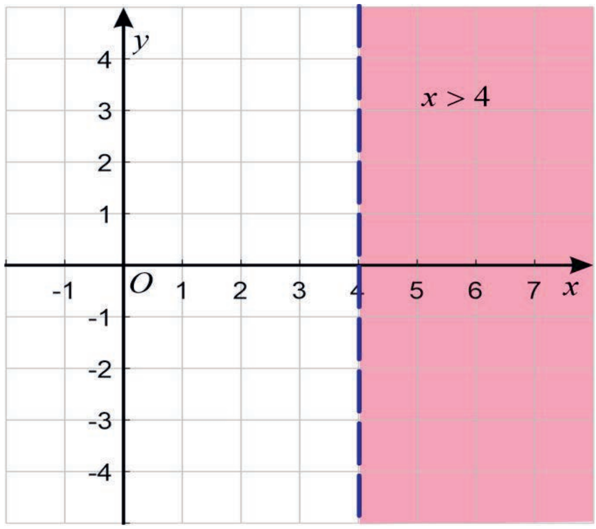
   a) $y > 5x + 3$



   b) $y \leq -\dfrac{1}{2}x + 7$

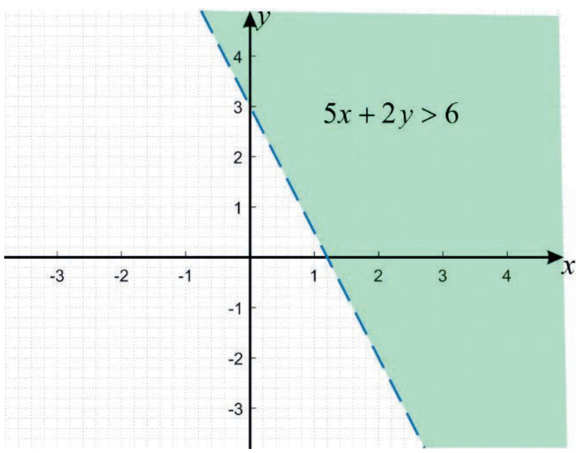


c) $x > 4$



d) $5x+2y>6$ solving for y you obtain $y > -\frac{5}{2}x + 3$, then draw the graph of

$$y > -\frac{5}{2}x + 3$$



2. Answer

To test $(2, 5)$ is in the solution of the inequality $y + 5x \geq 1$, by substituting the order pair in the inequality you obtain

(LHS) $5 + 5 * 2 = 15$

Thus, (LHS) $\geq (RHS)$. Is true so (2, 5) is in the solution of the inequality.

3. To test $(2, \frac{1}{2})$ is in the solution of the inequality $5x - 2y < 10$, by substituting the order pair in the inequality you obtained

$$(LHS) = 5 \times 2 - 2 \times (\frac{1}{2}) = 9$$

Thus, $(LHS) < (RHS)$. Is true so$(2, \frac{1}{2})$ is in the solution of the inequality.

4. The inequality that describes all points in:
   a. the upper half-plane above the x-axis is $y > 0$.
   b. the lower half-plane below the x-axis is $y < 0$.
   c. the half-plane left of the y-axis is $x < 0$.
   d. the half-plane right of the y-axis is $x > 0$.

4.1.4 Systems of Linear Inequalities in Two Variables and Graphical solutions

Answer to Activity 4.3

Let x be the number of cups of milk a student consumed.

y be the number of loaf of bread a student consumed.

The total available money is Birr 15.00. On this budget a grade 12 student able to buy

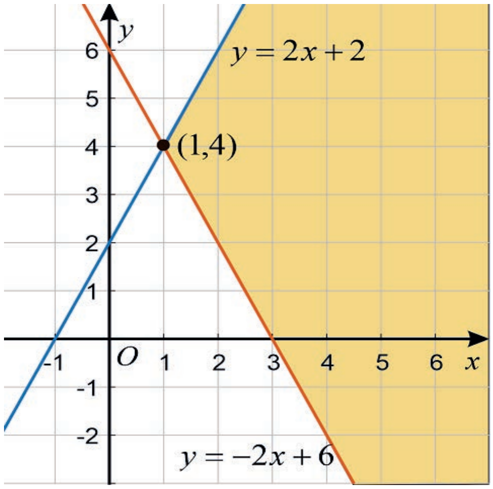
$$\begin{cases} 2.4x + 1.2y \leq 15 \\ x \geq 0, y \geq 0 \end{cases}$$

These are required inequalities.

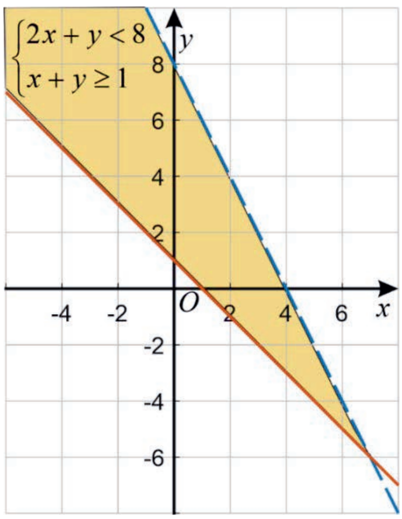
Answer to Exercise 4.4

1. Solve the following systems of inequalities graphically.

a. $\begin{cases} y \leq 2x + 2 \\ y \geq -2x + 6 \end{cases}$ by drawing the two lines $y = 2x + 2$ and $y = -2x + 6$ in the same coordinate axes. By shading below the line $y = 2x + 2$ and above the line $y = -2x + 6$. The intersection point (1, 4) is part of the solution.



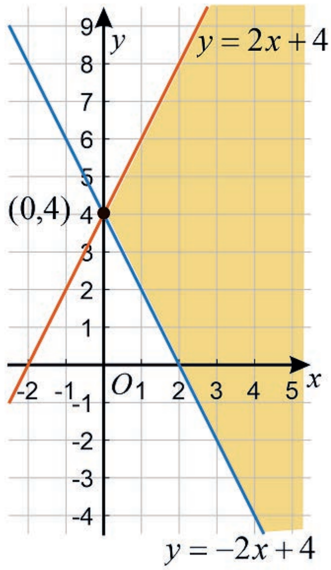
b. $\begin{cases} 2x + y < 8 \\ x + y \geq 1 \end{cases}$ The system of inequality is equivalent with $\begin{cases} y < -2x + 8 \\ y \geq -x + 1 \end{cases}$. The solution region is shown in the region shaded by blue color.



The two lines intersect at $(7, -6)$. The intersection point $(7, -6)$ is not part of the solution.

c. $\begin{cases} y - 2x \leq 4 \\ 2x + y \geq 4 \end{cases}$ The system of inequality is equivalent with $\begin{cases} y \leq 2x + 4 \\ y \geq -2x + 4 \end{cases}$.

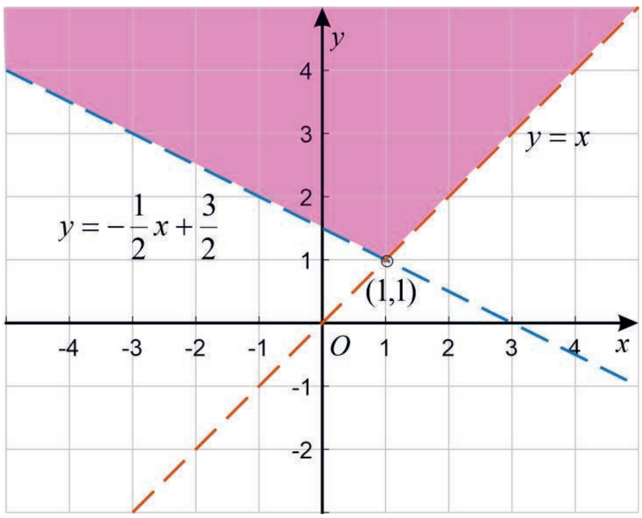
Similar to problem (a) the solution region is shown in the region shaded by yellow color.



The intersection point $(0,4)$ is part of the solution.

d. $\begin{cases} x - y < 0 \\ x + 2y > 3 \end{cases}$ The system of inequality is equivalent with $\begin{cases} y > x \\ y > -\frac{1}{2}x + \frac{3}{2} \end{cases}$.

Similar to problem (a) the solution region is shown in the region shaded by blue color.

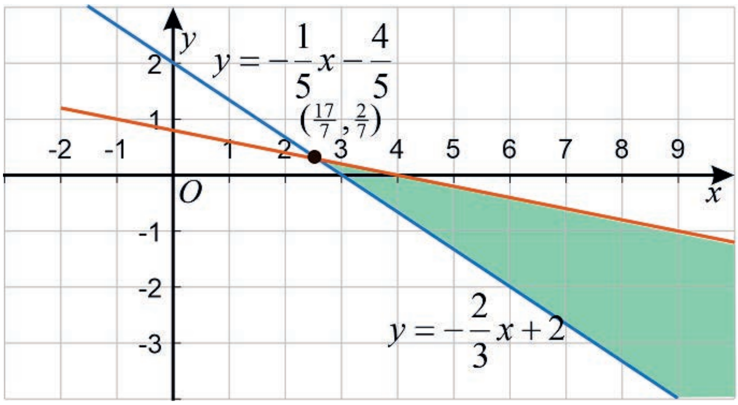


The intersection point $(1,1)$ is not part of the equation.

e. $\begin{cases} 2x + 3y \geq 6 \\ x + 5y \leq 4 \end{cases}$

The system of inequality is equivalent with $\begin{cases} y \geq -\frac{2}{3}x + 2 \\ y \leq -\frac{1}{5}x + \frac{4}{5} \end{cases}$.

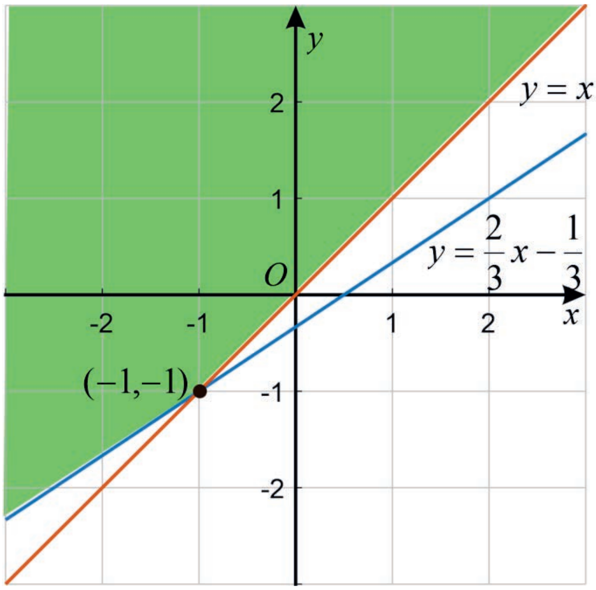
Similar to problem (a) the solution region is shown in the region shaded by green color.



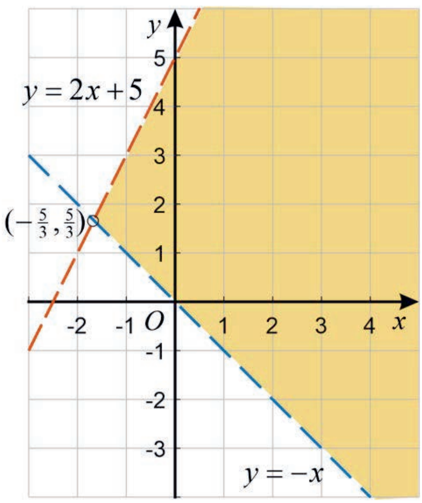
The intersection point $(\frac{17}{7}, \frac{2}{7})$ is part of solution.

f. $\begin{cases} y - x \geq 0 \\ 2x - 3y \leq 1 \end{cases}$ The system of inequality is equivalent with $\begin{cases} y \geq x \\ y \geq \frac{2}{3}x - \frac{1}{3} \end{cases}$.

Similar to problem (a) the solution region is shown in the region shaded by blue color. The two lines intersect at $(-1, -1)$. This point is part of the solution.



g. $\begin{cases} y + x > 0 \\ y - 2x < 5 \end{cases}$ The system of inequality is equivalent with $\begin{cases} y > -x \\ y < 2x + 5 \end{cases}$ the solution region is shown in the region shaded by blue color. The intersection point $(-\frac{5}{3}, \frac{5}{3})$.



2. To test each point are in the solution region of the system we use the following Table

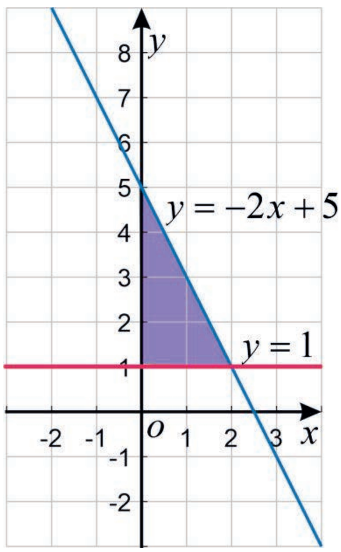
Order pairs	Satisfies $x + 4y \geq 10$	Satisfies $3x - 2y < 12$	In the overlapping shaded regions?
$(-2, 4)$	Yes	Yes	Yes
$(3, 1)$	No	Yes	No
$(1, 3)$	Yes	No	No
(0, 0)	No	Yes	No

Therefore, the only point in the solution region is (-2, 4). Thus, the answer is a.

4.1.5 Further on the system of inequalities

Answer to Exercise 4.5

1. To solve the system of the inequality $\begin{cases} y \leq -2x+5 \\ x \geq 0 \\ y \geq 1 \end{cases}$ first we draw the lines $\begin{cases} y = -2x+5 \\ y = 1 \end{cases}$ in $xy-$plane as follow:



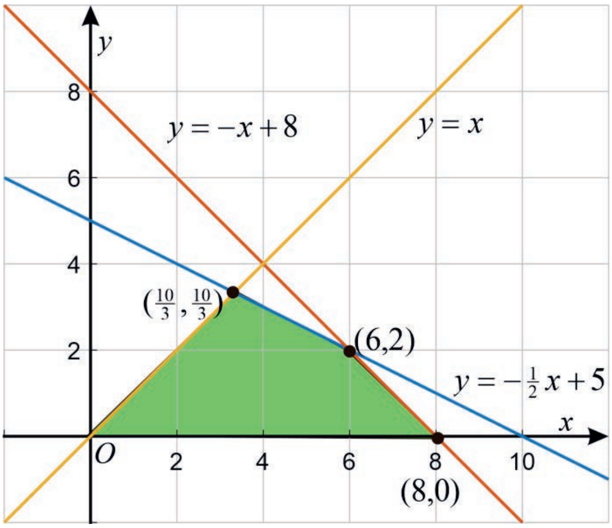
After drawing the lines, shade the region below the line $= -2x + 5$, the right of the y-axis and above the line $y = 1$.

All the points in the xy −plane shown by the shaded region is the solution of the system.

2. To solve the system graphically you can plot the graph of the inequality

$$\begin{cases} y \leq -\frac{1}{2}x + 5 \\ y \leq -x + 8 \\ y \leq x \\ x \geq 0 \\ y \geq 0 \end{cases}$$. To draw the graph first convert the system of inequality to a system of equation as

$$\begin{cases} y = -\frac{1}{2}x + 5 \\ y = -x + 8 \\ y = x \\ x \geq 0 \\ y \geq 0 \end{cases}$$ By plotting the first three lines and shading below each line in the first quadrant you will get the shaded region in as shown:



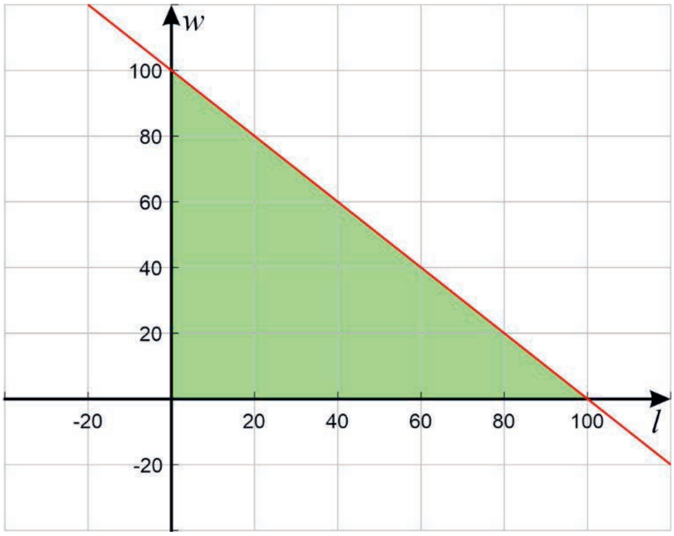
3. Let l be the length of the rectangular field, w be the width of the rectangular field. The perimeter p of the field is enclosed by 200 meter fencing material. i.e $p \leq 200$. Thus, we express this as an inequality

$$p = 2l + 2w \leq 200.$$

$$l + w \leq 100 \Leftrightarrow w \leq -l + 100.$$

$$l > 0, w > 0$$

Then the shaded part in Figure are all possible solutions to this problem.

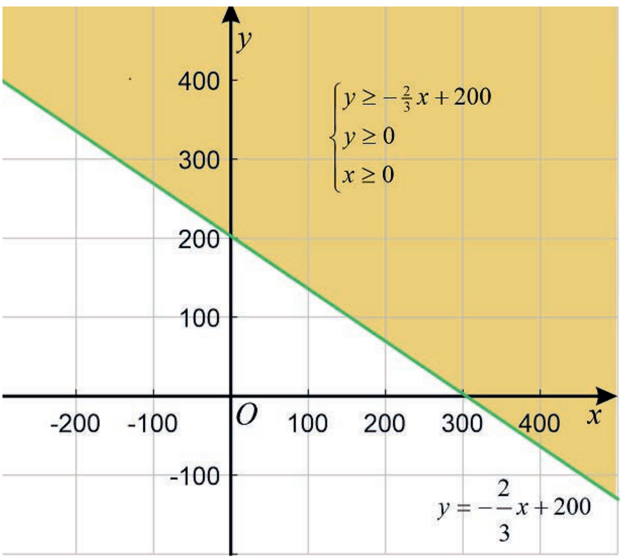


4. Letting Revenue is R, the number of first product sold is x unit, the number of second product sold is y unit. $R = 8x + 12y \geq 2400$. To graph the inequality first solve for y.

$12y \geq 2400 - 8x,$

$y \geq -\frac{2}{3}x + 200.$

Combining this inequality with the criteria $x \geq 0$ and $y \geq 0$ you obtain the following graph.



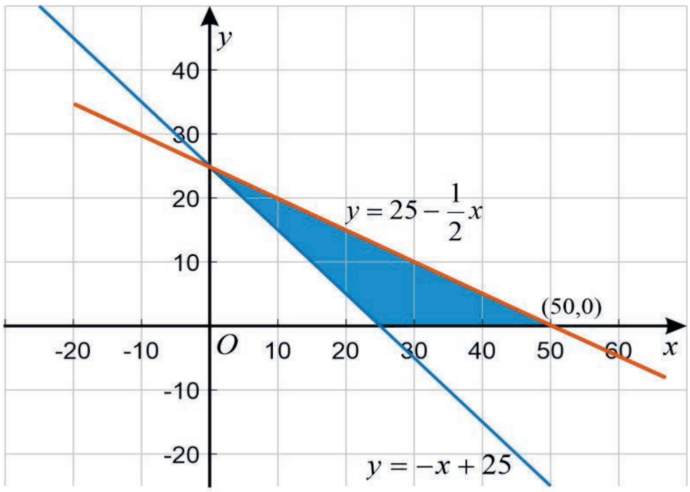
5. Let $x \geq 0$ and $y \geq 0$ be the number of small and large photo displayed in the street fair respectively.

a) To model the situation,

$x + y \geq 25$, (Yiseresh want at least 25 photos in her Gallery

$40x + 80y \leq 2000$. (Money constraint) The system of inequality which represent the situation was $\begin{cases} x + y \geq 25 \\ 40x + 80y \leq 2000 \end{cases}$

b) Draw the two inequality in the same xy −coordinate plane as follow:



c) Check the coordinate $(20, 10)$ is in the shaded region or not. Yes because the coordinate is inside the shaded region..

d) Check the coordinate $(10, 30)$ is in the shaded region or not. Not in the region.

4.2 Maximum and Minimum Value

Minimum Learning Competencies

- Find maximum and minimum values of a given objective function under given constraints.

Start this section by revising extreme value theorem of chapter two. How maximum and minimum values of a continuous function can be determined. In real life everyone wants to maximize revenue, profit, exam result, cumulative grade point average and so on. And also want to minimize loss, cost of producing a product, distance to reach somewhere, complaints, grievance and so on.

Mathematical formulation of the Problem

In solving real life problems the first step is Mathematical formulation of the problem. Formulation of an LPP refers to translating the real-world problem into the form of mathematical equations which could be solved. It usually requires a thorough understanding of the problem.

Steps Mathematical formulation of the problem

Step 1: Identify the 'n' number of decision variables which govern the behavior of the objective function (which needs to be optimized).

Step 2: Identify the set of constraints on the decision variables and express them in the form of linear equations /inequalities. This will set up our region within which the objective function needs to be optimized. Don't forget to impose the condition of non-negativity on the decision variables i.e. all of them must be positive since the problem might represent a physical scenario, and such variables can't be negative.

Step 3: Express the objective function in the form of a linear equation in the decision variables.

Step 4: Optimize the objective function graphically. (Find maximum and minimum)

Answer to Activity 4.4

In case of circular shape. Let the radius be r. Then, $2\pi r = 10$, $\Rightarrow r = \dfrac{100}{2\pi}$. Hence, the area to be formed,

$$A = \pi r^2 = \pi(\frac{100}{2\pi})^2 = \frac{10000}{4\pi} \approx 795.774 \approx 796m^2.$$

In case of rectangular shape, $A = l \times w$ since $2l + 2w = 100$, $w = 50 - l$.

Therefore, $A = l(50 - l) = 50l - l^2$.

$A = -(l-25)^2 + 625$ It is the quadratic function with downward open.

$$\begin{cases} l + w \leq 50 \\ l \geq 0 \\ w \geq 0 \end{cases}$$

Thus, the maximum area is $625m^2$ when $l = 25m$. The minimum area is $0m^2$ when $l = 0m$ or $50m$.

The area is maximum when the region is circular in shape.

Answer to Exercise 4.6

1. To solve the given problem first you can formulate the mathematical expression as follow:

   a) Let x be the number of table produced and sold and y be the number of chair produced and sold out.

   A dealer has two limitations finance and space,

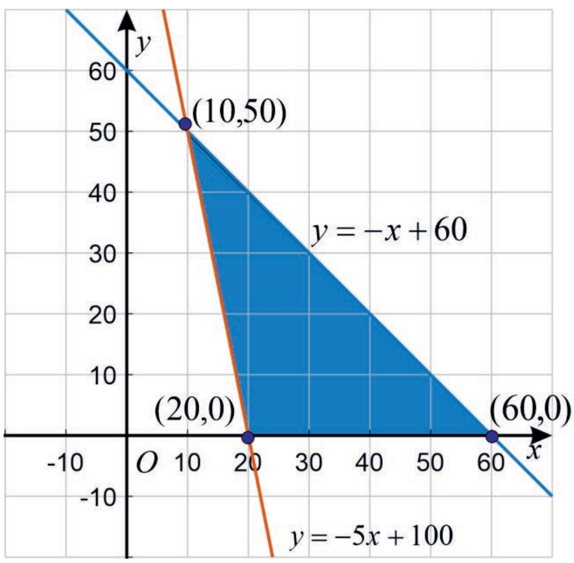
   $x + y \leq 60$, space limitation.

   $2500x + 500y \leq 50000$, finance limitation,

   The goal of the dealer is to maximize the profit function

   $Z = 200x + 50y$. This gives answer for part a.

   b) You can solve the problem graphically. Draw the limitation inequalities, $y \leq -x + 60$ and $y \leq -5x + 100$.



From the graph the possible optimal points are: (20, 0), (60, 0) and (10, 50). Testing each point in the objective functions as

At (20, 0), $Z = 200(20) + 50(0) = 4000$,

At (60, 0), $Z = 200(60) + 50(0) = 12000$.

At (10, 50) $Z = 200(10) + 50(50) = 2000 + 2500 = 4500$. The dealer earns maximum profit by producing and selling 60 tables.

2. The first thing to do is represent the problem in a tabular form for better understanding.

Type	Milk	Cream	Profit per unit (in Birr)
A	1	3	6
B	1	2	5
Total	5	12	

Let the total number of units produced by A be $= x$

Let the total number of units produced by B be $= y$

Now, the total profit is represented by Z

The total profit the company makes is given by the total number of units of A and B produced multiplied by its per-unit profit of Birr 6 and Birr 5 respectively.

Profit: $Max\ Z = 6x + 5y$, which means we have to maximize Z.

The company will try to produce as many units of A and B to maximize the profit. But the resources Milk and cream are available in a limited amount. As per the above table, each unit of A and B requires 1 unit of Milk. The total amount of Milk available is 5 units. To represent this mathematically,

$$x + y \leq 5$$

Also, each unit of A and B requires 3 units & 2 units of Cream respectively. The total amount of Choco available is 12 units. To represent this mathematically,

$$3x + 2y \leq 12$$

Also, the values for units of A can only be integers. So we have two more constraints, $x \geq 0$, $y \geq 0$. For the company to make maximum profit, the above inequalities have to be satisfied. This is called formulating a real-world problem into a mathematical model.

The instructor will plot the graph in class and get the corner points $(2,3), 4,0)$, $(0,5)$ and $(0,0)$. Testing each point on profit function

$$Z = 6x + 5y$$

At $(2,3), Z = 12 + 15 = 27$

At $(4,0)), Z = 24 + 0 = 24$

At $(0,\ 5), Z = 0 + 25 = 25$

At $(0,0), Z = 0 + 0 = 0$

By producing and selling 2 units of A and 3 units of B by using all available resource made a profit of Birr 27.

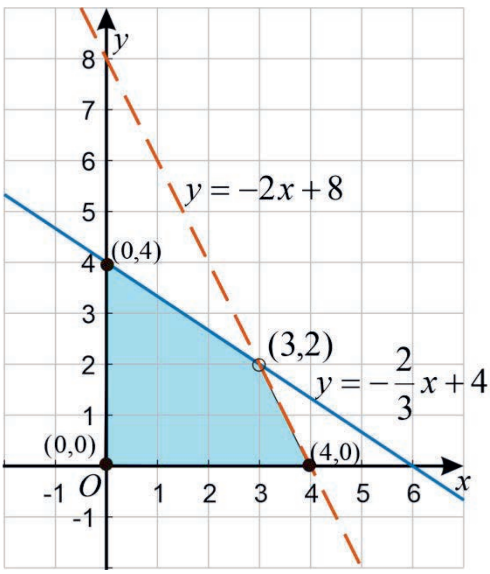
4.2.1 Steps to Solve Linear Programming Problem

Answer to Exercise 4.7

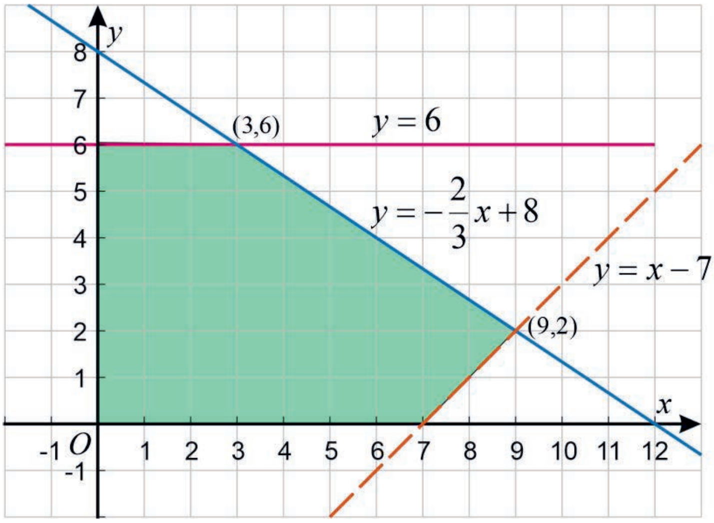
1. Solutions

   a. To maximize the objective function $Z = x + y$,

Subject to: $\begin{cases} 2x+3y \le 12 \\ 2x+y<8 \\ x \ge 0, y \ge 0 \end{cases}$



   b. To maximize the objective function $Z = x + 3y$, Subject to: $\begin{cases} 2x+3y \le 24 \\ x-y<7 \\ y \le 6 \\ x \ge 0, y \ge 0 \end{cases}$



c. Minimize: $Z = -40x + 20y$,

Subject to: $\begin{cases} 2x-y \ge -5 \\ 3x+y \ge 3 \\ 2x-3y \le 12 \\ x \ge 0, y \ge 0 \end{cases}$

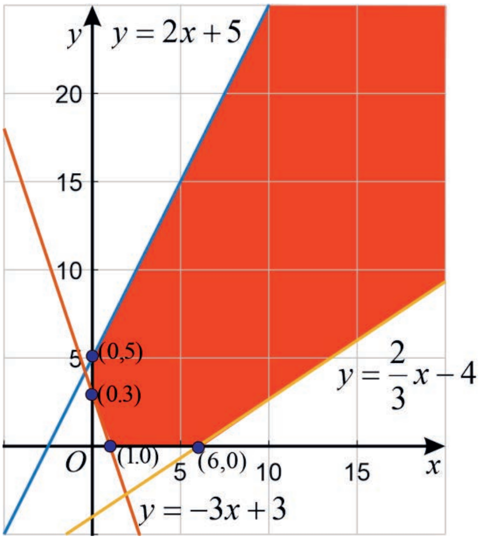
The problem is minimization and bounded below. The possible solutions are the corner points$(6,0), (1,0), (0,3), (0,5)$. Testing each point in the objective functions we obtain:

At $(6,0)$, $Z = -40(6) + 20(0) = -240$

At $(1,0)$, $Z = -40(1) + 20(0) = -40$

At $(0,3)$, $Z = -40(0) + 20(3) = 60$

At $(0,5)$. $Z = -40(0) + 20(5) = 100$. The minimum result is obtained at $(6,0)$ and its value is $Z = -240$.



Answer to Activity 4.5

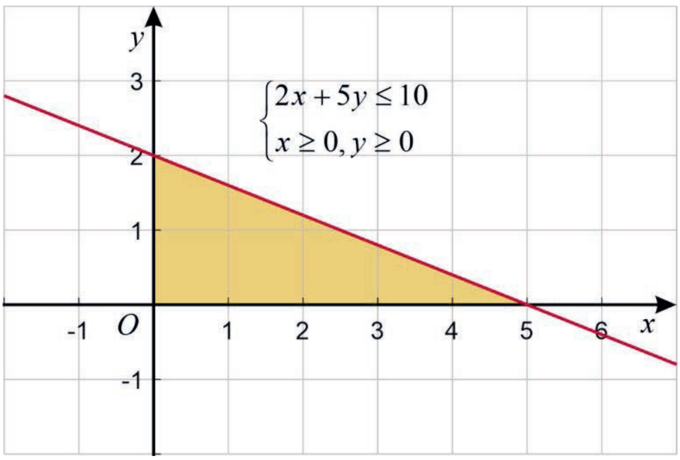
1. Explain for the student how to draw an inequality in xy-coordinate axes. On this activity you will show the closed region enclosed by the given inequality.

   a. $2x + 5y \leq 10, x \geq 0, y \geq 0$.

By rewriting the first inequality $2x + 5y \leq 10$ becomes

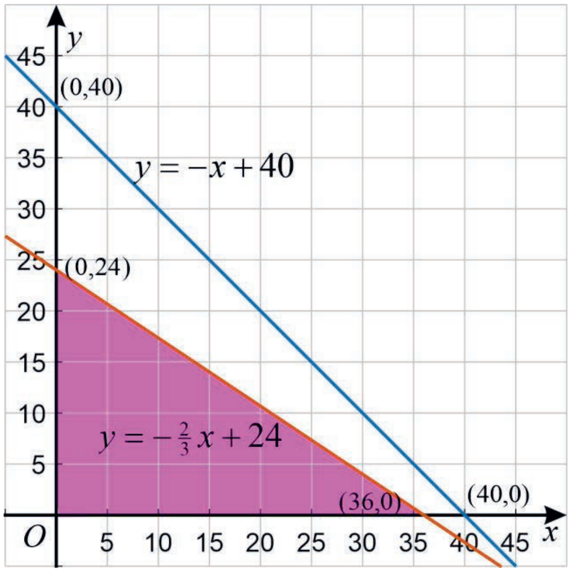
$$y \leq -\frac{2}{5}x + 2.$$

Drawing along $x \geq 0$ *and* $y \geq 0$, obtain the following figure.



b. $x + y \leq 40,\ 2x + 3y \leq 72, x \geq 0, y \geq 0$. Similar to a) by drawing the

inequality: $\begin{cases} y \leq -x + 40 \\ y \leq -\frac{2}{3}x + 24 \\ x \geq 0, y \geq 0 \end{cases}$



Answer to Exercise 4.8

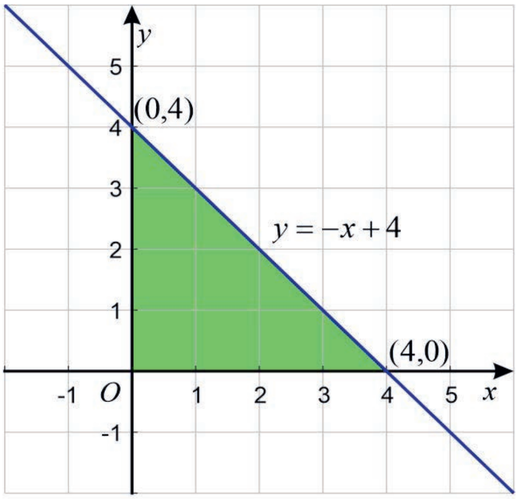
1. Solve the following Linear Programming Problems graphically:

a. Maximize $Z = 3x + 4y$

Subject to the constraints: $x + y \leq 4, x \geq 0, y \geq 0$.

Solution: by drawing the constraint inequality and testing each corner you can get the maximum value for the objective function.

Plot $\leq -x + 4$, $x \geq 0$, and $y \geq 0$ on the same coordinate as follow.



The corner points are $(0,0), (4,0), (0,4)$ testing each point on the objective function $Z = 3x + 4y$ you will get:

At $(0,0)$, $Z = 3(0) + 4(0) = 0$,

At $(4,0)$ $Z = 3(4) + 4(0) = 12$,

At $(0,4)$ $Z = 3(0) + 4(4) = 16$. From the result Z is maximum value at $(0,4)$.

b. Maximize or Minimize $Z = 5x + 6y$,

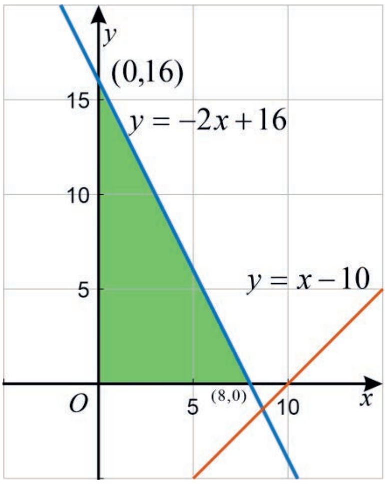
Subject to $2x + y \leq 16$

$x - y \leq 10$

$x \geq 0, y \geq 0$.

Solution: by drawing the constraint inequality and testing each corner you can get the maximum value and the minimum value of the objective function.

Plot $\leq -2x + 16$, $y \geq x - 10$, $x \geq 0$, and $y \geq 0$ on the same coordinate as follow.



The corner points of the solution region are $(0, 16), (8, 0), (0, 0)$ testing each point on the objective function $Z = 5x + 6y$ you will get:

At $(0, 0)$, $Z = 5(0) + 6(0) = 0$,

At $(8, 0)$ $Z = 5(8) + 6(0) = 40$,

At $(0, 16)$ $Z = 5(0) + 6(16) = 96$. From the result Z is maximum value at $(0, 16)$ and minimum at the origin.

c. Maximize or Minimize $Z = 2x + 3y$,

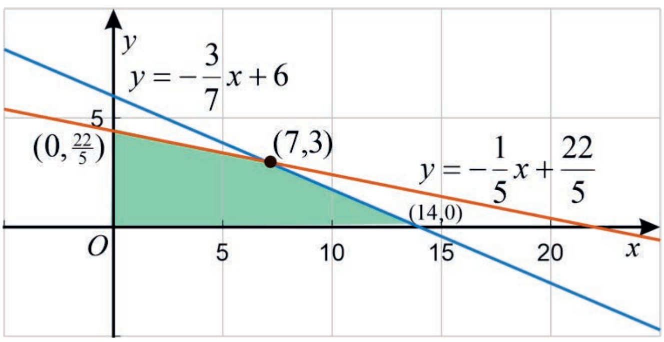
Subject to $3x + 7y \leq 42$

$x + 5y \leq 22$

$x \geq 0, y \geq 0$

by drawing the constraint inequality and testing each corner you can get the maximum value and the minimum value of the objective function.

Plot $\leq -\frac{3}{7}x + 6$, $y \leq -\frac{1}{5}x + \frac{22}{5}$, $x \geq 0$, and $y \geq 0$ on the same coordinate as follow.



The corner points of the solution region are $(0,0), \left(0, \frac{22}{5}\right), (7,3), (14,0)$ testing each point on the objective function $Z = 2x + 3y$ you will get:

At $(0,0)$, $Z = 2(0) + 3(0) = 0$,

At $\left(0, \frac{22}{5}\right)$ $Z = 2(0) + 3\left(\frac{22}{5}\right) = 13.2$,

At $(7,3)$ $Z = 2(7) + 3(3) = 23$,

At $(14,0)$ $Z = 2(14) + 3(0) = 28$. From the result Z is maximum value at $(14,0)$ and minimum at the origin.

4.3 Applications

Minimum Learning Competencies

Create inequalities from real life examples for linear programming and solve the problem

• Solve practical problems related to road traffic, customer production, tax, climate change

Start this section by explaining the application of mathematics in general and linear programming in particular. For example explain how household efficiently use monthly income for different expenditure. Take Mr. Alemu and his wife with their three children's live together as a family Mr. Alemu and his wife collectively got Birr 15000 monthly. Assume there is no other income for this family. How this family allocate the monthly income for food, house rent, utility bills, saving and so on. Discuss on it.

Answer to Exercise 4.9

1. **Solution** Let x represents the number of units of cereal that the person consumes a day and y represents the number of units of milk consumed.

	Content		Cost (Birr/unit)	Number of unit
	Iron (per unit)	Protein (per unit)		
Cereal	30	5	120	x
Milk	15	10	30	y
	≥ 60	≥ 70		

For the diet to meet the minimum requirements, we must have

Iron requirement: $30x + 15y \geq 60$,

Protein requirement: $5x + 10y \geq 70$, were $x \geq 0, y \geq 0$.

The cost of the diet is $120x + 30y$. Hence, the diet problem is

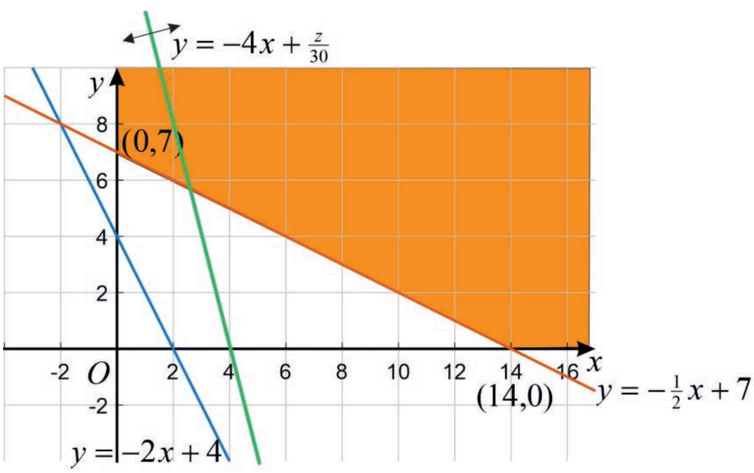
Minimize $z = 120x + 30y$

$$\text{Subject to } \begin{cases} 30x + 15y \geq 60 \\ 5x + 10y \geq 70 \\ x \geq 0, y \geq 0 \end{cases}$$

To solve the LPP graphically first draw the constraint inequality. Solving for y, you obtain:

$$\begin{cases} y \geq -2x + 4 \\ y \geq -\frac{1}{2}x + 14 \\ x \geq 0, y \geq 0 \end{cases}$$

By drawing the constraint inequality we get the feasible region as shown in the **Figure.**



Testing the vertices (0, 7) and (14, 0) of the feasible region we obtain the following result.

At $(0,7)$, $z=120(0)+30(7)=210$

At $(14,0)$, $z=120(14)+30(0)=1680$

To fulfill the minimum requirement 7 units of milk needed.

2. **Solution**

Let x be the number of packets of M food items developed, y be the number of N food item developed. After reading carefully we get the following constraint inequality

$15x+5y \geq 245$ Calcium requirement

$6x+22y \geq 460$ Iron requirement

$7x+4y \leq 300$ Cholesterol restriction

$x \geq 0, y \geq 0$ non negativity of the variables.

As you see the inequality why the 3rd constraint has maximum limitations and other two constraints have minimum requirement's discuss with your class (the reason is excess cholesterol may damage human health).

The objective of the dietician is to minimize the vitamin amount on food mix.

The LPP is

Minimize $Z = 9x+3y$

Subject to

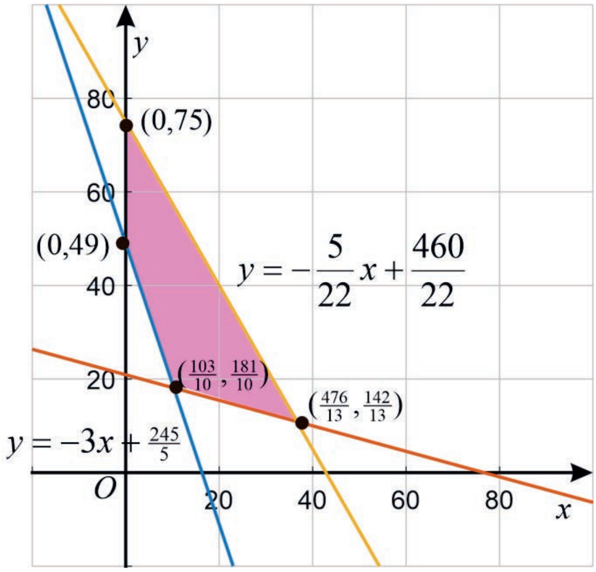
$$\begin{cases} 15x+5y \geq 245 \\ 6x+22y \geq 460 \\ 7x+4y \leq 300 \\ x \geq 0, y \geq 0 \end{cases}$$

To solve the LPP graphically first draw the constraint inequality. By drawing the constraint inequality we get the feasible region as shown in the **Figure.**

The coordinate of the corner points are $(0,49), (0,\ 75), \left(\frac{476}{13}, \frac{142}{13}\right), \left(\frac{103}{10}, \frac{181}{10}\right)$.

Solving for y

$$\begin{cases} y \geq -3x + 49 \\ y \geq -\dfrac{3}{11}x + \dfrac{230}{11} \\ y \leq -\dfrac{7}{4}x + 75 \\ x \geq 0, y \geq 0 \end{cases}$$



Testing each corner points you obtain the following result.

At$(0, 49)$, $Z = 9(0) + 3(49) = 147$

At $(0,\ 75)$, $Z = 9(0) + 3(75) = 225$

At $(\frac{476}{13}, \frac{142}{13})$, $Z = 9(\frac{476}{13}) + 3(\frac{142}{13}) = 362.31$

At $\left(\frac{103}{10}, \frac{181}{10}\right)$, $Z = 9\left(\frac{103}{10}\right) + 3\left(\frac{181}{10}\right) = 147.$

$Z = 147$, which are obtained at point $(0, 47)$ and $(\frac{103}{10}, \frac{181}{10})$. There are two points to make the value of Z minimum. This is because the objective function, $y = -3x + \frac{z}{3}$ and $y = -3x + 49$, which is the related equation of the 1st inequality has the same slope. This means that any points on the $y = -3x + 49$ can make the value of Z minimum.

Answer to Exercise 4.10

1. Solutions

Let x be the number of product P1 and y be the number of product 2 produced per day. There are three machines in the production line each have a limited hour per day. The number of available time for each machine is expressed in hour to model the problem we can convert hour to minute or vice versa.

Process time/ Product	M1 (min)	M2 (min)	M3 (min)	Selling price (birr)	Number
P1	9	6	10	12	x
P2	12	14	18	27	y
Capacity per day	15 hr = 900 min	14 hr = 840 min	12 h = 720 min		

The objective function can be written as follows: The Company want to maximize revenue or income from the sales of the two products.

Maximize $z = 12x + 27y$

Subject to the constrains

$9x + 12y \leq 900$ Available time in minute in M1

$6x + 14y \leq 840$ Available time in minute in M2

$10x + 18y \leq 720$ Available time in minute in M3

$x + y \geq 70$ Minimum sale

$x \geq 0, y \geq 0$ Non negativity

Simplifying the above constraints you get

$$\begin{cases} x+\frac{4}{3}y\leq 100 \\ x+\frac{7}{3}y\leq 140 \\ x+\frac{9}{5}y\leq 72 \\ x+y\geq 70 \\ x\geq 0, y\geq 0 \end{cases} \text{Solving for } y, \begin{cases} y\leq -\frac{3}{4}x+75 \\ y\leq -\frac{3}{7}x+60 \\ y\leq -\frac{5}{9}x+40 \\ y\geq -x+70 \\ x\geq 0, y\geq 0 \end{cases}$$

The figure shows the feasible region is the overlapping region in the graph with vertex A (70, 0), B (72, 0), C(67.5, 2.5)

Testing these vertices we get

At point A, $z = 120(70) + 27(0) = 840$,

At point B, $z = 12(72) + 27(0) = 864$

At point C, $z = 12(67.5) + 27(2.5) = 877.5$

Comparing the result the maximum possible daily revenue is Birr 877.50 can be made if 67.5 product P1 and 2.5 product P2 are produced and sold.

In practice, we can't produce materials like 67.5 units or 2.5 units. So, we have to find the figure in integer, which is in the region, such as $(67, 2), (68,2)$, but $(69, 2)$ is not. These two points meet all the inequalities, including $\frac{x}{72} + \frac{y}{40} \leq 1$.

When you calculate Z,

At $(67, 2)$, $z = 12(67) + 27(2) = 804 + 54 = 858$

At $(68, 2)$, $z = 12(68) + 27(2) = 816 + 54 = 870$

Therefore, the maximum possible daily revenue is Birr 870, that can be made if 69 pieces of P1 and 2 pieces of P2 are produced and sold.

2. **Solution**

a) In this situation there are two variables that you need to consider: let the number of Chopping board produced be x and let the number of Knife holder produced be, y. In each department the available finishing time already determined by manager. The total profit gain by firm is emanated from the two products and wants maximize it. The objective function is

Maximize $Z = 20x + 60y$

The firm has only one constrain in each department which is time

$1.4x + 0.8y \leq 56$	Cutting department
$5x + 13y \leq 650$	Gluing department
$12x + 3y \leq 360$	Finishing department
$x \geq 0, y \geq 0.$	

b) To determine the optimal quantity first we should draw graph of constrains and identify the feasible region. The inequalities are changed to:

$$\begin{cases} \dfrac{x}{40}+\dfrac{y}{70}\le 1 \\ \dfrac{x}{130}+\dfrac{y}{50}\le 1 \\ \dfrac{x}{30}+\dfrac{y}{120}\le 1 \\ x\ge 0, y\ge 0 \end{cases}$$

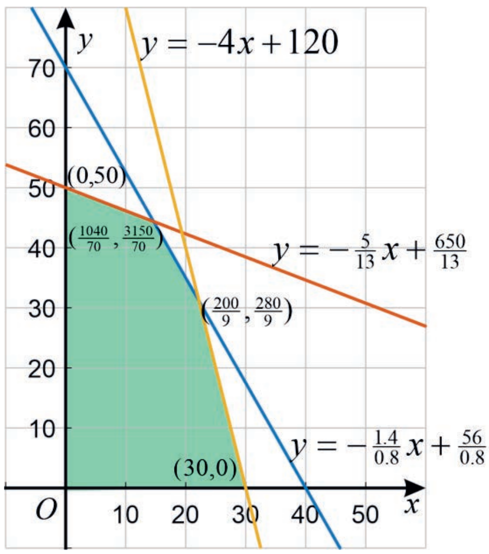
The corners of the visible region is $(0,0), (0,50), \left(\frac{1040}{70}, \frac{3150}{70}\right), \left(\frac{200}{9}, \frac{280}{9}\right)$ and $(30,0)$. Testing each point you get

At $(0, 0)$ $Z = 20(0) + 60(0) = 0$

At $(0, 50), Z = 20(0) + 60(50) = 3000$

At $\left(\frac{1040}{70}, \frac{3150}{70}\right)$, $Z = 20\left(\frac{1040}{70}\right) + 60\left(\frac{3150}{70}\right) = 2997.14$

At $\left(\frac{200}{9}, \frac{280}{9}\right)$, $Z = 20\left(\frac{200}{9}\right) + 60\left(\frac{280}{9}\right) = 2311.11$. Z is maximum at $(0, 50)$.



3. **Solution**

Let x be the number of product A and y be the number of product B manufactured per month. The agreement show that product B was supplied more than 200 units,

$$y \ge 200$$

Machine hour available for product A is 400 hours. One unit of product A require 1 machine hour.

$x \le 400$ Machine constraint for product A.

There are 500 labor hours are available for the two products

$x + y \le 500$ Labor hour's availability for the two products.

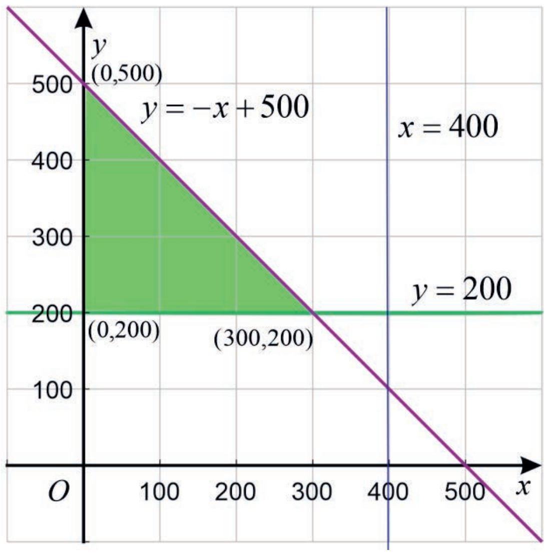
The objective is to minimize the cost of production,

Minimize $z = 60x + 80y$

The LPP is

Minimize $z = 60x + 80y$

Subject to $\begin{cases} y \ge 200 \\ x \le 400 \\ x + y \le 500 \\ x \ge 0, y \ge 0 \end{cases}$



The feasible set is the triangular region with vertex A (0, 200), B (300, 200) and C (0, 500). To solve graphically draw the constraint and the objective function for different value of z **as shown** below

When $z = 0, y = -\frac{3}{4}x$ this line passes through the origin. Drawing different lines parallel to this line (a slope of $-\frac{3}{4}$). The line intersect once the feasible set with a slope $-\frac{3}{4}$ is $3x + 4y = 1700$. multiplying by 20 both sides we obtain, $60x + 80y = 34000$

At vertex B, $z = 60(300) + 80(200) = 34000$.

At vertex A, $z = 80(200) = 16000$.

At vertex C, $z = 80(500) = 40000$.

The minimum possible monthly cost is Birr 16,000.00, at a production of 0 unit of product A and 200 unit of product B.

Answer to Exercise 4.11

1. Solution

Let x be the number of business class travelers and y be the number of economy class travelers.

From each business class ticket the airline get a profit of Birr 750.00 and from each economy class get Birr 450.00. The total profit $Profit = 750x + 450y$

So the airline wants to maximize profit function, the objective function become, maximize $z = 750x + 450y$

The airline reserved at least 20 seats for business class, i.e. $x \geq 20$.

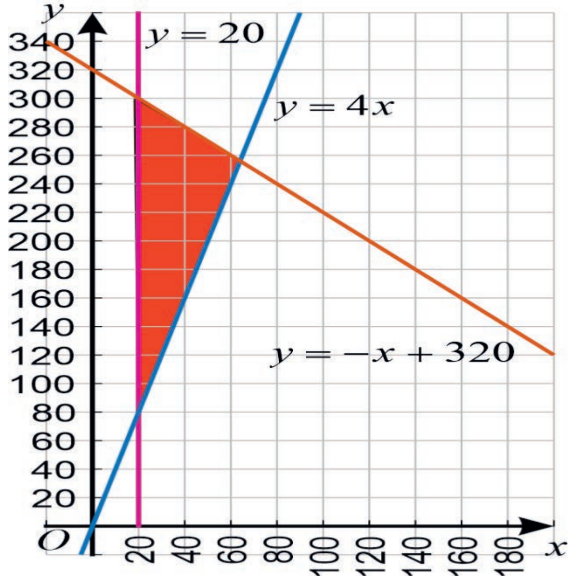
The economy class travelers prefer at least 4 times as many passengers than business class travelers, i.e. $y \geq 4x$, the total number of travelers is not more than 320, which is $x + y \leq 320$.

The LPP becomes,

Maximize $z = 750x + 450y$

Subject to $\begin{cases} x \geq 20 \\ y - 4x \geq 0 \\ x + y \leq 320 \\ x \geq 0, y \geq 0 \end{cases}$ Solving for y, you obtain $\begin{cases} x \geq 20 \\ y \geq 4x \\ y \leq -x + 320 \\ x \geq 0, y \geq 0 \end{cases}$

By plotting the constraint inequalities you obtain triangular feasible region with corner points (20, 80), (20, 300) and (64, 256).



Testing each vertices we have the following result

At $(20, 80), z = 750(20) + 450(80) = 15000 + 36000 = 51000$

At $(20, 300), z = 750(20) + 450(300) = 15000 + 135000 = 150000$

At$(64, 256), z = 750(64) + 450(256) = 48000 + 115200 = 163000$. The airline make a profit of Birr 163,000 when travelling 64 business class and 256 economy class travelers.

Answer to Exercise 4.12

1. Solution

Consider 10 liters of oil as 1 unit. Let x be the number of units transported from depot A.

y be the number of units transported from depot B. The transportation rate is 2 Birr per km for one unit of petrol.

The objective is to minimize the cost of distribution from the two depots to three stations.

Minimize $Z = 2x + 2y$

Subject to

$14x + 6y \geq 900$ Station D requirement

$12x + 8y \geq 600$ Station E requirement

$6x + 4y \geq 700$ Station F requirement

$x \leq 700, y \leq 400$ Available petrol in each depot

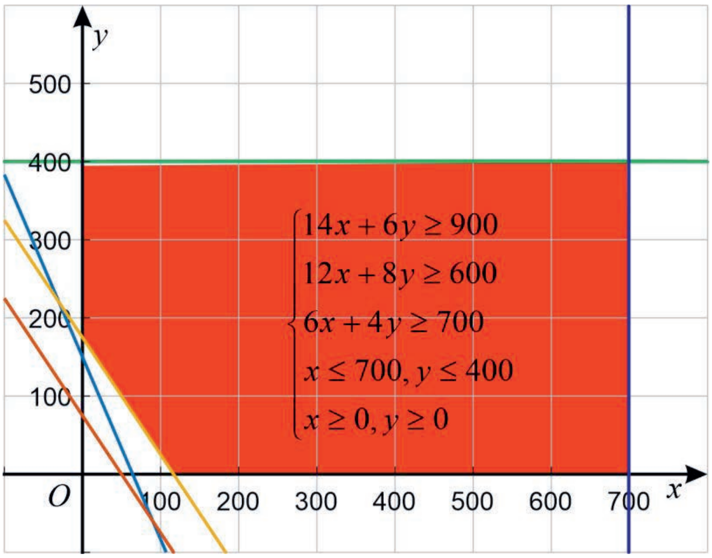
$x \geq 0, y \geq 0.$ Non negativity constraints

To find the optimal solution we **plot** the graph as follow

The vertices of the feasible region are as follows:

Point V: $(\frac{350}{3}\ 0)$, Point W: (700, 0), Point X:(700, 400), Point Y:(0, 400)

Point Z: (0, 175).



The cost at each vertex is as follows:

Point V: $Z = 2(116.67) + 2(0) = 233.33$

Point W: $Z = 2(700) + 2(0) = 1400$

Point X: $Z = 2(700) + 2(400) = 2200$

Point Y: $Z = 2(0) + 2(400) = 800$

Point Z: $Z = 2(0) + 2(175) = 350$

By transporting 117 unit from depot A the transportation cost will be minimized.

4.3.2 Solving Linear Programing Problems Using Spreadsheet

Before starting this lesson ask students' knowledge about application software's in particular Microsoft Excel. Arrange computer laboratory and then start the lesson

Answer to Activity 4.6

Main feature of spreadsheet or excel is listed below

Home

- Comprises options like font size, font styles, font colour, background colour, alignment, formatting options and styles, insertion and deletion of cells and editing options

Insert

- Comprises options like table format and style, inserting images and figures, adding graphs, charts and sparklines, header and footer option, equation and symbols

Page Layout

- Themes, orientation and page setup options are available under the page layout option

Formulas

- Since tables with a large amount of data can be created in MS excel, under this feature, you can add formulas to your table and get quicker solutions

Data

- Adding external data (from the web), filtering options and data tools are available under this category

Review

- Proofreading can be done for an excel sheet (like spell check) in the review category and a reader can add comments in this part

View

- Different views in which we want the spreadsheet to be displayed can be edited here. Options to zoom in and out and pane arrangement are available under this category

Answer to Exercise 4.13

Solve the Linear Programing Problem using spreadsheet

a. Maximize $z = 4x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 10$

$x_1 \leq 4$

$x_1,\ x_2 \geq 0$

Solution

This solution was made on Microsoft Excel 2013 version

Start by opening a new worksheet, then these are the first three steps

D6 | =B4*B6+C6*C4

	A	B	C	D	E
1					
2					
3					
4	Variables Value to manipulate	0	0		
5	Name of dicision variables	x1	x2	Total	RHS
6	Objective to maximize	4	5	0	
7	Constraint 1	2	1	0	0
8	Constraint 2	1	0	0	4

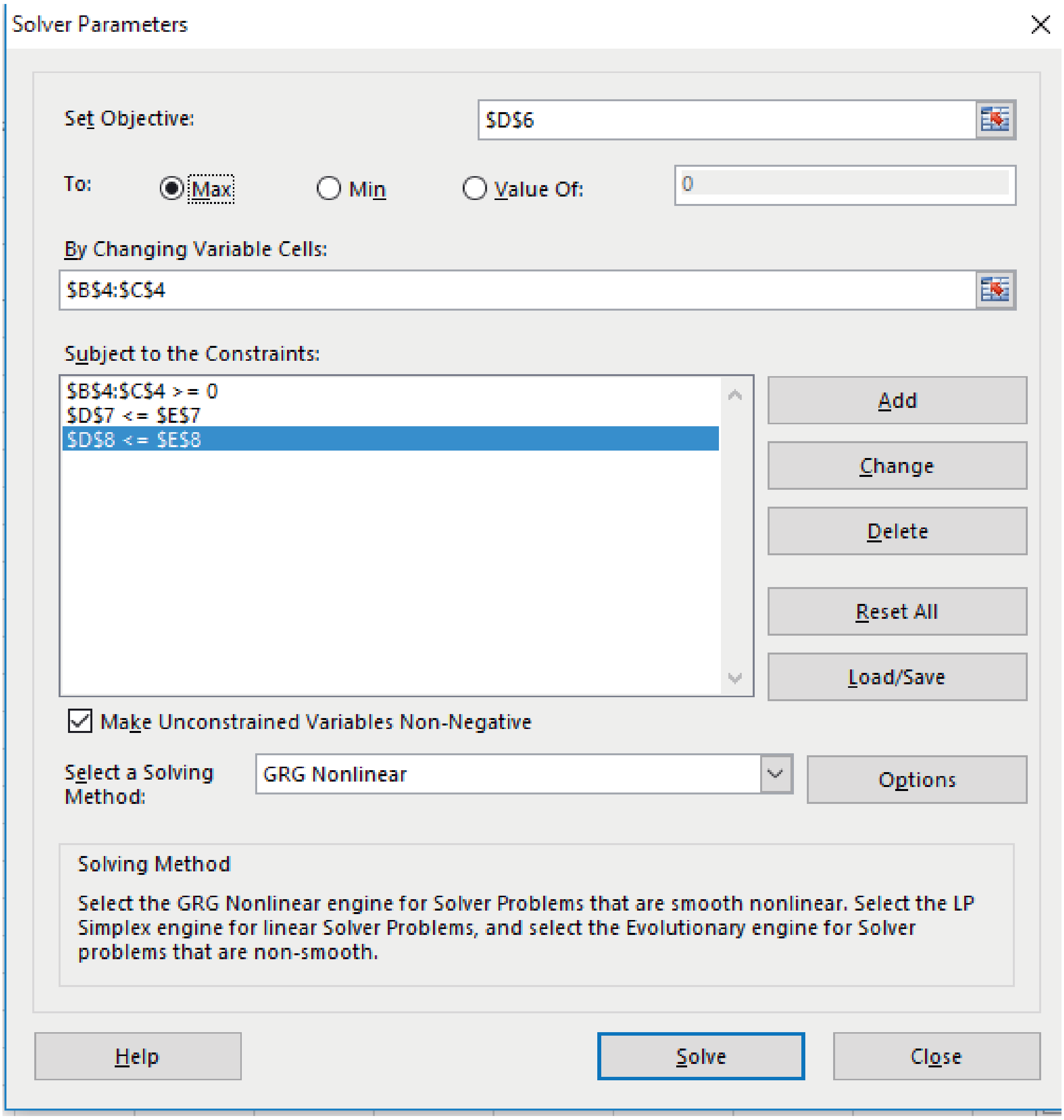
Step 4 Activate solver

The following steps are slightly vary with excel versions

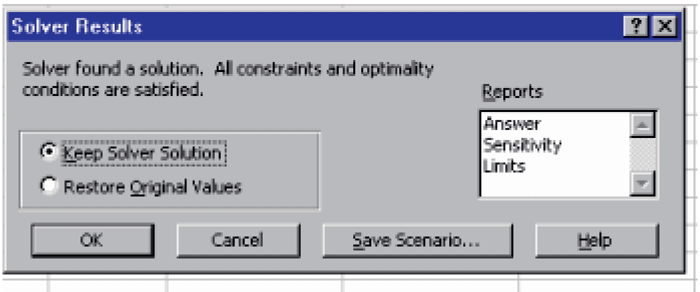
- **Excel 2002/03**: click Tools at the top, then Solver.
- **Excel 2007/10/13/16**: click the Data tab at the top, then click Solver in the Analysis section toward the top right (called "Analyze" in Excel 2016).
- Click in the "Set Target Cell" box for Excel 2002/03/07, or the "Set Objective" box for Excel 2010/13/16, then select the one cell containing the

objective function formula. Also click the Max or Min button as applicable (linear programming does not use the "Value of" option).

The solution was developed on Excel 2013



Then click solve, When Excel finds an optimal solution, the following appears.



Highlight both answer and sensitivity

Click on Keep Solver Solution and OK then the Reports will be generated. In this step Excel will add two new sheets, Answer report and sensitivity report.

Worksheet: [Excel 1, Ex442.xlsx]Sheet1

Report Created: 9/29/2021 4:19:04 AM

Result: Solver found a solution. All Constraints and optimality conditions are satisfied.

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.094 Seconds.

Iterations: 3 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
\$D\$6	Objective to maximize Total	0	50

Variable Cells

Cell	Name	Original Value	Final Value	Integer
\$B\$4	Variables Value to manipulate	0	0	Contin
\$C\$4	Variables Value to manipulate	0	10	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$D\$7	Constraint 1 Total	10	\$D\$7<=\$E\$7	Binding	0
\$D\$8	Constraint 2 Total	0	\$D\$8<=\$E\$8	Not Binding	4
\$B\$4	Variables Value to manipulate	0	\$B\$4>=0	Binding	0
\$C\$4	Variables Value to manipulate	10	\$C\$4>=0	Not Binding	10

b. Maximize $z = 5x_1 + 6x_2$

Subject to

$3x_1 + 4x_2 \leq 18$

$2x_1 + x_2 \leq 7$

$x_1,\ x_2 \geq 0$

Solution

Similar to question a. we solve this LPP by using Microsoft excel 2013 and obtain the following table on excel.

D6 =B4*B6+C6*C4

	A	B	C	D	E
1					
2					
3					
4	Variables Value to manipulate	2	3		
5	Name of dicision variables	x1	x2	Total	RHS
6	Objective to maximize	5	6	28	
7	Constraint 1	3	4	18	18
8	Constraint 2	2	1	7	7

Which means when $x_1 = 2$ *and* $x_2 = 3$, the objective function attain a maximum of $z = 28$. The answer report appear as follow

Microsoft Excel 15.0 Answer Report
Worksheet: [Excel 2, Ex442.xlsx]Sheet1
Report Created: 9/29/2021 4:37:24 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.078 Seconds.
Iterations: 3 Sub-problems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Sub problems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume Non Negative
Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	Objective to maximize Total	0	28

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$4	Variables Value to manipulate	0	2	Contin
$C$4	Variables Value to manipulate	0	3	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$7	Constraint 1 Total	18	$D$7<=$E$7	Binding	0
$D$8	Constraint 2 Total	7	$D$8<=$E$8	Binding	0
$B$4	Variables Value to	2	$B$4>=0	Not	2

	manipulate			Binding	
$C$4	Variables Value to manipulate	3	$C$4>=0	Not Binding	3

As you see on the constraint table all resources were used with zero slack.

c. Minimize $z = 2x_1 + 3x_2$

Subject to

$4x_1 + 2x_2 \geq 12$

$x_1 + 4x_2 \geq 6$

$x_1,\ x_2 \geq 0$

Solution

Similar to the above two exercise we solve by excel this minimization problem. Applying all the steps we get

	A	B	C	D	E
1					
2					
3					
4	Variables Value to manipulate	2.724745	0.55051		
5	Name of dicision variables	x1	x2	Total	RHS
6	Objective to minimize	2	3	7.10102	
7	Constraint 1	4	2	12	12
8	Constraint 2	1	4	5.999999	6

The answer report is shown as follow

Worksheet: [Excel 3, Ex442.xlsx]Sheet1

Report Created: 9/29/2021 4:55:51 AM

Result: Solver found a solution. All Constraints and optimality conditions are satisfied.

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.125 Seconds.

Iterations: 4 Sub problems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100,

Random Seed 0, Derivatives Forward, Require Bounds

Microsoft Excel 15.0 Answer Report

Max Sub problems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume Nonnegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$D$6	Objective to maximize Total	0	7.101020333

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$4	Variables Value to manipulate	0	2.724744917	Contin
$C$4	Variables Value to manipulate	0	0.550510166	Contin

Constraints

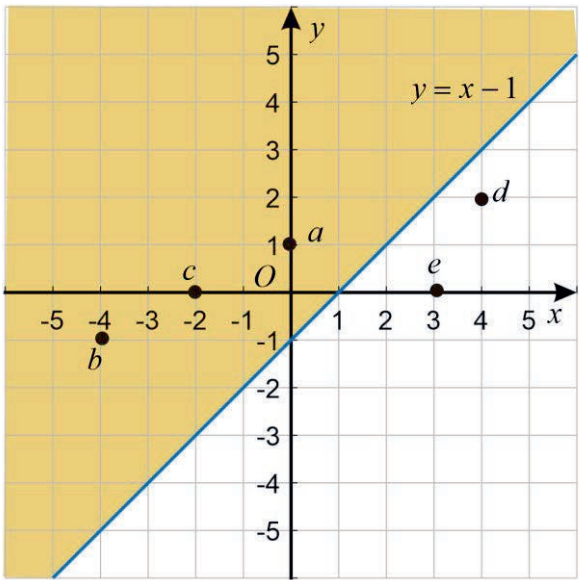
Cell	Name	Cell Value	Formula
$D$7	Constraint 1 Total	12	$D$7>=$E$7
$D$8	Constraint 2 Total	5.99999911	$D$8>=$E$8
$B$4	Variables Value to manipulate	2.724744917	$B$4>=0
$C$4	Variables Value to manipulate	0.550510166	$C$4>=0

Before closing this unit you can revise the main point of the unit. The main points are listed on the summery of the textbook. Motivate students to know those points listed on the summery.

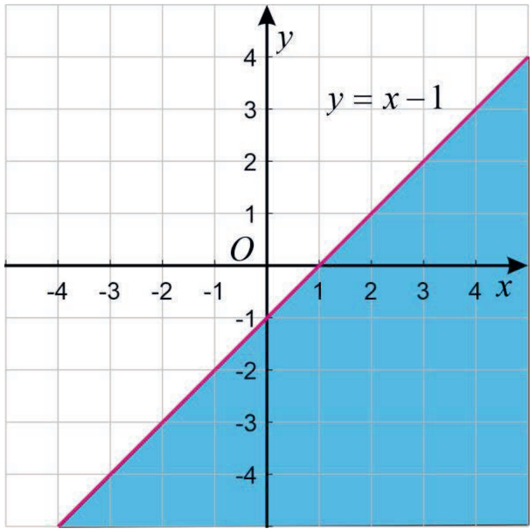
Answer to Review Exercise

1. **Solution**: Test each order pairs whether it satisfies the inequality $y > x-1$ or not.
   a. $(LHS) = 1, (RHS) = 0 - 1 = -1$, thus $(LHS) > (RHS)$. So the order pair is the solution of $y > x - 1$.
   b. $-1 > -4 - 1 = -5$, true,
   c. $0 > -2 - 1 = -3$, true.
   d. $2 > 4 - 1 = 3$, false,
   e. $0 > 3 - 1 = 2$, false

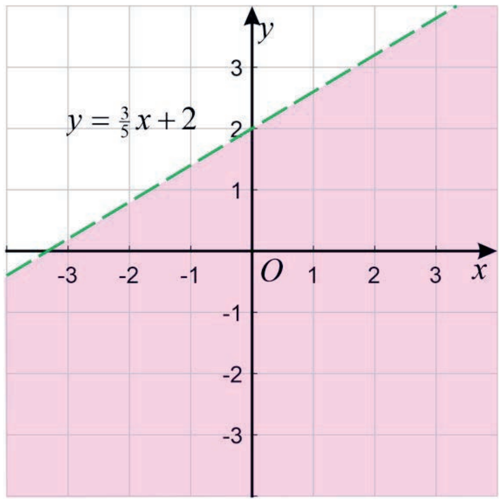
   The order pairs (4, 2) and (3. 0) don't satisfy the inequality. Alternatively you can answer by drawing the graph of the inequality $y > x - 1$ and pin the points in the xy- plane.



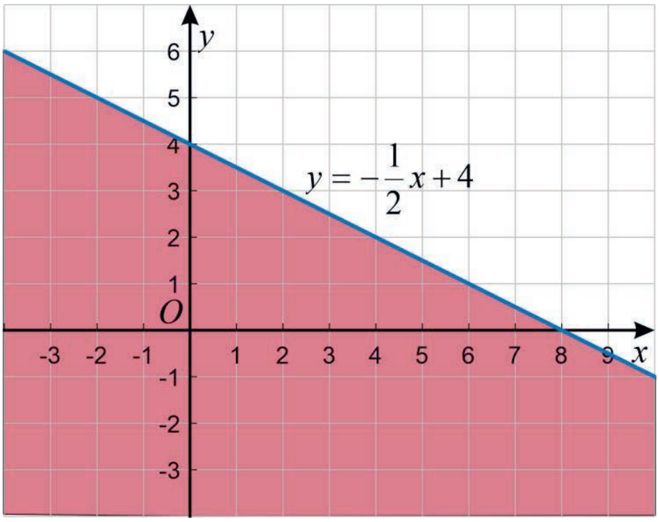
2. Graph each of the following linear inequalities
   a. The graph of $y \leq x-1$ is shown as follow



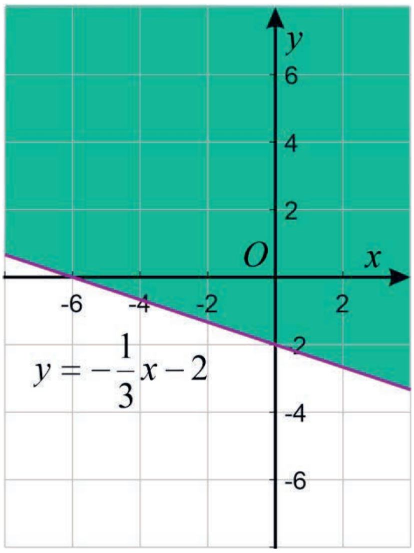
b. The graph of the inequality $y < \frac{3}{5}x + 2$ given as follow



c. The graph of $y \leq -\frac{1}{2}x + 4$ is:

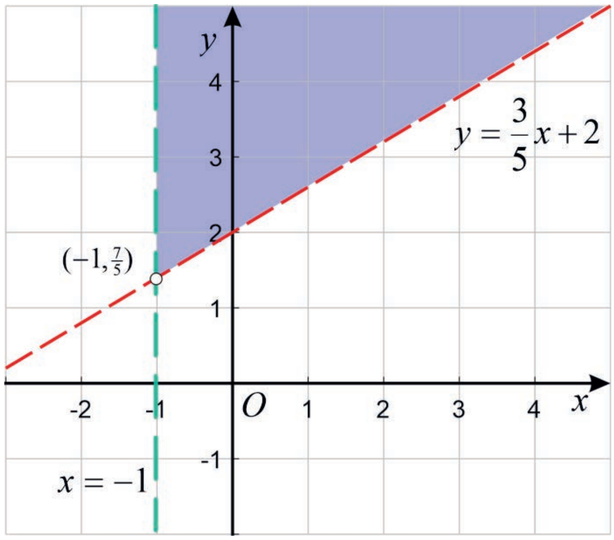


d. The graph of $y \geq -\frac{1}{3}x - 2$ is:

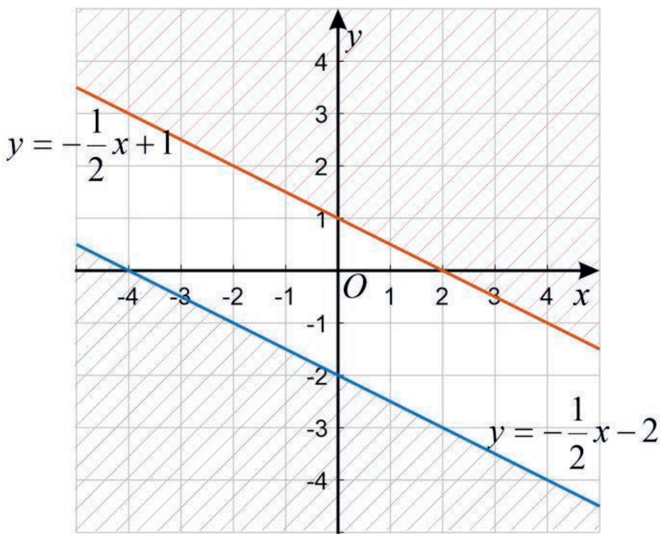


3. Solve each system by graphing

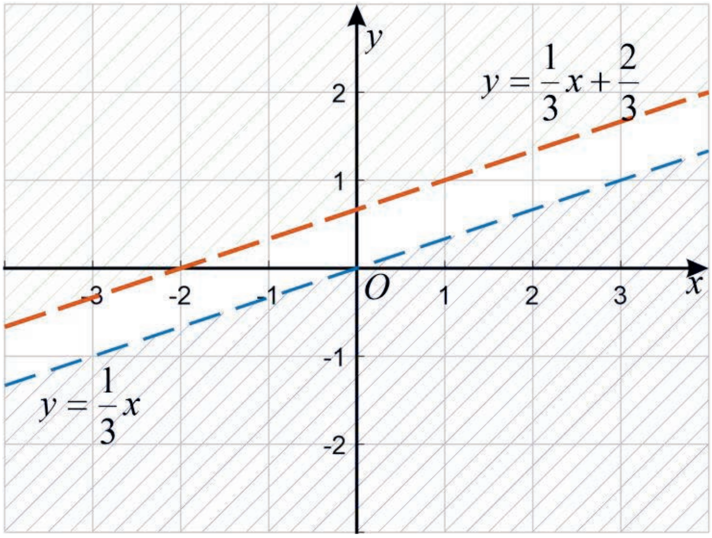
i. $\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$ it is equivalent to $\begin{cases} y > \frac{3}{5}x + 2 \\ x > -1 \end{cases}$



ii. $\begin{cases} 2x + 4y > 4 \\ y \le -\frac{1}{2}x - 2 \end{cases}$ equivalently, $\begin{cases} y > -\frac{1}{2}x + 1 \\ y \le -\frac{1}{2}x - 2 \end{cases}$ As shown in the following Figure the system has no solution.

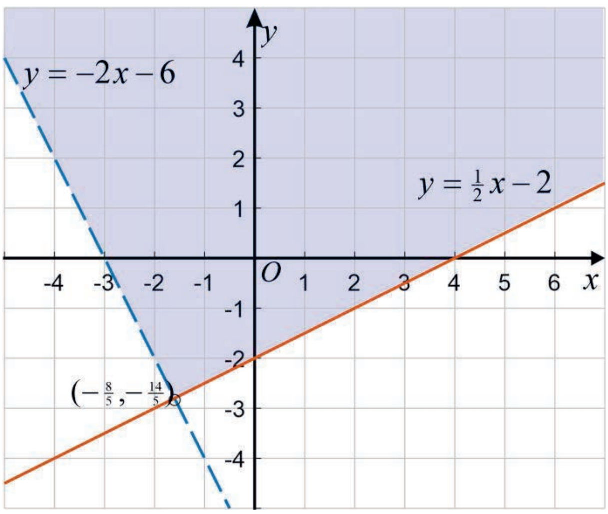


iii. $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$, it is equivalent to $\begin{cases} y < \frac{1}{3}x \\ x > \frac{1}{3}x + \frac{2}{3} \end{cases}$

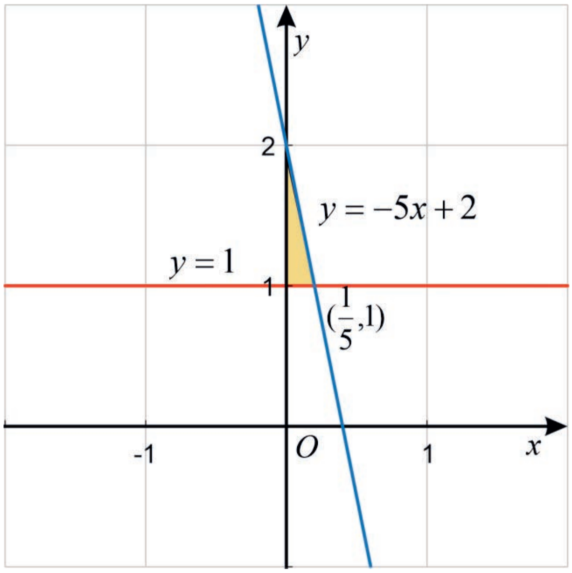


iv. $\begin{cases} 2x+y>-6 \\ -x+2y\geq -4 \end{cases}$ is equivalent to $\begin{cases} y<-2x-6 \\ y\geq \frac{1}{2}x-2 \end{cases}$ As shown in the figure the line $y=-2x-6$ and $y=\frac{1}{2}x-2$ intersect at $(-\frac{8}{5},-\frac{14}{5})$.

The solution is the shaded part in the Figure.



4. $\begin{cases} y\leq -5x+2, \\ x\geq 0, \\ y\geq 1. \end{cases}$ To solve the system graphically you can plot the graph of the inequalities on the same coordinate plane.



As shown in the figure, the shaded region is the solution of the system.

The region R is the collection of all points in $xy-$ plane which is equivalent to

$R = \{(x, y): 0 \leq x \leq \frac{1}{5}, 1 \leq y \leq 2\}$.

5. The steps are found on textbook. The instructor will give feedback.
6. **Solution:** Let x be the number of hours Sultan engaged in a gas station job and y be the number of hours Sultan works as a computer technician in four weeks. Sultan wants to earn Birr 3300. He get Birr $11x$ from gas station job and Birr $16.5y$ from computer job respectively.

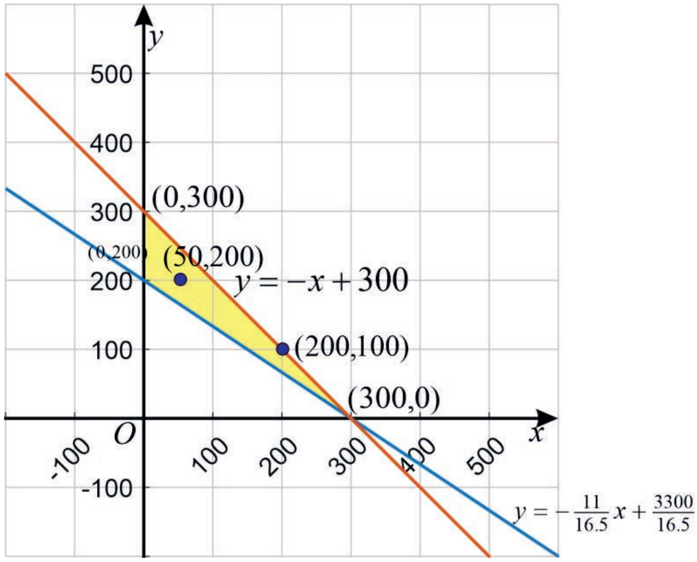
   $11x + 16.5y \geq 3300$ Personal goal,

Assume Sultan have 15 working hour per day and works 5 days per week. Sultan take one day per week for rest. Sultan have $15 \times 5 \times 4 = 300$ working hour per four week.

$x + y \leq 300$ Available time per four week.

Solving the system gives a required answer.

$$\begin{cases} 11x + 16.5y \geq 3300 \\ x + y \leq 300 \\ x \geq 0, y \geq 0 \end{cases}$$



Solve the system of equations $\begin{cases} 11x + 16.5y = 3300 \\ x + y = 300 \end{cases}$.

By using the substitution method you obtain $(x, y) = (300, 0)$ the point of intersection. The corner points are, $(300,0),(0,200),(0,300)$. Any points in the region meet the requirements and Sultan can achieve his goal

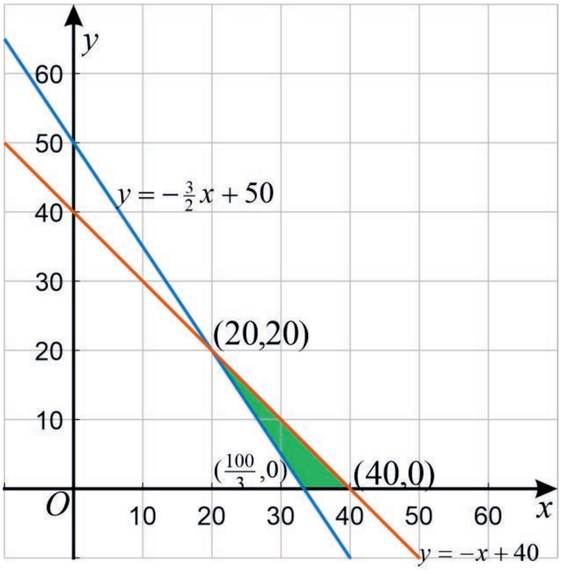
7. Solution

Let x be the noumber of minute Fikadu run and y be the number of minute Fikadu ride a bike.

The total calorie burn is $15x + 10y$ daily. To fulfil the order of the doctor Fikadu must exercise to burn at least 500 calorie. And the available time for biking and running is less than 40 minutes. So we get the following inequality

$$\begin{cases} 15x + 10y \geq 500 \\ x + y \leq 40 \\ x \geq 0, \quad y \geq 0 \end{cases}$$

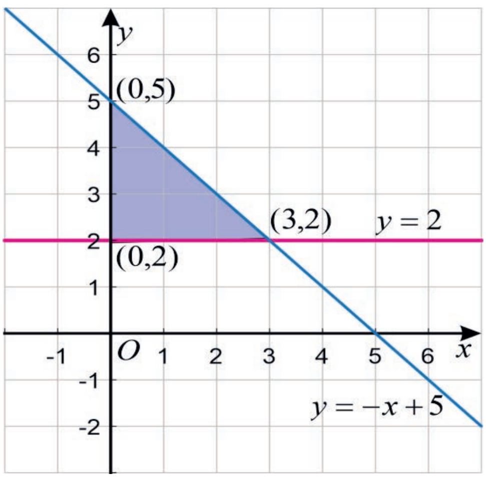
Solving for y, $y \geq -\frac{3}{2}x + 50$ and $y \leq -x + 40$. Drawing the graph, the coordinate of the corner points are $(20, 20)$, $(\frac{100}{3}, 0)$ and $(40, 0)$.



8. Find the maximum and minimum values of the objective functions subject to the given constraints

   i) Objective functions $z = 6x + 4y$

$$\text{Subject to} \begin{cases} x + y \leq 5 \\ y \geq 2 \\ x \geq 0, y \geq 0 \end{cases}$$



The feasible region is shaded in the above graph, the corner points are: $(0, 2), (3, 2), (0, 5)$. Testing each points we obtain

At $(0, 2)$, $z = 6(0) + 4(2) = 8$,

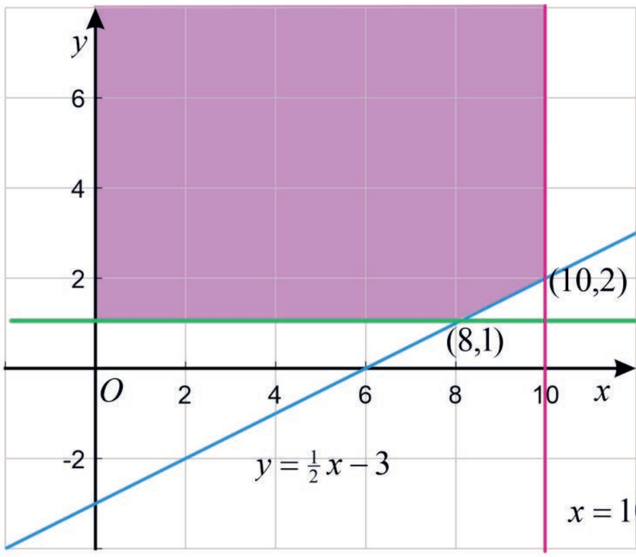
At $(3, 2)$, $z = 6(3) + 4(2) = 26$,

At $(0,5)$, $z = 6(0) + 4(5) = 20$. From the result $z = 26$ and $z = 8$ are the maximum and minimum value respectively.

ii) Objective functions $z = 3x + 5y$

Subject to $\begin{cases} x - 2y \le 6 \\ x \le 10 \\ y \ge 1 \\ x \ge 0, y \ge 0 \end{cases}$ equivalently, $\begin{cases} y > \frac{1}{2}x - 3 \\ x \le 10 \\ y \ge 1 \\ x \ge 0, y \ge 0 \end{cases}$

The feasible region is unbounded so they have only minimum value. Testing the corner points $(0,1)$, $(8,1)$ and $(10,2)$ we get the following result.



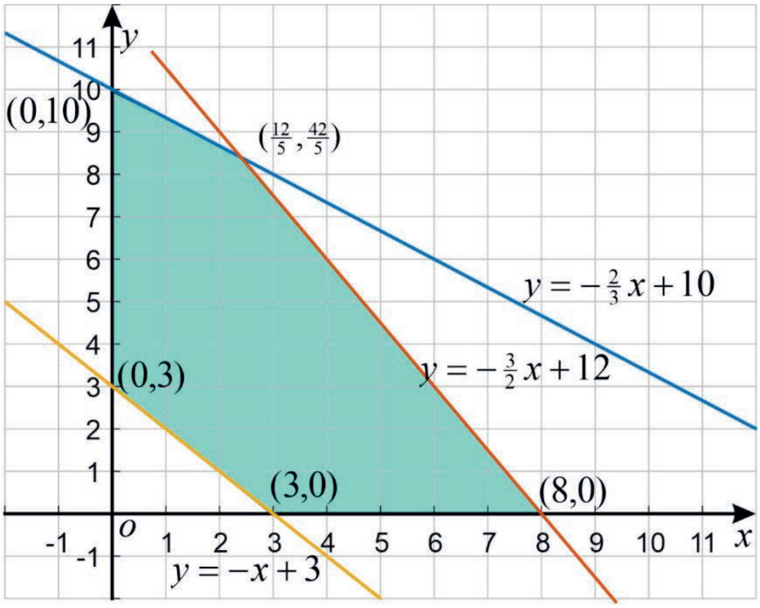
At $(0,1)$, $z = 3(0) + 5(1) = 5$,

At $(8,1)$, $z = 3(8) + 5(1) = 29$,

At $(10,2)$, $z = 3(10) + 5(2) = 40$. From the result the value z is minimum of 5 at $(0,1)$.

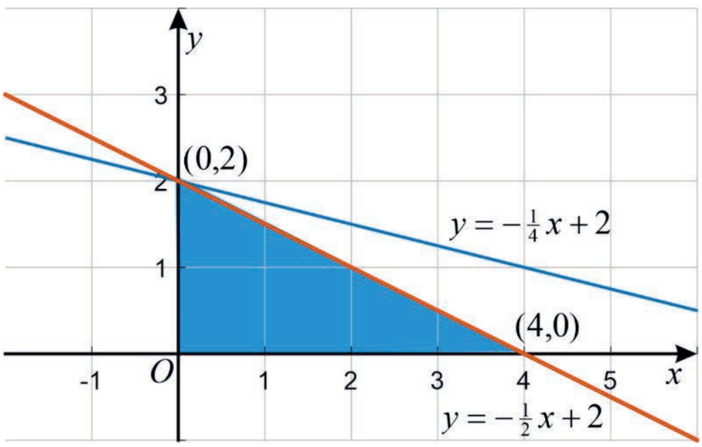
iii) Objective function $z = 6x + 4y$

Subject to: $\begin{cases} 2x + 3y \le 30 \\ 3x + 2y \le 24 \\ x + y \ge 3 \\ x \ge 0, y \ge 0 \end{cases}$ it is equivalent to $\begin{cases} y \le -\frac{2}{3}x + 10 \\ y \le -\frac{3}{2}x + 12 \\ y \ge -x + 3 \\ x \ge 0, y \ge 0 \end{cases}$



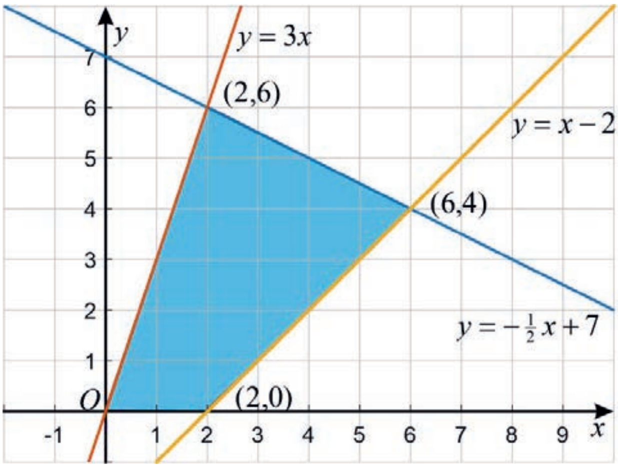
iv) Objective function: $z = 3x + 9y$

Subject to: $\begin{cases} x + 4y \le 8 \\ x + 2y \le 4 \\ x \ge 0, y \ge 0 \end{cases}$ solving for y, $\begin{cases} y \le -\frac{1}{4}x + 2 \\ y \le -\frac{1}{2}x + 2 \\ x \ge 0, y \ge 0 \end{cases}$



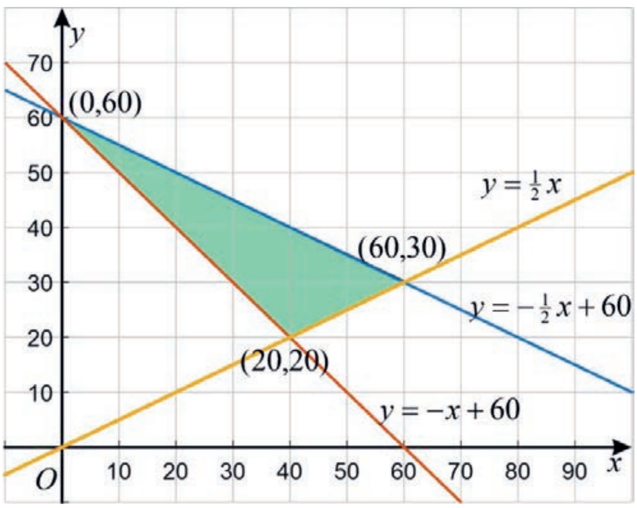
9. Maximize or Minimize, $Z = 3x + 4y$,

Subject to: $\begin{cases} x+2y \le 14 \\ 3x-y \ge 0 \\ x-y \le 2 \\ x \ge 0, y \ge 0 \end{cases}$ is equivalent to $\begin{cases} y \le -\frac{1}{2}x+7 \\ y \le 3x \\ y \ge x-2 \\ x \ge 0, y \ge 0 \end{cases}$



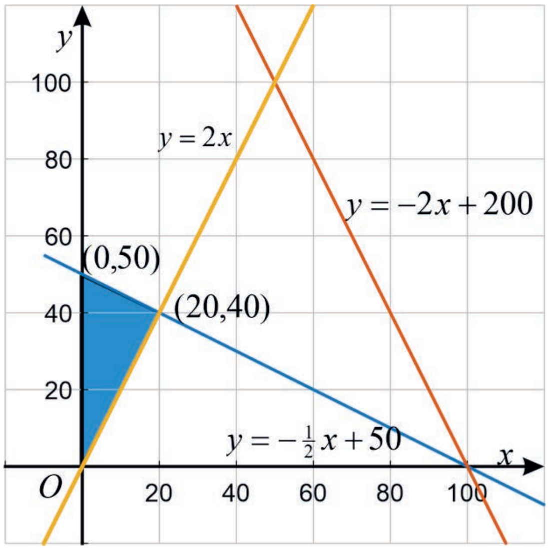
10. Minimize and Maximize $Z = 5x + 10y$,

Subject to: : $\begin{cases} x+2y \le 120 \\ x+y \ge 60 \\ x-2y \le 0 \\ x \ge 0, y \ge 0 \end{cases}$ This was changed to $\begin{cases} y \le -\frac{1}{2}x+60 \\ y \ge -x+60 \\ y \ge \frac{1}{2}x \\ x \ge 0, y \ge 0 \end{cases}$



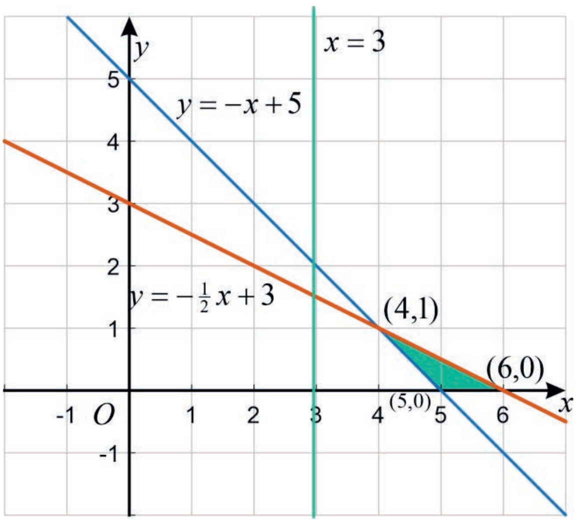
11. Minimize and Maximize $Z = x + 2y$,

Subject to: $\begin{cases} x+2y \le 100 \\ 2x-y \le 0 \\ 2x+y \le 200 \\ x \ge 0, y \ge 0 \end{cases}$ equivalently, $\begin{cases} y \le -\frac{1}{2}x+50 \\ y \ge 2x \\ y \le -2x+200 \\ x \ge 0, y \ge 0 \end{cases}$



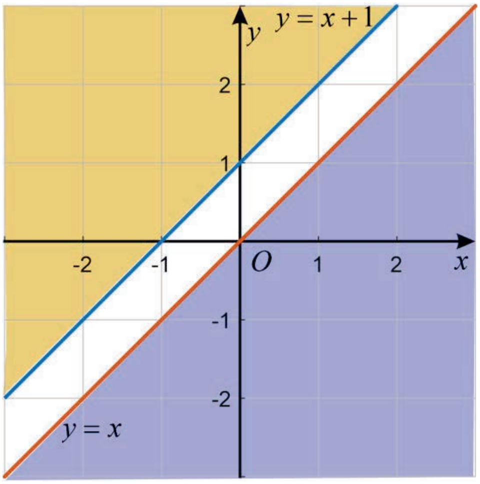
12. Maximize, $Z = -x + 2y$,

Subject to: $\begin{cases} x \ge 3 \\ x+y \ge 5 \\ x+2y \le 6 \\ y \ge 0 \end{cases}$ is equivalent to $\begin{cases} x \ge 3 \\ y \ge -x+5 \\ y \le -\frac{1}{2}x+3 \\ y \ge 0 \end{cases}$



13. Maximize $Z = x + y$,

Subject to $\begin{cases} x-y\le -1 \\ -x+y\le 0 \\ x\ge 0, y\ge 0 \end{cases}$ is changed to $\begin{cases} y\ge x+1 \\ y\le x \\ x\ge 0, y\ge 0 \end{cases}$

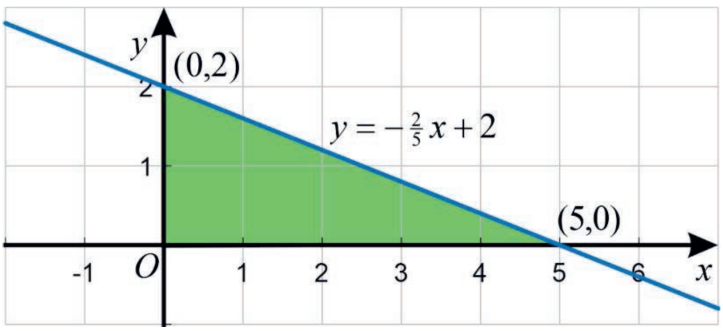


14. Find the maximum and minimum values of

a) Objective function $Z = 6x + 10y$

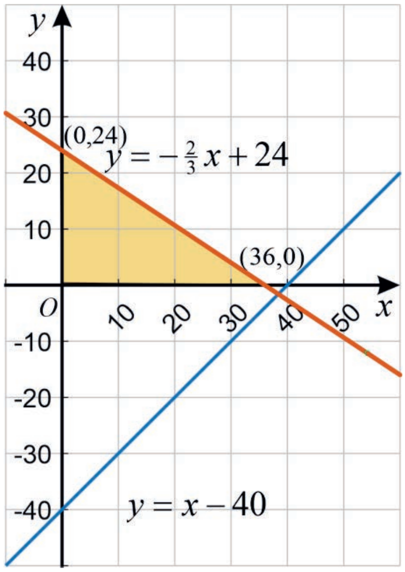
Subject to $\begin{cases} 2x+5y\le 10 \\ x\ge 0,\ y\ge 0 \end{cases}$ is written as $\begin{cases} y\le -\frac{2}{5}x+2 \\ x\ge 0, y\ge 0 \end{cases}$

Testing the corner points $(0,0), (0,2)$ and $(5,0)$ on the objective function you obtain $Z = 30$ at the point $(5,0)$ maximum value and $Z = 0$ minimum at the origin.



b) Objective function $Z = 4x + y$

Subject to: $\begin{cases} x - y \le 40 \\ 2x + 3y \le 72 \\ x \ge 0,\ y \ge 0 \end{cases}$ equivalently $\begin{cases} y \ge x - 40 \\ y \le -\frac{2}{3}x + 24 \\ x \ge 0, y \ge 0 \end{cases}$



15. Solution

a) Estimates show that no more than 10 000 copies of both items together will be sold:

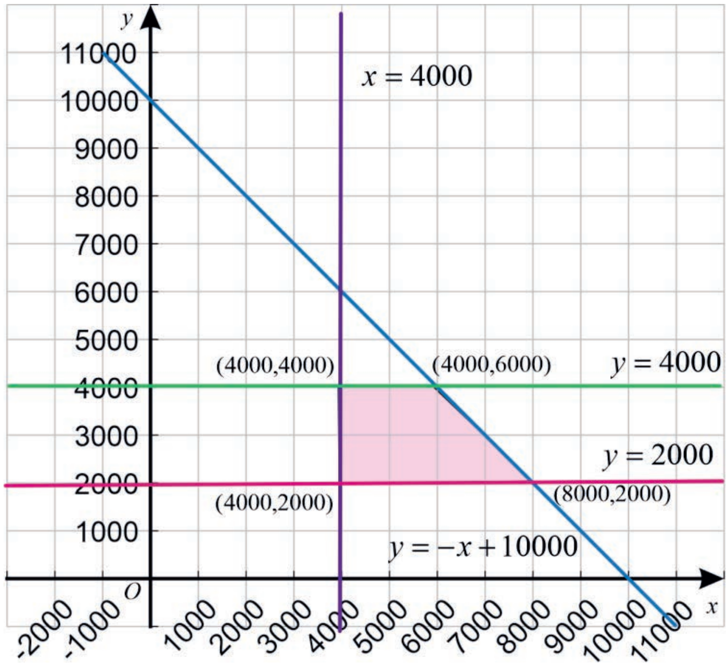
$$x + y \le 10\,000$$

At least 4000 copies of the video could be sold: $x \ge 4000$

At least 2000 copies of the book could be sold: $y \ge 2000$

Sales of the book are not expected to exceed 4000 copies: $y \le 4000$

b) We indicate the feasible regions by plotting the inequality listed above:



c) The objective function is

$$\text{Maximize } I = 50x + 30y$$

d) The vertices of the feasible region and the cost at each vertex is summarized in the table as

Vertices	$I = 50x + 30y$
(4000; 2000)	260,000
(4000, 4000)	320,000
(6000, 4000)	420,000
(8000, 2000)	460,000

The maximum profit of Birr 460,000 can be made if 8000 videos and 2000 books were sold.

e) We will solve the same problem by using Microsoft excel 2013 as follow.

16. Solution

Once you have the idea of using Microsoft excel to solve LPP the number of variables is not material. We proceed by opening a new Microsoft excel sheet Let x_1 is the number of convenience house constructed, x be the number of standard house constructed, and x_3 be the number of luxury houses constructed.

The total amount of money needed for constructing three types of houses are

$$4.125x_1 + 8.25x_2 + 12.375x_3 \leq 82.5 \text{ (Financial constraint)}$$

$$30x_1 + 15x_2 + 45\,x_3 \leq 300 \text{ (Labor limitation)}$$

$$x_1 + x_2 + x_3 \leq 11 \text{ (Construction plan of the year)}$$

The average revenue earned annually from each type of house was given.

Total Revenue, $R = 1.2x_1 + 2x_2 + 2.6x_3$

The LPP become

$$maximize\ , R = 1.2x_1 + 2x_2 + 2.6x_3$$

Subject to

$$4.125x_1 + 8.25x_2 + 12.375x_3 \leq 82.5$$

$$30x_1 + 15x_2 + 45x_3 \leq 300$$

$$x_1 + x_2 + x_3 \leq 11$$

Obviously, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

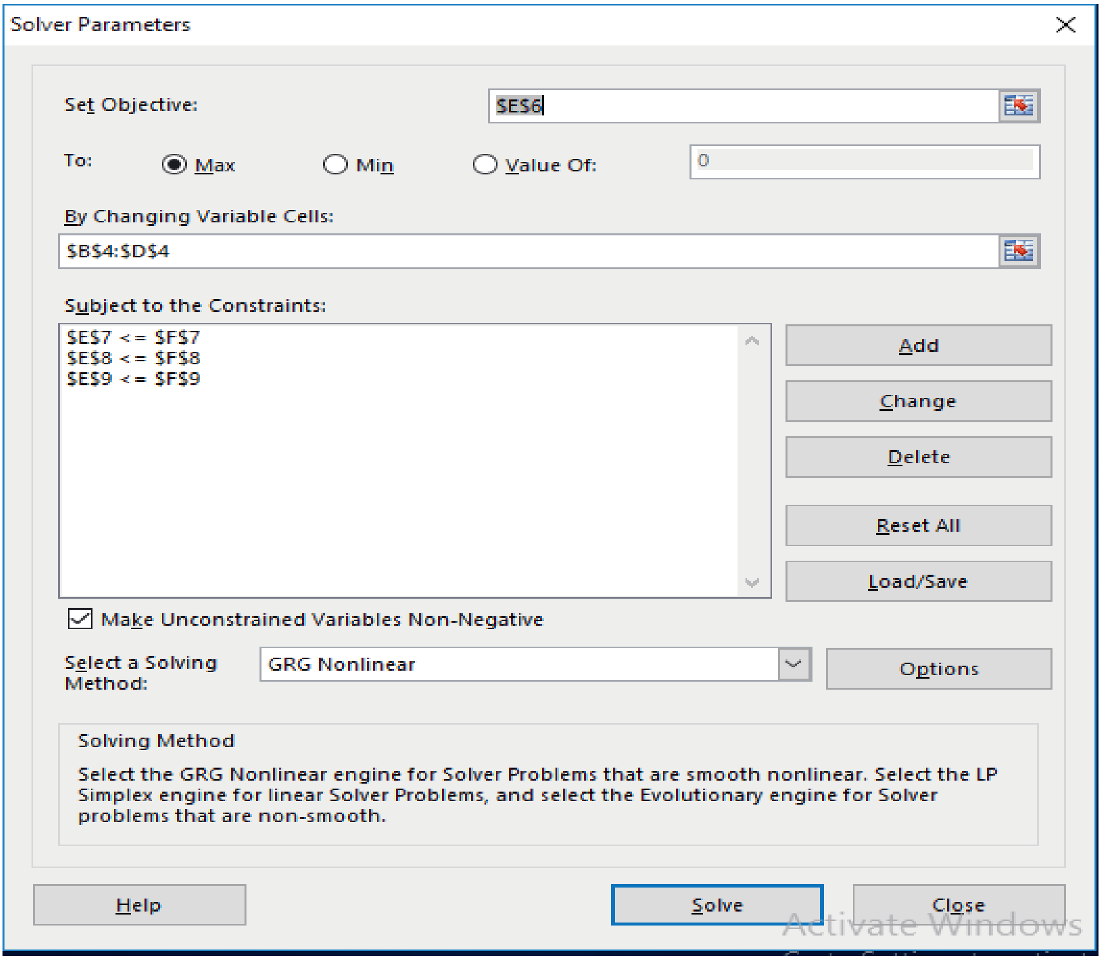
By applying the first three steps we obtain the following table

E6 | fx | =B4*B6+C4*C6+D4*D6

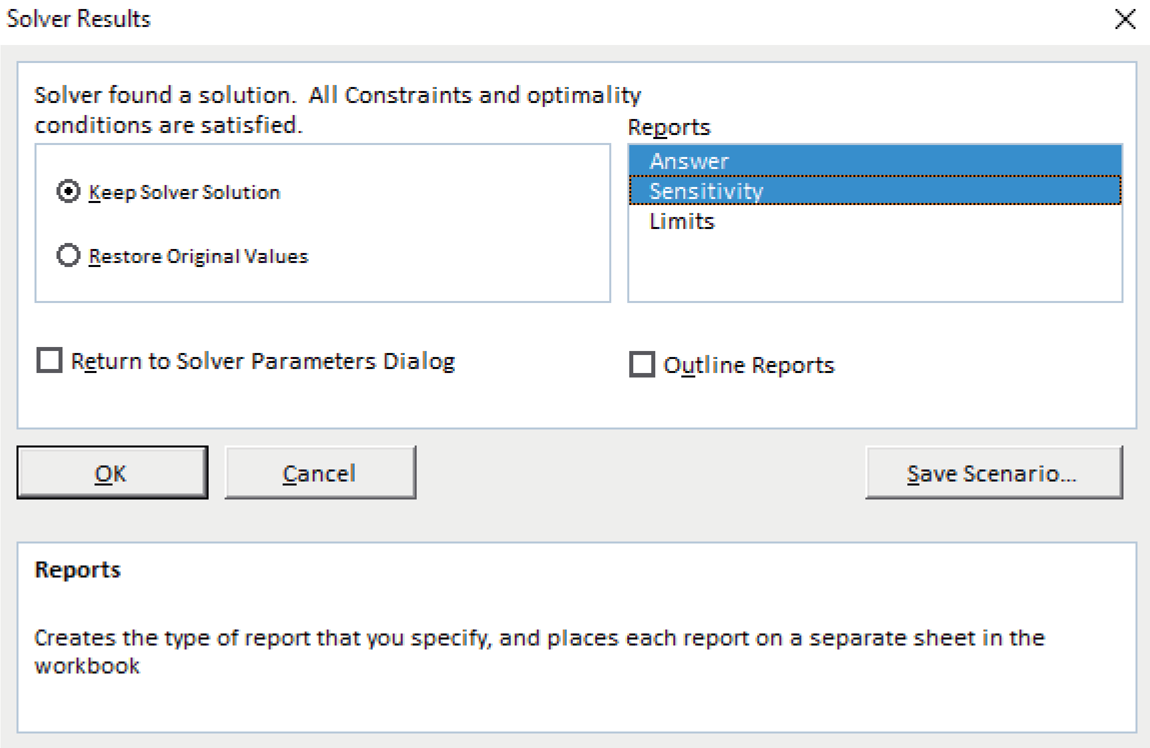
	A	B	C	D	E	F
1						
2						
3						
4	Variables to manipulate	0	0	0		
5	Variables name	x1	x2	x3	Total	RHS
6	objectives to be maximized	1.2	2	2.6	0	
7	Financial constraint	4.125	8.25	12.375	0	82.5
8	Labor	30	15	45	0	300
9		1	1	1	0	11

Observe line 6 column E, which is E6 is calculated as shown on the function bar.

The next step is to activate the solver



By click solve



Then Ok, we will get the result in Excel sheet

E6 =B4*B6+C4*C6+D4*D6

	A	B	C	D	E	F
1						
2						
3						
4	Variables Value to manipulate	2	9	0		
5	Name of dicision variables	x1	x2	x3	Total	RHs
6	Objective to maximize	1.2	2	2.6	20.4	
7	Financial constraint	4.125	8.25	12.375	82.5	82.5
8	Labor	30	15	45	195	300
9	number of houses	1	1	1	11	11

The result means the corporation build 2 convenience and 9 standard houses to maximize the revenue. The maximum revenue in this case is 20.4 million Birr.

Microsoft Excel 15.0 Answer Report
Worksheet: [review 8.xlsx]Sheet1
Report Created: 10/10/2021 6:51:13 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.047 Seconds.
Iterations: 0 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Iriginal Valu	inal Value
E(	Objective to maximize Total	20.4	20.4

Variable Cells

Cell	Name	Iriginal Valu	inal Valu	Integer
B	Variables Value to manipulat	2	2	Contin
C	Variables Value to manipulat	9	9	Contin
D	Variables Value to manipulat	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
E	Financial constraint Total	82.5	$E$7<=$F	Binding	0
E(	Labor Total	195	$E$8<=$F	Not Binding	105
E	number of houses Total	11	$E$9<=$F	Binding	0

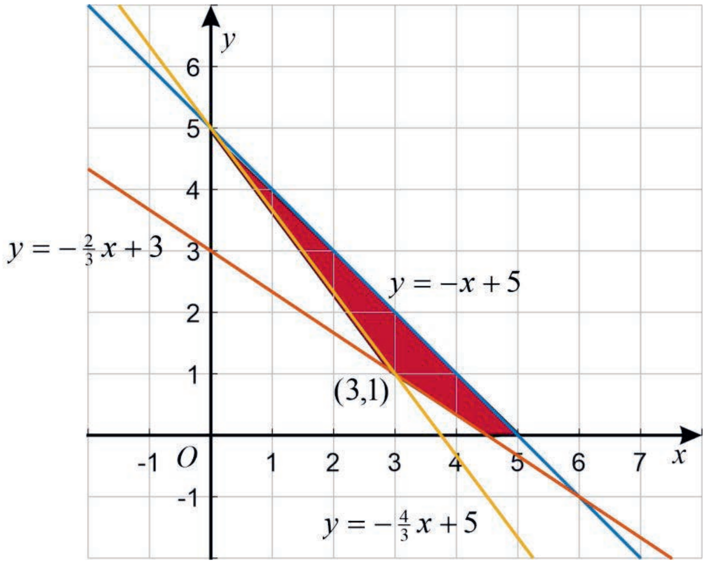
17. Solution

Let x be the number of packets of wheat, y be the number of packets of Sorghum

i) Adding the protein $4x+3y \geq 15$ and carbohydrate content in the two items $16x+24y \geq 72$, collectively 5 packets of food available, it is expressed as $x+y \leq 5$.

ii) The feasible region is shown in Figure. To plot the figure the constraints

are: $\begin{cases} 4x+3y \geq 15 \\ 16x+24y \geq 72 \\ x+y \leq 5 \\ x \geq 0, y \geq 0 \end{cases}$ changed to $\begin{cases} y \geq -\frac{4}{3}x+5 \\ y \geq -\frac{2}{3}x+3 \\ y \leq -x+5 \\ x \geq 0, y \geq 0 \end{cases}$



iii) First you can write the objective function as

$$\text{minimize or maximize, } Z = 40x + 40y$$

The vertices of the feasible region are as follows:

Vertices	The cost at each vertex, $Z = 40x + 40y$
Point (3, 1)	160
Point $(\frac{9}{2}, 0)$	180
Point (5, 0)	200
Point (0, 5)	200

To minimize the objective function select Point A. The minimum possible cost of Birr 160 can be made if 3 packets of Wheat and 1 packets of Sorghum are sold.

iv) To maximize the objective function select Point C, or Point D. The maximum possible cost of Birr 200 can be made if either 5 packets of Wheat and 0 packets of Sorghum are sold, or 0 packets of Wheat and 5 packets Sorghum of are sold.

Unit 5: Mathematical Applications in Business

Introduction

In this unit, you first need to give a brief discussion on applications of mathematics in different area of study. In particular the day to day activity of our world closely associated with money, money plays a central role in the world and in our lives, both professional and personal.

We all have to earn livings and pay bills, and to accomplish our goals, whatever they may be, reality requires us to manage the financing of those goals. After this revision, it is important to point out that the first section mainly focuses on introducing student's concept of ratio, rate, proportion, and percentage. Here you will see how these concepts are implemented in business. The second section deals with the time value of money and computation of simple and compound interest, annuity, amortization and depreciation of a fixed asset. The third section deals with the concepts of saving, investing, and financing business. The fourth section deals with taxation and different types of taxes commonly implemented in Ethiopia. In the last section we deal with spreadsheets, solving financial problems in Microsoft excel. Each section deal with solving problems that are associated with business activities. At the end of each main point, it is helpful to provide students with examples that are applicable to real life.

Unit Outcomes

After learning this unit students will be able to:

- Know common terms related to business.
- Know basic concepts in business.
- Understand time value of money.
- Apply mathematical principles and theories to practical situations.

5.1 Basic Mathematical Concepts in Business

Minimum Learning competencies

- Compare quantities in terms of ratio.
- Calculate the rate of increase and the rate of decrease in price of commodities
- Solve problems on proportional variation in business.
- Solve problems on compound proportion
- Find a required percentage of certain given amount
- Compute problems on percentage increase or percentage decrease

Answer to Activity 5.1

Give chance for the learners to react the activity. Give 3-5 minutes to answer. Assume the original price of one package vegetable is Birr 10.00. At S Super market after discount of 25% one package vegetable will be sold at Birr 7.50. The total money paid for five package will be Birr 37.50.

At Y Super Market from five package vegetable she will only pay the original price of four packages. The total money paid for five packages is Birr 40.00 only. Therefore, 25% offer is better offer. To get the benefit of the discount in Y you should buy at least four package but in S you will get the discount for any amount of purchase.

5.1.1 Ratio

Revise with the students about rational numbers and start the lesson by asking students what is the ratio of male to female students in the class or any other similar questions like, Doctor to patient ratio, car to population ratio, hospitals to population, and resident house to population ratio. What is their importance to know this ratio?

Answer to Activity 5.2

Yes there was an improvement on the expansion of commercial bank branches.

Answer to Exercise 5.1

1. Given a total of 200 pupils, 25 major in music, the remaining 200-25 =175 major in sport. The ratio of majoring in music=$\frac{25}{200}=\frac{1}{8}$, and ratio of majoring in sports=$\frac{175}{200}=\frac{7}{8}$. This means from a total of eight students in a class one student major music seven students major in sport. Or music to sport ratio is 1:7.
2. To divide Birr 19,560/00 for two individuals in the ratio of 3:1. First by adding 3 and 1 you get 4. Divide the total profit 19,560/00 by 4, i.e $\frac{19560}{4}=4890$. there are four Birr 4,890.00. From this the first individual took $3\times 4890=14690$ Birr and the second one collected Birr 4,890.00.
3. The larger shareholders share is ten times bigger than the smallest one.

Answer to Activity 5.3

To prepare concrete, you will mix cement, sand and gravel in the ratio 1:3:2. This means 1/6 of the mix is cement, 3/6 sand and 2/6=1/3 is gravel. You can add water also.

Answer to Exercise 5.2

1. Deborah, Kalid, and Mesfin divided a certain amount of money in the ratio of 5:3:1. Deborah received Birr 3,504.00. The given ratio 5:3:1 is part to part ratio converted to part to the whole ratio by adding $5+3+1=9$. Deborah's part to whole ratio is 5:9. Which means, let x be the total money

$$\frac{5}{9}x=3504,$$

$$x=\frac{9}{5}\times 3504=6307.20.$$

   At the beginning there was a sum of Birr 6,307.20.

2. Let L be length, W be width and H be height. $L=2W$, and $H=0.5W$ required is the ratio $L:W:H$.

$L:W:H=2W:W:0.5W$

Simplifying and multiplying by two, $4:2:1$, is the ratio of length to width to height of the Wooden block.

3. In the following exercises the unit of measurements is not identical first make it in the same unit of measurement.

   a. $1\frac{1}{2}$ hour to 30 minutes equivalent to 90 minutes to 30 minutes. They are in the ratio of 3:1(part to part ratio).

   b. 6 Birr to 50 cents. Convert either Birr to cents or cents to Birr. 6 Birr=600 cents because one Birr is hundred cents. $600\ cents: 50\ cents = 12:1$ ratio.

   c. Similar to the above problems. 1000:1 ratio.

4. The question asks part to whole ratio. From the total of 12 ran he won 7 gold which is the ratio of 7:12.

5. Water to acid ratio is 3:2 is part to part ratio. Converting part to whole ratio you get, $\frac{3}{5}$ is water, $\frac{2}{5}$ is acid. The amount of water is $\frac{3}{5} \times 15\ liters = 9\ liters$. The amount of acid is $\frac{2}{5} \times 15\ liters = 6\ liters$. She used 9 liters water and 6 liters of acid in the solution.

5.1.2 Rates

Start the lesson by asking students about the difference between rates and ratios. And relate the topic by asking the learners what is the growth rate of world population. And also relate the topic with the slope of the line. Like slope rate may be positive, negative or zero.

Answer to Activity 5.4

A farmer paid Birr 150/hour to collect 3 hectares of wheat. Assume the cost only depends on the area of the field. The cost of collecting wheat per hectares is $\frac{150}{3\text{hectares}} = 50$ Birr per hectares. To collect 16.5 hectares the farmer will pay $16.5 \times 50 = Birr\ 825$.

Answer to Exercise 5.3

1. To compare the ability of typing of the two students Yohannes and Semira first write the rate of typing per minute. Yohannes typing rate is 1225/15minutes=81words per minute. Semira's typing rate is 2700 words/30minutes=90words per minute. Semira is faster than Yohannes on entering words (typing).
2. Average Rate change of income $=\frac{\text{change in income}}{\text{change in unit produced}}=\frac{2400-1600}{40-20}=\frac{800}{20}=40$. The carpenter's income increase by Birr 40 for each chair produced.
3. $Rate\ of\ Change=\frac{amount\ of\ change}{original\ amount}=\frac{105-70}{70}=\frac{35}{70}=0.5$. This means in the given interval of time the import is increased by half.

5.1.3 Proportion

Start the lesson by asking students the concept of proportion from chemistry and physics courses. Allot enough time to raise what they know in other courses.

Answer to Exercise 5.4

1. Let x vary directly to y. Their ratio $\frac{x}{y}=k$ is constant. Here, $k=\frac{10}{15}=\frac{2}{3}$.

   Therefore, four corresponding pairs are given in the table:

   For instance when $x=12$, $y=\frac{3}{2}x=\frac{3}{2}\times 12=18$.

x	12	14	16	18
y	18	21	24	27

2. Two things vary inversely their product is constant. That is, $x=\frac{k}{y}$.

   In the given situation $k=xy=10\times 6=60$.

x	12	15	20	30
y	5	4	3	2

3. The price of the stamp increases when the number increase. Let x is the number of postal stamps and y is the price. When $=15$ $y=60$. The ratio

$\frac{y}{x} = \frac{60}{15} = 4$ is constant which is price per unit. The price of 72 stamp is $4 \times 72 = 288\ Birr$.

4. Interest earned and the amount deposited related directly. The ratio $\frac{interest}{deposit} = \frac{500}{2000} = 0.25$ is constant. Therefore, $\frac{interest}{36000} = 0.25$, $interest = 36000 \times 0.25 = 9000\ Birr$ The depositor earns Birr 9000 in three years.

5. First find mass per unit length of cm. $mass\ per\ length = \frac{192g}{16cm} = 12g/cm$ is constant. $\frac{105g}{y} = 12g/cm$. Thus $y = \frac{105g}{12g/cm} = 8.75\ cm$.

5.1.4 Percentage

Answer to Activity 5.5

Percentage	250	100	625	2000
10%	$250 \times 0.1 = 25$	10	62.5	200
25%	$250 \times 0.25 = 62.5$	25	156.25	500
53%	132.5	53	331.25	1060
15%	37.5	15	93.75	300
7.2%	18	7.2	45	144

Answer to Exercise 5.5

1. Find each indicated percentages
   - i. To calculate 2.5% of 400 is the same as multiplying 400 by 0.025. $0.025 \times 400 = 10$.
   - ii. 12.5% of 175 is the same as $0.125 \times 175 = 21.875$.
   - iii. 35% of 1500 is the same as $0.35 \times 1500 = 525$.
   - iv. 100% of 77 is simply 77.

2. Solution
   a. To answer this question. Let x be the unknown number whose 8% is 12. Hence, $0.08x = 12$. From this, $x = \frac{12}{0.08} = 150$. Conversely, 8% of 150 is 12. In real world 8% discount on the price of a certain item with Birr 150 is Birr 12.
   b. Let x be the unknown number whose 75% is 24. Thus, $0.75x = 24$. From this, $x = \frac{24}{0.75} = 32$.
   c. Let x be the unknown number whose 33% is 200. Thus, $0.33x = 200$. This implies$x = \frac{200}{0.33} = 606.06$.
   d. Let x be the unknown number whose 15% is 1.91.Thus, $0.15x = 1.91$. This implies, $x = \frac{1.91}{0.15} = 12.73$. This result interpreted as a client will pay Birr 1.91 Value added Tax (VAT) for Birr 12.73 purchased.
3. An alloy is a metal made by combining two or more metallic elements. In 80kg of an alloy 8% is silver. Getting the mass of silver is the same as knowing 8% of the total mass 80kg. $mass\ of\ silver(\text{Ag}) = 0.08 \times 80 = 6.4\text{kg}$.
4. Two compare the achievement of Bontu, express her achievement in percent form. In the first test she answered $\frac{30}{36} \times 100 = 83.33\%$. In the second test she answered, $\frac{20}{24} \times 100 = 83.33$. In both case her performance is the same.
5. Let T is the salary of Mr. Tilahun and J is the salary of Mr. Jemal. 26% of Mr. Jemal salary is $0.26 \times 15396 = 4{,}002.96$.

   So Mr. Tilahun salary

   $$T = 15{,}396 + 0.26 \times 15{,}396 = 15{,}396 + 4{,}002.96 = 19{,}398.96.$$

6. 25% in road symbol means for every 100 m horizontal distance the road up 25 meter vertically. Alternatively, $\tan\theta = 0.25$. $\theta = tan^{-1}(0.25) = 14.03$. The road made 14.03 degree vertically.

Answer to Activity 5.6

The main reason to offer a discount is to attract new customers, to foster long term customer loyalty, to improve customer lifetime value. The second reason for discount is to meet sale goals.

Answer to Exercise 5.6

1. Let the discount price of the car $P_1 = 510{,}000.00$.

   Discount amount $D = 90{,}000.00$.

   Original price $P_0 = 600{,}000.00$.

   Percentage of the discount $\dfrac{90000}{600000} \times 100 = 0.15 \times 100 = 15\%$

2. Let x be the total volume of the business. From the given information, $0.08 \times x = 520000$. Hence, $x = \frac{520000}{0.08} = 6{,}500{,}000.00$. The total volume of the business is Birr 6.5 million.

3. Percent increase is $\frac{15-12}{12} \times 100 = \frac{3}{12} \times 100 = 25\%$.

4. $Markup = Selling\ price - cost = 155 - 110 = 45$.

   Markup percentage on retail, $\frac{Markup}{Selling\ price} \times 100 = \frac{45}{155} \times 100 = 29.03\%$.

5. Markup percentage on retail, $\frac{Markup}{Selling\ price} = 0.37$.

   From this $Markup = selling\ price - cost = 0.37 Selling\ price$.

   Markup percentage on cost $\frac{Markup}{Cost} \times 100 = \frac{0.37\ Selling\ Price}{0.63\ Selling\ price} \times 100 = \frac{37}{63} \times 100 = 58.73\%$.

6. Driving the formula for cost $markup\ percent\ on\ selling\ price = \frac{markup}{selling\ price} \times 100 = 45\%$

$$0.45\ selling\ price = selling\ price - cost$$

$$cost = selling\ price(1 - 0.45) = 0.55\ \times selling\ price$$

$$= 0.55 \times 5250 = 2887.50$$

7. Markup percentage on cost, $\frac{0.3}{0.7} \times 100 = 42.85\%$.

8. Regular price is given then, the required is discount amount and selling price.

$$Discount = Discount\ rate \times regular\ price = 0.18 \times 1700 = Birr\ 306.00$$

$$selling\ price = regular\ price - discount = 1700 - 306 = Birr\ 1394$$

9.

$$Discount\ rate = \frac{discount}{regular\ price} \times 100\%$$

$$regular\ price = \frac{discount}{discount\ rate} = \frac{2000}{0.25} = 8000.00$$

The original price of the television set is Birr 8000.00 and its selling price is Birr 6000.00.

5.2 Time Value of Money

Minimum Learning Competencies

- Calculate payment by installment for a given simple interest arrangement.
- Calculate the compound interest of a certain amount invested for a given period of time.
- Apply the formula for compound interest to solve practical problems in business.
- Compute annuity for a give arrangement
- Define annuity
- Calculate annuity
- Classify annuity
- Interpret annuity concept to solve real life problems
- Define different terms related to amortization

- Formulate home (any) payment schedule
- Calculate monthly payment
- Describe what is depreciation mean and some of its causes
- Compute depreciation by using either of the two methods appropriately

Start the lesson by asking, why house renter pays house rent? Why not for money? You may begin the lesson with a brief revision of "simple interest" that the students had learnt in the previous grade ; in doing so give emphasis on the notions conveyed by terms like "principal" "rate of interest" and "interest period" or simply "Time". Use several examples to clarify and remind students about them. Introduce the notion of "Payment by Installment" (or deferred terms) and by using examples discuss how this arrangement of payment is carried out and assist students to solve related problems.

5.2.1 Simple Interest

Explain the terms like, Principal, Interest Rate and Time, term in detail. Ask if students have any real life experience related to simple interest.

Discuss with the class how simple interest formula is related to Arithmetic sequence discussed in unit 1.

Answer to Exercise 5.7

1. Loaned amount=Birr 12,000,

    Return amount= Birr 12,600,

    Duration=6 months=0.5 year.

    Interest paid=Returned amount-loaned amount=12600-12000=600.

    $$Interest\ rate = \frac{Interest}{Principal \times time} \times 100 = \frac{600}{12000 \times 0.5} \times 100 = 10\%.$$

2. Principal=800

    Return amount=900

    Interest received =900-800=100

    $$Interest\ rate = \frac{Interest}{Principal \times time} \times 100 = \frac{100}{800 \times 2} \times 100 = 6.25\%.$$

3. $Amount\ paid = pit + p = 7829.14 \times 1 \times 0.0975 + 7829.14 = 8592.48.$

   Semira needs Birr 8,592.48 to repay the loan.

Answer to Exercise 5.8

1. Given, Time, t=3 years

   Annual interest rate, $r = 6.09\%$

   Loan amount, $P = 14{,}043.43$

   Required, simple interest, $I = Prt = 14{,}043.43 \times 0.0609 \times 3 = 2{,}565.73.$

2. Given, Time, t=2 years

   Annual interest rate, $r = 9\%$

   Simple interest rate, $I = 63.00$

   Required, Principal, P. Substituting the given value on the formula $P = \frac{I}{rt}$, we obtain, $P = \frac{63}{0.09\times 2} = 350.00.$

3. From the problem you get the following information.
   $I = 450.00,$
   $r = 7.5\%$
   $t = \frac{1}{12}\ year$
   Required the initial investment or P, $P = \frac{450}{0.075\times\frac{1}{12}} = 72{,}000.00$. Kedija should invest Birr 72,000 to collect Birr 450 on monthly base.

5.2.2 Compound interest

Answer to Activity 5.7

a. 1. On cell C2 , we write "=6000",(i.e.) the principal

2. On cell B3 , we write" =C$2*0.08", C2 contains the principal and multiply it by the rate , 0.08.

3. Press enter and then drag from B3 down a column up to the desired row.

4. On cell C3 , we write" =C2+B3"and press enter and then drag from C3 down a column up to the desired row.

b. 1. On cell C2 , we write "=6000",(i.e.) the principal

2. On cell C3 , we write" =C2*1.08", C2 contains the principal and multiply it by the (1+rate) , 1.08.

3. Press enter and then drag from C3 down a column up to the desired row.

The formula that provides the above table for the interest (compound):

Answer to Exercise 5.9

1. Collecting the given information, $P = Birr\ 10{,}000.00$. The

$$r = 9.5\%$$

$$t = 5\ years$$

In this problem students are expected to answer two things. The first one is the amount A after five years and the compound interest (C.I) earned in the given period of time.

$$A = P\left(1 + \frac{r}{100}\right)^5 = 10000(1 + 0.095)^5 = 10000(1.095)^5$$

$$= 10000 \times 1.5742 = 15{,}742.38$$

The instructor will support learners to use calculators to get the result.

Compound interest, $C.I = A - P = 15742.38 - 10000 = 5742.38$. Finally the depositor will collect a compound interest of Birr 5,742.38.

2. Let P be the original amount deposited. After n years these amount grow to triplet i.e $3P$.

   $A = P(1 + \frac{14}{100})^n = 3P$. Dividing both sides by P you obtain

   $(1 + 0.14)^n = 3$. To solve for n you can use your knowledge of logarithmic function.

$$n \log 1.14 = \log 3$$

$$n = \frac{\log 3}{\log 1.14} = 8.385$$

The deposit must be left for 8.385 years but as interest is paid yearly, it would have to be left for 9 years.

3. The time is 5 years, the required is the annual interest rate r to double the original value. $A = P(1 + \frac{r}{100})^5 = 2P$. $(1 + \frac{r}{100})^5 = 2$

Dividing both sides by P. Solve the equation $\left(1+\frac{r}{100}\right)^5 = 2$ for r

$$r = 100\sqrt[5]{2}$$

$$r \approx 14.8698$$

The annual interest rate is approximately $r = 14.87\%$ in rounding two decimal place.

4. In this problem P is unknown. The given values are $t = 3\ years,$

$$r = 9.5\%$$

$A = 4000$. By using the formula $A = P\left(1+\frac{r}{100}\right)^t$.

$P(1+0.095)^3 = 4000$. Solving for P, $P = \frac{4000}{(1.095)^3} = 3046.615$.

5. In this problem both initial deposit P and the value after t years A given. The required is the time t. By using the formula $A = P\left(1+\frac{r}{100}\right)^t$. Substituting the given values you obtain $3000 = 2000 \times (1+0.08)^t$

Solve for t

$$\frac{3}{2} = 1.08^t$$

$$tlog(\frac{27}{25}) = \log(\frac{3}{2})$$

$$t = \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{27}{25}\right)}$$

$$t = log_{\frac{27}{25}}^{(\frac{3}{2})}$$

$$t \approx 5.26845$$

To reach Birr 3000.00 after depositing Birr 2000.00 at 8% interest rate compounded annually approximately it takes 5.27 years.

Answer to Activity 5.8

CBE pays for the depositors an interest on monthly basis.

Most of the banks do the same. Credit and saving unions calculate on yearly basis.

- The teacher accepts student's response and discuss on it.
- Which one is beneficiary for the depositor?
- Which one is good for the bank? Discuss in a group of 5 to 7 students.

Answer to Exercise 5.10

1. The value of i and n that would be used in the compound interest formula

a. $r = 8\%$, $m = 4$. So $i = \frac{r}{m} = \frac{0.08}{4} = 0.02 \text{ or } 2\%$

And the total number of compounding in ten years is n, $n = 10 \times m = 10 \times 4 = 40$.

b. $r = 9\%, m = 12, \quad t = 7 \text{ years}$. So $i = \frac{r}{m} = \frac{0.09}{12} = 0.0075$. Equivalently, 0.75%. $n = t \times m = 7 \times 12 = 84$. This means the institute can compute 84 times in 0.75% interest rate.

c. $r = 15\%, \ m = 2, t = 15 \text{ years}$. So $i = \frac{r}{m} = \frac{0.015}{2} = 0.0075$ that is 0.75%. Number of computation period, $n = t \times m = 15 \times 2 = 30$.

d. $r = 9\%, \ m = 360$, and $t = 8 \text{ years}$. $i = \frac{r}{m} = \frac{0.09}{360} = 0.00025$, that is 0.025% daily. The total number of computation n is $n = m \times t = 360 \times 8 = 2880$.

2. The PV is Birr 4,500, but the values of i and n require some work. Since the interest is compounded monthly, the 6% (per year) needs to be divided by 12 (since each month is 1/12 of a year) to make it monthly. By the same token, the term of 7 years must be expressed in months, so we multiply it by 7).

So $i = \frac{r}{12} = \frac{0.06}{12} = 0.005$, and $n = mt = 12 \times 7 = 84$.

Therefore, $FV = PV(1+i)^n = 4500(1+0.005)^{84} = 6{,}841.66.$

Alternatively, using calculators,

4500 × (1.005) ^ 84 = **6,841.6633**

3. Here, $P = 100, r = 12\%, m = 2$ and $A = 1200$ was given. The time t is unknown.

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Substituting the given values, $1200 = 100\left(1+\frac{0.12}{2}\right)^{2t}$

$$12 = \left(1+\frac{0.12}{2}\right)^{2t}$$

Solve for t, $t = \frac{1}{2} \times log_{\frac{53}{50}}^{(12)}$

$t \approx 21.32275$ or by using natural logarithm

$$t = \frac{\ln(12)}{2\ln(1..06)}$$

$$t \approx 21.32275.$$

4. Given values are, $r = 13.5\%, m = 12, A = 100000.00$ and $t = 3\ years$. The initial deposit P is unknown. Using

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Note: From the future value of a compound investment, you can get the present value. The formula for present value P when future value A compounded m times a year with annual interest rate of r is given by

$$P = A(1+i)^{-n},$$

where $i = \frac{r}{m}$, and $n = mt$.

Since interest compounds monthly, $i = \frac{0.135}{12} = 0.01125\ and\ n = 3(12) = 36$

$$A = P \times (1 + i)^n$$

$$100{,}000 = P(1 + \frac{0.135}{12})^{36}$$

Solving for P, we divide both sides through to get the present value P. Using calculator may make things easier.

$$P = \frac{100{,}000 \times 800^{36}}{809^{36}} = 66{,}848.667$$

So I need to deposit approximately Birr 66,848.67.

5. By using the same step as problem 4, from the problem P is unknown, $A = 600{,}000.00, m = 4, t = 2\ years$ and $r = 9\%$.

Using the formula

$P = A(1 + i)^{-n}$, , $i = \frac{r}{m} = \frac{0.09}{4} = 0.0225$, and $n = m \times t = 4 \times 2 = 8$

substituting you get

$$P = 600000 \times (1 + 0.0225)^{-8} = 600000 \times \frac{400^8}{409^8}$$

$$P \approx 502{,}163.0078$$

The investor should invest about Birr 502,163.01.

6. Let P be the amount deposited by Mr. Degalla at $r = 8.4\%$ interest rate per year, $m = 12$. The question in this problem is to find the time t to double the original deposit, i.e $A = 2P$.

$$2P = P \times (1 + i)^n$$

Where, $i = \frac{0.084}{12} = 0.007$ and $n = m \times t = 12t$.
Substituting the given values,

$$2P = P \times (1 + 0.007)^{12t}$$

$$1.007^{12t} = 2$$

Solve for t, $t = \frac{1}{12} log_{\frac{1007}{1000}}^{(2)}$

$$t \approx 8.2806$$

Dear teacher please allow students to use calculators (digital tools) to calculate.

Effective Interest Rate

Answer to Activity 5.9

Dear teacher please facilitate for the students to report and force them to bring the required information.

Answer to Exercise 5.11

1. Late $P = 100$ Birr, calculate the amount A after a year

   $A = 100 \times (1 + \frac{0.125}{2})^2$

   $A = 100 \times (1.0625)^2$

   $A = 100 \times 1.128906 = 112.891$ or $A = \left(\frac{85}{8}\right)^2$. Suppose r_e is the rate compounded annually and give a future value of Birr 112.891.

   $112.891 = 100(1 + r_e)^1 = 100(1 + r_e)$

   Dividing by 100 both sides, you obtain $1 + r_e = 1.1289$. Subtracting 1 from both sides we have

   $r_e = 0.1289 = 12.89\%$.

   This result mean 12.5% interest rate compounded semi-annually is equivalent with 12.89% rounded in two decimal places. Or by using the formula

   $r_e = (1 + \frac{r}{m})^m - 1 = (1 + \frac{0.125}{2})^2 - 1 = 0.1289$. Hence, $r_e = 12.89\%$.

   Remark: r_e a rate to convert it to percentage you can multiply by 100.

2. All the three banks offer the same interest rate annually. For the depositor the banks with higher frequencies paid better interest. Bank C preferable.
3. The three institute offer different interest rate and different compounding frequencies. To choose the better you can calculate the effective interest rate for each institute.

For Bank B: Let the principal $P = 100$, $A = 100 \times (1 + \frac{0.0898}{12})^{12}$

$$A = 100 \times (1 + 0.0074833)^{12}$$

$$A = 100 \times (1.0074833)^{12} = 100 \times 1.09358978$$

$$A \approx 109.359$$

For Credit and Saving Association: $A = 100 \times (1 + \frac{0.0905}{4})^{4}$

$$A = 100 \times (1 + 0.022625)^{4}$$

$$A = 100 \times 1.022625^{4}$$

Computing for A by using calculators, you can get

$$A \approx 109.3618$$

For XY-microfinance, $A = 100 \times (1 + \frac{0.091}{2})^{2}$

$$A = 100 \times (1 + 0.0455)^{2}$$

$$A = 100 \times (1.0455)^{2}$$

$$A \approx 109.307$$

Comparing the above three results 9.05% interest rate compounded quarterly gave a good result in one year.

4. Using the same approach as before, we find the future value using any amount and any term that we like. Like the previous one, we will make "nice" choices of Present Value, P=Birr 100 and time t=1 year,

$$\text{Daily Dividend: } F.V = 100(1 + \frac{0.08}{360})^{360} = 108.33$$

$$\text{Annual advancement: } F.V = 100(1 + 0.0833)^{1} = 108.33$$

This one ends in a draw. Both options pay the same result, and so it makes no difference rate. Thus, there is no difference which option Dechase chooses. When two rates and compounding's give the same result, we say that they are equivalent. Since both options result in the same amount of interest being paid, we can use the two interchangeably. Even if Dechase chose the Daily Dividends option paying 8% compounded daily, the company could just as well credit his interest using the 8.33% with annual compounding.

5. The interest paid is the difference between the amount A after and the original deposit P.

   $A = 475 \times (1 + 0.07) = 475 \times 1.07 = 508.25$

   $I = 508.25 - 475 = 33.25$. The credit union would paid Birr 33.25 annually. But the problem asked for 300 days. By considering 365 days per year we obtain $\frac{33.25}{365} \approx 0.0911$ per day. Hence, the union pay

   $I = 0.0911 \times 300 = 27.33$ in 300 days.

6. $P = 372{,}000$, $R = 15.25\%$

   $A = 372000 \times (1 + 0.1525) = 428730$

   The value after one year is Birr 428,730.00. The required is the amount after 9 months. The value after nine months **is** $\boldsymbol{A_9}$**,** $A_{9months} = \frac{428730}{12} \times 9 = 321{,}547.50$

5.2.3 Annuity

Discuss with the class when house rent is paid. When other utility bills like Electric, telephone, water bills our family pay?

Answer to Activity 5.10

Iqub is an annuity because the payment is constant. While it has no interest.

Answer to Exercise 5.12

1. Classifying as either future value or present value
   a. Birr 1,000,000.00 is future value for Mr. Worku
   b. Birr 835,000.00 is Present value
   c. Future value

2. Classifying as Annuity due or ordinary annuity
   a. Annuity due because Beshir gets the payment always at the beginning of the term.
   b. Ordinary Annuity because dividend always at the end of the term.
   c. Ordinary annuity

Answer to Activity 5.11

Consider $S_{\overline{n}|i} = (1+0.075)^9 + (1+0.075)^8 + \ldots + (1+0.075)^1 + (1+0.075)^0$. This is a sum of a series with first term $a_1 = (1+0.075)^0 = 1$ and common ratio, $r = 1.075$. By using the n[th] partial sum of a geometric series formula, $S_n = \dfrac{a_1(1-r^n)}{1-r}$ for $r \neq 1$. Substituting the values we will get the same result as Equation (5.6).

Answer to Exercise 5.13

1. Solution:
   a. periodic deposit $R = 0.25 \times 10460 = 2615$ the payment was done regularly at the end of the month. So it is ordinary annuity.

   Annual interest rate $r = 0.07$

   Compounding frequency $m=12$

   Since the payment made monthly, $i = \dfrac{r}{m} = \dfrac{0.07}{12} = 0.00583$. And $n = m \times 3 = 36$

   By using annuity factor formula, $FV = RS_{\overline{n}|i} = 2615\left(\frac{(1+0.00583)^{36}-1}{0.00583}\right) = 2615 \times 39.93.$

   Ato Tesfaye will have a balance of, $FV = 104{,}416.95$.

   b. In 3 years there are a total of 36 payments of Birr 2,615.00. Which is a sum of Birr 94,140.00. The total interest earned $I = 104416.95 - 94140 = 10{,}276.95$.

2. Solution: Given R= Birr 15,000.00

   r=8% , n=40 years.

a. This is an ordinary annuity. By using annuity factor approach, and substituting the given values you obtain

$$FV = RS_{\frac{n}{i}} = R\frac{(1+i)^n - 1}{i} = 15000 \times \frac{(1+0.08)^{40} - 1}{0.08} \text{ Simplifying}$$

$$FV = \frac{7500 \times 27^{40} - 7500 \times 25^{40}}{25^{39}} \approx 3.88585 \times 10^6$$

In his retirement age the retiree will have approximately Birr 3,885,850.00

b. Now the number of years reduced by 10 years an n=30 years. Using the same approach as of a. $FV = 15000\frac{(1+0.08)^{30} - 1}{0.08}$

$FV = \frac{7500 \times 27^{30} - 7500 \times 25^{30}}{25^{29}} \approx 1.69925 \times 10^6$. In this case the retiree will have Birr 1,699,250.00 in his retire age 65. This means who served long will get more in his retire age.

c. The saving schedule is shown in the following Table .

Year of payment	Payment Amount	Years of interest	Future value
26	15,000.00	39	301,729.4652
27	15,000.00	38	279,379.1345
28	15,000.00	37	258,684.3838
29	15,000.00	36	239,522.5776
30	15,000.00	35	221,780.1644
31	15,000.00	34	205,352.0041
32	15,000.00	33	190,140.7445
33	15,000.00	32	176,056.2449
34	15,000.00	31	163,015.0416
35	15,000.00	30	150,939.8533
36	15,000.00	29	139,759.1235
37	15,000.00	28	129,406.5958
38	15,000.00	27	119,820.9220
39	15,000.00	26	110,945.2982
40	15,000.00	25	102,727.1279
41	15,000.00	24	95,117.7111
42	15,000.00	23	88,071.9547

43	15,000.00	22	81,548.1062
44	15,000.00	21	75,507.5057
45	15,000.00	20	69,914.3572
46	15,000.00	19	64,735.5159
47	15,000.00	18	59,940.2925
48	15,000.00	17	55,500.2708
49	15,000.00	16	51,389.1397
50	15,000.00	15	47,582.5367
51	15,000.00	14	44,057.9044
52	15,000.00	13	40,794.3559
53	15,000.00	12	37,772.5518
54	15,000.00	11	34,974.5850
55	15,000.00	10	32,383.8750
56	15,000.00	9	29,985.0694
57	15,000.00	8	27,763.9532
58	15,000.00	7	25,707.3640
59	15,000.00	6	23,803.1148
60	15,000.00	5	22,039.9212
61	15,000.00	4	20,407.3344
62	15,000.00	3	18,895.6800
63	15,000.00	2	17,496.0000
64	15,000.00	1	16,200.0000
65	15,000.00	0	15,000.0000
Total			3,885,847.78

The table was constructed by using Microsoft excel. The result agreed with the answer on part a.

3. The required is annuity factor $S_{\frac{n}{i}}$, $n = 15 \times 12 = 180$, $i = \dfrac{0.111}{12} = 0.00925$. By using annuity factor formula Equation 5.5, $S_{\frac{n}{i}} = \dfrac{(1+i)^n - 1}{i}$ Substiting the given values

$$S_{\frac{180}{0.00925}} = \frac{(1+0.00925)^{180} - 1}{0.00925} = 458.959$$

4. Given, monthly deposit R=250, interest rate r=12%, t=20 years m=12. $n = 20 \times 12 = 240$ and $i = \frac{0.12}{12} = 0.01$

   The value after 20 years is the future value of ordinary annuity.

   $FV = R\frac{(1+i)^n - 1}{i} = 250 \times \frac{(1+0.01)^{240} - 1}{0.01} \approx 247{,}313.84$. Rekike will have approximately Birr 247,313 after 20 years, The interest earned $I = 247313.84 - 250 \times 240 = 187{,}313.84$

5. Please refer the note from the text.

6. You can find the ordinary annuity factor $S_{\frac{n}{i}} = \frac{(1+i)^n - 1}{i}$

   a. Substituting the given values we obtain

   $$S_{\frac{20}{0.025}} = \frac{(1+0.025)^{20} - 1}{0.025} \approx 25.545$$

   b. $S_{\frac{1}{0.075}} = \frac{(1+0.075) - 1}{0.075} = 1$

   c. $S_{\frac{60}{0.0025}} = \frac{(1+0.0025)^{60} - 1}{0.0025} \approx 64.6467$

   d. $S_{\frac{1}{0.1}} = \frac{(1+0.1) - 1}{0.1} = 1$ from b and d you assert in year 1 annuity factor ia independent of the interest rate.

Answer to Exercise 5.14

1. The problem here is annuity due with a periodic payment $R = 3500.00$ and $interest\ rate\ r = 12.6\%$ the time $t = 30\ years$. The deposit taker compounded monthly.$n = m \times t = 12 \times 30 = 360$. $i = \frac{0.125}{12} = 0.0105$. By using the annuity due formula the maturity value or future value is

$F.V = R \times S_{\frac{n}{i}}(1+i)$ where $S_{\frac{n}{i}} = \left(\frac{(1+i)^n - 1}{i}\right)$

$S_{\frac{n}{i}} = \left(\frac{(1+i)^n - 1}{i}\right) = \left(\frac{(1+0.0105)^{360} - 1}{0.0105}\right)$ Calculating using calculators you obtain

$S_{\frac{n}{i}} \approx 3,996.29$ in two decimal place.

The future value of annuity due

$F.V = R \times S_{\frac{n}{i}} \times (1+i) = 3500 \times 3996.29 \times 1.0105 = 14,133,864.9$

2. For “Iqub” Abdulkadir made a payment $T = 12 \times 5 \times 1000 = 60{,}000$.

Traditionally there is no interest in Iqub.

3. This one is again annuity due. The future value of annuity due

$$F.V = R \times S_{\frac{n}{i}} \times (1+i) = 1000 \times \frac{(1+0.007)^{60} - 1}{0.007} \times 1.007 = 1000 \times 74.25 \times 1.007$$

$F.V = 74769.75$

When you see the result of no. 2 which was Birr 60,000.00, there is a big difference between them. $Difference = 74769.75 - 60000 = 14769.75$.

Answer to Exercise 5.15

1. To calculate present value annuity factor, first calculate future value annuity factor. $i = \frac{0.09}{4} = 0.0225$ and $n = 5 \times 4 = 20$ Substituting this on future value annuity formula you get

   $S_{\frac{n}{i}} = \frac{(1+i)^n - 1}{i} = \frac{(1+0.0225)^{20} - 1}{0.0225} = 24.912$. The present value annuity

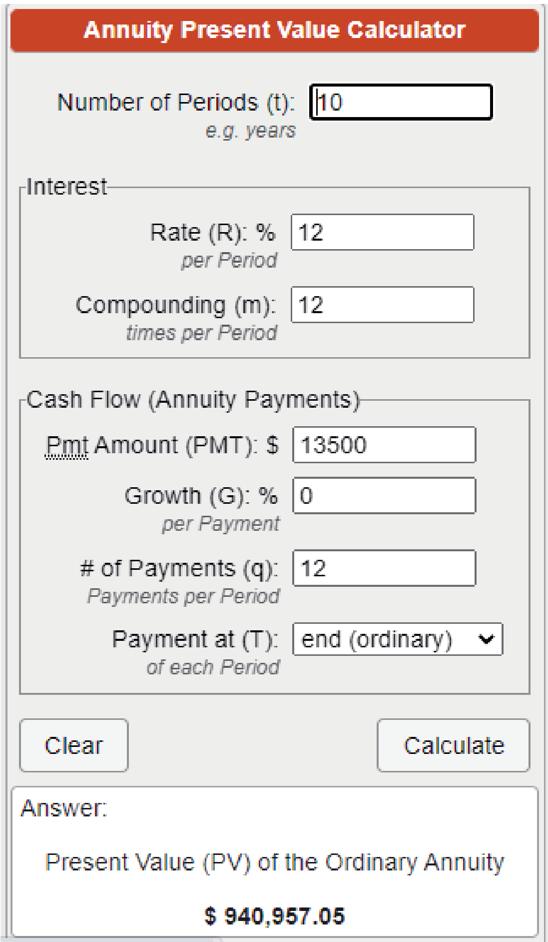
   $$a_{\frac{n}{i}} = \frac{S_{\frac{n}{i}}}{(1+i)^n} = \frac{24.912}{1.0225^{20}} = 15.964$$

2. On this problem the required is the monthly payment R. $PV = R \times a_{\frac{n}{i}}$

   $650000 = R \times 61.1838$

$R = \dfrac{650000}{61.1838} = 10623.73$, so the monthly payment for the loan Birr 650,000 is around Birr 10,623.73.

3. By using annuity present value calculator we get the following result



5.2.4 Amortization

Answer to Activity 5.12

The borrower will return the total sum once or may return by dividing the total sum 12 parts or 4 part or twice.

Answer to Exercise 5.16

1. To calculate monthly payment first identify the given information. Total loan $PV = 600,000.00$

$r = 14\%, \text{period } t = 30 \text{ years}$ $\quad i = \frac{0.15}{12} = 0.0125$, By using the amortization formula $R = PV\left(\frac{i}{1-(1+i)^{-n}}\right)$ where $n = 12 \times 30 = 360$.

$$R = 600000\left(\frac{0.0125}{1-(1+0.0125)^{-360}}\right) = 600000 \times \frac{0.0125}{1-0.01142} = 600000 \times \frac{0.0125}{0.98858} = 7586.66$$

Dear learner there are different types of Excel template to calculate loan amortization.

Repayment Number	Opening Balance	Loan Repayment	Interest Charged	Capital Repaid	Closing Balance	% Capital Outstanding
1	600,000.00	7,586.66	7,500.00	86.66	599,913.34	100.0%
2	599,913.34	7,586.66	7,498.92	87.75	599,825.59	100.0%
3	599,825.59	7,586.66	7,497.82	88.84	599,736.74	100.0%
4	599,736.74	7,586.66	7,496.71	89.95	599,646.79	99.9%
5	599,646.79	7,586.66	7,495.58	91.08	599,555.71	99.9%
6	599,555.71	7,586.66	7,494.45	92.22	599,463.49	99.9%
7	599,463.49	7,586.66	7,493.29	93.37	599,370.12	99.9%
8	599,370.12	7,586.66	7,492.13	94.54	599,275.58	99.9%
9	599,275.58	7,586.66	7,490.94	95.72	599,179.86	99.9%
10	599,179.86	7,586.66	7,489.75	96.92	599,082.95	99.8%
11	599,082.95	7,586.66	7,488.54	98.13	598,984.82	99.8%
12	598,984.82	7,586.66	7,487.31	99.35	598,885.47	99.8%

2. Similar to problem 1. You can obtain the periodic payment R= 7,990.45. The amortization schedule also shown as follow

Repayment Number	Opening Balance	Loan Repayment	Interest Charged	Capital Repaid	Closing Balance	% Capital Outstanding
1	600,000.00	7,990.45	7,000.00	990.45	599,009.55	99.8%
2	599,009.55	7,990.45	6,988.44	1,002.00	598,007.55	99.7%
3	598,007.55	7,990.45	6,976.75	1,013.69	596,993.85	99.5%
4	596,993.85	7,990.45	6,964.93	1,025.52	595,968.33	99.3%
5	595,968.33	7,990.45	6,952.96	1,037.48	594,930.85	99.2%
6	594,930.85	7,990.45	6,940.86	1,049.59	593,881.26	99.0%
7	593,881.26	7,990.45	6,928.61	1,061.83	592,819.43	98.8%
8	592,819.43	7,990.45	6,916.23	1,074.22	591,745.21	98.6%
9	591,745.21	7,990.45	6,903.69	1,086.75	590,658.45	98.4%
10	590,658.45	7,990.45	6,891.02	1,099.43	589,559.02	98.3%
11	589,559.02	7,990.45	6,878.19	1,112.26	588,446.76	98.1%
12	588,446.76	7,990.45	6,865.21	1,125.24	587,321.52	97.9%

3. To fill the table first export the table to excel and add one column next to column 1. The loan amount is Birr 25,000.00 and the borrower pays Birr 500.00 in a fixed rate monthly. This Birr 500 covers both principal and interest.

   a. To amortize the loan as shown in the following table took 63 months.

   b. On the loan term the borrower paid Birr 6451.095= 31451.09503-25000 as an interest.

Payment No.	Beginning balance	Payment amount	interest amount	Principal	Remaining balance	Interest rate
1	25,000.00	500.00	187.50	312.50	24,687.50	0.09
2	24,687.50	500.00	185.16	314.84	24,372.66	0.09
3	24,372.66	500.00	182.79	317.21	24,055.45	0.09
4	24,055.45	500.00	180.42	319.58	23,735.87	0.09

5	23,735.87	500.00	178.02	321.98	23,413.89	0.09
6	23,413.89	500.00	175.60	324.40	23,089.49	0.09
7	23,089.49	500.00	173.17	326.83	22,762.66	0.09
8	22,762.66	500.00	170.72	329.28	22,433.38	0.09
9	22,433.38	500.00	168.25	331.75	22,101.63	0.09
10	22,101.63	500.00	165.76	334.24	21,767.39	0.09
11	21,767.39	500.00	163.26	336.74	21,430.65	0.09
12	21,430.65	500.00	160.73	339.27	21,091.38	0.09
13	21,091.38	500.00	158.19	341.81	20,749.56	0.09
14	20,749.56	500.00	155.62	344.38	20,405.19	0.09
15	20,405.19	500.00	153.04	346.96	20,058.23	0.09
16	20,058.23	500.00	150.44	349.56	19,708.66	0.09
17	19,708.66	500.00	147.81	352.19	19,356.48	0.09
18	19,356.48	500.00	145.17	354.83	19,001.65	0.09
19	19,001.65	500.00	142.51	357.49	18,644.16	0.09
20	18,644.16	500.00	139.83	360.17	18,283.99	0.09
21	18,283.99	500.00	137.13	362.87	17,921.12	0.09
22	17,921.12	500.00	134.41	365.59	17,555.53	0.09
23	17,555.53	500.00	131.67	368.33	17,187.20	0.09
24	17,187.20	500.00	128.90	371.10	16,816.10	0.09
25	16,816.10	500.00	126.12	373.88	16,442.22	0.09
26	16,442.22	500.00	123.32	376.68	16,065.54	0.09
27	16,065.54	500.00	120.49	379.51	15,686.03	0.09
28	15,686.03	500.00	117.65	382.35	15,303.68	0.09
29	15,303.68	500.00	114.78	385.22	14,918.45	0.09
30	14,918.45	500.00	111.89	388.11	14,530.34	0.09
31	14,530.34	500.00	108.98	391.02	14,139.32	0.09
32	14,139.32	500.00	106.04	393.96	13,745.37	0.09
33	13,745.37	500.00	103.09	396.91	13,348.46	0.09
34	13,348.46	500.00	100.11	399.89	12,948.57	0.09
35	12,948.57	500.00	97.11	402.89	12,545.68	0.09
36	12,545.68	500.00	94.09	405.91	12,139.78	0.09
37	12,139.78	500.00	91.05	408.95	11,730.82	0.09
38	11,730.82	500.00	87.98	412.02	11,318.81	0.09
39	11,318.81	500.00	84.89	415.11	10,903.70	0.09
40	10,903.70	500.00	81.78	418.22	10,485.47	0.09
41	10,485.47	500.00	78.64	421.36	10,064.12	0.09

42	10,064.12	500.00	75.48	424.52	9,639.60	0.09
43	9,639.60	500.00	72.30	427.70	9,211.89	0.09
44	9,211.89	500.00	69.09	430.91	8,780.98	0.09
45	8,780.98	500.00	65.86	434.14	8,346.84	0.09
46	8,346.84	500.00	62.60	437.40	7,909.44	0.09
47	7,909.44	500.00	59.32	440.68	7,468.76	0.09
48	7,468.76	500.00	56.02	443.98	7,024.78	0.09
49	7,024.78	500.00	52.69	447.31	6,577.46	0.09
50	6,577.46	500.00	49.33	450.67	6,126.79	0.09
51	6,126.79	500.00	45.95	454.05	5,672.75	0.09
52	5,672.75	500.00	42.55	457.45	5,215.29	0.09
53	5,215.29	500.00	39.11	460.89	4,754.41	0.09
54	4,754.41	500.00	35.66	464.34	4,290.06	0.09
55	4,290.06	500.00	32.18	467.82	3,822.24	0.09
56	3,822.24	500.00	28.67	471.33	3,350.91	0.09
57	3,350.91	500.00	25.13	474.87	2,876.04	0.09
58	2,876.04	500.00	21.57	478.43	2,397.61	0.09
59	2,397.61	500.00	17.98	482.02	1,915.59	0.09
60	1,915.59	500.00	14.37	485.63	1,429.96	0.09
61	1,429.96	500.00	10.72	489.28	940.68	0.09
62	940.68	500.00	7.06	492.94	447.74	0.09
63	447.74	451.10	3.36	447.74	-	0.09
Total		31,451.10	6,451.10	25,000.00	835,146.00	

5.2.5 Depreciation

Answer to Activity 5.13

Yes there was a price change. The change is more significant as the time goes on.

Answer to Exercise 5.17

1. Current price, *P=Birr 1,275,356.00*, appreciation rate *r=6.3%.* Appreciated value, $A = P(1+i)^n = 1275356(1+0.063)^5 = 1275356(1.063)^5 = 1731002.72$.

2. Current price *P=Birr 640.00*, appreciation rate *r=15%, time =5 years*. Required is the price of gold after 5 years. $A = P(1+i)^n = 640(1+0.15)^5 = 640(1.15)^5 = 1,287.27$.

3. Original value, P=Birr 34,750.00 depreciation rate, r=8%. Required is the boat's value, A after year n=5 years and n=10 years.

   $A_5 = P(1-r)^n = 34750(1-0.08)^5 = 34750(0.92)^5 = 22,903.083$ The value of the boat after five year is Birr 22,903.083. And the value of the boat further reduced to Birr 15,095 in ten years.

   $A_{10} = P(1-r)^n = 34750(1-0.08)^{10} = 34750(0.92)^{10} = 15,095.00$

4. Current price, P=Birr 950,409.00 depreciation rate, r=11% and the time n=3. Required is the price after 3 years,

   $A = P(1-r)^n = 950,409(1-0.11)^3 = 950,409(0.89)^3 = 670,008.88$ in principle the price of the car reduced to Birr 670,008.88 after three years.

Straight-Line Depreciation

Answer to Exercise 5.18

1. We use the formula

   $$Deperciation(D) = \frac{Initial\,Value - Residual\,Value}{Useful\,life\,in\ \ years}$$

   $$= \frac{621450-411050}{5} = Birr\,42,080\ per\ year$$

   To get the percentage you can take the ratio of annual depreciated and original values. That is, $Depreciation\ percent = \frac{42080}{621450} \times 100 = 6.77\%$. The answer tells us the car depreciated 6.77% of original value yearly.

2. Original price=Birr16,000.00, Useful life=8 years, Salvage value=Birr 1,500.00.

   a. $D = \frac{original\,value - Salvage\,value}{Useful\,life} = \frac{16000-1500}{8} = 1,812.50$ the refrigerator depreciates Birr 1,812.50 annually.

   b. To answer this question you can develop a table with constant annual depreciation Birr 1,812.50.

Year	Starting price	Decrease	Ending price	Cumulated depreciation
1	16000	1812.50	14187.50	1812.50
2	14187.50	1812.50	12375.00	3625.00
3	12375.00	1812.50	10562.50	5437.50
4	10562.50	1812.50	8750.00	7250.00
5	8750.00	1812.50	6937.50	9062.50
6	6937.50	1812.50	5125.00	10875.00
7	5125.00	1812.50	3312.50	12687.50
8	3312.50	1812.50	1500.00	14500.00

At the end of 4 years the refrigerator depreciates a Birr value 7,250.00 and the depreciated value is obtained after subtracting Birr 7,250 from the original price which is Birr 8750.00.

3. Using the given information, Initial cost=Birr 47,500.00, Useful life=8 years, Salvage value=Birr 7,500.00.

$$D = \frac{original\,value - Salvage\,value}{Useful\,life} = \frac{47500 - 7500}{8} = 5000\ per\ year\,.$$

The straight line depreciation summarized by the following table

Year	Starting price	Decrease	Ending price	Cumulated depreciation
1	47500.00	5000.00	42500.00	5000.00
2	42500.00	5000.00	37500.00	10000.00
3	37500.00	5000.00	32500.00	15000.00
4	32500.00	5000.00	27500.00	20000.00
5	27500.00	5000.00	22500.00	25000.00
6	22500.00	5000.00	17500.00	30000.00
7	17500.00	5000.00	12500.00	35000.00
8	12500.00	5000.00	7500.00	40000.00

5.3 Saving, Investing and Borrowing Money

Minimum Learning Competencies

- List five valid reasons for savings.
- Explain how savings become investment.
- List three saving plans.
- Identify four kinds of financial institutions.
- Describe three main factors in choosing a particular institution for saving
- Compute and solve numerical problems on saving
- Identify the four factors that should guide consumers in planning an investment strategy.
- Explain the differences between stocks and bond.
- Describe ways to invest in stock and bond.
- Compute and solve numerical problems on investment
- Describe the advantages and disadvantages of borrowing money
- Identify the usual sources of cash loan.
- Compute the amount and time on the return of loan based on the given agreement.

Answer to Activity 5.14

Discuss the reason of saving. The main reasons of saving money are:

a. **Save for emergency funds**: An emergency fund is money that you have set aside to cover any financial emergencies or unexpected expenses that may come up. Those can include anything that you haven't planned for, such as unexpected car repairs, medical bills, unemployment or other income loss, property damage, or family emergencies, etc.

b. **Save for a new car, vacations and other luxury items:** Your savings account isn't only for things you need—it can be for things you want, too. Saving up for a big purchase beforehand means you won't pay extra in finance costs such as interest and fees, the way you would if you put these purchases on credit. Perhaps you're saving for a once-in-a-lifetime vacation or trip abroad. Having an exciting goal like this can make it easier to motivate yourself to put money away.

c. **Save to Maximize Interest Rates** : Where you save your money matters, too. Use a regular savings account, high-yield savings account, money market account, savings bond, or certificate of deposit (CD) to earn interest on your savings. When interest rates go up, your yield will go up as well.

d. **Saving for retirement is essential**. When you save for retirement, you are saving for your future. When you neglect to do so, you run the risk of not being able to take care of yourself when you are older.

e. **Save for sinking funds** A sinking fund is a strategic way to save money by setting aside a little bit each month. Sinking funds work like this: Every month, you'll set money aside in one or multiple categories to be used at a later date. With a sinking fund, you save up a small amount each month for a certain block of time before you spend.

f. **Save for a Down Payment for a House:** Save money for a down payment on a house. If you can save up 20% of the purchase price, you can avoid private mortgage insurance (PMI) and receive better interest rates on a home loan. It can also reduce the amount you need to borrow, making your mortgage payments more affordable.

g. **Save for your education:** Don't neglect to save money for your education. Higher education can improve your career prospects.

In this world you can save anything which can be used in future.

Answer to Activity 5.15

Money is said to be money because

- **Money is a medium of exchange**. This means that money is widely accepted as a method of payment. When I go to the market, I am confident that the cashier will accept my payment of money. In fact, Ethiopian, paper money carries this statement: "Payable to the bearer on demand" or " ላምጨው እንዲከፈል ሕግ ያስገድዳል. This means that the Ethiopian government protects my right to pay with Ethiopian Birr.

- **Money is a store of value**. If I work today and earn Birr 250, I can hold on to the money before I spend it because it will hold its value until tomorrow, next week, or even next year. In fact, holding money is a more effective way of storing value than holding other items of value such as corn, which might rot. Although it is an efficient store of value, money is not a perfect store of value. Inflation slowly erodes the purchasing power of money over time.
- **Money is a unit of account.** You can think of money as a yardstick - the device we use to measure value in economic transactions.

Answer to Activity 5.16

Dear teacher please collect the information from the student and group the response.

Planning a saving program

Sometimes the hardest thing about saving money is just getting started. This step-by-step guide for how to save money can help you develop a simple and realistic strategy, so you can save for all your short- and long-term savings goals.

I. **Record your expense**: The first step to start saving money is to figure out how much you spend. Keep track of all your expenses—that means every coffee, household item and cash tip. Once you have your data, organize the numbers by categories, such as gas, groceries and mortgage, and total each amount. Use your credit card and bank statements to make sure you're accurate—and don't forget any.

II. **Budget for saving**: Once you have an idea of what you spend in a month, you can begin to organize your recorded expenses into a workable budget. Your budget should outline how your expenses measure up to your income—so you can plan your spending and limit overspending. Be sure to factor in expenses that occur regularly but not every month, such as car maintenance.

Tip: Include a savings category—aim to save 10 to 15 percent of your income.

III. **Find ways you can cut your spending**: If your expenses are so high that you can't save as much as you'd like, it might be time to cut back. Identify nonessentials that you can spend less on, such as entertainment and dining out. Look for ways to save on your fixed monthly expenses like television and your cell phone, too.

Here are some ideas for trimming everyday expenses:

- Use resources such as community event listings to find free or low-cost events to reduce entertainment spending.
- Cancel subscriptions and memberships you don't use—especially if they renew automatically.
- Commit to eating out only once a month and trying places that fall into the "cheap eats" category.
- Give yourself a "cooling off period": When tempted by a nonessential purchase, wait a few days. You may be glad you passed—or ready to save up for it.

IV. **Set saving goals** One of the best ways to save money is to set a goal. Start by thinking of what you might want to save for—perhaps you're getting married, planning a vacation or saving for retirement. Then figure out how much money you'll need and how long it might take you to save it.

V. **Decide on your priorities** After your expenses and income, your goals are likely to have the biggest impact on how you allocate your savings. Be sure to remember long-term goals—it's important that planning for retirement doesn't take a back seat to shorter-term needs.

VI. Pick the right tools If you're saving for short-term goals, consider using these government insured deposit accounts: Savings account, Certificate of deposit (CD), which locks in your money for a fixed period of time at a rate that is typically higher than savings accounts. For long-term goals consider: insured individual retirement accounts, which are tax-efficient savings accounts, Securities, such as stocks or mutual funds. These investment products are available through investment accounts with a broker-dealer.

VII. **Make saving automatics** Almost all banks offer automated transfers between your checking and savings accounts. You can choose when, how much and where to transfer money or even split your direct deposit so a portion of every paycheck goes directly into your savings account.

VIII. **Watch your saving grows** Review your budget and check your progress every month. Not only will this help you stick to your personal savings plan, but it also helps you identify and fix problems quickly. Understanding how to save money may even inspire you to find more ways to save and hit your goals faster.

Answer to Activity 5.17

a. Public and private company save for their retirement in the following in Ethiopia.
   - Retirement or pension fund is collected from employers and employee on a monthly basis.
   - The rate depends on the income of the individuals.
   - For a person who run private business their retirement saving is his or her wealth.

b. Down payment is common in condominium housing saving

c. Auto saving and loan means to cover some portion from saving and the remaining by loan and so on.

Savings as investment

Answer to Activity 5.18

Saved money is converted to investment through financial institutions such as banks. Banks collect saving from households and lend it to investors.

Other than saving there are different sources of capital to start a new business. Some are equity financing and debt financing. Equity means selling part of the company to others. Debt financing means taking loan from different lenders.

Answer to Activity 5.19

1) The major categories of financial institutions are central banks, commercial banks, credit unions, savings and loan associations, investment banks and companies, brokerage firms, insurance companies, and mortgage companies.
2) The three main factors to choose for saving are, scope of the bank, location advantage, benefits like interest rate, service quality.

Saving Institutions

Answer to Exercise 5.19

1. The main differences between saving institutions are listed as follows:
    - Ownership, commercial banks are owned by state like Commercial Bank of Ethiopia or by shareholders like private banks in Ethiopia. Credit unions and saving and credit associations are owned by depositors
    - Scope
    - Products
2. Certificate of deposit earns more interest income than others. In Ethiopian banking industry there is no interest on checking account.
3. Beneficiary for the depositor means it is costly for the financial institutions and vice versa. Certificate of deposit account is expensive than other types of deposits.

4. Step-by-step guide for how to save money can help you develop a simple and realistic strategy, so you can save for all your short- and long-term savings goals.

5.3.2 Investment

Answer to Activity 5.20

Yes it is an investment. The bond issuer pay's the predefined interest for bond holder. On the maturity date Mr. Tesfaye will collect the principal and the interest collectively. For GERD bond, here investors are anyone who buy bonds, and borrower (issuer of the bond) is Ethiopian Government.

Answer to Activity 5.21

Discuss with your students.

Answer to Exercise 5.20

1. The main difference between preferred and common stock is that preferred stock gives no voting rights to shareholders while common stock does. Preferred shareholders have priority over a company's income, meaning they are paid dividends before common shareholders.
2. Voting, the right to get dividends, liquidation, and preemption.
3. Number of share=3000, Dividend amount=Birr 37,560.00.

   a) $Divinded\ per\ share = \dfrac{Dividend\ amount}{number\ of\ shares} = \dfrac{37560}{3000} = Birr\,12.52$

   b) Fantu receive, $12.52 \times 1445 = Birr\,18,091.40$.

Answer to Activity 5.22

Answer to Exercise 5.21

1. The main difference between Bond and Stock is listed as follow:
   - Both have face value
   - Bond owners are not owners of the company but stock holders own part of the company on proportionate with the amount of the stock.

2. Given, face value =Birr 1,000.00, coupon rate =7.5%. Semiannual interest payment I is $I = Face\,value \times rate \times time = 1000 \times 0.075 \times 0.5 = Birr\,37.5$ In each six months the bond owner collects Birr 37.50.

3. Given, *face value=Birr 1,000.00, coupon rate =6%, bank interest rate=9%, compounding frequency, m=12.* Monthly interest payment of the bond is: $I = Face\,value \times rate \times time = 1000 \times 0.06 \times 1/12 = 5$. At the end of the month the bank paid Birr 5. But this amount again is deposited on the saving account at the rate of 5% for the next 5 years or 60 months. By using ordinary annuity concept at maturity date the owner of the bond collects the bond face value Birr 1,000.00 and Birr 377.12.

$$A = 1000 + 5 \times \frac{1.0075^{60} - 1}{0.0075} = 1000 + 5 \times \frac{1.56568 - 1}{0.0075}$$
$$= 1000 + 5 \times 75.424 = 1000 + 377.12 = 1377.12$$

4. In five years,

$$ROI = \frac{\text{Current Value of investment-cost of investment}}{\text{cost of investment}}$$
$$= \frac{3500000 - 2000000}{2000000} = 75\%$$

This means simple annual ROI is $\frac{75}{5} = 15\%$ and let i be the compound annual ROI,

$$2000000 \times (1+i)^5 = 3500000$$
$$(1+i)^5 = 1.75 \qquad ,$$
$$i = 0.1184$$

the compound annual ROI is 11.84%. In Birr the ROI is Birr 1,500,000.00.

Answer to Activity 5.23

Dear teacher please facilitate students to discuss in a group. Write student's response for each activity and check whether or not the answer is similar with the following.

a. The person may finance the business by themselves, by taking from family and relatives, taking loan from individual, taking loan from bank or selling properties and so on.

b. Source of finance are loan, own saving, from family and relatives….

c. Banks, credit unions, saving and credit associations, other organization.

d. Different sources of finance have their own advantage and disadvantage.

Answer to Exercise 5.22

1. The company needs, $F = 20,0000,000.00$.

a) The number of shares issued and sold is 10,000 which is a total of $10000 \times 1000 = Birr\,10,000,000.00$, the company will borrow Birr 10,000,000.00 from commercial bank. The total outstanding shares of the company after selling 10,000 shares is 40,000. The profit per share in this case is,

$$Earning\ per\ share(EPS) = \frac{5000000}{40000} = Birr\,125.00$$

b) When the company borrows all from the financial institution the outstanding share will stay at 30,000.00.

$Earning\ per\ share(EPS) = \dfrac{5000000}{30000} = Birr\,166.67$. Taking a loan is a better decision.

2. Advantages and disadvantages of debt and equity financing

Advantages of debt financing	Advantages of equity financing
You won't give up your business ownership	No loan to repay, Learn and gain from partners
There are tax deduction, Low interest rate may be available.	With equity financing, you might form informal partnerships with more knowledgeable or experienced individuals.
disadvantages of debt financing	disadvantages of equity financing

You must repay the lender (even if your business goes bust), You'll need collateral.	You have to give investors an ownership percentage of your company, You have to share your profits with investors, You give up some control over your company It may be more expensive than borrowing

3. Open answer
4. Common sources of loan in Ethiopia are
   - Saving and credit associations,
   - Commercial banks,
   - Families and relatives.
5. The most common form of secured loan is called a 'further advance' and is made against your home by borrowing extra on your mortgage. Unsecured loans are often more expensive and less flexible than secured loans.

5.4 Taxation

Minimum Learning Competencies

- Give name three types of activities that government performs and examples of each.
- Explain why governments collect taxes.
- Describe the basic principles of taxation.
- Describe the various kinds of taxes.
- Give their opinion about "income taxes" mean for them in relation to their future first job.
- Calculate different types of taxes based on the "rate of tax" in Ethiopia.
- Solve problems related to road traffic, tax education, customer protection, climate change, and anti-doping.

Discuss with the student how the general public think about tax.

Answer to Activity 5.24

Duty of the Federal government of Ethiopia are listed as follows:

- Allocation of resource rationally
- Distribution of income, welfare program, tax structure
- Stabilization and growth. Yes all activity cost the government.

The government sources of income are: different tax incomes, export of raw materials and industrial products from different sectors, tourism, donation, remittance and various fee.

Answer to Exercise 5.23

1. The activity of a government varies from country to country. But according to Richard Musgrave, allocation, Distribution and stabilization and growth are main activities of a government.
2. Objectives of taxation are:
   - Removal of inequalities in income and wealth:
   - Ensuring economic stability:
   - Changing people's behaviors:
   - Beneficial diversion of resources:
   - Promoting economic growth:
3. A tax is said to be direct or indirect based on its impact and incidence on the tax payers.

5.4.1 Direct Taxes

Answer to Exercise 5.24

1. Gross salary=Birr 14,500.00. By using progression method $Income\ tax = 14{,}500 = 0 \times 600 + 0.1 \times 1050 + 0.15 \times 1550 + 0.2 \times 2050 + 0.25 \times 2550 + 0.3 \times 3100 + 0.35 \times 3600 = 3{,}575.00$

Net Pay=Gross salary-income tax=14500-3575=10,925.00. Again by deduction method: Birr 14,500.00 is on tax bracket 35%.

$Income\ tax = (14500 \times 0.35) - 1500 = 3{,}575.00$. The result is the same in both cases.

2. Sources of taxable incomes are:
   - Income from employment;
   - Income from business activities;
   - Income derived by an entertainer, musician, or sports person from his personal activities;
   - Income from entrepreneurial activities carried on by a non-resident through a permanent establishment in Ethiopia;
   - Income from immovable property and appurtenances thereto, income from livestock and inventory in agriculture and forestry, and income from usufruct and other rights deriving from immovable property situated in Ethiopia;
   - Dividends distributed by a resident company;
   - Profit shares paid by a resident registered partnership;
   - Interest paid by the national, a regional or local Government or a resident of Ethiopia, or paid by a non-resident through a permanent establishment that he/she maintains in Ethiopia;
   - License fees, including lease payments, and royalties paid by a resident or paid by a nonresident through a permanent establishment that he/she maintains in Ethiopia.
3. The progression method computes income tax sequentially on each tax brackets while deduction method computes the income tax based on where the gross amount lies on the tax bracket and deduct the predefined amount.
4. Calculate income tax on both cases. The result will be summarized in the following table.

Descriptions	Initial salary	final salary	Percentage change
Salary	10,000.00	21000	110%
Income tax	2,045.00	5850	186%
net income	7,955.00	15,150.00	90%

The table tells a 110% increase on gross salary brings 186% increase of income tax for the government and a 90% increase for the employee.

Answer to Activity 5.25

The difference is found on the textbook note.

Answer to Activity 5.26

Facilitate student to discuss the issue.

Answer to Exercise 5.25

1. According to tax proclamation 979/2016, dividend income is subject to 10% tax rate. The share of a company can be expressed as in amount money or on number of shares. In this exercise it is in amount.
   a) $Dividned\ income = .25 \times 300000 = 75{,}000.00$. Birr 75,000.00 is subject to 10% dividend income tax. Which is $0.1 \times 75000 = 7{,}500.00$.
   b) Similarly, $Dividned\ income = .25 \times 100000 = 25{,}000.00$. Dividend income tax $0.1 \times 25000 = 2500.00$.
   c) Similarly, $Dividned\ income = .25 \times 450000 = 112{,}500.00$. Dividend income tax $0.1 \times 112500 = 11{,}250.00$.
2. According to tax proclamation 979/2016 , income from game of chances subject to 15% income tax rate. Kelbessa shall pay income tax of $0.15 \times 150000 = 22{,}500.00$.
3. Renting a machine is casual rental income which is subject to income tax 15% on gross income. $Gross\ income = 15000 \times 100 = 1{,}500{,}000.00$. he income tax shall be $Income\ tax\ payable = 0.15 \times 1500000 = 225{,}000.00$.

4. For banking business:
   a. The banking sector categorized in Schedule C: income from business.
   b. The rate of business income tax is 30%. The income tax is calculated in the following table

Bank	Profit before tax ETB'000	Share capital ETB'000	Income Tax ETB'000
Awash	4,823,110	8,188,948	1,446,933
Dashen	2,426,804	4,387,996	728,041
Abyssinia	2,051,544	5,182,212	615,463

   c. The amount earned by each bank is shown in fifth column as follows:

Bank	Profit before tax ETB'000	Share capital ETB'000	Income Tax ETB'000	Income after tax ETB'000
Awash	4,823,110	8,188,948	1,446,933	3,376,177
Dashen	2,426,804	4,387,996	728,041	1,698,763
Abyssinia	2,051,544	5,182,212	615,463	1,436,081

   d. Earnings per share is the amount money earned by the company for each capital invested.

Bank	Profit before tax ETB'000	Share capital ETB'000	Income Tax ETB'000	Income after tax ETB'000	Earnings per share
Awash	4,823,110	8,188,948	1,446,933	3,376,177	0.412285
Dashen	2,426,804	4,387,996	728,041	1,698,763	0.387139
Abyssinia	2,051,544	5,182,212	615,463	1,436,081	0.277117

0.412285 earnings per share means for Birr 1000.00 capital invested the shareholder get Birr 412.285. This is equivalent to 41.23% rounding to two decimal place.

5. List and classify.

Answer to Activity 5.27

This is because of many taxes are included on the price of goods and services.

Answer to Exercise 5.26

1. The total amount A paid by the suit buyer is, $A = 85{,}100$. Let x be the cost the suit without value added tax (VAT). Thus, $85{,}100 = x + 0.15x$.

$$85{,}100 = x(1.15)$$

From these, $x = \frac{85100}{1.15} = 74{,}000$. Hence, the amount of VAT payable is, $VAT = 0.15x = 0.15(74000) = 11{,}100$. The suit seller will pay Birr 11,100 for the tax authority.

2. Let x be the price of the goods sold without VAT. $x = \frac{166750}{1.15} = 145000$. The amount of VAT is, $VAT = 0.15 \times 145000 = 21750$.

3. By introducing one column in the table given you obtain the following table:

i. The complete table is given below

Item	Quantity	Unit price before VAT (in Birr)	VAT Paid	Unit price including VAT	Total	Total VAT
Laptop computers	3	36,000.00	5,400.00	41,400.00	124,200.00	16,200.00
Desktop computers	5	24,000.00	3,600.00	27,600.00	138,000.00	18,000.00
Hard disk	2	9,000.00	1,350.00	10,350.00	20,700.00	2,700.00
Mathematics reference book	10	180	27.00	207.00	2,070.00	270.00
Total					284,970.00	37,170.00

ii. The total VAT paid is Birr 37,170.00 on all items.

iii. The total price is Birr 284,970.00.

iv. The price without VAT is $284970 - 37170 = 247{,}800$. Therefore, withholding tax is $0.02 * 247800 = 4{,}956.00$.

The company will pay $284970 - 4956 = 280{,}014$.

4. Given: Purchasing price=8000, Selling price =12000. The dealer is liable on net VAT value. Which was calculated as the difference between the two VAT values.

$VAT\ on\ purchasing = 8000 \times \frac{0.15}{1.15} = 1043.48$ and $VAT\ on\ selling = 12000 \times \frac{0.15}{1.15} = 1565.22$. So the dealer is liable only on $1565.22 - 1043.48 = 521.74$.

5. The royalty tax is 5% on gross amount. So

$Royalty\ tax = 0.05 \times 1350000 = 67500$. The artist will pay Birr 67,500.00.

6. You will get the answer from the note on textbooks.

7. A "zero-rated good," the government doesn't tax its retail sale but allows credits for the value-added tax (VAT) paid on inputs. This reduces the price of a good. Governments commonly lower the tax burden on low-income households by zero rating essential goods, such as food and utilities or prescription drugs.

8. The difference between Zero rated VAT and VAT exempted is explained as follows. Ethiopian tax system applies preferential rates to some goods and services, making them either "zero rated" or "exempt." For a "zero-rated good," the government doesn't tax its retail sale but allows credits for the value-added tax (VAT) paid on inputs. If, by contrast, a good or business is "exempt," the government doesn't tax the sale of the good, but producers cannot claim a credit for the VAT they pay on inputs to produce it. Because exempting breaks the VAT's chain of credits on input purchases, it can sometimes raise prices and revenues. Hence, governments generally only use exemptions when value added is hard to define, such as with financial and insurance services.

9. Article 3 of the stump duty proclamation exhaustively lists instruments chargeable with stamp duty as follow:

i. Memorandum and articles of association of any business organization cooperative or any other form of association.

ii. award

iii. bonds

iv. warehouse bond

v. contractor agreements and memoranda thereof

vi. security deeds

vii. collective agreement

viii. contract of employment

ix. Lease, including sub-lease and transfer of similar rights.

x. natural acts

xi. power of attorney

xii. documents

Answer to Review Exercise

1. To calculate the ratio the two values must be in the same unit of measurements. Either convert kilometer to meter or vice versa. Let convert kilometer to meter. 1.8km= $1.8 \times 1000 = 1800m$. So the ratio is $1800: 900$ which is equivalent to $2: 1$.

2. Assume that in this family there are father and mother. Three daughters and one son. A total of six family members four female and two male.

   i. Female to the number of people in the family, $\frac{2}{3}$ of the family is female,

   ii. Male to female ratio is $2: 4$, which is one male for two female.

3. The share of the three partners are $1: 2: 3$. this means one of the shareholders has a share of $\frac{1}{6}$ of the company, the second one has $\frac{2}{6}$ of the company and the third one has $\frac{3}{6} = \frac{1}{2}$ of the company. Based on ownership of the company shareholders share gain and loss of the company. The income Birr 21,300 is divided as $\frac{1}{6} \times 21300 = 3550$. The first share holder earns a dividend of Birr 3,550. The second one earns Birr 7,100 and the third one earns half of the profit Birr 10,650.

4. To accomplish the job 15 workers used 28 days. For the job $15 \times 28 =$ 420 days needed. Eight days less means 20 days. The number of workers needed is $\frac{420}{20} = 21$. 21 workers needed to complete the job in 20 days by assuming the rate is the same.

5. To know the percentage, $\frac{3.12}{52} \times 100 = 6$. So Birr 3.12 is 6% of Birr 52.

6. Let x be the known number whose 8.35% is 18.37.

$0.0835 \times x = 18.37$

$$x = \frac{18.37}{0.0835} = 220$$

7. Let x be the price of the shoes without tax. So 6% tax levied on the shoe which is equal to Birr 102. Thus, $0.06x = 102$

$$x = \frac{102}{0.06} = 1700$$

The price of the shoes is Birr 1700.00.

8. $Rate\ of\ change = \frac{(final\ value - original\ value)}{original\ vlaue} = \frac{21.64-16}{16} = \frac{5.64}{16} = 0.3525.$

The rate of increase in three year is 35.25%.

9. Markup percent $= \frac{\text{markup}}{\text{selling price}} \times 100\%$.

$$25 = \frac{mark\ up}{210} \times 100$$

$$\text{mark up} = 210 \times 0.25 = 52.5.$$

Therefore, cost of the radio recorder is $210 - 52.5 = 157.5$.

10. Given, $P = 3000, r = 6\%, m = 4.\ n = 7 \times 4 = 28.\ i = \frac{0.06}{4} = 0.015$

$$A_7 = P(1+i)^{28} = 3000(1+0.015)^{28} = 3000(1.015)^{28}$$

$$\approx 3000(1.517) = 4551.67$$

The interest earned can be $4551.67 - 3000 = 1551.67$.

11. Let P the amount deposited at a time 0. $r = 6\%, m = 12.\ i = \frac{0.06}{12} = 0.005$.

$$3P = P(1+0.005)^n$$

$$3 = 1.005^n$$

By taking the natural logarithm for both sides you obtain,

$\ln 3 = n \ln 1.005$

$n = \frac{\ln 3}{\ln 1.005} = 220.27.$

Therefore, $n = 12 \times t = 220.27$

$$t = \frac{220.27}{12} = 18.35$$

It takes about 18 years to triple the deposited amount by 6% compounded interest monthly.

12. This is ordinary annuity problem. Monthly deposit $R = 230$. $n = 12 \times 3 = 36, r = 9\%$ and $i = \frac{0.09}{12} = 0.0075.$

By using the annuity formula, $A_n = R\left(\frac{(1+i)^n - 1}{i}\right)$. Substituting the given values you obtain the future value A.

$$A = 230\left(\frac{(1+0.0075)^{36} - 1}{0.0075}\right) = 230\left(\frac{1.0075^{36} - 1}{0.0075}\right) = 230 \times 41.1527$$

$$A = 9465.1247$$

13. In this problem you can evaluate the future value of the deposit as A_1 and A_2.

$$A_1 = 180\left(\frac{(1+0.005)^{12} - 1}{0.005}\right) = 180 \times \left(\frac{1.005^{12} - 1}{0.005}\right) = 180 \times 12.3357$$

$$A_1 = 2220.40$$

For the three years Ato Mohammed deposited $0.15 \times 1800 = 270$ monthly.

$$A_2 = 270\left(\frac{(1+0.005)^{36} - 1}{0.005}\right) = 270(\frac{1.005^{36} - 1}{0.005})$$

$$A_2 = 270 \times 39.3361 = 10620.75$$

Totally, Ato Mohammed will have Birr 12,841.15 at the end of four years.

14. Future value annuity factor calculated by the formula $S_{\frac{n}{i}} = \left(\frac{(1+i)^n - 1}{i}\right)$

$S_{\frac{180}{0.00625}} = \left(\frac{(1+0.00625)^{180} - 1}{0.00625}\right) = 331.11.$

15. $FV = R\left(\frac{(1+i)^n - 1}{i}\right) = 857.35\left(\frac{(1+0.105)^{20} - 1}{0.105}\right) = 51{,}981.82$

16. Given: $initial\ value = 50{,}000, residual\ value = 7{,}000,$

$Useful\ life = 8\ years.$

i. By straight line method:

$$Depreciation = \frac{initial\ value - residul\ value}{useful\ life\ in\ years} = \frac{50000-7000}{8} = 5375.$$

ii. Assume the machinery depreciate 10% annually. The depreciated value of the year can be: $Depreciation = 0.1 \times 50{,}000 = 5000.$

17. All the three cases have their own advantage and disadvantages. It depends on the performance of the company. If the company perform well and is able to pay loan without any difficulty financing, by loan is good. If the company will have a chance of loss selling, a share is suitable.

TABLE OF RANDOM SAMPLING

11164	36318	75061	37674	26320	75100	10431	20418	19228	91792
21215	91791	76831	58678	87054	31687	93205	43685	19732	08468
10438	44482	66558	37649	08882	90870	12462	41810	01806	02977
36792	26236	33266	66583	60881	97395	20461	36742	02852	50564
73944	04773	12032	51414	82384	38370	00249	80709	72605	67497
49563	12872	14063	93104	78483	72717	68714	18048	25005	04151
64208	48237	41701	73117	33242	42314	83049	21933	92813	04763
51486	72875	38605	29341	80749	80151	33835	52602	79147	08868
99756	26360	64516	17971	48478	09610	04638	17141	09227	10606
71325	55217	13015	72907	00431	45117	33827	92873	02953	85474
65285	97198	12138	53010	94601	15838	16805	61004	43516	17020
17264	57327	38224	29301	31381	38109	34976	65692	98566	29550
95639	99754	31199	92558	68368	04985	51092	37780	40261	14479
61555	76404	86210	11808	12841	45147	97438	60022	12645	62000
78137	98768	04689	87130	79225	08153	84967	64539	79493	74917
62490	99215	84987	28759	19177	14733	24550	28067	68894	38490
24216	63444	21283	07044	92729	37284	13211	37485	10415	36457
16975	95428	33226	55903	31605	43817	22250	03918	46999	98501
59138	39542	71168	57609	91510	77904	74244	50940	31553	62562
29478	59652	50414	31966	87912	87154	12944	49862	96566	48825
96155	95009	27429	72918	08457	78134	48407	26061	58754	05326
29621	66583	62966	12468	20245	14015	04014	35713	03980	03024
12639	75291	71020	17265	41598	64074	64629	63293	53307	48766
14544	37134	54714	02401	63228	26831	19386	15457	17999	18306
83403	88827	09834	11333	68431	31706	26652	04711	34593	22561
67642	05204	30697	44806	96989	68403	85621	45556	35434	09532
64041	99011	14610	40273	09482	62864	01573	82274	81446	32477
17048	94523	97444	59904	16936	39384	97551	09620	63932	03091
93039	89416	52795	10631	09728	68202	20963	02477	55494	39563
82244	34392	96607	17220	51984	10753	76272	50985	97593	34320

96990	55244	70693	25255	40029	23289	48819	07159	60172	81697
09119	74803	97303	88701	51380	73143	98251	78635	27556	20712
57666	41204	47589	78364	38266	94393	70713	53388	79865	92069
46492	61594	26729	58272	81754	14648	77210	12923	53712	87771
08433	19172	08320	20839	13715	10597	17234	39355	74816	03363
10011	75004	86054	41190	10061	19660	03500	68412	57812	57929
92420	65431	16530	05547	10683	88102	30176	84750	10115	69220
35542	55865	07304	47010	43233	57022	52161	82976	47981	46588
86595	26247	18552	29491	33712	32285	64844	69395	41387	87195
72115	34985	58036	99137	47482	06204	24138	24272	16196	04393
07428	58863	96023	88936	51343	70958	96768	74317	27176	29600
35379	27922	28906	55013	26937	48174	04197	36074	65315	12537
10982	22807	10920	26299	23593	64629	57801	10437	43965	15344
90127	33341	77806	12446	15444	49244	47277	11346	15884	28131
63002	12990	23510	68774	48983	20481	59815	67248	17076	78910
40779	86382	48454	65269	91239	45989	45389	54847	77919	41105
43216	12608	18167	84631	94058	82458	15139	76856	86019	47928
96167	64375	74108	93643	09204	98855	59051	56492	11933	64958
70975	62693	35684	72607	23026	37004	32989	24843	01128	74658
			85812	61875	23570	75754			