Robustness of Synchrony in Complex Networks, Generalized Kirchhoff Indices and Resistance Centralities

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We define measures to assess the excursion from the operational state of power grids due to perturbations. We relate these measures to new topological indices and centralities related to the resistance distance.

1: Swing Equations

The transient dynamics of a power grid is governed by the Swing equations

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j \Big(b_{ij}|V_i||V_j|\sin(\theta_i-\theta_j) + g_{ij}|V_i|[|V_i| - |V_j|\cos(\theta_i-\theta_j)]\Big)\,, \qquad (1)$$

- m_i, d_i the inertia and the damping at node i.
- $V_j = |V_i|e^{i\theta_j}$ complex voltage at node j.
- $P_i \geq 0\,(\leq 0)$ injected (consumed) active power at node i.
- b_{ij} and g_{ij} susceptance and conductance of the line connecting nodes i and j.

Working assumptions

- Consider PV-nodes and neglect voltage fluctuations $|V_i| = 1$.
- High voltage AC transmission lines have $g_{ij}/b_{ij} \approx 5\% - 10\%$.
  Neglecting the conductance (lossless approximation), the active power flow on a line is $P_{ij} = b_{ij}\sin(\theta_i - \theta_j)$.
  In this limit, analogy with the DC Josephson current between superconducting islands.

The considered model

$$m_i\ddot{\theta} + d_i\dot{\theta}_i = P_i - \sum_j b_{ij}\sin(\theta_i - \theta_j)\,. \qquad (2)$$

- Operational states of the power grid are stable fixed points of Eq. (2).
- In the following, we assume homogeneous damping and inertia, $d_i = d$, $m_i = m$ with $d/m = \gamma$.
- For $m = 0$, we recover the celebrated Kuramoto model.

2: Linearization around a stable fixed point

We consider the power grid at a stable fixed point $\{\theta_i^{(0)}\}$ for the injections $\{P_i^{(0)}\}$. Adding a perturbation in the injections such that $P_i(t) = P_i^{(0)} + \delta P(t)$, phases become time dependent $\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$ and we get the vectorial equation,

$$m\delta\ddot{\boldsymbol{\theta}} + d\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{P} - \mathbb{L}(\{\theta_i^{(0)}\})\,\delta\boldsymbol{\theta}\,, \qquad (3)$$

where we introduced the weighted Laplacian matrix $\mathbb{L}(\{\theta_i^{(0)}\})$ with matrix elements

$$\mathbb{L}_{ij} = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,,\\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,. \end{cases} \qquad (4)$$

$\mathbb{L}(\{\theta_i^{(0)}\})$ has a single eigenvalue $\lambda_1 = 0$ with eigenvector $\mathbf{u}_1 = (1,1,1,...1)/\sqrt{n}$, and $\lambda_i > 0$, $i = 2,3,...n$. Eq. (3) can be solved by expanding the angle deviation over the eigenstates of $\mathbb{L}(\{\theta_i^{(0)}\})$,

$$\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha\,, \qquad (5)$$

with

$$c_\alpha(t) = e^{\frac{-\gamma-\Gamma_\alpha}{2}t}\int_0^t e^{\Gamma_\alpha t'}\int_0^{t'}\delta\boldsymbol{P}(t'')\cdot\mathbf{u}_\alpha e^{\frac{\gamma-\Gamma_\alpha}{2}t''}\mathrm{d}t''\mathrm{d}t'\,, \qquad (6)$$

and $\Gamma_\alpha = \sqrt{\gamma^2 - 4\lambda_\alpha/m}$.

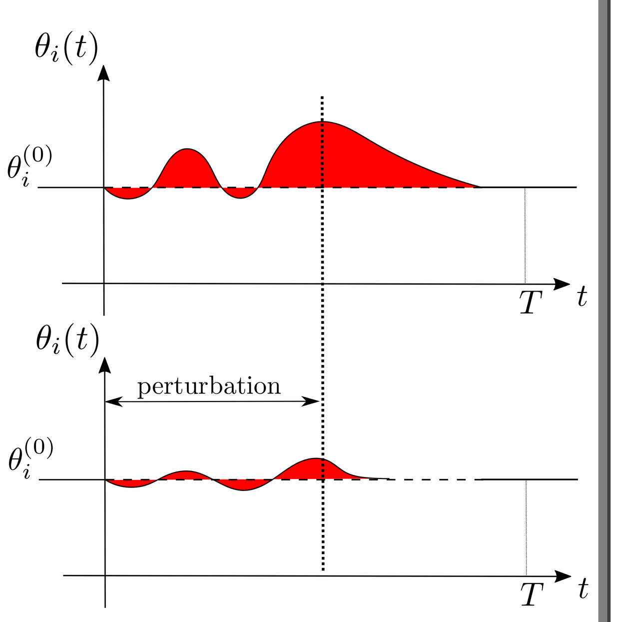
3: Fragility Performance Measures

To assess the magnitude of this excursion in the spirit of Ref. [3] we consider two fragility performance measures

$$\mathcal{P}_1(T) = T^{-1}\sum_i\int_0^T|\delta\theta_i(t) - \Delta(t)|^2\mathrm{d}t\,,$$

$$\mathcal{P}_2(T) = T^{-1}\sum_i\int_0^T|\delta\dot{\theta}_i(t) - \dot{\Delta}(t)|^2\mathrm{d}t\,.$$

with $\Delta(t) = n^{-1}\sum_j\delta\theta_j(t)$ and $\dot{\Delta}(t) = n^{-1}\sum_j\delta\dot{\theta}_j(t)$. In power grids $\mathcal{P}_1(T)$ is known as the coherence of the operational state and $\mathcal{P}_2(T)$ is proportional to the primary control effort. Below we consider $\mathcal{P}_{1,2} = \mathcal{P}_{1,2}(T\to\infty)$.



4: Generalized Kirchhoff Indices and Resistance Centralities

The Kirchhoff index originally followed from the definition of the resistance distance in a graph [4]. To a connected graph, one associates an electrical network where each edge is a resistor given by the inverse edge weight in the original graph. The resistance distance is the resistance Ω_{ij} between any two nodes i and j on the electrical network. The Kirchhoff index is then defined as [4, 5, 6]

$$Kf_1 \equiv \sum_{i<j}\Omega_{ij} = n\sum_{\alpha\geq 2}\lambda_\alpha^{-1}\,, \qquad (8)$$

where the first sum runs over all pairs of nodes in the graph and $\{\lambda_\alpha\}$ is the spectrum of the graph Laplacian $\mathbb{L}$. Up to a normalization prefactor, Kf_1 gives the the mean resistance distance $\overline{\Omega}$ over the whole graph. Higher moments of $\{\Omega_{ij}\}$ are encoded in generalized Kirchhoff indices Kf_p which we define as

$$Kf_p = n\sum_{\alpha\geq 2}\lambda_\alpha^{-p}\,, \qquad (9)$$

for integers p. Using resistance distance, we can define the resistance centrality of node i,

$$C_1(i) = \Big[n^{-1}\sum_j\Omega_{ij}\Big]^{-1} = \Big[\sum_{\alpha\geq 2}\frac{u_{\alpha,i}^2}{\lambda_\alpha} + n^{-2}Kf_1\Big]^{-1}\,,$$

which we generalize to,

$$C_p(i) = \Big[n^{-1}\sum_j\Omega_{ij}^{(p)}\Big]^{-1} = \Big[\sum_{\alpha\geq 2}\frac{u_{\alpha,i}^2}{\lambda_\alpha^p} + n^{-2}Kf_p\Big]^{-1}\,,$$

where $\Omega_{ij}^{(p)}$ is the resistance distance between node i and j in a graph whose Laplacian is $\mathbb{L}^{(p)} = \mathbb{L}^p$. Below we show that $\mathcal{P}_{1,2}$ can be expressed as linear combinations of the Kf_p's and C_p's corresponding to $\mathbb{L}$ in Eq. (4).

5: Global Averaged Robustness and Kf_p's, Local Specific Vulnerabilities and C_p's

Fragility measures $\mathcal{P}_1$ and $\mathcal{P}_2$ can be computed for various types of $\delta\boldsymbol{P}(t)$ [1, 2]. Here we show results for a noisy time-correlated perturbation with second moment $\overline{\delta P_i(t_1)\,\delta P_j(t_2)} = \delta_{ij}\delta P_{0i}^2\exp[-|t_1 - t_2|/\tau_0]$ and vanishing average. We have,

$$\mathcal{P}_1 = \sum_{\alpha\geq 2}\frac{\sum_i\delta P_{0i}^2u_{\alpha,i}^2(\tau_0 + m/d)}{\lambda_\alpha(\lambda_\alpha\tau_0 + d + m/\tau_0)}\,,$$

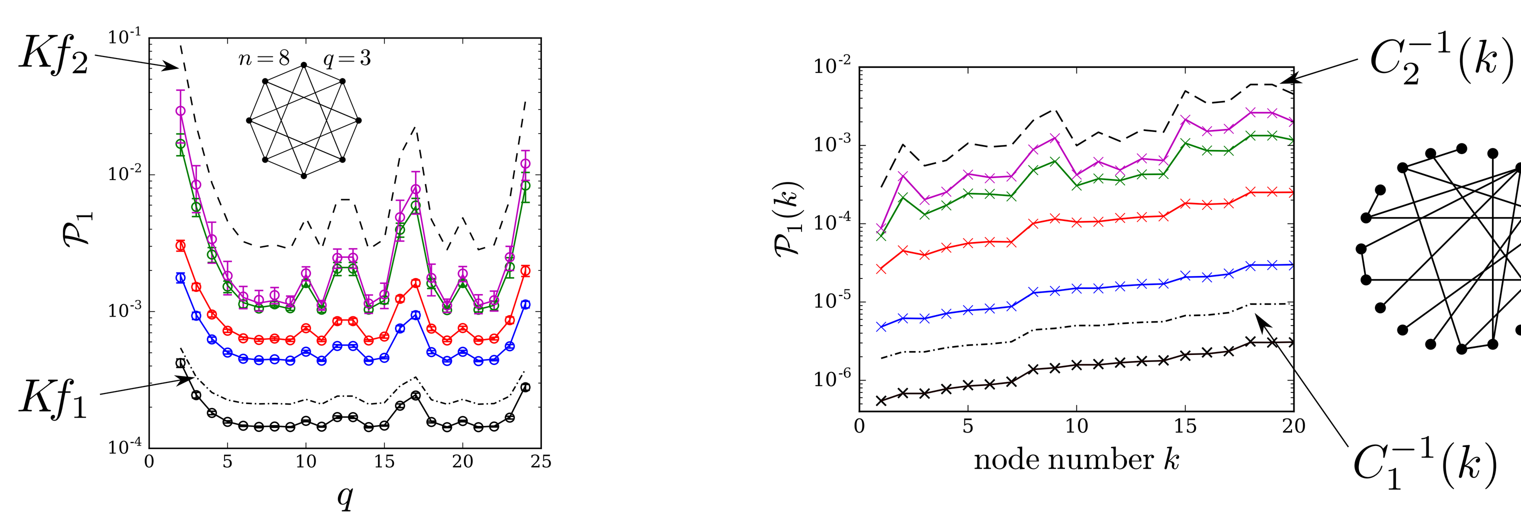
$$\mathcal{P}_2 = \sum_{\alpha\geq 2}\frac{\sum_i\delta P_{0i}^2u_{\alpha,i}^2}{d(\lambda_\alpha\tau_0 + d + m/\tau_0)}\,.$$

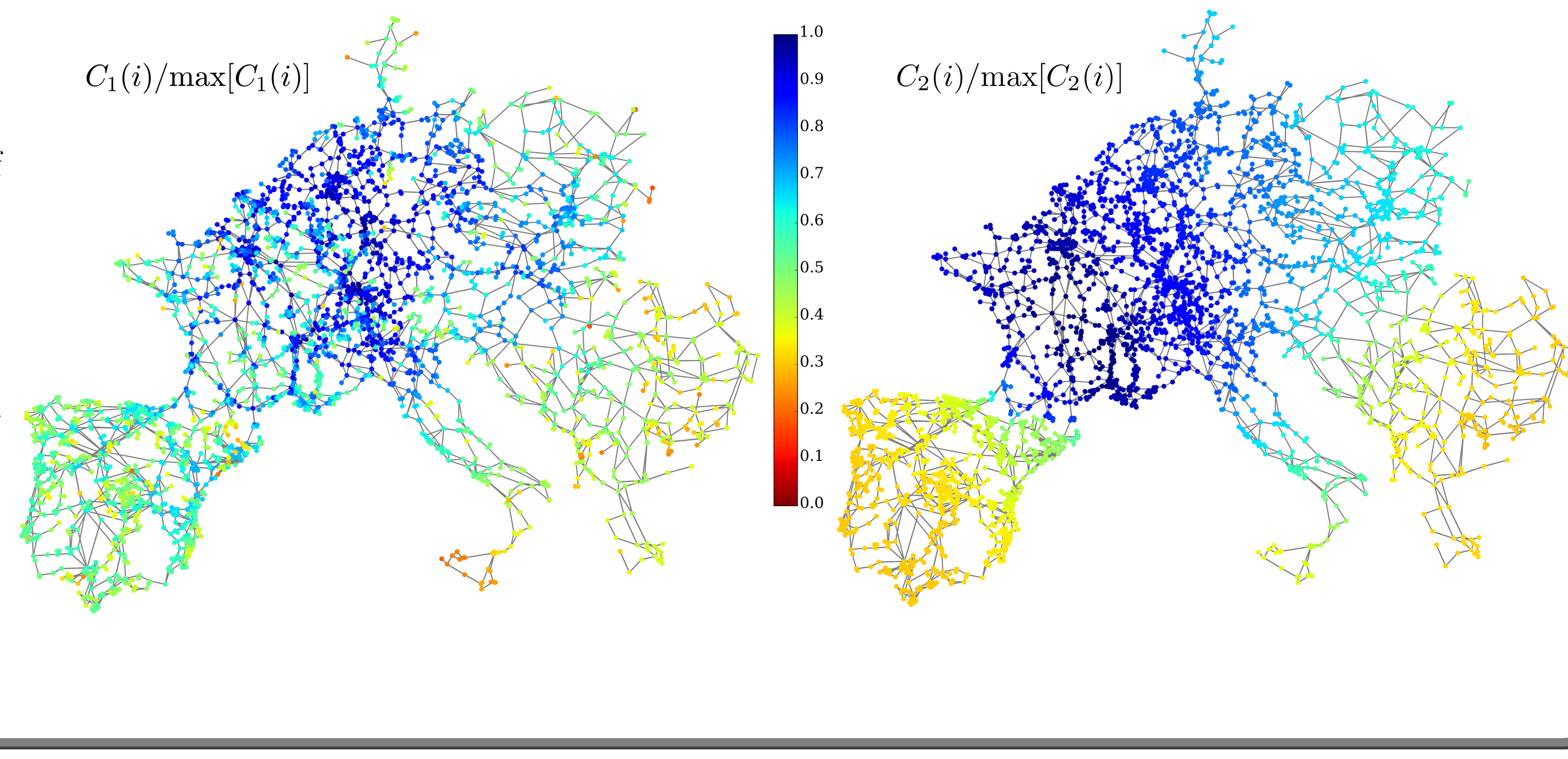
- **Global Robustness [1]** : Averaging over an ensemble of perturbation vectors,

$$\langle\mathcal{P}_1\rangle = \begin{cases}\langle\delta P_0^2\rangle\tau_0\,Kf_1/nd\,, & \tau_0 \ll d/\lambda_\alpha, \gamma^{-1}\,,\\ \langle\delta P_0^2\rangle\,Kf_2/n\,, & \tau_0 \gg d/\lambda_\alpha, \gamma^{-1}\,.\end{cases}$$

- **Local Vulnerabilities [2]** : Choosing a specific perturbation vector acting on node k,

$$\mathcal{P}_1 = \begin{cases}\frac{\delta P_0^2\tau_0}{d}\Big(C_1^{-1}(k) - n^{-2}Kf_1\Big)\,, & \tau_0 \ll d/\lambda_\alpha, \gamma^{-1}\,,\\ \delta P_0^2\Big(C_2^{-1}(k) - n^{-2}Kf_2\Big)\,, & \tau_0 \gg d/\lambda_\alpha, \gamma^{-1}\,.\end{cases}$$





6: Conclusion

- Global averaged robustness determined by topological indices Kf_p's.
- Local specific vulnerabilities determined by resistance centralities C_p's.
- Kf_p's and C_p's are easy to compute as they only require the eigenvalues and eigenmodes of the weighted Laplacian $\mathbb{L}$.

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