A Laplacian approach to stubborn agents and their role in opinion formation on influence networks

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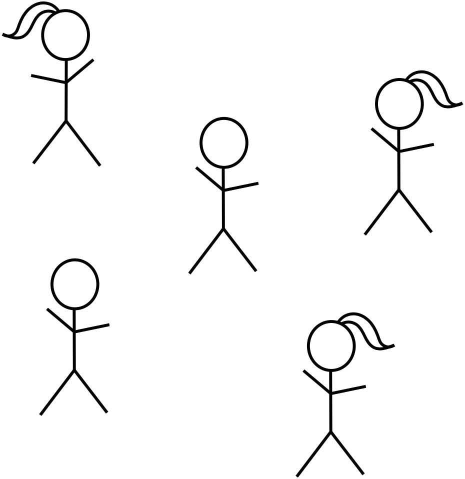
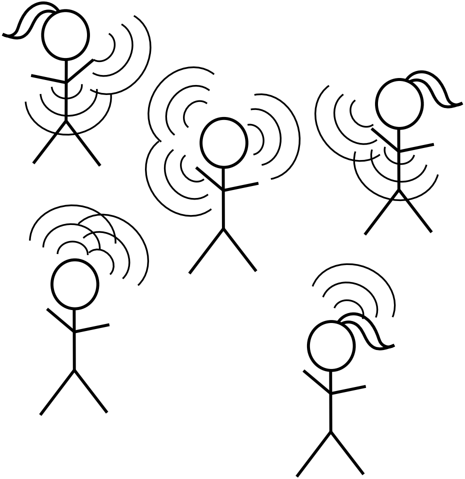
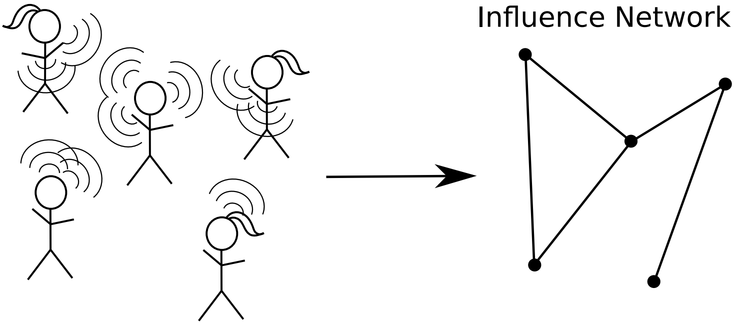
Opinion Formation on Influence Networks



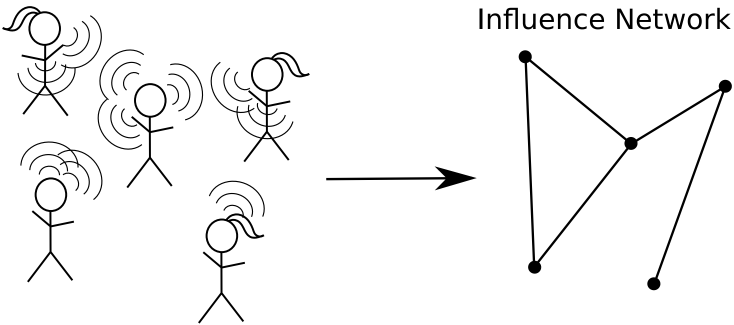
Opinion Formation on Influence Networks



Opinion Formation on Influence Networks



Modeling Opinion Formation on Influence Networks

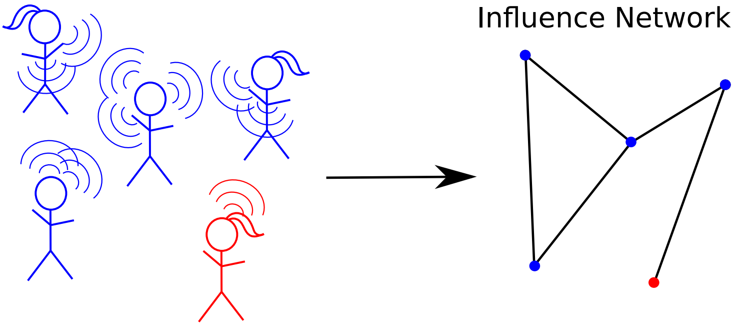


Opinion Dynamics (Taylor model 1968)

$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) - \kappa(x_i - P_i) \qquad (1)$$

M. Taylor, *Human Relations* **21** (1968).

Modeling Opinion Formation on Influence Networks

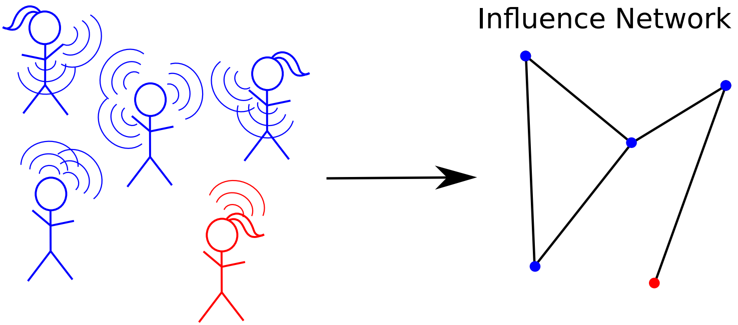


Opinion Dynamics (Taylor model 1968)

$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) - \kappa(x_i - P_i)\,, \quad i \in V_s\,, \tag{2}$$

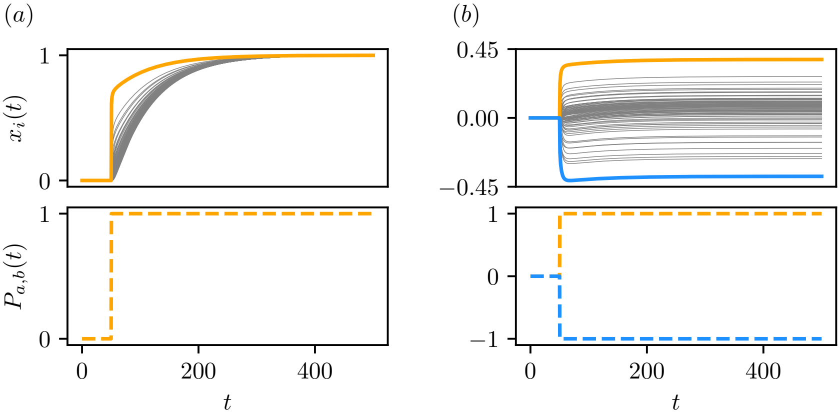
$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j)\,, \quad i \in V_f\,, \tag{3}$$

Opinion Formation on Influence Networks



- Single stubborn agent: Which node is the most efficient to change the overall opinion?
- Pair of stubborn agents: How does the network structure relates to the distribution of final opinions?

Two Cases of Interest



Single stubborn agent

Pair of stubborn agents

Laplacian Approach

Laplacian approach

$$\dot{\mathbf{x}} = -(\mathbb{L} + K)\,\mathbf{x} + \kappa \mathbf{P}\ . \tag{4}$$

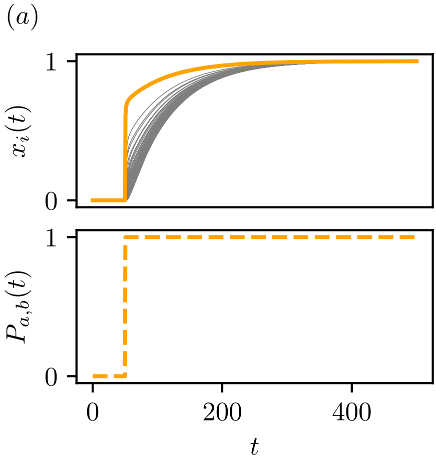
Modified Laplacian matrix

$$\mathbb{L}^{\kappa} = \mathbb{L} + K \tag{5}$$

$$\mathbb{L}_{ij} = \begin{cases} -b_{ij}\,, & i \neq j\,, \\ \sum_k b_{ik}\,, & i = j\,. \end{cases} \qquad K_{ij} = \begin{cases} \kappa\,, & i = j \in V_s\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

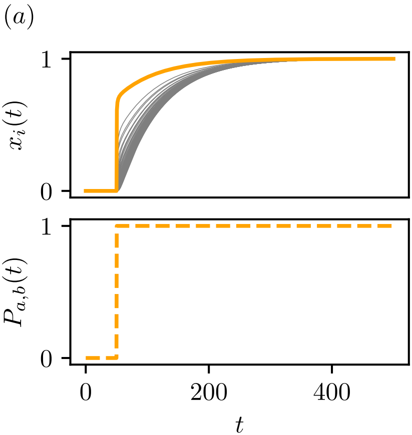
Eigenvectors: $u_{\alpha,i}^{\kappa}$.
Eigenvalues: $\lambda_{\alpha}^{\kappa}$.

Single Stubborn Agent



Effort vs. network structure for a single stubborn agent?

Single Stubborn Agent: Coherence



Coherence

$$\mathcal{C}(a) = \sum_i \int_0^\infty |x_a(t) - x_i(t)| \mathrm{d}t \quad (6)$$

Effort vs. network structure for a single stubborn agent?

Single Stubborn Agent: Coherence

Solutions for trajectories

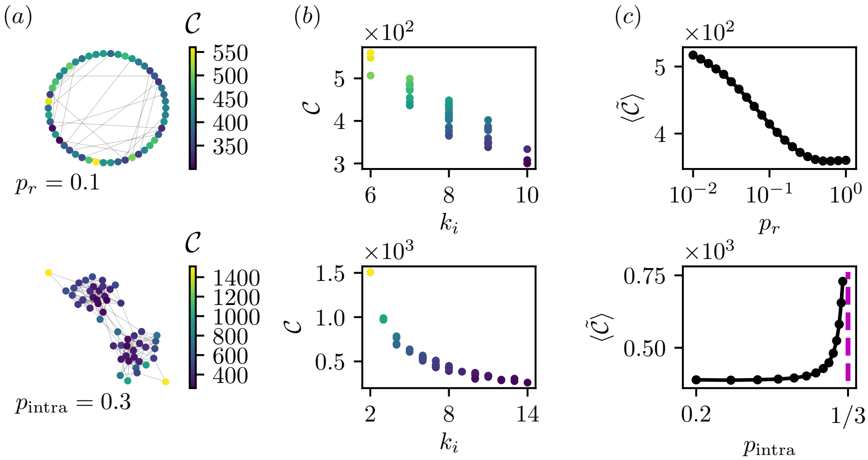
$$x_i(t) = \kappa P \sum_\alpha \frac{u^\kappa_{\alpha,a}}{\lambda^\kappa_\alpha}\left(1 - e^{-\lambda^\kappa_\alpha (t-t_0)}\right) u^\kappa_{\alpha,i} \ \ , \tag{7}$$

$$x_i(t\to\infty) = \kappa P \underbrace{\sum_\alpha \frac{u^\kappa_{\alpha,a} u^\kappa_{\alpha,i}}{\lambda^\kappa_\alpha}}_{=1/\kappa} = P \ , \tag{8}$$

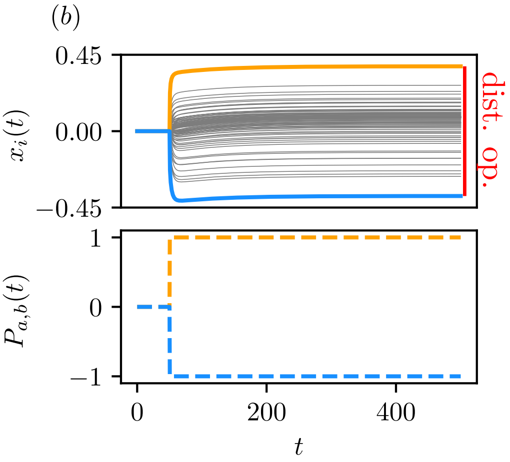
which ensures the emergence of a new consensus at $x_i(t\to\infty) = P$ as final state of the system. Integrating Eq. (7), $\mathcal{C}$ can be expressed as

$$\mathcal{C}(a) = -\kappa P \sum_i \sum_\alpha \frac{{u^\kappa_{\alpha,a}}^2 - u^\kappa_{\alpha,a} u^\kappa_{\alpha,i}}{{\lambda^\kappa_\alpha}^2} \ . \tag{9}$$

Single Stubborn Agent: Coherence



Pair of Stubborn Agents



Final opinions distribution vs. network structure for a pair of stubborn agents?

Pair of Stubborn Agents

Solutions for trajectories

$$x_i(t) = \kappa P \sum_\alpha \frac{u^\kappa_{\alpha,a} - u^\kappa_{\alpha,b}}{\lambda^\kappa_\alpha}(1 - e^{-\lambda^\kappa_\alpha(t-t_0)})u^\kappa_{\alpha,i} \quad , t > t_0 \, . \tag{10}$$

$$x_i^\infty = \kappa P \sum_\alpha \frac{u^\kappa_{\alpha,a} - u^\kappa_{\alpha,b}}{\lambda^\kappa_\alpha} u^\kappa_{\alpha,i} . \tag{11}$$

Using a complex network distance

$$x_i^\infty = \frac{\kappa P}{2}[\Omega^{(\kappa,1)}_{bi}(\{a,b\}) - \Omega^{(\kappa,1)}_{ai}(\{a,b\})] \tag{12}$$

Modified Resistance distances

Resistance Distance

$$\Omega_{ij} = {\mathbb{L}_{ii}}^{-1} + {\mathbb{L}_{jj}}^{-1} - {\mathbb{L}_{ij}}^{-1} - {\mathbb{L}_{ji}}^{-1} \,. \tag{13}$$

Modified Resistance Distance (MRD)

$$\Omega_{ij}^{(\kappa,p)}(V_s) = [\mathbb{L}^{\kappa}]_{ii}^{-p} + [\mathbb{L}^{\kappa}]_{jj}^{-p} - [\mathbb{L}^{\kappa}]_{ij}^{-p} - [\mathbb{L}^{\kappa}]_{ji}^{-p} \,, \tag{14}$$

$$= \sum_{\alpha} \frac{(u_{\alpha,i}^{\kappa} - u_{\alpha,j}^{\kappa})^2}{{\lambda_{\alpha}^{\kappa}}^{p}} \,. \tag{15}$$

Closeness centrality

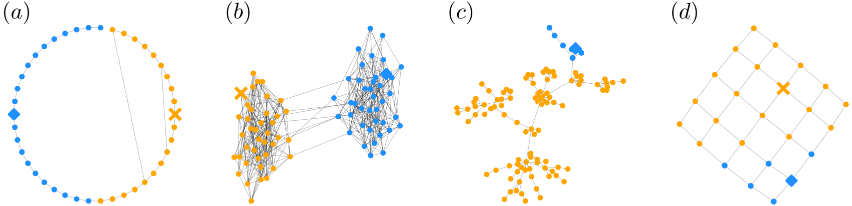
$$C_p(i, V_s) = \left[n^{-1} \sum_j \Omega_{ij}^{(\kappa,p)}(V_s) \right]^{-1} \,, \tag{16}$$

Klein and Randìc *J. Math. Chem.* **12**, 81 (1993).

MT, Pagnier, Jacquod *Sci. Adv.* **11**(5) eaaw8359 (2019).

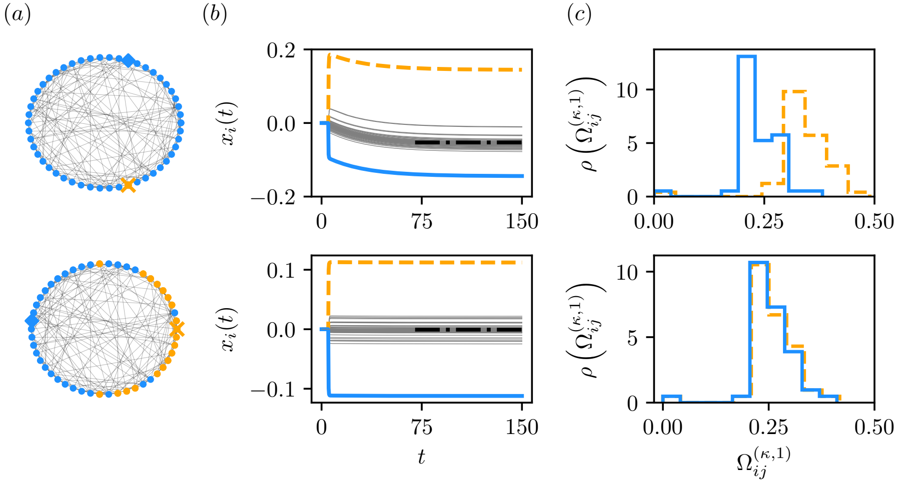
Opinion association $x_i^\infty > 0 \to$ agent a, $x_i^\infty < 0 \to$ agent b.

$$x_i^\infty = \frac{\kappa P}{2}[\Omega_{bi}^{(\kappa,1)}(\{a,b\}) - \Omega_{ai}^{(\kappa,1)}(\{a,b\})] \tag{17}$$

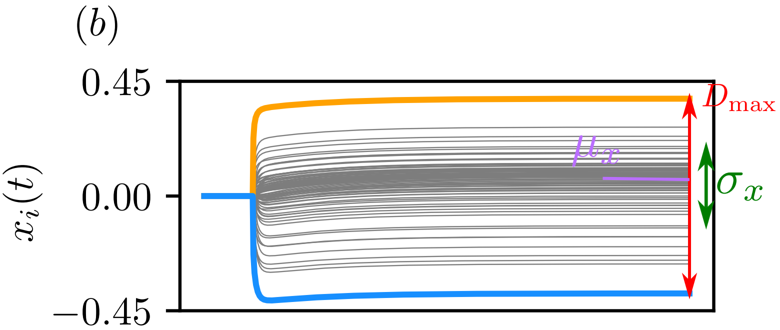


Pair of Stubborn Agents: Opinion Association

Opinion association From modified resistance distances.



Opinion heterogeneity



Opinion heterogeneity

$$D_{\max}(\{a,b\}) = \kappa P \sum_{\alpha} \frac{(u^{\kappa}_{\alpha,a} - u^{\kappa}_{\alpha,b})^2}{\lambda^{\kappa}_{\alpha}} = \kappa P \Omega^{(\kappa,1)}_{ab}. \tag{18}$$

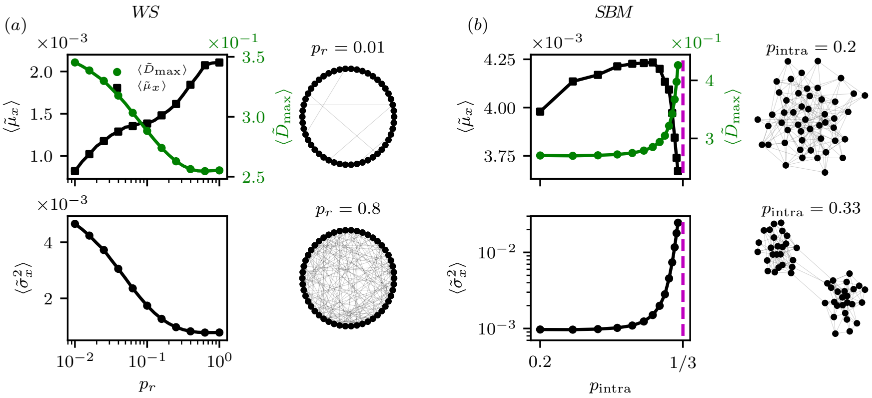
$$\mu_x(\{a,b\}) = \frac{\kappa P}{2}\left[C_1^{-1}(b) - C_1^{-1}(a)\right] . \tag{19}$$

$$\sigma_x^2(\{a,b\}) = \left(\frac{\kappa P}{2}\right)^2 \left(\frac{4\Omega^{(\kappa,2)}_{ab}}{n} - \left[C_1^{-1}(b) - C_1^{-1}(a)\right]^2\right), \tag{20}$$

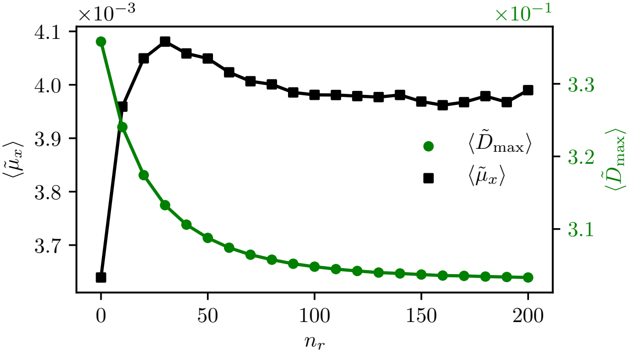
with

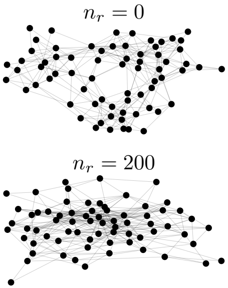
$$\Omega^{(\kappa,2)}_{ab} = \frac{1}{4}\sum_{i}\left(\Omega^{(\kappa,1)}_{bi} - \Omega^{(\kappa,1)}_{ai}\right)^2 \tag{21}$$

Opinion heterogeneity



Opinion heterogeneity





Conclusion

Laplacian approach to relate network structure to opinion formation

Single stubborn agent

- Coherence during a change of consensus given by network spectral properties.

Pair of stubborn agents

- Opinion association in terms of MRDs.
- Distribution of the final opinions as functions of MRDs.

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Opinion heterogeneity