# Detecting Majorana Fermions through Persistent Currents in Normal-Superconducting Rings

Philippe Jacquod

EPFL - March 2014

# Outline

Symmetries (i) conservation laws
(ii) degeneracies
(iii) universality classes

Symmetry classes and classification of topological states
-topological insulators and superconductors
-Z and Z2 topological invariants

Majoranas : Why ? How ?

Detecting Majoranas with persistent currents

Final remarks : reciprocity, symmetry of the free energy
and Friedel sum rule

# Continuous symmetries and conservation laws

Noether's theorem

"If a system has a continuous symmetry, then there are corresponding quantities whose values are conserved in time."

"To every differentiable symmetry generated by local actions corresponds a conserved current."

Ex.: spatial translational symmetry -> momentum conservation
rotational symmetry -> angular momentum conservation
time translational symmetry -> energy conservation

....

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 \rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{dp_i}{dt} = 0$$

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time translational symmetry -> energy conservation
....

In quantum mechanics : symmetries as **unitary operators -> $U^{+}HU=H$**
Ex.: spatial translational symmetry -> $U = \exp[-i\, P\, x]$
rotational symmetry -> $U = \exp[-i\, J\, \phi]$
....

# Discrete symmetries and degeneracies

C : charge conjugation, q -> -q

P : parity, (x,y,z) -> -(x,y,z)

T : time reversal, t -> -t

PH : Particle <-> Hole symmetry (with SC)

SLS : Chiral / sublattice symmetry (Dirac, graphene...)

Ex.: time-reversal symmetry **-> twofold "Kramers" degeneracy**

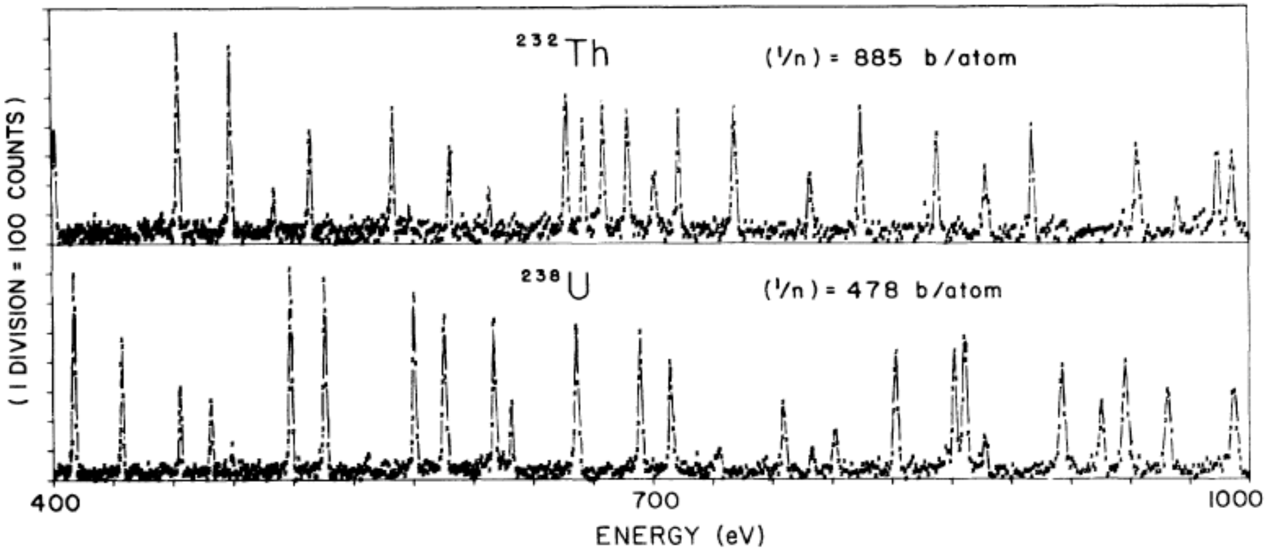
(i) no spin : invert momentum -> complex conjugation, T = -i K, $\mathbf{T^2=1}$

(ii) spin 1/2 : invert momentum and spin -> T = -i $\sigma^y$ K, $\mathbf{T^2=-1}$

!! T and PH are **antiunitary operators** :

- $T[a\,|\phi_1\rangle + b\,|\phi_2\rangle] = a^*\,|T\phi_1\rangle + b^*\,|T\phi_2\rangle$
- $|\langle T\phi_2\,|T\phi_1\rangle| = |\langle\phi_2\,|\phi_1\rangle|$

# Random Matrix Theory: (i) Wigner




Problem : excitation spectrum of heavy nuclei
many-body problem; do not know Hamiltonian

Solution : write Hamiltonian as random matrix

Example : $\langle H_{ij} \rangle = 0$, $P(H) = \exp\{-\beta\ \mathrm{Tr}[H^2]\}$ ~ Gaussian ensembles

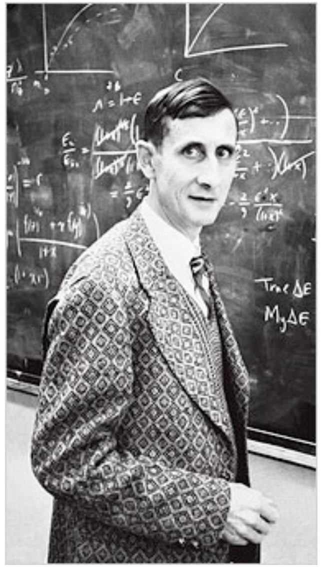
ask that the ensemble is stationary under unitary transformation $H \to H' = U^{+} H U$

# Random Matrix Theory: (ii) Dyson's 3-fold way

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER 1962

**The Threefold Way.**
**Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics**

FREEMAN J. DYSON



"The irreducible representations of a group by unitary matrices fall into three classes (...) real, complex and pseudoreal (quaternionic)."

-> three ensembles that are stationary under

$$H \to H' = U^+ H U$$

$$P(H) = \exp\{-\beta \, \mathrm{Tr}[H^2]\}$$

$\beta=1$: orthogonal ensemble of real symmetric H

$\beta=2$: unitary ensemble of complex hermitian H

$\beta=4$: symplectic ensemble of real quaternionic H

# Random Matrix Theory: (ii) Dyson 3-fold way

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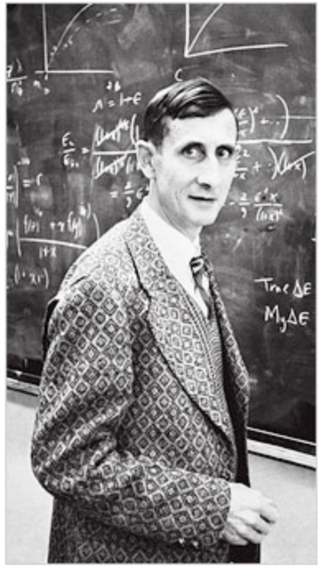


- Extend the theory to unitary matrices S
- Three ensembles defined by invariance under

| | |
|---|---|
| $S \to U^T S U$ | $\beta$=1: Circular orthogonal ensemble |
| $S \to U S V$ | $\beta$=2: Circular unitary ensemble |
| $S \to W^R S W$ | $\beta$=4: Circular symplectic ensemble |

U, V are arbitrary unitary matrices, W is a real quaternionic unitary matrix (symplectic), $W^R = \sigma^y W^T \sigma^y$ is the dual of W

# Random Matrix Theory: (iii) Chiral symmetry

Chiral symmetric Hamiltonian operator

$$H = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \qquad H = -\sigma^z H \sigma^z$$

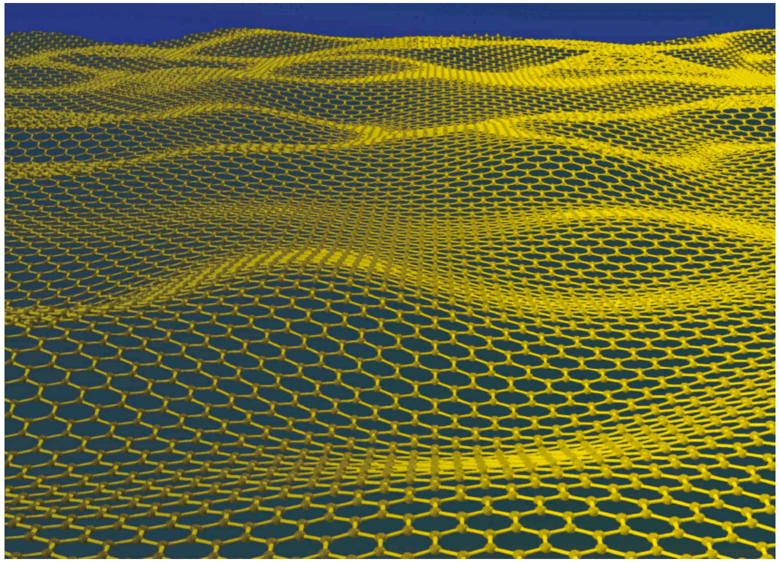
J Verbaarschot

-> three new ensemble of RMT (also vs. TRS/SRS or $T^2$=-1,0,1)

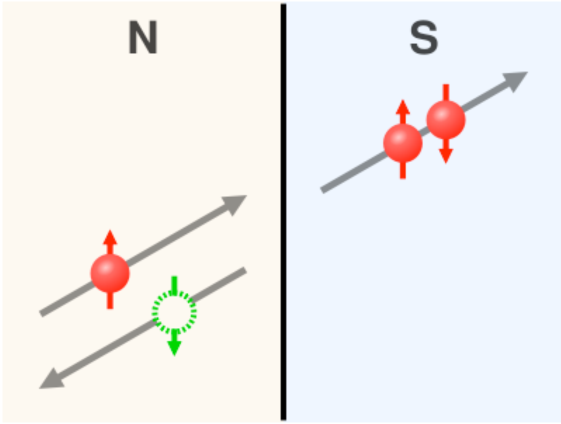
Examples :

QCD Dirac operator

Lattice models with sublattice symmetry,
e.g. graphene



# Random Matrix Theory: (iv) Particle-hole symmetry




- Quantum coherent metal (N) in contact with a superconductor (S)
- Andreev reflection of electron into hole
- Fermi energy of S sets E=0 -> breaking of energy translational symmetry

-> Four new ensembles of Bogoliubov-de Gennes H

$$H = \begin{pmatrix} h - \mu_{\rm sc} & \Delta \\ \Delta^* & \mu_{\rm sc} - \sigma^y h^* \sigma^y \end{pmatrix}$$

defined by presence/absence of TRS and/or SRS

- Particle (E>0) - hole (E<0) symmetry

$$H = -PHP^{-1} \qquad P = -i\sigma^y \tau^y K$$

spin space

Nambu space

M Zirnbauer

A Altland

# Symmetry classes - the 10-fold way

"Dyson's 3-fold way + particle-hole symmetry"

Time-reversal symmetry
Particle-hole symmetry

↔

Antiunitary symmetries
$P^2, T^2 = -1, 0, 1$

3x3=9 and two possibilities for P=T=0 -> 10-fold way

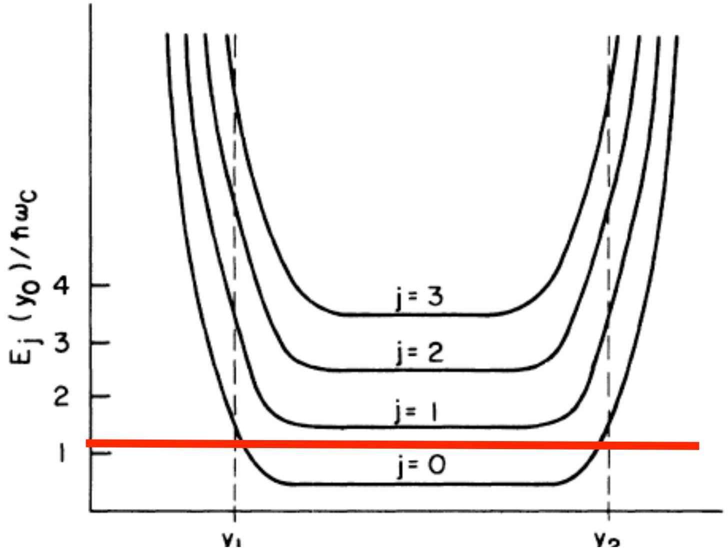
| Symmetry class | | TRS | PHS | SLS |
|---|---|---|---|---|
| Wigner-Dyson | A (unitary) | 0 | 0 | 0 |
| | AI (orthog.) | +1 | 0 | 0 |
| | AII (sympl.) | −1 | 0 | 0 |
| Chiral | AIII (unitary) | 0 | 0 | 1 |
| | BDI (orthog.) | +1 | +1 | 1 |
| | CII (sympl.) | −1 | −1 | 1 |
| Altland-Zirnbauer | D | 0 | +1 | 0 |
| | C | 0 | −1 | 0 |
| | DIII | −1 | +1 | 1 |
| | CI | +1 | −1 | 1 |

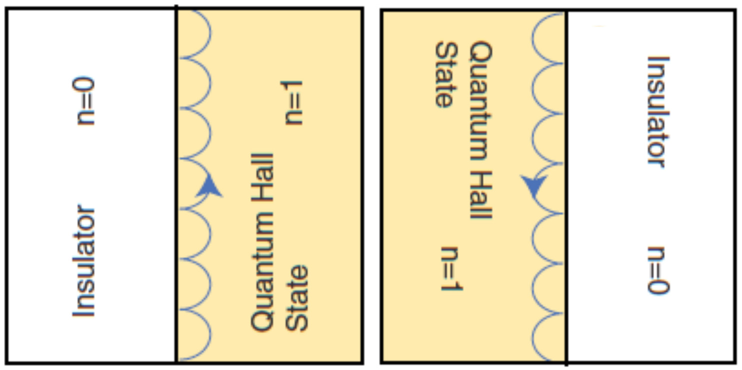
# Topological phase of matter : the integer quantum hall effect

## Chiral edge states and the IQHE

(Halperin '82, Büttiker '88)

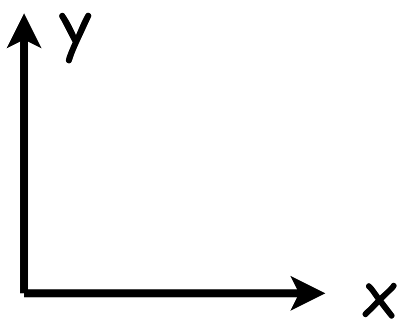
~2D electrons in magnetic field + confining potential




- Landau levels "move up" at edge
- Number of edge states fixed by chemical potential
- Not sensitive to perturbation
- Velocity of edge states $v_y \sim d_x V$

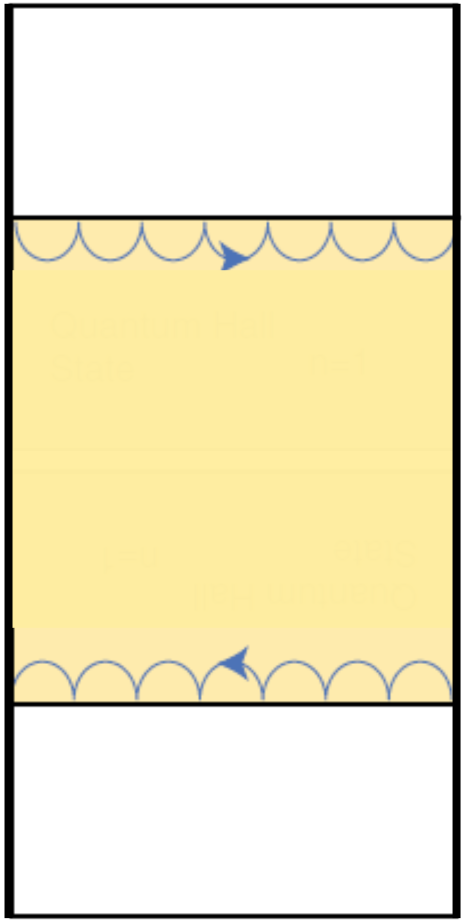
  -> states are chiral (broken TRS)



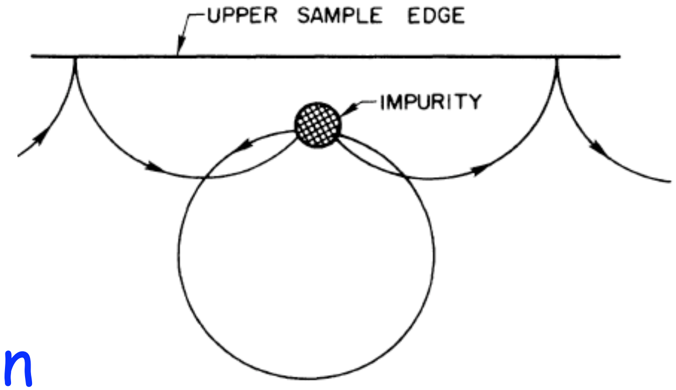
# Topological phase of matter : the integer quantum hall effect

## Chiral edge states and the IQHE

(Halperin '82, Büttiker '88)



Edge states do not backscatter !
(where can they go ?)




-> perfect transmission

-> Hall conductance quantized as $G=n\ 2e^2/h$

-> n=0,1,2,3,...: Z-topological insulator

topological # = # of edge states

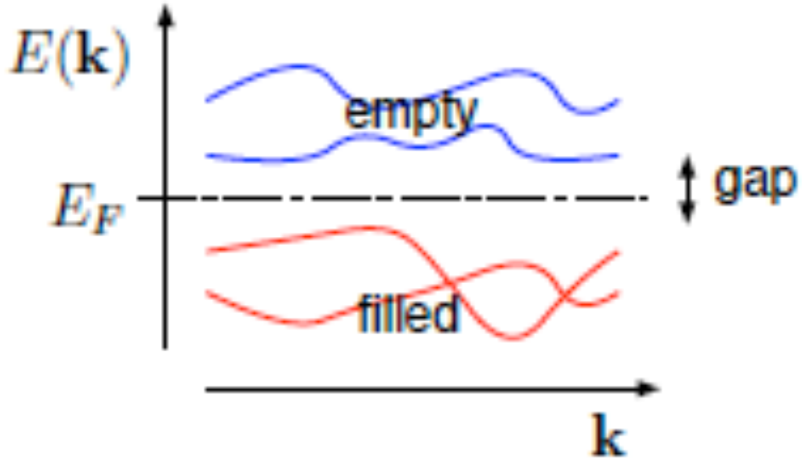
= # of occupied LL in bulk

"Topological protection from gap between LL's"

Topological insulator : definition

"Material that is insulating in the bulk but carries metallic (i.e. extended) states at its boundary"

Band theory predicts
a gap at the Fermi energy
-> bulk band insulator




Yet, the bulk carries a
topological quantum number n

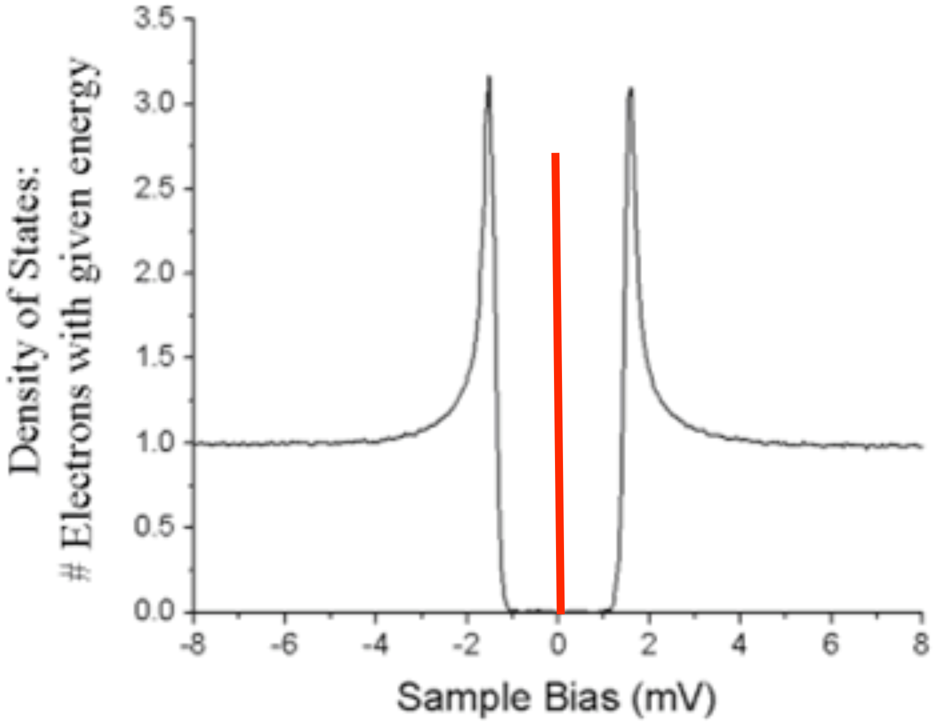
Connecting the bulk to the
outside (with n=0) generates
gapless edge states

F Bloch

Topological superconductor : definition

"Material that is superconducting in the bulk - with a quasiparticle excitation gap - but with zero-energy gapless surface states."

Those surface states are Majorana fermions




E Majorana

# Classification of nontrivial topological states vs. symmetries

PHYSICAL REVIEW B **78**, 195125 (2008)

**Classification of topological insulators and superconductors in three spatial dimensions**

Andreas P. Schnyder,[1] Shinsei Ryu,[1] Akira Furusaki,[2] and Andreas W. W. Ludwig[3]

| | | TRS | PHS | SLS | $d=1$ | $d=2$ | $d=3$ |
|---|---|---|---|---|---|---|---|
| Standard | A (unitary) | 0 | 0 | 0 | - | $Z$ | - |
| (Wigner-Dyson) | AI (orthogonal) | +1 | 0 | 0 | - | - | - |
| | AII (symplectic) | −1 | 0 | 0 | - | $Z_2$ | $Z_2$ |
| Chiral | AIII (chiral unitary) | 0 | 0 | 1 | $Z$ | - | $Z$ |
| (sublattice) | BDI (chiral orthogonal) | +1 | +1 | 1 | $Z$ | - | - |
| | CII (chiral symplectic) | −1 | −1 | 1 | $Z$ | - | $Z_2$ |
| BdG | D | 0 | +1 | 0 | $Z_2$ | $Z$ | - |
| | C | 0 | −1 | 0 | - | $Z$ | - |
| | DIII | −1 | +1 | 1 | $Z_2$ | $Z_2$ | $Z$ |
| | CI | +1 | −1 | 1 | - | - | $Z$ |

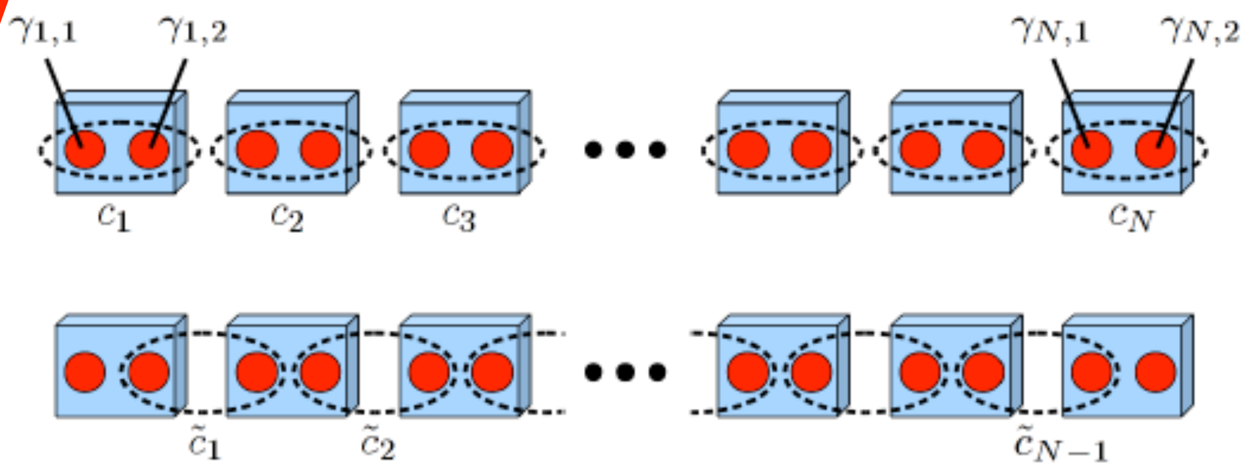
Answer given by homotopy group of band structure as map from the Brillouin Zone to Hamiltonian space

# Kitaev's chain : 1D p-wave superconductor

$$\mathcal{H}_{\text{chain}} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left( t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c. \right)$$

Majorana operators

$$\boxed{\begin{aligned} \gamma_{i,1} &= c_i^\dagger + c_i, \\ \gamma_{i,2} &= i\left(c_i^\dagger - c_i\right) \end{aligned}}$$




$$\mathcal{H}_{\text{chain}} = -it \sum_{i=1}^{N-1} \gamma_{i,2}\gamma_{i+1,1}$$

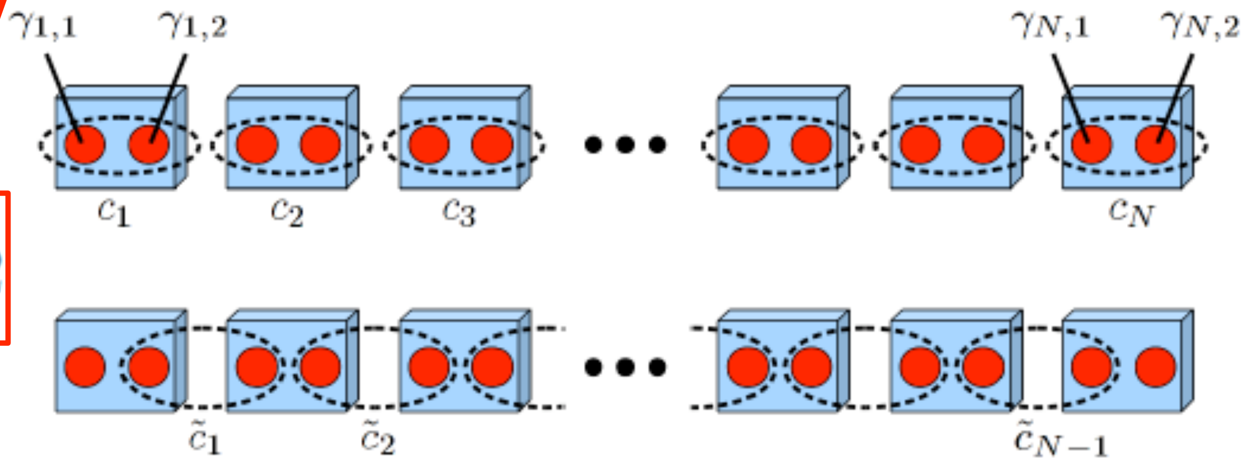
(special case μ=0 and Δ=t)

# Kithaev's chain : 1D p-wave superconductor

$$\mathcal{H}_{\text{chain}} = -it \sum_{i=1}^{N-1} \gamma_{i,2}\gamma_{i+1,1}$$

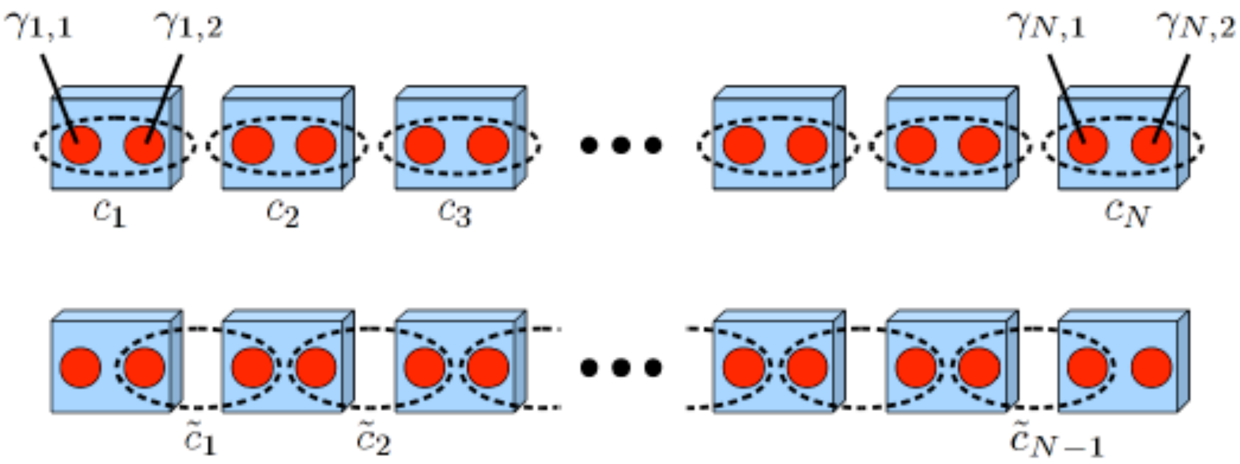
Fermion operators

$$\boxed{\tilde{c}_i = (\gamma_{i+1,1} + i\gamma_{i,2})/2}$$




$$\mathcal{H}_{\text{chain}} = 2t \sum_{i=1}^{N-1} \tilde{c}_i^\dagger \tilde{c}_i$$

# Kithaev's chain : 1D p-wave superconductor




$$\mathcal{H}_{\text{chain}} = 2t \sum_{i=1}^{N-1} \tilde{c}_i^\dagger \tilde{c}_i$$

Absence of $\tilde{c}_M = (\gamma_{N,2} + i\gamma_{1,1})/2$ !!!

-> zero-energy state

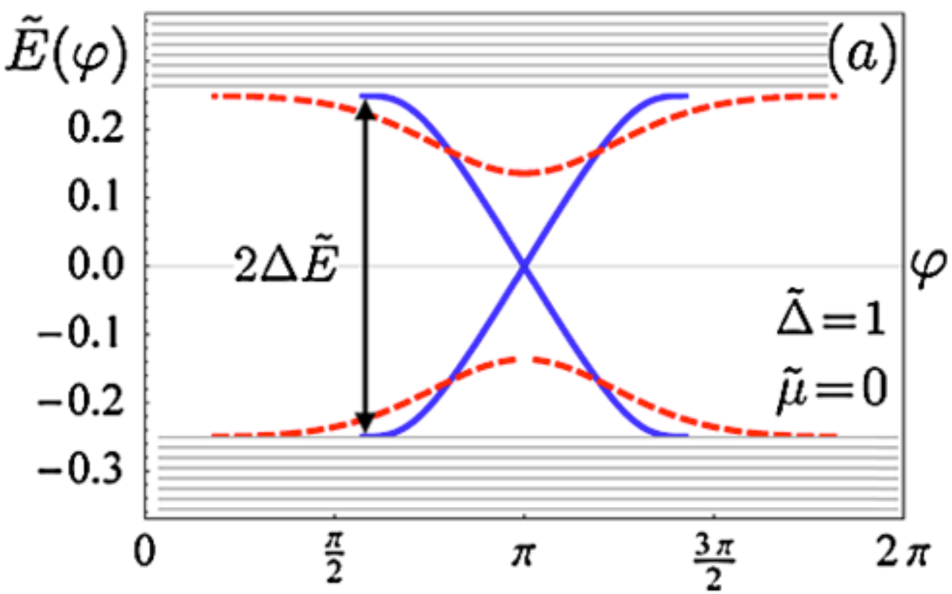
("two half-fermions")

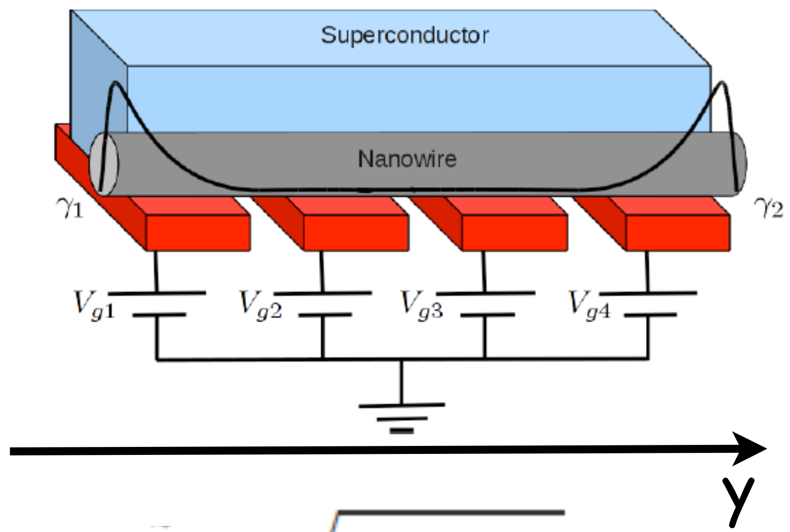
## Majorana platforms : how to generate a 1D p-wave superconductor

- Fu and Kane: Interface between SC and TI

- Lutchyn, Sau, Sarma; Oreg, Refael, von Oppen :
  Proximity-coupled nanowire with spin-orbit

$$H = (p_y^2/2m - \mu)\tau_z + u p_y \sigma_z \tau_z + B_Z \sigma_x + \Delta \tau_x$$

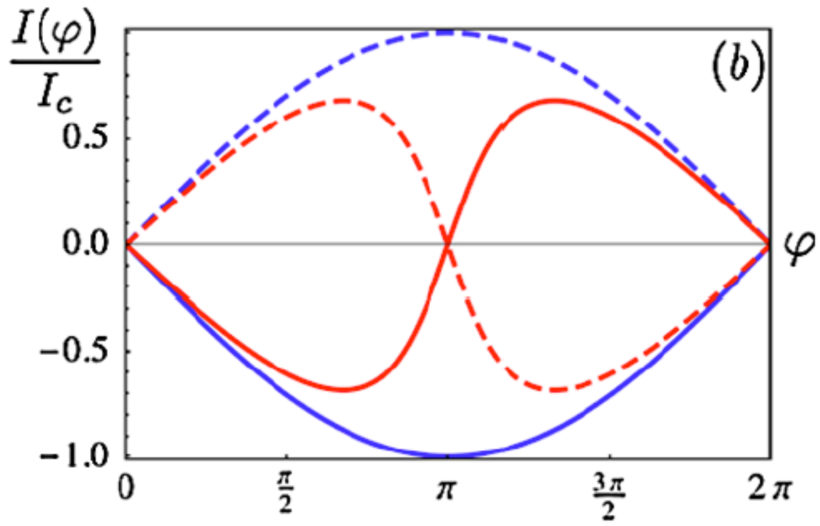


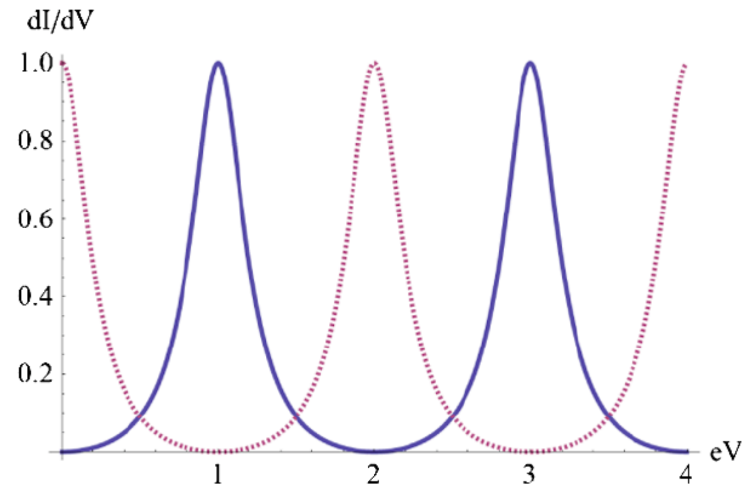

$$|B| > \sqrt{\Delta^2 + \mu^2}$$

# Signatures of Majorana fermions

•Fu and Kane; Lutchyn, Sau, Sarma:
"Period-doubling of the Josephson effect"

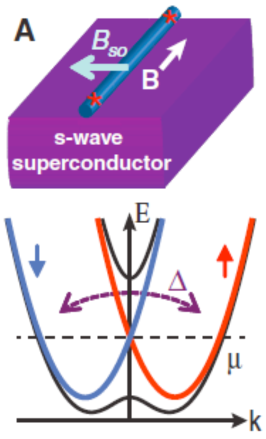


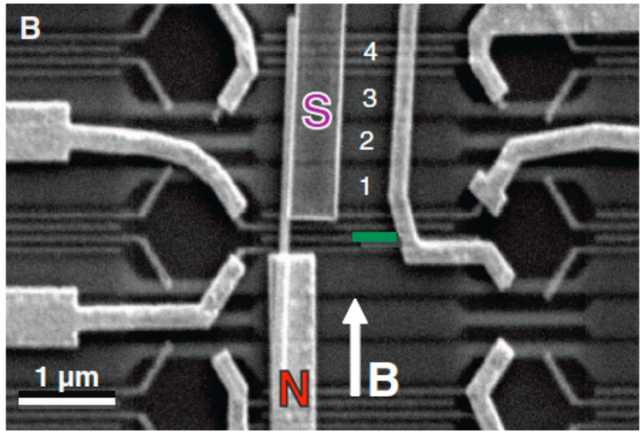

•Law, Lee, Ng; Flensberg; Sau, Tewari, Lutchyn, Stanescu, Sarma:
"T=1 resonance peak in tunneling conductance"

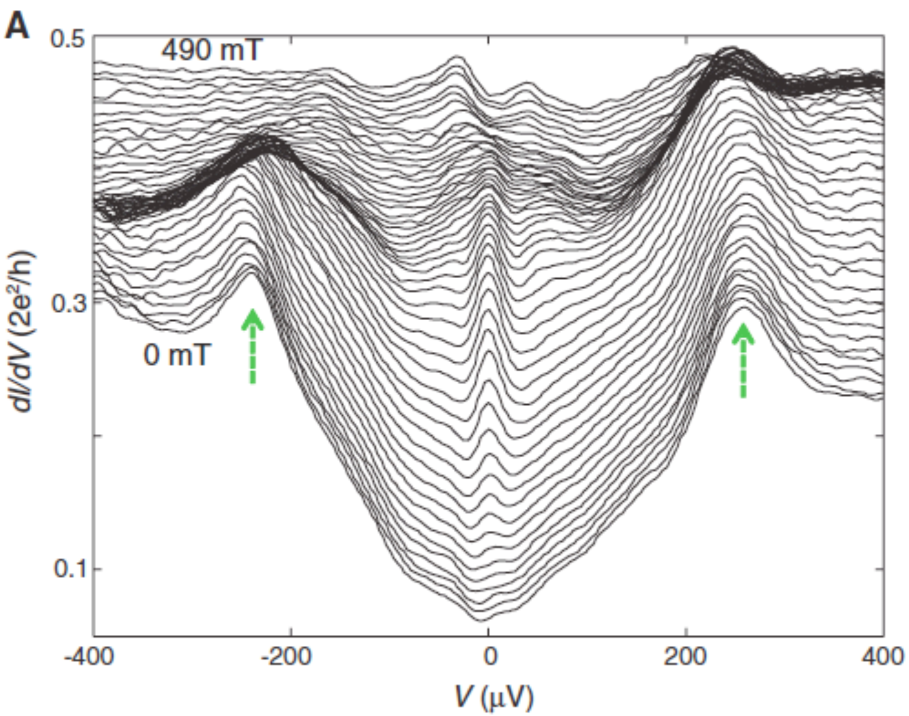


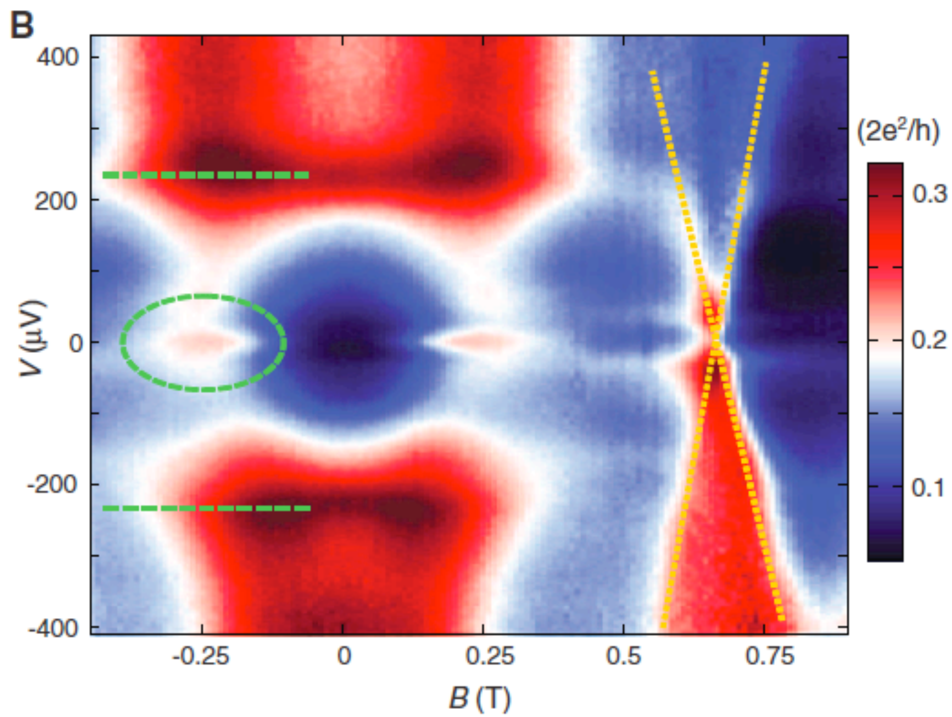
# Signatures of Majorana fermions ?

• Mourik et al. (Kouwenhoven)
Zero-bias anomaly



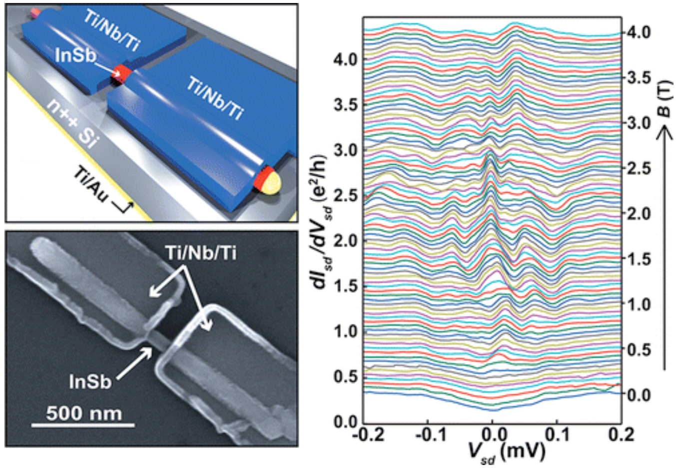



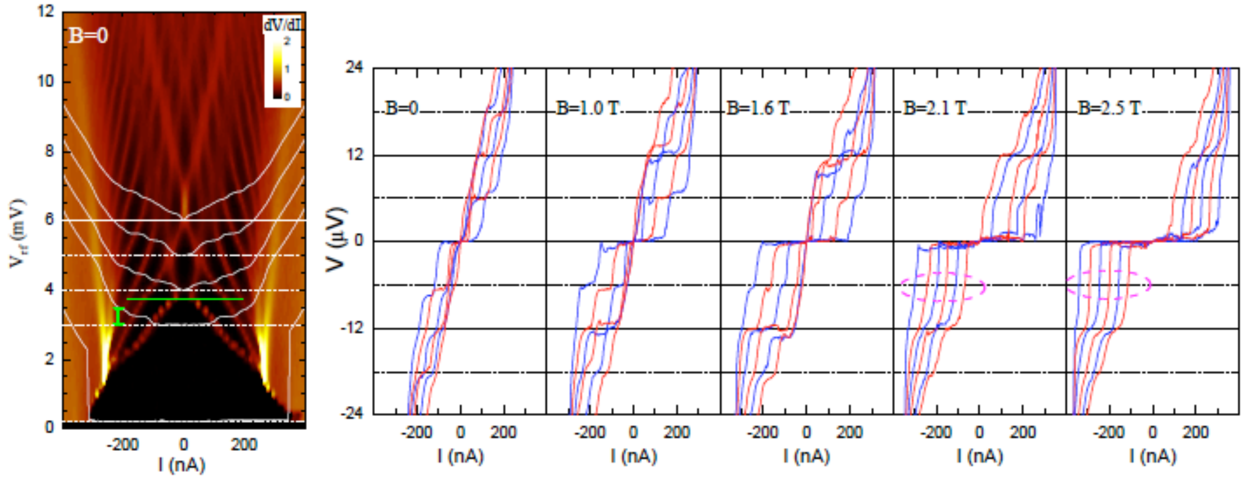
# Signatures of Majorana fermions ?

- Mourik et al. (Kouwenhoven)
- Deng et al. (Xu - Beijing)
- Das et al. (Heiblum)
- Finck et al. (van Harlingen)
- Churchill et al. (Marcus)




Zero-bias anomaly

- Rokhinson et al.

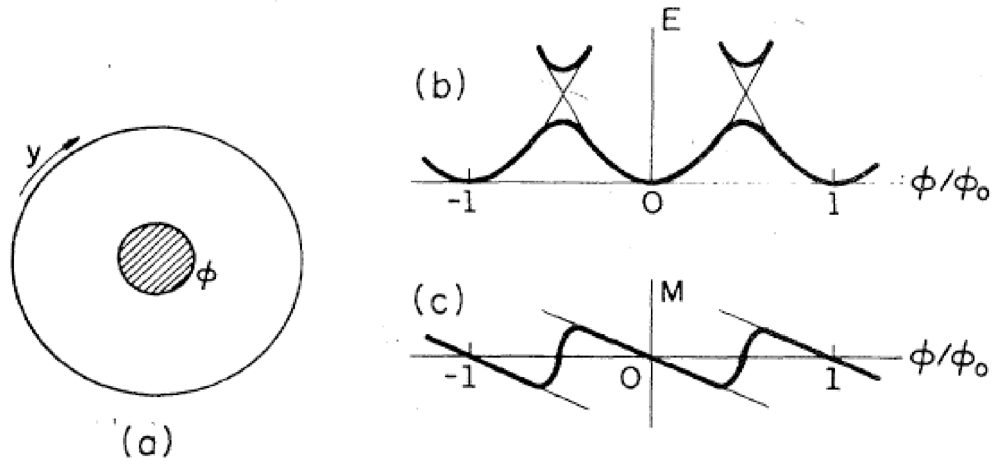



Josephson effect (?)

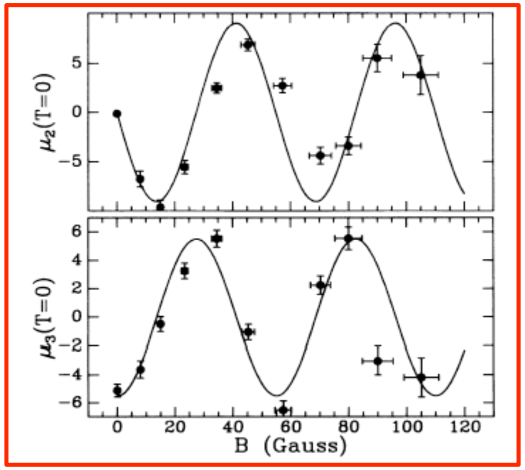
## Signatures of Majorana fermions in persistent currents

General idea :

-ring pierced by B-flux
-low enough T
-QM: p -> p-eA

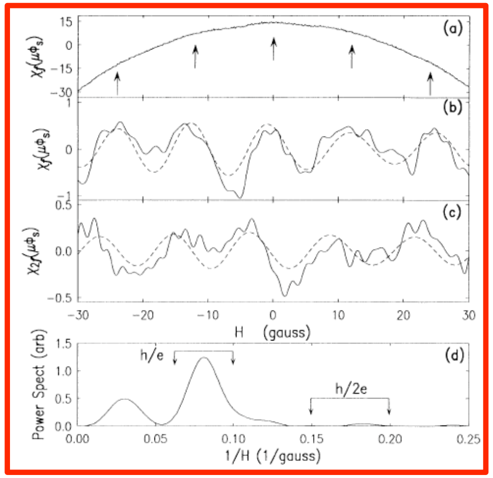



-> ground-state current with period h/p e (p integer)




Lévy et al.

Chandrasekhar et al.



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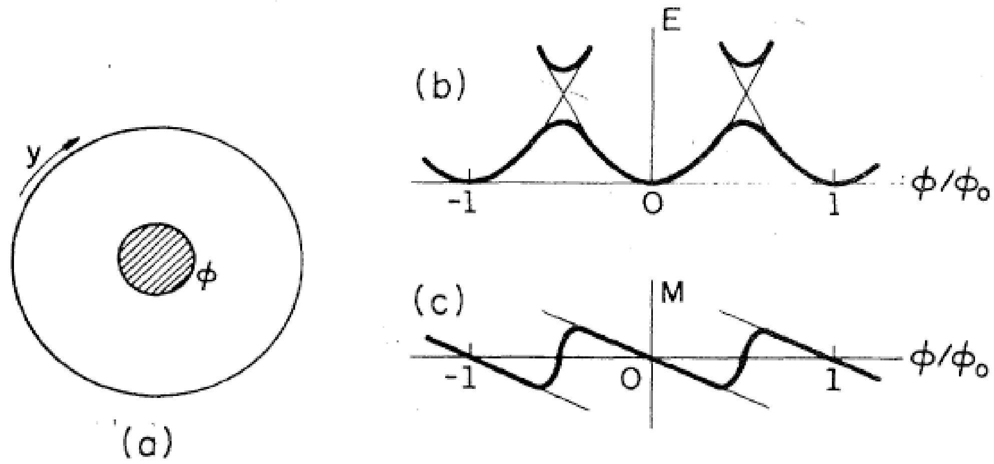



-> ground-state current with period h/p q
(p integer; q charge of the transferred particle)

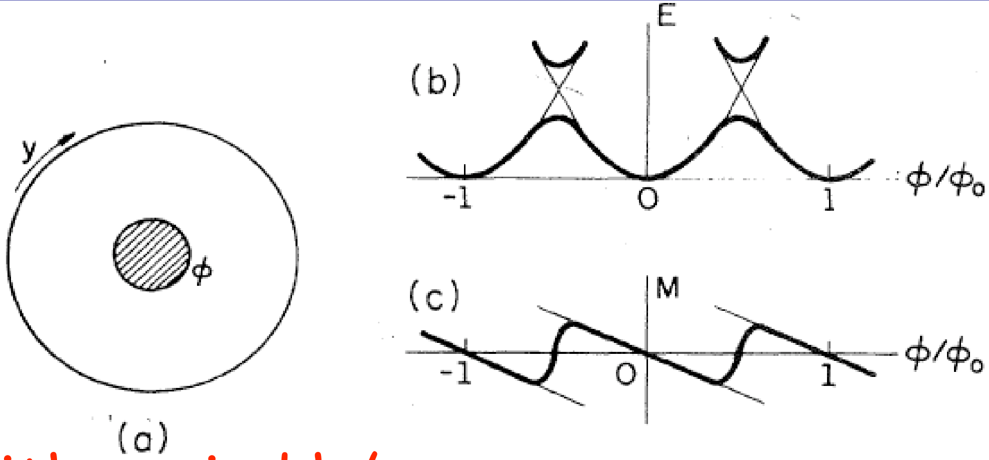
Büttiker and Klapwijk : ring with superconducting segment

$L \gg \xi$ : h/pq=h/2e, h/4e ..

$L < \xi$ : h/pq=h/e, h/2e ...

## Signatures of Majorana fermions in persistent currents

General idea :



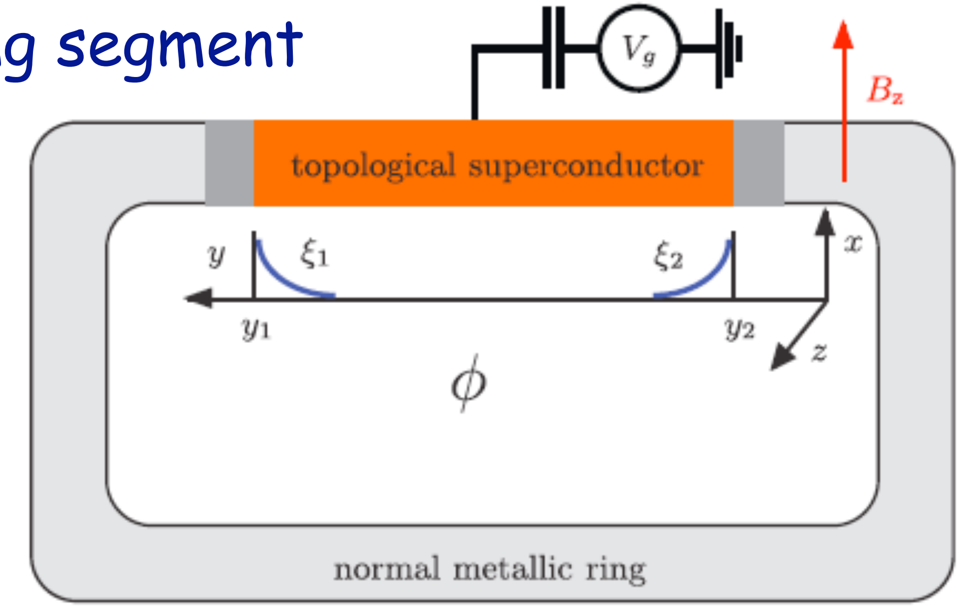

-> ground-state current with period h/p q
(p integer; q charge of the transferred particle)

**Can one detect the presence of Majorana bound states via the periodicity of persistent currents ?**

## Signatures of Majorana fermions in persistent currents

The setup :

Normal metallic ring interrupted by a Coulomb blockaded superconducting segment




-> fix parity on the SC :

- n, n+2, n+4... electrons in the trivial phase
- n, n+1, n+2, n+3... electrons in the topological phase

# Signatures of Majorana fermions in persistent currents

**Electron Teleportation via Majorana Bound States in a Mesoscopic Superconductor**

Liang Fu

*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 23 October 2009; published 2 February 2010)

Zero-energy Majorana bound states in superconductors have been proposed to be potential building blocks of a topological quantum computer, because quantum information can be encoded nonlocally in the fermion occupation of a pair of spatially separated Majorana bound states. However, despite intensive efforts, nonlocal signatures of Majorana bound states have not been found in charge transport. In this work, we predict a striking nonlocal phase-coherent electron transfer process by virtue of tunneling in and out of a pair of Majorana bound states. This teleportation phenomenon only exists in a mesoscopic superconductor because of an all-important but previously overlooked charging energy. We propose an experimental setup to detect this phenomenon in a superconductor–quantum-spin-Hall-insulator–-magnetic-insulator hybrid system.

I.e. Green's function

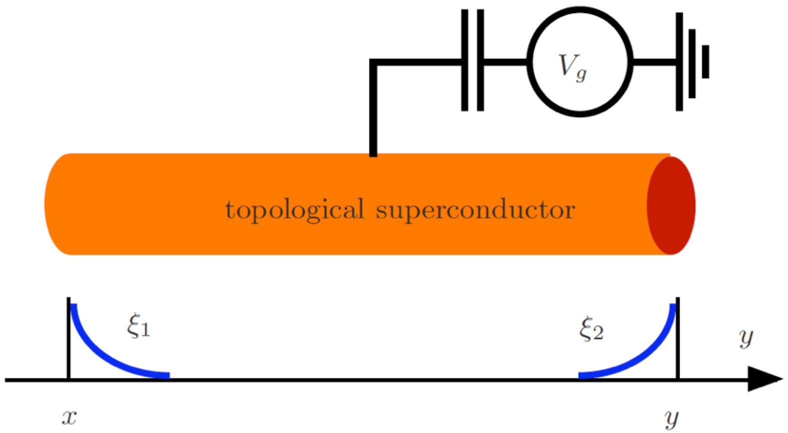
sign change with n->n+1

$$G^{e,o}(x, t \rightarrow \infty; y, 0) = \mp i\xi_2^*(y)\xi_1(x) \sim O(1)$$

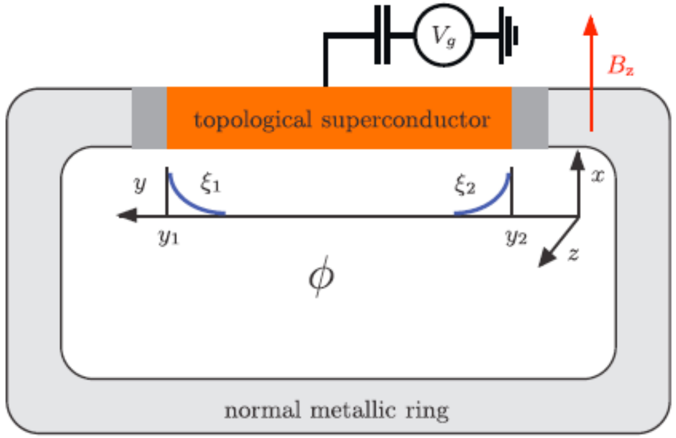
"long-range coherence"

-> tunneling of single electron

-> needs fixed parity



# Signatures of Majorana fermions in persistent currents




Effective Hamiltonian

$$H = H_{\text{ring}} + \delta(f^\dagger f - 1/2) + (\lambda_1 c_{\text{L}}^\dagger f + \text{H.c.}) + [-i\lambda_2(-1)^{f^\dagger f} c_{\text{R}}^\dagger f \exp(i\varphi) + \text{H.c.}].$$

C : fermions on the ring    f : fermion on the topological SC

λ : hopping on SC from left/right

δ : energy difference between N and N+1 states (tunable)

$\phi = \hbar\varphi/e$

# Signatures of Majorana fermions in persistent currents

Additional projection onto
|# e in ring, # e on SC> = |M,n> and |M-1,n+1>

$$H_{\mathrm{red}} = \begin{pmatrix} \epsilon_M & \tilde{\lambda}_1 - i\tilde{\lambda}_2(-1)^{n_0}e^{i\varphi} \\ \tilde{\lambda}_1 + i\tilde{\lambda}_2(-1)^{n_0}e^{-i\varphi} & \delta \end{pmatrix}$$

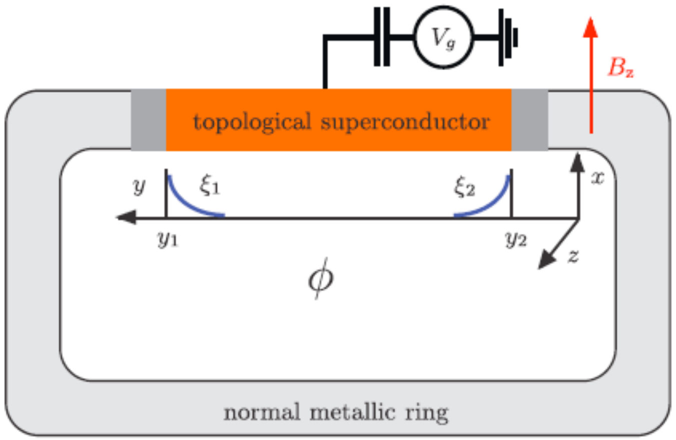
$$I(\varphi) = -(e/\hbar)\partial_\varphi E_-$$

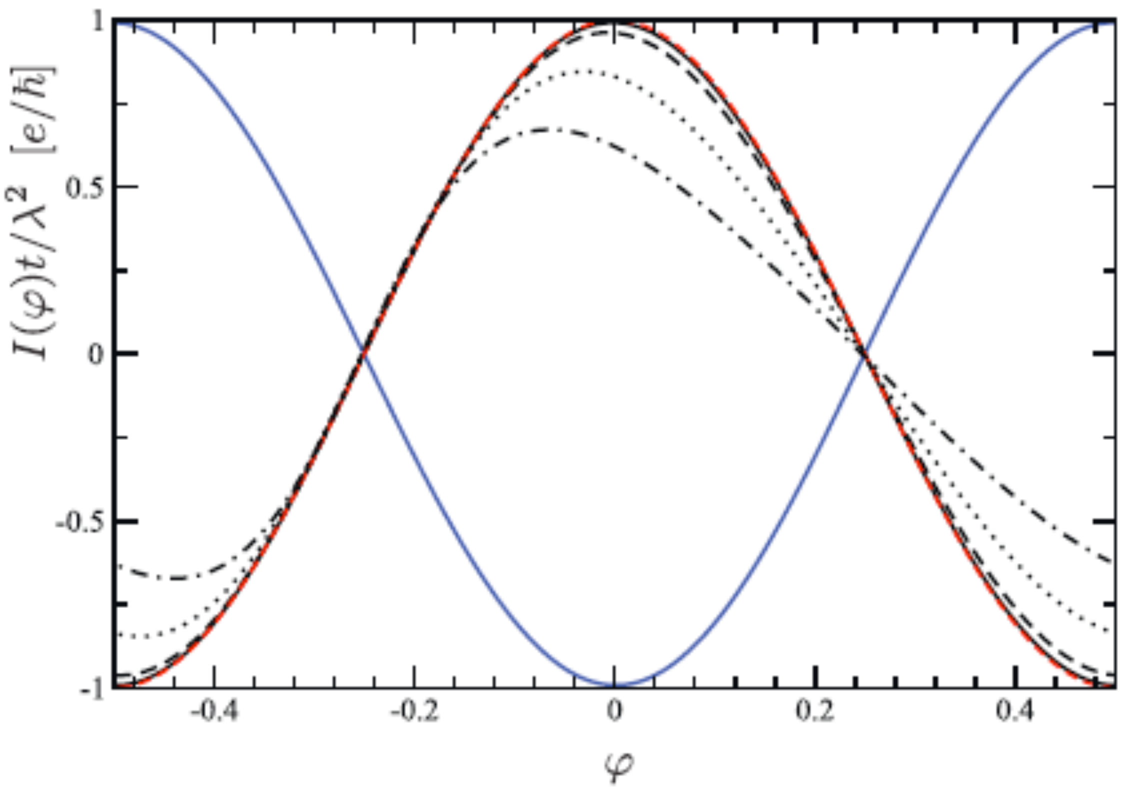
$$I(\varphi) = \frac{e}{\hbar}\frac{(-1)^{n_0}\tilde{\lambda}_1\tilde{\lambda}_2\cos\varphi}{\sqrt{(\epsilon_M-\delta)^2/4+\tilde{\lambda}_1^2+\tilde{\lambda}_2^2+2\tilde{\lambda}_1\tilde{\lambda}_2(-1)^{n_0}\sin\varphi}}$$

# Signatures of Majorana fermions in persistent currents

$$I(\varphi) = \frac{e}{\hbar} \frac{(-1)^{n_0} \tilde{\lambda}_1 \tilde{\lambda}_2 \cos\varphi}{\sqrt{(\epsilon_M - \delta)^2/4 + \tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + 2\tilde{\lambda}_1 \tilde{\lambda}_2 (-1)^{n_0} \sin\varphi}}$$

(i) finite current at zero flux
(ii) parity-dependence
(iii) h/e harmonics despite SC



## Free energy symmetry

Generally: $I(\phi) = -\partial_\phi \mathcal{F}$ with free energy even in B-field

How can one get finite I(0) ?

Answer : $\mathcal{F}(\phi, B_Z) = \mathcal{F}(-\phi, -B_Z)$
i.e. F even in total field (flux + Zeeman)

Proof : take wire hamiltonian

$$H = (p_y^2/2m - \mu)\tau_z + u p_y \sigma_z \tau_z + B_Z \sigma_x + \Delta \tau_x$$

$B_Z \to -B_Z$ is equivalent to space inversion in y-direction
~ interchanges Majorana operators
~ $\phi \to -\phi$

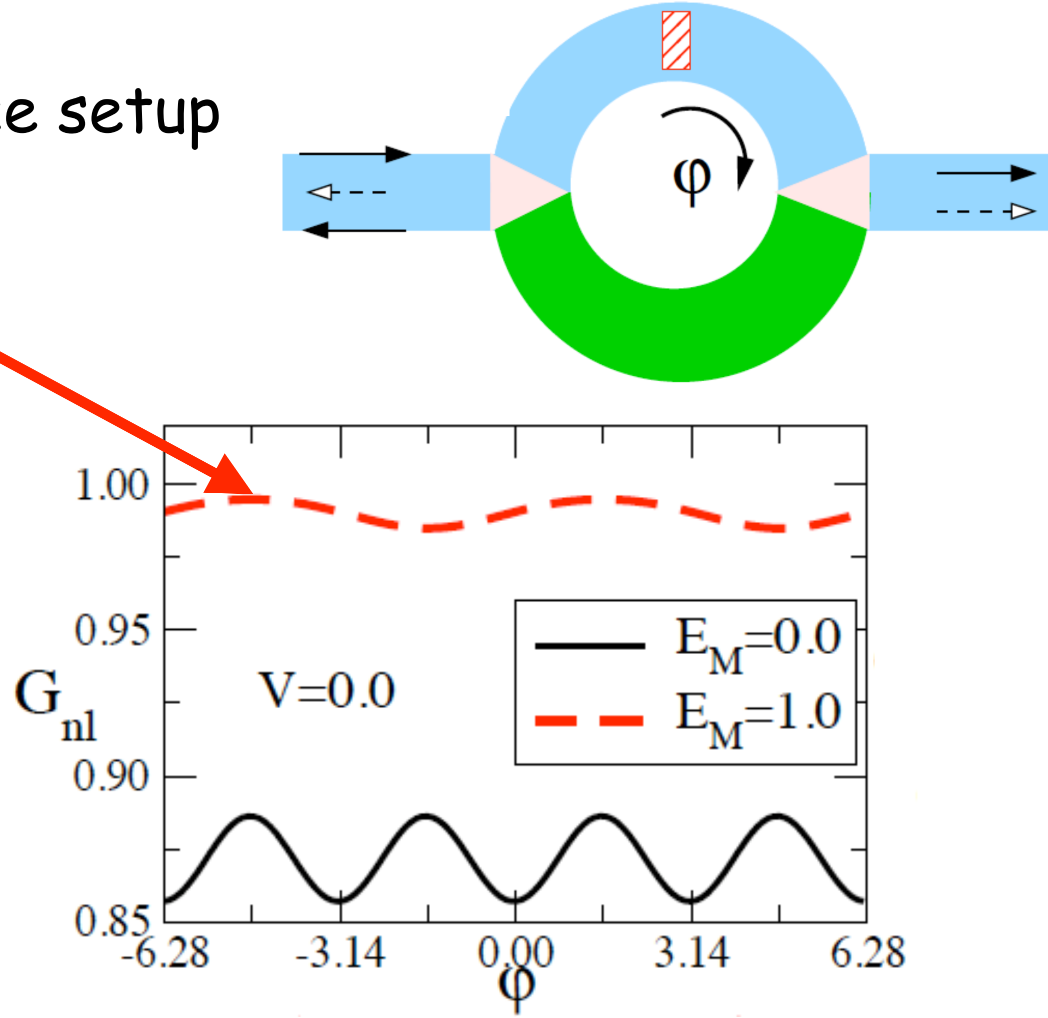
## Onsager symmetry

Aharonov-Bohm conductance setup

Fixed BZ -> antisymmetric conductance

But NOT a violation of Onsager, i.e.

$$G(\phi, B_Z) = G(-\phi, -B_Z)$$



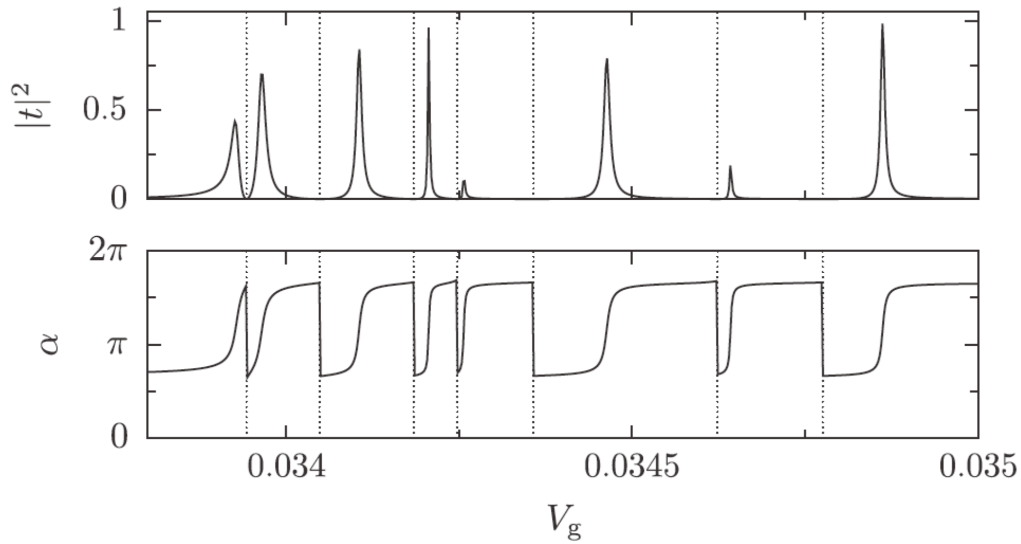

From Benjamin and Pachos prb 2010

# Friedel sum rule

"Friedel sum rule" : -connection between a scattering phase and the number of occupied states in the scatterer
-for Coulomb blockade, this works for the transmission phase

$$N_{dot} = \frac{1}{\pi} \theta_F(E_F)$$

i.e. additional phase of π when n->n+1




From Molina et al. prl 2012

# Friedel sum rule

"Friedel sum rule" : -connection between a scattering phase and the number of occupied states in the scatterer

-for Coulomb blockade, this works for the transmission phase

$$N_{dot} = \frac{1}{\pi} \theta_F(E_F)$$

i.e. additional phase of $\pi$ when n->n+1

-not observable for trivial superconductor

-observable for topological superconductor

$$G^{e,o}(x, t \longrightarrow \infty; y, 0) = \mp i \xi_2^*(y) \xi_1(x) \sim O(1)$$

e.g.via Fisher-Lee connection $t_{ab} = -2\, i\, (\Gamma_a \Gamma_b)^{1/2}\, G_{ab}$

# In Memoriam Markus Büttiker (1950-2013)