Network Reconstruction with Ambient Noise

Melvyn Tyloo,[1] Robin Delabays,[1,2] and Philippe Jacquod[1,3]
[1] *School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, CH-1951 Sion, Switzerland.*
[2] *Automatic Control Laboratory, ETH Zurich, CH-8092 Zurich, Switzerland*
[3] *Department of Quantum Matter Physics, University of Geneva, CH-1211 Geneva, Switzerland*
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The dynamics of systems of coupled agents is determined by the structure of their coupling network. Often, the latter is not directly observable and a fundamental, open question is how to reconstruct it from system measurements. We develop a novel approach to identify the network structure underlying dynamical systems of coupled agents based on their response to homogeneous, ambient noise. We show that two-point frequency signal correlators contain all the information on the network Laplacian matrix. Accordingly, when all agents are observable, the full Laplacian matrix can be reconstructed. Furthermore, when only a fraction of the agents can be observed, pairs of observable agents can be ranked in order of their geodesic distance when the noise correlation time is short enough. The method is computationally light and we show numerically that it is accurate and scalable to large networks.

Introduction. Network science – the field that studies complex, networked systems [1] – has seen an enormous growth of activity in recent years. More and more diverse systems of physical, life and human sciences are analyzed through larger and larger models of agents connected to one another [2], thanks in large part to the ever-increasing capacity for data mining and processing [3]. Network science draws both on analytical methods, for instance from graph theory or statistical mechanics, and on purely data-based approaches. Analytical approaches are arguably much more powerful than data-based approaches, however they rely on the precise knowledge of the underlying network and become harder to apply in large networks. Data-based approaches rely only on observations, but the collected data are algorithmically processed with little gain in physical knowledge. Combining the two approaches may compensate for the weaknesses of one with the strengths of the other. In this manuscript we connect data-based to analytical approaches. We show that the topology of a coupling network can be precisely reconstructed from sufficiently long sets of measurement data on systems subjected to uncontrolled, homogeneous noise from their environment. Accordingly, one big advantage of our reconstruction method is that it requires only the ability to measure the dynamics of the n agents, but does not require to precisely control its input signal. The approach will be precious to infer the structure of unknown, noisy networks such as, e.g. social networks which change over short time scales [4], interconnected power grids whose topology is determined by line faults and disconnections that are not systematically reported [5], or gene regulatory networks made of such huge numbers of proteins and genes that the exact structure of their interaction network cannot be known a priori [6].

A widely used method to reconstruct unknown interaction networks is to inject a probe signal somewhere in the network and to measure the response dynamics of the agents [7–13]. The successful reconstruction of the network topology, through e.g., the Laplacian or adjacency matrix, requires however that one is able not only to measure agents everywhere in the network, but additionally that one can control and inject specially tailored probe signals. Another difficulty is that the probe signals may modify the network dynamics. Soft approaches have been used in power grids, with injected signals with small amplitude and in frequency ranges outside the network bandwidth [13–15]. Because of the restricted probe frequency range, the method is then limited and identifies only certain network modes [14, 15] or requires a large number of probings [13].

A different approach relies on passive observation – i.e., without probe signal injection – of the agents dynamics. A brute-force approach is then to minimize a cost function over all the network parameters to identify the network edges [16–18]. While the approach works in principle, the required computation time scales at least as $\mathcal{O}(n^4)$, with the number n of agents [17, 18], and it quickly becomes computationally prohibitive in large networks. Lighter approaches identify edges between pairs of agents through trajectory correlators and Granger causality [19, 20], or leverage a Bayesian approach to determine the most likely network structure, given a set of data [21, 22]. Ref. [23] extracts spectral moments of the network from its dynamics, but cannot directly reconstruct its network matrix. Recent works leverage the response of a system to some noisy signal to improve the inference accuracy [24]. Rather interestingly, Refs. [25, 26] noticed that the accuracy of their machine learning network inference was improved by dynamical noise. So far, this is however only an observation and no formal understanding has been provided. Reviews of network reconstruction in different scientific domains are given in Refs. [27] and [28]. It is commonly accepted that measurements on all nodes are prerequisite for full network reconstruction.

All existing reconstruction methods we know of suffer from at least one of several weaknesses. They either require a large degree of control over the investigated system, are approximate or computationally prohibitive, or they deliver only limited information on the network structure. In this paper, we propose a novel technique for network reconstruction which does not suffer from any of these deficiencies. To fully reconstruct the network, it requires the passive observation of the dynamics of the n agents subjected to unavoidable ambient noise. Because it relies on the computation of two-point correlators of the agent dynamics, the method requires sets of measurements on all n nodes and a computation time scaling as $\mathcal{O}(n^2)$ for high frequency ambient noise, or $\mathcal{O}(n^3)$ for low frequency ambient noise for which the method requires a Laplacian matrix inversion.

Measuring all nodes may turn out to be impractical or even unreachable in large networks, and one often has to settle for partial measurements on only a fraction of the agents. In that case, our method is still able to identify the geodesic distance between pairs of such observable nodes, when the noise has a sufficiently small correlation time.

Network-coupled dynamical systems. We consider an ensemble of $i = 1, \ldots, n$ agents with coordinates $x_i \in \mathbb{R}$, with a dynamics governed by a set of coupled ordinary differential equations,

$$\dot{x}_i = \omega_i - \sum_j a_{ij}\, f_{ij}(x_i - x_j) + \delta\omega_i(t)\,, \qquad (1)$$

where $\omega_i \in \mathbb{R}$ are constant natural frequencies with $\sum_i \omega_i = 0$, and $\delta\omega_i(t)$ is the unavoidable ambient noise due to the environment. The interaction between agents is a differentiable function $f_{ij}\colon \mathbb{R} \to \mathbb{R}$, that is even in its indices i and j and odd in its argument, and $a_{ij} \geq 0$ are the unknown elements of the adjacency matrix of the interaction network. When the nonzero a_{ij} are sufficiently large and numerous, Eq. (1) with $\delta\omega_i(t) = 0$ has a stable fixed point which we denote $\boldsymbol{x}^* \in \mathbb{R}^n$. We observe the dynamics generated by $\delta\omega_i(t) \neq 0$ in the vicinity of this fixed point. Accordingly, we consider small deviations $\boldsymbol{\delta x} = \boldsymbol{x} - \boldsymbol{x}^*$ and linearize Eq. (1) in $\boldsymbol{\delta x}$ to obtain

$$\boldsymbol{\delta\dot{x}} = -\mathbb{J}(\boldsymbol{x}^*)\, \boldsymbol{\delta x} + \boldsymbol{\delta\omega}\,, \qquad (2)$$

where the Jacobian matrix reads

$$\mathbb{J}_{ij}(\boldsymbol{x}^*) = \begin{cases} -a_{ij}\, \partial_x\, f_{ij}(x)\big|_{x = x_i^* - x_j^*}\,, & i \neq j\,,\\ \sum_k a_{ik}\, \partial_x\, f_{ik}(x)\big|_{x = x_i^* - x_k^*}\,, & i = j\,. \end{cases} \qquad (3)$$

The matrix $\mathbb{J}$ is a real, symmetric Laplacian matrix of the interaction network, that is weighted by the derivative of f_{ij} at the fixed point. It contains information on both the interaction network and the fixed point. This Laplacian is the matrix we want to reconstruct. Under our assumption of a stable fixed point, it has real, nonnegative eigenvalues, $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ and an orthonormal basis of eigenvectors $\{\boldsymbol{u}_\alpha\}_{\alpha=1}^n$, with a constant-component zero-mode $u_{1i} = n^{-1/2}$, $\forall i$.

Equation (2) is solved by expanding $\boldsymbol{\delta x}$ over the eigenbasis $\{\boldsymbol{u}_\alpha\}$,

$$\boldsymbol{\delta x}(t) = \sum_\alpha c_\alpha(t) \boldsymbol{u}_\alpha\,, \qquad (4)$$

which yields a set of coupled Langevin equations for $c_\alpha(t)$ whose solution reads [29]

$$c_\alpha(t) = e^{-\lambda_\alpha t} \int_0^t e^{\lambda_\alpha t'} \boldsymbol{u}_\alpha \cdot \boldsymbol{\delta\omega}(t') \mathrm{d}t'\,. \qquad (5)$$

Network reconstruction from ambient noise. It is reasonable to expect that the network's randomly fluctuating environment generates noise with spatial correlations decaying fast with distance. Accordingly we define $\delta\omega_i(t)$ in Eq. (1) as a spatially uncorrelated noise with its first two moments given by

$$\langle \delta\omega_i(t) \rangle = 0\,, \qquad (6a)$$

$$\langle \delta\omega_i(t_1) \delta\omega_j(t_2) \rangle = \delta\omega_0^2\, \delta_{ij} \exp\left(-|t_1 - t_2|/\tau_0\right)\,, \qquad (6b)$$

where $\delta\omega_0$ is the noise standard deviation, τ_0 its correlation time, and the brackets denote averaging over noise realizations or a large enough observation time, $\langle \ldots \rangle = T^{-1} \int_0^T \ldots \mathrm{d}t$, $T \gg \tau_0$.

The equal time two-point frequency correlator is straightforwardly calculated in the limit of long observation times ($\lambda_\alpha T \gg 1$). From Eqs. (4-6), one obtains

$$\langle \delta\dot{x}_i(t) \delta\dot{x}_j(t) \rangle = \delta\omega_0^2 \Big(\delta_{ij} - \sum_{\alpha \geq 2} u_{\alpha,i} u_{\alpha,j} \frac{\lambda_\alpha \tau_0}{1 + \lambda_\alpha \tau_0} \Big)\,. \qquad (7)$$

Two regimes are of particular interest. First, in the limit of short correlation time, $\lambda_\alpha \tau_0 < 1$, Taylor-expanding Eq. (7) and realizing that the matrix elements of the k^{th} power of the Laplacian read $(\mathbb{J}^k)_{ij} = \sum_\alpha \lambda_\alpha^k\, u_{\alpha,i} u_{\alpha,j}$ gives

$$\langle \delta\dot{x}_i \delta\dot{x}_j \rangle = \delta\omega_0^2 \left[\delta_{ij} + \sum_{k=1}^{\infty} (-\tau_0)^k (\mathbb{J}^k)_{ij} \right]\,. \qquad (8)$$

Second, in the other limit of long correlation time, $\lambda_\alpha \tau_0 > 1$, a similar series expansion of Eq. (7) gives this time

$$\langle \delta\dot{x}_i \delta\dot{x}_j \rangle = \delta\omega_0^2 \left[n^{-1} - \sum_{k=1}^{\infty} (-\tau_0)^{-k} (\mathbb{J}^{-k})_{ij} \right]\,, \qquad (9)$$

where $\mathbb{J}^{-k}$ stands for the k^{th} power of the pseudo-inverse $\mathbb{J}^\dagger$ of the Laplacian.

Equations (8) and (9) form the basis of our network reconstruction approach. In both deep asymptotic limits

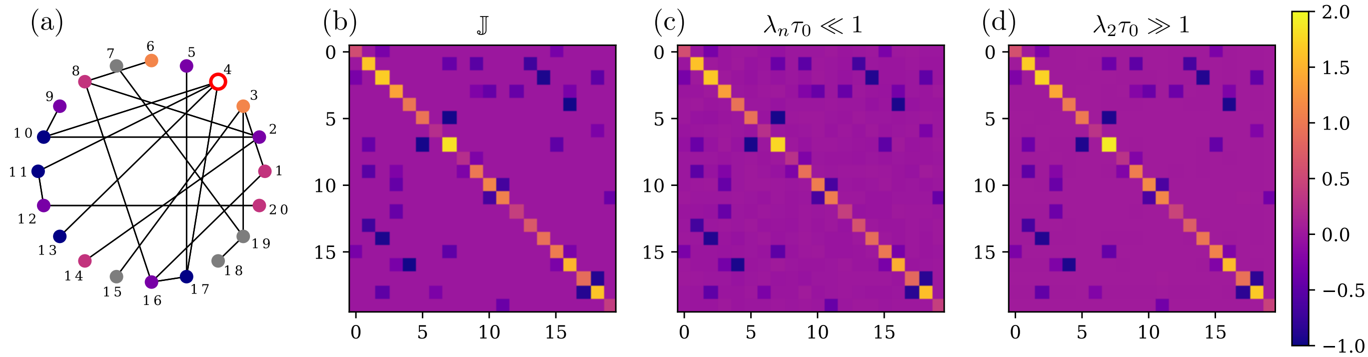

Figure 1. (a): Random interaction network used for reconstruction, with color-coded geodesic distances from node #4. Edge weights are randomly distributed as $a_{ij} \in [0.2, 1]$. (b): True network Laplacian matrix corresponding to the network of panel (a). (c) and (d): Reconstructed Laplacian matrix using (c) Eq. (10a) with $\lambda_n\tau_0 = 0.014$ and (d) Eq. (10b) with $\lambda_2\tau_0 = 4.97$. Noise averages are performed over observation time T with $\lambda_2 T = 500$ and over 40 different noise realizations.

of either very short noise correlation time τ_0 or very long τ_0 the network Laplacian is reconstructed as

$$\hat{\mathbb{J}}_{ij} = (\delta_{ij} - \langle \delta\dot{x}_i \delta\dot{x}_j \rangle / \delta\omega_0^2)\, \tau_0^{-1}\,, \quad \lambda_\alpha\tau_0 \to 0\,, \tag{10a}$$

$$\hat{\mathbb{J}}^\dagger_{ij} = (\langle \delta\dot{x}_i \delta\dot{x}_j \rangle / \delta\omega_0^2 - n^{-1})\, \tau_0\,, \quad \lambda_\alpha\tau_0 \to \infty\,. \tag{10b}$$

The short correlation time asymptotics of Eq. (10a) allows for a direct reconstruction of the Laplacian matrix. Based on the calculation of two-point correlation functions it requires a computation time scaling as $\mathcal{O}(n^2)$ and therefore allows to reconstruct large networks as we numerically illustrate below. The long correlation time asymptotic of Eq. (10b) requires on the other hand to first completely reconstruct the pseudo-inverse of the Laplacian and second to invert that matrix to obtain the network Laplacian, thereby requiring a computation time scaling as $\mathcal{O}(n^3)$.

More generally, ambient noise may be given by a superposition of different, uncorrelated noises $\delta\omega_i(t) = \sum_{\alpha=1}^k \delta\omega_i^{(\alpha)}(t)$, with noise sequences $\delta\omega_i^{(\alpha)}$ each with its own standard deviation $\delta\omega_\alpha$ and correlation time τ_α. Our approach remains applicable in that case as long as either $\max_{\alpha,j} \lambda_\alpha\tau_j < 1$ or $\min_{\alpha,j} \lambda_\alpha\tau_j > 1$. Equation (10) becomes

$$\hat{\mathbb{J}}_{ij} = \Big(\delta_{ij} \sum_\alpha \delta\omega_\alpha^2 - \langle \delta\dot{x}_i \delta\dot{x}_j \rangle\Big) \Big/ \sum_\alpha \delta\omega_\alpha^2 \tau_\alpha\,, \tag{11a}$$

$$\hat{\mathbb{J}}^\dagger_{ij} = \Big(\langle \delta\dot{x}_i \delta\dot{x}_j \rangle - n^{-1} \sum_\alpha \delta\omega_\alpha^2\Big) \Big/ \sum_\alpha \delta\omega_\alpha^2 \tau_\alpha^{-1}\,, \tag{11b}$$

in the short and and long correlation time asymptotics, respectively. Up to noise-dependent but spatially homogeneous factors $\sum_\alpha \delta\omega_\alpha^2$, $\sum_\alpha \delta\omega_\alpha^2\tau_\alpha$, and $\sum_\alpha \delta\omega_\alpha^2\tau_\alpha^{-1}$, the Laplacian matrix can be reconstructed as before.

Geodesic distances. In the short correlation time limit of Eq. (8), it is furthermore possible to partially reconstruct the network even with a limited number of measurements on only a fraction of the nodes. This is so, because the geodesic distance between any two nodes, i and j, is given by the minimal exponent q for which $(\mathbb{J}^q)_{ij} \neq 0$. Therefore,

$$\langle \delta\dot{x}_i \delta\dot{x}_j \rangle = \delta\omega_0^2 \left[\delta_{ij} + \sum_{k=q}^\infty (-\tau_0)^k \left(\mathbb{J}^k\right)_{ij}\right]\,, \tag{12}$$

which makes it possible to determine the geodesic distance q between any measurable pair of nodes (i,j) as long as

$$\min_{l,m}(\mathbb{J}^{q-1})_{lm}\tau_0^{-1} \gg (\mathbb{J}^q)_{ij} \gg \max_{l,m}(\mathbb{J}^{q+1})_{lm}\tau_0\,, \tag{13}$$

where the minimum (resp. maximum) is taken over pairs (l,m) of nodes with geodesic distance $\leq q-1$ (resp. $\geq q+1$). When Eq. (13) holds, pairs of nodes with geodesic distance q have noise correlators sufficiently away from those with geodesic distances $q-1$ and $q+1$ that one can easily identify them.

Numerical illustrations. We first validate our approach on a random network with $n = 20$ agents. The time evolution is given by Eq. (1), with interaction network edges shown in Fig. 1(a), randomly distributed as $a_{ij} \in [0.2, 1]$. Couplings are taken as $f_{ij}(x) = \sin(x)$, corresponding to Kuramoto oscillators [30].

Figures 1(c) and (d) display the Laplacian matrix with elements $\hat{\mathbb{J}}_{ij}$ inferred through Eqs. (10a) and (10b) respectively. Figure 1(b) shows the original Laplacian matrix $\mathbb{J}$. The relation between inferred and real elements of the Laplacian is shown in Fig. 2(a). The agreement between real and reconstructed network Laplacian is almost perfect, with both short and long correlation time methods.

We next demonstrate the scalability of our method and apply it to the PanTaGruEl network model of the European electric power grid [31, 32]. The network has $n = 3809$ agents and $m = 4944$ edges. Figure 2(b) shows the inferred vs. real Laplacian matrix elements for this large network. The short correlation time method

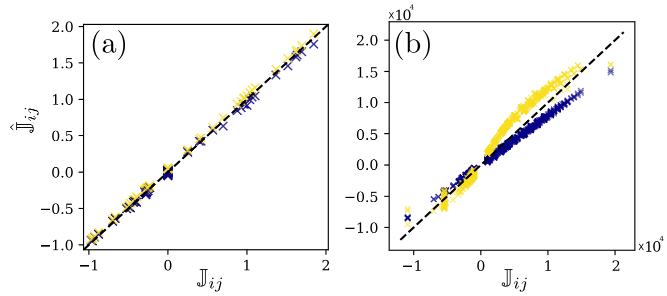

Figure 2. (a) Comparison between inferred, $\hat{\mathbb{J}}$, and true, $\mathbb{J}$, Laplacian of the weighted network shown in Fig. 1(a). Dark blue crosses are obtained using Eq. (10a) with $\lambda_n\tau_0 = 0.014$ and yellow crosses using Eq. (10b) with $\lambda_2\tau_0 = 4.97$. (b) Comparison between inferred and true Laplacian for a network corresponding to the European high-voltage electrical grid [31, 32]. Dark blue crosses are obtained using Eq. (10a) with $\lambda_n\tau_0 = 0.029$ and yellow crosses using Eq. (10b) $\lambda_2\tau_0 = 94$.

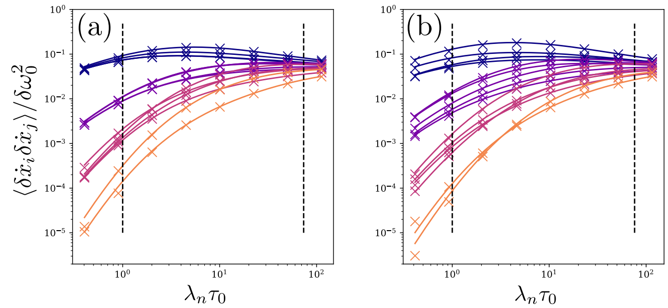

Figure 3. Two point correlator of Eq. (7) between agent #4 shown in red in Fig. 1(a) and the other nodes from 1st to 4th neighbors with corresponding colors in Fig. 1(a) as a function of the noise correlation time. (a) Uniform edge weights with $a_{ij} = 1$ if and only if i and j are connected. (b) Inhomogeneous edge weights with randomly distributed $a_{ij} \in [0.42, 1.9]$. Vertical dashed lines indicate $\tau_0 = \lambda_n^{-1}$ (left) and $\tau_0 = \lambda_2^{-1}$ (right).

of Eq. (10a) correctly reconstructs the network matrix, with a systematic rescaling factor of $\sim$1.27. We attribute this to the finite correlation time $\tau_0 > 0$ used. As a matter of fact, Eq. (10) gives only the asymptotic result for $\tau_0 \to 0$. Including the next order correction from Eq. (8) into Eq. (10) gives

$$\hat{\mathbb{J}}_{ij} = \mathbb{J}_{ij} - (\mathbb{J}^2)_{ij}\tau_0 + \mathcal{O}(\tau_0^2)\,. \tag{14}$$

Because for most networks with $\mathbb{J}_{ij} > 0$, $i \neq j$, and in particular for the PanTaGruEl network $\mathbb{J}_{ij}$ and $(\mathbb{J}^2)_{ij}$ have the same sign, the error in reconstructing off-diagonal matrix elements goes systematically towards smaller absolute values, in agreement with the data shown in Figs. 2(a) and 2(b). It is furthermore proportional to τ_0 as we numerically checked, but do not show. The error between the reconstructed and the real matrix elements in the long correlation time also appears to be systematic. A similar argument as just made indicates that it is proportional to τ_0^{-1}. We attribute its clearly nonlinear behavior to the fact that reconstruction in the large correlation time case requires a matrix inversion. We have found that, in the limit $\tau_0 \to 0$ our method captures vanishing matrix elements with probability one, provided enough and sufficiently long noise realizations are considered. In the other limit $\tau_0 \to \infty$, errors in detecting vanishing matrix elements occur due to the necessary matrix inversion. For the data shown in Fig. 2(b), we found only ten spurious edges, compared to 4944 real ones and more than seven millions possible ones. This is particularly satisfying, given that coupling strengths vary by an order of magnitude in PanTaGruEl [31, 32].

One of our main results is that partial network reconstruction can still occur when one has access to partial measurements over a fraction of the network agents. The method is illustrated in Fig. 3 which plots noise correlators vs. τ_0 for the network of Fig. 1(a) with homogeneous (left panel) and inhomogeneous (right panel) edge weights. For small enough τ_0, so that $\lambda_n\tau_0 < 1$, indicated by the left vertical dashed line in Fig. 3, noise correlators come in well defined bunches, indicating that Eq. (13) is satisfied. Geodesic distances are then clearly identified, as is seen in Fig. 3 where data colors correspond to the color-coded geodesic distances in Fig. 1(a).

Figure 4 further illustrates partial network reconstruction by showing a colorplot histogram of Eq. (8) for each agent $i = 1, \ldots 20$. Bands corresponding to geodesic distances are clearly identified, at least up to the third neighbor. While full reconstruction of the network requires observability of all n nodes, partial structures can be inferred from geodesic distances between the observable nodes via the two-point noise correlator. The partial reconstruction method works well, however it is in practice limited to identifying the first few neighbors. This is so because it requires short correlation times, and is based on Eq. (8) which states that pairs of $k^{\rm th}$ neighbor agents (i, j) have a noise correlator given by

$$\langle\delta\dot{x}_i\delta\dot{x}_j\rangle = \delta\omega_0^2(-\tau_0)^k(\mathbb{J}^k)_{ij} + \mathcal{O}[\tau_0^{k+1}(\mathbb{J}^{k+1})_{ij}]\,. \tag{15}$$

For distant pairs of agents with large values of k, the two-point correlator therefore becomes smaller and smaller, until it becomes smaller than its statistical standard deviation. For the networks we investigated we have found that geodesic distances at least up to $k = 4$ can be inferred in practice.

Conclusion. We have presented a network reconstruction method based on the dynamics of its agents under unavoidable ambient noise from the network's environment. Our approach is scalable to large networks, it has good accuracy, and has the significant advantage of being nonintrusive. In particular, compared to earlier approaches [7–13], it does not require the ability to inject signal at each network node. To fully reconstruct the

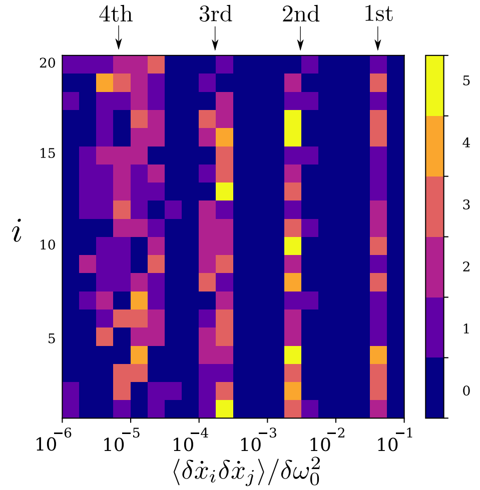

Figure 4. Reconstruction of the 1st to 4th neighbors from the two point correlator of Eq. (7) for the network shown in Fig. 1(a). For each agent index i, and each correlator value, the color plot gives the number of agents $j \neq i$ with that correlator value at the smallest value of τ_0 in Fig. 3(a).

network, the approach requires that one is able to measure the dynamics of all agents, however, when not all those measurements are accessible, the method is still able to determine the geodesic distance between pairs of measurable agents. This still provides precious information on the network structure. We trust that our method could significantly improve existing approach of network reconstruction with partial measurements based on most likely network estimates [21, 22].

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