Optimal placement of inertia and primary control: a matrix perturbation theory approach

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Swiss National Science Foundation

Power system dynamics: grid frequency and power imbalance

Swing equation:

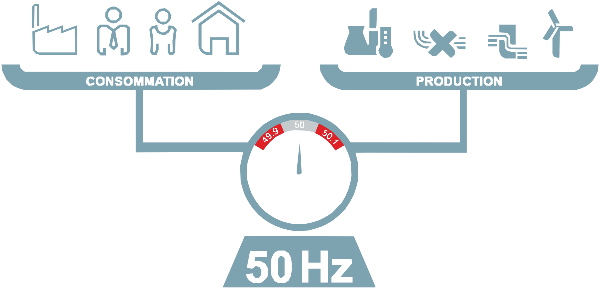
$$M \frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{sys}} = P_{\mathrm{gen}} - P_{\mathrm{cons}}$$

M: system inertia

$\omega_{\mathrm{sys}} \equiv 2\pi f_{\mathrm{sys}}$: system frequency

P_{gen}: generation

P_{cons}: consumption



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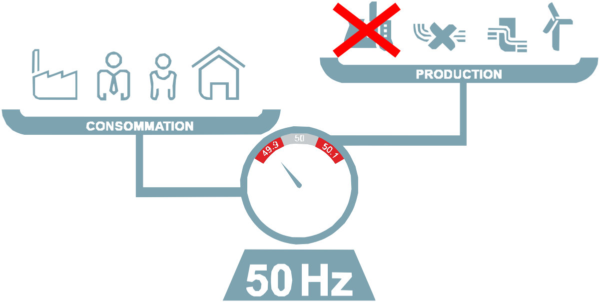

Power system dynamics: grid frequency and power imbalance

Swing equation:

$$M\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{sys}} + D\,\omega_{\mathrm{sys}} = P_{\mathrm{gen}} - P_{\mathrm{cons}}$$

frequency deviation: $\omega_{\mathrm{sys}} - \omega_0 \rightarrow \omega_{\mathrm{sys}}$

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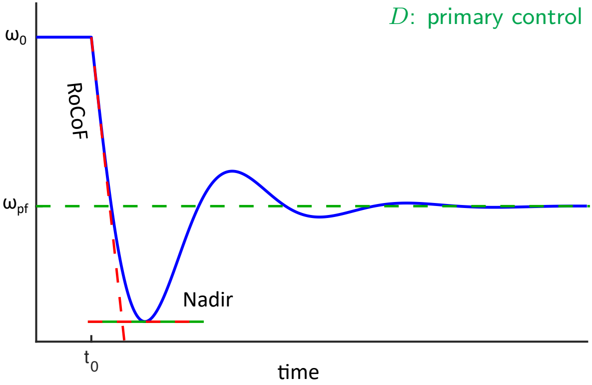
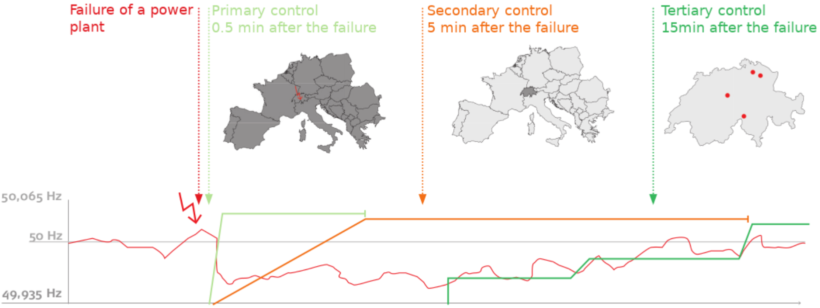


Scope of validity of the swing equation

Simple scheme of the frequency control:



Source: adapted from Swissgrid.

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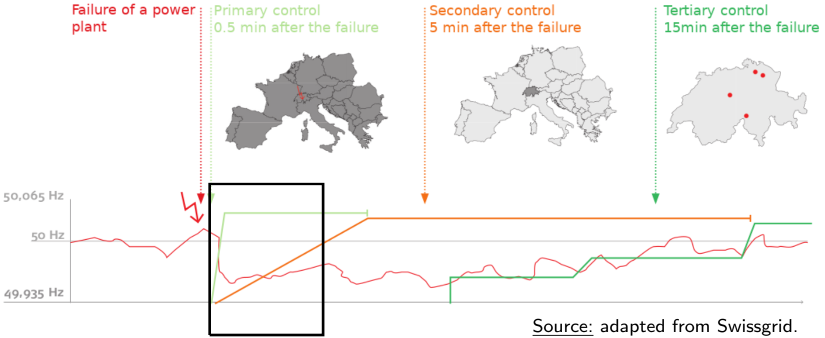

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up to 20-30s after the fault.

New renewable sources are substituting conventional ones

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New renewable sources:

- Distributed generation
- Non-dispatchable & fluctuating
- Negligible marginal cost
- Power inverters

Conventional sources:

- Power plants
- Dispatchable (in most cases)
- Fuel cost
- Rotating generators

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New renewable sources have no rotational inertia.

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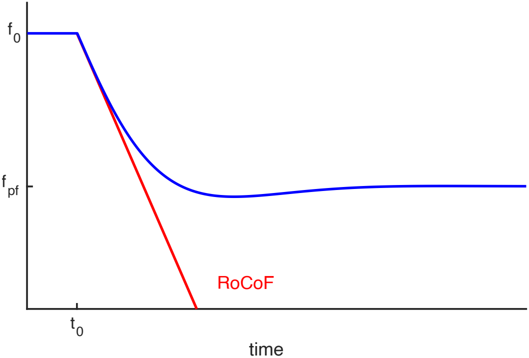
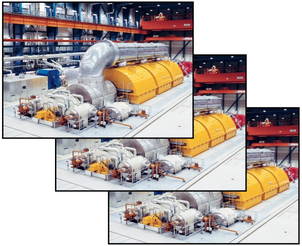


RoCoF: rate of change of frequency



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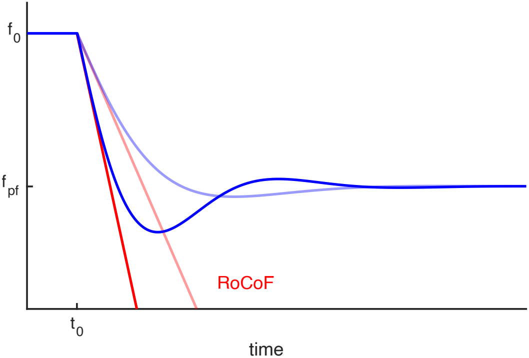

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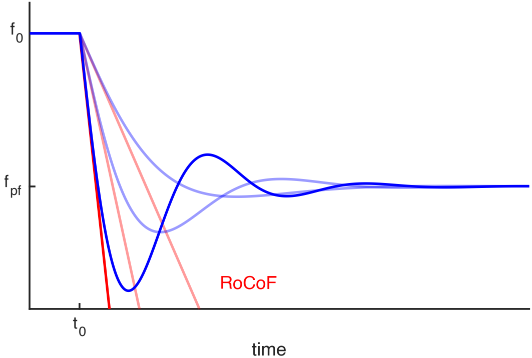

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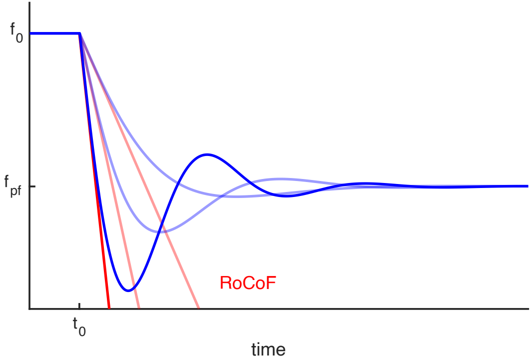

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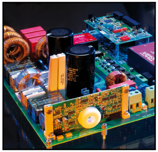

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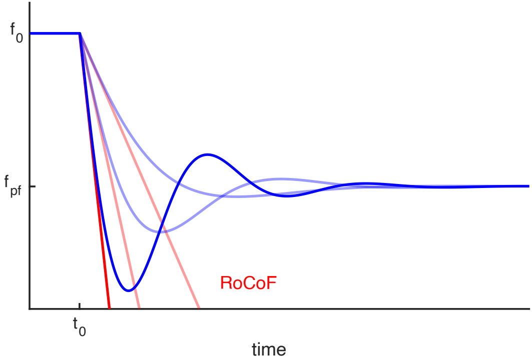

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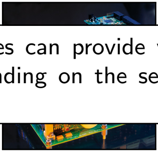

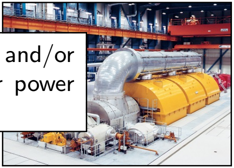

New renewable sources can provide virtual inertia and/or primary control depending on the settings of their power electronics.

Definition of the problem

Analytical $\rightleftharpoons$ Numerical

Approximations $\rightleftharpoons$ "TSO friendly" parameters

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Disturbance measures:

- Rate of change of frequency (RoCoF)
- Frequency nadir/overshoot
- Performance measures based on θ_i and $\omega_i \equiv \dot{\theta}_i$

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Generic expression:

$$\mathcal{M} = \int_0^\infty \left[\boldsymbol{\theta}^\top \boldsymbol{N} \boldsymbol{\theta} + \boldsymbol{\omega}^\top \boldsymbol{S} \boldsymbol{\omega}\right] \mathrm{d}t$$

Bamieh, Jovanovic, Mitra, Patterson (2012)
Bamieh and Gayme (2013)
Siami and Motee (2014)
Grunberg and Gayme (2016)
Poolla, Bolognani, Dörfler (2017)
Paganini and Mallada (2017)
Tyloo, Coletta, Jacquod (2018)
Coletta, Bamieh, Jacquod (2018)

Steady-state operation

Power system dynamics:

$$\boxed{m_i\dot{\omega}_i + d_i\omega_i = P_i(t) - \sum_j b_{ij}\sin(\theta_i - \theta_j)}$$

Before a disturbance, a power systems is in a stable steady-state solution.

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Power flow equations:

$$P_i = \textstyle\sum_j V_iV_j\Big[B_{ij}\sin\big(\theta_i - \theta_j\big) + G_{ij}\cos\big(\theta_i - \theta_j\big)\Big],$$
$$Q_i = \textstyle\sum_j V_iV_j\Big[G_{ij}\sin\big(\theta_i - \theta_j\big) - B_{ij}\cos\big(\theta_i - \theta_j\big)\Big].$$

P_i: active power injections

Q_i: reactive power injections

V_i: voltage magnitudes

θ_i: voltage phases

B_{ij}: susceptances

G_{ij}: conductances

$\omega \equiv \dot{\theta}_i$: phase frequencies

Steady-state operation

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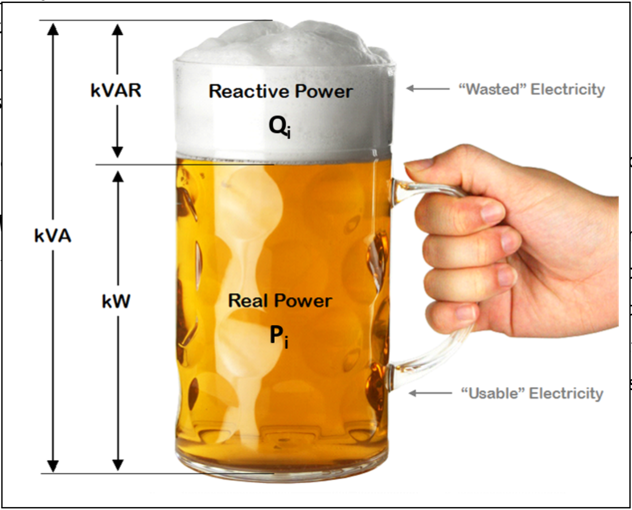
$m_i\dot{\omega}_i + d_i$

Before a dis .

Power flow

$P_i = \sum_j V_i$

$Q_i = \sum_j V_i$



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In transmission grids, $B_{ij} \gg G_{ij}$. In the lossless line approximation ($G_{ij} = 0$ and $V_i = V_i^{\mathrm{R}}$)

$$\boxed{P_i = \sum\nolimits_j b_{ij}\sin\left(\theta_i - \theta_j\right),} \qquad \text{with } b_{ij} = V_i^{\mathrm{R}}V_j^{\mathrm{R}}B_{ij}.$$

Network Laplacian

Diagonalization of the network Laplacian:

$$\boldsymbol{L} = \boldsymbol{U}^{(0)\top}\boldsymbol{\Lambda}^{(0)}\boldsymbol{U}^{(0)}$$

$$\boldsymbol{L}_{ij} = \begin{cases} \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}), \text{ if } i = j, \\ -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}), \text{ otherwise.} \end{cases}$$

$\theta_i^{(0)}$: a stable solution

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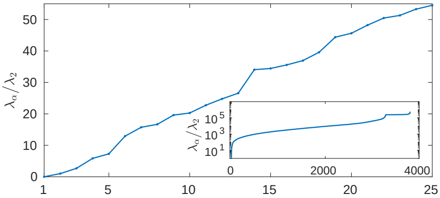

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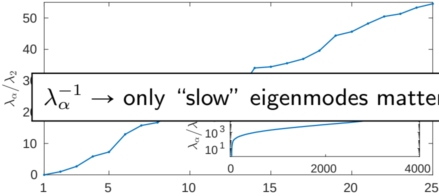

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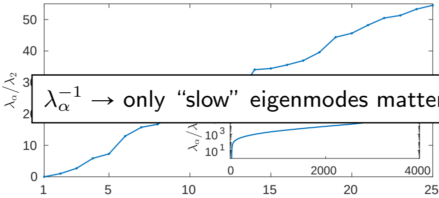

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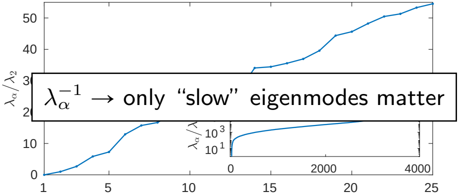

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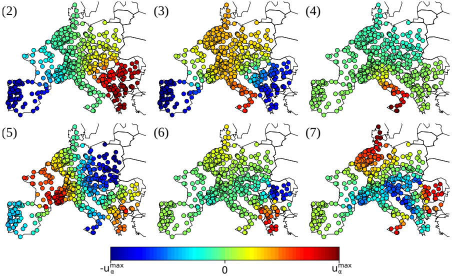
Slowest eigenmodes:



Pagnier and Jacquod to appear in IEEE-Access

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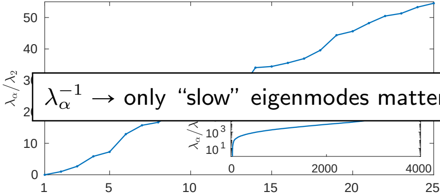

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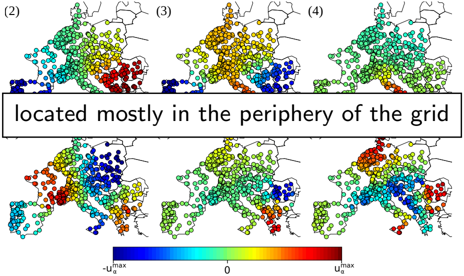

located mostly in the periphery of the grid

Pagnier and Jacquod to appear in IEEE-Access

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Scheme:

1) quantify the disturbance $\Rightarrow$ introduction of a measure
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3) evaluate our performance measure
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Heaviside function: $\Theta(t) = \begin{cases} 0, \text{ if } t < 0, \\ 1, \text{if } t \geqslant 0 \end{cases}$

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- linearized dynamics $[\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)]$

$$\boxed{m_i\delta\dot{\omega}_i + d_i\delta\omega_i = \delta P_i(t) - \sum_j b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i - \delta\theta_j)}$$

$P_i(t) = P_i^{(0)} + \delta P_i(t)$, where $P_i^{(0)} = \sum_j b_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)})$

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$$\boxed{\boldsymbol{M}\dot{\boldsymbol{\omega}} + \boldsymbol{D}\boldsymbol{\omega} = \delta\boldsymbol{P}(t) - \boldsymbol{L}\boldsymbol{\theta}}$$

$\boldsymbol{M} = \mathrm{diag}(\{m_i\})$, $\boldsymbol{D} = \mathrm{diag}(\{d_i\})$ and $\boldsymbol{L}_{ij} = \begin{cases} \sum\limits_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}), \text{ if } i = j, \\ -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}), \text{ otherwise.} \end{cases}$

Problem resolution for an uniform inertia-damping ratio

$$\boldsymbol{M}\dot{\boldsymbol{\omega}} + \boldsymbol{D}\boldsymbol{\omega} = \delta\boldsymbol{P}(t) - \boldsymbol{L}\boldsymbol{\theta}$$

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$$\dot{\boldsymbol{\omega}} + \boldsymbol{\Gamma}\boldsymbol{\omega} = \boldsymbol{M}^{-1}\delta\boldsymbol{P}(t) - \boldsymbol{M}^{-1}\boldsymbol{L}\boldsymbol{\theta}$$

1) $\boldsymbol{\Gamma} \equiv \boldsymbol{M}^{-1}\boldsymbol{D} = \gamma\mathbb{1}_N$ (assumption)

Problem resolution for an uniform inertia-damping ratio

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\theta}\\ \boldsymbol{\omega}\end{bmatrix} = \begin{bmatrix}\mathbb{0}_{N\times N} & \mathbb{1}_N\\ -\boldsymbol{M}^{-1}\boldsymbol{L} & -\boldsymbol{\Gamma}\end{bmatrix}\begin{bmatrix}\boldsymbol{\theta}\\ \boldsymbol{\omega}\end{bmatrix} + \begin{bmatrix}\mathbb{0}\\ \boldsymbol{M}^{-1}\boldsymbol{\delta P}\end{bmatrix},$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{array}\right]=\left[\begin{array}{cc}\mathbb{0}_{N\times N} & \mathbb{1}_N\\ -\boldsymbol{\Lambda} & -\gamma\mathbb{1}_N\end{array}\right]\left[\begin{array}{c}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{array}\right]+\left[\begin{array}{c}\mathbb{0}\\ \boldsymbol{\mathcal{P}}\end{array}\right],$$

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In this basis, $\mathcal{M} = \int_0^\infty \sum_{\alpha>1} \dot{\xi}_\alpha^2(t)\mathrm{d}t$ and $\ddot{\xi}_\alpha + \gamma\dot{\xi}_\alpha + \lambda_\alpha\xi_\alpha = \mathcal{P}_\alpha\,,\ \forall\alpha$

$\boxed{\dot{\xi}_\alpha(t) = \frac{2\mathcal{P}_\alpha}{f_\alpha}e^{-\gamma t/2}\sin\left(\frac{f_\alpha}{2}t\right)}$ where $f_\alpha = \sqrt{4\lambda_\alpha - \gamma^2}$

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For a fault localized at bus $\#b$, $\delta P_i(t) = \delta P\delta_{ib}\Theta(t)$,

$$\boxed{\mathcal{M}_b = \frac{\delta P^2}{2\gamma m}\sum_{\alpha>1}\frac{u_{\alpha b}^2}{\lambda_\alpha}}$$

Perturbation theory

Mild inhomogoeneity
Parametrization:

$$m_i = m(1 + \mu r_i)\,,$$
$$d_i = m_i\gamma_i = m\gamma(1 + \mu r_i)(1 + g a_i)\,,$$

r_i, a_i: inhomogeneity parameters

$\mu, g \ll 1$: small dimensionless parameters

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Fixed total resources:

$\sum_i a_i = \sum_i r_i = 0,$

Bounded increases/decreases:

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$$\boxed{\mathcal{M}_b = \mathcal{M}_b^{(0)} + \textstyle\sum_i \rho_i r_i + \sum_i \alpha_i a_i + \mathcal{O}(\mu^2, g^2)}$$

where $\rho_i = \dfrac{\partial\mathcal{M}_b}{\partial r_i} = -\dfrac{\mu\delta P^2}{\gamma N}\displaystyle\sum_{\alpha>1}\frac{u_{\alpha i}^{(0)}u_{\alpha b}^{(0)}}{\lambda_\alpha^{(0)}}\,,$

$$\alpha_i = \frac{\partial\mathcal{M}_b}{\partial a_i} = -\frac{g\delta P^2}{2\gamma m}\Bigg[\sum_{\alpha>1}\frac{u_{\alpha i}^{(0)2}u_{\alpha b}^{(0)2}}{\lambda_\alpha^{(0)}} + \sum_{\substack{\alpha>1,\\ \beta\neq\alpha}}\frac{u_{\alpha i}^{(0)}u_{\beta i}^{(0)}u_{\alpha b}^{(0)}u_{\beta b}^{(0)}}{(\lambda_\alpha^{(0)}-\lambda_\beta^{(0)})^2 + 2\gamma(\lambda_\alpha^{(0)}+\lambda_\beta^{(0)})}\Bigg].$$

Optimization of the global vulnerability

There is no certainty on when and where a fault will occur.

Global "vulnerability" measure:

$$\mathcal{V} = \sum_b \eta_b \mathcal{M}_b$$

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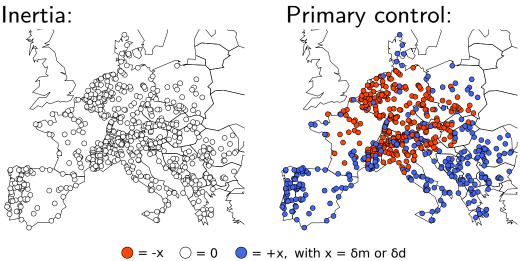

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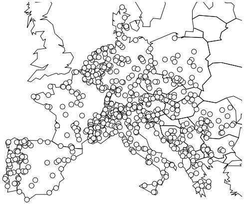
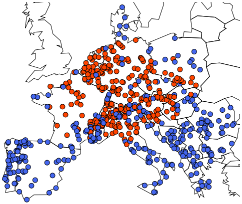
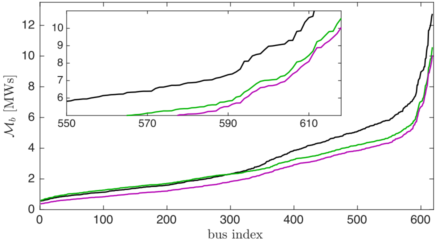


Primary control:



● = -x ○ = 0 ● = +x, with x = δm or δd



uniform, $\eta_b = 1$, max bound

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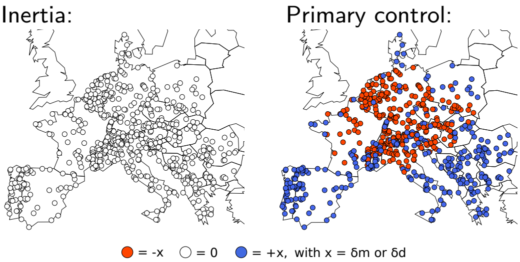

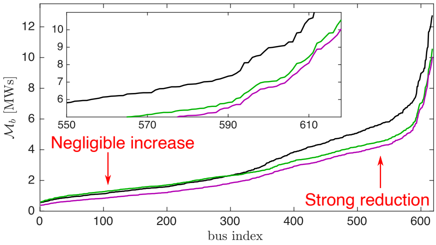

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Some other choices:

$\eta_b = \mathcal{M}_b^{(0)2}$

$\eta_b = \begin{cases} 1\,, & \text{if } \mathcal{M}_b^{(0)} > \mathcal{M}_{\text{thres}}\,, \\ 0\,, & \text{otherwise} \end{cases}$

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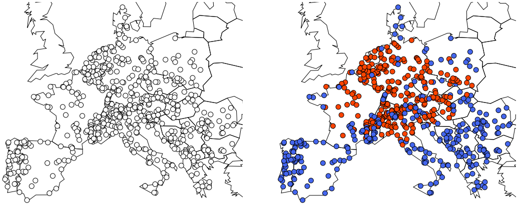
Inertia: Primary control:



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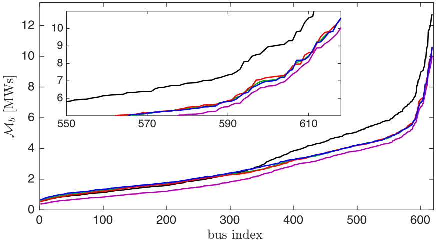

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Economical optimization of realistic configurations

For inhomogeneous initial configurations (still with $\Gamma = \gamma \mathbb{1}_N$), our procedure remains valid: $\boldsymbol{L} \rightarrow \boldsymbol{L}_M^{(0)} = \boldsymbol{M}_0^{-1/2} \boldsymbol{L} \boldsymbol{M}_0^{-1/2}$

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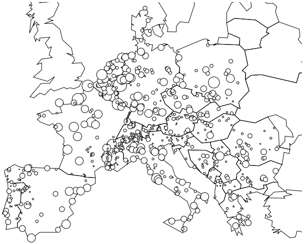

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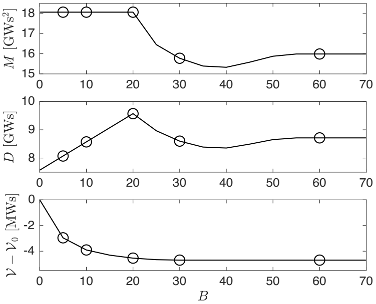

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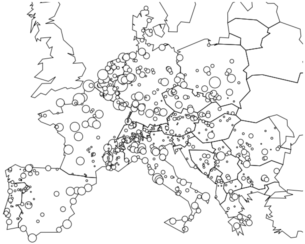

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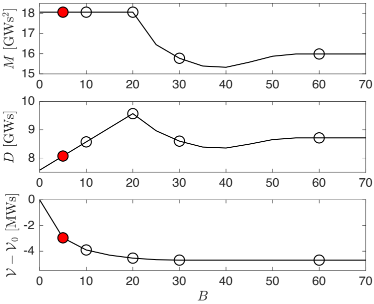

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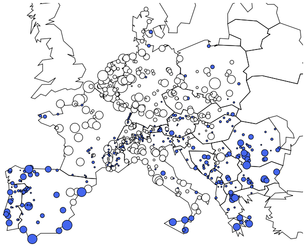

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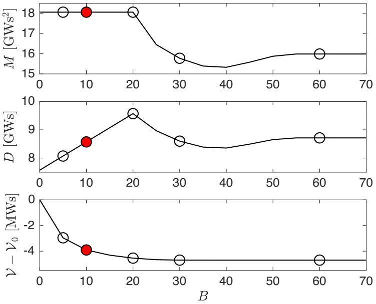

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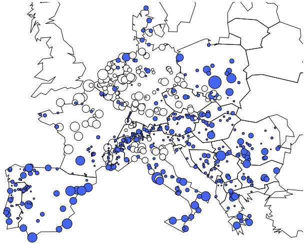

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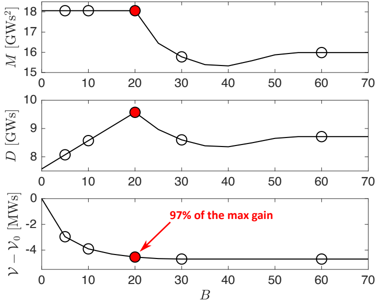

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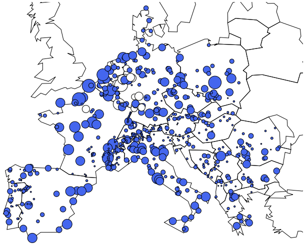

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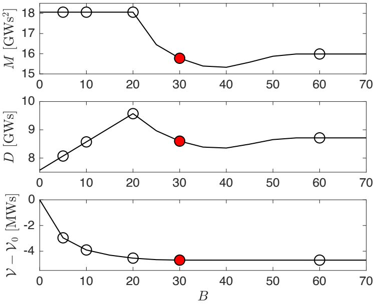

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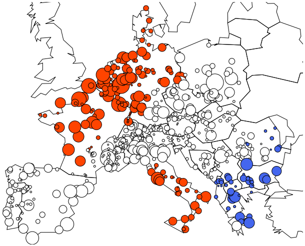

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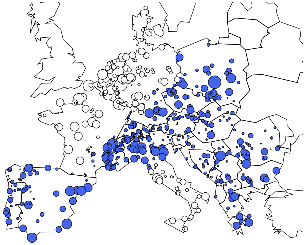

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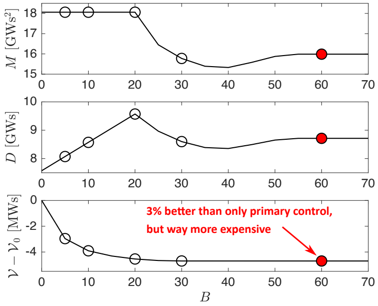

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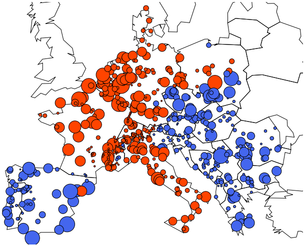

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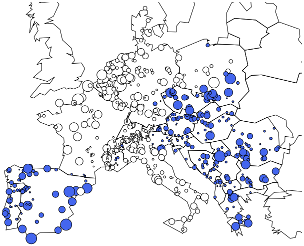

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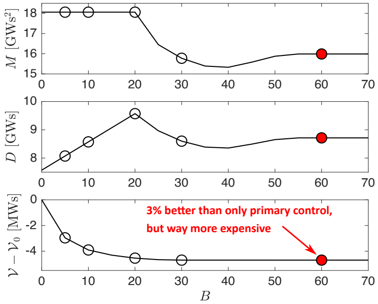

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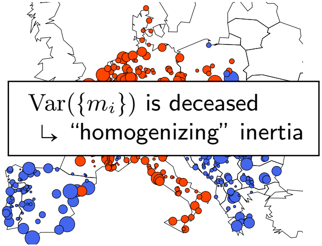
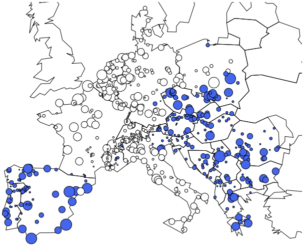


$\mathrm{Var}(\{m_i\})$ is deceased

↳ "homogenizing" inertia

Primary control:



Conclusion

We applied perturbation theory to optimally placed inertia and primary control.

In terms of global system vulnerability:

- Primary control is the key element to mitigate disturbances.
- If resources are limited, primary control must first be increased in the periphery of the grid.
- Inertia has a really limited effect compared to primary control.
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second order perturbation theory $\rightarrow$ quadratic programming

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Some reading:

L. Pagnier, P. Jacquod, *Optimal placement of inertia and primary control : a matrix perturbation theory approach*, to appear in IEEE-Access

L. Pagnier, P. Jacquod, *Inertia location and slow network modes determine disturbance propagation in large-scale power grids*, PloS-ONE (2019)

M. Tyloo, L. Pagnier, P. Jacquod, *The Key Player Problem in Complex Oscillator Networks and Electric Power Grids: Resistance Centralities Identify Local Vulnerabilities*, in press (Science Advances)