Robustness of Synchrony in Complex Networks and Generalized Kirchhoff Indices.

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MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018).
MT,Pagnier and Jacquod submitted (2018).
MT and Jacquod in preparation (2018).

Kuramoto model :

$$\dot{\theta}_i = P_i - \sum_j b_{ij} \sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n. \tag{1}$$

$b_{ij} = b_{ji} \geq 0$.

Steady-state solutions : Synchronous state $\{\theta_i^{(0)}\}$ such that :

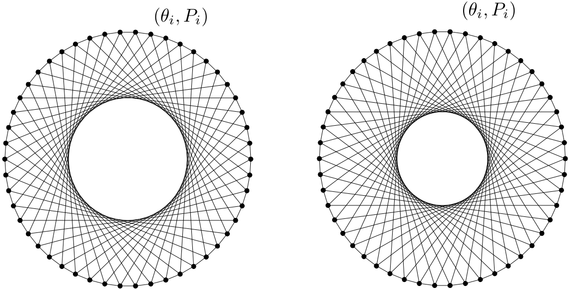
$$P_i = \sum_j b_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n. \tag{2}$$

$\sum_i P_i = 0.$

Kuramoto model :

$$\dot{\theta}_i = P_i - \sum_j b_{ij} \sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n, \tag{3}$$

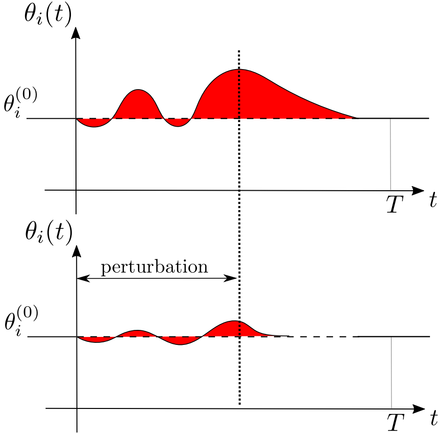
Examples :



Perturbations : $P_i \to P_i^{(0)} + \delta P_i(t)$.

Quantifying Robustness

Performance measures :



$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t \,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t \,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty)\,.$$

Noisy disturbances $\rightarrow$ divide by T.

Perturbations : $P_i \rightarrow P_i^{(0)} + \delta P_i(t)$.

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Response to Perturbations

Linear response : Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \rightarrow \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ , \tag{4}$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & i \neq j \, , \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & i = j \, . \end{cases} \tag{5}$$

Topology $\rightarrow b_{ij}$.

Steady state $\rightarrow \{\theta_i^{(0)}\}$.

Expanding on the eigenvectors $\mathbf{u}_\alpha$ of $\mathbb{L}$, we have $\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha$.

$\rightarrow \mathcal{P}_1(T)$, $\mathcal{P}_2(T)$ for specific perturbations

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Averaged Global Robustness and Kf_m's

Box perturbation $\delta \boldsymbol{P}(t) = \delta \boldsymbol{P}_0 \Theta(t)\Theta(\tau_0 - t)$,

$$\mathcal{P}_1^{\infty} = \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha^3} \left(\lambda_\alpha \tau_0 - 1 + e^{-\lambda_\alpha \tau_0}\right) \; . \tag{6}$$

Averaged Global Robustness : Averaging over an ensemble of perturbation vectors ,

$$\langle \mathcal{P}_1^{\infty} \rangle \simeq \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2n} Kf_1 \; , \; \lambda_\alpha \tau_0 \ll 1 \; . \tag{7}$$

Generalized Kirchhoff indices :

$$Kf_1 = \sum_{i<j} \Omega_{ij} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \; , \quad Kf_m = n \sum_{\alpha \geq 2} \lambda_\alpha^{-m} \; , \tag{8}$$

Ω_{ij} : Resistance distance between nodes i, j.

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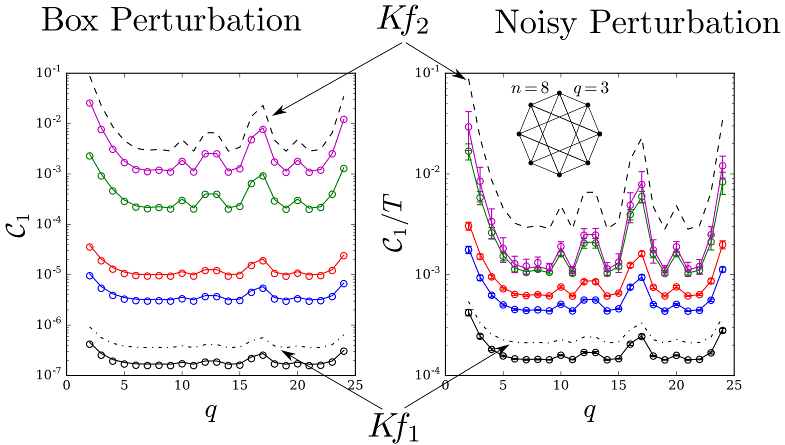
Averaged Global Robustness and Kf_m's



MT, Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

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Local Vulnerability : Perturbing a specific node i.e. $\delta P_{0i} = \delta_{ik} \delta P_0$,

$$\mathcal{P}_1^{\infty} \simeq \frac{\delta P_0^2 \tau_0^2}{2} \left(C_1^{-1}(k) - n^{-2} K f_1 \right) \ , \ \lambda_\alpha \tau_0 \ll 1 \quad . \tag{10}$$

Resistance Centralities

$$C_1(k) = \left[n^{-1} \sum_j \Omega_{kj} \right]^{-1} \quad , \quad C_m(k) = \left[\sum_{\alpha \geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^m} + n^{-2} K f_m \right]^{-1} . \tag{11}$$

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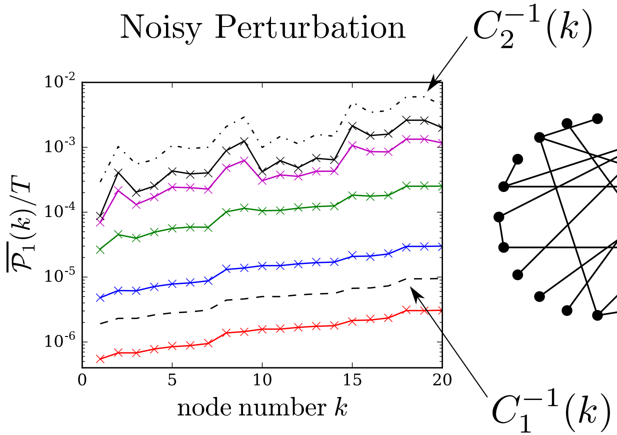
MT and Jacquod in preparation (2018).

Specific Local Vulnerabilities and C_m's



MT and Jacquod in preparation (2018).

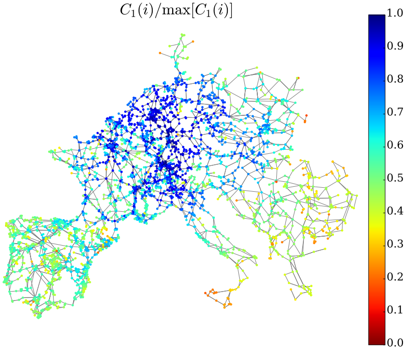
Power Grids : Lossless line approximation of the Swing Equations.

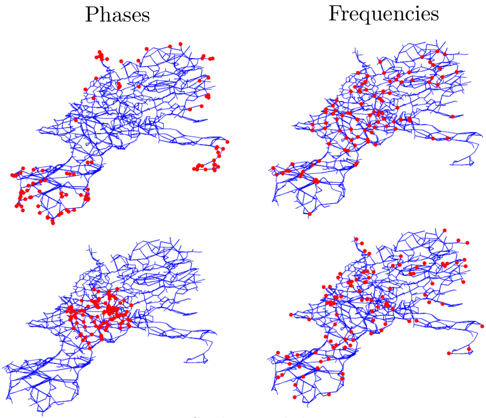
$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \underbrace{P_i}_{\text{inj./cons. power}} - \sum_j \underbrace{b_{ij}\sin(\theta_i - \theta_j)}_{\text{flux of power from } i \text{ to } j} \quad , \; i = 1, ..., n. \quad (12)$$

1. m_i, d_i : inertia and damping of the rotative mass at node i ($m_i = 0$ for consumers).
2. b_{ij} : line capacity between nodes i and j.
3. θ_i : phase of the complex voltage at node i.

MT, Pagnier and Jacquod submitted (2018).

Physical Realization : European Electric Grid





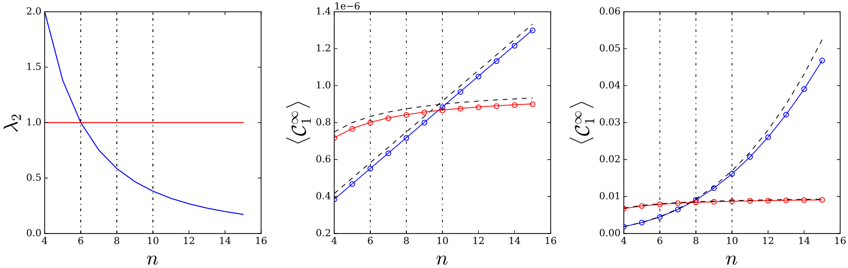
$$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0^2}{2}\left(C_1^{-1}(k) - n^{-2} K f_1\right) \quad \tau_0 \ll d/\lambda_\alpha \,, m/d$$

$$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2 \tau_0}{dm}\frac{(n-1)}{n} \quad \tau_0 \ll d/\lambda_\alpha \,, m/d$$

L. Gambuzza and al.

Analysis of Dynamical Robustness to Noise in Power Grids, IEEE Journal on Emerging and Selected Topics in Circuits and Systems, vol. 7, no 3 (2017).

MT, Pagnier and Jacquod submitted (2018).



blue : cycle graph
red : star graph

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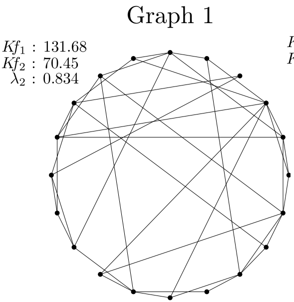
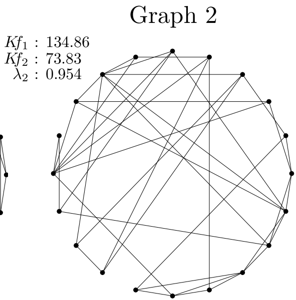
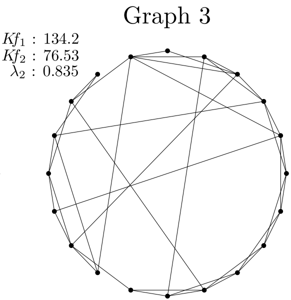
Supplemental Material







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