Optimal placement of inertia and primary control in high voltage power grids

Philippe Jacquod*†, Laurent Pagnier*‡

*School of Engineering, University of Applied Science of Western Switzerland HES-SO CH–1950 Sion, Switzerland

Email: {philippe.jacquod,laurent.pagnier}@hevs.ch

†Department of Quantum Matter Physics, University of Geneva, CH–1211 Geneva, Switzerland

‡ Institute of Physics, EPF Lausanne, CH-1015 Lausanne, Switzerland

***Abstract*—The energy transition's ultimate goal is to meet energy demand from human activities sustainably. Accordingly, the penetration of new renewable energy sources (RES) such as photovoltaic panels and wind turbines is increasing in most power grids. In their current configuration, RES are essentially inertialess, therefore, low inertia situations are more and more common, in periods of high RES production, making grid stability a high concern in power grids with high share of RES. It has been suggested that the resulting reduction of overall inertia can be compensated to some extent by the deployment of substitution inertia - synthetic inertia, flywheels, synchronous condensers aso. Of particular importance is to optimize the placement of the limited available substitution inertia. Here, we construct a matrix perturbation theory to optimize inertia and primary control placement under the assumption that both are moderately heterogeneous. Armed with that efficient tool, we construct simple but efficient algorithms that independently determine the optimal geographical distribution of inertia and primary control.**

I. Introduction

The increasing penetration of inertialess new RES raises issues of power grid stability. RES are connected to the grid via inverters and they essentially lack inertia in their current configuration. In the European power grid, the penetration is already high enough that the resulting reduction of overall inertia can be temporarily quite significant [1], which is a much harder problem to tackle than absorbing the volatility of RES [2], [3]. The lack of inertia of new RES can in principle be compensated by the deployment of substitution inertia such as flywheels, synthetic inertia or synchronous condensers. Because of the limited amount of such resources, a question that naturally arises is : where is it optimal to distribute the available substitution inertia ? This question has been investigated in power grid models with not much more than a dozen buses [4], [5], [6], [7], [8]. On larger power grids, numerical simulations have emphasized the competition between inertia and the location of slow network modes [9]. So far, analytical results have been obtained under assumptions on the homogeneity of damping and inertia parameters or of their ratio, which is a significant constraint when dealing with optimized geographical distributions of inertia and primary control. Here we construct an analytical method able to tackle moderate inhomogeneities in inertia and/or in damping to inertia ratios.

We introduce *matrix perturbation theory* [10], a method that is standardly used, e.g. in quantum physics, where it delivers approximate solutions to complex, perturbed problems, extrapolated from known, exact solutions of simpler, integrable problems [11]. The approximation is valid as long as the difference between the two problems is small and it makes sense to consider the full, complex problem as a small perturbation of the exactly soluble, simpler problem. The procedure identifies a dimensionless parameter in which eigenvalues and eigenvectors of the perturbed problem can be systematically expanded in a power series. When the dimensionless parameter is small, the power series converges fast and the behavior of the perturbed problem is captured by the first few terms in the expansion. We apply matrix perturbation theory [10] to calculate performance measures following an abrupt power loss. The main step forward brought about by our approach is that we are able to derive analytical results without relying on the often used homogeneity assumptions that damping, inertia or their ratio is constant - assumptions which are not satisfied in real electric power grids. Our main results are given in Theorems 1 and 2 below, which formulate algorithms for optimal placement of inertia and damping parameters. The spectral decomposition approach used here has been recently used to calculate performance measures in power grids and consensus algorithms e.g. in [7], [12], [13], [14], [15].

In Section II, we start by solving exactly the simple problem where the damping to inertia ratio is homogeneous. In Section III we apply matrix perturbation theory to approximate the solution of the nonhomogeneous problem from the exact solution of the homogeneous problem. Section IV discusses the optimal placement of inertia and primary control for moderate inhomogeneities. In Section V we apply our optimal placements to the continental European grid. Section VI concludes our article.

II. Homogeneous case

We consider power system dynamics in the lossless line approximation, appropriate to deal with high voltage transmission grids [16]. The dynamics is governed by the swing equations,

$$m_i\dot{\omega}_i + d_i\omega_i = P_i - \sum_j B_{ij}\sin(\theta_i - \theta_j)\,, \tag{1}$$

which determine the time-evolution of the voltage angles θ_i and frequencies $\omega_i = \dot{\theta}_i$ at each of the N buses in a rotating frame such that ω_i measures the angle frequency deviation to the rated grid frequency of 50 or 60 Hz. Each bus is characterized by an inertia parameter m_i and a damping ($\sim$ primary control) parameter d_i, and P_i is the active power injected ($P_i > 0$) or extracted ($P_i < 0$) at bus i. We define the damping ratio $\gamma_i \equiv d_i/m_i$. Buses are connected to one another via lines with susceptance B_{ij}. Stationary solutions $\{\theta_i^{(0)}\}$ are power flow solutions determined by $P_i = \sum_j B_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)})$. Under a change in active power $P_i \to P_i + \delta P_i$, linearizing the dynamics about such a solution with $\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$ gives, in matrix form,

$$\boldsymbol{M}\dot{\boldsymbol{\omega}} + \boldsymbol{D}\boldsymbol{\omega} = \boldsymbol{\delta P} - \boldsymbol{L}\boldsymbol{\delta\theta}\,, \tag{2}$$

where $\boldsymbol{M} = \mathrm{diag}(\{m_i\})$, $\boldsymbol{D} = \mathrm{diag}(\{d_i\})$ and voltage angles and frequencies are cast into vectors $\boldsymbol{\delta\theta}$ and $\boldsymbol{\omega}$. The Laplacian matrix $\boldsymbol{L}$ has matrix elements $L_{ij} = -B_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})$, for $i \neq j$ and $L_{ii} = \sum_k B_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})$.

A. Integrating the dynamics for homogeneous damping ratio

For constant damping ratio, $d_i/m_i = \gamma_i \equiv \gamma$, (2) can be integrated exactly [7], [17], [18]. We rewrite (2) in a new coordinate system $\boldsymbol{\delta\theta} = \boldsymbol{M}^{-1/2}\boldsymbol{\delta\theta_M}$ to obtain

$$\dot{\boldsymbol{\omega}}_{\boldsymbol{M}} + \underbrace{\boldsymbol{M}^{-1}\boldsymbol{D}}_{\boldsymbol{\Gamma}}\boldsymbol{\omega}_{\boldsymbol{M}} + \underbrace{\boldsymbol{M}^{-1/2}\boldsymbol{L}\boldsymbol{M}^{-1/2}}_{\boldsymbol{L_M}}\boldsymbol{\delta\theta_M} = \boldsymbol{M}^{-1/2}\boldsymbol{\delta P}\,, \tag{3}$$

with $\boldsymbol{\Gamma} = \mathrm{diag}(\{d_i/m_i\}) \equiv \mathrm{diag}(\{\gamma_i\})$. The inertia-weighted Laplacian matrix $\boldsymbol{L_M}$ is real and symmetric, therefore it can be diagonalized as $\boldsymbol{L_M} = \boldsymbol{U}^\top\boldsymbol{\Lambda}\boldsymbol{U}$ with an orthogonal matrix $\boldsymbol{U}$, the α^{th} row of which gives the components $u_{\alpha,i}$, $i = 1, \ldots N$ of the α^{th} eigenvector $\boldsymbol{u}_\alpha$ of $\boldsymbol{L_M}$. The diagonal matrix $\boldsymbol{\Lambda} = \mathrm{diag}(\{\lambda_1 = 0, \lambda_2, \cdots, \lambda_N\})$ contains the eigenvalues of $\boldsymbol{L}_M$ with $\lambda_\alpha < \lambda_{\alpha+1}$ and $\lambda_1 = 0$ following from the Laplacian property of $\boldsymbol{L_M}$. We make a change of basis and rewrite (3) in the basis diagonalizing $\boldsymbol{L}_M$. We obtain

$$\ddot{\boldsymbol{\xi}} + \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{U}^\top\dot{\boldsymbol{\xi}} + \boldsymbol{\Lambda}\boldsymbol{\xi} = \boldsymbol{U}\boldsymbol{M}^{-1/2}\boldsymbol{\delta P}\,, \tag{4}$$

where $\boldsymbol{\delta\theta_M} = \boldsymbol{U}^\top\boldsymbol{\xi}$. Equation (4) makes it clear that, if $\boldsymbol{\Gamma}$ is a multiple of identity, the problem can be recast as a diagonal ordinary differential equation problem that can be exactly integrated. This is done below in (10).

Proposition 1. *For homogeneous damping ratio and under an abrupt power loss, $\boldsymbol{\delta P}(t) = \boldsymbol{\delta P}\,\Theta(t)$ with the Heaviside step function defined by $\Theta(t > 0) = 1$, $\Theta(t < 0) = 0$, $\dot{\xi}_\alpha$ evolve independently from one another as*

$$\dot{\xi}_\alpha(t) = \frac{2\mathcal{P}_\alpha}{f_\alpha} e^{-\gamma t/2} \sin\left(\frac{f_\alpha t}{2}\right),\ \forall\alpha > 1, \tag{5}$$

where $f_\alpha = \sqrt{4\lambda_\alpha - \gamma^2}$ and $\mathcal{P}_\alpha = \sum_i u_{\alpha i}\,\delta P_i/m_i^{1/2}$.

This result generalizes Theorem III.3 of [14].

Proof: The proof goes along the lines of the diagonalization procedure proposed in [7], [17], [18], [19], [20]. Equation (4) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbb{0} & \mathbb{1}\\ -\boldsymbol{\Lambda} & -\gamma\mathbb{1}\end{bmatrix}}_{\boldsymbol{H}_0}\begin{bmatrix}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{bmatrix} + \begin{bmatrix}\mathbb{0}\\ \boldsymbol{\mathcal{P}}\end{bmatrix}, \tag{6}$$

where $\boldsymbol{\mathcal{P}} = \boldsymbol{U}\boldsymbol{M}^{-1/2}\boldsymbol{\delta P}$ and $\mathbb{0}$ is the $N \times M$ matrix of zeroes. The matrix $\boldsymbol{H}_0$ is block-diagonal up to a permutation of rows and columns [17], [18], and can easily be diagonalized block by block, where each 2×2 block corresponds to one of the eigenvalues λ_α of $\boldsymbol{L_M}$. The α^{th} block is diagonalized by the transformation

$$\begin{bmatrix}\chi_{\alpha+}\\ \chi_{\alpha-}\end{bmatrix} = \boldsymbol{T}_\alpha^L\begin{bmatrix}\xi_\alpha\\ \dot{\xi}_\alpha\end{bmatrix},\quad \boldsymbol{T}_\alpha^L \equiv \frac{i}{f_\alpha}\begin{bmatrix}\mu_{\alpha-}^{(0)} & -1\\ -\mu_{\alpha+}^{(0)} & 1\end{bmatrix}, \tag{7}$$

$$\begin{bmatrix}\xi_\alpha\\ \dot{\xi}_\alpha\end{bmatrix} = \boldsymbol{T}_\alpha^R\begin{bmatrix}\chi_{\alpha+}\\ \chi_{\alpha-}\end{bmatrix},\quad \boldsymbol{T}_\alpha^R \equiv \begin{bmatrix}1 & 1\\ \mu_{\alpha+}^{(0)} & \mu_{\alpha-}^{(0)}\end{bmatrix}, \tag{8}$$

with the eigenvalues $\mu_{\alpha\pm}^{(0)}$ of the α^{th} block,

$$\mu_{\alpha\pm}^{(0)} = -\frac{1}{2}(\gamma \mp i f_\alpha)\,. \tag{9}$$

The two rows (columns) of $\boldsymbol{T}_\alpha^L$ ($\boldsymbol{T}_\alpha^R$) give the nonzero components of the two left (right) eigenvectors $\boldsymbol{t}_{\alpha\pm}^L$ ($\boldsymbol{t}_{\alpha\pm}^R$) of $\boldsymbol{H}_0$. Following this transformation, (6) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\chi_{\alpha+}\\ \chi_{\alpha-}\end{bmatrix} = \begin{bmatrix}\mu_{\alpha+}^{(0)} & 0\\ 0 & \mu_{\alpha-}^{(0)}\end{bmatrix}\begin{bmatrix}\chi_{\alpha+}\\ \chi_{\alpha-}\end{bmatrix} + \frac{i}{f_\alpha}\begin{bmatrix}-\mathcal{P}_\alpha\\ \mathcal{P}_\alpha\end{bmatrix}. \tag{10}$$

Equation (10) is diagonal and its solutions are easily obtained. Transforming back to $\boldsymbol{\xi}$-coordinates, one finds (5). ■

B. Performance measure

We use performance measures which evaluate the overall disturbance magnitude following an abrupt power loss. We focus on frequency deviations and inspired by [6], [7], [17], [19], [21], [22], [23], [24], [13] we introduce the following quadratic performance measure

$$\mathcal{M} = \int_0^\infty (\boldsymbol{\omega}^\top - \bar{\boldsymbol{\omega}}^\top)\boldsymbol{M}(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})\mathrm{d}t\,, \tag{11}$$

where $\bar{\boldsymbol{\omega}} = (\omega_{\mathrm{sys}}, \omega_{\mathrm{sys}}, \ldots \omega_{\mathrm{sys}})^\top$ is the instantaneous average frequency vector with components

$$\omega_{\mathrm{sys}}(t) = \sum_i m_i\omega_i(t) \Big/ \sum_i m_i\,. \tag{12}$$

It is straightforward to check that $\mathcal{M}$ reads

$$\mathcal{M} = \int_0^\infty \sum_{\alpha>1} \dot{\xi}_\alpha^2(t)\mathrm{d}t\,, \tag{13}$$

when rewritten in the eigenbasis of $\boldsymbol{L_M}$, once one notices that the first eigenvector of $\boldsymbol{L_M}$ (the one with zero eigenvalue) has components $u_{1i} = \sqrt{m_i}/\sqrt{\sum_j m_j}$.

Proposition 2. *For an abrupt power loss, $\boldsymbol{\delta P}(t) = \boldsymbol{\delta P}\,\Theta(t)$ on a single bus labeled b, $\delta P_i = \delta_{ib}\,\delta P$, and with an homogeneous damping ratio, one has*

$$\mathcal{M}_b = \frac{\delta P^2}{2\gamma m_b} \sum_{\alpha>1} \frac{u_{\alpha b}^2}{\lambda_\alpha}\,, \tag{14}$$

in terms of the eigenvalues λ_α and the components $u_{\alpha b}$ of the eigenvectors $\boldsymbol{u}_\alpha$ of $\boldsymbol{L_M}$.

Note that we introduced the subscript b to indicate that the fault is localized on that bus only.

Proof: Equation (5) straightforwardly gives

$$\int_0^\infty \dot{\xi}_\alpha^2(t)\mathrm{d}t = \frac{u_{\alpha b}^2\,\delta P^2}{2\gamma\, m_b\, \lambda_\alpha}\,, \alpha > 1\,, \tag{15}$$

which, once inserted in (13) gives (14). ∎

Remark 1. *For homogeneous inertia coefficients, the eigenvectors and eigenvalues of $\boldsymbol{L_M}$ are given by $\boldsymbol{u}_\alpha = \boldsymbol{u}_\alpha^{(0)}$, and $\lambda_\alpha = m^{-1}\lambda_\alpha^{(0)}$, in terms of the eigenvectors $\boldsymbol{u}_\alpha^{(0)}$ and eigenvalues $\lambda_\alpha^{(0)}$ of $\boldsymbol{L}$. The performance measure then reads*

$$\mathcal{M}_b^{(0)} = \frac{\delta P^2}{2\gamma} \sum_{\alpha>1} \frac{u_{\alpha b}^{(0)2}}{\lambda_\alpha^{(0)}}\,, \tag{16}$$

where the superscript $^{(0)}$ refers to inertia homogeneity.

III. Perturbation Theory and the Nonhomogeneous Case

We next lift the restriction that the damping ratio is homogeneous. To parametrize inhomogeneities we write

$$m_i = m + \delta m\, r_i\,, \tag{17}$$
$$d_i = m_i\gamma_i = (m + \delta m\, r_i)(\gamma + \delta\gamma\, a_i)\,, \tag{18}$$

with the average m and γ and the maximum deviation amplitudes δm and $\delta\gamma$ of inertia and damping ratio. The coefficients $-1 \le a_i, r_i \le 1$ with $\sum_i r_i = \sum_i a_i = 0$ are determined following a minimization of $\mathcal{M}_b$. In the following two paragraphs we construct a matrix perturbation theory to linear order in the inhomogeneity parameters δm, and $\delta\gamma$ to calculate the performance measure $\mathcal{M}_b = \mathcal{M}_b^{(0)} + r_i\rho_i + a_i\alpha_i + \mathcal{O}(\delta m^2, \delta\gamma^2)$. This requires to calculate the *susceptibilities* $\rho_i \equiv \partial\mathcal{M}_b/\partial r_i$ and $\alpha_i \equiv \partial\mathcal{M}_b/\partial a_i$.

A. Inhomogeneity in inertia

When inertia is inhomogeneous, but $\gamma_i \equiv \gamma$, the system dynamics and $\mathcal{M}_b$ are still given by (6) and (14). However, the eigenvectors of the inertia-weighted Laplacian matrix $\boldsymbol{L_M}$ differ from those of $\boldsymbol{L}$ and consequently $\mathcal{M}_b$ is no longer equal to $\mathcal{M}_b^{(0)}$.

Assumption 1 (Moderate inhomogeneity in inertia)**.** *The deviations $\delta m\, r_i$ of the local inertias m_i are all small compared to their average m. We write $\boldsymbol{M} = m\big[\mathbb{1} + \mu\,\mathrm{diag}\big(\{r_i\}\big)\big]$, where $\mu \equiv \delta m/m \ll 1$ is a small, dimensionless parameter.*

To linear order in μ, the series expansion of $\boldsymbol{L_M}$ reads

$$\boldsymbol{L_M} = \boldsymbol{M}^{-1/2}\boldsymbol{L}\boldsymbol{M}^{-1/2} = m^{-1}\big[\boldsymbol{L} + \mu\boldsymbol{V}_1 + \mathcal{O}(\mu^2)\big]\,, \tag{19}$$

with $\boldsymbol{V}_1 = -\big(\boldsymbol{RL} + \boldsymbol{LR}\big)/2$ and $\boldsymbol{R} = \mathrm{diag}(\{r_i\})$. In this form, the inertia-weighted Laplacian matrix $\boldsymbol{L_M}$ is given by the sum of an easily diagonalizable matrix, $m^{-1}\boldsymbol{L}$, and a small perturbation matrix, $(\mu/m)\,\boldsymbol{V}_1$. Truncating the expansion of $\boldsymbol{L_M}$ at this linear order gives an error of order $\sim\mu^2$, which is small under Assumption 1.

Matrix perturbation theory gives approximate expressions for the eigenvectors and eigenvalues of $\boldsymbol{L_M}$ in terms of those of $\boldsymbol{L}$ [10]. To leading order in μ one has

$$\lambda_\alpha = m^{-1}\big[\lambda_\alpha^{(0)} + \mu\lambda_\alpha^{(1)} + \mathcal{O}(\mu^2)\big]\,, \tag{20}$$
$$\boldsymbol{u}_\alpha = \boldsymbol{u}_\alpha^{(0)} + \mu\boldsymbol{u}_\alpha^{(1)} + \mathcal{O}(\mu^2)\,, \tag{21}$$

with

$$\lambda_\alpha^{(1)} = \boldsymbol{u}_\alpha^{(0)\top}\boldsymbol{V}_1\boldsymbol{u}_\alpha^{(0)}\,, \tag{22}$$
$$\boldsymbol{u}_\alpha^{(1)} = \sum_{\beta\neq\alpha} \frac{\boldsymbol{u}_\beta^{(0)\top}\boldsymbol{V}_1\boldsymbol{u}_\alpha^{(0)}}{\lambda_\alpha^{(0)} - \lambda_\beta^{(0)}}\boldsymbol{u}_\beta^{(0)}\,. \tag{23}$$

From (14), (20) and (21), we approximate $\mathcal{M}_b$ as

$$\mathcal{M}_b = \mathcal{M}_b^{(0)} + \frac{\mu\delta P^2}{2\gamma}\sum_{\alpha>1}\lambda_\alpha^{(0)-1}\Big(2u_{\alpha b}^{(0)}u_{\alpha b}^{(1)} - r_b u_{\alpha b}^{(0)2} - u_{\alpha b}^{(0)2}\lambda_\alpha^{(0)-1}\lambda_\alpha^{(1)}\Big) + \mathcal{O}(\mu^2)\,. \tag{24}$$

Proposition 3. *For an abrupt power loss, $\boldsymbol{\delta P}(t) = \boldsymbol{\delta P}\,\Theta(t)$ on a single bus labeled b ($\delta P_i = \delta_{ib}\,\delta P$), and under Assumption 1, the susceptibilites $\rho_i \equiv \partial\mathcal{M}_b/\partial r_i$ are given by*

$$\rho_i = -\frac{\mu\delta P^2}{\gamma N}\sum_{\alpha>1}\frac{u_{\alpha b}^{(0)}u_{\alpha i}^{(0)}}{\lambda_\alpha^{(0)}}\,. \tag{25}$$

Proof: Take the derivative of (24) with respect to r_i, with $\lambda_\alpha^{(1)}$ and $u_{\alpha b}^{(1)}$ given in (22) and (23). Equation (25) is obtained after some algebra. ∎

B. Inhomogeneity in damping ratios

Equations (5) give exact solutions to the linearized dynamical problem defined in (4), under the assumption of homogeneous damping ratio, $d_i/m_i \equiv \gamma$. In this section we lift that constraint and write $\gamma_i = \gamma + \delta\gamma\, a_i$. With inhomogeneous damping ratios, (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbb{0} & \mathbb{1}\\ -\boldsymbol{\Lambda} & -\gamma\mathbb{1} - \delta\gamma\,\boldsymbol{V}_2\end{bmatrix}}_{\boldsymbol{H}}\begin{bmatrix}\boldsymbol{\xi}\\ \dot{\boldsymbol{\xi}}\end{bmatrix} + \begin{bmatrix}\mathbb{0}\\ \boldsymbol{\mathcal{P}}\end{bmatrix}, \tag{26}$$

which differs from (6) only through the additional term $-\delta\gamma\boldsymbol{V}_2$ with $\boldsymbol{V}_2 = \boldsymbol{UAU}^\top$, $\boldsymbol{A} = \mathrm{diag}(\{a_i\})$. Under the assumption that $g \equiv \delta\gamma/\gamma \ll 1$, this additional term gives only small corrections to the unperturbed problem (6).

Assumption 2 (Moderate inhomogeneity in damping ratios)**.** *The deviations $\delta\gamma\, a_i$ of the damping ratio γ_i from their average γ are all small compared to their average. We write $\boldsymbol{\Gamma} = \gamma\big[\mathbb{1}+$*

$g\,\mathrm{diag}\big(\{a_i\}\big)\big]$, *where* $g \equiv \delta\gamma/\gamma \ll 1$ *is a small, dimensionless parameter.*

We want to integrate (26) using the spectral decomposition described in the proof of Proposition 1. In principle this requires to know the eigenvalues and eigenvectors of $\boldsymbol{H}$ in (26), which is impossible analytically, because $\boldsymbol{V}_2$ does not commute with $\boldsymbol{\Lambda}$. When g is small enough, the eigenvalues and eigenvectors are only slightly altered [10] and can be systematically calculated order by order in a polynomial expansion in g [10]. Formally, one has, for the eigenvalues $\mu_{\alpha s}(g)$ and for the left and right eigenvectors $\boldsymbol{t}_{\alpha s}^{L,R}(g)$ of $\boldsymbol{H}$, $s = \pm$,

$$\mu_{\alpha s} = \sum_{m=1}^{\infty} g^m\,\mu_{\alpha s}^{(m)}\,, \qquad \boldsymbol{t}_{\alpha s}^{L,R}(g) = \sum_{m=1}^{\infty} g^m\,\boldsymbol{t}_{\alpha s}^{(m)L,R}\,, \quad (27)$$

where the $m = 0$ terms are given by the eigenvalues and the left and right eigenvectors of the unperturbed matrix $\boldsymbol{H}_0$ in (6), corresponding to homogeneous inertia. In order for the sums in (27) to converge, a necessary condition is that $g < 1$. The task is to calculate the terms $\mu_{\alpha s}^{(m)}$ and $\boldsymbol{t}_{\alpha s}^{(m)L,R}$ with $m = 1, 2, \ldots$. When $g \ll 1$, one expects that only few, low order terms already give a good estimate of the eigenvalues and eigenvectors of $\boldsymbol{H}$. The $m = 1$ corrections are given by formulas similar to (22) and (23),

$$g\,\mu_{\alpha s}^{(1)} = \boldsymbol{t}_{\alpha s}^{(0)L} \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & -\delta\gamma\,\boldsymbol{V}_2 \end{bmatrix} \boldsymbol{t}_{\alpha s}^{(0)R}\,, \quad (28)$$

$$g\,\boldsymbol{t}_{\alpha s}^{(1)R} = \overline{\sum_{\beta,s'}} \frac{\boldsymbol{t}_{\beta s'}^{(0)L} \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & -\delta\gamma\,\boldsymbol{V}_2 \end{bmatrix} \boldsymbol{t}_{\alpha s}^{(0)R}}{\mu_{\alpha s}^{(0)} - \mu_{\beta s'}^{(0)}} \boldsymbol{t}_{\beta s'}^{(0)R}\,, \quad (29)$$

$$g\,\boldsymbol{t}_{\alpha s}^{(1)L} = \overline{\sum_{\beta,s'}} \frac{\boldsymbol{t}_{\alpha s}^{(0)L} \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & -\delta\gamma\,\boldsymbol{V}_2 \end{bmatrix} \boldsymbol{t}_{\beta s'}^{(0)R}}{\mu_{\alpha s}^{(0)} - \mu_{\beta s'}^{(0)}} \boldsymbol{t}_{\beta s'}^{(0)L}\,, \quad (30)$$

where $\overline{\sum}$ indicates that the sum runs over $(\beta, s') \neq (\alpha, s)$. One obtains

$$g\,\mu_{\alpha s}^{(1)} = -\delta\gamma\Big(\frac{1}{2} + is\frac{\gamma}{2f_\alpha}\Big)\boldsymbol{V}_{2;\alpha\alpha}\,, \quad (31)$$

$$g\,\boldsymbol{t}_{\alpha s}^{(1)R} = 2\,\delta\gamma \overline{\sum_{\beta,s'}} \frac{\boldsymbol{V}_{2;\alpha\beta}\,\mu_{\alpha s}^{(0)}}{f_\beta(ss'\,f_\alpha - f_\beta)} \boldsymbol{t}_{\beta s'}^{(0)R}\,, \quad (32)$$

$$g\,\boldsymbol{t}_{\alpha s}^{(1)L} = 2\,\delta\gamma \overline{\sum_{\beta,s'}} \frac{\boldsymbol{V}_{2;\alpha\beta}\,\mu_{\beta s'}^{(0)}}{f_\alpha(f_\alpha - ss'\,f_\beta)} \boldsymbol{t}_{\beta s'}^{(0)L}\,, \quad (33)$$

with $\boldsymbol{V}_{2;\alpha\beta} = \sum_i a_i\,u_{\alpha i}^{(0)}\,u_{\beta i}^{(0)}$.

Remark 2. *By definition,* $-1 \leq \boldsymbol{V}_{2;\alpha\alpha} \leq 1$. *Therefore, from* (31) *one sees that when the parameters* $\{a_i\}$ *are correlated (anticorrelated) with the square components* $\{u_{\alpha i}^2\}$ *then the mode* α *is more strongly (more weakly) damped.*

Proposition 4. *For an abrupt power loss,* $\boldsymbol{\delta P}(t) = \boldsymbol{\delta P}\,\Theta(t)$ *on a single bus labeled* b ($\delta P_i = \delta_{ib}\,\delta P$)*, and under Assumption* 2*,* $\dot{\xi}_\alpha(t)$ *reads, to leading order in* g*,*

$$\begin{aligned}\dot{\xi}_\alpha(t) =& \frac{\mathcal{P}_\alpha}{f_\alpha} e^{-\gamma t/2}\Big[2s_\alpha\left(1 + g\frac{\gamma^2}{f_\alpha^2}\boldsymbol{V}_{2;\alpha\alpha}\right) \\ &\qquad\qquad - g\gamma t\boldsymbol{V}_{2;\alpha\alpha}\Big(s_\alpha + \frac{\gamma}{f_\alpha}c_\alpha\Big)\Big] \\ &+ g\gamma\sum_{\beta\neq\alpha}\frac{\boldsymbol{V}_{2;\alpha\beta}\mathcal{P}_\beta}{\lambda_\alpha - \lambda_\beta} e^{-\gamma t/2}\Big[\frac{\gamma}{f_\beta}s_\beta - \frac{\gamma}{f_\alpha}s_\alpha + c_\alpha - c_\beta\Big] \\ &+ \mathcal{O}(g^2)\,, \end{aligned} \quad (34)$$

where $s_\alpha = \sin(f_\alpha t/2)$ *and* $c_\alpha = \cos(f_\alpha t/2)$*, and* $\mathcal{P}_\alpha$ *and* f_α *are defined below* (5)*.*

The proof follows the same steps as for Proposition 1. One first performs a unitary transformation to rewrite (26) in the left and right eigenbasis of $\boldsymbol{H}$. The unitary matrix that does this transformation has elements given by components of the left and right eigenvectors of $\boldsymbol{H}$, which we calculate to first order in g using (27) and (32)–(33). In this basis, one has an equation similar to (10), with perturbed eigenvalues which one also calculates perturbatively to first order in g, using (31). The expansion has to be transformed back to the original ξ-basis and this is again done in linear order in g. It is rather tedious, though algebraically straightforward to obtain (34). Details will be given elsewhere.

Proposition 5. *For an abrupt power loss,* $\boldsymbol{\delta P}(t) = \boldsymbol{\delta P}\,\Theta(t)$ *on a single bus labeled* b ($\delta P_i = \delta_{ib}\,\delta P$)*, and under Assumption* 2*, the susceptibilities* $\alpha_i \equiv \partial\mathcal{M}_b/\partial a_i$ *are given by*

$$\begin{aligned}\alpha_i \quad = \quad & -\frac{g\delta P^2}{2\gamma m_b}\Bigg[\sum_{\alpha>1}\frac{u_{\alpha i}^2 u_{\alpha b}^2}{\lambda_\alpha} \\ &+ \sum_{\alpha>1,\beta\neq\alpha}\frac{u_{\alpha i}\,u_{\alpha b}u_{\beta i}\,u_{\beta b}}{(\lambda_\alpha-\lambda_\beta)^2 + 2\gamma^2(\lambda_\alpha+\lambda_\beta)}\Bigg]\end{aligned} \quad (35)$$

The calculation is straightforward though somehow tedious. We have found numerically that the second term is generally much smaller than the first one.

IV. Optimal placement of inertia and primary control

Theorems 1 and 2 below give two simple algorithms for optimal distribution of the parameters a_i and r_i under Assumption 1 that inertia is moderately inhomogeneous at constant damping ratio and under Assumption 2 that the damping ratio is moderately inhomogeneous at constant inertia, respectively. Additionally, Conjecture 1 proposes an algorithm for combined optimization under the assumption that both inertia and damping ratio are moderately inhomogeneous.

Theorem 1. *For an abrupt power loss, under Assumption 1 and with homogeneous damping, the optimal distribution of parameters* $\{r_i\}$ *that minimizes* $\mathcal{M}_b$ *is obtained as follows.*

1) *Sort the sensitivities* $\rho_i = \partial\mathcal{M}_b/\partial r_i$ *as in* (25) *in ascending order.*

2) *Set $r_i = 1$ for $i = 1, \dots \mathrm{Int}[N/2]$ and $r_i = -1$ for $i = N - \mathrm{Int}[N/2] + 1, \dots N$.*

The optimal placement of inertia and primary control is given by

$$m_i = m + \delta m\, r_i\,, \qquad d_i = \gamma(m + \delta m\, r_i)\,. \tag{36}$$

Details of the proof will be given elsewhere.

Theorem 2. *For an abrupt power loss, under Assumption 2 and with homogeneous inertia, the optimal distribution of parameters $\{a_i\}$ that minimizes $\mathcal{M}_b$ is obtained as follows.*

1) *Sort the sensitivities $\alpha_i = \partial \mathcal{M}_b / \partial r_i$ as in (35) in ascending order.*
2) *Set $a_i = 1$ for $i = 1, \dots \mathrm{Int}[N/2]$ and for $i = N - \mathrm{Int}[N/2] + 1, \dots N$.*

The optimal placement of primary control is given by

$$d_i = m(\gamma + \delta\gamma\, a_i)\,. \tag{37}$$

The proof is the same as the one for Theorem 1 and will be given elsewhere.

We next conjecture an algorithmic combined linear optimization treating simultaneously Assumptions 1 and 2. The difficulty is that for fixed total inertia and damping, one must have $\sum_i m_i = N\, m$, $\sum_i d_i = N\, d$. From (18), the second condition requires $\sum_i a_i r_i = 0$. This is a quadratic, nonconvex constraint, which makes the problem nontrivial to solve. The following conjecture presents an algorithm that starts from the distribution $\{a_i\}$ and $\{r_i\}$ from Theorems 1 and 2 and orthogonalizes them while trying to minimize the associated increase in $\mathcal{M}_b$.

Conjecture 1 (Combined linear optimization)**.** *For an abrupt power loss, under Assumptions 1 and 2, the optimal placement of a fixed total amount of inertia $\sum_i m_i = mN$ and primary control $\sum_i d_i = dN$ that minimize $\mathcal{M}_b$ is obtained as follows.*

1) *Compute the parameters r_i and a_i.*
   a) *If N is odd, align the zeros of $\{r_i\}$ and $\{a_i\}$. Let i_{r0} and i_{a0} be the indexes of these zeros. Their new common index is*

   $$i_{\rm align} = \underset{i}{\rm argmin}(r_i \rho_{i_{r0}} + a_i \alpha_{i_{a0}} - r_i \rho_i - a_i \alpha_i)\,.$$

   Interchange the parameter values $r_{i_{r0}} \leftrightarrow r_{i_{\rm align}}$ and $a_{i_{r0}} \leftrightarrow a_{i_{\rm align}}$.
   b) *If N is even, do nothing*
2) *If $n \equiv \sum_i r_i a_i = 0$, the optimization is done.*
3) *Find the set $\mathcal{I} = \{i\,|\,\mathrm{sgn}(r_i a_i) = \mathrm{sgn}(n)\}$. To reach $\sum_i r_i a_i \to 0$, our strategy is to set to zero some elements $r_i a_i$ for some $i \in \mathcal{I}$. Since however $\sum_i a_i = \sum_i r_i = 0$ must be conserved, this must be accompanied by a simultaneous change of some other parameter.*
4) *Find the pair $(a_{i1}, a_{i2} = -a_{i1})$ or $(r_{i1}, r_{i2} = -r_{i1}) \in \mathcal{I} \times \mathcal{I}$ which, when sent to $(0,0)$, induce the smallest increase of the objective function $\mathcal{M}_b$. Send it to $(0,0)$. Because the two parameters in the pair have opposite sign, this does not affect the condition $\sum_i a_i = \sum_i r_i = 0$.*
5) *go to step # 2.*

It is not at all guaranteed that the algorithm presented in Conjecture 1 is optimal, however numerical results to be presented below show that it goes in the right direction.

Because we are interested in finding the optimal distribution of inertia and/or primary control for all possible faults, we introduce the following global vulnerability measure

$$\mathcal{V} = \sum_b \eta_b\, \mathcal{M}_b\,, \tag{38}$$

where the sum runs over all generator buses. The vulnerability measure $\mathcal{V}$ gives a weighted average over all possible fault positions. The weight η_b accounts for (i) the probability that a fault occurs at b and (ii) its potential severity as given, e.g. by the rated power of the generator at bus b. In the preliminary investigations to be presented below, we set $\eta_b \equiv 1$.

V. Numerical simulations

We illustrate our main results on a model of the synchronous power grid of continental Europe. The network has 3809 nodes, among them 618 generators, connected through 4944 lines. For details of the model and its construction we refer the reader to [25], [9]. To connect to the theory presented above, we remove inertialess buses through a Kron reduction [26] and uniformize the distribution of inertia to $m_i = 29.22\mathrm{MWs}^2$, and primary control $d_i = 12.25\mathrm{MWs}$. This guarantees that the total amounts of inertia and primary control are kept at their initial levels.

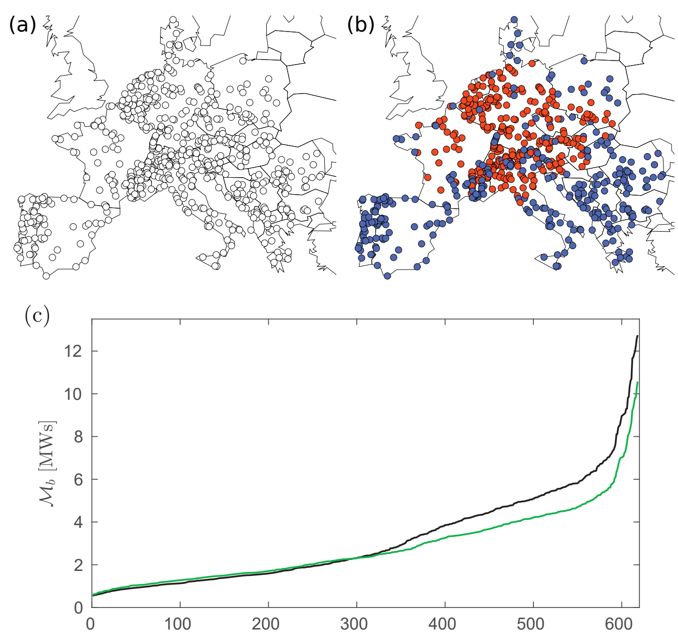

Fig. 1. Deviation from homogeneous inertia (a) and primary control (b) following the minimization of $\mathcal{V}$ in (38) with homogeneous weight $\eta_b \equiv 1$. $r_i = -1, 0, 1$ (left) and $a_i = -1, 0, 1$ (right) are displayed in red, white and blue respectively. (c) Vulnerability $\mathcal{M}_b$ vs. fault location (in increasing order of $\mathcal{M}_b$) for the homogeneous model (black) and the optimized model corresponding to panels (a) and (b) (green).

Fig. 1 (a) and (b) show the optimal inertia and primary control distribution that minimize $\mathcal{V}$ of (38) with constant weights, $\eta_b = 1$. With that choice of global vulnerability measure, an homogeneous distribution of inertia is a local optimum. This directly follows from (25), with $\sum_b u_{\alpha b}^{(0)} = 0$, $\forall\alpha > 1$. Primary control on the other hand needs to be distributed primarily on peripheral buses. Fig. 1 (c) furthermore shows that the minimization of $\mathcal{V}$ significantly reduces the performance measure $\mathcal{M}_b$ for fault location b where it is largest, i.e. for faults leading to the largest transient response, while not affecting much $\mathcal{M}_b$ where it is small. The optimal placement of primary control displayed in Fig. 1 (b) decreases $\mathcal{V}$ by more than 12% with respect to the homogeneous case.

VI. Conclusion

This manuscript presents a first step towards an analytical optimization of inertia and primary control location in electric power grids with limited inertia and primary control. Our treatment is so far limited to leading (linear) order in inertia and damping ratio inhomogeneities, and the next step is obviously to extend it to the next order in inhomogeneities, which would both allow to explore a wider range of inhomogeneities and go beyond the linear optimization problem discussed here. Additionally, exploring different, nonconstant weights η_b in the global vulnerability measure (38) should give nontrivial inertia distributions.

Acknowledgment

This work has been supported by the Swiss National Science Foundation under an AP Energy Grant.

References

[1] A. Ulbig, T. S. Borsche, and G. Andersson, "Impact of low rotational inertia on power system stability and operation," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 7290–7297, 2014.

[2] M. Milligan, B. Frew, B. Kirby, M. Schuerger, K. Clark, D. Lew, P. Denholm, B. Zavadil, M. O'Malley, and B. Tsuchida, "Alternatives no more: Wind and so- lar power are mainstays of a clean, reliable, affordable grid," *IEEE Power and Energy Magazine*, no. 13, pp. 78–87, 2015.

[3] W. Winter, K. Elkington, G. Bareux, and J. Kostevc, "Pushing the limits: Europe's new grid: Innovative tools to combat transmission bottlenecks and reduced inertia," *IEEE Power and Energy Magazine*, no. 13, pp. 60–74, 2015.

[4] T. S. Borsche, T. Liu, and D. J. Hill, "Effects of rotational inertia on power system damping and frequency transients," in *Decision and Control (CDC), 2015 IEEE 54th Annual Conference on*. IEEE, 2015, pp. 5940–5946.

[5] A. Mešanović, U. Münz, and C. Heyde, "Comparison of H_∞, H_2, and pole optimization for power system oscillation damping with remote renewable generation," *IFAC-PapersOnLine*, vol. 49, no. 27, pp. 103–108, 2016.

[6] B. K. Poolla, S. Bolognani, and F. Dörfler, "Optimal placement of virtual inertia in power grids," *IEEE Transactions on Automatic Control*, vol. 62, no. 12, pp. 6209–6220, 2017.

[7] F. Paganini and E. Mallada, "Global performance metrics for synchronization of heterogeneously rated power systems: The role of machine models and inertia," in *55th Annual Allerton Conference on Communication, Control, and Computing*, Oct 2017, pp. 324–331.

[8] T. S. Borsche and Dörfler, "On placement of synthetic inertia with explicit time-domain constraints," *arXiv:1705.03244*, 2017.

[9] L. Pagnier and P. Jacquod, "Disturbance propagation, inertia location and slow modes in large-scale high voltage power grids," *arXiv:1810.04982*, 2018.

[10] G. W. Stewart and J.-G. Sun, *Matrix perturbation theory*. Academic Press, Boston, 1990.

[11] D. McIntyre, *Quantum Mechanics*. Pearson Addison-Wesley, San Francisco, 2012.

[12] M. Pirani, J. Simpson-Porco, and B. Fidan, "System-theoretic performance metrics for low-inertia stability of power networks," in *Decision and Control (CDC), 2017 IEEE 56th Annual Conference on*. IEEE, 2017, pp. 5106–5111.

[13] M. Tyloo, T. Coletta, and P. Jacquod, "Robustness of synchrony in complex networks and generalized kirchhoff indices," *Phys. Rev. Lett.*, vol. 120, p. 084101, 2018.

[14] L. Guo, C. Zhao, and S. Low, "Graph laplacian spectrum and primary frequency regulation," *arXiv preprint arXiv:1803.03905*, 2018.

[15] M. Porfiri and M. Frasca, "Robustness of synchronization to additivie noise: how vulnerability depends on dynamics," *to appear in IEEE*, 2018.

[16] J. Machowski, J. Bialek, and J. R. Bumby, *Power system dynamics: stability and control*, 2nd ed. John Wiley & Sons, 2008.

[17] T. Coletta and P. Jacquod, "Performance measures in electric power networks under line contingencies," *arXiv preprint arXiv:1711.10348*, 2017.

[18] ——, "Performance measures in electric power networks under line contingencies," *IFAC PapersOnLine*, no. 51–23, pp. 337–342, 2018.

[19] T. Coletta, B. Bamieh, and P. Jacquod, "Transient performance of electric power networks under colored noise," *arXiv preprint arXiv:1807.09048*, 2018.

[20] ——, "Transient performance of electric power networks under colored noise," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 6163–6167.

[21] E. Tegling, B. Bamieh, and D. F. Gayme, "The price of synchrony: Evaluating the resistive losses in synchronizing power networks." *IEEE Trans. Control of Network Systems*, vol. 2, no. 3, pp. 254–266, 2015.

[22] M. Fardad, F. Lin, and M. R. Jovanovic, "Design of optimal sparse interconnection graphs for synchronization of oscillator networks," *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2457–2462, 2014.

[23] T. W. Grunberg and D. F. Gayme, "Performance measures for linear oscillator networks over arbitrary graphs," *IEEE Transactions on Control of Network Systems*, vol. PP, no. 99, p. 1, 2016.

[24] M. Siami and N. Motee, "Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 4055–4062, 2016.

[25] M. Tyloo, L. Pagnier, and P. Jacquod, "The key player problem in complex oscillator networks and electric power grids: Resistance centralities identify local vulnerabilities," submitted.

[26] F. Dörfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks." *IEEE Trans. on Circuits and Systems*, vol. 60, no. 1, pp. 150–163, 2013.