Robustness of electoral systems to external attack

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Dynamics

The dynamics with *natural opinions* ($\boldsymbol{x}_0 \in [-1,1]^N$)

$$\dot{\boldsymbol{x}} = -D^{-1}\mathbb{L}_\epsilon \boldsymbol{x} - (\boldsymbol{x} - \boldsymbol{x}_0)\,, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^N$ and $\mathbb{L}_\epsilon$ is the Laplacian matrix,

$$\mathbb{L}_{\epsilon,ij} := \begin{cases} -1 & i \neq j\,,\ |x_{0i} - x_{0j}| < \epsilon\,, \\ -\sum_k \mathbb{L}_{ik} & \text{if } i = j\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

$\mathbb{I}$ is the identity matrix and D is the degree matrix.

$$D_i := \begin{cases} \sum_k \mathbb{L}_{\epsilon,ik} & \text{if } i = j\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Dynamics

Natural opinion :

$$f(x_{0i}) = \frac{p}{\sigma\sqrt{2\pi}} e^{-\frac{(x_{0i}-\mu-\Delta)^2}{2\sigma^2}} + \frac{(1-p)}{\sigma\sqrt{2\pi}} e^{-\frac{(x_{0i}-\mu+\Delta)^2}{2\sigma^2}} , \tag{4}$$

where μ : bias
Δ : polarization
σ : standard deviation
$p \in [0, 1]$: proportion of negatively opinionated agents .

Outcome :

$$\boldsymbol{x}^* = (D^{-1}\mathbb{L}_\epsilon + \mathbb{I})^{-1}\boldsymbol{x}_0 . \tag{5}$$

External Attack

Dynamics with the external attack :

$$\dot{\boldsymbol{x}} = -D^{-1}\mathbb{L}_\epsilon \boldsymbol{x} - (\boldsymbol{x} - \boldsymbol{x}_0 - \boldsymbol{\omega})\,. \tag{6}$$

Outcome with external attack :

$$\boldsymbol{x}^{**} = (D^{-1}\mathbb{L}_\epsilon + \mathbb{I})^{-1}(\boldsymbol{x}_0 + \boldsymbol{\omega})\,. \tag{7}$$

Effort needed to change the outcome of election :

$$\xi := \|\boldsymbol{\omega}\|_1\,. \tag{8}$$

Example

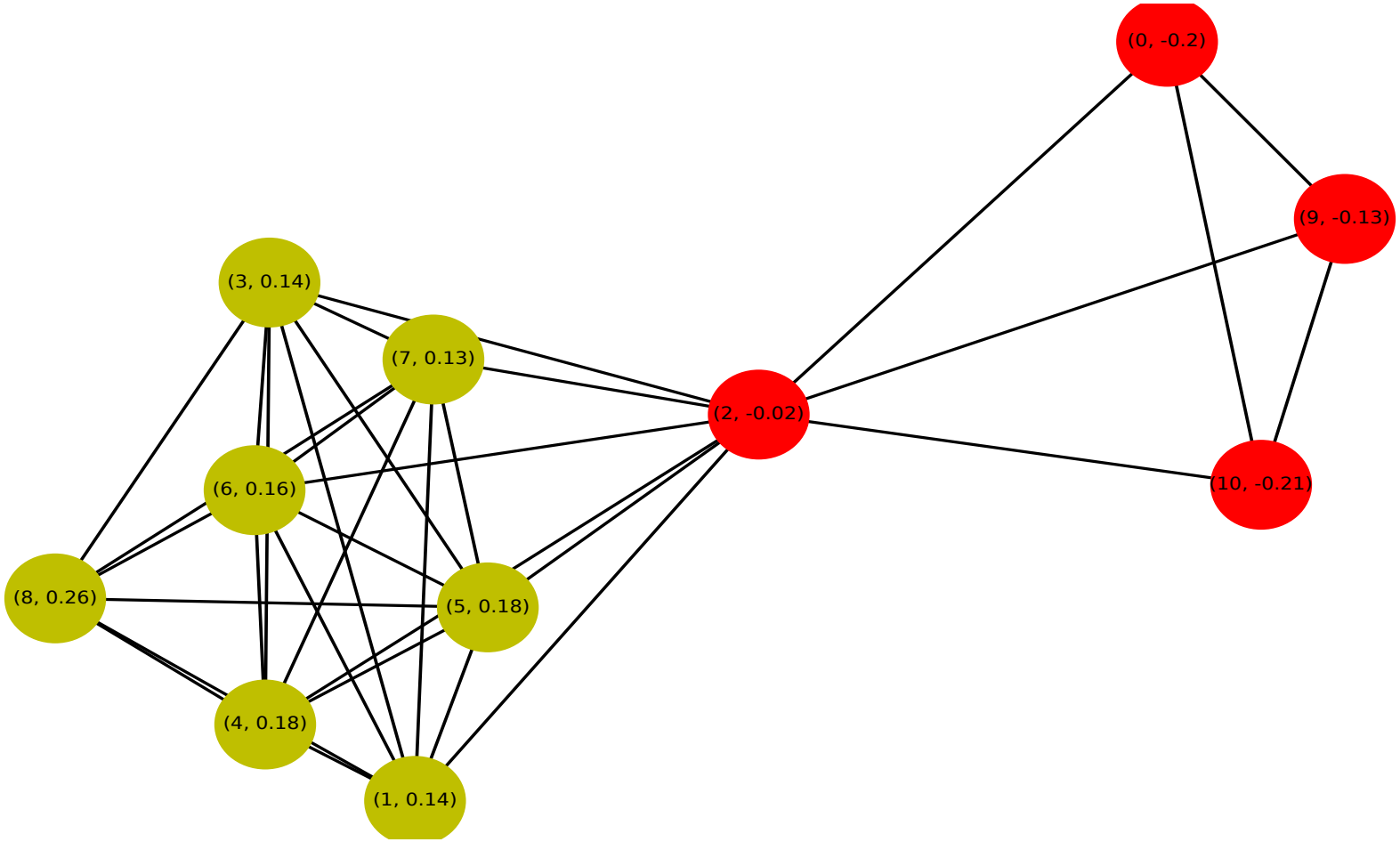


Figure: Interaction graph of x_0

Example

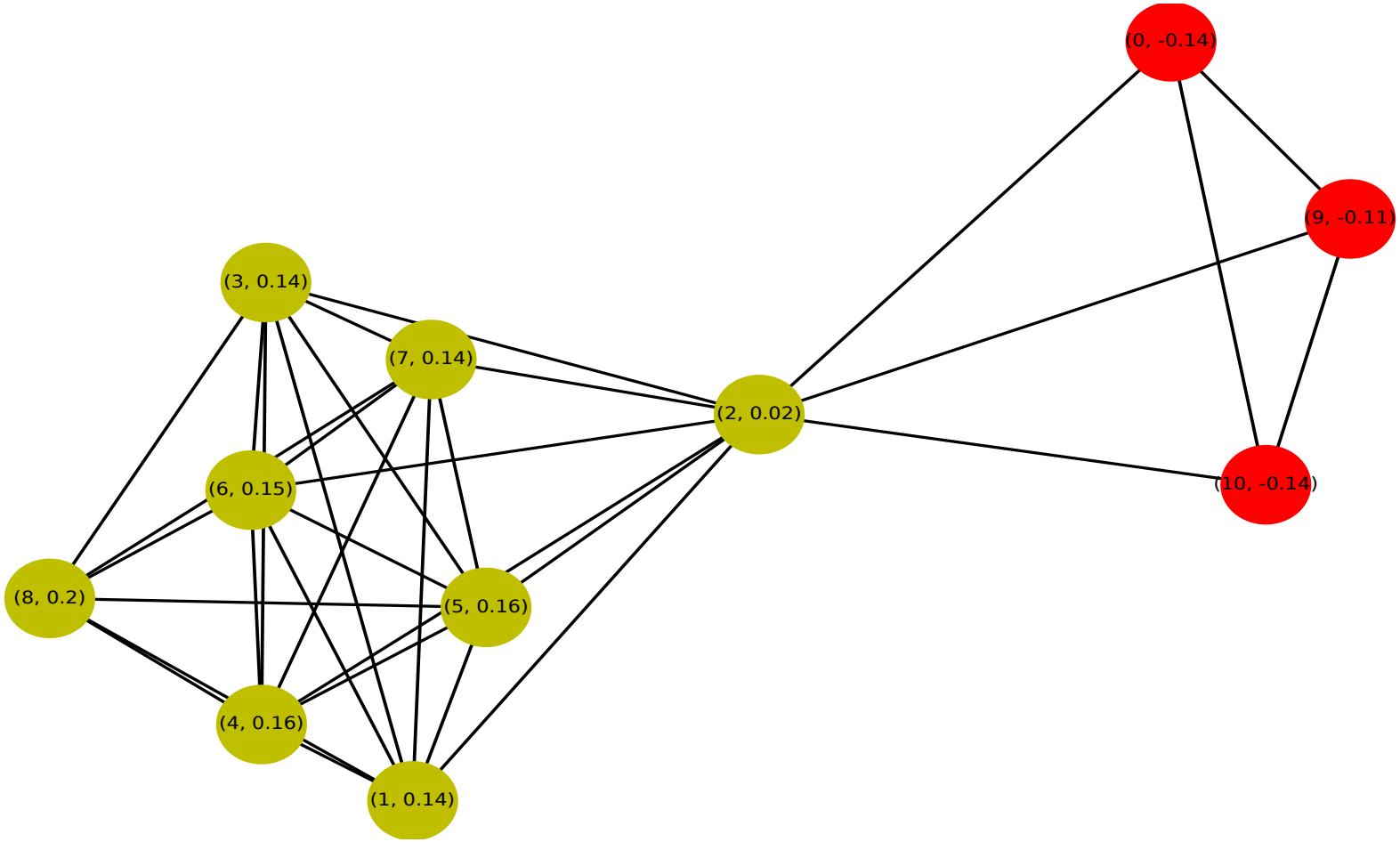


Figure: Interaction graph of x^*

Example

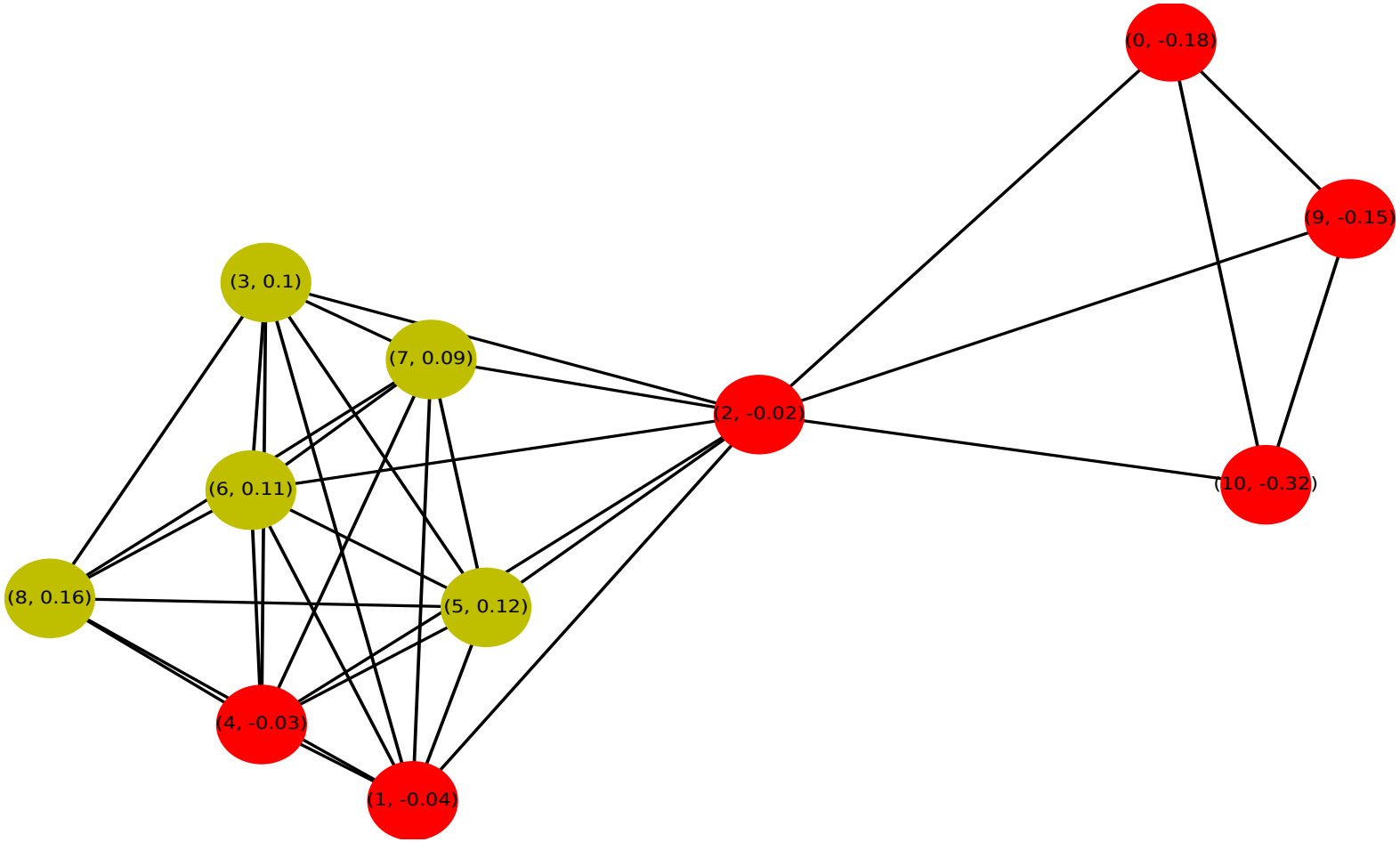


Figure: Interaction graph of x^{**} after changing the outcome of the election.

Strategies and Electoral Unit

Strategies :

- *Random :* Random nodes are selected for influence.
- *Minimum :* The agents with opinion close to zero are selected to influence first.

Electoral Unit :

- Country
- States
- Districts

Minimum vs Random

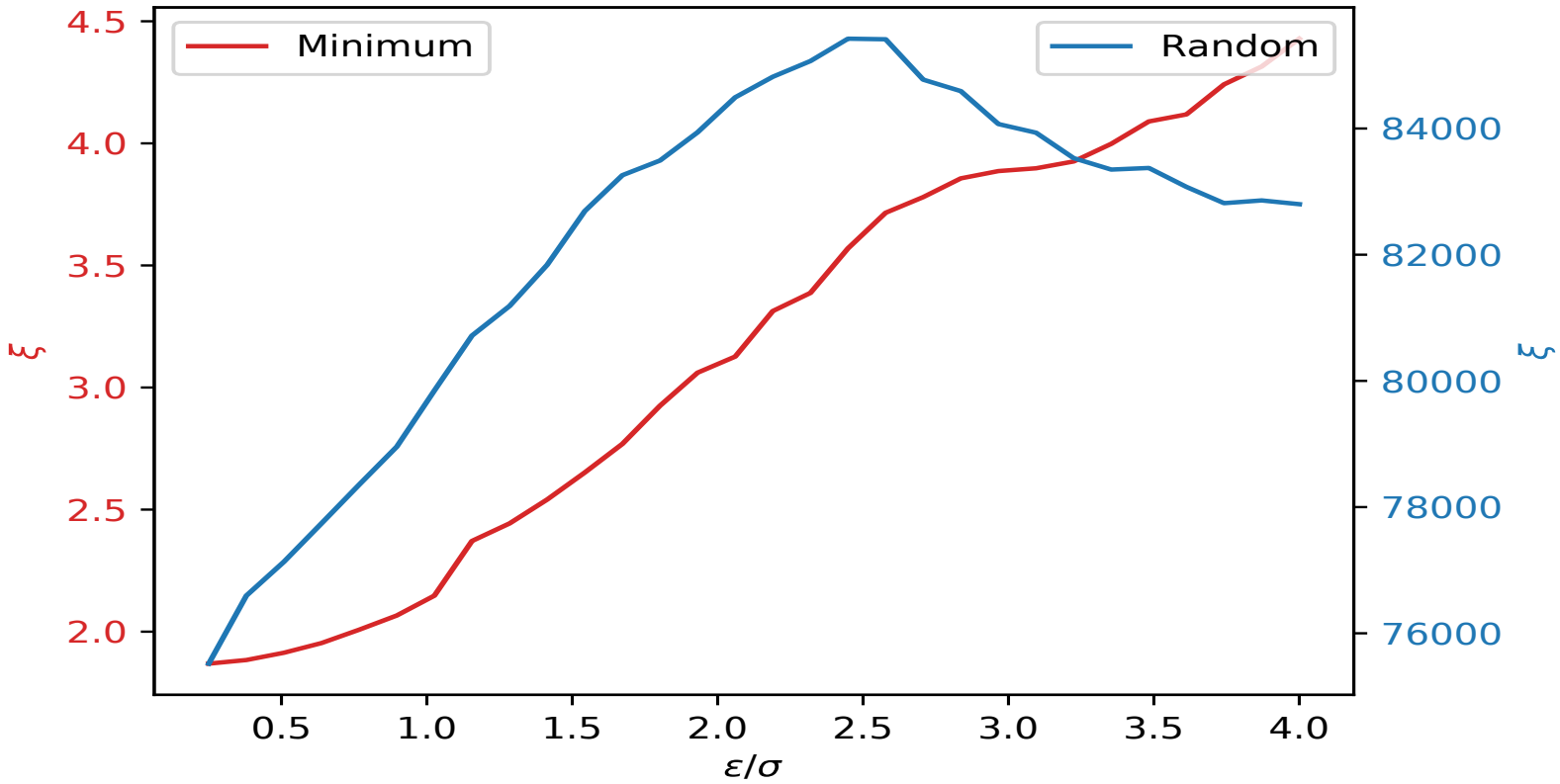


Figure: Minimum vs Random strategy

Effect of change in Polarization (Δ)

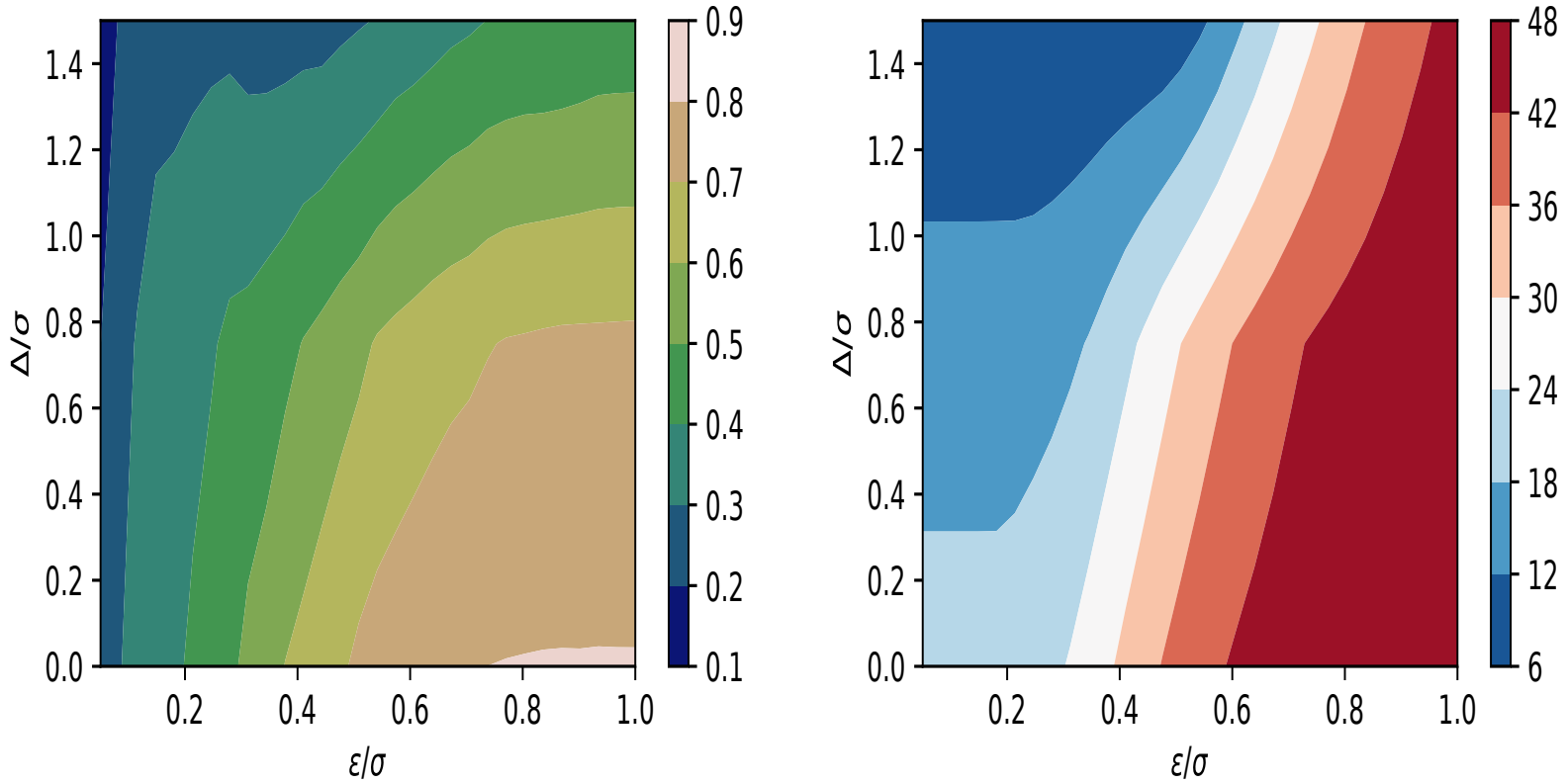


Figure: Effect of change in polarization on the effort needed to change the outcome of the election with $\mu = 0$(Left) and $\mu = 0.05$ (Right).

Effect of change in proportion of votes (p)

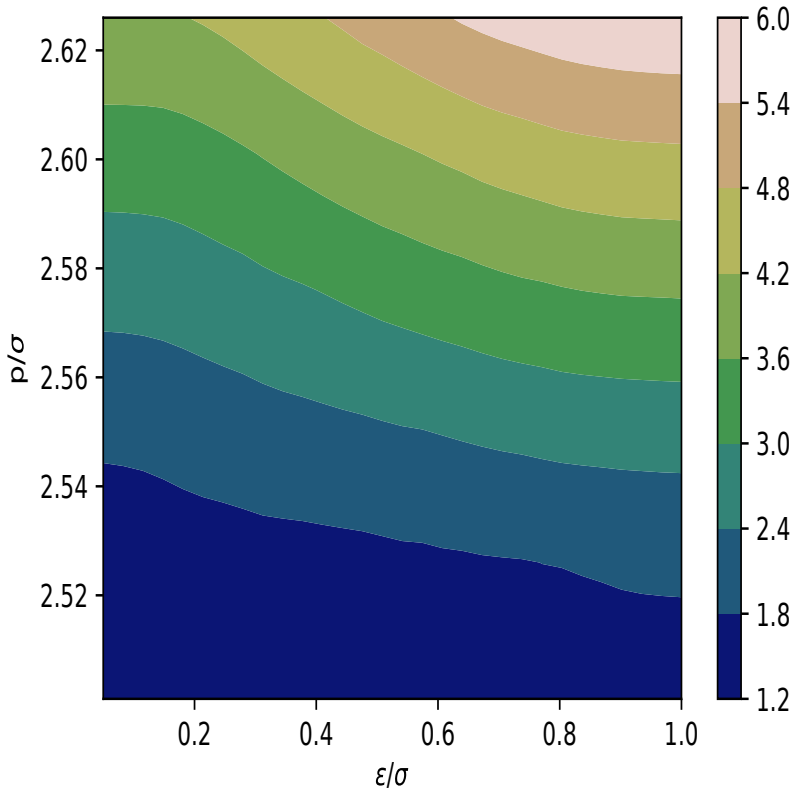


Figure: Effect of change in polarization on the effort needed to change the outcome of the election with $\mu = 0$.

Electoral Systems

- Single Representative (SR) / House of Representative (HOR):

$$o(\boldsymbol{x}^*) = \sum_i \mathrm{sign}(x_i^*)\,, \tag{9}$$

- Proportional Representative (PR) :

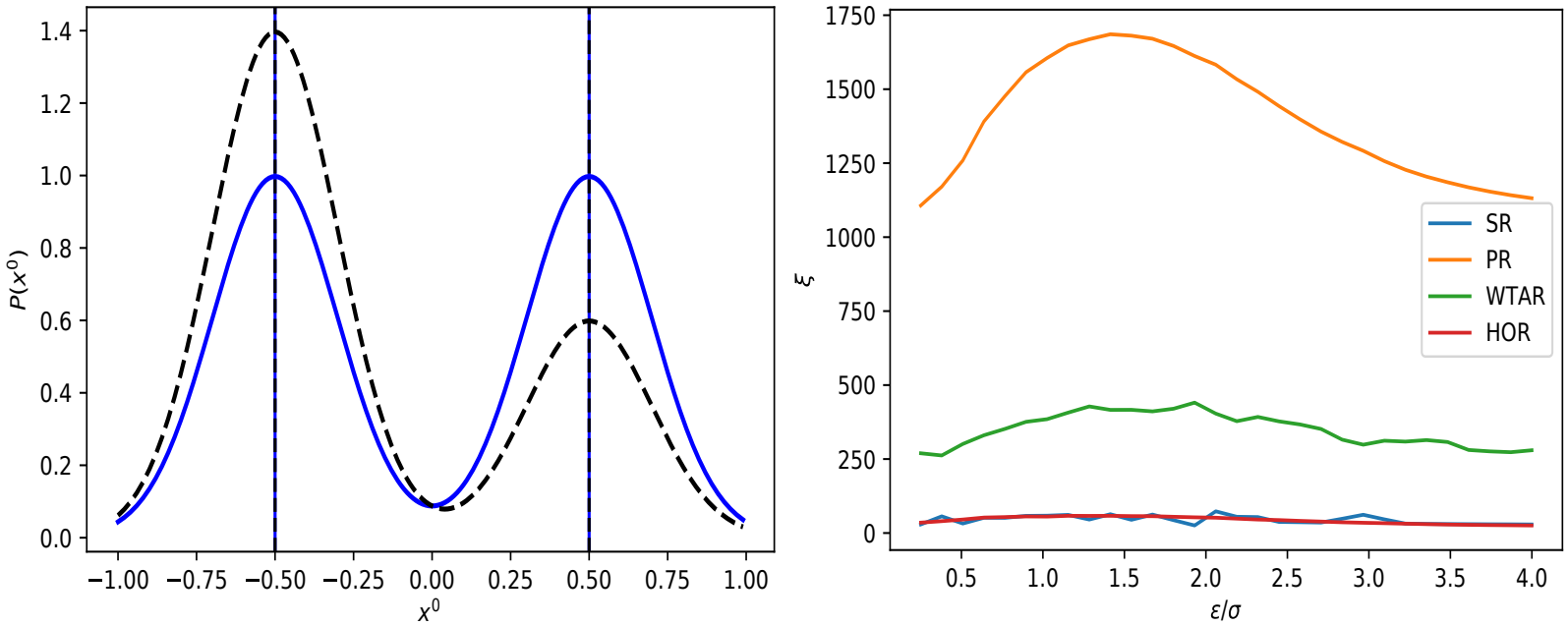
$$m_A = Round\left[\frac{m}{n} \sum_{i;x_i^* \in [x_A]} |\mathrm{sign}(x_i^*)|\right], \tag{10}$$

- Winner Takes All Representative (WTAR) :

$$m_A := \begin{cases} m\,, & \text{if } \sum_i (x_i^* > 0) > \sum_i (x_i^* < 0)\,, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

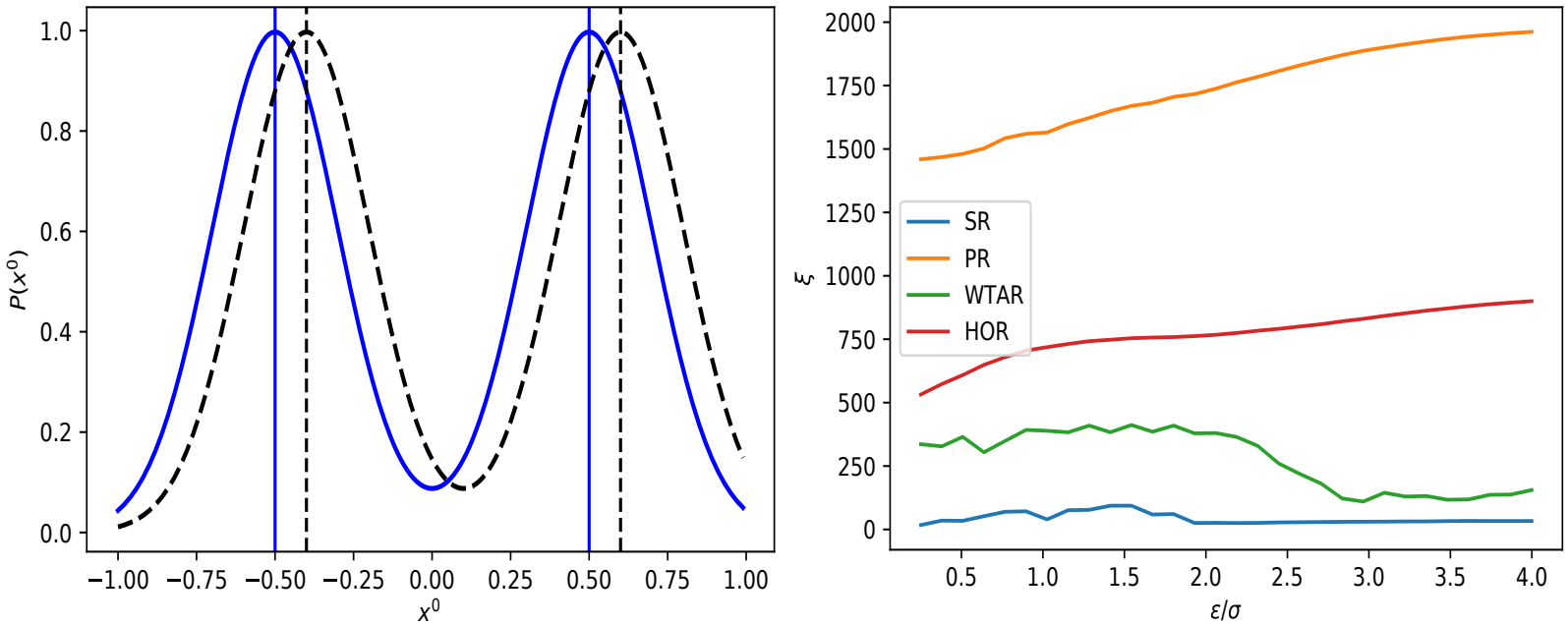
assuming that party A is positively opinionated.

Robustness of electoral systems



At left : Type of Natural opinion
At Right : The parameter p is varied in order to impose the majority as per the data of results in HOR election in US with $\mu = 0$ and $\Delta/\sigma = 1.5$.

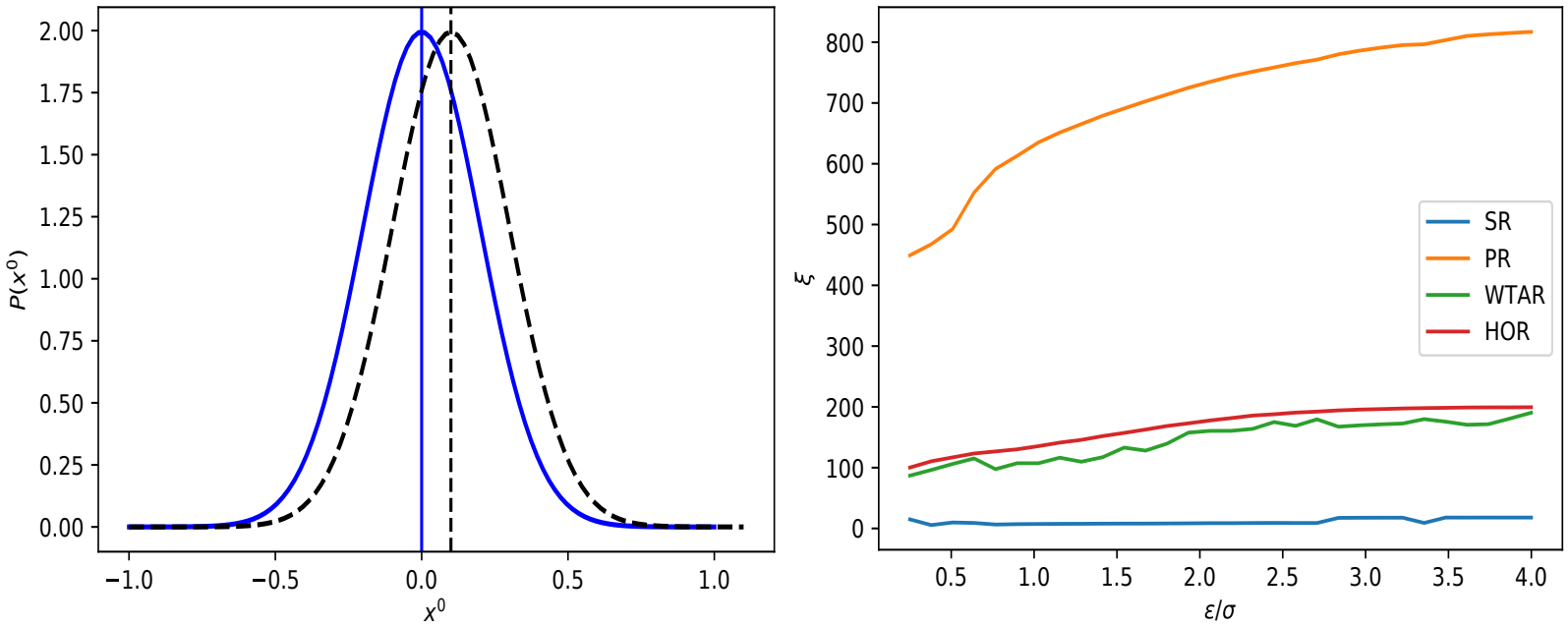
Robustness of electoral systems



At left : Type of Natural opinion

At Right : The parameter μ is varied in order to impose the majority as per the data of results in HOR election in US with $p = 0.5$ and $\Delta/\sigma = 1.5$

Robustness of electoral systems



At left : Type of Natural opinion
At Right : The parameter μ is varied in order to impose the majority as per the data of results in HOR election in US with $p = 0$ and $\Delta/\sigma = 0$

Summary

- Highly robust : Proportional Representative in states
  Least robust : Single Representative in states
- Society is highly robust when the society is opinionated (parameter μ) with no polarization (Δ)