Novel Results on Slow Coherency in Consensus and Power Networks

Julian Fritzsch

June 29, 2021

Network of Coupled Oscillators

Swing Equations

Networks of generators are modeled as

$$\underbrace{m_i}_{\text{inertia}} \ddot{\theta}_i + \underbrace{d_i}_{\text{control}} \dot{\theta}_i = \underbrace{P_i}_{\text{power}} - \sum_j \underbrace{B_{ij}}_{\text{susceptance}} \sin(\theta_i - \theta_j)$$

Linearize around fixed points

$$M\delta\ddot{\theta} + D\delta\dot{\theta} = \delta P - L\delta\theta,$$

where

$$L_{ij} = \begin{cases} -B_{ij}\cos\left(\theta_i^{(0)} - \theta_j^{(0)}\right) & i \neq j \\ \sum_k B_{ik}\cos\left(\theta_i^{(0)} - \theta_k^{(0)}\right) & i = j \end{cases}$$

Machowski, Bialek, Bumby: "Power System dynamics: stability and control"

Stability

Assume homogeneous damping and inertia
The stability matrix takes the form

$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -d/m\mathbf{1}_{n\times n} & -L/m \end{bmatrix}$$

The eigenvalues are given by

$$\lambda_{\alpha\pm} = -\frac{d}{2m} \pm \frac{i}{2}\sqrt{4\lambda_{L\alpha} - \frac{d}{m}^{2}},$$

where $\lambda_{L\alpha}$ is αth eigenvalue of L

Pagnier et al. 2019

Laplacian Matrix

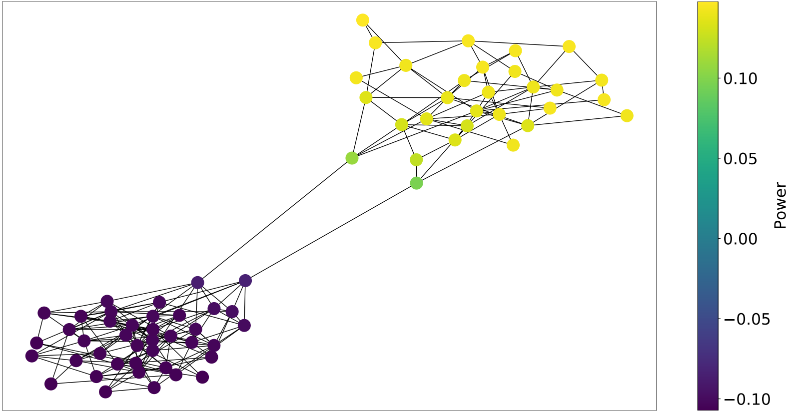
Laplacian has positive eigenvalues and one zero eigenvalue
Zero eigenvalue corresponds to constant shift in angles

Laplacian Matrix

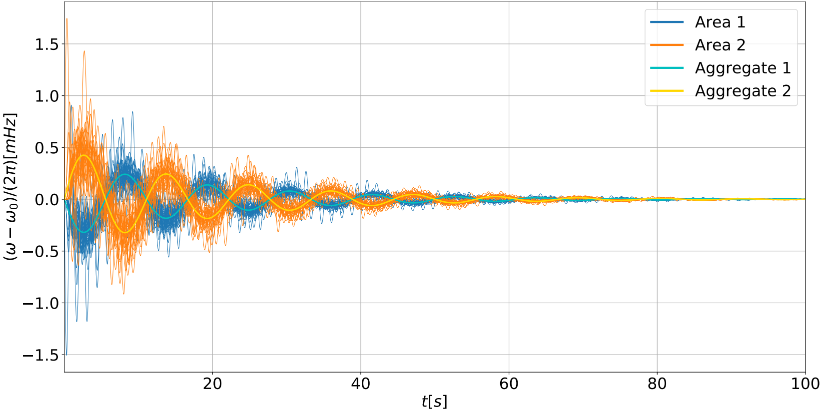
Laplacian has positive eigenvalues and one zero eigenvalue
Zero eigenvalue corresponds to constant shift in angles
$\rightarrow$ All fixed points are either sinks or stable spirals

Motivation

Simple Two Area Network



2 Areas With Equal Injection / Consumption



Separation of Time Scales

Idea

System with two time scales
$\rightarrow$ separation into fast and slow variable

Dynamical system:

$$\dot{y} = f(t, y, z, \epsilon)$$
$$\epsilon\dot{z} = g(t, y, z, \epsilon)$$

$\epsilon \ll 1$

Khalil '02

Slow Subsystem

- take limit $\epsilon \to 0$
- fast system converges infinitely fast
- from $g(t, \bar{y}, z, 0) = 0$ find $z = h(t, \bar{y})$

Slow Subsystem

- take limit $\epsilon \to 0$
- fast system converges infinitely fast
- from $g(t, \bar{y}, z, 0) = 0$ find $z = h(t, \bar{y})$

$\rightarrow$ slow subsystem is solution to

$$\dot{\bar{y}} = f(t, \bar{y}, h(t, \bar{y}), 0)$$

Fast Subsystem

- fast variable $z_f = z - h(t, \bar{y})$
- fast time scale $t_f = (t - t_0)/\epsilon$
- $\frac{dz_f}{dt_f} = g(t_0, \bar{y}(t_0), z_f + h(t_0, \bar{y}(t_0)), 0)$
- $z_f(0) = z(t_0) - h(t_0, \bar{y}(t_0))$

Fast Subsystem

- fast variable $z_f = z - h(t, \bar{y})$
- fast time scale $t_f = (t - t_0)/\epsilon$
- $\frac{dz_f}{dt_f} = g(t_0, \bar{y}(t_0), z_f + h(t_0, \bar{y}(t_0)), 0)$
- $z_f(0) = z(t_0) - h(t_0, \bar{y}(t_0))$

The full solution is:

$$y(t) = \bar{y}(t) + \mathcal{O}(\epsilon)$$
$$z(t) = h(t, \bar{y}) + z_f(t_f) + \mathcal{O}(\epsilon)$$

Aggregation of Power Networks

Setup

Linearized swing equation

$$M\ddot{\theta} + D\dot{\theta} = -L\theta$$

Separate Laplacian into internal and external part

$$L = L^I + L^E$$

Node parameter

$$d = \frac{\text{largest inter-area weight of all nodes}}{\text{smalles intra-area weight of all nodes}} = \frac{c^E}{c^I}$$

Area parameter

$$\delta = \frac{\text{strongest connection between two areas}}{\text{nodes in smallest area} \times \text{weakest intra-area connection}} = \frac{\gamma^E}{\underline{n}c^I}$$

Chow '82, Chow '85, Date '91, Romeres '13

Slow and Fast Variables

slow variables: center of inertia angle of area

$$y_\alpha = \frac{\sum_i m_i \theta_i}{\sum_i m_i} = (C_a \theta)_\alpha$$

Slow and Fast Variables

slow variables: center of inertia angle of area

$$y_\alpha = \frac{\sum_i m_i \theta_i}{\sum_i m_i} = (C_a \theta)_\alpha$$

fast variables: weighted difference between nodes in area

$$z_\alpha = Q_\alpha \theta^\alpha$$

$$Q_\alpha = \begin{bmatrix} -1 + (n_\alpha - 1)v & 1 - v & -v & \cdots & -v \\ -1 + (n_\alpha - 1)v & -v & 1 - v & \cdots & -v \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 + (n_\alpha - 1)v & -v & -v & \cdots & 1 - v \end{bmatrix}$$

$$v = \frac{n_\alpha - \sqrt{m_\alpha}}{m_\alpha (m_\alpha - 1)}$$

Applying transformation

After applying the transformations we find

$$\frac{d}{dt_s}\begin{bmatrix} \bar{y} \\ \dot{\bar{y}} \\ \sqrt{\delta}z \\ \sqrt{\delta}\dot{z} \end{bmatrix} = \begin{bmatrix} 0 & I/c^I & 0 & 0 \\ \tilde{A}_{11} & -\tilde{D}_1 & -\tilde{A}_{12} & 0 \\ 0 & 0 & 0 & I/c^I \\ d\tilde{A}_{21} & -\sqrt{\delta}QM^{-1}DU/c^I & \tilde{A}_{22} & -QM^{-1}DQ^T/c^I \end{bmatrix}\begin{bmatrix} \bar{y} \\ \dot{\bar{y}} \\ z \\ \dot{z} \end{bmatrix}$$

where $[\bar{y}, \dot{\bar{y}}] = [y, \dot{y}/\sqrt{\delta}]$ and $t_s = \sqrt{\delta}c^I t$

Slow System

- the system can be separated
- taking the limit $\delta \to 0$ yields

$$\frac{d}{dt_s}\begin{bmatrix}\bar{y}\\ \dot{\bar{y}}\end{bmatrix} = \begin{bmatrix}0 & I/c^I\\ \tilde{A}_0 & -D_1\end{bmatrix}\begin{bmatrix}\bar{y}\\ \dot{\bar{y}}\end{bmatrix}$$

For an exact definition of the matrices see Romeres et al. 2013

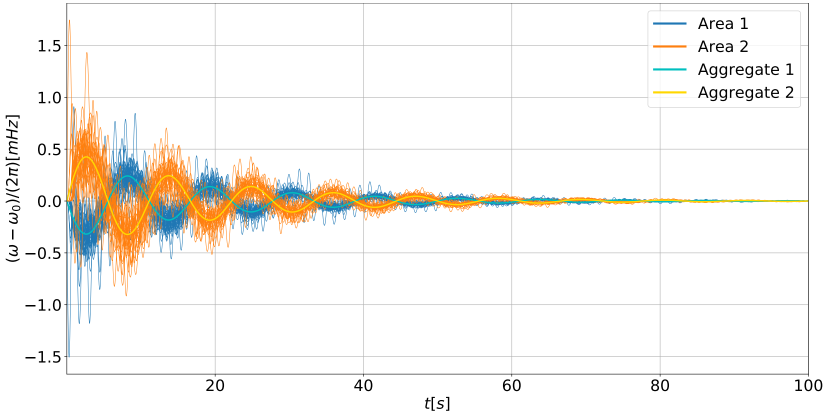
Fast System

the fast variables are given by: $z(t) = z_f(t_f) \underbrace{-\tilde{A}_{22}^{-1}(d\tilde{A}_{21}\bar{y} + \frac{QP}{c^I})}_{h}$

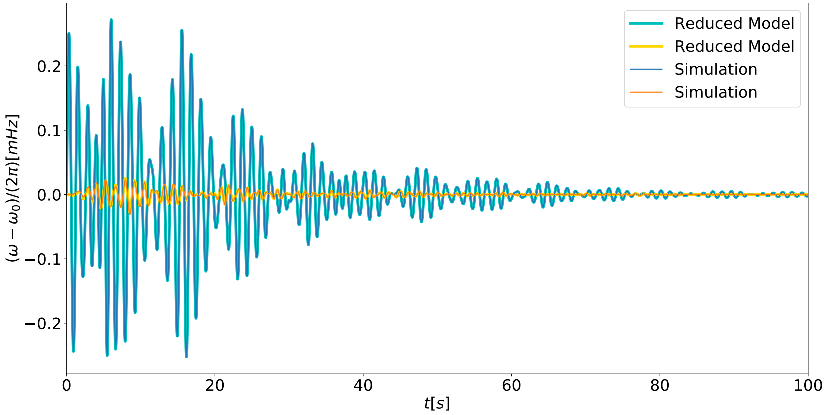
$t_f = t_s/\sqrt{\delta}$

$$\frac{d}{dt_f}\begin{bmatrix} z_f \\ \dot{z}_f \end{bmatrix} = \begin{bmatrix} 0 & \frac{I}{c^I} \\ \tilde{A}_{22} & \frac{-QM^{-1}DQ^T}{c^I} \end{bmatrix}\begin{bmatrix} z_f \\ \dot{z}_f \end{bmatrix}$$

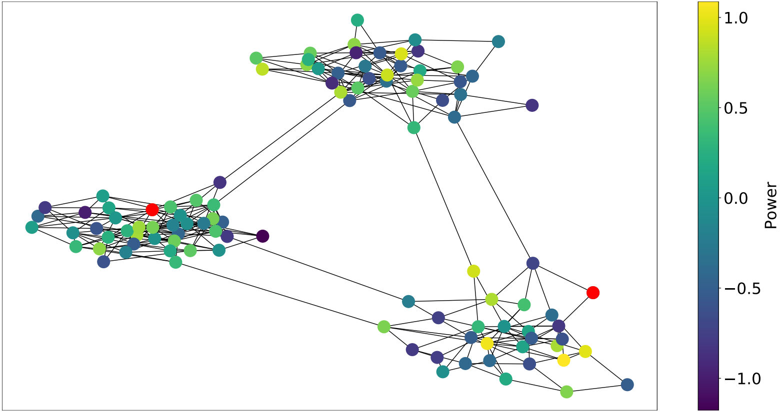
2 Areas With Equal Injection / Consumption



2 Areas With Equal Injection / Consumption



3 Areas With Equal Injection / Consumption



3 Areas With Equal Injection / Consumption