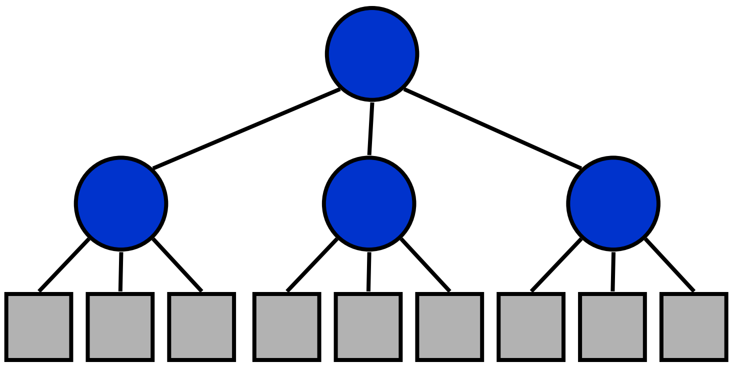
Divide-and-Conquer Technique: Merge Sort, Quick Sort

Divide-and-Conquer

- Divide-and-Conquer is a general algorithm design paradigm:
  - Divide the problem into a number of subproblems that are smaller instances of the same problem
  - Conquer the subproblems by solving them recursively
  - Combine the solutions to the subproblems into the solution for the original problem
- The base case for the recursion are subproblems of constant size
- Analysis can be done using **recurrence equations**



Merge Sort and Quick Sort

Two well-known sorting algorithms adopt this divide-and-conquer strategy

- Merge sort
  - Divide step is trivial – just split the list into two equal parts
  - Work is carried out in the conquer step by merging two sorted lists

- Quick sort
  - Work is carried out in the divide step using a pivot element
  - Conquer step is trivial

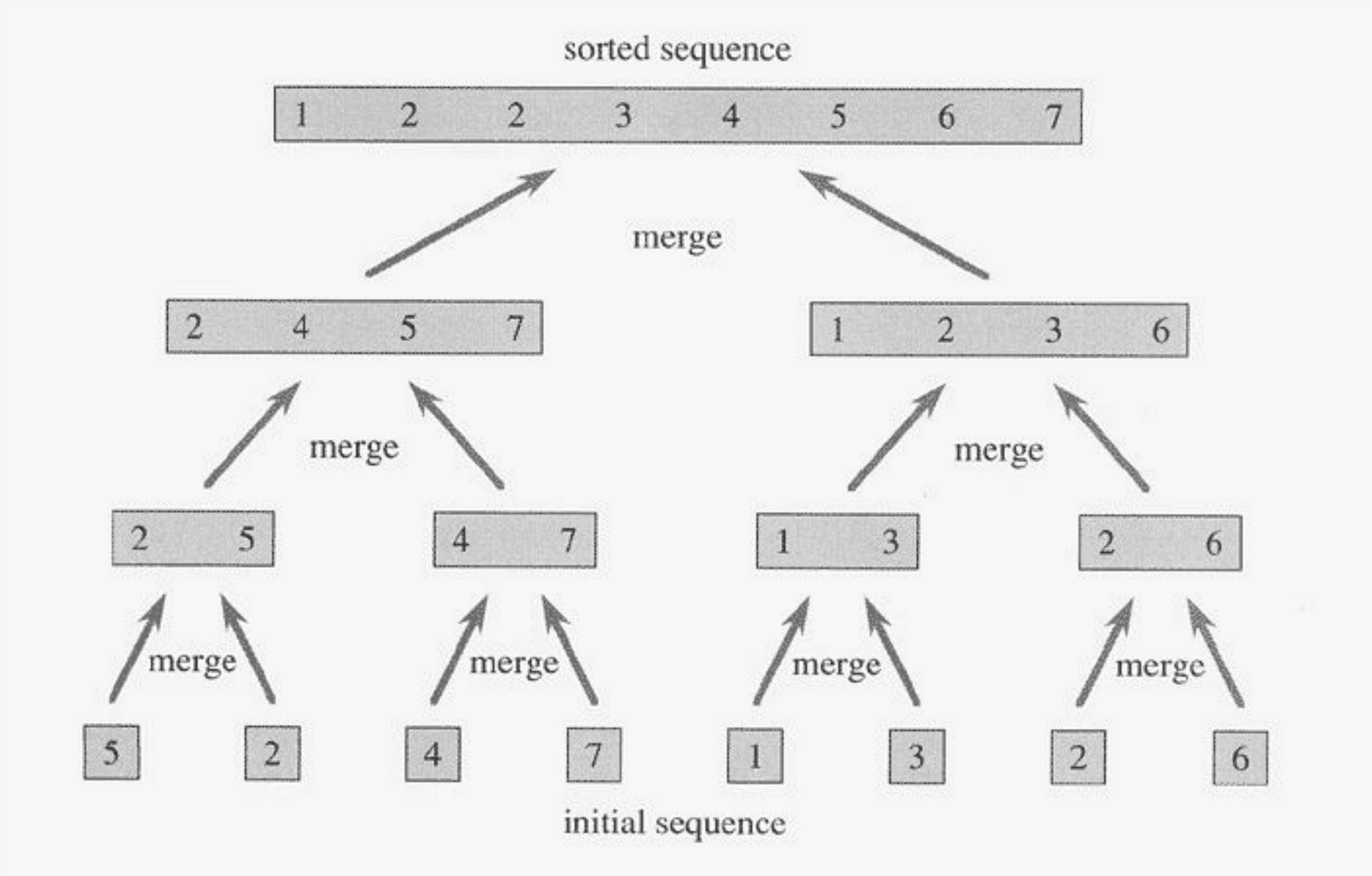
Merge Sort: Algorithm

MERGE-SORT(A, p, r)
1 **if** $p < r$
2 **then** $q \leftarrow \lfloor (p+r)/2 \rfloor$
3 MERGE-SORT(A, p, q)
4 MERGE-SORT($A, q+1, r$)
5 MERGE(A, p, q, r)

Merge Sort: Algorithm

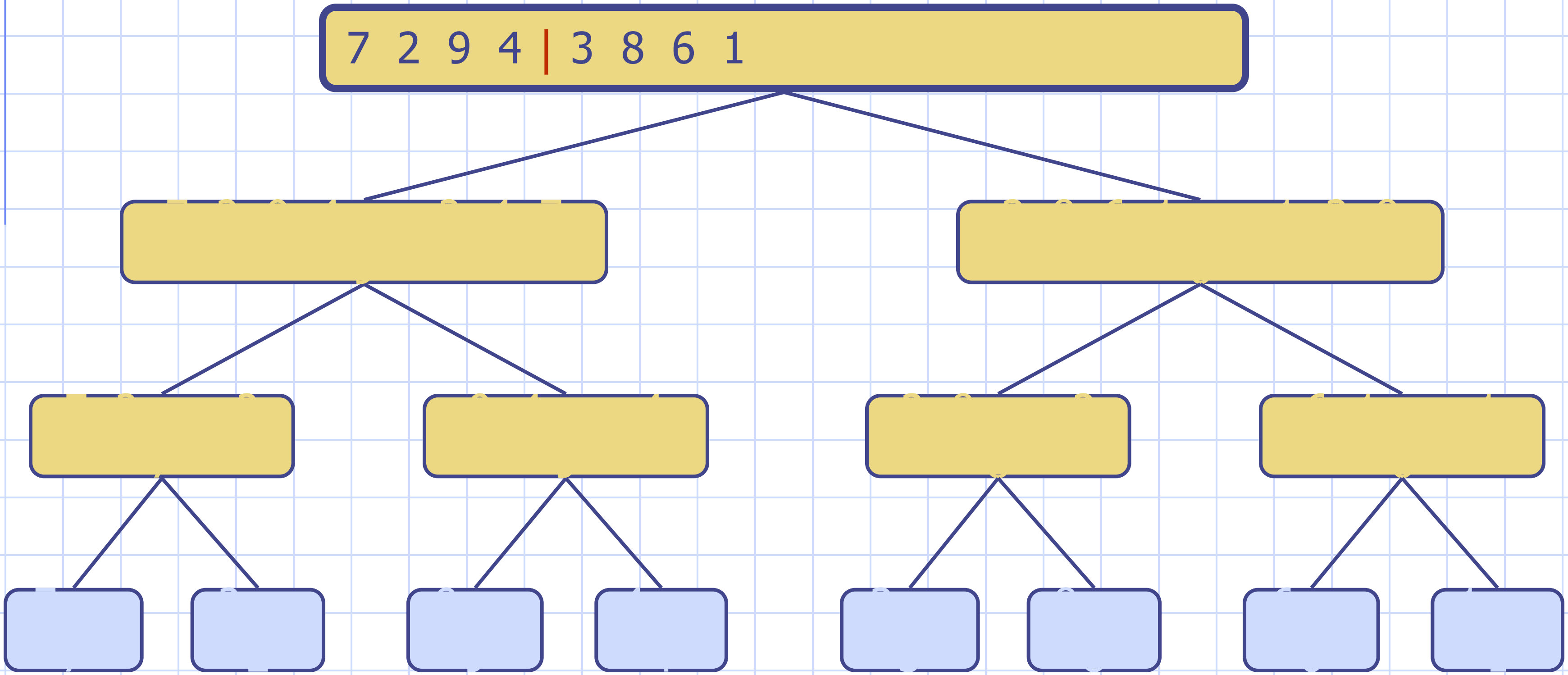
```
MERGE(A, p, q, r)
n1 ← q − p + 1
n2 ← r − q
create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
for i ← 1 to n1
        do L[i] ← A[p + i − 1]
for j ← 1 to n2
        do R[j] ← A[q + j]
L[n1 + 1] ← ∞
R[n2 + 1] ← ∞
i ← 1
j ← 1
for k ← p to r
        do if L[i] ≤ R[j]
              then A[k] ← L[i]
                   i ← i + 1
              else A[k] ← R[j]
                   j ← j + 1
```

Merge Sort: Example



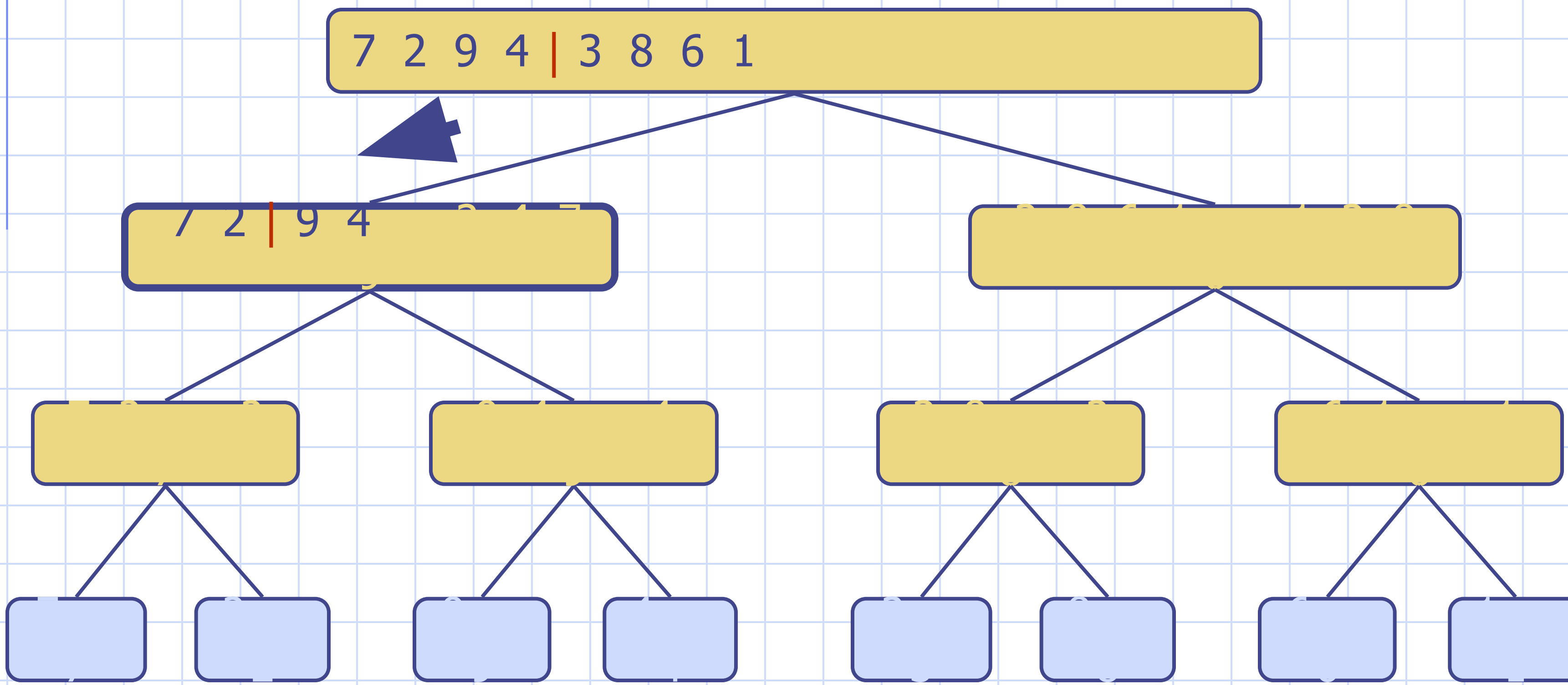
Execution Example

- Partition



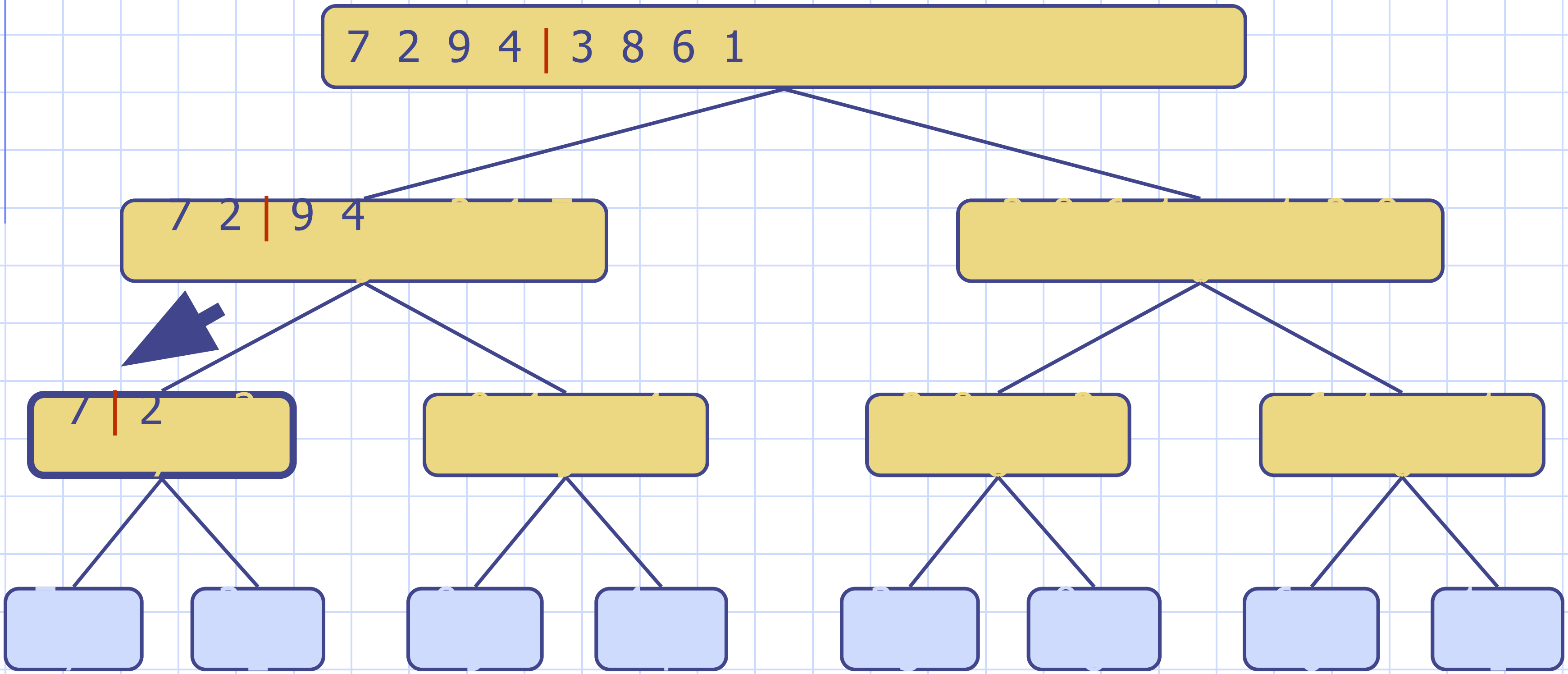
Execution Example

- Recursive call, partition



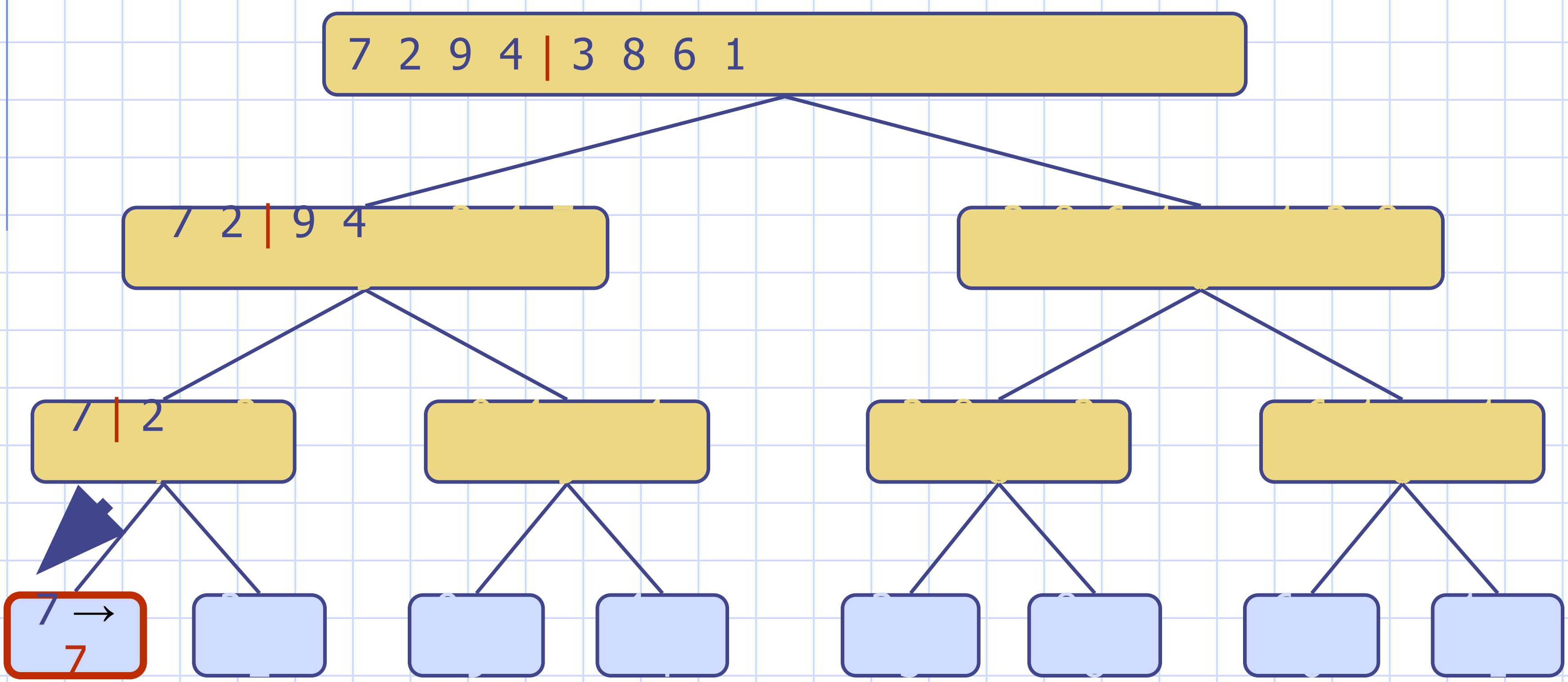
Execution Example

- Recursive call, partition



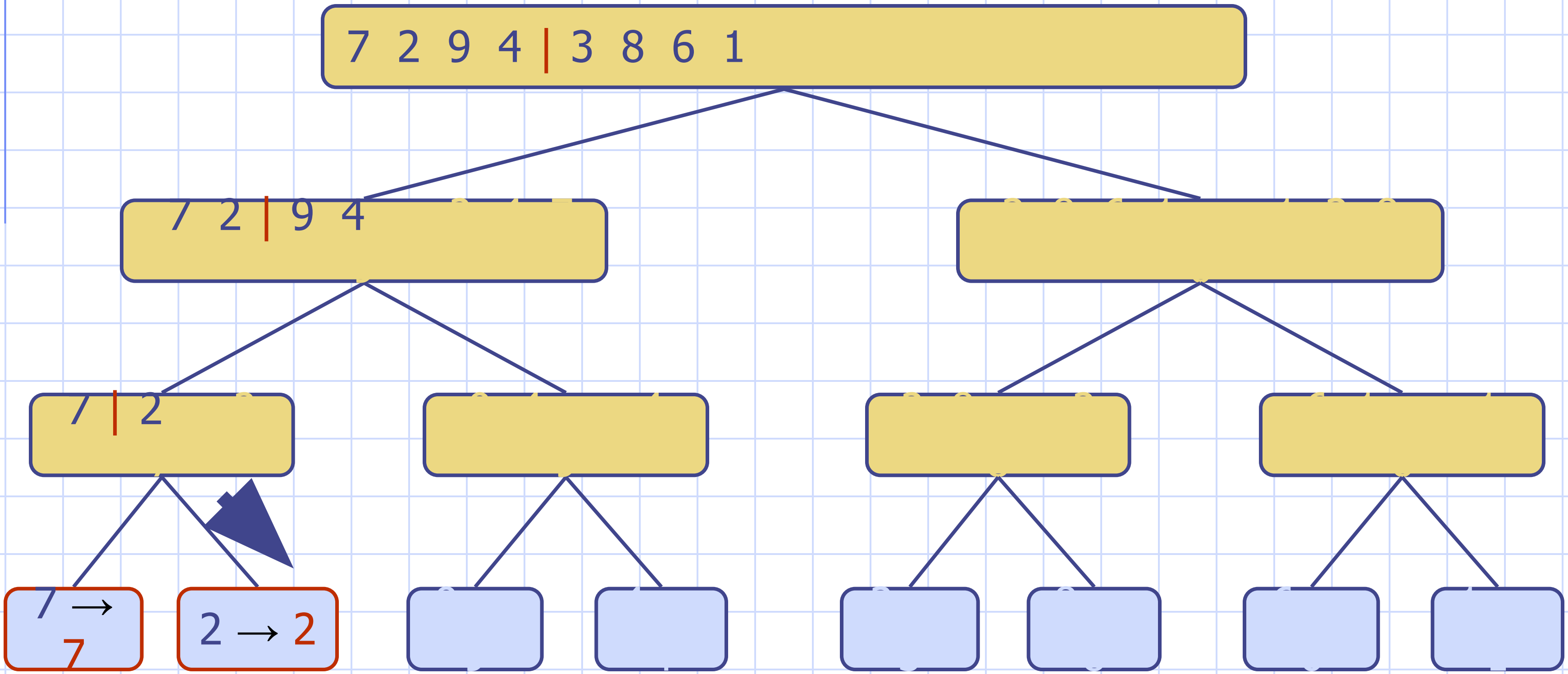
Execution Example

- Recursive call, base case



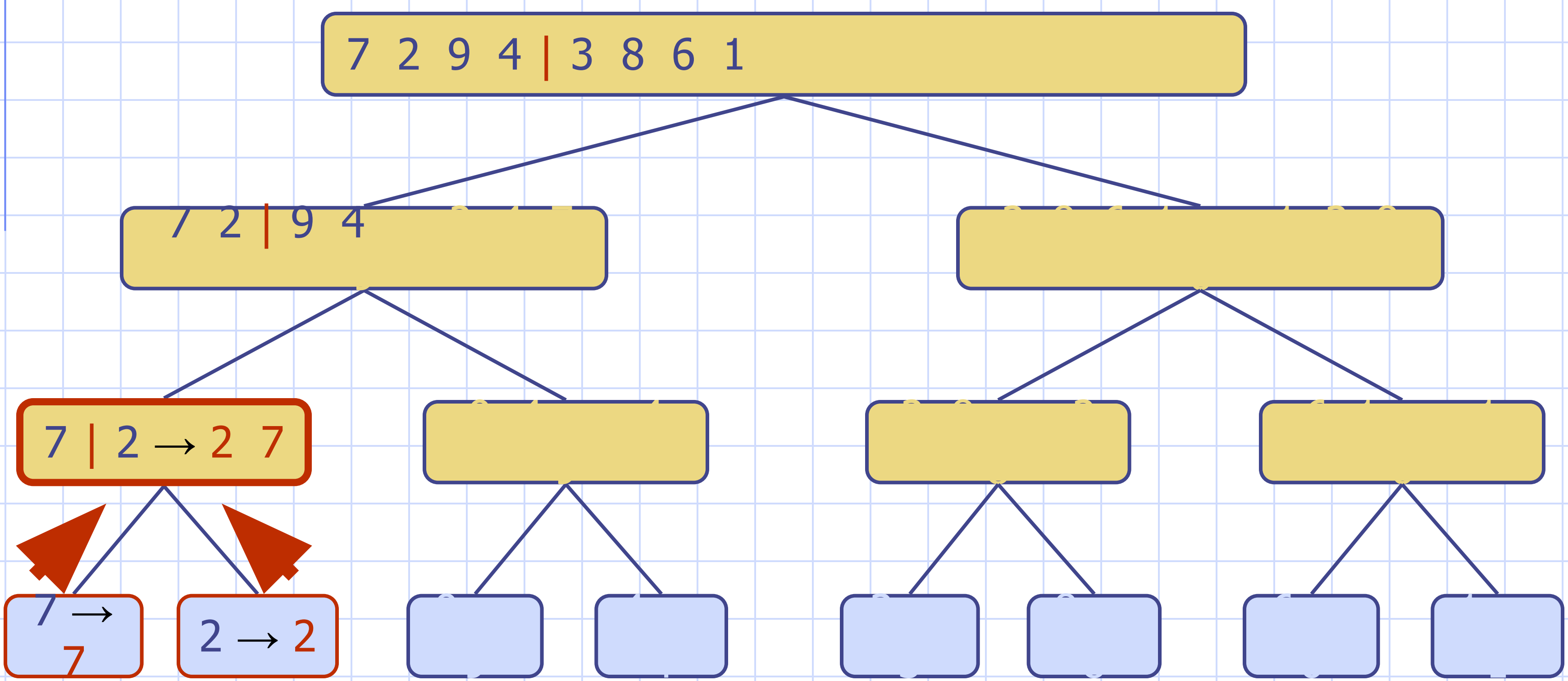
Execution Example

- Recursive call, base case



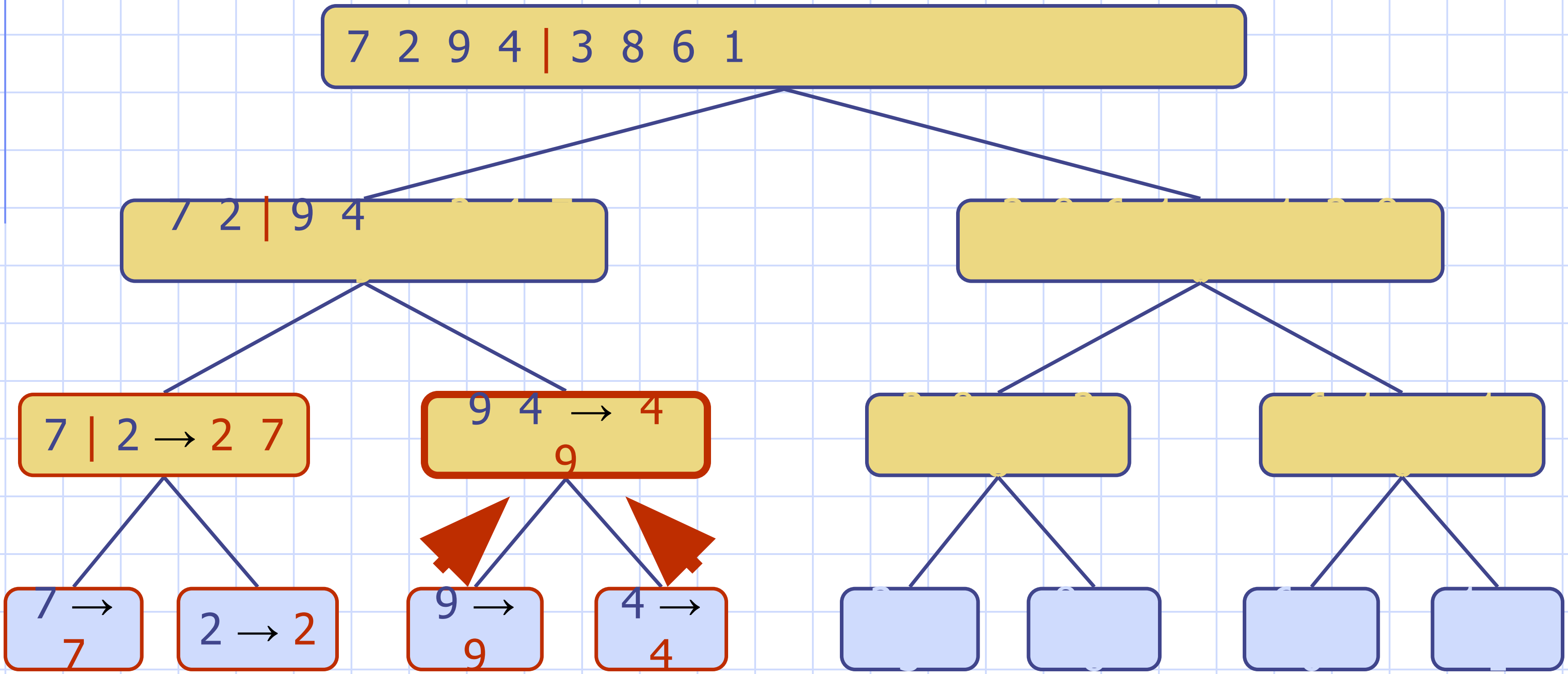
Execution Example

- Merge



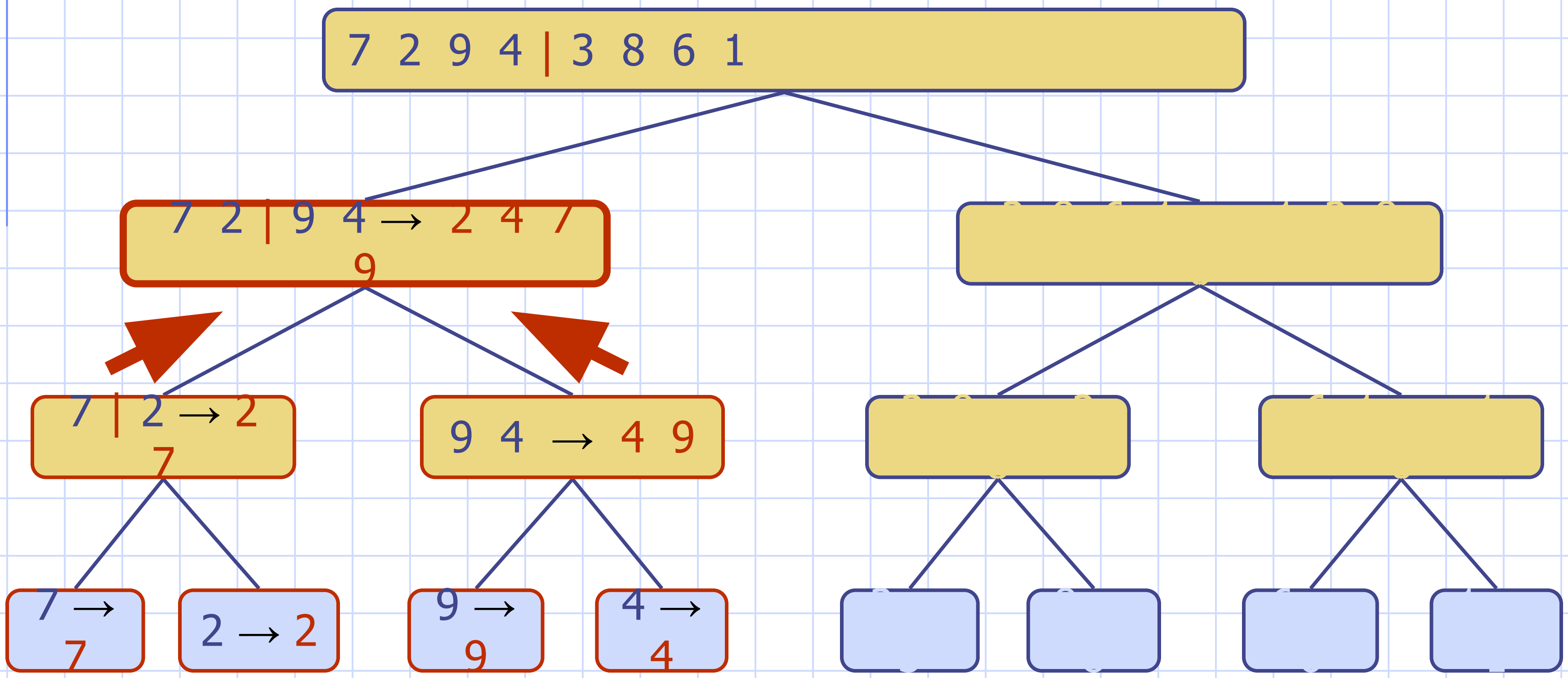
Execution Example

- Recursive call, …, base case, merge



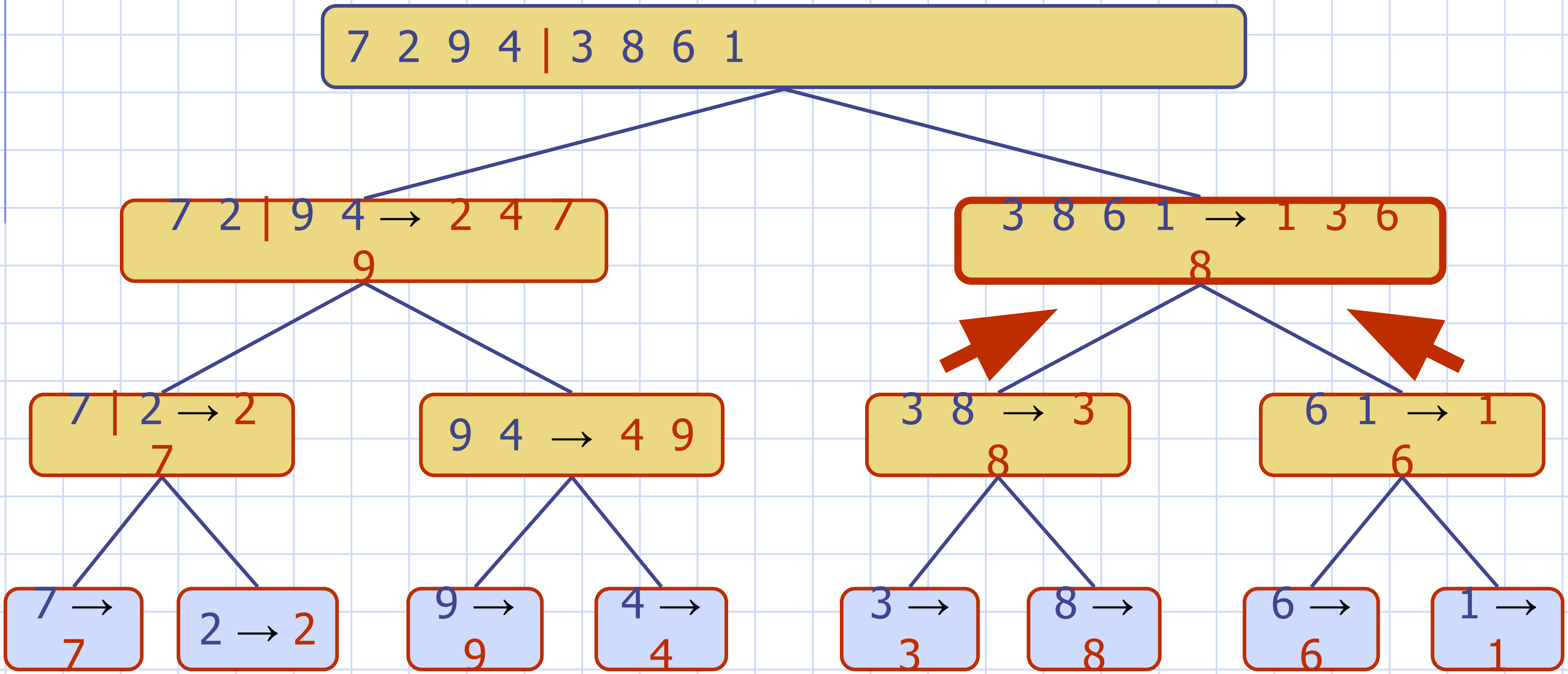
Execution Example

- Merge



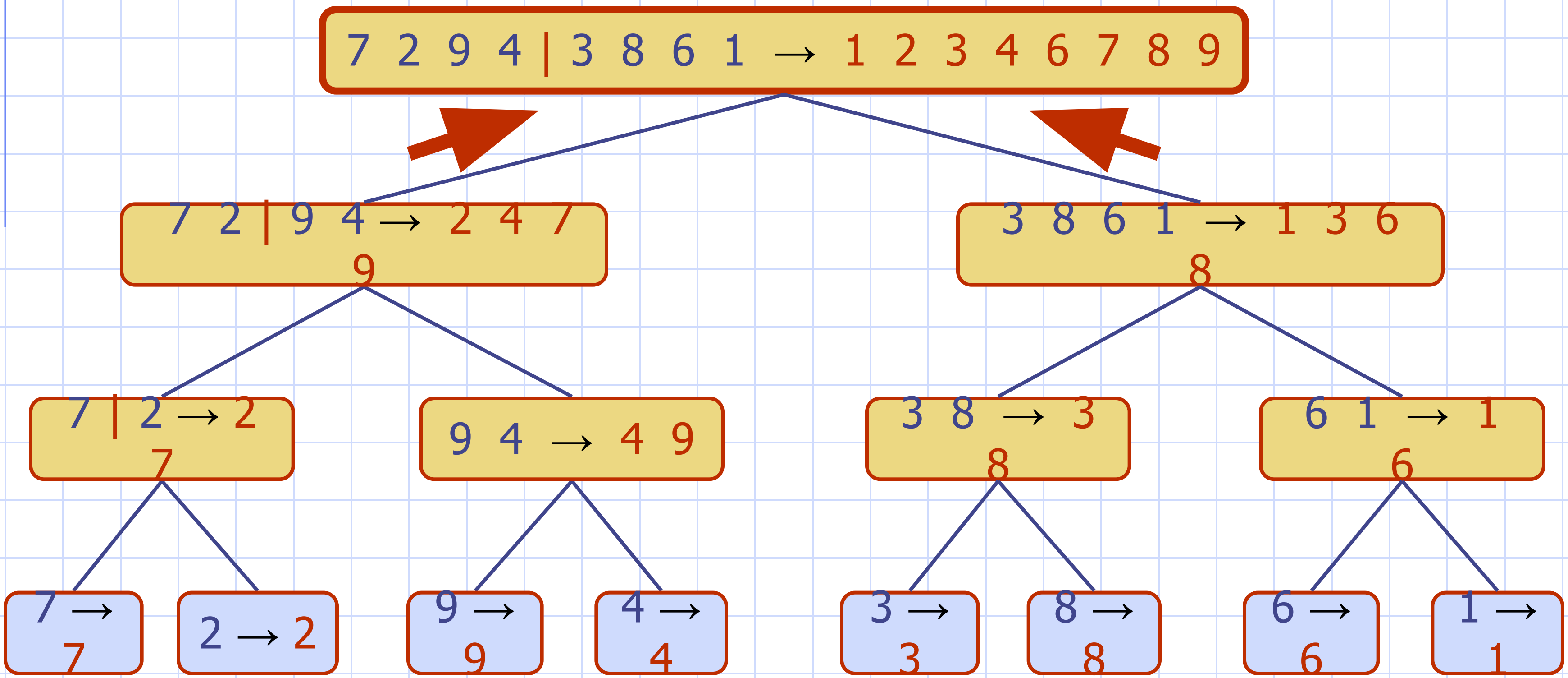
Execution Example

- Recursive call, …, merge, merge



Execution Example

- Merge



Merge Sort: Running Time

The recurrence for the worst-case running time T(n) is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

equivalently

$$T(n) = \begin{cases} b & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n > 1 \end{cases}$$

Solve this recurrence by

(1) iteratively expansion

(2) using the recursion tree

Merge Sort: Running Time (Iterative Expansion)

$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2 T(n/2^2) + 2bn \\
&= 2^3 T(n/2^3) + 3bn \\
&= 2^4 T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^i T(n/2^i) + ibn
\end{aligned}
$$

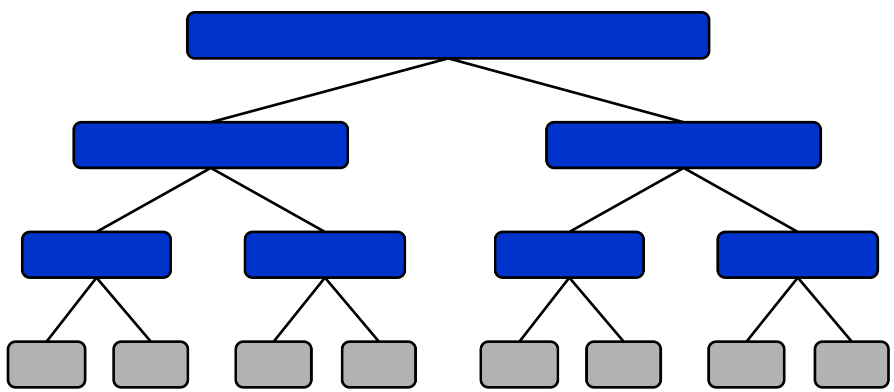
- Note that base, $T(n) = b$, case occurs when $2^i = n$.
  That is, $i = \log n$.
- So, $T(n) = bn + bn \log n$
- Thus, $T(n)$ is $O(n \log n)$.

Merge Sort: Running Time (Recursion Tree)

- Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

depth	T's	size		time
0	1	n		bn
1	2	$n/2$		bn
i	2^i	$n/2^i$		bn
…	…	…		…



Total time = $bn + bn \log n$

(last level plus all previous levels)

Quick Sort: Algorithm

- Another divide-and-conquer algorithm
  - The array A[p..r] is *partitioned* into two non-empty subarrays A[p..q] and A[q+1..r]
    - Invariant: All elements in A[p..q] are less than all elements in A[q+1..r]
  - The subarrays are recursively sorted by calls to quicksort
  - Unlike merge sort, no combining step: two subarrays form an already-sorted array

Quick Sort: Algorithm

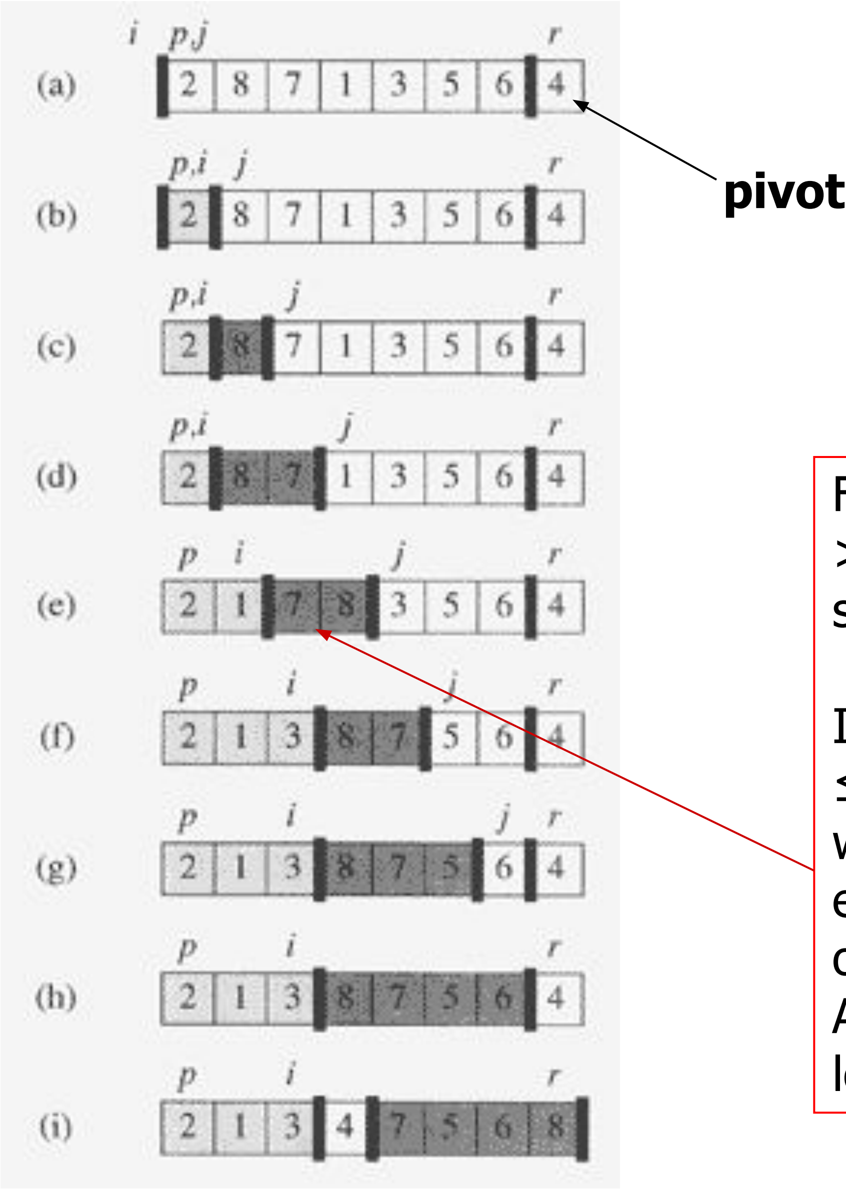
```
QUICKSORT(A, p, r)
if p < r
   then q ← PARTITION(A, p, r)
        QUICKSORT(A, p, q − 1)
        QUICKSORT(A, q + 1, r)
```

```
PARTITION(A, p, r)
x ← A[r]
i ← p − 1
for j ← p to r − 1
     do if A[j] ≤ x
           then i ← i + 1
                exchange A[i] ↔ A[j]
exchange A[i + 1] ↔ A[r]
return i + 1
```

Quick Sort: Algorithm (Partition)

- Clearly, all the actions take place in the **`partition()`** function
  - Rearranges the subarrays in place
  - End result:
    - Two subarrays
    - All values in first subarray $\leq$ all values in the second
  - Returns the index of the "pivot" element separating the two subarrays

Quick Sort: Algorithm



From i +1 to j is a window of elements > A[*r*]. The cursor *j* moves right one step at a time.

If the cursor *j* "discovers" an element ≤ A[*r*], then this element is swapped with the front element of the window, effectively moving the window right one step; if it discovers an element > A[*r*], then the window simply becomes longer one unit.

Quick Sort: Algorithm

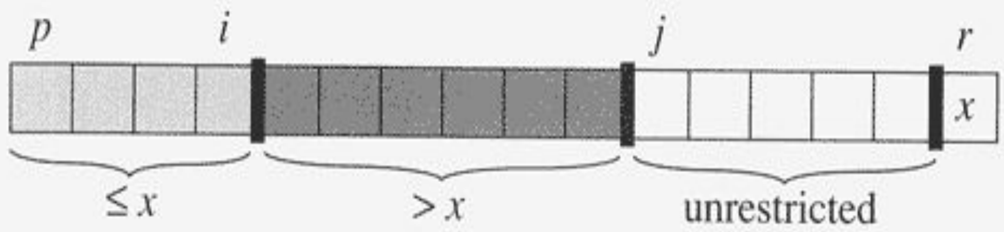


Figure 7.2 The four regions maintained by the procedure PARTITION on a subarray $A[p..r]$. The values in $A[p..i]$ are all less than or equal to x, the values in $A[i+1..j-1]$ are all greater than x, and $A[r] = x$. The values in $A[j..r-1]$ can take on any values.

Quick Sort: Analysis

- *What will be the worst case for the algorithm?*
  - Partition is always unbalanced
- *What will be the best case for the algorithm?*
  - Partition is perfectly balanced

- *Which is more likely?*
  - The latter, by far, except...
- *Will any particular input elicit the worst case?*
  - Yes: Already-sorted input

Quick Sort: Analysis

- **In the worst case:**

  $T(1) = \Theta(1)$

  $T(n) = T(n - 1) + \Theta(n)$

  Works out to

  $T(n) = \Theta(n^2)$

- **In the best case:**

  $T(1) = \Theta(1)$

  $T(n) = 2T(n/2) + \Theta(n)$

  Works out to

  $T(n) = \Theta(n \lg n)$

Quick Sort: Analysis

- The real liability of quicksort is that it runs in $O(n^2)$ on already-sorted input

- Book discusses two solutions:
  - Randomize the input array, OR
  - *Pick a random pivot element*

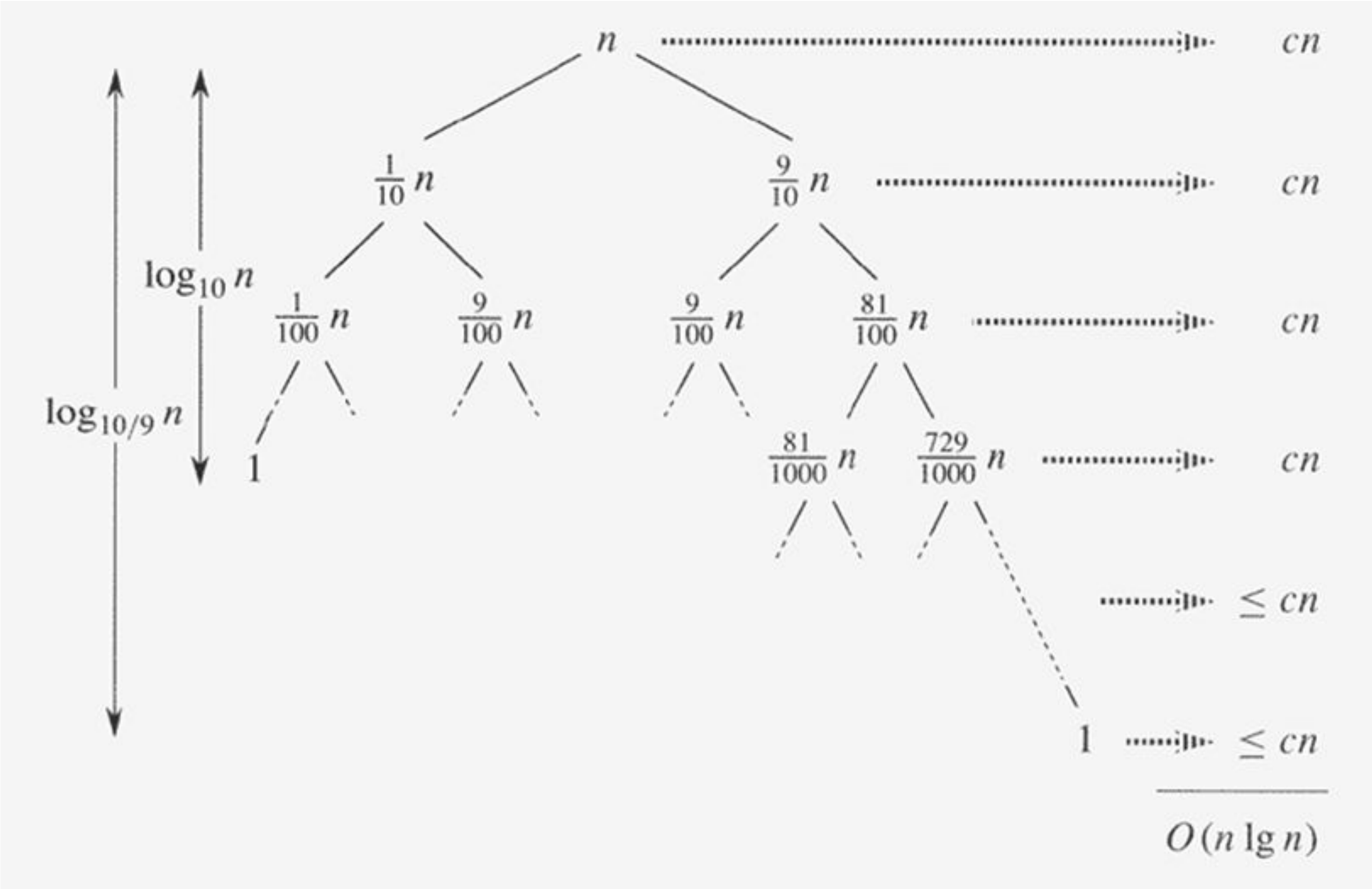
- *How will these solve the problem?*
  - By ensuring that no particular input can be chosen to make quicksort run in $O(n^2)$ time

Analyzing Quicksort: Average Case

- Assuming random input, average-case running time is much closer to $O(n \lg n)$ than $O(n^2)$
- First, a more intuitive explanation/example:
  - Suppose that partition() always produces a 9-to-1 split. This looks quite unbalanced!
  - The recurrence is thus:

    $T(n) = T(9n/10) + T(n/10) + n$
  - *How deep will the recursion go?* (draw it)

Analyzing Quicksort: Average Case



Analyzing Quicksort: Average Case

- Intuitively, a real-life run of quicksort will produce a mix of "bad" and "good" splits
  - Randomly distributed among the recursion tree
  - Pretend for intuition that they alternate between best-case (n/2 : n/2) and worst-case (n-1 : 1)
  - *What happens if we bad-split root node, then good-split the resulting size (n-1) node?*

Analyzing Quicksort: Average Case

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  - Randomly distributed among the recursion tree
  - Pretend for intuition that they alternate between best-case (n/2 : n/2) and worst-case (n-1 : 1)
  - *What happens if we bad-split root node, then good-split the resulting size (n-1) node?*
    - We end up with three subarrays, size 1, (n-1)/2, (n-1)/2
    - Combined cost of splits = n + n -1 = 2n -1 = O(n)
    - No worse than if we had good-split the root node!

Analyzing Quicksort: Average Case

- Intuitively, the O(n) cost of a bad split (or 2 or 3 bad splits) can be absorbed into the O(n) cost of each good split
- Thus running time of alternating bad and good splits is still O(n lg n), with slightly higher constants
- How can we be more rigorous?