Military Institute of Science and Technology

Department of Computer Science and Engineering

Subject: Numerical Methods Sessional (**CSE – 214)**

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Name of the Exp.: Solutions to Ordinary Differential Equations by Euler's Method

Introduction:

In mathematics, a differential equation is an equation in which the derivatives of a function appear as variables. Differential equations have many applications in physics and chemistry, and are widespread in mathematical models explaining biological, social, and economic phenomena.

Differential equations are divided into two types:

- An **Ordinary Differential Equation (ODE)** only contains functions of one variable, and derivatives in that variable.
- A **Partial differential Equation (PDE)** contains multivariate functions and their partial derivatives.

The **order** of a Differential equation is that of the highest derivative that it contains. For instance, a first-order Differential equation contains only first derivatives.

A general linear differential equation is of the following form:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \dots\dots\dots + a_n(x)y^{(n)} + b(x) = 0$$

Where $a_0(x), a_1(x), \dots\dots\dots a_n(x)$and $b(x)$ are arbitrary functions and $y', y'' \dots\dots\dots y^{(n)}$ are the successive derivatives of the unknown function y of variable x.

Why Numerical Solutions for ODE:

- For many of the differential equations we need to solve in the real world, there is no "nice" closed form solutions.
- As a result, we need to resort to using numerical methods for solving such DE's
- Numerical Approach gives us a good approximate solution.

Initial value problem:

A problem in which we are looking for the unknown function of a differential equation where the values of the unknown function and its derivatives at some point are known is called an initial value problem (in short IVP).

Methods of solving the Ordinary Differential Equations - Euler's Method:

Let $\boldsymbol{y'(x) = f(x, y(x))}$

$$\boldsymbol{f(x_0) = y_0}$$

Here, $\boldsymbol{f(x,y)}$ is a given function, $\boldsymbol{y_0}$ is the given initial value at $\boldsymbol{x = x_0}$. The unknown in the problem is the function $\boldsymbol{y(x)}$. The goal here is to determine approximately the unknown function $\boldsymbol{y(x)}$ for $\boldsymbol{x > x_0}$. The given value is $\boldsymbol{y_0}$ i.e. $\boldsymbol{y(x_0)}$. Using the given differential equation, we can also determine the derivative of $\boldsymbol{y}$ at $\boldsymbol{x_0}$ which is simply $\boldsymbol{f(x_0, y_0)}$.

$$\boldsymbol{y'(x_0) = f\big(x_0, y(x_0)\big) = f(x_0, y_0)}$$

If the rate of change of $\boldsymbol{y(x)}$ were to remain $\boldsymbol{f(x_0, y_0)}$ for all time, then $\boldsymbol{y(x)}$ would be exactly $\boldsymbol{y_0 + f(x_0, y_0)(x - x_0)}$. The rate of change of, however, does not remain constant for all values of $\boldsymbol{x}$. But it is reasonable to expect that it remains close to $\boldsymbol{f(x_0, y_0)}$ for $\boldsymbol{x}$ close to $\boldsymbol{x_0}$. If this is the case, then the value of $\boldsymbol{y(x)}$ will remain close to $\boldsymbol{y_0 + f(x_0, y_0)(x - x_0)}$ for $\boldsymbol{x}$ close to $\boldsymbol{x_0}$. Let us define $\boldsymbol{h}$ as follows:

$$\boldsymbol{x_1 = x_0 + h}$$
$$\boldsymbol{y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h}$$

By the above argument,

$$\boldsymbol{y(x_1) \approx y_1}$$

Now defining

$$\boldsymbol{x_2 = x_1 + h = x_0 + 2h}$$
$$\boldsymbol{y_2 = y_1 + f(x_1, y_1)(x_2 - x_1) = y_1 + hf(x_1, y_1)}$$
$$\boldsymbol{y(x_2) \approx y_2}$$

We then repeat this argument for all $\boldsymbol{x}$.
Let us define for $\boldsymbol{n = 0, 1, 2, 3}$ … … …

$$\boldsymbol{x_n = x_0 + nh}$$

Suppose that for some value of $\boldsymbol{n}$, the approximated value of $\boldsymbol{y_n(x)}$ has been calculated as $\boldsymbol{y_n}$. Then the rate of change of $\boldsymbol{y(x)}$ for $\boldsymbol{x}$ close to $\boldsymbol{x_n}$ is

$$\boldsymbol{f\big(x, y(x)\big) \approx f\big(x_n, y(x_n)\big) \approx y(x_n, y_n)}$$

And then

$$\boldsymbol{y_{n+1} = y_n + f(x_n, y_n)(x - x_n)}$$

Hence

$$\boldsymbol{y(x_{n+1}) \approx y_{n+1} + f(x_n, y_n)h}$$

This algorithm is called **Euler's method**. The parameter $\boldsymbol{h}$ is called **step size.**

Problem:

Consider a first order differential equation given by the following expression:

$$\boldsymbol{y = x - y}$$

And the initial conditions are:

$$\boldsymbol{y(0) = 1}$$