Section 4.4

Prove that n! increases exponentially.

n! = 1.2.3…………n = $\prod_{k=1}^{n} K$

n! = (n-1)! n

n	0	1	2	3	4	5	6	7	8	9	10
n!	1	1	2	6	24	120	720	5040	40320	362880	3628800

Prove that: n! increases exponentially using Gauss's Trick

$n!^2$ = (1.2……n) (n………2.1) = $\prod_{k=1}^{n} k\,(n+1-k)$

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Note:

(1.2.3…….n) ((n-0) (n-1) (n-2)…….2.1)

(1. ((n-0) (n-1) (n-2)…….2.1) . 2. ((n-0) (n-1) (n-2)…….2.1) ……………. n ((n-0) (n-1) (n-2)…….2.1))

Now

1 x (n-0) = 1 (n+1-1)
2 x (n-1) = 2 (n+1-2)
So, k(n+1-k)

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Now we can say,

n <= k(n+1-k) <= ¼ $(n+1)^2$

because of quadratic polynomial

Let's see

From, k(n+1-k) suppose k= a and n+1-k =b

As,

ab = $\{(a+b)^2 - (a-b)^2\}/4$

So,

$k(n+1-k) = \{ (k+n+1-k)^2 - (k-n-1+k)^2 \}/ 4$

$= ¼ (n+1)^2 - \{2k- (n+1) \}^2 /4$

$= ¼ (n+1)^2 - \{4k^2 - 2.2k (n+1) + (n+1)^2 \}/4$

$= ¼ (n+1)^2 - (k - ½ (n+1)\}^2$ …………………………… (1)

So we can say,

Smallest value of k = 1

And largest value of k = ½ (n+1)

So, $\prod_{k=1}^{n} n$ <= $n!^2$ <= $\prod_{k=1}^{n} \frac{1}{4} (n+1)^2$ …………………… (2)

When k=1 from equ (1)

$k(n+1-k) = ¼ (n+1)^2 - (1 - ½ (n+1)\}^2$

$= \{(n+1)^2 - 4 + 4.2.1/2 . (n+1) - 4.1/4 (n+1)^2 \} /4$

$= \{n^2+2n+1 -4 + 4 (n+1) - (n^2+2n+1) \} /4$

$= (-4 + 4n +4) /4$

$= n$

When k= ½ (n+1)

$k(n+1-k) = ¼ (n+1)^2 - (½ ((n+1) - ½ (n+1)\}^2$

$= ¼ (n+1)^2$

Now from eqn (2)

n^n <= $n!^2$ <= $(n+1)^{2n} / 2^{2n}$

$n^{n/2}$ <= $n!$ <= $(n+1)^n / 2^n$

So we can say that n! increases exponentially.