Newton - Cotes Integration Formula

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

$f(x) = \sin(x)$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$f_n(x)$ = a polynomial of the form

$f(x) = \cos(x)$

$= 1 - x^2/2! + x^4/4! - \frac{x^6}{6!} + \cdots$

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots \; a_{n-1}x^{n-1} + a_n x^n$$

Trapezoidal Rule

$I = \int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx$ polynomial is first order.

We know, straight line connecting points a and b

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Area under the straight line is the integral of $f(x)$ between a and b.

So, $I = \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]dx$

$$= \int_a^b \frac{f(b)-f(a)}{b-a}\,x\,dx + \int_a^b \frac{f(a)b - a\cancel{f(a)} - f(b)a + \cancel{f(a)}a}{b-a}\,dx$$

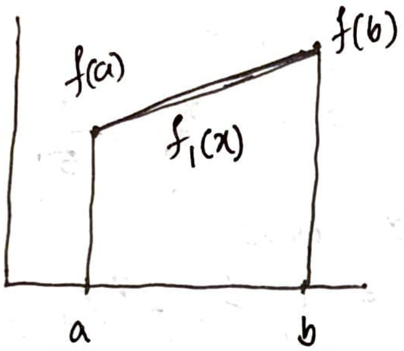
$$= \left[\frac{f(b)-f(a)}{b-a}\frac{x^2}{2} + \frac{bf(a) - af(b)}{b-a}x\right]_a^b$$

$$= \frac{f(b)-f(a)}{\cancel{b-a}}\,\frac{(b+a)\cancel{(b-a)}}{2} + bf(a) - af(b)$$

$$= \frac{bf(b) + \cancel{af(b)} + bf(a) - af(a) + \cancel{2bf(a)} - \cancel{2}af(b)}{2}$$

$$= \frac{b\{f(b)+f(a)\} - a\{f(a)+f(b)\}}{2}$$

$$= (b-a)\,\frac{f(a)+f(b)}{2}$$



$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$$\frac{x - a}{a - b} = \frac{f_1(x) - f(a)}{f(a) - f(b)}$$

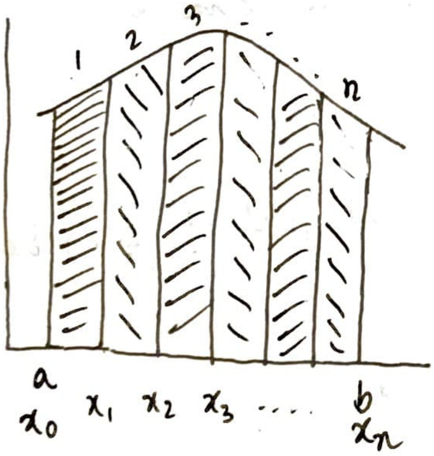
$$\frac{x-a}{a-b}\left[f(a) - f(b)\right] = f_1(x) - f(a)$$

$$\frac{x-a}{b-a}\left[f(b) - f(a)\right] = f_1(x) - f(a)$$

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

$$\text{Area} = \frac{1}{2} \times \frac{f(a) + f(b)}{\cancel{(b-a)}}(b-a)$$

= width × avg height



n segments of equal width

$$h = \frac{b-a}{n}$$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= h\,\frac{f(x_0) + f(x_1)}{2} + h\,\frac{f(x_1) + f(x_2)}{2} + \cdots h\,\frac{f(x_{n-1}) + f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$= (b-a)\,\frac{\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}{2n}$$

(Avg

= width × avg height

Trapezoidal Rule, $I = \int_a^b f_1(x)\,dx = (b-a)\,\frac{f(b)+f(a)}{2}$

Problem-1

$I = \int_0^1 e^{-x}\,dx \quad h = 0.1 \qquad I = \int_0^{\pi/2} \sin x\,dx \quad h = \pi/4$

$$I = \frac{h}{2}\left\{ f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right\}$$

$$I = \int_0^{\pi/2} \sin(x)\,dx$$

$$= \left[-\cos(x)\right]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2} + \cos(0)\right]$$

$$= 0 + 1 = 1$$

Simpson's Rule

Take not straight lines but polynomials for ~~rep~~ connecting point a and b.

$$I = \int_a^b f(x)\,dx = \int_a^b f_2(x)\,dx$$

$f(x)$ will be approximated using a second order polynomial.

~~Second order~~ lagrange Interpolation formula

$$f(x) = \sum_{j=1}^{n} P_j(x) \quad \text{where,}$$

$$P_j(x) = y_j \prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{x - x_i}{x_j - x_i}$$

second order lagrange Interpolation Formula

$$f(x) = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$\int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0)\, dx$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} \int_{x_0}^{x_2} (x^2 - xx_1 - xx_2 + x_1x_2)\, dx$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} \left(\frac{x^3}{3} - \frac{x_1x^2}{2} - \frac{x_2x^2}{2} + x_1x_2x\right)$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} \left[\frac{(x_2-x_0)^3}{3} - \frac{x_1(x_2-x_0)^2}{2} - \frac{(x_2-x_0)^2x_2}{2} + x_1x_2(x_2-x_0)\right]$$

$$= \frac{f(x_0)}{(x_1-x_0)\cancel{(x_2-x_0)}} \left[\frac{(x_2-x_0)^2}{3} - \frac{1}{2}(x_2-x_0)(x_1-x_2) + x_1x_2\right]$$

$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$ = Simpson's 1/3 Rule.

$h = (b-a)/2$

I = width x Avg height

$$= (b-a)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right]$$

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$h = \frac{b-a}{n}$ $\qquad I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots \int_{x_{n-2}}^{x_n} f(x)\,dx$

$$= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right)$$

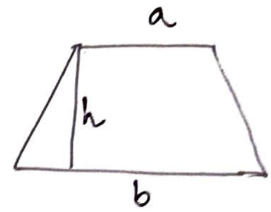
$$+ \dots + \frac{h}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5\ldots}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right]$$

$$= (b-a) \times \frac{f(x_0) + 4\sum_{i=1,3,5}^{n-2} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

width

Avg height

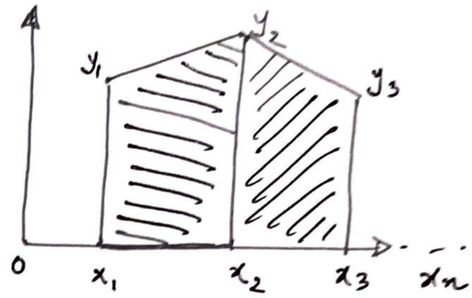


Area of ~~trapezoi~~ trapezium $= \frac{1}{2} \times (a+b) \times h$

$h = (x_2 - x_1)$

$a = y_1$ So, $Area_1 = \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$b = y_2$ $= \frac{1}{2} \Delta x (y_1 + y_2)$



$Area_2 = \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

$= \frac{1}{2}(y_2 + y_3)\Delta x$

Area $= \frac{1}{2} \Delta x (y_1 + y_2 + y_2 + y_3)$

$= \frac{1}{2} \Delta x (y_1 + 2y_2 + y_3)$ $Area_n = \frac{1}{2} \Delta x (y_1 + 2y_2 + 2y_3 + 2y_4 + \dots + y_n)$

$f(x) = y$ $y = \sin x$ $(x_0, y_0), (x_1, y_1) (x_2, y_2) \dots (x_n, y_n)$

$$I = \int_a^b f(x)\,dx$$

Newton's forward difference Interpolation formula:

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

✓ $p = \frac{x - x_0}{h}$ $h =$ step size $\Delta^i y_j = \Delta^{i-1} y_{j+1} - \Delta^i y_j$

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \right] dx \,(?)$$

$\Delta y_i = y_{i+1} - y_i$

We know, $p = \frac{x - x_0}{h}$ ~~59~~ ~~59~~

$ph = x - x_0$ ~~$x_0 = x - ph$~~ ~~$x_n =$~~

$x = x_0 + ph$ $[\, 1 = 0 + \frac{d}{dx}(ph)$

$dx = h.dp$

$$I = h \int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \right] dp$$

$$= h \left[y_0 n + \Delta y_0 \frac{n^2}{2} + \Delta^2 y_0 \left(\frac{n^3}{12} - \frac{n^2}{4}\right) + \frac{n^2(n-2)^2}{24}\Delta^3 y_0 + \dots \right]$$

$$+ \Delta^2 y_0 \frac{2n^3 - 3n^2}{12}$$

General quadrature formula for numerical Integration

$$\int \frac{p(p-1)}{2}\,dp \qquad \frac{1}{2}\times \frac{p^2(p-3p)}{6} = \frac{1}{12}$$

$$\int \frac{p^2-p}{2}\,dp$$

$$\int \frac{p^2}{2}dp - \int \frac{p}{2}\,dp$$

$$\frac{1}{2}\,\frac{p^3}{3} - \frac{p^2}{4} \qquad \frac{n(2n-3)}{12}$$

$$\frac{p^3}{6} - \frac{p^2}{4}$$

$$\frac{2p^3-3p^2}{12} \qquad \frac{n^3-3n^2}{6} = \frac{n^2(n-3)}{6}$$

$$\frac{p(p-1)(p-2)}{6}$$

$$(p^2-p)(p-2)$$

$$\frac{p^3-2p^2-p^2+2p}{6}$$

$$\frac{\frac{p^4}{4} - \cancel{2}\frac{p^3}{3} \; \cancel{3}\frac{p^3}{\cancel{3}} + \cancel{2}\frac{p^2}{\cancel{2}}}{6}$$

$$\frac{p^4-4p^3+4p^2}{24}$$

$$n^2(n^2-4n+4)$$

$$\frac{n^2(n-2)^2}{24}$$

put, $n=1$

$$I = \int_{x_0}^{x_n} y\,dx = h\left[y_0 + \Delta y_0 \frac{1}{2} + \Delta^2 y_0 \left(-\frac{1}{12}\right) + \Delta^3 y_0 + \frac{1}{24} + \dots\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_2-y_1) + \right.$$

$$I = hn\left[y_0 + \Delta y_0 \frac{1}{2} + \Delta^2 y_0 \frac{2n^3-3n^2}{12} + \Delta^3 y_0 \frac{n^2(n-2)^2}{24}\Delta^3 y_0 + \dots\right]$$

$$= \frac{1}{2}h\left[2y_0 + y_1 - y_0 - \frac{y_2-2y_1+y_0}{\cancel{12}\,6} + \frac{y_3-3y_2+3y_1-y_0}{\cancel{24}\,12} + \dots\right]$$

$$= \frac{1}{2}h\left[y_0 + y_1\right.$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= y_2 - y_1 - y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= \Delta y_2 - \Delta y_1 - \Delta y_1 + \Delta y_0$$
$$= \Delta y_2 - 2\Delta y_1 + \Delta y_0$$
$$= y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$