

Numerical Method for Solving Ordinary Differential Equation

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Differential Equation (DE)

- In Mathematics, a **differential equation** is an equation with one or more derivatives of a function. The derivative of the function is given by dy/dx .
- In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to the one or more independent variables.
- For example, $dy/dx = 5x$

Order of Differential Equation

- The order of the differential equation is the highest order derivative present in the equation. Here some of the examples for different orders of the differential equation are given.

$dy/dx = 3x + 2$ The order of the equation is 1

$(d^2y/dx^2) + 2(dy/dx) + y = 0$ The order is 2

$(dy/dt) + y = kt$ The order is 1

Order of Differential Equation

- **First Order Differential Equation**

[first-order differential equation](#) has only the first derivative such as dy/dx , where x and y are the two variables and is represented as:

$$dy/dx = f(x, y) = y'$$

- **Second-Order Differential Equation**

The equation which includes [second-order derivative](#) is the second-order differential equation. It is represented as:

$$d/dx(dy/dx) = d^2y/dx^2 = f''(x) = y''$$

Degree of Differential Equation

- The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as y' , y'' , y''' , and so on.
- Suppose $(d^2y/dx^2) + 2(dy/dx) + y = 0$ is a differential equation, so the degree of this equation here is 1. More examples:

➤ $dy/dx + 1 = 0$, degree is 1

➤ $(y''')^3 + 3y'' + 6y' - 12 = 0$, degree is 3

Ordinary Differential Equation (ODE)

- An **Ordinary Differential Equation** is differential equation which contains one or more functions of **one independent** variable and one or more of its derivative with respect to the variable.
- A general linear differential equation is of the following form

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \dots + a_n(x)y^{(n)} + b(x) = 0 \text{ where}$$

$a_0(x), \dots, a_n(x)$ and $b(x)$ are arbitrary functions and $y', y'', \dots, y^{(n)}$ are the successive derivatives of the unknown function y of the variable x .

Partial Differential Equation (PDE)

- A **Partial Differential Equation** is commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with **more than one** independent variable. A PDE for a function $u(x_1, \dots, x_n)$ is an equation of the following form and denoted using subscripts:

$$f\left(x_1, \dots, x_n; u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \dots\right) = 0$$

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right)$$

Application

1. Differential equations describe various exponential growths and decays.
2. They are also used to describe the change in return on investment over time.
3. They are used in the field of medical science for modelling cancer growth or the spread of disease in the body.
4. Movement of electricity can also be described with the help of it.
5. They help economists in finding optimum investment strategies.
6. The motion of waves or a pendulum can also be described using these equations.

The General Initial Value Problem

In this course, we'll try to solve the problems presented in the following way:

$$\frac{dy}{dx} = f(x, y)$$

$y(a)$ (the initial value) is known. Here, $f(x, y)$ is some function of the variables x and y .

Examples of Initial Value Problems

$$\begin{aligned} \text{(a)} \quad \frac{dy}{dx} &= 6 - 2\frac{y}{x} \\ y(3) &= 1 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{dy}{dx} &= \frac{y \ln y}{x} \\ y(2) &= e \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{dy}{dx} &= \frac{50x^2 - 10y}{3} \\ y(0) &= 0 \end{aligned}$$

Euler's Method

Euler's method assumes the solution of the ODE can be written in the form of a Taylor Series i.e. the function is of the form:

$$y(x + h) \approx y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \frac{h^3 y'''(x)}{3!} + \frac{h^4 y^{iv}(x)}{4!} + \dots$$

For Euler's method, we just take the first 2 terms only

The last term is just h time $y(x + h) \approx y(x) + hy'(x)$

$$y(x + h) \approx y(x) + hf(x, y)$$

Euler's Method: Graphical Representation

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\begin{aligned}\text{Slope} &= \frac{\text{Rise}}{\text{Run}} \\ &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0)\end{aligned}$$

$$\begin{aligned}y_1 &= y_0 + f(x_0, y_0)(x_1 - x_0) \\ &= y_0 + f(x_0, y_0)h\end{aligned}$$

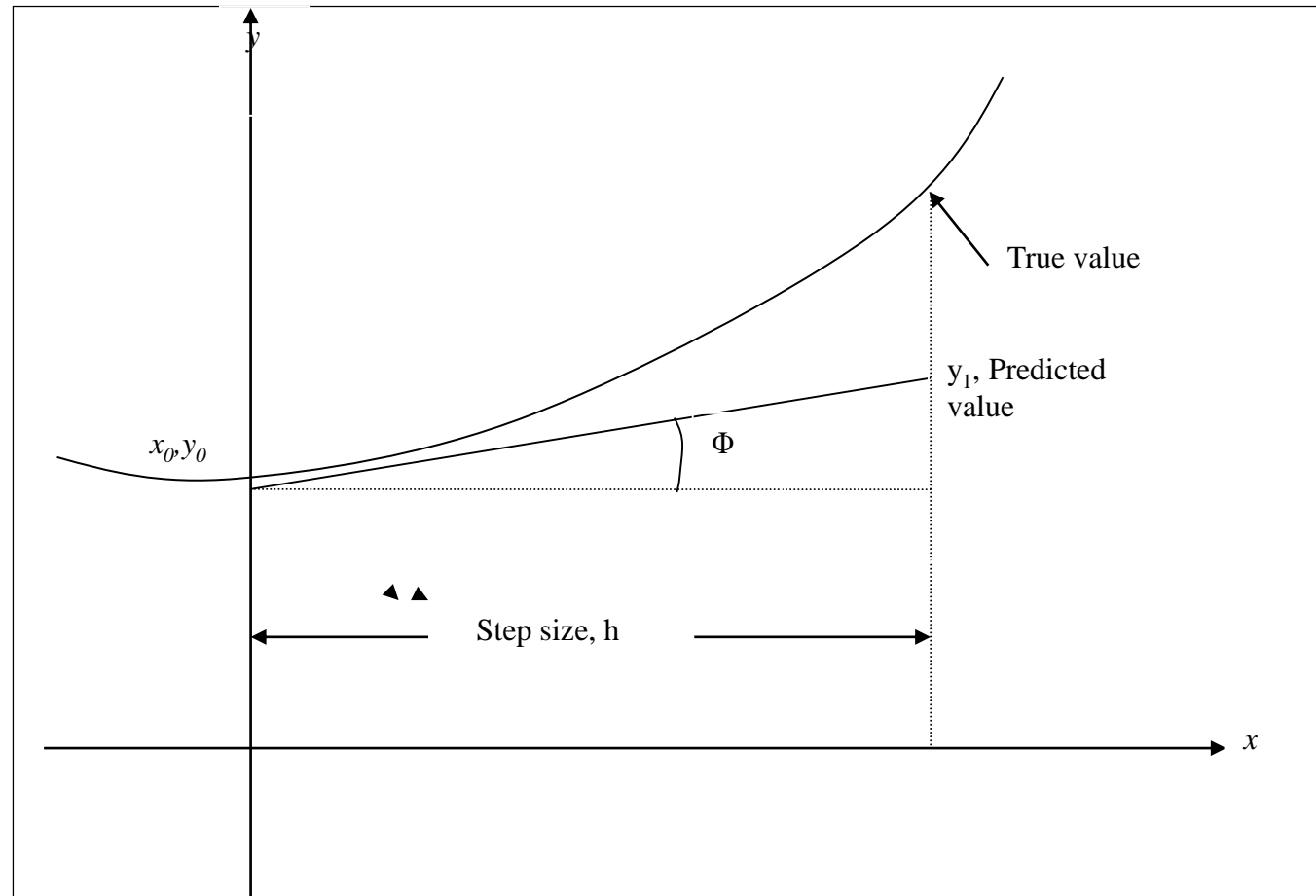


Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method: Graphical Representation

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

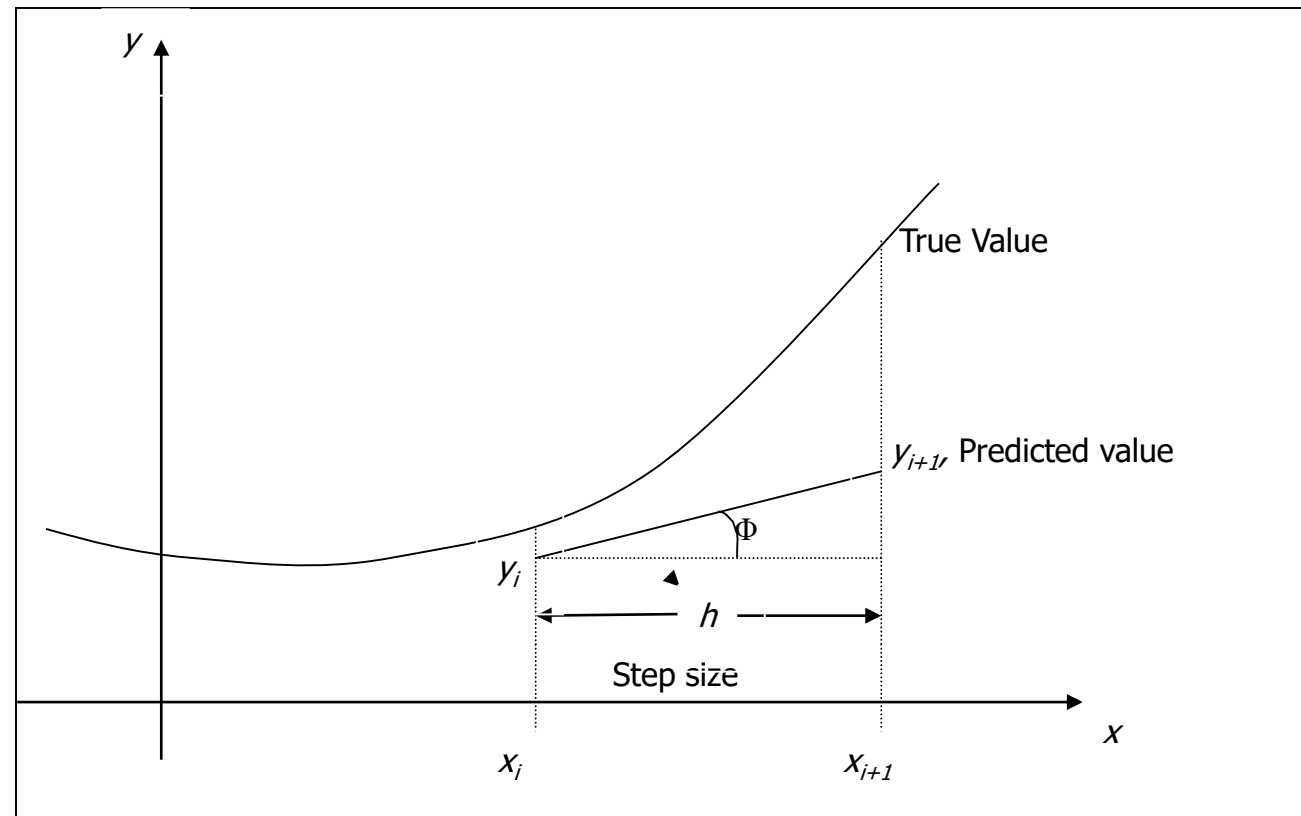


Figure 2. General graphical interpretation of Euler's method

How to use this Formula:

We start with some known value for y . This is actually the given initial value y_0 when $x = x_0$

h is the step size. It can be tuned for accuracy consideration.

Using Euler's formula, we calculate new value of y which we denote by y_1 ,

$$y_1 \approx y_0 + hf(x_0, y_0)$$

Here, y_1 is the next estimated solution value

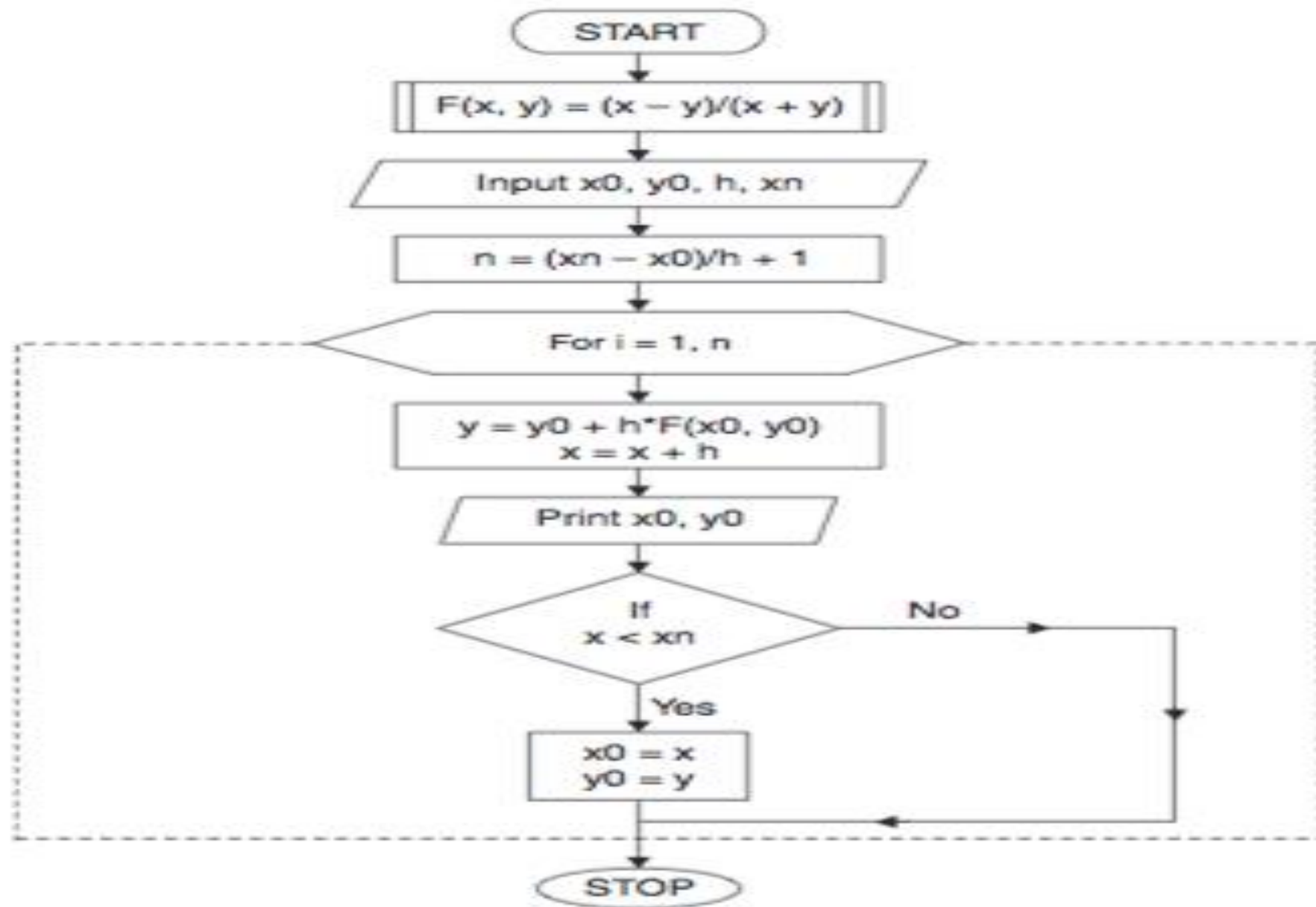
y_0 is the current value,

h is the step size

$f(x_0, y_0)$ is the value of the derivative at point (x_0, y_0)

Thus we calculate the further values of y with the following recursive formula:

$$y_i = y_{i-1} + h * f(x_{i-1}, y_{i-1})$$



Flow Chart for Euler's Method

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$$x_0 = 0$$

$$y_0 = 1$$

$$f(x, y) = x - y$$

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$$x_0 = 0$$

$$y_0 = 1$$

$$f(x, y) = x - y$$

$$h = 0.2$$

$$x_1 = 0.2$$

$$x_2 = 0.4$$

$$x_3 = 0.6$$

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$$x_0 = 0$$

$$y_0 = 1$$

$$f(x, y) = x - y$$

$$h = 0.2$$

$$x_1 = 0.2$$

$$x_2 = 0.4$$

$$x_3 = 0.6$$

Iteration : 1

$$\begin{aligned} f(x_0, y_0) &= f(0, 1) \\ &= 0 - 1 = -1 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.2(-1) \\ &= 0.8 \end{aligned}$$

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$$x_0 = 0$$

$$y_0 = 1$$

$$f(x, y) = x - y$$

$$h = 0.2$$

$$x_1 = 0.2; y_1 = 0.8$$

$$x_2 = 0.4$$

$$x_3 = 0.6$$

Iteration : 2

$$\begin{aligned} f(x_1, y_1) &= f(0.2, 0.8) \\ &= 0.2 - 0.8 = -0.6 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.8 + 0.2(-0.6) \\ &= 0.68 \end{aligned}$$

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$$x_0 = 0$$

$$y_0 = 1$$

$$f(x, y) = x - y$$

$$h = 0.2$$

$$x_1 = 0.2; y_1 = 0.8$$

$$x_2 = 0.4; y_2 = 0.68$$

$$x_3 = 0.6$$

Iteration : 3

$$\begin{aligned} f(x_2, y_2) &= f(0.4, 0.68) \\ &= 0.2 - 0.68 = -0.28 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.68 + 0.2(-0.28) \\ &= 0.624 \end{aligned}$$

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

- $x_0 = 0 ; y_0 = 1$
- $x_1 = 0.2 ; y_1 = 0.8$
- $x_2 = 0.4 ; y_2 = 0.68$
- $x_3 = 0.6 \quad y_3 = 0.624$

Code & Output

```
1  F = @(x,y) x-y;  
2  
3  
4  x0 = 0;  
5  y0 = 1;  
6  xfinal = 0.6;  
7  h = 0.2;  
8  
9  Xout = [];  
10 y = y0;  
11 Yout = y;  
12  
13 for x = x0:h:xfinal-h  
14     Xout = [Xout;x];  
15     s = F(x,y);  
16     y = y + h*s;  
17     Yout = [Yout;y];  
18 end  
19 Xout = [Xout;xfinal];  
20 display(Xout);  
21 display(Yout);  
22 plot(Xout,Yout);
```

