Military Institute of Science and Technology

Department of Computer Science and Engineering

Subject: Numerical Methods Sessional (**CSE 214)**

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Name of the Expt.: Numerical differentiation for equidistant x by Newton's Interpolating Formulae

Introduction:

We are familiar with the analytical method of finding the derivative of a function when the functional relation between the dependent variable y and the independent variable x is known. However, in practice, most often functions are defined only by tabulated data or the values of y for specified values of x can be found experimentally. Also in some cases, it is not possible to find the derivative of a function by analytical method. In such cases, the analytical process of differentiation breaks down and some numerical process have to be invented. The process of calculating the derivative of a function by means a set of given values of that function is called numerical differentiation. This process consists in replacing a complicated or an unknown function by an interpolation polynomial and then differentiating this polynomial as many times as desired.

Theory:

Let there are n+1 number of data points $(x_0, y_0), (x_1, y_1)........(x_n, y_n)$ that are given. To find out the values of $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ at various given points of x of the table, the methods given below are followed.

- **Points at the beginning of the table:**

To find out $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ for the given points of x at the beginning of the table, use Numerical Forward Differentiation formulae using Newton's forward difference table.

Numerical Differentiation formulae using forward difference table

The Newton's Forward Interpolation formula is given by:

$$y = f(x) = y_o + p\Delta y_o + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ...(1)$$

According to the chain rule of differentiation, $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx}$

Differentiating equation (1) with respect to ***x*,** we get,

$$f'(x) = \frac{1}{h}[\Delta y_o + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + ...] \quad .... (2)$$

Note here that $\frac{dp}{dx} = \frac{1}{h}$

Differentiating Eq. (2) with respect to ***x***, we get

$$f''(x) = \frac{1}{h^2}[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2+36p+22}{4!}\Delta^4 y_0 + ...] \quad .... (3)$$

Equation (2) and (3) give the approximate derivatives of $f(x)$ at arbitrary point $x = x_0 + ph$.

When $x = x_0$, *p=0,* then

$$f'(x_0) = \frac{1}{h}\left[\Delta y_o - 1/2 * \Delta^2 y_o + 1/3 * \Delta^3 y_o - 1/4 * \Delta^4 y_o + 1/5 * \Delta^5 y_o - \frac{1}{6} * .\Delta^6 y_0 + ...\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_o - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_o - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

Where**,** h=difference between two successive values of *x*.

$$\Delta y_0 = y_1 - y_0; \Delta y_1 = y_2 - y_1; \Delta y_2 = y_3 - y_2; \Delta^2 y_0 = \Delta y_1 - \Delta y_0; \Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

❖ **Points at the end of the table:**

To find out $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ for the given points of x at the end of the table use Numerical Backward Differentiation formulae using Newton's backward difference table.

Numerical Differentiation formula using Backward Difference Table

The Newton's backward interpolation formula is given by:

$$y = f(x) = y_o + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Differentiating Eq. (4) with respect to ***x***, we get

$$f'(x) = \frac{1}{h}[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_n + \ldots\ldots\ldots] \quad \ldots\ldots\ldots\ldots (5)$$

Differentiating Eq. (5) with respect to ***x***, we get

$$f''(x) = \frac{1}{h^2}[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{12p^2+36p+22}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\ldots] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Equation (5) and (6) give the approximate derivatives of $f(x)$ at arbitrary point $x = x_0 + ph$

When $x = x_n$, $p=0$, then

$$f'(x_n) = \frac{1}{h}\left[\nabla y_n + 1/2*\nabla^2 y_n + 1/3*\nabla^3 y_n + 1/4*\nabla^4 y_n + 1/5*\nabla^5 y_n + \frac{1}{6}*.\nabla^6 y_n + \ldots\ldots\right]$$

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \ldots\ldots\ldots\ldots\ldots\right]$$

Where, h=difference between two successive values of x. $\nabla y_1 = y_1 - y_0$; $\nabla y_2 = y_2 - y_1$; $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$; $\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$ and so on.

Finding out the location of a point (beginning/end);

Now, It is very important to find out in program that whether a point "i", (where i represents the index of the x or y vector), is at the beginning, or end or at the middle point of the table because according to it forward or backward formula should be applied respectively.

[Hints: now if i < (length(x) + 1)/2, use forward differentiation formula and when i > (length(x) + 1)/2, use backward differentiation formula.]

Problems

1. Find the value of f′(x) and f″(x) at x=1.0 and 5.0 for the following tabular data using Numerical Differentiation.

x	$x_0=1$	$x_1=2$	$x_2=3$	$x_3=4$	$x_4=5$
y	$y_0=2$	$y_1=5$	$y_2=9$	$y_3=12$	$y_4=20$

2. Find the value of f′(x) and f″(x) at x=1.0, 1.2, 2.0, 2.2 using Numerical Differentiation.

x	x0=1.0	x1=1.2	x2=1.4	x3=1.6	x4=1.8	x5=2.0	x6=2.2
y	y0=2.7183	y1=3.3201	y2=4.0552	y3=4.9530	y4=6.0496	y5=7.3891	y6=9.0250

3. Find f′(2) , f″(2), f′(6) , f″(6) ,f′(7) , f″(7) for the following function using Numerical Differentiation formulae, when , $2 \le \zeta \le 7$

$$f(x)=y=x^3-3x^2+2x-1$$