Numerical Method for Solving Ordinary Differential Equation

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Differential Equation (DE)

- In Mathematics, a differential equation is an equation with one or more derivatives of a function. The derivative of the function is given by dy/dx.

- In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to the one or more independent variables.

- For example, $dy/dx = 5x$

Order of Differential Equation

- The order of the differential equation is the highest order derivative present in the equation. Here some of the examples for different orders of the differential equation are given.

$dy/dx = 3x + 2$ The order of the equation is 1

$(d^2y/dx^2) + 2\,(dy/dx) + y = 0$ The order is 2

$(dy/dt) + y = kt$ The order is 1

Order of Differential Equation

- **First Order Differential Equation**

first-order differential equation has only the first derivative such as *dy/dx*, where x and y are the two variables and is represented as:

$$dy/dx = f(x, y) = y'$$

- **Second-Order Differential Equation**

The equation which includes second-order derivative is the second-order differential equation. It is represented as:

$$d/dx(dy/dx) = d^2y/dx^2 = f''(x) = y''$$

Degree of Differential Equation

- The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as y’, y”, y”’, and so on.
- Suppose $(d^2y/dx^2)+ 2\,(dy/dx)+y = 0$ is a differential equation, so the degree of this equation here is 1. More examples:

  - $dy/dx + 1 = 0$, degree is 1
  - $(y''')^3 + 3y'' + 6y' - 12 = 0$, degree is 3

Ordinary Differential Equation (ODE)

- An Ordinary Differential Equation is differential equation which contains one or more functions of one independent variable and one or more of its derivative with respect to the variable.

- A general linear differential equation is of the following form

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \ldots\ldots\ldots + a_n(x)y^{(n)} + b(x) = 0 \text{ where}$$

$a_0(x), \ldots\ldots\ldots a_n(x)$ and $b(x)$ are arbitrary functions and $y', y'' \ldots\ldots\ldots y^{(n)}$ are the successive derivatives of the unknown function y of the variable x.

Partial Differential Equation (PDE)

- A Partial Differential Equation is commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with more than one independent variable. A PDE for a function $u(x_1, \ldots\ldots x_n)$ is an equation of the following form and denoted using subscripts:

$$f\left(x_1, \ldots . x_n; u, \frac{\partial u}{\partial x_1}, \ldots\ldots, \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots\ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}; \ldots .\right) = 0$$

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

Application

1. Differential equations describe various exponential growths and decays.
2. They are also used to describe the change in return on investment over time.
3. They are used in the field of medical science for modelling cancer growth or the spread of disease in the body.
4. Movement of electricity can also be described with the help of it.
5. They help economists in finding optimum investment strategies.
6. The motion of waves or a pendulum can also be described using these equations.

The General Initial Value Problem

In this course, we'll try to solve the problems presented in the following way:

$$\frac{dy}{dx} = f(x, y)$$

$y(a)$ (the initial value) is known. Here, $f(x, y)$ is some function of the variables x and y.

Examples of Initial Value Problems

(a) $\frac{dy}{dx} = 6 - 2\frac{y}{x}$

$y(3) = 1$

(b) $\frac{dy}{dx} = \frac{y \ln y}{x}$

$y(2) = e$

(c) $\frac{dy}{dx} = \frac{50x^2 - 10y}{3}$

$y(0) = 0$

Euler’s Method

Euler’s method assumes the solution of the ODE can be written in the form of a Taylor Series i.e. the function is of the form:

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \frac{h^3 y'''(x)}{3!} + \frac{h^4 y^{\text{iv}}(x)}{4!} + \cdots$$

For Euler’s method, we just take the first 2 terms only

The last term is just h time $y(x+h) \approx y(x) + hy'(x)$

$$y(x+h) \approx y(x) + hf(x,y)$$

Euler’s Method: Graphical Representation

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

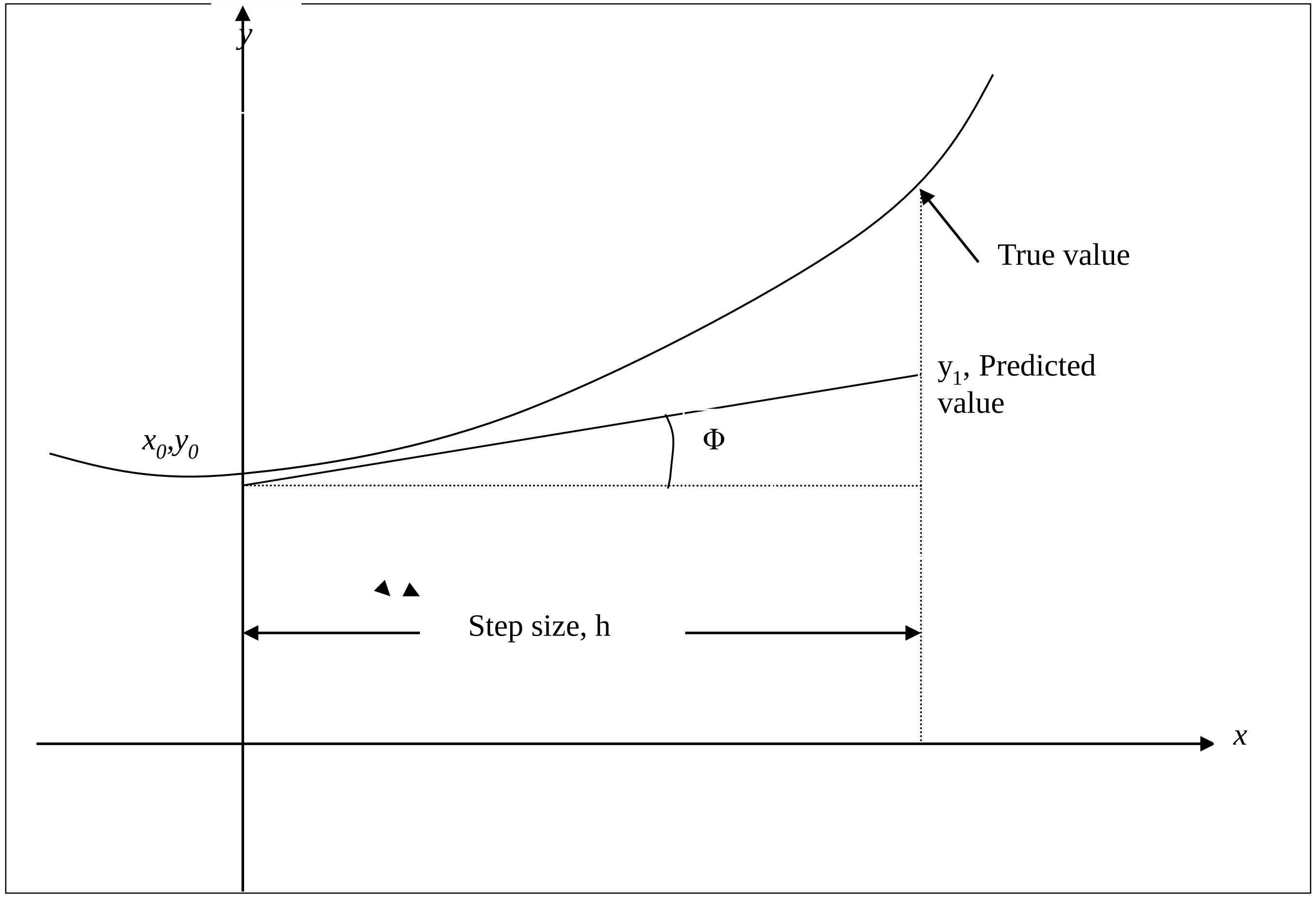


Figure 1 Graphical interpretation of the first step of Euler’s method

Euler’s Method: Graphical Representation

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

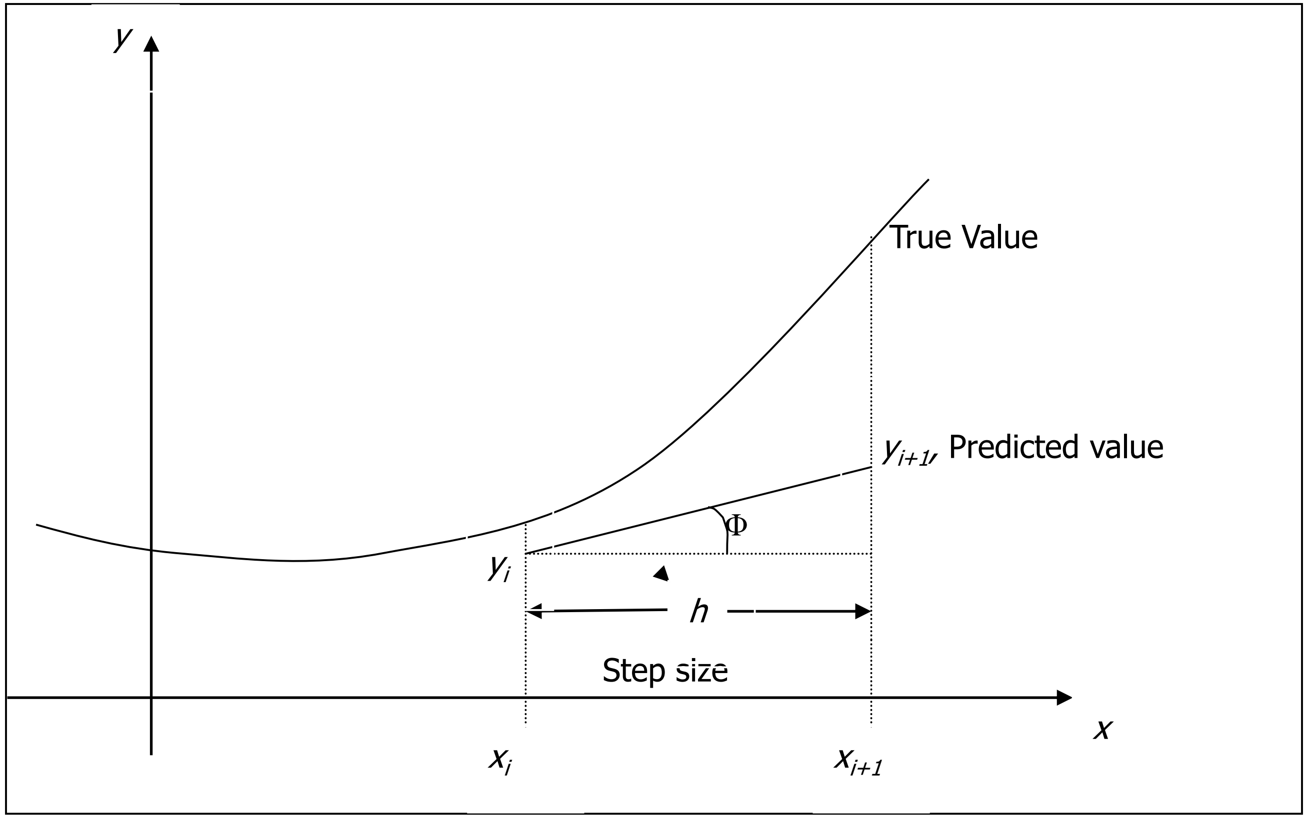


Figure 2. General graphical interpretation of Euler’s method

How to use this Formula:

We start with some known value for y . This is actually the given initial value y_0 when $x = x_0$

h is the step size. It can be tuned for accuracy consideration.

Using Euler's formula, we calculate new value of y which we denote by y_1,

$$y_1 \approx y_0 + hf(x_0, y_0)$$

Here, y_1 is the next estimated solution value

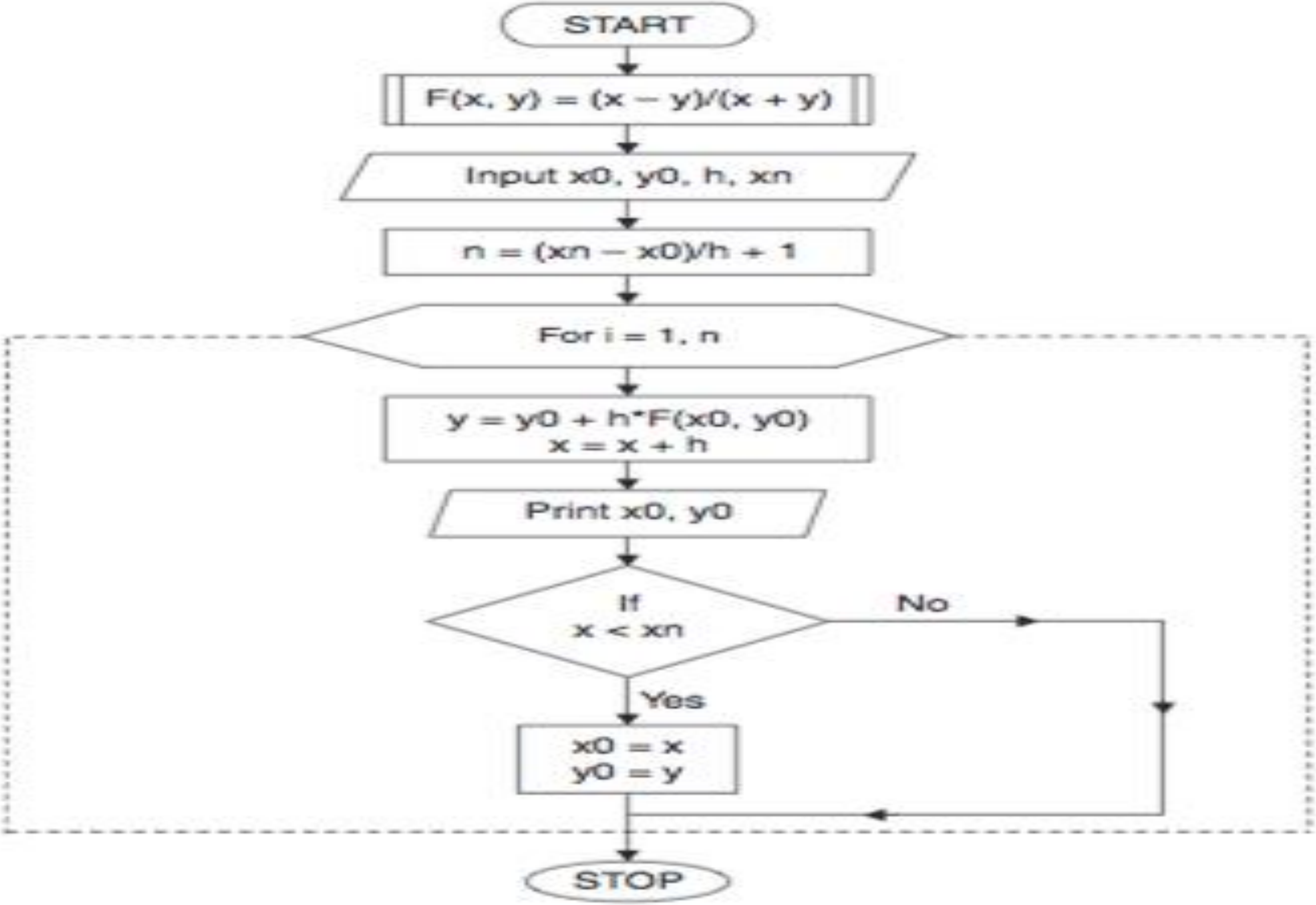
y_0 is the current value,

h is the step size

$f(x_0, y_0)$ is the value of the derivative at point (x_0, y_0)

Thus we calculate the further values of y with the following recursive formula:

$$y_i = y_{i-1} + h * f(x_{i-1}, y_{i-1})$$



Flow Chart for Euler's Method

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x - y$

Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x - y$

$h = 0.2$

$x_1 = 0.2$

$x_2 = 0.4$

$x_3 = 0.6$

Example Problem

$$\frac{dy}{dx} = x - y; \;\; y(0) = 1$$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x - y$

$h = 0.2$

$x_1 = 0.2$

$x_2 = 0.4$

$x_3 = 0.6$

Iteration : 1

$$f(x_0, y_0) = f(0,1)$$
$$= 0 - 1 = -1$$

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.2(-1)$$
$$= 0.8$$

Example Problem

$$\frac{dy}{dx} = x - y; \;\; y(0) = 1$$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x - y$

$h = 0.2$

$x_1 = 0.2\,;\, y_1 = 0.8$

$x_2 = 0.4$

$x_3 = 0.6$

$Iteration : 2$

$$f(x_1, y_1) = f(0.2, 0.8)$$
$$= 0.2 - 0.8 = -0.6$$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.8 + 0.2(-0.6)$$
$$= 0.68$$

Example Problem

$$\frac{dy}{dx} = x - y; \;\; y(0) = 1$$

$x_0 = 0$

$y_0 = 1$

$f(x, y) = x - y$

$h = 0.2$

$x_1 = 0.2\,; y_1 = 0.8$

$x_2 = 0.4\,; y_2 = 0.68$

$x_3 = 0.6$

Iteration : 3

$$f(x_2, y_2) = f(0.4, 0.68)$$
$$= 0.2 - 0.68 = -0.28$$

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.68 + 0.2(-0.28)$$
$$= 0.624$$

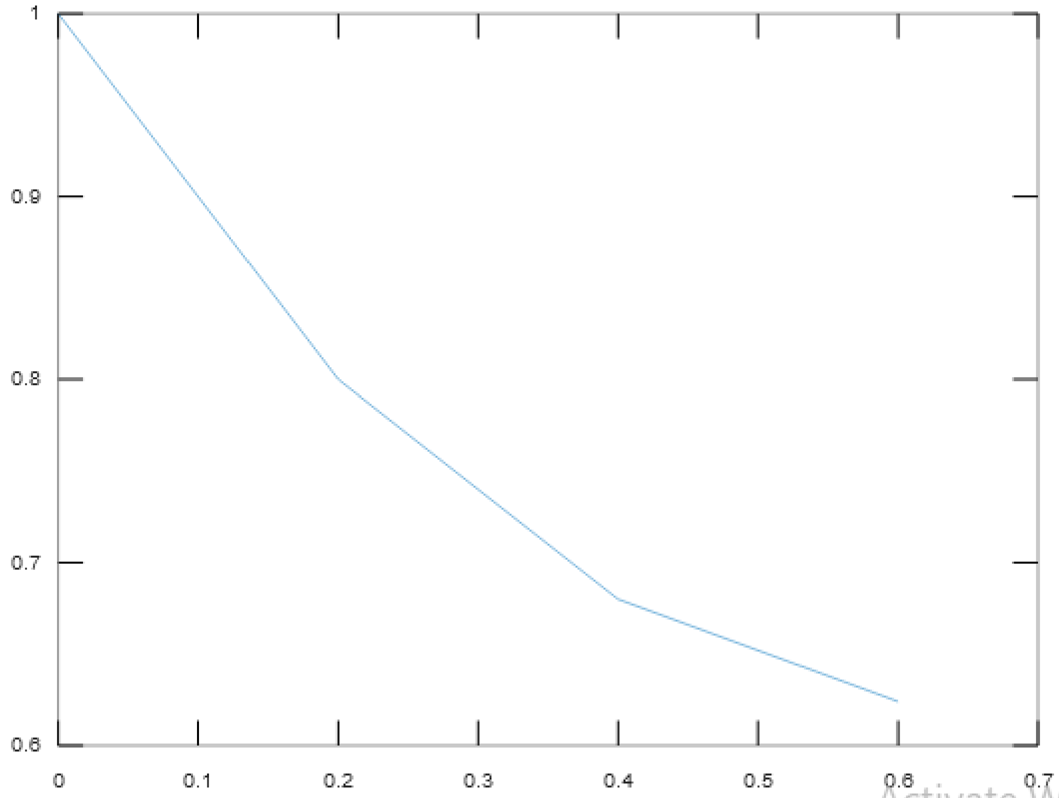
Example Problem

$$\frac{dy}{dx} = x - y; \quad y(0) = 1$$

- $x_0 = 0\ ; y_0 = 1$
- $x_1 = 0.2\ ; y_1 = 0.8$
- $x_2 = 0.4\ ; y_2 = 0.68$
- $x_3 = 0.6\ y_3 = 0.624$

Code & Output

```
F = @(x,y) x-y;

x0 = 0;
y0 = 1;
xfinal = 0.6;
h = 0.2;

Xout = [];
y = y0;
Yout = y;

for x = x0:h:xfinal-h
    Xout = [Xout;x];
    s = F(x,y);
    y = y + h*s;
    Yout = [Yout;y];
end
Xout = [Xout;xfinal];
display(Xout);
display(Yout);
plot(Xout,Yout);
```



Picard’s Method

Picard’s Method

The **Picard’s** method is an iterative method and is primarily used for approximating solutions to differential equations.

Picard’s Method

- This method of solving a differential equation approximately is one of successive approximation; that is, it is an iterative method in which the numerical results become more and more accurate, the more times it is used.
- The Picard’s iterative method gives a sequence of approximations Y1(x), Y2(x), …Yk(x) to the solution of differential equations such that the nth approximation is obtained from one or more previous approximations.

Picard’s Method: **Steps involved**

- Step 1: An approximate value of y (taken, at first, to be a constant) is substituted into the right hand side of the differential equation: $dy/dx= f(x, y)$.
- Step 2: The equation is then integrated with respect to x giving y in terms of x as a second approximation, into which given numerical values are substituted and the result rounded off to an assigned number of decimal places or significant figures.
- Step 3: The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

Example 1

Find the successive approximate solution of the differential equation

$$y' = y \, , \quad y(0) = 1$$

and compare it with the infinite series expansion of the exact solution $y = e^x$.

solution ⇒

$$y' = f(x,y) = y$$

$x_0 = 0$

$y_0 = 1$

Example 1

picards iterative formula →

$$y_n = y_0 + \int_0^x y_{n-1}\, dx \qquad (n = 1, 2, 3 \ldots)$$

$$y_1 = 1 + \int_0^x y_0\, dx = 1 + \int_0^x (1)\, dx = 1 + x$$

$$y_2 = 1 + \int_0^x y_1\, dx = 1 + \int_0^x (1+x)\, dx = 1 + x + \frac{x^2}{2}$$

$$y_3 = 1 + \int_0^x y_2\, dx = 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Example 1

The exact solution of the given differential equation is

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The successive approximations $y_1, y_2, y_3 \cdots$ have the same terms as in the infinite series expansion of the exact solution trucated after 2 terms, 3 terms, 4 terms etc.

Example 2

Given that

$$\frac{dy}{dx} = x + y^2$$

and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

solution ⇒

$$\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} (x + y^2)\, dx$$

$x_0 = 0$
$y_0 = 0$

$$y - y_0 = \int_{x_0}^{x} (x + y^2)\, dx$$

$$y = \int_{0}^{x} (x + y^2)\, dx$$

Example 2

1st Iteration:

$$Y_1 = Y_0 + \int_0^x (x + Y_0^2)\, dx$$

$$= 0 + \int_0^x x\, dx + \int_0^x 0\, dx \qquad \Big|\, Y_0 = 0$$

$$= \frac{x^2}{2}$$

at $x = 0 \cdot 3$, $\quad Y_1 \simeq 0 \cdot 0450$

Example 2

2nd Iteration:

$$y_2 = y_0 + \int_0^x (x + y_1^2)\,dx$$

$$= 0 + \int_0^x \left(x + \frac{x^4}{4}\right)dx$$

$$= \int_0^x x\,dx + \int_0^x \frac{x^4}{4}\,dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

$$y_1^2 = \left(\frac{x^2}{2}\right)^2 = \frac{x^4}{4}$$

at $x = 0.3$ $\quad Y_2 \simeq 0.0451$

Example 2

3rd Iteration:

$$y_3 = y_0 + \int^{x} (x + y_2^2)\, dx$$

$$= 0 + \int_0^{x} \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

$$y_2^2 = \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2$$

$$= \frac{x^4}{4} + 2 \cdot \frac{x^2}{2} \cdot \frac{x^5}{20} + \frac{x^{10}}{400}$$

$$= \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}$$

at $x = 0.3$ $\quad y_3 \simeq 0.0451$

Hence, $y = 0.0451$, correct to four decimal places, at $x = 0.3$.

Code

```
clc

syms x y
dydx = y;
x0 = 0;
y0 = 1;

f = subs(dydx, y, y0);
y1 = y0 + int(f,x,x0,x);
disp(y1);
```

Code

```
f = subs(dydx, y, y1);
y2 = y0 + int(f,x,x0,x);
disp(y2);

f = subs(dydx, y, y2);
y3 = y0 + int(f,x,x0,x);
disp(y3);

f = subs(dydx, y, y3);
y4 = y0 + int(f,x,x0,x);
disp(y4);
```

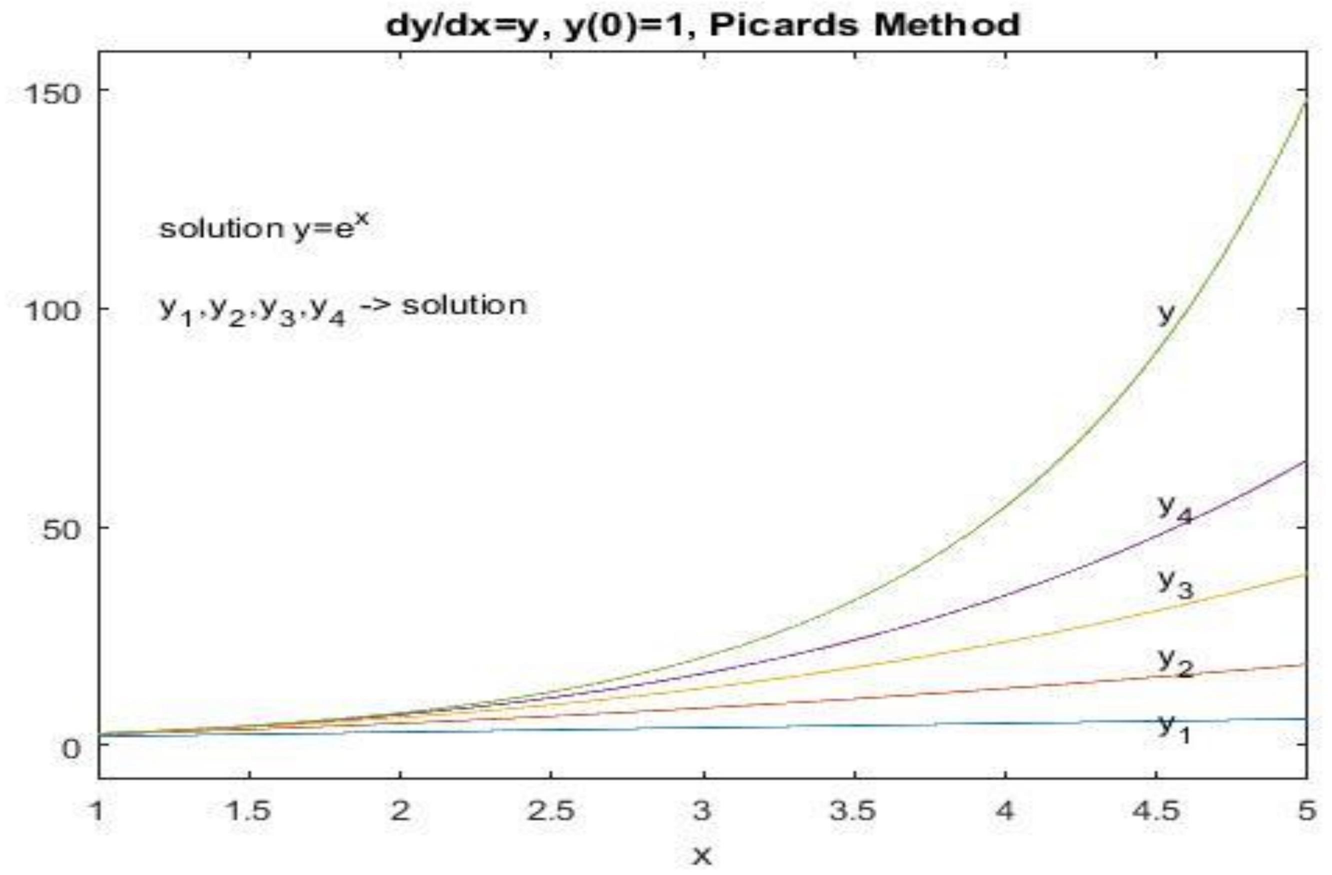
Code

```

ysol = exp(x);

ezplot(y1,1,5);
hold on;
ezplot(y2,1,5);
ezplot(y3,1,5);
ezplot(y4,1,5);
ezplot(ysol,1,5);
hold off;
```

Code

```
title('dy/dx=y, y(0)=1, Picards Method')
text(1.2, 120,'solution y=e^x')
text(1.2, 100,'y_1,y_2,y_3,y_4 -> solution')
text(4.5,100,'y')
text(4.5,55,'y_4')
text(4.5,38,'y_3')
text(4.5,20,'y_2')
text(4.5,5,'y_1')
```

Output



dy/dx=y, y(0)=1, Picards Method
solution y=e^x
y1,y2,y3,y4 -> solution
y
y4
y3
y2
y1
150
100
50
0
1
1.5
2
2.5
3
3.5
4
4.5
5
x