Topic 3: Homework problems

Example 1

All x, y, z, t have to be different so we can consider the biggest number x (WLOG). When x is the biggest number then $x > z > t > y$. Otherwise one side of equation would be bigger than other.

Let $x = z + a$ where a is a positive integer. According to the binomial theorem we get:

$$x^x = (z + a)^x = z^x + x \cdot z^{x-1} \cdot a + \cdots + a^x > z^z + 1 \cdot t^t + 0 + \cdots + 0$$

$$x^x > z^z + t^t$$

We can see that the left side of the equation is bigger than right side.

All x, y, z, t have to be positive integers. We already know that $x^x > z^z + t^t$. When you add some positive number to the left side, it is still bigger than right side. Therefore $x^x + y^y = z^z + t^t$ has no solution for pairwise different positive integers.

Example 2

Square of any integer is a positive integer. Left side is always positive so $2xyz$ has to be positive too. Number $2xyz$ is positive when x, y, z are positive or when two of them are negative. That means we can try to find just positive x, y, z and if they exist then for finding another solutions we will just change sign.

First of all we will look at the parity of $x^2 + y^2 + z^2 = 2xyz$. There are just two possibilities for the parity – all numbers are even or one is even and two are odd.

<u>One number is even and two odd</u>

When one number is even and two are odd, then $2xyz$ is divisible by 4 so the other side of the equation ($x^2 + y^2 + z^2$) has to be divisible by 4 too.

Even numbers on the square are always $0 \ (mod\ 4)$ and odd numbers on the square are always $1 \ (mod\ 4)$.

Consider z an even number (without loss of generality). The square of the z^2 is $0 \ (mod\ 4)$, that means it is divisible by 4. Number x and y are odd. That means $x^2 \equiv 1 \ (mod\ 4)$ and $y^2 \equiv 1 \ (mod\ 4)$.

We can use formulas using modulo. Suppose $a \equiv u \ (mod\ n)$ and $b \equiv v \ (mod\ n)$ and conditions $n \in \mathbb{N}; a, b, u, v \in \mathbb{Z}$, then is valid $a + b \equiv u + v \ (mod\ n)$.

According to the formula:

$$x^2 + y^2 + z^2 \equiv 1 + 1 + 0 \ (mod\ 4)$$

$$x^2 + y^2 + z^2 \equiv 2 \ (mod\ 4)$$

Now we can see that $x^2 + y^2 + z^2$ is not divisible by 4. So there is no positive integer solution when one number is even and two odd.

<u>All numbers are even</u>

Consider the smallest even x, y, z.

$$(2a)^2 + (2b)^2 + (2c)^2 = 2 \cdot 2a \cdot 2b \cdot 2c$$

$$4a^2 + 4b^2 + 4c^2 = 8abc$$

It is possible to divide equation by 4.

$$a^2 + b^2 + c^2 = 4abc$$

Let´s look at parity of numbers a, b, c – all numbers have to be even or one number is even and two are odd.

When one number is even and two are odd. Then $a^2 + b^2 + c^2$ would have to be divisible by 4 and as we already proved it is not possible.

When all numbers are even, then that is contradiction because at the beginning we considered the smallest x, y, z and now we found smaller number.

That is contradiction and therefore x, y, z cannot be all even.

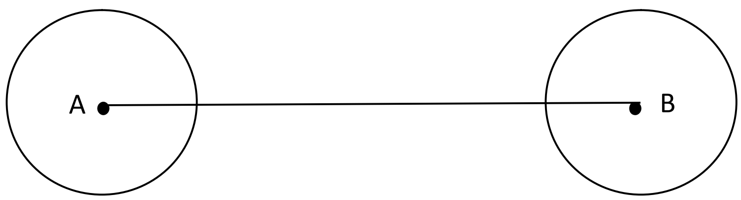
<u>Conclusion</u>

As there is no solution in any positive integers there is not solution in any negative integers neither. There is only a trivial solution $(a, b, c) = (0,0,0)$.

Example 3

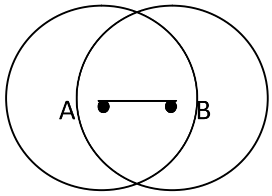
Consider two points A and B in the plane whose distance is the biggest. We have two possibilities:

1. Distance between A and B is more than 1.
   Among any three points in the plane there have to be two whose distance is less or equal to 1. When $|A, B| > 1$ then every other point has to be in one of the circles of radius 1 around point A or B. Otherwise there would not be two points among any other three points whose distance is less or equal to 1.



2. Distance between A and B is less or equal to 1.

When $|A, B| \leq 1$ and $|A, B|$ is the biggest distance then there is no bigger distance between points in the plane. That is why any other point has to be in one of the circles of radius 1 around A or B.



Example 4

It is possible to put students to the line A, B, C, D, E, F, G. Student A has least coins and student G most coins: $A < B < C < D < E < F < G$.

Students E, F, G have most coins. To get the lowest sum of coins of these students we need to keep numbers of coins as close as possible (we need to find integers with the lowest differential between them). When we would have for example six low numbers and one big number, then we get higher sum of E, F, G.

Consider case of 100 coins. When we prove it for 100 coins then it is valid for any number higher than 100.

We get numbers with the lowest differentials when we count average number of coins of one student. Average number a is $\frac{100}{7}$, that is 14 coins and 2 coins left.

When there would not be 2 coins left, numbers of coins would be like that:

$$a-3, a-2, a-1, a, a+1, a+2, a+3$$

However we have to give 2 coins to some student or students.

$$a-3, a-2, a-1, a, a+1, a+2, a+5$$

$$a-3, a-2, a-1, a, a+1, a+3, a+4$$

There is not any other solution because then two students would have same number of coins.

There are two solutions which give us the lowest sum of coins of E, F, G:

- 11,12,13,14,15,16,19
  Sum of E, F, G is $15 + 16 + 19$ and that is 50.
- 11,12,13,14,15,17,18
  Sum of E, F, G is $15 + 17 + 18$ and that is 50.

Minimal sum of three students is 50 so we proved that among 7 students there are always three students who have together at least 50 coins.