Topic 1: Homework problems

Example 1

First proof

It is possible to simplify problem to related problem: there is a possitive integer n which satisfies the condition $3^n \equiv 1 \ (mod\ 100\ 000)$.

There is infinity powers of 3 but only $100\ 000$ reminders modulo $100\ 000$. According to The Pigeon Hole Principle there exist numbers 3^n and 3^m, where $n > m$ (W.L.O.G.), with same reminders modulo $100\ 000$.

$$3^n - 3^m = 100\ 000k$$

$$3^m(3^{n-m} - 1) = 100\ 000k$$

When $100\ 000$ is decompose into prime numbers, it is $2^5 \cdot 5^5$. Powers of 3 can end only with digits $1, 3, 7, 9$. Number is dividend by 2 and 5 only when its unit digit is 0. 3^m is a power of 3 so it doesn´t have 0 as unit digit. That is why 3^m is not divisible by $100\ 000$.

When 3^m is not divisible, then $3^{n-m} - 1$ is divisible by $100\ 000$.

$$3^{n-m} - 1 = 100\ 000a$$

$$3^{n-m} - 1 \equiv 0 \ (mod\ 100\ 000)$$

This means 3^{n-m} ends with 00001. Q.E.D.

Second proof

It is possible to simplify problem to related problem: there is a possitive integer n which satisfies the condition $3^n \equiv 1 \ (mod\ 100\ 000)$.

There is infinity powers of 3 but only $100\ 000$ reminders modulo $100\ 000$. According to The Pigeon Hole Principle there exist numbers 3^n and 3^m, where $n > m$ (W.L.O.G.), with same reminders modulo $100\ 000$.

It is possible to write these numbers and their reminders modulo to equations.

$$3^n \equiv a \ (mod\ 100\ 000)$$

$$3^m \equiv b \ (mod\ 100\ 000)$$

Now we divide first equation by second equation.

$$\frac{3^n}{3^m} \equiv \frac{a}{b} \ (mod\ 100\ 000)$$

When 3^n and 3^m have same reminder modulo $100\ 000$, then $a = b$.

$$\frac{3^n}{3^m} \equiv \frac{a}{a} \ (mod\ 100\ 000)$$

$$\frac{3^n}{3^m} \equiv 1 \ (mod\ 100\ 000)$$

$$3^{n-m} \equiv 1 \ (mod\ 100\ 000)$$

That means 3^{n-m} end with 00001. Q.E.D.

Example 2

Take one pair of numbers (f_n, f_{n+1}) of the Fibonacci sequence. Modulo 100 will help to ignore digits beyond the decimal place.

$$(f_n, f_{n+1})\ mod\ 100$$

(It is necessary to take one pair, not just one number f_n, because repeating of two digits from one number doesn´t mean that the sequence is periodic.)

Among the first $10\ 001 = 100 \cdot 100 + 1$ pairs there are two which are identical. The recursion of the Fibonacci numbers now implies the desired periodicity and one full period must be between these two identical pairs.

According to The Pigeon Hole Principle that means the sequence of the last two digits of the Fibonacci sequence is periodic and the period length is less than $10\ 000$.

Example 3

Choose any two of nine lattice points in tree dimensional space. Name their coordinates $[x_1, y_{1,} z_1]$ and $[x_2, y_{2,} z_2]$.

The mid point of two point is given by $\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right]$. These coordinates are all integers if (x_1, x_2), (y_1, y_2) and (z_1, z_2) have the same parity – both are even or both are odd.

Since there are only eight parity pattern, namely (even, odd, odd), (even, even, odd), (even, odd, even), (even, even, even), (odd, even, even), (odd, odd, even), (odd, even, odd) and (odd, odd, odd) among nine lattice points you can find two with the same parity pattern.

According to The Pigeon Hole Principle it is always possible to choose two lattice points among nine lattice points in tree dimensional space so that mid point of the corresponding line segment is also a lattice point.

Example 4

Choose any of the 18 persons and call him/her P. Among the other 17 persons P knows less than six persons or P knows at least six persons.

1. When P knows less than six persons, then there exist at least twelve persons who P doesn´t know. Choose any six of them. Among these six persons exist at least one pair of persons who don´t know each other or everybody know each other.

   a. If there is at least one pair of persons who don´t know each other, then there are always tree persons who all don´t know each other (persons in pair don´t know each other and P doesn´t know persons in pair, so both of them don´t know P neither).
   b. If everybody of six persons know each other, then there are always four persons who all know each other.
2. When P knows at least six persons, then there exist at least tree persons who all don´t know each other or at least tree persons who all know each other.
   a. If there are tree persons who don´t know each other, then condition of the example is satisfied.
   b. If there are tree persons who all know each other, then there are always four persons who all know each other (tree persons know each other and all of them know P).

Among 18 persons there are always four who all know each other or tree who all don´t know each other. Q.E.D.