Topic 11: Polynomials

Example 1

According to Vieta formulas we have:

$$3 = -(a + b + c)$$

$$4 = ab + bc + ac$$

$$-11 = -abc$$

$$t = -(a + b)(b + c)(a + c)$$

Use equations above and get:

$$t = -(a + b)(b + c)(a + c) = -(ab + bc + ac)(a + b + c) + abc = -(4) \cdot (-3) + 11 = 23$$

We found out that t is equal to 23.

Example 2

Let a, b, c be roots of the polynomial. According to Vieta formulas we have:

$$a + b + c = -p$$

$$ab + bc + ac = q$$

$$abc = -r$$

Without loss of generality assume $a = b + c$. Thus:

$$2b + 2c = -p$$

$$b(b + c) + bc + c(b + c) = b^2 + 3bc + c^2 = (b + c)^2 + bc = q$$

$$bc(b + c) = -r$$

From the first equation we know $b + c = -\frac{p}{2}$. Thus:

$$\left(-\frac{p}{2}\right)^2 + bc = q$$

$$-bc\frac{p}{2} = -r$$

Now we can get this relation $-\frac{p}{2}\left(q - \frac{p^2}{4}\right) = -r$ from two equations above.

The relation between p, q, r is $p(4q - p^2) = 8r$.

Example 3

The polynomial has two double roots. Name them a, b.

From the Vieta formulas we know:

$$16 = 2a + 2b$$

$$94 = a^2 + 4ab + b^2$$

$$-p = 2a^2b + 2ab^2$$

$$q = a^2b^2$$

We need to find $p + q$. From equations above we have relation:

$$p + q = a^2b^2 - 2a^2b - 2ab^2 = a^2b^2 - ab(2a + 2b)$$

From the equation $94 = a^2 + 4ab + b^2$ we know:

$$2ab = 94 - (a + b)^2$$

$$ab = 15$$

We have relations $ab = 15$ and $16 = 2a + 2b$, so we can count:

$$p + q = a^2b^2 - ab(2a + 2b)$$

$$p + q = 15^2 - 15 \cdot 16 = -15$$

We found out $p + q$ is equal to -15.

Example 4

Define a new polynomial:

$$Q(x) = P(x) + 1$$

$$Q(x) = (x - a)(x - b)(x - c) \cdot R(x)$$

Thus:

$$P(x) = (x - a)(x - b)(x - c) \cdot R(x) - 1$$

$$(x - a)(x - b)(x - c) \cdot R(x) = 1$$

Left hand side of equation above is product of 4 integers, at least 3 distinct. Right hand side is a product at most 2 distinct integers $(-1, -1; 1, 1)$. That is a contradiction.

Thus we can see $P(x)$ has no integral roots in this case.