Problem 1

Let p be a prime number and n a positive integer. The decimal expansion of p^n has 20 digits. Prove that some digit occurs more than twice.

Assume that there is number p^n which satisfies conditions from task that has in its decimal expansion every digit exactly twice.

Find sum of digits of p^n. Thus:

$$2 \bullet (1 + 2 + \ldots + 9) = 90$$

Sum of digits is 90. Number 90 is divisible by 3. Thus number p^n is divisible by 3.

Number p^n is divisible by 3 only when $p = 3$. So it is in form 3^n.

There are only two numbers which have 20 digits in its decimal expansion that are in form 3^n. They are:

- $3^{40} = 12\ 157\ 665\ 459\ 056\ 928\ 801$
- $3^{41} = 36\ 472\ 996\ 377\ 170\ 786\ 403$

However, these two numbers do not have in its decimal expansion every digit exactly twice. Thus our assumption was not right.

Number p^n has in its decimal expansion some digit more than twice.

Problem 2

On a board are written the numbers $1, 2, 3, \dots, 2009$. A move consists of choosing some of the numbers on the board, erasing them and replacing them with one new number namely the remainder mod 11 of the sum of the numbers chosen. After carrying out these moves a certain number of times only two numbers are left on the board. One of them is 1000. What is the other number?

Let x be the wanted number.

Notice that summation and congruence are commutative. So order of these operations does not influence the result.

When we sum up all numbers on the board and find their reminder modulo 11, we will find out the same number as when we would sum up x and $1\,000$ and find the reminder modulo 11. Thus:

$$\frac{(1+2009) \bullet 2009}{2} = 2\,019\,045 \equiv 6\ (mod\ 11)$$

Now we have to find x from congruence bellow.

$$x + 1\,000 \equiv 6\ (mod\ 11)$$

$$x + 10 \equiv 6\ (mod\ 11)$$

We know x can be bigger than 11. Thus x is equal to 7.

The other number that is on the board is 7.

Problem 3

What is the 2009th decimal after the decimal point of $(\sqrt{50}+7)^{10000}$?

Note that $\left(\sqrt{50}+7\right)^{10\,000}+\left(\sqrt{50}-7\right)^{10\,000}$ is an integer because when we expand it, all odd powers of $\sqrt{50}$ cancel.

Number $\left(\sqrt{50}-7\right)^{10\,000}$ is very small. It is approximately $(0{,}071)^{10\,000}$. Thus $\left(\sqrt{50}-7\right)^{10\,000}<(0{,}072)^{10\,000}<(0{,}1)^{10\,000}$.

So, the decimal expansion of $\left(\sqrt{50}-7\right)^{10\,000}$ has at least $10\,000$ zeros after the decimal point. Therefore number $\left(\sqrt{50}+7\right)^{10\,000}$ has at least $10\,000$ nines after decimal point. Thus the 2009th decimal after decimal point is 9.

We found that the 2009th decimal of number $\left(\sqrt{50}+7\right)^{10\,000}$ after the decimal point is 9.

Problem 4

Find all quadruples of positive integers (m, n, p, q) such that $p^m q^n = (p+q)^2 + 1$.

Divide the equation by pq. So:

$$p^{m-1}q^{n-1} = \frac{p^2+q^2+1}{pq} + 2$$

Left hand side of the equation is an integer, so right hand side has to be an integer too. Thus $\frac{p^2+q^2+1}{pq}$ has to be an integer. We can find out that it is an integer just in case $\frac{p^2+q^2+1}{pq} = 3$. Thus:

$$p^{m-1}q^{n-1} = 3+2$$

$$p^{m-1}q^{n-1} = 5$$

There are two possible solutions of the equation. They are $p^{m-1} = 5$ and $q^{n-1} = 1$ or $p^{m-1} = 1$ and $q^{n-1} = 5$.

Now we need to solve these two cases:

1. $p^{m-1} = 5$ and $q^{n-1} = 1$

   When $p^{m-1} = 5$, then $p = 5$ and $m = 2$. Now we have two possibilities:

   - $$n = 1$$

     When we put it to the original equation, we get:

     $$25q = (5+q)^2 + 1$$

     $$q^2 - 15q + 26 = 0$$

     Solutions of this quadratic equation are 2 and 13.

     We have two solutions: $[m, n, p, q]$ is equal to $[2,1,5,2]$ or $[2,1,5,13]$.

   - $$n \neq 1$$

     When $n \neq 1$, then $q = 1$. However, when we put these solutions to the original equation, we can see they are not real solutions.

2. $p^{m-1} = 1$ and $q^{n-1} = 5$

   When $q^{n-1} = 5$, then $q = 5$ and $n = 2$. Now we have two possibilities:

   - $$m = 1$$

     When we put it to the original equation, we get:

     $$25p = (5+p)^2 + 1$$

$$p^2 - 15p + 26 = 0$$

Solutions of this quadratic equation are 2 and 13.

We have two solutions $[m, n, p, q]$ is equal to $[1,2, 2,5]$ or $[1,2, 13,5]$.

- $$m \neq 1$$

  When $m \neq 1$, then $p = 1$. However, when we put these solutions to the original equation, we can see they are not real solutions.

We found there are four quadruples of positive integers m, n, p, q of the equation. They are $[m, n, p, q] = [2,1, 5,2]$, $[m, n, p, q] = [2,1, 5,13]$., $[m, n, p, q] = [1,2, 2,5]$ and $[m, n, p, q] = [1,2, 13,5]$.

Problem 5

How many solutions in positive integers ≥ 3 are there to the equation $x_1 + x_2 + x_3 = 19$?

Define y_1, y_2, y_3 like that:

$$y_1 = x_1 - 3$$

$$y_2 = x_2 - 3$$

$$y_3 = x_3 - 3$$

Thus:

$$y_1 + y_2 + y_3 + 9 = 19$$

$$y_1 + y_2 + y_3 = 10$$

When y_1, y_2, y_3 are all non-negative, it is possible to use combinatorics formula:

$$\binom{10 + 3 - 1}{3 - 1} = \binom{12}{2} = \frac{12 \bullet 11}{2} = 66$$

We found there are 66 solutions of the equation $x_1 + x_2 + x_3 = 19$ such x_1, x_2, x_3 are positive integers bigger than 3.

Problem 6

Compute the probability that a randomly chosen divisor of 10^{99} is an integer multiple of 10^{88}.

Let p be probability. We can count probability when we divide number of multiples 10^{88} (that divide 10^{99}) by divisors of 10^{99}. Thus:

$$p = \frac{\text{number of multiples of } 10^{88}}{\text{number of divisors of } 10^{99}}$$

First count the number of multiples 10^{88} that divide 10^{99}. Let $x \in \mathbb{N}$ be the multiple of 10^{88}. Thus:

$$\frac{10^{99}}{x \bullet 10^{88}} = \frac{10^{11}}{x} = \frac{2^{11} \bullet 5^{11}}{x}$$

We simplified the problem to how many x there are that divide $2^{11} \bullet 5^{11}$. From combinatorics formula we know there are 144 such numbers x. Thus there are 144 multiples of 10^{88} that divide 10^{99}.

Now we have to count how many divisors of 10^{99} there are. Number 10^{99} is possible to rewrite to form $2^{99} \bullet 5^{99}$. From combinatorics formula we know there are $10\,000$ divisors of 10^{99}.

Now we use formula above. Thus:

$$p = \frac{\text{number of multiples of } 10^{88}}{\text{number of divisors of } 10^{99}} = \frac{144}{10\,000} = \frac{9}{625}$$

The probability that a randomly chosen divisor of 10^{99} is an integer multiple of 10^{88} is $\frac{9}{625}$. That is $0{,}0144$.

Problem 7

If p and q are primes and $x^2 - px + q = 0$ has distinct integer roots, find p and q.

Let x_1 and x_2 be roots of the equation. According to Vieta Formulas:

$$q = x_1 \bullet x_2$$

$$p = x_1 + x_2$$

We know q has to be a prime number. This condition is satisfied only when one of factors is equal to 1 and second factor is a prime number. Thus without loss of generality assume x_1 is equal to 1. So:

$$q = x_2$$

$$p = 1 + x_2$$

We can see that $p > 2$ because x_2 has to be prime number. Every prime number bigger than 2 is odd. Thus p is odd number. So x_2 has to be even. There is just one even prime number and it is 2. Thus $x_2 = 2$.

We founded $x_1 = 1$ and $x_2 = 2$. From equations $q = x_1 \bullet x_2$ and $p = x_1 + x_2$ we can count that:

$$q = 2$$

$$p = 3$$