Topic 9: Combinatorics II

Example 1

Every polygon that has three or more vertices on the circle is convex.

There are 10 points on the circle. Number of convex polygons of n vertices (polygon with n sides has n vertices) in circle with 10 potential vertices is $\binom{10}{n}$, where $n \in \langle 3,10\rangle$. Now we have to sum up numbers of polygons of 3 sides, 4 sides, … 10 sides. Thus:

$$\binom{10}{3}+\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}=968$$

Number of convex polygons of three or more sides is 968.

Example 2

Cube has 6 faces so there are 6 colors.

Color any face (name it A) of cube by any color. There are only 5 possibilities for color on the face opposite to face A.

Choose any of reminding faces (name the face B) of the cube and color it by any reminding color. There are only 3 possibilities how to color the face opposite to face B.

Choose any of reminding faces of the cube and color it by any reminding color. There are only 2 possibilities for color on that face.

There is one face left. It can be colored by only one reminding color.

There are $5 \cdot 3 \cdot 2 = 30$ colorings of the cube that are district.

Example 3

Part 1

For squares of side k there are $n-(k-1)$ squares in a row and $n-(k-1)$ squares in a column. Thus in chessboard of side n there are $(n-(k-1))^2$ squares of side k.

We have to use form above for every integer k from interval $\langle 1, n\rangle$ and then sum up results. After that we get:

$$n^2+(n-1)^2+(n-2)^2+\cdots+(n-(n-1))^2=n^2+\cdots+3^2+2^2+1^2=\frac{n(n+1)(2n+1)}{6}$$

On the chessboard of side n can be formed $\frac{n(n+1)(2n+1)}{6}$ squares of side k, where $k \in \langle 1, n\rangle$.

Part 2

From the practice problem number three we know that there are $\binom{n+1}{2}\cdot\binom{n+1}{2}$ rectangles on the chessboard of side n. In our case $n=8$. Thus there are $\binom{9}{2}\cdot\binom{9}{2}=1296$ rectangles.

We can easily count how many squares there are using formula $\frac{n(n+1)(2n+1)}{6}$. In our case $n = 8$. There are 204 squares on the chessboard of side 8.

The proportion of squares and rectangles is $204{:}\,1296$. Thus on the chessboard of side 8 there is approximately $15{,}74\%$ of rectangles that are squares.

Example 4

Use Euler´s polyhedron formula to count how many edges there are.

$$v - e + f = 2$$

$$26 - e + 36 = 2$$

$$e = 60$$

The convex polyhedron contains 60 edges.

The line segment connects two vertices so in total there are $\binom{26}{2}$ line segments. To get number of space diagonals we need to subtract number of diagonals of triangles and quadrilaterals (Triangle does not have diagonals. Quadrilateral contains two diagonals.), number of edges from total number of the line segment.

$$\binom{26}{2} - 24 \cdot 0 - 12 \cdot 2 - 60 = 241$$

In total there are 241 space diagonals in the polyhedron.