Topic 4: The Induction Principle

Example 1

First number of the form $a_1 = 12\ 008$ is divisible by 19.

Let´s see relation between numbers of the form. We can see that every number is multiplied by 10 and add up 228 to get following number.

$$a_{n+1} = 10 \cdot a_n + 228$$

Number a_n is divisible by 19. When we multiplied a_n by 10, it will be still divisible by 19. Number 228 is also divisible by 19. As we already know $a_{n+1} = 10 \cdot a_n + 228$. When right side of equation is divisible by 19, then a_{n+1} is also divisible by 19. It is also possible to write it as below.

$$19/\ a_n$$

$$19/\ 228$$

$$19/\ a_n \wedge 19/\ 228 \Rightarrow 19/\ (10 \cdot a_n + 228) \Rightarrow 19/a_{n+1}$$

We proved that all numbers of the form are divisible by 19.

Example 2

First of all we have to prove that it is valid for $n = 3$.

$$a_3 = \frac{a_2^2 + 2}{a_1} = \frac{1^2 + 2}{1} = 3$$

We showed that a_3 is an integer, so it is valid.

Now we have to prove it for $n + 1$.

We have definition of sequence for n. Write definition of sequence for $n + 1$.

$$a_n = \frac{a_{n-1}^2 + 2}{a_{n-2}}$$

$$a_{n+1} = \frac{a_n^2 + 2}{a_{n-1}}$$

It is possible to add first sequence to second sequence as below.

$$a_n \cdot a_{n-2} - a_{n-1}^2 - 2 = a_{n+1} \cdot a_{n-1} - a_n^2 - 2$$

$$a_{n-1}(a_{n+1} + a_{n-1}) = a_n(a_n + a_{n-2})$$

$$\frac{a_{n+1} + a_{n-1}}{a_n} = \frac{a_n + a_{n-2}}{a_{n-1}}$$

After a few adjustments of the sequence, we can see we got an invariant (above). It does not change for all triplets of consecutive members.

This invariant does not change so it is constant. We can count this constant c with integers a_1, a_2, a_3 (for $n = 2$) – or generally with triplet of consecutive members.

$$\frac{a_{n+1} + a_{n-1}}{a_n} = \frac{3 + 1}{1} = 4$$

Our invariant is that if you take any triplet of consecutive members and put them to the formula below, they will always give us constant c.

$$a_{n+1} = 4 \cdot a_n - a_{n-1}$$

From this equation we can see that if a_n, a_{n-1} are integers, then a_{n+1} is integer too.

We have to show that for a_1, a_2 (for $n = 2$).

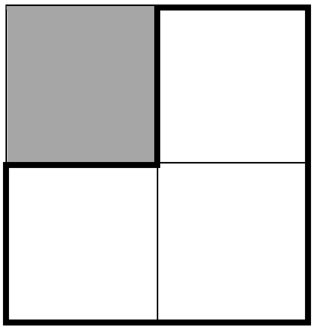
$$a_3 = 4 \cdot a_2 - a_1$$

$$a_3 = 4 \cdot 1 - 1 = 3$$

We can see that a_3 is an integer so every other term of sequence will be an integer. So it is also valid for $n \geq 3$.

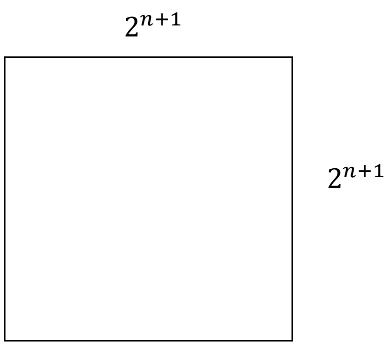
Example 3

First of all we need to prove it is valid for $n = 1$. From the picture below we can see that it is possible to tile quadratic board with sides 2 when we remove one unit square.

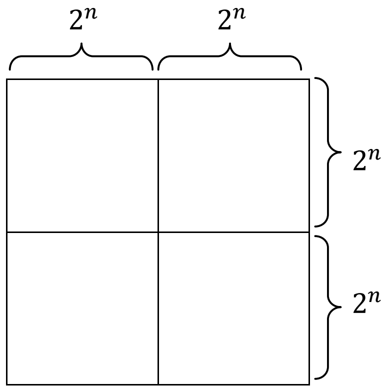


When we already know, it is valid for $n = 1$, then we have to prove it for $n + 1$.

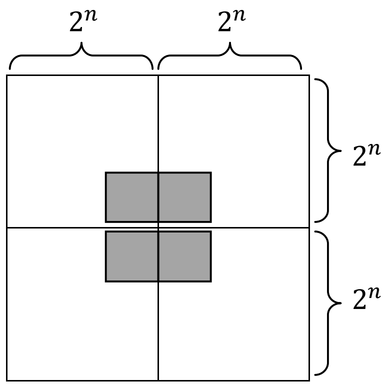
We have quadratic board with sides 2^{n+1}.



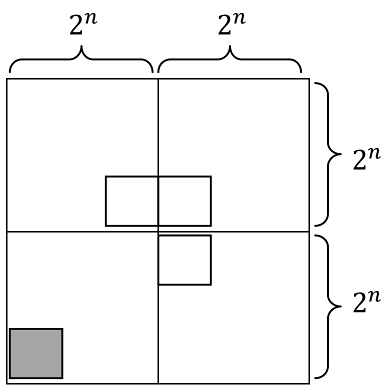
Let´s divide every side of quadratic board by 2. Half of side is $\frac{2^{n+1}}{2} = 2^n$.



We know it is possible to tile quadratic board with sides 2^n with L-shaped tiles when we remove one unit square (grey square) from one corner. It is necessary to put all removed square in the middle because it will help in next step.



But actually the grey square in the middle with sides 2 can be tiled with L-shaped tiles when we remove one unit square. That is why we do not need to removed all 4 unit squares but only one of them.



We proved it is valid for $n+1$ so we proved it is valid for every positive n.

Example 4

For $n = 1$ there is number $a_1 = 2$. This number is divisible by $2^1 = 2$, it has 1 digit and its decimal representation consists of only 1´s or 2´s.

Now we have to prove it is also valid for $n + 1$.

To get new number a_{n+1} we need to add up number $1 \cdot 10^n$ or $2 \cdot 10^n$ to a_n because number a_{n+1} has to be divisible by 2^{n+1} and has to have $n + 1$ digits.

Every a_n can be written like that $a_n = 2^n \cdot b$ because a_n is divisible by 2^n. There are 2 possibilities – b is even or b is odd.

***b* is even**

When b is even then it is possible to write it like that $b = 2x$.

$$a_n = 2^n \cdot b$$

$$a_n = 2^n \cdot 2x$$

To get a_{n+1} we need to add up number $1 \cdot 10^n$ or $2 \cdot 10^n$ to a_n but we do not know with which one it will work. Let´s have constant c which can be only 1 or 2 so we will add up number $c \cdot 10^n$.

$$a_{n+1} = c \cdot 10^n + a_n$$

$$a_{n+1} = c \cdot 10^n + 2^n \cdot 2x$$

$$a_{n+1} = c \cdot 2^n \cdot 5^n + 2^{n+1} \cdot x$$

As we know a_{n+1} has to be divisible by 2^{n+1}. Member $2^{n+1} \cdot x$ is divisible by 2^{n+1}. Member $c \cdot 2^n \cdot 5^n$ has to be divisible 2^{n+1} too so $c = 2$.

$$a_{n+1} = 2^{n+1} \cdot 5^n + 2^{n+1} \cdot x$$

$$a_{n+1} = 2^{n+1}(5^n + x)$$

We proved that when b is even and we add up $2 \cdot 10^n$ to a_n (by this step we just add 2 in front of number a_n), then a_{n+1} is always divisible by 2^{n+1}. As we can see a_{n+1} also has $n + 1$ digits because of adding up $2 \cdot 10^n$.

***b* is odd**

When b is odd, then we can write it like that $b = 2x + 1$.

$$a_n = 2^n \cdot 2x + 2^n$$

Let´s again have constant c which can be equal only to 1 or 2 as in previous case.

$$a_{n+1} = c \cdot 10^n + a_n$$

$$a_{n+1} = c \cdot 2^n \cdot 5^n + 2^{n+1} \cdot x + 2^n$$

$$a_{n+1} = 2^n(5^n \cdot c + 1) + 2^{n+1} \cdot x$$

Member $2^{n+1} \cdot x$ is divisible by 2^{n+1}. Member $2^n(5^n \cdot c + 1)$ has to be divisible by 2^{n+1} and this can only be when $c = 1$ because then $5^n \cdot c + 1$ is even that means divisible by 2. When $5^n \cdot c + 1$ is divisible by 2, then $2^n(5^n \cdot c + 1)$ is divisible by 2^{n+1}.

We proved that when b is odd and we add up number $1 \cdot 10^n$ to a_n, then a_{n+1} is always divisible by 2^{n+1}. As we can see a_{n+1} also has $n + 1$ digits because of adding up $1 \cdot 10^n$.

<u>Conclusion</u>

We proved that for every n there is a number what is divisible by 2^n, has n digits and his decimal presentation contains only 1´s or 2´s.