Topic 2: Homework problems

Example 1

There are $4n-1$ integers. In one move we replace any two integers by their diference. So every step reduce two integers to one integer. That means after $4n-2$ moves disappeare $4n-2$ integers from $4n-1$ integers. Let x be number of integers which will remain after $4n-2$ moves.

$$x = 4n - 1 - (4n - 2)$$

$$x = 1$$

We can see that after $4n-2$ moves will remain just one integer. Now we have to prove that last remaining integer is even.

Suppose S is the sum of all remaining numbers. Initially it is $S = 1 + 2 + \cdots + 4n - 1 = 2n(4n-1)$ so it is an even number.

Choose integer a and b. Let $a > b$ (without loss of generality).

$$[1,2,3, \dots, b, \dots, a, \dots]$$

Replace a and b by their possitive integer $a-b$.

$$[1,2,3, \dots, a-b, \dots]$$

It is also possible to write it as below. We can leave there original set and subtract $2b$. Sum of integers will not change because $a+b-2b = a-b$.

$$[1,2,3, \dots, b, \dots, a, \dots] - 2b$$

It is possible to continue in this way until we get just one integer.

$2b$ is always an even integer. That means parity of S does not change (even – even = even).

Parity of S is an invariant because during whole reduction process S remain an even number. If S remains an even number then sum S of last integer will be also even. That means an even integer will be left after $4n-2$ moves.

Example 2

First invariant:

We have a definition of sequence for n. Write definition of sequence for $n+1$.

$$a_{n+1}a_{n-1} = a_n^2 + 2007$$

$$a_{n+2}a_n = a_{n+1}^2 + 2007$$

It is possible to add first sequence to second sequence as below.

$$a_{n+1}a_{n-1} + a_{n+1}^2 + 2007 = a_n^2 + 2007 + a_{n+2}a_n$$

$$a_{n+1}(a_{n-1}+a_{n+1}) = a_n(a_n+a_{n+2})$$

$$\frac{a_{n-1}+a_{n+1}}{a_n} = \frac{a_n+a_{n+2}}{a_{n+1}}$$

After a few adjustments of the sequence, we got our invariant (above). It does not change for all triplets of consecutive members.

This invariant does not change so it is constant. We can count this constant c with integers a_0, a_1, a_2 (for $n = 1$) – or generally with triplet of consecutive members.

$$a_2 = \frac{a_1^2+2007}{a_0} = \frac{1^2+2007}{1} = 2008$$

$$c = \frac{a_0+a_2}{a_1} = \frac{1+2008}{1} = 2009$$

Our invariant is that if you take any triplet of consecutive members and put them to the formula below, they will always give us constant c.

$$\frac{a_{n-1}+a_{n+1}}{a_n} = 2009$$

$$\frac{a_n+a_{n+2}}{a_{n+1}} = 2009$$

<u>Second invariant (just for interest):</u>

If we get couple of integer from sequence, we can see that their reminder modulo 2007 is 1. So other invariant is that every a_n gives reminder 1 modulo 2007.

$$a_0 = 1 \equiv 1 \ (mod\ 2007)$$

$$a_1 = 1 \equiv 1 \ (mod\ 2007)$$

$$a_2 = 2008 \equiv 1 \ (mod\ 2007)$$

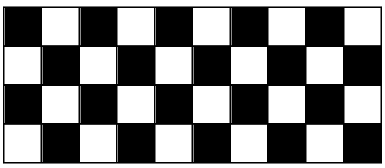
$$a_3 = 2008^2 + 2007 \equiv 1 \ (mod\ 2007)$$

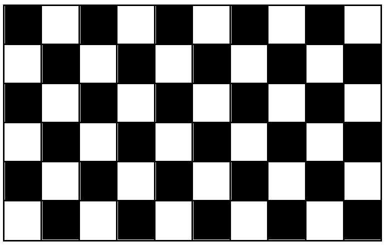
$$a_4 = \frac{(2008^2+2007)^2+2007}{2008} \equiv 1 \ (mod\ 2007)$$

<u>Note:</u> We probably can prove it with mathematical induction but I am afraid I can not do that because I am not able to put sequence to the general formula.

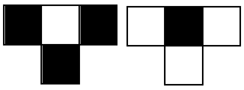
Example 3

Color 10x10 chessboard as below.





Every T-tetrominoe covers one white square and tree black squares or tree white squares and one black square. There is not any other possibility how to place T-tetrominoe on the chessboard.



There are 50 white squares and 50 black squares on the chessboard. 25 T-tetrominoes have to be place on the chessboard.

First possition of T-tetraminoe covers tree black squares and second possition covers one black square. First possition of T-tetraminoe covers one white square and second possition covers tree white squares.

It is possible to put it to system of equations where x is for number of T-tetraminoes in first possition and y is for number of T-tetraminoes in second possition.

$$50 = 3x + y$$

$$50 = x + 3y$$

Roots of equations are not integers $\left(x = \frac{25}{2}, y = \frac{25}{2}\right)$. That means we would have to place there $\frac{25}{2}$ T-tetraminoes in first possition and $\frac{25}{2}$ T-tetreminoes in second possition. However, it is not possible to divide T-tetraminoes. Therefore it is not possible to cover 10x10 chessboard with 25 T-tetraminoes.

Example 4

Sum S of all 10 digits from set $\{0,1, \dots 8,9\}$ is 45. 2^{29} has nine digits all different. As we can see below 45 and 0 are congruent modulo 9.

$$45 \equiv 0 \ (mod\ 9)$$

It is possible to use this fact to get to know what digit is missing. That means we have to count modulo 2^{29}.

2^{29} can also be written as below.

$$2^{29} = (2^6)^4 \cdot 2^5$$

When 2^{29} is in this form, we can use formulas using modulo. Suppose $a \equiv u \ (mod\ n)$ and $b \equiv v \ (mod\ n)$ and conditions $n \in \mathbb{N}; a, b, u, v \in \mathbb{Z}$, then is valid $a \cdot b \equiv u.v \ (mod\ n)$ and $a^k \equiv a^k \ (mod\ n)$. According to these formulas we can do operations below and count 2^{29}.

$$2^6 \equiv 1 \ (mod\ 9)$$

$$1^4 \equiv 1 \ (mod\ 9)$$

$$2^5 \equiv 5 \ (mod\ 9)$$

$$(2^6)^4 \cdot 2^5 \equiv 1 \cdot 5 \ (mod\ 9) \equiv 5 \ (mod\ 9)$$

We found that 2^{29} and 5 are congruent modulo 9.

Number si divisible by 9 when sum of its digits is divisible by 9. It is possible to generalize this rule: Number and sum of its digits has same reminder modulo 9.

<u>Proof:</u> Let have number abc.

$$abc \equiv x \ (mod\ 9)$$

We can write it in form as below.

$$a \cdot 100 + b \cdot 10 + c \cdot 1 \equiv x \ (mod\ 9)$$

10^n has always reminder 1 modulo 9. Thus it is possible to simplify it to form below.

$$a + b + c \equiv x \ (mod\ 9)$$

As we can see number and sum of its digits has same reminder (x) modulo 9.

Now we are looking for number from set $\{0,1,2,\dots,9\}$ which gives us reminder 5 modulo 9.

$$45 - z \equiv 5 \ (mod\ 9)$$

$$z = 4$$

Solutions is 4. We can get other numbers for sure, however they are not in our set so 4 must be a number what we are looking for.

That also mean that missing number from 2^{29} is 4.