Topic 8: Combinatorics I.

Example 1

Number of all possible combinations is $\binom{20}{2} \cdot \binom{18}{2}$ – Terry picks two candies and then Mary picks two candies from reminding 18 candies.

Same color combination is when both of them have just red candies or both of them have just blue candies or when Terry has one blue and one red candy and Mary has the same. It is possible to write it as $2 \cdot \binom{10}{2} \cdot \binom{8}{2} + \binom{10}{1} \cdot \binom{9}{1} \cdot \binom{10}{1} \cdot \binom{9}{1}$.

To get the probability we need to divide number of same color combinations by number of all possible combinations.

$$\frac{2 \cdot \binom{10}{2} \cdot \binom{8}{2} + \binom{10}{1} \cdot \binom{9}{1} \cdot \binom{10}{1} \cdot \binom{9}{1}}{\binom{20}{2} \cdot \binom{18}{2}} = \frac{118}{323}$$

Probability of that they get same color combination is $\frac{118}{323}$.

Example 2

It is possible to write an odd number in form $2k - 1$. Thus:

$$2k - 1 + 2l - 1 + 2m - 1 = 2009$$

$$2(k + l + m) = 2012$$

$$k + l + m = 1006$$

Use Theorem 1 from lecture.

$$\binom{1006 - 1}{3 - 1} = \binom{1005}{2} = \frac{1005 \cdot 1004}{2} = 504\,510$$

We found that there are $504\,510$ ways how to write 2009 as a sum of three positive odd integers.

Example 3

Part 1

When positive integer a has exactly two proper divisors then it has to be in form below. One of divisors in 1 and second one is x (x^2 is not proper divisor because $a = x^2$).

$$a = 1 \cdot x^2$$

Number x has to be a prime number (otherwise there would be more than two proper divisors). We know that divisors have to be less than 50. Number one is definitely less than 50. Number x can be equal to every prime number that is less than 50. There are 15 numbers which satisfies this condition.

When there are 15 numbers x, then there are 15 numbers a (because of $a = x^2$) which have exactly two proper divisors less than 50.

Part 2

When positive integer a has exactly three proper divisors, then it has to be in one of forms below.

$$a = 1 \cdot x \cdot y$$

$$a = 1 \cdot x^3$$

Form $a = 1 \cdot x \cdot y$

In this case there are conditions $x \neq y$ and x, y have to be prime numbers (otherwise there would be more than three proper divisors).

From the previous case we know there are 15 prime numbers less than 50. Use combinatorics to count how many ways it is possible to choose 2 numbers from 15 numbers.

$$\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$$

There are 105 possibilities for number a.

Form $a = 1 \cdot x^3$

In this case divisors are numbers $1, x, x^2$. We know divisor has to be less than 50. This conditions satisfies only numbers 2,3,5,7. So there are 4 possibilities for number a.

Conclusion

We need to sum up results:

$$105 + 4 = 109$$

There are 109 positive integers which have exactly three proper divisors less than 50.

Example 4

Choose child A.

Child A sits on his seat. When one child does not move, then actually rest of children $(n-1)$ sit in the "line". We can take result from practice problem number four. It say for n children in a line there are $a_n = f_{n+1} = a_{n-1} + a_{n-2}$ arranges (f_n is number on n^{th} place in Fibonacci sequence). In this case we have $n-1$ children sit in line so we have $a_{n-1} = a_{n-2} + a_{n-3} = f_n$ arranges.

Child A moves to next seat of child B. Child B has to move to seat of child A. Thus there is a "line" with $n-2$ children. Number of arranges is $a_{n-2} = a_{n-3} + a_{n-4} = f_{n-1}$. Child A can move to right or left side thus we have to multiply it by 2. We get $2 \cdot a_{n-2} = 2 \cdot a_{n-3} + 2 \cdot a_{n-4} = 2 \cdot f_{n-1}$.

There is also a possibility that child A moves to next seat of child B, child B moves to seat of child C… child X moves to seat of child A. Child A can move to right side or left side thus there are two possibilities.

Total number of arranges is $b_n = f_n + 2 \cdot f_{n-1} + 2$.