Topic 5: Systematic Checking

Example 1

Fraction is in lowest term when numerator and denominator are relatively prime.

Write 30 as below.

$$30 = 2 \cdot 3 \cdot 5$$

We need to find all numbers from interval $\langle 1,300 \rangle$ what are not divisible by 2,3 or 5.

In interval $\langle 1,30 \rangle$ there are numbers $1,7,11,13,17,19,23,29$. Now we need to find all numbers which satisfies congruence below.

$$a \equiv 1,7,11,13,17,19,23 \text{ or } 29 \ (mod\ 30)$$

Sum of numbers $1,7,11,13,17,19,23,29$ is 120.

To get numbers from the interval $\langle 30,60 \rangle$ we need to add up 30 to every number from set $\{1,7,11,13,17,19,23,29\}$. Set has 8 members so sum of numbers from the interval $\langle 30,60 \rangle$ is $120 + 240 = 360$.

Sum of numbers from interval $\langle 60,90 \rangle$ is $120 + 240 + 240 = 600$.

Now we can see that sum of numbers a from interval $\langle 1,300 \rangle$ is $120 \cdot 9 + 240 \cdot (1 + 2 + \cdots + 9) = 120 \cdot 10 + 240 \cdot 45 = 12\ 000$.

We found that sum of numbers a from interval $\langle 1,300 \rangle$ is $12\ 000$ and now we have to divide it by 30. So it is $\frac{12\ 000}{30} = 400$.

Sum of all fraction in lowest form is 400.

Example 2

First of all let´s look at digit on the unit place of n.

$$a^3 \equiv 8 \ (mod\ 10)$$

We can see that the only possibility is $a = 2$ (a has to have just one digit because it is digit in number n).

Let´s look at decimal place of number n.

$$(10b + 2)^3 \equiv 88 \ (mod\ 100)$$

$$1\ 000b^3 + 600b^2 + 120b + 8 \equiv 88 \ (mod\ 100)$$

We can see that members $1\ 000b^3$ and $600b^2$ are congruent 0 modulo 100 so we can cancel them out. Now we need to get $120b$ to other form to make it easier to count so $120b = 100b + 20b$. Member 100b is congruent 0 modulo 100 so we can cancel it out. Now we have to count:

$$20b + 8 \equiv 88 \ (mod\ 100)$$

We get two solutions for b. First of them is $b_1 = 4$ and second one $b_2 = 9$. It is necessary to count digit on hundreds place for both solutions.

1. $b_1 = 4$
$$(100c + 40 + 2)^3 \equiv 888 \ (mod\ 1\,000)$$
$$1\,000\,000c^3 + 3 \cdot 42 \cdot 10\,000c^2 + 42^2 \cdot 300c + 42^3 = 888 \ (mod\ 1\,000)$$
Let´s cancel out members $1\,000\,000c^3$ and $3 \cdot 42 \cdot 10\,000c^2$ because they are congruent 0 modulo $1\,000$.
$$42^2 \cdot 300c + 42^3 = 888 \ (mod\ 1\,000)$$
$$200c + 88 = 888 \ (mod\ 1\,000)$$
There are two solutions for c. It is $c_1 = 4$ and $c_2 = 9$.
2. $b_2 = 9$
$$(100c + 90 + 2)^3 \equiv 888 \ (mod\ 1\,000)$$
After we do exactly the same as in previous case we get form below.
$$200c + 688 \equiv 888 \ (mod\ 1\,000)$$
We get again two solutions for c. It is $c_3 = 1$ and $c_4 = 6$.

We get four numbers: $[c_1 b_1 a] = [442]$, $[c_2 b_1 a] = [942]$, $[c_3 b_2 a] = [192]$ and $[c_4 b_2 a] = [692]$. We can see the smallest number of these numbers is 192.

The smallest positive number ending in 888 is 192.

Example 3

Let´s assume that number n is a square. That means its reminder modulo 4 has to be 1 or 0. Look at last two digits of positive integer n (other digits are not important when we want to know just reminder modulo 4).

There are three possibilities of two last digits of number n. It can be in form $55, 5a, a5$ where a is digit different from 5.

Form	$0 \ (mod\ 4)$	$1 \ (mod\ 4)$
55	no	no
$5a$	$a = 2{,}6$	$a = 3{,}7$
$a5$	no	$a = 0{,}4{,}8$

Modulo 10

Let´s look at squares modulo 10.

x	0	1	2	3	4	5	6	7	8	9
$x^2 \ (mod\ 10)$	0	1	4	9	6	5	6	9	4	1

We found that number n can have last two digits equal to $52, 56, 53, 57, 05, 45, 85$.

However we can see no square has reminder $2{,}3{,}7$. So last two digits of number n cannot be equal to $52{,}53{,}57$.

Modulo 3

Now there are four possibilities of last digits of number n. It is 56,05,45,85.

Let´s look at reminder modulo 3. Number is divisible by 3 when its sum of digits is divisible by 3. Squares are always 0 or 1 modulo 3.

Let´s again put it to the table.

Last two digits of n	56	05	45	85
Sum of digits of n	$999 \cdot 5 + 6$	$999 \cdot 5 + 0$	$999 \cdot 5 + 4$	$999 \cdot 5 + 8$
$n \ (mod\ 3)$	0	0	1	2

After this step there are only three possibilities for last two digits: 56,05,45.

When number n is divisible by 3 and we assume it is a square then it has to be divisible by 9 too (otherwise it would not be a square – it has to be possible to write square as $(x \cdot 3)^2 = 9x^2$). Let´s again make a table.

Last two digits of n	56	05	45
Sum of digits of n	$999 \cdot 5 + 6$	$999 \cdot 5 + 0$	$999 \cdot 5 + 4$
$n \ (mod\ 9)$	6	0	4

We can see there is just one possibility for last two digits of n and it is 05. If number n ending with 05 is a square, then from the table of squares above we can see it is some square which contains 5. When it contains 5, then it has to be divisible by 25 because of $(y \cdot 5)^2 = 25y^2$.

Number is divisible by 25 when its last two digits are divisible by 25. However we can see 05 is not divisible by 25 so number n ending with 05 is not a square.

Conclusion

We discussed all possibilities and by contradiction we found that number n what in its decimal representation contains 999 fives and one different number cannot be a square.

Example 4

Let´s discuss possibilities:

$n = 1$

B always wins because every number is divisible by 1.

n is even number

Number is divisible by 2 when its last digit is divisible by 2. When n is even, then player *A* always wins because he just need to put some odd digit on the unit place.

$n = 3, 9$

Number is divisible by 3 or 9 when its sum of digits is also divisible by 3 or 9. *B* wins because he makes last move – so he always can choose digit which will make number z divisible by 3 or 9.

$n = 5, 15$

Number is divisible by 5 or 15 when its last digit is 0 or 5. A always wins because he makes first move - he put some digit on unit place of z what is different from 0 or 5.

Other numbers

There are three numbers left: $7{,}11{,}13$.

Let´s have number $abcabc$ when a, b, c represent digits of number z. We can see that $abcabc = 1001 \cdot abc = 7 \cdot 11 \cdot 13 \cdot abc$.

So number $abcabc$ is divisible by $7{,}11{,}13$.

When A starts for example with number a then B can always write same number a on position in which belong according to $abcabc$. Player B can do this in every case because he has last turn in the game.

Table

For better orientation there is a table of winners according to number n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
winner	B	A	B	A	A	A	B	A	B	A	B	A	B	A	A