Topic 7: Number Theory II

Example 1

Use Binomial Theorem.

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = x^{2001} + \left(-x^{2001} + \frac{2001}{2}x^{2000} - 500\,250x^{1999} \dots\right)$$

We get:

$$\frac{2001}{2}x^{2000} - 500\,250x^{1999} \dots$$

It is possible to use Vieta formulas just in case coefficient in front of x^{1999} is equal to 1. Thus we have to multiply it by $\frac{2}{2001}$.

$$x^{2000} - 500x^{1999} \dots$$

According to Vietas formulas we get:

$$500 = x_1 + x_2 + \cdots + x_{2000}$$

We found that sum of the roots is 500.

Example 2

Rewrite equation as below.

$$2^{333x-2} - 2^{222x+1} + 2^{111x+2} - 1 = 0$$

Have substitution $a = 2^{111x}$.

$$\frac{1}{4}a^3 - 2a^2 + 4a - 1 = 0$$

Multiply equation by 4 because it is necessary for applying Vieta formulas.

$$a^3 - 8a^2 + 16a - 4 = 0$$

There are three roots a_1, a_2, a_3. Vieta formulas tell us:

$$a_1 \cdot a_2 \cdot a_3 = 4$$

We can rewrite it as below. Roots of original equation are x_1, x_2, x_3.

$$2^{111x_1} \cdot 2^{111x_2} \cdot 2^{111x_3} = 2^2$$

Exponential function is injective so we can compare only exponents because of the same base.

$$111x_1 + 111x_2 + 111x_3 = 2$$

$$x_1 + x_2 + x_3 = \frac{2}{111}$$

We found that sum of three real roots. It is equal to $\frac{2}{111}$.

Example 3

Use Sophie Germain´s identity to factorize $a^4 + 4b^4$.

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

It is multiple of 5 when at least one bracket is multiple of 5.

There are two cases - a is multiple of 5 or a is not multiple of 5.

1. Assume that a is multiple of 5 and b is not. Have substitution $a = 5k$.
$$(25k^2 + 2b^2 + 10ak)(25k^2 + 2b^2 - 10ak)$$
In first bracket member $2b^2$ is not divisible by 5 and $25k^2$, $10ak$ are divisible 5. Thus their sum is not divisible by 5. Second bracket is not divisible by 5 because of the same reason as in previous case.
2. Assume that b is multiple of 5 and a is not. Have substitution $b = 5k$.
$$(a^2 + 50k^2 + 10ak)(a^2 + 50k^2 - 10ak)$$
In first bracket member a^2 is not divisible by 5 and $50k^2$, $10ak$ are divisible 5. Thus their sum is not divisible by 5. Second bracket is not divisible by 5 because of the same reason as in previous case.

We showed that $a^4 + 4b^4$ is not multiple of 5 when exactly one positive integer a or b is divisible by 5.

Example 4

In case $a = b$ we have $q = \frac{2a^2}{a^2-1}$. Number q is not an integer in this case. Thus we can assume $a < b$.

When we multiply $q = \frac{a^2+b^2}{ab-1}$ by $ab - 1$, we get $b^2 - qab + (a^2 + q) = 0$. By Vieta formula there is b' which satisfy $b + b' = qa$ and $b \cdot b' = a^2 + q$. Consider the transformation $(a, b) \to (b', a)$.

- $b' \epsilon \mathbb{N}$: From equation $b' = qa - b$ we see b' is an integer because a, q, b are integers too. From equation $b' = \frac{a^2+q}{b}$ we see b' is a positive number because a, q, b are positive. Thus b' is positive integer.
- Now we have to prove this $b' < a$. Assume it is true.
$$qa - b < a$$
$$q < \frac{a + b}{a}$$
$$\frac{a^2 + b^2}{ab - 1} < \frac{a + b}{a}$$
$$a^3 < a^2b - a - b$$
$$a^3 + a < a^2b - b$$
$$a(a^2 + 1) < b(a^2 - 1)$$

Note: I think I made a mistake somewhere – brackets should be same. Then it would be possible to divide it by bracket and we get $a < b$. So $b' < a$ would be right. It is just an idea – I do not know where is mistake.

Thus 1^{st} $coordinate < 2^{nd} coordinate.$

- From $b' < a < b$ we can see that both coordinates decreased.

We know a is positive integer so $a \geq 1$.

For $a = 1$ we have $q = \frac{1+b^2}{b-1}$. We can see q is positive integer when $b = 2$ or $b = 3$ and it is also valid $q = 5$.

So there is a starting pair $(a + b)$ so that q is an integer then after finitely many transformations we get another pair with the second coordinate equal to 1, then the first coordinate is equal to 2 or 3 and in this case $q = 5$.

Example 5

When $(n^2 - m^2 + 1)/(n^2 + 1)$ then have to be valid also $(n^2 - m^2 + 1)/m^2$.

Use hint $a = \frac{n+m}{2}$ and $b = \frac{n-m}{2}$. We can see that $0 < b < a$ because m, n are positive integers and $n > m$.

Have $q = \frac{m^2}{n^2-m^2+1}$. We can see $a - b = \frac{n+m-n+m}{2} = m$ and $4ab + 1 = n^2 - m^2 + 1$. Thus it is possible to rewrite it to form $q = \frac{(a-b)^2}{4ab+1}$.

After multiplying it by $4ab + 1$ we get quadratic equation $a^2 - 2b(1 - 2q)a + (b^2 - q) = 0$. By Vieta formulas we have $a + a' = 2b(1 - 2q)$ and $a \cdot a' = b^2 - q$. Consider transformation $(a, b) \rightarrow (b, a')$.

- $a' \in \mathbb{N}$: From this equation we can see it is a positive number $a' = \frac{b^2-q}{a}$. From this equation we can see a' is an integer $a' = 2b(1 - 2q) - a$. So a' is a positive integer.
- $1^{st} coord. > 2^{nd} coord.$: We can see $a' = \frac{b^2-q}{a} < \frac{b^2}{a} < \frac{b^2}{b} = b$ so $a' < b$.
- Both coordinates have decreased

We know $a' \geq 0$.

For $a' = 0$ we get $q = \frac{(a'-b)^2}{4a'b+1} = \frac{b^2}{1} = b^2$. We can see q is a square.

So we started with pair (a, b) so that q is an integer then after finitely many transformations we get another pair with second coordinate equal to 0 and q is a square of first coordinate.