Topic 10: Sequences

Example 1

First of all count more members of the sequence:

$$a_2 = 2, a_3 = 3, a_4 = 7, a_5 = 22, a_6 = 155$$

Use modulo 4 to count reminders of founded members of the sequence:

$$a_2 = 2\ (mod\ 4), a_3 = 3\ (mod\ 4), a_4 = 3\ (mod\ 4), a_5 = 2\ (mod\ 4), a_6 = 3\ (mod\ 4)$$

Assume that the sequence is periodic in modulo 4 and the period is 2,3,3 (except first two members of sequence).

Proof

We need to prove that:

$$a_{3n} = 3\ (mod\ 4)$$

$$a_{3n+1} = 3\ (mod\ 4)$$

$$a_{3n+2} = 2\ (mod\ 4)$$

Let´s prove it by mathematic induction so $n = k$.

For $k = 1$ it is valid: $a_3 = 3\ (mod\ 4), a_4 = 3\ (mod\ 4), a_5 = 2\ (mod\ 4)$.

Now we have to prove it for $k + 1$. Thus:

- $a_{3(k+1)} = a_{3k+3} = a_{3k+2} \cdot a_{3k+1} + 1 = 2 \cdot 3 + 1 \equiv 3\ (mod\ 4)$
- $a_{3(k+1)+1} = a_{3k+4} = a_{3k+3} \cdot a_{3k+2} + 1 = (a_{3k+2} \cdot a_{3k+1} + 1)a_{3k+2} + 1 = (2 \cdot 3 + 1) \cdot 2 + 1 \equiv 3\ (mod\ 4)$
- $a_{3(k+1)+2} = a_{3k+5} = a_{3k+4} \cdot a_{3k+3} + 1 = \big((a_{3k+2} \cdot a_{3k+1} + 1)a_{3k+2} + 1\big) \cdot (a_{3k+2} \cdot a_{3k+1} + 1) + 1 = \big((2 \cdot 3 + 1) \cdot 2 + 1\big) \cdot (2 \cdot 3 + 1) + 1 \equiv 2\ (mod\ 4)$

We proved that $a_{3n} = 3\ (mod\ 4)$, $a_{3n+1} = 3\ (mod\ 4)$, $a_{3n+2} = 2\ (mod\ 4)$.

Conclusion

Member a_{2008} is in form a_{3n+1}. Thus $a_{2008} \equiv 3\ (mod\ 4)$. We can see that its reminder after dividing by 4 is 3 so it is not divisible by 4.

Example 2

We have a definition of sequence for n. Write definition of sequence for $n - 1$.

$$a_n a_{n+2} = a_{n+1}^2 + 2$$

$$a_{n-1} a_{n+1} = a_n^2 + 2$$

It is possible to add first sequence to second sequence as below.

$$a_{n+2}a_n - a_{n+1}^2 = a_{n-1}a_{n+1} - a_n^2$$

$$a_n(a_{n+2} + a_n) = a_{n+1}(a_{n-1} + a_{n+1})$$

$$\frac{a_{n+2} + a_n}{a_{n+1}} = \frac{a_{n-1} + a_{n+1}}{a_n}$$

After a few adjustments of the sequence, we got our invariant (above). It does not change for all quaternions of consecutive members.

This invariant does not change so it is constant. We can count this constant c with integers a_1, a_2, a_3 (for $n = 1$) – or generally with triplet of consecutive members.

$$c = \frac{a_3 + a_1}{a_2} = \frac{3 + 1}{1} = 4$$

So we have:

$$\frac{a_{n+2} + a_n}{a_{n+1}} = 4$$

$$a_{n+2} = 4a_{n+1} - a_n$$

Let´s prove it by mathematic induction so $n = k$. Assume that all term of sequence $a_{n+2} = 4a_{n+1} - a_n$ are integers.

For $k = 1$ it is valid: $a_3 = 4a_2 + a_1$ – all numbers are integers.

For $k + 1$ we have:

$$a_{k+3} = 4a_{k+2} - a_{k+1}$$

From the induction supposition we know that $4a_{k+2}$ and a_{k+1} are integers. When we multiply an integer by 4 and subtract from it another integer we will always get an integer. Thus a_{k+3} is an integer.

We proved that all terms of sequence are integers.

Example 3

Count more members of the sequence.

$$a_1 = 1, a_2 = 1, a_3 = -1, a_4 = -1, a_5 = 1, a_6 = -1, a_7 = -1, a_8 = 1, a_9 = -1$$

Assume that the sequence is periodic (period $1, -1, -1$). Thus we have to prove this:

$$a_{3n} = -1$$

$$a_{3n+1} = -1$$

$$a_{3n+2} = 1$$

Prove it by mathematical induction. Let $n = k$.

For $k = 1$ it is valid: $a_3 = -1, a_4 = -1, a_5 = 1$.

For $k + 1$ we have:

$$a_{3k+3} = a_{3k+2}a_{3k} = -1$$

$$a_{3k+4} = a_{3k+3}a_{3k+1} = a_{3k+2}a_{3k}a_{3k+2}a_{3k} = -1$$

$$a_{3k+5} = a_{3k+4}a_{3k+2} = a_{3k+2}a_{3k}a_{3k+2}a_{3k}a_{3k+2} = 1$$

We proved that $a_{3n} = -1$, $a_{3n+1} = -1$, $a_{3n+2} = 1$.

Number a_{2009} is in form a_{3n+2}. Thus $a_{2009} = -1$.

Example 4

Count more members of the sequence:

$$a_1 = 1, a_2 = 12, a_3 = 20, a_4 = 63, a_5 = 165, a_6 = 455$$

Let $b_n = 1 + 4a_n a_{n+1}$.

Count more numbers b_n: $b_1 = 49$, $b_2 = 961$. Note that the square root of these numbers is an integer. Assume that a square root of every b_n is an integer. Call the square root c_n. Thus $c_n = \sqrt{b_n}$.

Assume that $c_{n+1} - c_n = 2a_{n+1}$ and $c_n = a_{n+1} + a_n - a_{n-1}$.

We need to prove this $b_n = c_n^2$.

Prove it by mathematical induction.

For $n = 1$ it is valid:

$$b_1 = c_1^2$$

$$49 = 7^2$$

For $n + 1$ we have:

$$\begin{aligned} b_{n+1} = 1 + 4a_{n+1}a_{n+2} &= 1 + 4a_{n+1}(2a_{n+1} + 2a_n - a_{n-1}) = 1 + 8a_{n+1}^2 + 8a_{n+1}a_n - 4a_{n+1}a_{n-1} \\ &= 1 + 4a_{n+1}a_n + 8a_{n+1}^2 + 4a_{n+1}a_n - 4a_{n+1}a_{n-1} \\ &= b_n + 4a_{n+1}(a_{n+1} + a_n - a_{n-1}) + 4a_{n+1}^2 = c_n^2 + 4a_{n+1}c_n + 4a_{n+1}^2 \\ &= (c_n + 2a_{n+1})^2 = (c_{n+1})^2 \end{aligned}$$

From the relation $b_{n+1} = c_{n+1}^2$ we can see b_{n+1} is square.

We proved that $b_n = c_n^2$. Thus b_n is a square for every positive integer n.