CS 70 Discrete Mathematics and Probability Theory
Summer 2019 James Hulett and Elizabeth Yang

DIS 1B

1 Contraposition

Prove the statement "if $a+b<c+d$, then $a<c$ or $b<d$".

Solution:

The implication we're trying to prove is $(a+b<c+d) \implies ((a<c) \vee (b<d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

2 Perfect Square

A *perfect square* is an integer n of the form $n=m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

We will proceed with a direct proof. Let $n=m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m=2l+1$ for some integer l. Then, $m^2=4l^2+4l+1=4l(l+1)+1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ for some integer k and $n=m^2=8k+1$.

3 Infinite Primes

Prove by contradiction that there are an infinite number of primes.

Solution:

We assume there are a finite number n of primes, $p_1, \ldots, p_n$. Let $m=p_1 \cdots p_n+1$. We know m is either prime or divisible by a prime; m is not divisible by a prime by construction, since we will have remainder 1. Clearly, $m>p_n$, so m can not be prime because p_n is the largest prime. Thus we have a contradiction, and there must be an infinite number of primes.

4 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n>m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, ..., n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.