1 Venn Diagram

Out of 1000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

(b) What is the probability that the student belongs to a club?

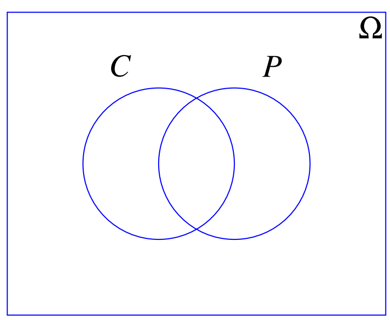
(c) What is the probability that the student works part time?

(d) What is the probability that the student belongs to a club AND works part time?

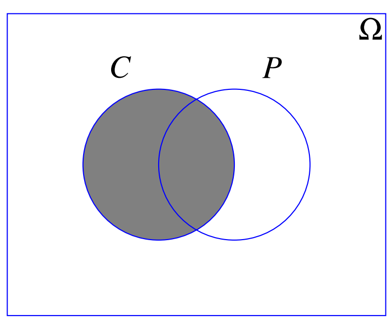
(e) What is the probability that the student belongs to a club OR works part time?

Solution:

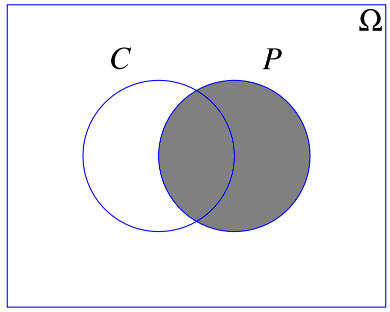
(a) The sample space will be illustrated by a Venn diagram.



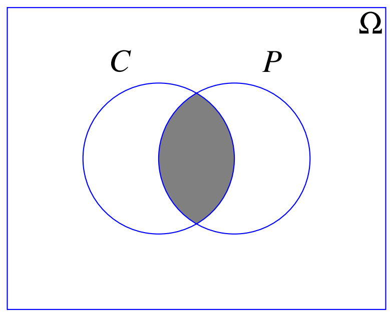
(b) $\mathbb{P}[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \frac{2}{5}$.



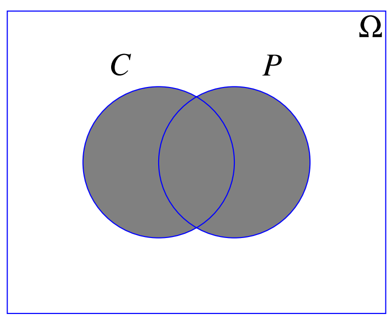
(c) $\mathbb{P}[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \frac{1}{2}$.



(d) $\mathbb{P}[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \frac{1}{20}$.



(e) $\mathbb{P}[P \cup C] = \mathbb{P}[P] + \mathbb{P}[C] - \mathbb{P}[P \cap C] = \frac{1}{2} + \frac{2}{5} - \frac{1}{20} = \frac{17}{20}$.



2 Parking Lots

Some of the CS 70 staff members founded a start-up company, and you just got hired. The company has twelve employees (including yourself), each of whom drive a car to work, and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

(a) On any given day, what is the probability that you park next to Professor Rao, who is working there for the summer?

(b) What is the probability that there are exactly three cars between yours and Professor Rao's?

(c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

Solution:

(a) There are 11 ways to choose two adjacent spaces, and 2 ways to choose which of those spaces is yours and which of those spaces is Professor Rao's. After the positions of your car and Professor Rao's car are determined, the remaining cars can park in any of 10! arrangements. Therefore, the probability is

$$\frac{11 \cdot 2 \cdot 10!}{12!} = \frac{2}{12} = \frac{1}{6}.$$

(b) Again we have to count the number of arrangements and divide by 12!. Either your car is parked to the left of Professor Rao's or to his right. In the first case your car can be in any of these spots (assuming they're numbered from left to right) $1, 2, \ldots, 8$, and Professor Rao's car will always be in the spot whose number is higher by 4. This gives your car and Professor Rao's car 8 ways to be parked with your car being on the left. By symmetry, there are also 8 ways in which your car is to the right of Professor Rao's car with three cars in between. So in total there are 16 ways for your car and Professor Rao's car to be parked with 3 spaces in between. After the spaces for these two cars are determined, the remaining ones can park in any of the 10! arrangements. So the total number of arrangements is $16 \times 10!$. The probability is therefore $(16 \times 10!)/12! = 16/(12 \cdot 11) = 4/33$.

(c) We know that 5 spots from the 11 that are not occupied by your car have been freed. All choices of these 5 spots are equally likely to happen. So we can count the arrangements of the 5 free spots to compute the probability. In total there are $\binom{11}{5}$ possible ways to choose the free spots. To count to number of ways that free both spaces next to your car, we pick those two spaces and then choose 3 more spots from the remaining 9. So there are $\binom{9}{3}$ such arrangements. Therefore the desired probability is $\binom{9}{3}/\binom{11}{5}$ which is equal to

$$\frac{9!/(3!6!)}{11!/(5!6!)} = \frac{5 \times 4}{11 \times 10} = \frac{2}{11}.$$

3 Calculate These... or Else

(a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order, i.e. $\{8, 9, 10, J, Q\}$ is a straight. Values do not loop around, so $\{Q, K, A, 2, 3\}$ is not a straight. However, an ace counts as both a low card and a high card, so both $\{A, 2, 3, 4, 5\}$ and $\{10, J, Q, K, A\}$ are considered straights. When drawing a 5 card hand, what is the probability of drawing a straight from a standard 52-card deck?

(b) What is the probability of drawing a straight or a flush? (A flush is five cards of the same suit.)

(c) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?

(d) Two distinct squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

(e) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column.)

Solution:

(a) The probability space is uniform over all possible 5-card hands, so we can use counting to solve this problem. There are $\binom{52}{5}$ possible hands, so that is our denominator. To count the number of possible straights, note that there are 4 choices of suit for each of the cards for a total of 4^5 suit choices. Also, observe that once we pick a starting card for the straight, the rest of the cards are determined (e.g. if we choose 3 as the first card, then our straight must be $\{3,4,5,6,7\}$). Therefore, we need to multiply by the number of possible starting cards.

$$\frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394.$$

(b) From part (a), we already know the probability of drawing a straight. We count the number of flushes in the following way: there are 4 choices for the suit, and $\binom{13}{5}$ choices for the cards within the suit, for a total of $4\binom{13}{5}$ flushes. Therefore, the probability of a flush is $4\binom{13}{5}/\binom{52}{5}$.

However, by the inclusion-exclusion principle, we must subtract the number of straight flushes. We now count the number of straight flushes: there are 4 choices for the suit and 10 choices for the starting card, for a total of 40 straight flushes.

The probability of drawing a straight or a flush is therefore:

$$\frac{10 \cdot 4^5 + 4 \cdot \binom{13}{5} - 40}{\binom{52}{5}} \approx 0.00591.$$

(c) $13^4 \cdot 48$ counts twice the total number of combinations of 1 card from each suit. So the final probability is $13^4 \cdot 24/\binom{52}{5} \approx 0.264$.

(d) In 64 squares, there are:

(1) 4 at-corner squares, each shares ONLY 2 sides with other squares.

(2) $6 \cdot 4 = 24$ side squares, each shares ONLY 3 sides with other squares.

(3) $6 \cdot 6 = 36$ inner squares, each shares 4 sides with other squares.

Notice that the three cases are mutually exclusive and we cannot choose the same square twice. So we just sum up the probabilities.

$$\frac{4}{64} \cdot \frac{2}{63} + \frac{24}{64} \cdot \frac{3}{63} + \frac{36}{64} \cdot \frac{4}{63} = \frac{1}{18} \approx 0.0556.$$

Alternatively, there are $\binom{64}{2}$ total pairs of squares, and for every pair of adjacent squares there is a unique edge associated with the pair (the edge that they share). Therefore, we can count the total number of edges in the chessboard, which is $8 \cdot 7 \cdot 2$ (if we only look at the horizontal edges, there are 8 edges per row, and 7 rows of edges, and then we multiply by 2 for the vertical edges too). Thus the probability is $8 \cdot 7 \cdot 2 / \binom{64}{2} = 1/18$ as before.

(e) $8!/\binom{64}{8} \approx 9.11 \cdot 10^{-6}$. This counts safe arrangements (8 choices in first row, 7 in second row, etc.) over total arrangements, and since this is a uniform probability space, this gives the probability no rooks are threatening one another.