CS 70 Discrete Mathematics and Probability Theory
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DIS 1C

1 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

→ base case: $n=3$, $2^3 > 2\cdot 3+1$ is $8>7$ which is true
→ inductive step: Suppose statements holds for value k. We want to prove that for $n=k+1$, $2^n > 2n+1$ holds

$2^k > 2k+1$ (holds)
$2\cdot 2^k > 4k+2$ since $k\geq 3$, $4k \geq 2k+6$
$2\cdot 2^k > 2k+6$ - therefore $4k+2 > 2k+6$
$2\cdot 2^k > 2k+3$ we have shown that $2^{k+1} > 2(k+1)+1$
$2^{k+1} > 2(k+1)+1$ which completes inductive step

✓

(b) For all positive integers n, $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

→ base case, $n=1$
$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{6}{6} = 1$ holds
→ inductive step: Suppose statements hold for value k. We want that for $n=k+1$, statement $1^2+\ldots+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ holds

$1^2+\ldots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\left(\frac{k(2k+1)}{6}+k+1\right)$
$= (k+1)\frac{2k^2+7k+6}{6} = (k+1)\frac{2(k+2)(k+\frac{3}{2})}{6} = (k+1)\frac{(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$
$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

✓

(c) For all positive natural numbers n, $\frac{5}{4}\cdot 8^n + 3^{3n-1}$ is divisible by 19.

→ base case: $n=1$, $\frac{5}{4}\cdot 8^1 + 3^{3-1} = \frac{40}{4}+9 = 10+9 = 19$ is divisible by 19
→ inductive: Suppose statement holds for value k. We need to show that for $k+1$, expression $\frac{5}{4}\cdot 8^{k+1}+3^{3(k+1)-1}$ is divisible by 19

$19 \mid \frac{5}{4}\cdot 8^k + 3^{3k-1}$
$19 \mid 8\cdot 27\left(\frac{5}{4}\cdot 8^k + 3^{3k-1}\right)$
$19 \mid 27\left(\frac{5}{4}\cdot 8\cdot 8^k\right) + 8(27\cdot 3^{3k-1})$
$19 \mid 27\left(\frac{5}{4}\cdot 8^{k+1}\right) + (27-19)(3^{3k+2})$
$19 \mid 27\left(\frac{5}{4}\cdot 8^{k+1} + 3^{3k+2}\right) - 19(3^{3k+2})$

→ since $19 \mid 19(3^{3k+2})$, $19 \mid 27\left(\frac{5}{4}\cdot 8^{k+1}+3^{3k+2}\right)$
→ since 19 and 27 don't have any factor in common
$19 \mid \frac{5}{4}\cdot 8^{k+1} + 3^{3k+2}$
$19 \mid \frac{5}{4}\cdot 8^{k+1} + 3^{3(k+1)-1}$
↳ that is what we wanted to proof

✓

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

Prove that $a_n \leq 3^{2^n}$

→ base case: $n=1$

$a_1 = 1 \leq 3^{2^1} = 9$ ✓

→ inductive step

✓

- Assume $a_n \leq 3^{2^n}$
- we need to show $a_{n+1} \leq 3^{2^{n+1}}$

$a_{n+1} = 3(a_n^2) \leq 3 \cdot (3^{2^n})^2 = 3 \cdot 3^{2 \cdot 2^n} = 3^{2^{n+1}+1}$

→ we got $a_{n+1} \leq 3^{2^{n+1}+1}$ which is not what we want

extra component

(b) Try to instead prove the statement $\boxed{a_n \leq 3^{2^n-1}}$ using induction. Does this statement imply what you tried to prove in the previous part?

⊗

$n=1$	$a_1 = 1 = 3^0$
$n=2$	$a_2 = 3 \cdot 1 = 3^1$
$n=3$	$a_3 = 3^3$
$n=4$	$a_4 = 3^7$
$n=5$	$a_5 = 3^{15}$

Sequence is equivalent to $a_n = 3^{2^{n-1}-1}$

→ base case: $n=1$

$a_1 = 1$, $a_1 = 3^{2^0-1} = 1$ ✓

→ inductive step

$a_{n+1} = 3(a_n)^2 = 3(3^{2^{n-1}-1})^2 = 3 \cdot 3^{2(2^{n-1}-1)} = 3^{2^n-2+1} = 3^{2^{(n+1)-1}-1}$

⇒ $\boxed{\text{Prove that } a_n = 3^{2^{n-1}-1} \leq 3^{2^n}}$

→ or without pattern prove that $a_n \leq 3^{2^n-1}$

different than task

→ base case: $n=1$

$3^{2^0-1} = 3^0 = 1 \leq 3^{2^1} = 9$ ✓

→ inductive step

- Assume $3^{2^{n-1}-1} \leq 3^{2^n}$

$3^{2^{n+1}} \geq 3^{2^n} \geq 3^{2^{n-1}} = 3^{2^{(n+1)-1}-1}$

- therefore $3^{2^{n+1}} \geq 3^{2^{(n+1)-1}-1}$

$a_{n+1} = 3(a_n)^2 \leq 3 \cdot (3^{2^n-1})^2$
$= 3 \cdot 3^{2^{n+1}-2} = 3^{2^{n+1}-1}$

3 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0,1\}$.

→ base case: $n=1$

$1 = 1 \cdot 2^0$

→ inductive step

- Assume n can be written as $n = c_k \cdot 2^k + \ldots + c_0 \cdot 2^0$, where $c_{k,\ldots,1} \in \{0,1\}$
- $n+1 = (c_k \cdot 2^k + \ldots + c_0 \cdot 2^0) + 1 = (c_k \cdot 2^k + \ldots + c_0 \cdot 2^0) + 1 \cdot 2^0$

→ inductive step

- hypothesis: Assume the statement is true for all $1 \leq k \leq n$
- if $n+1$ is divisible by 2

$\frac{n+1}{2} = c_k \cdot 2^k + \ldots + c_0 \cdot 2^0$ (by hypothesis)

$n+1 = 2\left(\frac{n+1}{2}\right) = c_k \cdot 2^{k+1} + \ldots + c_0 \cdot 2^1$

- if $n+1$ is not divisible by 2
  - then n must be divisible by 2 ($c_0 = 0$)

$n = c_k \cdot 2^k + \ldots + 0 \cdot 2^0$

$n+1 = (c_k \cdot 2^k + \ldots + 0 \cdot 2^0) + 1 \cdot 2^0 = c_k \cdot 2^k + \ldots + 1 \cdot 2^0$