CS 70 Discrete Mathematics and Probability Theory
Summer 2019 James Hulett and Elizabeth Yang **DIS 4B**

# 1 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program $P$ on input $x$ prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program $P$ prints "Hello World!" before running the $k$th line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program $P$ prints "Hello World!" in the first $k$ steps of its execution. Is there a computer program that can perform this task? Justify your answer.

**Solution:**

(a) Uncomputable. We will reduce `TestHalt` to `PrintsHW`$(P,x)$.

```
TestHalt(P, x):
  P'(x):
    run P(x) while suppressing print statements
    print("Hello World!")
  if PrintsHW(P', x):
    return true
  else:
    return false
```

If `PrintsHW` exists, `TestHalt` must also exist by this reduction. Since `TestHalt` cannot exist, `PrintsHW` cannot exist.

(b) Uncomputable. Reduce `PrintsHW`$(P,x)$ from part (a) to this program `PrintsHWByK`$(P,x,k)$.

```
PrintsHW(P, x):
  for i in range(len(P)):
    if PrintsHWByK(P, x, i):
      return true
  return false
```

(c) Computable. You can simply run the program until $k$ steps are executed. If $P$ has printed "Hello World!" by then, return true. Else, return false.

The reason that part (b) is uncomputable while part (c) is computable is that it's not possible to determine if we ever execute a specific line because this depends on the logic of the program, but the number of computer instructions can be counted.

# 2 Clothes and Stuff

(a) Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, etc.). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

(b) It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

(c) If we own $k$ different items of clothing, with $n_1$ varitions of the first item, $n_2$ variations of the second, $n_3$ of the third, and so on, how many outfits can we make?

(d) We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters.)

(e) Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of $d$, your answer from the previous part.)

(f) Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

**Solution:**

(a) $3^5$

(b) $14 \cdot 3^4$

(c) $n_1 \cdot n_2 \cdot n_3 \cdots n_k$

(d) $14!/10!$

(e) $\binom{14}{4}$ or written as a function of the previous part, $d/4!$

(f) Within each color category, the hats are indistinguishable, so we will use the stars-and-bars method. Specifically, consider the six choices you have to make to be six "stars", and you must place your stars in one of three color categories. Since there are three categories, we need two "bars" (dividers) to separate the categories. In total, we have $6+2=8$ stars and bars, and so there are $\binom{8}{6}$ ways to choose the positions of the stars (which corresponds to $\binom{8}{6}$ ways to choose our floppy hats). Equivalently, there are $\binom{8}{2}$ ways to choose the positions of the bars.

But let's be serious, you should just keep the black ones – so much more versatile.

# 3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

**Solution:**

One counting strategy is based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the $i^{th}$ digit, then the $i-1^{th}$ digit must be a 1, and the other $(10?5?1=4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$. possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.