1 Counting on Graphs

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct cycles are there in a complete graph with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

Solution:

(a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$.

(b) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

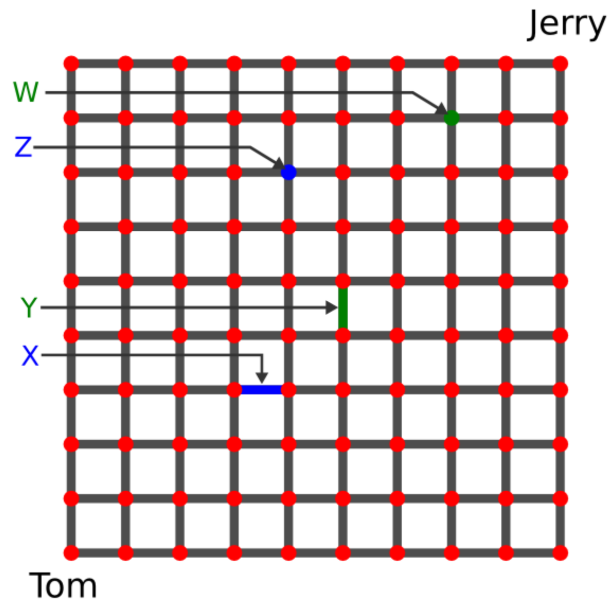
(c) The number of vertices k in a cycle is at least 3 and at most n. Without accounting for duplicates, there are $n!/(n-k)!$ cycles. Due to inversions and rotations, the number of cycles equivalent to any given cycle is $2k$. Hence the total number of distinct cycles is

$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}.$$

(d) Without considering symmetries there are $6!$ ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

2 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled X?

(c) The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(d) How many shortest paths pass through the vertex labeled Z? The vertex labeled W? Both the vertices Z and W? Neither vertex Z nor vertex W?

Solution:

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

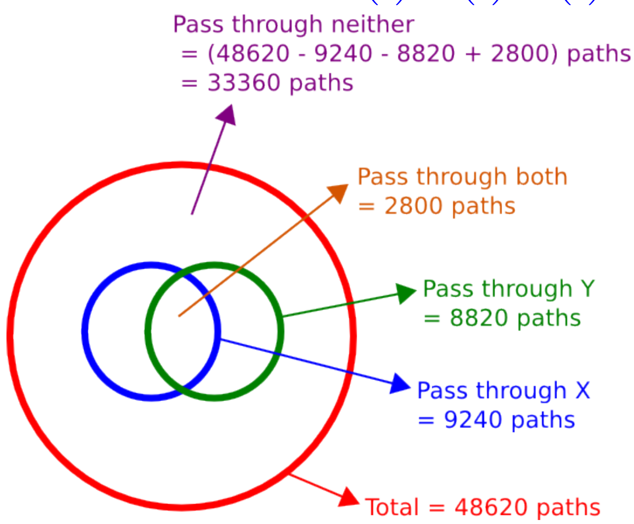
Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHVVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.

(b) For a shortest path to pass through the edge X, it has to first get to the left vertex of X. So the first portion of the path has to start at the origin, and end at the left vertex of X. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of X to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge X is $20 \times 462 = 9240$.

(c) Using similar logic, any shortest path that passes through Y has to consist of 2 legs, the first leg going from the origin to the bottom vertex of Y, and the second leg going from the top vertex of Y to the top right corner of the maze. The first leg will consist of exactly 5 "H"s and 4 "V"s, while the second leg will consist of exactly 4 "H"s and 4 "V"s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

By a similar argument, let's try to figure out how many paths will pass through both X and Y. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 "H"s and 3 "V"s (going from the origin to the left edge of X), the second leg consisting of 1 "H" and 1 "V" (going from the right vertex of X to the bottom vertex of Y), and the third leg consisting of 4 "H"s and 4 "V"s (going from the top vertex of Y to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.



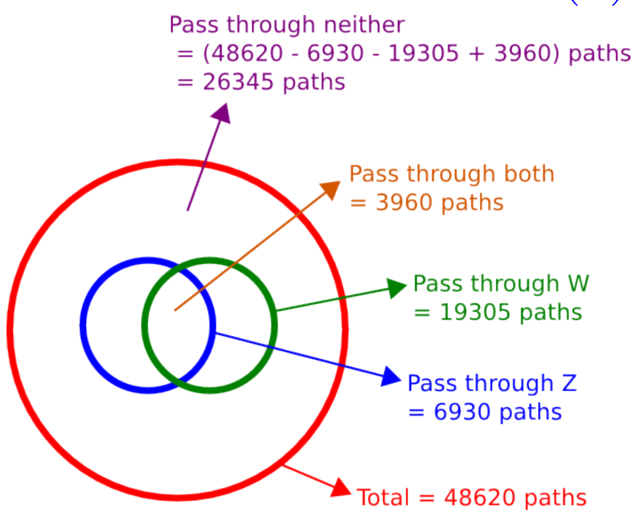
Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through X, 8820 pass through Y, and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).

(d) This part is very similar in spirit to the previous one, except that in this case, each leg of the path we consider begins exactly where the previous leg ended, and *not* to the right or to the top of where the previous leg ended.

For concreteness, let's find out how many shortest paths pass through vertex Z. Observe that for a shortest path to pass through Z, it has to first get to Z. So the first portion of the path has to start at the origin, and end at Z. Using the same logic as above, there are exactly $\binom{11}{4} = 330$ ways to complete this "first leg" of the path (consisting of 4 "H" edges and 7 "V" edges). Having chosen one of these 330 ways, the path has to then go from Z to the top right corner of the maze. This second leg will consist of 5 "H" edges and 2 "V" edges, and so there are exactly $\binom{7}{2} = 21$ possibilities. Therefore, the total number of shortest paths that pass through the vertex Z is $330 \times 21 = 6930$.

Using similar logic, any shortest path that passes through W has to consist of 2 legs, the first leg going from the origin to W, and the second leg going from W to the top right corner of the maze. The first leg will consist of exactly 7 "H"s and 8 "V"s, while the second leg will consist of exactly 2 "H"s and 1 "V". So the total number of such shortest paths will be $\binom{15}{7} \times \binom{3}{1} = 19305$.

By a similar argument, let's try to figure out how many paths will pass through both Z and W. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 4 "H"s and 7 "V"s (going from the origin to Z), the second leg consisting of 3 "H"s and 1 "V" (going from Z to W), and the third leg consisting of 2 "H"s and 1 "V" (going from W to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{11}{4} \times \binom{4}{1} \times \binom{3}{1} = 3960$.



Finally, we know that there are 48620 shortest paths in all, of which 6930 pass through Z, 19305 pass through W, and 3960 pass through both. So the number of paths that pass through neither is 26345 (see the figure above for an intuitive explanation).

3 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(b) $\binom{n}{i} = \binom{n}{n-i}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$.

Solution:

(a) For each i on the LHS, we can think of selecting a team of i members out of a pool of n players, and susequently choosing a captain out of the i team members. The RHS does the same by first choosing the captain out of the n players, and then a subset of the remaining $n-1$ players to constitute the team.

(b) Choosing i players out of n to play on a team is the same as choosing $n-i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

(c) Assume we have n women and n men. Using part (b) we can rewrite the LHS as $\sum_{i=1}^{n} i\binom{n}{i}\binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n-i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n-1$ players from all remaining men and women to form the team.