Note 4: STABLE MARRIGE

- **THE STABLE MARRIAGE PROBLEM**
  - match n men and n women
    - every person has preference list of n people of opposite sex
  - find pairing such that nobody can realistically hope to benefit by switching partners
- **THE STABLE MARRIAGE ALGORITHM**
  - → every morning
    - every man proposes to the most preferred woman on his list who has not yet rejected him
  - → every afternoon
    - each woman collects all the proposals she received
    - to the man she likes the best among these: "maybe come back tomorrow"
    - to other: "never"
  - → every evening
    - each rejected man crosses off woman who rejected him from his list
  - loop is repeated until every woman has man
  - = Gale-Shapley algorithm
- **THE RESIDENCY MATCH**
  - application of the algorithm (National Residency Matching Program)
  - graduates and hospitals submit their preference list
  - in 2012, Nobel Prize in economic was won for extending the stable marriage algorithm
- **PROPERTIES OF THE ALGORITHM**
  - ⇒ The algorithm always halts (ends)
    - on each day that the algorithm doesn't halt, at least one man must eliminate some woman from his list
    - since each list has n elements and there are n lists, it must terminate in at most n^2 interactions
  - ⇒ STABILITY
    - pairing is unstable if there is man and woman who prefer each other to their current partner
      - = rogue couple

- pairing of n men and n women is stable if it has no rogue couples
- the stable pairings always exist
  - in The Roommates Problem, there doesn't have to be stable pairing
    - you have 2n people, any person can be paired with (2n-1) people

=> ANALYSIS

- prove that the algorithm always outputs stable pairing
- each man begins with his first choice as a possibility
  - as the algorithm proceeds, his options can only get worse over time
- woman's options can only get better over time

(lemma 1) - If man M proposes to woman W on k^{th} day, then on every subsequent day W has someone on string whom she likes at least as much as M

- PROOF
  - proceed by induction on day j, $j \geq k$
  - base case: $j = k$
    - on day k, W receives at least one proposal (from M)
    - at the end of day k, she will therefore have on string M or better man (she chooses the best)
  - hypothesis
    - suppose the claim is true for $j \geq k$
  - inductive step
    - we prove the claim for $j+1$
    - by hypothesis, on day j, W had M' on string whom she likes at least as much as M (M' may be M)
    - according to algorithm, M' proposes to W again on day (j+1)
      - -> at the end of day (j+1), W will have on string either M' or someone better than M'
      - in both cases, she likes this person at least as much as M

- proof by induction
  - prove base case
  - prove inductive step
    - -> $P(j) \Rightarrow P(j+1)$
    - -> $\neg(P(j) \Rightarrow P(j+1))$ doesn't hold, therefore $P(j) \Rightarrow P(j+1)$ holds
- => well-ordering principle
  - Any non-empty set of natural numbers contains a "smallest" element
  - If $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then S has a smallest element
- we can use alternate approach to prove the previous lemma
  - Suppose... that on j^{th} day for $j > k$ is the first counterexample
  - PROOF
    - Suppose that the j^{th} day for $j > k$ is the first counterexample where W has either nobody or has M^* interfior to M on string
    - on day $(j-1)$, she has M' on string and likes M' at least as much as M
    - according to algorithm, M' still proposes to W on j^{th} day since she said "maybe" the previous day
      - so W has choice of at least one man on j^{th} day
      - moreover, her best choice is at least as good as M'
        - -> according to the algorithm, she will choose him over M^*
      - this contradicts our initial assumption

(lemma 2) -> The stable marriage algorithm always terminates with a pairing.
- suppose there is man M left unpaired
  - he must have proposed to all n woman on his list
- by lemma 1, each of n woman has had someone on string since M proposed to her
- when algorithm terminates, n women have n men on string not including M
  - so there must be at least $(n+1)$ men
    - contradiction since there are only n men

(lemma 3) -> The pairings produced by the algorithm is always stable.
- consider any couple (M, W) in the pairing
- suppose M prefers W^* to W
- since W^* occurs before W in M's list, M must have proposed to W
- by lemma 1, W^* likes her final partner at least as much as M and therefore prefers him to M
-> no man M can be involved in rogue couple
- pairing is stable

- OPTIMALITY
- there can be multiple stable pairing
- for a given man M, the optimal woman for M is the highest woman on M's preference list that M is paired with in any stable pairing
- male (female) optimal pairing
- each man (woman) is paired with his (her) optimal woman (man)
- pessimal woman for man is the lowest ranked woman whom he is ever paired with in some stable pairings
- The pairings output by the Stable marriage algorithm is male optimal.
- PROOF
- suppose that pairing is not male optimal
- there exists a day on which some man was rejected by his optimal woman W^* in favor of M^*
- by definition of optimal woman, there must be a stable pairing T in which M and W^* are paired
- suppose $T = \{\ldots, (M, W^*), \ldots, (M^*, W') \ldots\}$
- we will show that (M^*, W^*) is rogue couple
- W^* prefers M^* to M
- since day k was the first day when some man got rejected by his optimal woman, before day k, M^* hasn't been yet been rejected by his optimal woman
- since he proposed to W^* on day k, this implies that M^* likes W^* at least as much as his optimal woman
- therefore at least as much as W'

- therefore (M^*, W^*) form a rogue couple in T
  - $\rightarrow$ not stable
- this implies that pairing is male optimal

- If pairing is male optimal, then it is also female pessimal
  - PROOF
    - let $T = \{\ldots, (M, W), \ldots\}$ be male optimal pairing output by algorithm
    - suppose there is stable pairing $S = \{\ldots, (M^*, W), \ldots, (M, W'), \ldots\}$ such that M^* is lower on W's list than M
      - M is not her pessimal man
    - S can't be stable because (M, W) is rogue couple in S
      - by assumption, W prefers M to M^* (lower on her list)
      - M prefers W to W' in S because W is his partner in male optimal pairing T (contradiction)