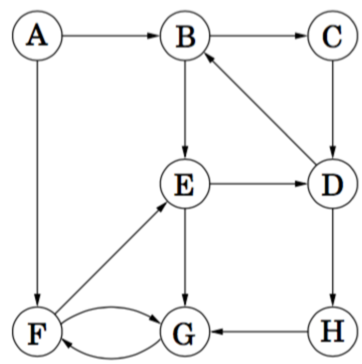
CS 70 Discrete Mathematics and Probability Theory
Summer 2019 James Hulett and Elizabeth Yang **DIS 1D**

1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $(B,C),(C,D),(D,B)$
ii. $(F,G),(G,F)$
iii. $(A,B),(B,C),(C,D),(D,B)$
iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

i. (E,G)
ii. $(E,G),(G,F)$
iii. $(F,G),(G,F)$
iv. $(A,B),(B,C),(C,D),(H,G)$
v. $(E,G),(G,F),(F,G),(G,C)$
vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

i. (E,G)

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$V = \{A,B,C,D,E,F,G,H\}$,
$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),(H,G)\}$.

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$(B,C),(C,D),(D,H),(H,G),(G,F)$
$(B,E),(E,D),(D,H),(H,G),(G,F)$
$(B,E),(E,G),(G,F)$
The first two have length 5, while the last one has length 3, so the last one is the shortest path.

(d) A cycle is a path that starts and ends at the same point. This means that (iii) is not a cycle, since it starts at A but ends at B. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)$ should be distinct, so (iv) is not a cycle. The correct answers are (i) and (ii).

(e) A walk consists of any sequence of edges such that the endpoint of each edge is the same as the starting vertex of the next edge in the sequence. Example (iv) does not fit this definition–even though it uses only valid edges, the endpoint of the second to last edge in D, while the start point of the next edge is H. Example (v) also is not a walk, since it tries to walk from G to C as its last step, but there is no such edge. All the rest are walks.

(f) A tour is simply a walk that has the same start and end vertex. Only (iii) satisfies this definition. Note in part (d), we already said that (iii) was a cycle–and indeed, all cycles are also tours.

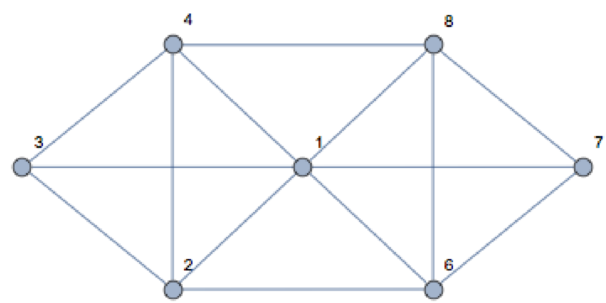
(g) True. In order for there to be a cycle in G starting and ending at some vertex v, we would need at least two edges incident to v: one to leave v at the start of the cycle, and one to return to v at the end. If every vertex has degree at most 1, no vertex has two or more edges incident on it, so no vertex is capable of acting as the start and end point of a cycle.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If this process terminates, the part of our walk from the first time we visited w until the second time is a cycle. Thus, it remains only to argue this process always terminates.

Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree at least 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above?

(b) Is there an Eulerian walk in the graph above?

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

Solution:

(a) No. Two vertices have odd degree.

(b) Yes. One of the two vertices with odd degree must be the starting vertex, and the other one must be the ending vertex. For example: $3,4,2,1,3,2,6,1,4,8,1,7,8,6,7$ will be an Eulerian walk (the numbers are the vertices visited in order). Note that there are 14 edges in the graph.

(c) An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

Note: There is no graph with only one odd degree vertex.

Justifications: *Only if.* Given an Eulerian walk, say starting at u and ending at v (possibly the same), clearly the graph is connected. Moreover, every other intermediate visit to a vertex is being paired with two edges, and therefore, except for u and v, all other vertices must be of even degree.

If. For a connected graph with no odd degree vertices, we have shown in lectures that there is an Eulerian tour.

Solution 1: If it has two odd degree vertices, say u and v, then one can first find a path from u to v, and remove the edges of the path from the graph. Next for each connected component of the residual graph, we find an Eulerian tour. Observe that an Eulerian walk is simply an edge-disjoint walk that covers all the edges. What we just did is decomposing all the edges into a path from u to v and a bunch of edge-disjoint Eulerian tours. A path is clearly an edge-disjoint walk. Then, given an edge-disjoint walk and an edge-disjoint tour such that they share at least one common vertex, one can combine them into an edge-disjoint walk simply by augmenting the walk with the tour at the common vertex. Therefore we can combine all the edge-disjoint Eulerian tours into the path from u to v to make up an Eulerian walk from u to v.

Solution 2: Alternatively, take the two odd degree vertices u and v, and add a vertex w with two edges (u,w) and (w,v). The resulting graph G' has only vertices of even degree (we added one to the degree of u and v and introduced a vertex of degree 2) and is still connected. So, we can find an Eulerian tour on G'. Now, delete the component of the tour that uses edges (u,w) and (w,v). The part of the tour that is left is now an Eulerian walk from u to v on the original graph, since it traverses every edge on the original graph.

3 Odd Degree Vertices

Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

Solution:

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\mathsf{c}}$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the right-hand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the left-hand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u, v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u, v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u, v\} & \text{if } u, v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u, v\} & \text{if } u, v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\mathsf{c}}$ had even degree in

G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (ii), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

4 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learned from the note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge (not already in the graph) between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies this property then it must be a tree:

(c) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

Solution:

(a) Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a simple cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a simple cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If $\{x,y\}$ is an edge, then clearly there is a path from x to y. Otherwise, if $\{x,y\}$ is not an edge, then by assumption, adding the edge $\{x,y\}$ will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge $\{x,y\}$ from this cycle. Therefore, we conclude the graph is a tree.