CS 70 Discrete Mathematics and Probability Theory
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1 Mutually Independent Events

There are three mutually independent events: A, B, and C. The probability that event A occurs is 0.4, the probability that event B occurs is 0.6, and the probability that event C occurs is 0.3. Calculate the following.

(a) $\mathbb{P}([A|B]$.

(b) $\mathbb{P}[A \cap B]$.

(c) $\mathbb{P}[A \cup C]$.

(d) $\mathbb{P}[B \cap C]$.

(e) $\mathbb{P}[A \cap B \cap C]$.

(f) $\mathbb{P}[A \cup B \cup C]$.

Solution: We note that all events A, B, and C are independent.

(a) 0.4.

(b) $0.4 \cdot 0.6$.

(c) $\mathbb{P}[A \cup C] = \mathbb{P}[A] + \mathbb{P}[C] - \mathbb{P}[A \cap C] = 0.4 + 0.3 - 0.4 \cdot 0.3$.

(d) $0.6 \cdot 0.3$.

(e) $0.4 \cdot 0.6 \cdot 0.3$.

(f) There are 2 ways to do this: First: Using inclusion-exclusion, we have

$$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + Pr[C] - \mathbb{P}[A \cap B] - \mathbb{P}[B \cap C] - \mathbb{P}[A \cap C] + \mathbb{P}[A \cap B \cap C]$$

$$= 0.4 + 0.6 + 0.3 - 0.4 \cdot 0.6 - 0.6 \cdot 0.3 - 0.4 \cdot 0.3 + 0.4 \cdot 0.6 \cdot 0.3$$

Second: Using complement event, we have

$$\mathbb{P}[A \cup B \cup C] = 1 - \mathbb{P}[\neg A \cap \neg B \cap \neg C] = 1 - (1 - 0.4)(1 - 0.6)(1 - 0.3).$$

2 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general rule that always holds?

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B)$ true? What is the general rule that always holds?

(c) If A and B are disjoint, does that imply they're independent?

Solution:

(a) In general, we know $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ holds.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ holds if and only if A and B are independent (by definition). The general rule that always holds is $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B)$.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

3 Unlikely Events

(a) Toss a fair coin x times. What is the probability that you never get heads?

(b) Roll a fair die x times. What is the probability that you never roll a six?

(c) Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

(d) How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

Solution:

(a) 0.5^x.

(b) $(1-1/6)^x$.

(c) $(1-1/10^6)^x$.

(d) (a) For the coin, we want $0.5^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log 0.5} \approx 3.32.$$

(b) For the die, we want: $(5/6)^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log 5/6} \approx 12.6.$$

(c) For the lottery, we want $(1-1/10^6)^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log(1-1/10^6)} \approx 2 \cdot 10^6.$$

(The answer is approximately equal to $(\log 0.1)/(-1/10^6)$, using the approximation for small values $(1-x) \approx e^{-x}$, where $x = 10^{-6}$.)

4 Balls and Bins

You have n empty bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

(a) What is the probability that the first ball thrown will cause the first collision?

(b) What is the probability that the second ball thrown will cause the first collision?

(c) What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?

(d) What is the probability that the third ball thrown will cause the first collision?

(e) What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?

(f) What is the probability that the m^{th} ball thrown will cause the first collision?

Solution:

(a) 0

(b) $\frac{1}{n}$

(c) $\frac{2}{n}$

(d) $\mathbb{P}(\text{Ball 3 collides} \mid \text{Balls 1, 2 do not collide}) \cdot \mathbb{P}(\text{Balls 1, 2 do not collide})$, which is $\frac{2}{n} \cdot \frac{n-1}{n}$

(e) $\frac{m-1}{n}$

(f) Similar to (d), $\frac{m-1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-m+2}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$

5 Pairwise Independence

The events A_1, A_2, A_3 are *pairwise independent* if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}(A_1, A_2, A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

Try to construct an example where three events are pairwise independent but not mutually independent.

Here is one potential starting point: Let A_1, A_2 be the respective results of flipping two fair coins. Can you come up with an event A_3 that works?

Solution:

A_1: the first result is Head; A_2: the second result is Head; A_3: both results are the same.