1 Countability Basics

1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (one-to-one)? Briefly justify.
2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ a surjection (onto)? Briefly justify.

Solution:

1. Yes. One way to illustrate is by drawing the one-to-one mapping from n to n^2. More formally we can show that the preimage is unique by showing that $m \neq n \implies f(m) \neq f(n)$.

   We'll do proof by contraposition. $f(m) = f(n) \implies m = n$.

   $$f(m) = f(n) \implies m^2 = n^2 \implies m^2 - n^2 = 0 \implies (m-n)(m+n) = 0 \implies m = \pm n$$

   Since n can't be negative, we have an injection.

2. Yes. For any value of y, there always exists a corresponding input x. If $y = x^3 + 1$, we know that $x = \sqrt[3]{y-1}$. Thus for any value of y, there exists this value of x which maps to it.

2 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) $\mathbb{N}$, the set of all natural numbers.

(b) $\mathbb{Z}$, the set of all integers.

(c) $\mathbb{Q}$, the set of all rational numbers.

(d) $\mathbb{R}$, the set of all real numbers.

(e) The integers which divide 8.

(f) The integers which 8 divides.

(g) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(h) Computer programs that halt.

(i) Numbers that are the roots of nonzero polynomials with integer coefficients.

Solution:

(a) Countable and infinite. See Lecture Note 10.

(b) Countable and infinite. See Lecture Note 10.

(c) Countable and infinite. See Lecture Note 10.

(d) Uncountable. This can be proved using a diagonalization argument, as shown in class. See Lecture Note 10.

(e) Finite. They are $\{-8,-4,-2,-1,1,2,4,8\}$.

(f) Countably infinite. We know that there exists a bijective function $f:\mathbb{N}\to\mathbb{Z}$. Then function $g(n)=8f(n)$ is a bijective mapping from $\mathbb{N}$ to integers which 8 divides.

(g) Uncountably infinite. We use the Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can represent a function $f\in\mathscr{F}$ as an infinite sequence $(f(0),f(1),\cdots)$, where the i-th element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$:

$$\begin{aligned}
0 &\longleftrightarrow (f_0(0),f_0(1),f_0(2),f_0(3),\dots)\\
1 &\longleftrightarrow (f_1(0),f_1(1),f_1(2),f_1(3),\dots)\\
2 &\longleftrightarrow (f_2(0),f_2(1),f_2(2),f_2(3),\dots)\\
3 &\longleftrightarrow (f_3(0),f_3(1),f_3(2),f_3(3),\dots)\\
&\vdots
\end{aligned}$$

Consider the function $g:\mathbb{N}\to\mathbb{N}$ where $g(i)=f_i(i)+1$ for $i\in\mathbb{N}$. We claim that the function g is not in our finite list of functions. Suppose for contradiction that it did, and that it was the n-th function $f_n(\cdot)$ in the list, i.e., $g(\cdot)=f_n(\cdot)$. However, $f_n(\cdot)$ and $g(\cdot)$ differ in the n-th number, i.e. $f_n(n)\neq g(n)$, because by our construction $g(n)=f_n(n)+1$ (Contradiction!).

(h) Countably infinite. The total number of programs is countably infinite, since each can be viewed as a string of characters (so for example if we assume each character is one of the 256 possible values, then each program can be viewed as number in base 256, and we know these numbers are countably infinite). So the number of halting programs, which is a subset of all programs, can be either finite or countably infinite. But there are an infinite number of halting programs, for example for each number i the program that just prints i is different for each i. So the total number of halting programs is countably infinite. (Note also that this result together with the previous one in (g) implies that not every function from $\mathbb{N}$ to $\mathbb{N}$ can be written as a program.)

(i) Countably infinite. Polynomials with integer coefficients themselves are countably infinite. So let us list all polynomials with integer coefficients as $P_1, P_2, \ldots$. We can label each root by a pair (i, j) corresponding to the j-th root of polynomial P_i (we can have an arbitrary ordering on the roots of each polynomial). This means that the roots of these polynomials can be mapped in an injective manner to $\mathbb{N} \times \mathbb{N}$ which we know is countably infinite. So this set is either finite or countably infinite. But every natural number n is in this set (it is the root of $x - n$). So this set is countably infinite.

3 Hilbert's Paradox of the Grand Hotel

Consider a magical hotel with a countably infinite number of rooms numbered according to the natural numbers where all the rooms are currently occupied. Assume guests don't mind being moved out of their current room as long as they can get to their new room in a finite amount of time. In other words, guests can't be moved into a room that's infinitely far from the current one.

1. Suppose one new guest arrived in their car, how would you shuffle guests around to accommodate them? What if k guests arrived, where k is a constant positive integer?
2. Suppose a countably infinite number of guests arrived in an infinite length bus with seat numbers according to the natural numbers, how would you accommodate them?
3. Suppose a countably infinite number of infinite length buses arrive carrying countably infinite guests each, how would you accommodate them? (*Hint*: There are infinitely many prime numbers.)

Solution:

1. Shift all guests into the room number k greater than their current room number. So for a guest in room i move them to room $i+k$. Then place the k new guests in the k first rooms in the hotel which will now be unoccupied.
2. Place all existing guests in the room $2i$ where i is their current room number. Place all the new guests in the room $2j+1$ where j is their seat number on the bus.
3. **Solution 1**: Notice that each guest has an "address": what car s/he is in and what seat number s/he is. Through the spiral method, we know there is a bijection between $\mathbb{N}^2 \leftrightarrow \mathbb{N}$. This reduces to the same exact problem as the previous part! Therefore, we can accommodate these guests.

   Solution 2: Place all existing guests in the room 2^i where i is their current room number. Assign the $(k+2)$th prime, p_{k+2}, to the kth bus (e.g. the 0th bus will be assigned the 2nd prime, 3). We then place each new guest in the room p_{k+2}^{j+1} where j is the seat number of the new guest in their bus.

   This works because any power of a prime p will not have any other prime factors than p.

Yes, there will be plenty of empty rooms, but that's okay because every guest will still have somewhere to stay.