CS 70 Discrete Mathematics and Probability Theory
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1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number x can equivalently be written as $x + 0i$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take $a = x$ and $b = 0$.

(Aside: this is a refererence to the very weak Goldback Conjecture (`https://xkcd.com/1310/`).)

2 DeMorgan's Law

Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as DeMorgan's Law.

Solution:

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A \wedge \neg B$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

3 XOR

The truth table of XOR is as follows.

A	B	A XOR B
F	F	F
F	T	T
T	F	T
T	T	F

1. Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.
2. Does $(A \text{ XOR } B)$ imply $(A \vee B)$? Explain briefly.
3. Does $(A \vee B)$ imply $(A \text{ XOR } B)$? Explain briefly.

Solution:

1. These are all correct:
   - $A \text{ XOR } B = (A \wedge \neg B) \vee (\neg A \wedge B)$
     Notice that there are only two instances when A XOR B is true: (1) when A is true and B is false, or (2) when B is true and A is false. The clause $(A \wedge \neg B)$ is only true when (1) is, and the clause $(\neg A \wedge B)$ is only true when (2) is.
   - $A \text{ XOR } B = (A \vee B) \wedge (\neg A \vee \neg B)$
     Another way to think about XOR is that exactly one of A and B needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \vee B)$ tells us *at*

least one of A and B needs to be true. In order to ensure that one of A or B is also false, we need the clause $(\neg A \vee \neg B)$ to be satisfied as well.

- A XOR $B = (A \vee B) \wedge \neg(A \wedge B)$
  This is the same as the previous, with De Morgan's law applied to equate $(\neg A \vee \neg B)$ to $\neg(A \wedge B)$.

2. Yes. $(A \text{ XOR } B) = T \implies ((A = T) \wedge (B = F)) \vee ((A = F) \wedge (B = T)) \implies (A \vee B = T)$.

3. No. When $(A = T) \wedge (B = T)$, then $(A \vee B) = T$ but $(A \text{ XOR } B) = F$.

4 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$.

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(d) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.