1 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

In this problem, we apply proof by contradiction, therefore we assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since $v_1 v_2$ is not an edge, by property T it must be that $v_1 v$ and $v_2 v$ where $v \in \{v_3, v_4, v_5, v_6, v_7\}$ are edges. Similarly for v_3, v_4, we have that $v_3 v$ and $v_4 v$ are edges where $v \in \{v_1, v_2, v_5, v_6, v_7\}$ are edges. So now any three vertices in $\{v_1, v_2, v_3, v_4\}$ on one side and $\{v_5, v_6, v_7\}$ on the other form an instance of $K_{3,3}$. Contradiction.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with > 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no tours of odd length.

Suppose there is a tour in the bipartite graph. Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

3 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

Solution:

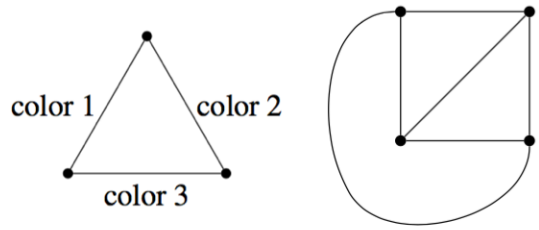
(a) The three hypercubes are a line, a square, and a cube, respectively. See also p12 on lecture notes 5.

(b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let L be the set of the vertices with an even number of 0 bits and let R be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

Alternate solution: We can also prove that the hyper-cube can be 2-colorable through induction. Base case: When $n = 1$, there are only 2 vertices and it is 2-colorable. Induction step: Assume that the hypercube can be 2-colored in the case of n. We will show that in the case of $n+1$, the hypercube is also 2-colorable: Suppose we have 2 already 2-colored n-dimensional hypercube G_1, G_2 (which we know can be done from our induction hypothesis). We add corresponding edges to the two n-dimensional hypercube to form the $n+1$-dimensional hypercube. Every newly added edge connects a vertex u in G_1 to a vertex v in G_2. For each (u,v) that we add, flip the color of the vertex v in G_2. By doing this, we've successfully found a way that the $n+1$-dimensional hypercube can be 2-colorable. And therefore, it must be a bipartite.

4 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Add the fourth vertex, notice that each edge has a different color available from the set of three colors.

(b) By induction on the number of edges. We will use a set of $2d-1$ colors. Remove an edge and $2d-1$ color the remaining graph from our set. This can be done by the induction hypothesis as the remaining graph's degree is no bigger than d and the graph has fewer edges. The edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts.

(c) By induction on the number of vertices. Base case is a single vertex, which has no edges to color, and thus can be colored with 0 colors. For the inductive step, we start by removing any leaf v from the tree. We can then color the remaining tree with d colors. Note that vertex v's neighboring vertices has degree at most $d-1$ without the edge to v and thus its incident edges use at most $d-1$ colors. Thus, there is a color available for coloring the edge incident to this vertex.