Lecture 2 – Linear Regression and Extensions

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Linear Regression

We start with a linear regression model:

$$y_i = x_i'\beta + \varepsilon_i, i = 1, \dots, n,$$

where x_i is a p-dimensional vector of regressors for observation i, β a p-dimensional coefficient vector, and ε_i iid error terms with $\mathbb{E}[\varepsilon_i | x_i] = 0$.
The ordinary least squares (ols) estimator for β is defined as

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} (y_i - x_i'\beta)^2.$$

Linear Regression

If the Gram matrix $\sum_{i=1}^n x_i x_i'$ is of full rank, the ols estimate is given by

$$\hat{\beta} = (\sum_{i=1}^n x_i x_i')^{-1} (\sum_{i=1}^n x_i y_i).$$

The residuals $\hat{\varepsilon}_i$ are defined as

$$\hat{\varepsilon}_i = y_i - x_i' \hat{\beta}.$$

For an observation x the *fitted* or *predicted* values are given by

$$\hat{y} = x' \hat{\beta}.$$

Linear Regression

In matrix notation we can write

$$Y = X\beta + \varepsilon$$

with $Y = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}$, $\varepsilon = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \end{pmatrix}$ and X is a $n \times p$-matrix with observation i forming the ith row of the matrix X. The ols estimate $\hat{\beta}$ can then be written as

$$\hat{\beta} = (X'X)^{-1}X'y.$$

Linear Regression

Under homoscedastic errors, i.e. $\mathbb{V}\varepsilon_i = \sigma^2$, we have that

$$\mathbb{V}(\hat{\beta}) = (X'X)^{-1}\sigma^2.$$

Asymptotically, the ols estimate is normal distributed:

$$\hat{\beta} \sim N(\beta, (X'X)^{-1}\sigma^2).$$

This can be used for testing hypotheses and construction of confidence intervals.

Linear Regression

$$z_j = \frac{\hat{\beta}_j}{\hat{\sigma}^2 \sqrt{v_j}}$$

where v_j is the jth diagonal element of $(X'X)^{-1}$.
Under the null hypothesis $\beta_j = 0$ the *Z-score* / *t-statistic* z_j is t_{n-p-1}-distributed.

Linear Regression

Remark: In the high-dimensional-setting, i.e. $p >> n$ the Gram Matrix is rank deficient and the ols estimate is not uniquely defined and the variance of the parameter estimate is unbounded.

Extensions

- Polynomial Regression
- Step Functions
- Basis Functions
- Regression Splines
- Smoothing Splines

Extensions | Remarks

- Although the linear regression model looks quite simple, it can be extended / modified to model complex relations.
- For the extensions we consider without loss of generality univariate regressions:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

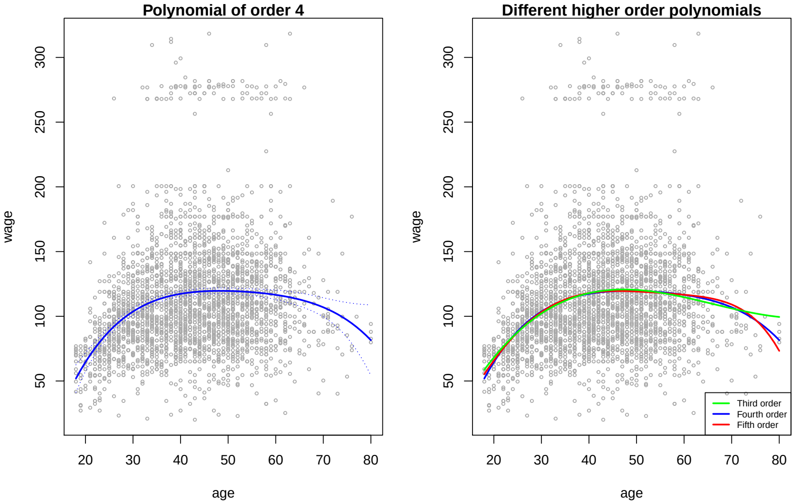
Extensions | Polynomial Regression

To make the linear specification more flexible, we might include higher-order polynomials:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \ldots \beta_p x_i^p + \varepsilon_i$$

- Estimation by ols
- Quite flexible, but usually p=3 or p=4
- Higher order polynomials ($p > 5$) might lead to strange fits (overfitting), especially at the boundary.

Extensions | Polynomial Regression - Example



Extensions | Step Functions

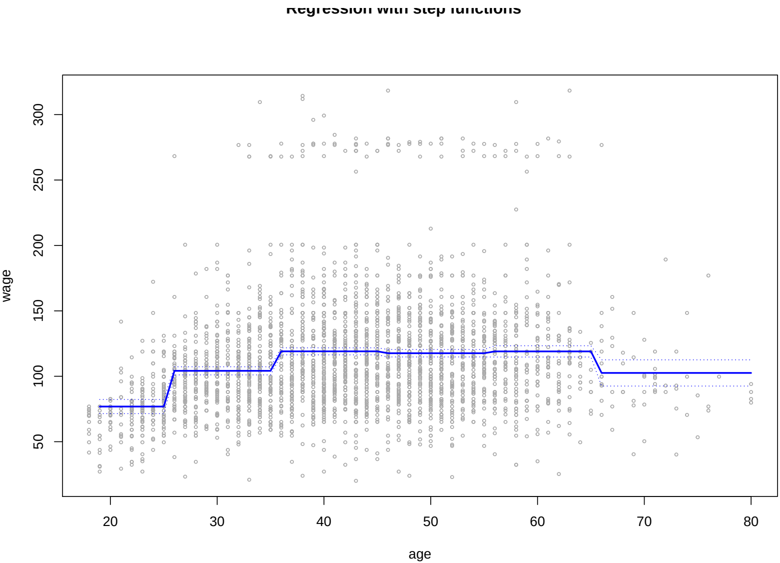
- Definition: Step functions are functions which are constant on each part of a partition of the domain.
- Univariate Regression: choosing K cut points $c_1, \ldots, c_K$ and defining new auxiliary variables:
  $C_0(x) = 1(x < c_1)$, $C_1(x) = 1(c_1 \leq x < c_2)$, …,
  $C_K(x) = 1(c_K \leq x)$
- $1(\cdot)$ is the so-called indicator function which is 1 is the condition is true and 0 otherwise.
- This gives us the following regression:

$$y_i = \beta_0 = \beta_1 * C_1(x_i) + \ldots + \beta_K * C_K(x_i) + \varepsilon_i$$

Extensions | Step Functions

- Note: $C_0(x) + \ldots + C_K(x) = 1$ and hence we drop $C_0 = (\cdot)$ to avoid multicollinearity.
- Interpretation β_0
- Example: wage regression

Extensions | Step Functions



Extension | Basis Functions

- Idea: family of functions or transformations that can be applied to a variable: $b_1(x), \ldots, b_K(x)$ (basis functions)
- Regression:

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \ldots + \beta_K b_K(x_i) + \varepsilon_i$$

- Examples
- Polynomial regression: $b_j(x_i) = x_i^j$
- Piecewise constant functions (step functions): $b_j(x_i) = 1(c_j \leq x_i < c_{j+1})$
- Regressions splines (coming next)