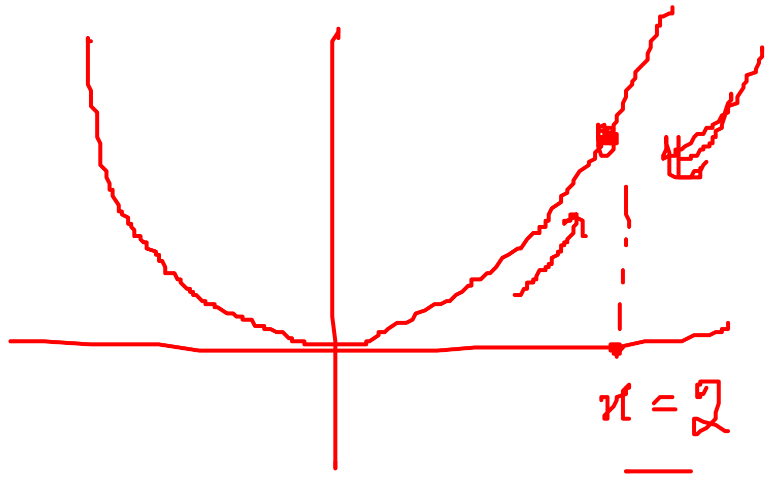
Limits

If LHL = RHL
limit exists.

$f(x) = \begin{cases} x^2, & x \neq 2 \\ 3, & x = 2 \end{cases}$

$f(x) = x^2$



LHL = $\lim_{x \to 2^-} f(x) = 4$

$x = 2$

$f(2) = 2^2 = 4$

$x = 1.9$	$f(x) = 3.61$
$x = 1.99$	$f(x) = 3.9601$
$x = 1.999$	$f(x) = 3.9960$

$x = 2.1$	$f(x) = 4.41$
$x = 2.01$	$f(x) = 4.0401$
$x = 2.001$	$f(x) = 4.004001$

RHL = $\lim_{x \to 2^+} f(x) = 4$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ 2x, & x < 0 \end{cases}$$

$$\lim_{x \to 0}$$

$$LHL = \lim_{x \to 0^-} 2x = 2 \times 0 = 0$$

$$RHL = \lim_{x \to 0^+} x^2 = 0^2 = 0$$

$$LHL = RHL.$$

limit exist

$$\lim_{x \to 0} f(x) = 0$$

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ x+1, & x < 0 \end{cases}$$

$$LHL = \lim_{x \to 0} x+1 = 1$$

$$RHL = \lim_{x \to 0^+} x^2 = 0.$$

$$LHL \neq RHL.$$

limit doesn't exists.

Q $\lim_{x \to \cancel{\pi}} \sin x = 0$

Q $\lim_{x \to 0} |x| = 0$

L'Hospital Rule –

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 1^{\infty},\ 0^{\infty},\ \infty^{0},\ 0^{0},\ \infty^{\infty},$$ indeterminate forms

$$0^0 \to \cancel{0} \to 1$$

$$\to \lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}$$

$$\to \lim_{x\to 2} \frac{2x}{1} = 2\times 2 = 4$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 2+2 = 4.$$

$$\lim_{x\to 0} \frac{\sin x}{x} \ \text{---}\ \left(\frac{0}{0}\right) \text{ form.}$$

$$= \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^5}{5!} + \dots$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \cos x = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
- $\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$
- $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
- $\lim_{x\to\infty} (1+\frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to 0} \frac{(a^x-1)}{x} = \ln a$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$

e^x

$\lim_{x\to 0} \frac{\sin^2 x}{x^2}$ --- $\left(\frac{0}{0}\right)$

$\longrightarrow \lim_{x\to 0} \frac{\cancel{2}\sin x \cos x}{\cancel{2}x}$ --- $\left(\frac{0}{0}\right)$ $= \lim_{x\to 0} \frac{\sin x(-\sin x) + \cos x(\cos x)}{1} = 1$

GATE-CS-2016 (Set 1)

$$\lim_{x \to 4} \frac{\sin(x-4)}{x-4} = \underline{\hspace{4cm}}.$$

$- - \left(\frac{0}{0}\right)$ form.

Answer: 1

$\lim_{x \to 4} \frac{\cos(x-4)}{1} = \frac{1}{-}.$

Example – Evaluate $\lim_{x\to 0} \frac{x\cos(x)-\sin(x)}{x^2\sin(x)}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^2\left(x - \frac{x^3}{3!} + \cdots\right)}$$

$$\lim_{x\to 0} \frac{-x\sin x + \cos x - \cancel{\cos x}}{x^2\cos x + 2x\sin x}$$

$$\lim_{x\to 0} \frac{-x\sin x}{x^2\cos x + 2x\sin x}$$

$$\lim_{x\to 0} \frac{-\cancel{x}\sin x}{\cancel{x}(x\cos x + 2\sin x)} \quad - \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{-\cos x}{\cos x + x(-\sin x) + 2\cos x}$$

$$-\frac{1}{3}$$

GATE-CS-2015 (Set 3)

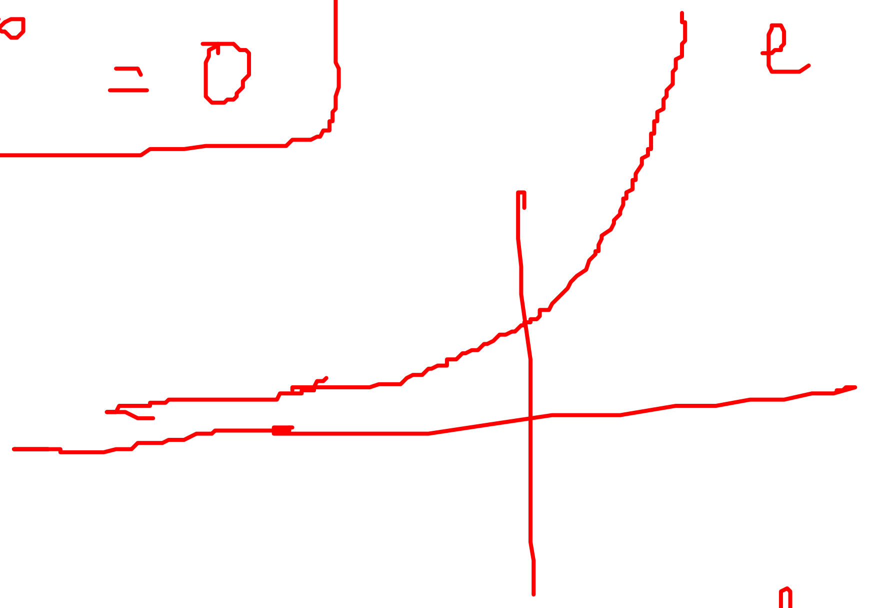
The value of $\lim_{x \to \infty} (1 + x^2) e^{-x}$ is

(A) 0
(B) 1/2
(C) 1
(D) ∞

$\frac{1}{0} = \infty$

$\boxed{\infty \cdot 0} = \infty \cdot \frac{1}{\infty} = \frac{\infty}{\infty}$

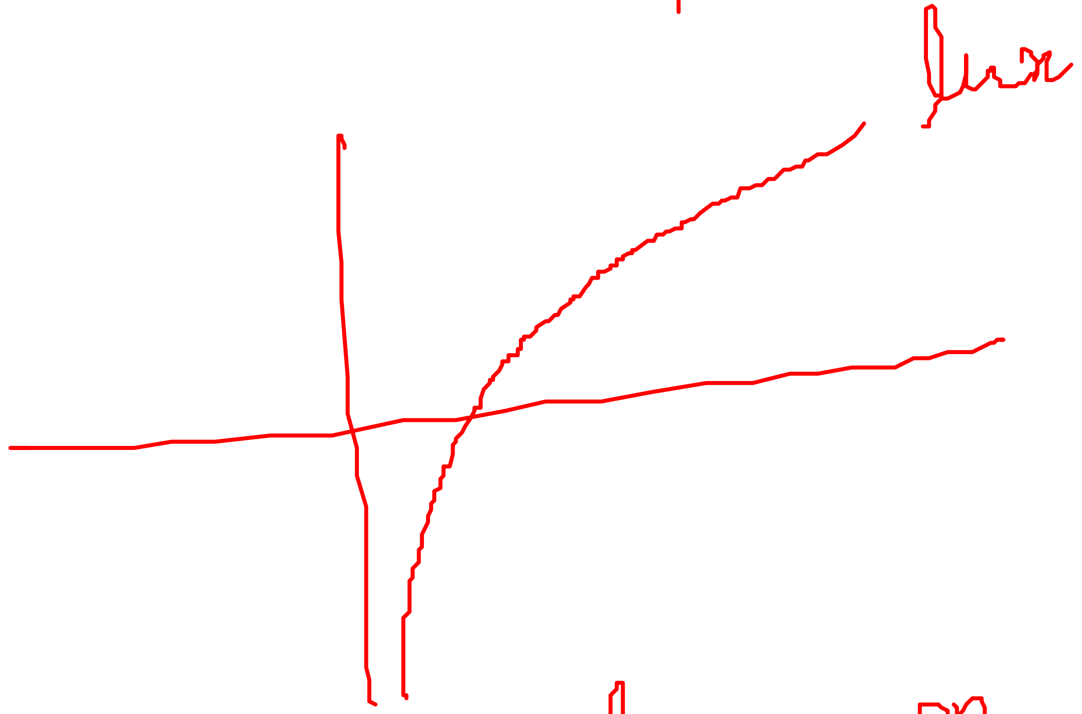
$\boxed{e^{\infty} = \infty \quad e^{-\infty} = 0}$



$\lim_{x \to \infty} \frac{1 + x^2}{e^x}$ --- $\left(\frac{\infty}{\infty}\right)$

Answer: (A)

$\lim_{x \to \infty} \frac{2x}{e^x}$ --- $\left(\frac{\infty}{\infty}\right)$



$\lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

$\ln \infty = \infty$

$\ln 0 = -\infty$

GATE-CS-2015 (Set 1)

$\ln a^b = b \ln a$

$\lim_{x\to\infty} \boxed{x^{1/x}}$ *is*

(A) ∞

(B) 0

(C) 1

(D) Not Defined

∞^0

$\lim_{x\to\infty} y$

$y = x^{1/x}$.

Taking ln on both sides

$\ln y = \frac{1}{x} \ln x$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x}$ ---- $\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{1/x}{1}$

$\lim_{x\to\infty} \ln y = 0$

$\lim_{x\to\infty} y = e^0 = 1$.

Answer: (C)

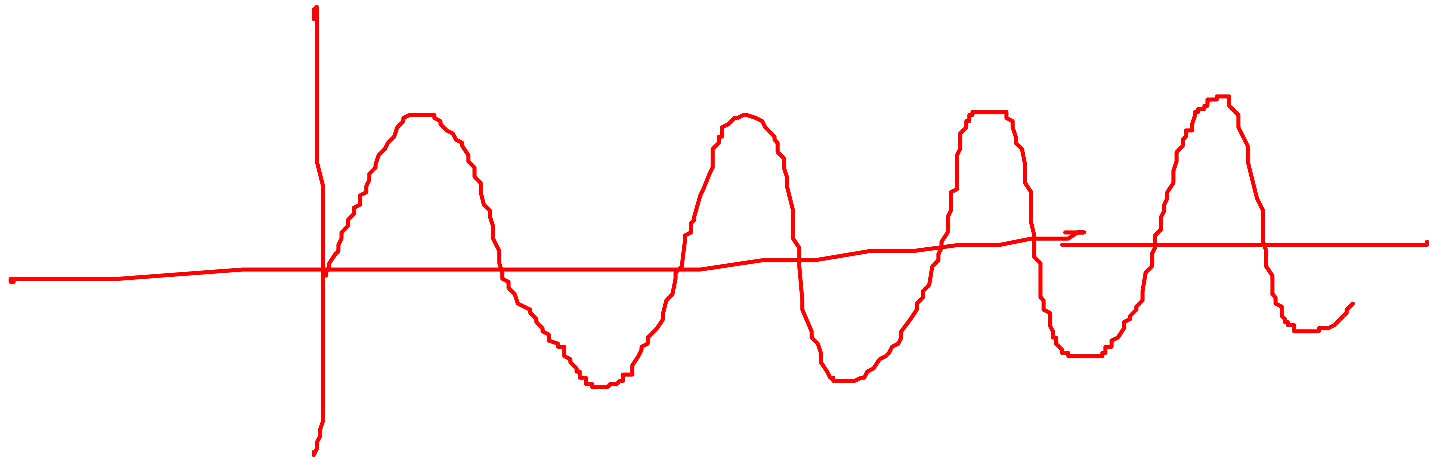
GATE CS 2008

$\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x}$ equals

$- - - \left(\frac{\infty}{\infty}\right)$

(A) 1
(B) -1
(C) INF
(D) -INF

$\sin x$



$\cancel{\lim_{x \to \infty} \frac{1 - \cos x}{1 - \sin x}}$

$\lim_{x \to \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$

$\lim_{x \to \infty} \frac{\sin x}{x} = 0$

$\lim_{x \to \infty} \frac{\cos x}{x} = 0$

Answer: (A)

GATE CS 2010

What is the value of $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$?

(A) 0

(B) e^{-2}

(C) $e^{-1/2}$

(D) 1

Answer: (B)

$$\lim_{x\to\infty}\left(1+f(x)\right)^{g(x)} = e^{\lim_{x\to\infty} f(x)\cdot g(x)}$$

$$= e^{\lim_{x\to\infty}\left(-\frac{1}{x}\right)2x}$$

$$= e^{-2}$$

$$y = \left(1-\frac{1}{n}\right)^{2n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} 2n \ln\left(1-\frac{1}{n}\right) \quad ---- \quad \infty \cdot 0$$

$$= 2\lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n}\right)}{1/n} \quad --- \quad \left(\frac{0}{0}\right)$$

$$= 2\lim_{n\to\infty} \frac{\frac{1}{\left(1-\frac{1}{n}\right)}\left(\frac{1}{n^2}\right)}{-1/n^2} = -2$$

$$\lim_{n\to\infty} y = e^{-2}$$

GATE CSE 2021 Set 1 | Question: 20

Consider the following expression.

$$\lim_{x \to -3} \frac{\sqrt{2x+22}-4}{x+3} \quad \text{---} \left(\frac{0}{0}\right)$$

The value of the above expression (rounded to 2 decimal places) is ____________.

$$\lim_{x \to -3} \frac{1}{\cancel{2}\sqrt{2x+22}} \times \cancel{2} = \frac{1}{4} = 0.25$$

Answer : 0.25

$$\lim_{x \to -3} \frac{\sqrt{2x+22}-4}{x+3} \times \frac{\sqrt{2x+22}+4}{\sqrt{2x+22}+4} = \frac{2x+22-16}{(x+3)\left(\sqrt{2x+22}+4\right)}$$

$$= \frac{2\cancel{(x+3)}}{\cancel{(x+3)}\left(\sqrt{2x+22}+4\right)} = \frac{2}{8} = \frac{1}{4}.$$

GATE CS Mock 2018

Find the value of

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

(A) 1/3
(B) -1/6
(C) 1/2
(D) None of these

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} \quad \text{----} \left(\frac{0}{0}\right)$$

Answer: (A)

$$\lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$\lim_{x \to 0} \frac{2}{6} \times 1 \left(\frac{\sec^2 x}{1}\right) = \frac{1}{3}$$

GATE CSE 2003

$$\lim_{x \to 0} \frac{\sin^2 x}{x}$$

(a) 0
(b) inf
(c) 1
(d) -1

Answer (a)

GATE CS Mock 2018

What is the value of the limit –

$$\lim_{x \to 0} \frac{a^{mx} - b^{mx}}{\sin(kx)}$$

$\left(\frac{0}{0}\right)$

(A) $\frac{1}{k} \ln \frac{a^m}{b^m}$

(B) $\frac{1}{k} \ln \frac{b^m}{a^m}$

(C) $\frac{2}{k} \ln \frac{a^m}{b^m}$

(D) $\frac{2}{k} \ln \frac{b^m}{a^m}$

Answer: (A)

$$= \lim_{x \to 0} \frac{m a^{mx} \ln a - m b^{mx} \ln b}{k \cos(kx)}$$

$$= \frac{m \ln a - m \ln b}{k}$$

$$= \frac{\ln a^m - \ln b^m}{k}$$

$$= \frac{1}{k} \ln\left(\frac{a^m}{b^m}\right) = \ln\left(\frac{a^m}{b^m}\right)^{1/k} = \ln\left[\left(\frac{a}{b}\right)^{m/k}\right]$$

Example – Evaluate $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$

$$\times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$\infty - \infty \neq 0$$

$$\lim_{x \to \infty} \sqrt{x^2+1} = \infty$$

$$\lim_{x \to \infty} \frac{\cancel{x^2}+1 - \cancel{x^2}}{\sqrt{x^2+1} + x}$$

$$\frac{1}{\infty} = 0.$$

$$\lim_{x \to \infty} x^2 = \infty$$

$$\lim_{x \to 0} \frac{1}{x} = \infty$$

GATE CSE 1995

$$\lim_{x \to \infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2} = \_\_\_\_\_\_\_.$$

$$\lim_{x \to \infty} \frac{x - \frac{\cos x}{x^2}}{1 + \frac{(\sin x)^2}{x^2}} = \frac{\infty - 0}{1 + 0} = \infty$$

Answer: Infinity

Continuity

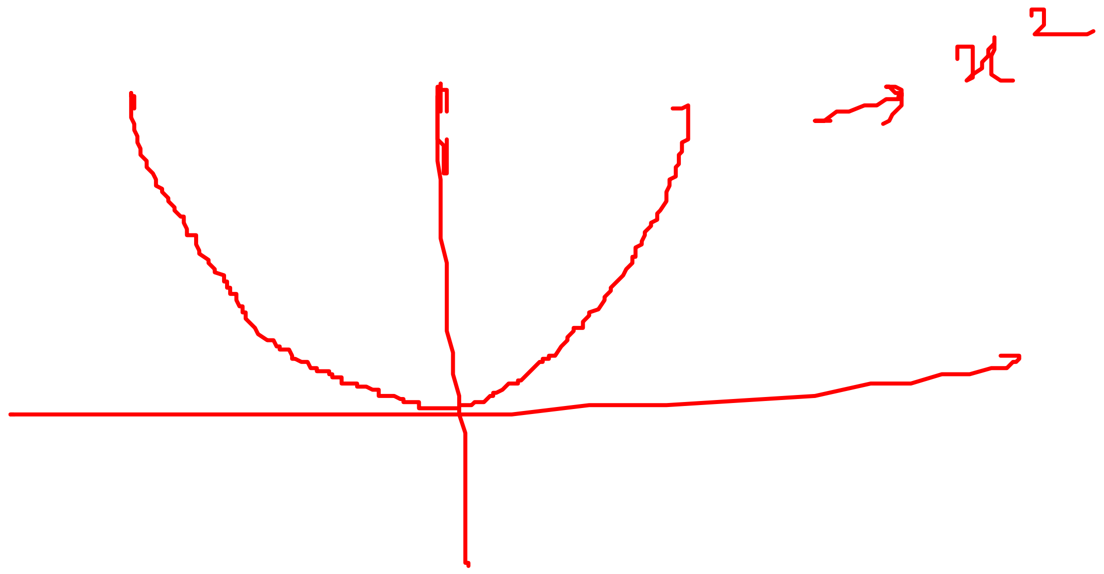
A function is said to be continuous over a range if it's graph is a single unbroken curve.

$x = a$

$LHL = \lim_{x \to a^-} f(x)$

$RHL = \lim_{x \to a^+} f(x)$

$LHL = RHL = f(a)$



$$f(x) = \begin{cases} x^2 + 1, & x \neq 1 \\ 5, & x = 1 \end{cases}$$

$LHL = 2$, $RHL = 2$

$f(1) = 5$

Formally,

A real valued function f(x) is said to be continuous at a point x = x_0 in the domain if – $\lim_{x \to x_0} f(x)$ exists and is equal to f(x_0).

If a function f(x) is continuous at x = x_0 then-

$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$ = f(x_0)

Functions that are not continuous are said to be discontinuous.

Example –

$$f(x) = \begin{cases} 0; & x = 0 \\ \frac{1}{2} - x; & 0 < x < \frac{1}{2} \\ \frac{1}{2}; & x = \frac{1}{2} \\ \frac{3}{2} - x; & \frac{1}{2} < x < 1 \\ 1; & x \geq 1 \end{cases}$$

At $x = 1/2$

$f(\frac{1}{2}) = 1/2$

$LHL = \lim_{x \to 1/2^-} \frac{1}{2} - x = \cancel{0}$

$RHL = \lim_{x \to \frac{1}{2}^+} \frac{3}{2} - x = \frac{3}{2} - \frac{1}{2} = 1$

At $x = 0$

$RHL \neq f(0)$

Let x be a real number.

The **floor function** of x, denoted by $\lfloor x \rfloor$ or floor(x), is defined to be the greatest integer that is less than or equal to x.

The **ceiling function** of x, denoted by $\lceil x \rceil$ or ceil(x), is defined to be the least integer that is greater than or equal to x.

For example,

$\lfloor \pi \rfloor = 3$, $\lceil \pi \rceil = 4$, $\lfloor 5 \rfloor = 5$, $\lceil 5 \rceil = 5$.

$\lfloor -e \rfloor = -3$, $\lceil -e \rceil = -2$, $\lfloor -1 \rfloor = -1$, $\lceil -1 \rceil = -1$.

6

$\lfloor 5 \rfloor = 5$

$5 \le x < 6$

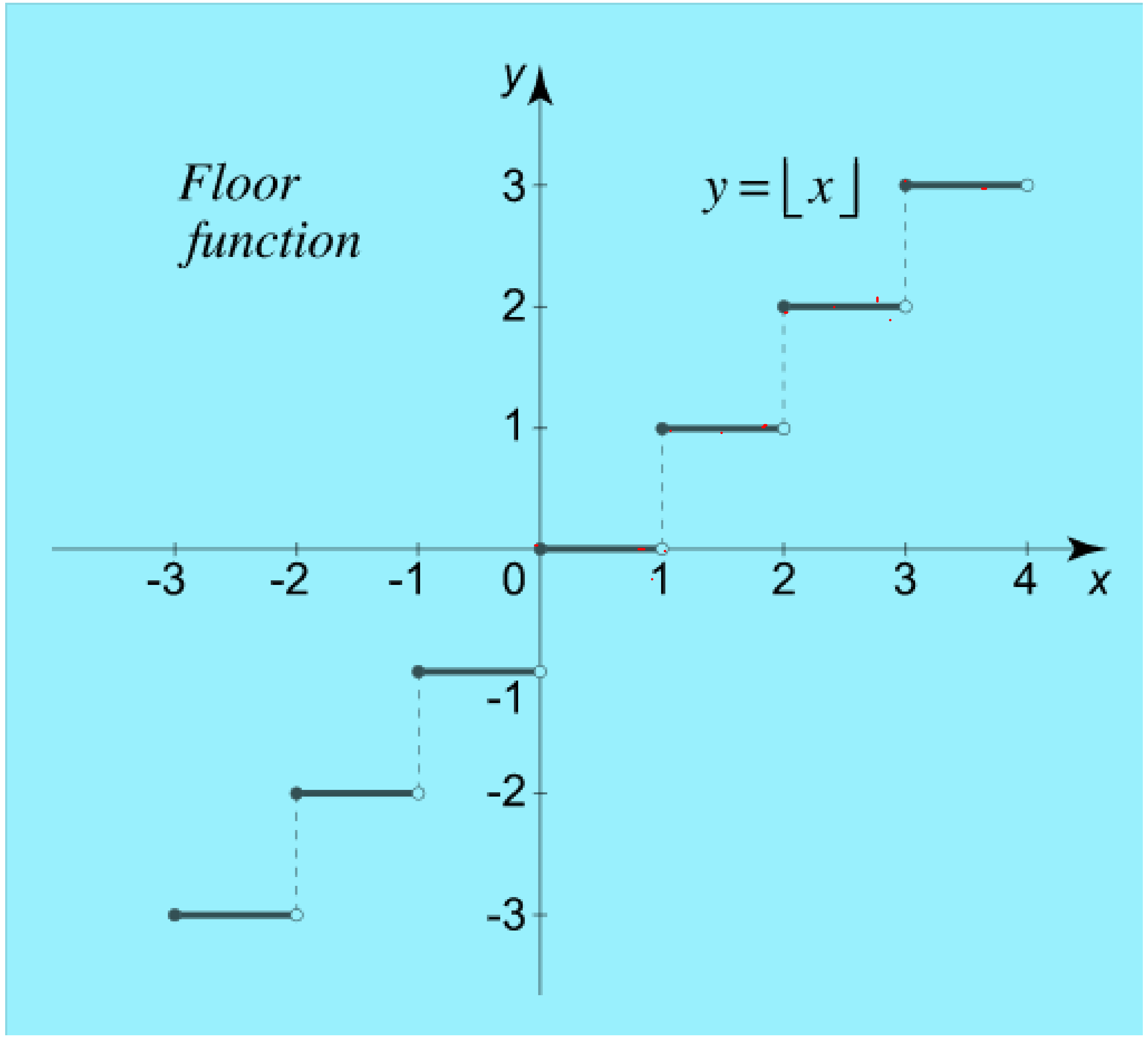
$\lfloor 5.1 \rfloor = 5$

5

$6 \le x < 7$

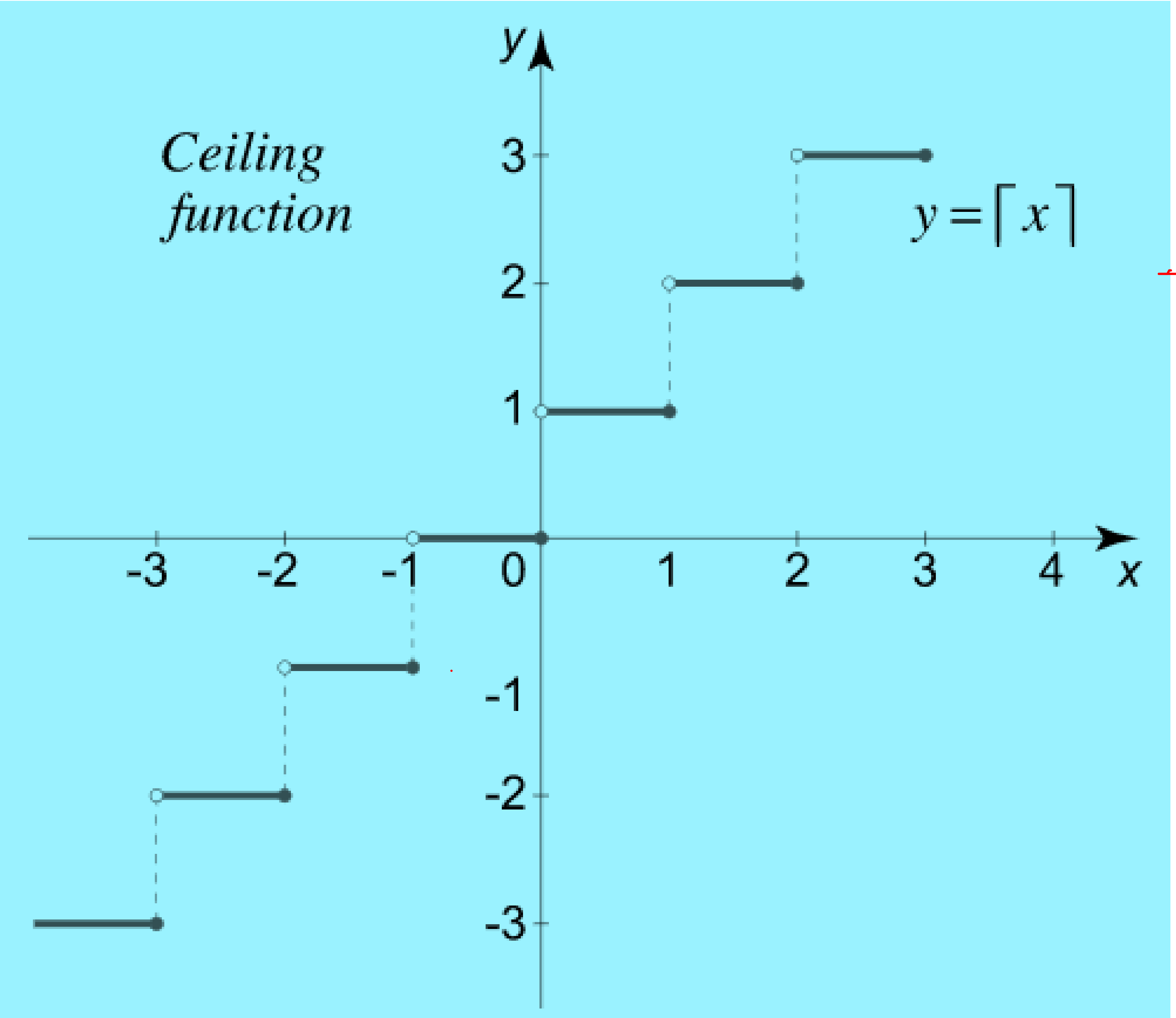
$\lfloor 5.2 \rfloor = 5$

6



Cont ?

floor f^n is continuous on real numbers except integers


Ceiling function
$y = \lceil x \rceil$
y
x
3
2
1
0
-1
-2
-3
1
2
3
4

Example – For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2), & \text{if } x \leq 0 \\ 4x+1, & \text{otherwise} \end{cases}$$

continuous at $x = 0$?

LHL = RHL.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0^-} \lambda(x^2 - 2) = \lim_{x \to 0^+} 4x + 1$$

$$\lambda(0 - 2) = 0 + 1$$

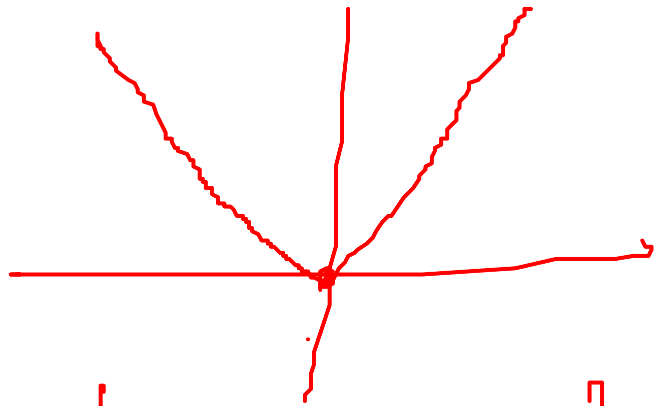
$$\boxed{\lambda = -\frac{1}{2}}$$

Example – Find all points of discontinuity of the function f(x) defined by –

$$f(x) = |x| - |x - 1|$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|x| - |x-1| = \begin{cases} -x - (1-x) & x < 0 \\ x - (1-x) & 0 \leq x \leq 1 \\ x - (x-1) & x > 1 \end{cases}$$



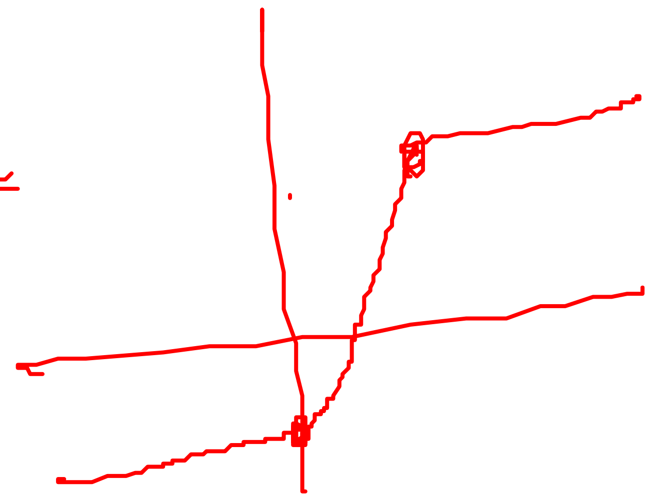
$$|x-1| = \begin{cases} 1-x & x < 1 \\ x-1 & x \geq 1 \end{cases}$$

$$= \begin{cases} -1 & x < 0 \\ 2x - 1 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$\lim_{x \to 0^-} |x| = 0$$

$$\lim_{x \to 0^+} |x| = 0$$

$$f(0) = 0$$



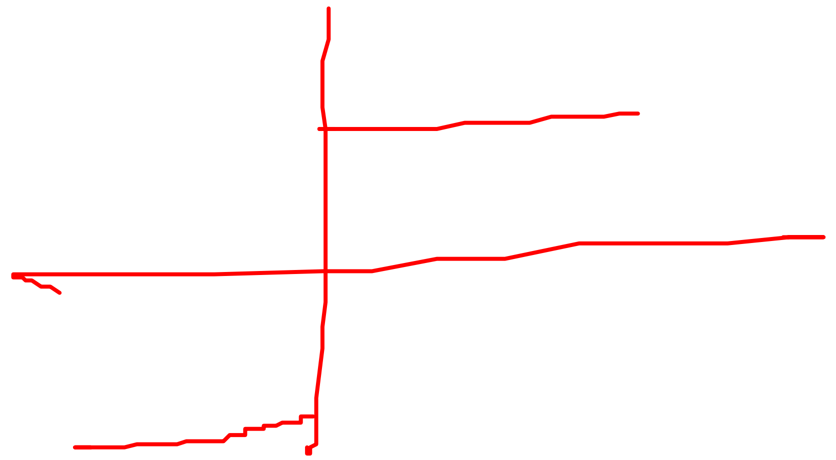
Example $- f(x) = \frac{|x|}{x}$

f(x) is not

cont. at x = 0

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



Example – $f(x) = \frac{x^2-3x-4}{x^2+3x-4}$ is not continuous at x = ?

$(x-4)(x+1)$

$LHL = RHL = f(a)$

$x^2+3x-4=0$

$x=-4, +1$

$f(x) \to \infty$ $\quad$ $x = a$

not cont at $x = a$

$f(x) = \frac{(x+4)}{x^2+3x-4}$

$\frac{1}{x-4}$

$= \frac{\cancel{x+4}}{\cancel{(x+4)}(x-1)} = \frac{1}{x-1}$

$\rightarrow x = 1$

GATE CS 2013

Which one of the following functions is continuous at x = 3?

(A) $f(x)=\begin{cases} 2, & \text{if} \quad x=3 \longrightarrow 2 \\ x-1, & \text{if} \quad x>3 \longrightarrow 2 \\ \dfrac{x+3}{3}, & \text{if} \quad x<3 \longrightarrow 2 \end{cases}$

(B) $f(x)=\begin{cases} 4, & \text{if} \quad x=3 \longrightarrow 4 \\ 8-x & \text{if} \quad x\neq 3 \quad 5 \end{cases}$

(C) $f(x)=\begin{cases} x+3, & \text{if} \quad x\leq 3 \longrightarrow 6 \\ x-4 & \text{if} \quad x>3 \longrightarrow -1 \end{cases}$

(D) $f(x)=\dfrac{1}{x^3-27}, \quad \text{if} \quad x \neq 3$

Answer: (A)

Example: What should be the value of λ such that the function defined below is continuous at $x = \frac{\pi}{2}$?

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\frac{\pi}{2} - x}, & if \;\; x \neq \dfrac{\pi}{2} \\ 1, & if \;\; x = \dfrac{\pi}{2} \end{cases}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\lambda \cos x}{-x + \pi/2} = 1$$

$$\lim_{x \to \pi/2} \frac{\lambda(-\sin x)}{-1} = 1$$

$$\lambda = 1$$

(A) 0

(B) $\frac{2}{\pi}$

(C) 1

(D) $\frac{\pi}{2}$

Answer : (C)

GATE-CS-2015 (Set 1)

If $g(x) = 1 - x$ and $h(x) = \dfrac{x}{x-1}$, then $\dfrac{g(h(x))}{h(g(x))}$ is

$$= \frac{g\left(\frac{x}{x-1}\right)}{h(1-x)} = \frac{1 - \frac{x}{x-1}}{\frac{1-x}{1-x-1}} = \frac{\frac{x-1-x}{x-1}}{\frac{1-x}{-x}} = \frac{x}{(x-1)(1-x)}$$

(A) $\dfrac{h(x)}{g(x)} = \dfrac{x}{(x-1)(1-x)}$

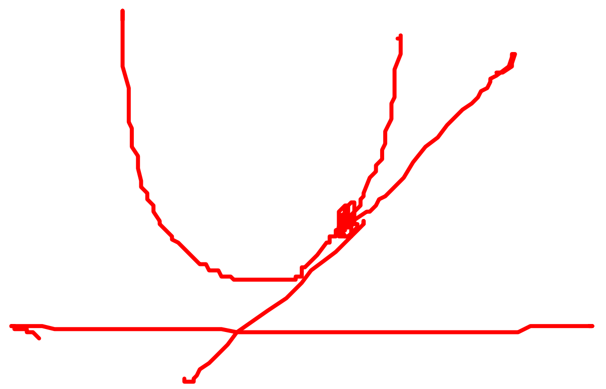
(B) $\dfrac{-1}{x}$

(C) $\dfrac{g(x)}{h(x)}$

(D) $\dfrac{x}{(1-x)^2}$

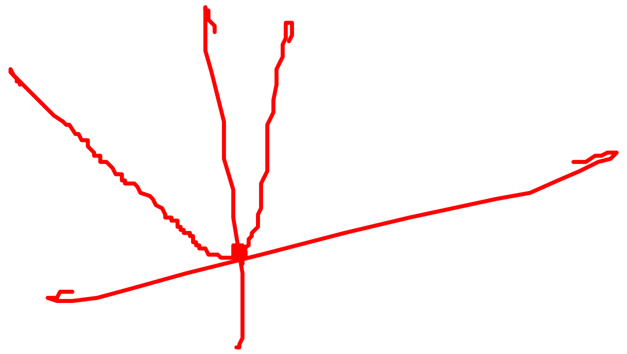
Answer: (A)

Differentiability



$x^2 \longrightarrow 2x$

A function f(x) is differentiable at the point x = a if the following limit exists.



$$\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \longrightarrow \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$\lim_{x \to a} \frac{(x-a)(x+a)}{(x-a)} = 2a$$

$$LHD = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \cancel{\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}}$$

$$\text{R}HD = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} = \cancel{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

$$f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$LHD = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x} = -1$$

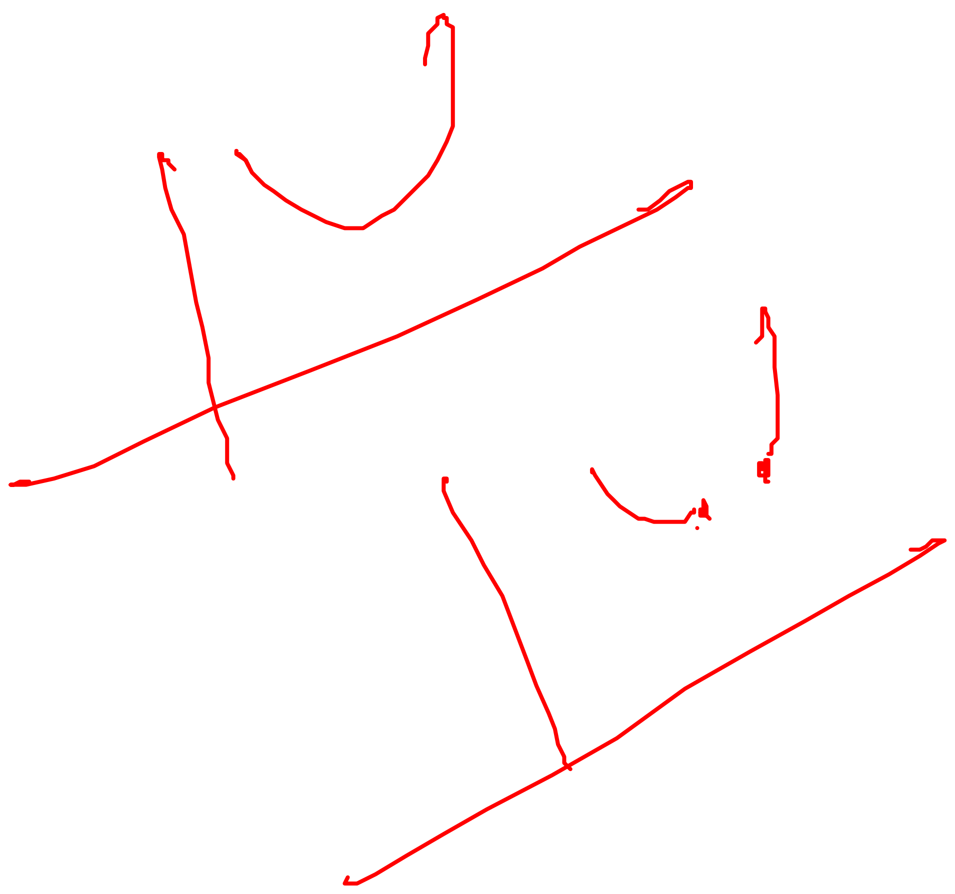
$$RHD = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x} = 1$$

→ Derivative doesn't exists at $x=0$

Example: $f(x) = x^2$

Example: $$g(x) = \begin{cases} x+1, & x \le 1 \\ 3x-1, & x > 1 \end{cases}$$

$$g'(x) = \begin{cases} 1, & x \le 1 \\ 3, & x > 1 \end{cases}$$



$$f(x) = \begin{cases} 2a+bx, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

f is cont

$2a+b = 3$

$a = 0$

$f \to$ diff

$$f'(x) = \begin{cases} b, & x \le 1 \\ 3, & x > 1 \end{cases}$$

$b = 3$

Differentiable

A function is said to be differentiable if the derivative of the function exists at all points in its domain.

Particularly, if a function f(x) is differentiable at x=a, then f′(a) exists in the domain.

Example : f(x) = $\mathbf{x^2 + 6x}$

When not stated we assume that the domain is the Real Numbers.

For $\mathbf{x^2 + 6x}$, its derivative of $\mathbf{2x + 6}$ exists for all Real Numbers.

$x^2 + 6x$ is differentiable.

What Is the Difference Between Differentiable and Continuous Function?

We say that a function is continuous at a point if its graph is unbroken at that point.

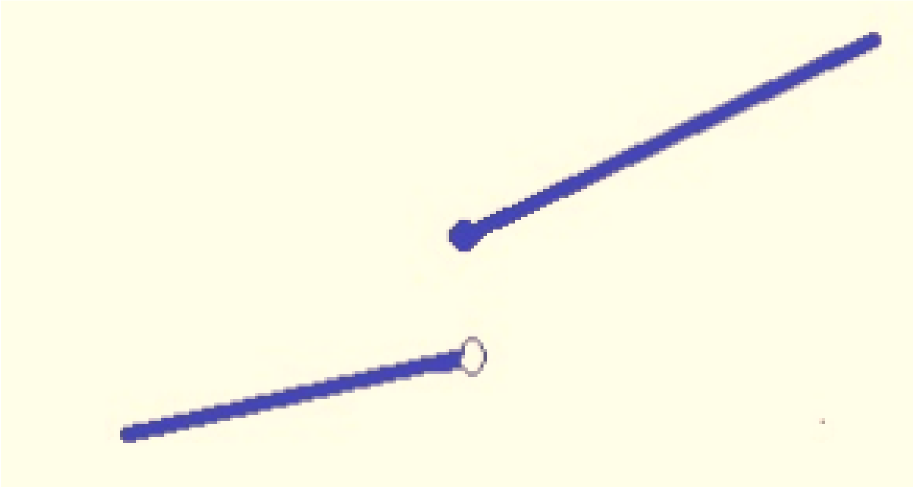
A differentiable function is always a continuous function but a continuous function is not necessarily differentiable.

Example

The absolute function is continuous at $x=0$ but not differentiable at $x = 0$.

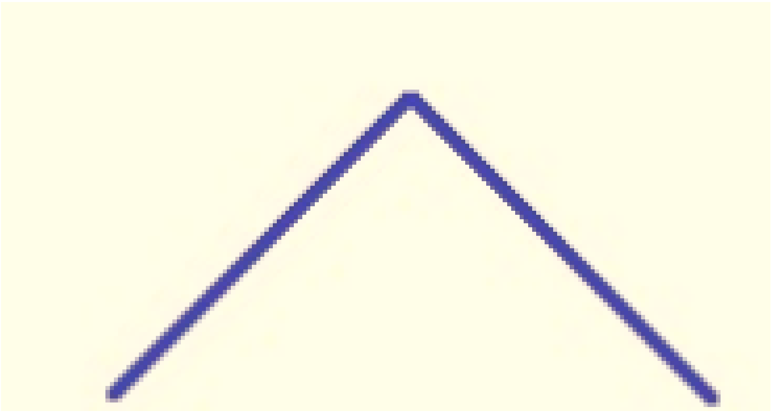
A function can fail to be differentiable at point if:

1. The function is not continuous at the point.



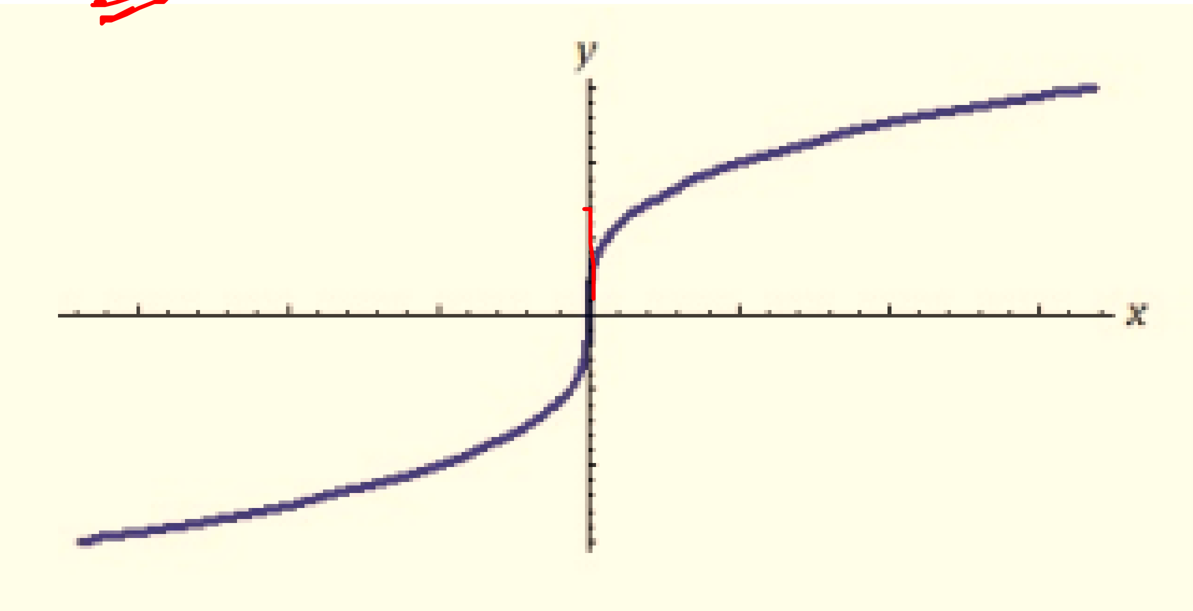
How can we make a tangent line here?

2. The graph has a sharp corner at the point.



3. The graph has a vertical line at the point.

Example : g(x) = $x^{1/3}$



The graph is smooth at x = 0, but does appear to have a vertical tangent.

$g(x) = x^{1/3}$

$g'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

At $x = 0$

GATE-CS-2007

Consider the following two statements about the function $f(x)=|x|$

P. f(x) is continuous for all real values of x

Q. f(x) is differentiable for all real values of x

Which of the following is TRUE?

(A) P is true and Q is false.

(B) P is false and Q is true.

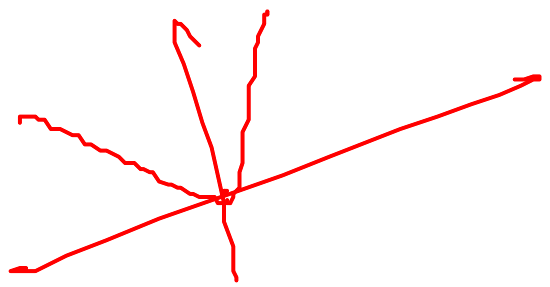
(C) Both P and Q are true

(D) Both P and Q are false.

Answer: (A)

Rolle’s Mean Value Theorem

$f \rightarrow$ cont. on $[a, b]$
diff. on (a, b)
$f(a) = f(b)$



Suppose f(x) be a function satisfying three conditions:

1) f(x) is continuous in the closed interval $a \leq x \leq b$
2) f(x) is differentiable in the open interval $a < x < b$
3) $f(a) = f(b)$

Then according to Rolle’s Theorem, there exists **at least one** point ‘c’ in the open interval (a, b) such that:

$$f'(c) = 0$$

$\exists\, c \in (a, b)$ s.t $f'(c) = 0$

Question : $f(x) = \frac{\sin x}{e^{x}}$ in $[0, \pi]$

$= e^{-x} \sin x$

$f(0) = 0$, $f(\pi) = 0$

Find (i) 'c' by RMVT → $c = \pi/4$

(ii) Point at which slope of tangent to function f(x) is zero.

(iii) Point at which tangent is parallel to x-axis

$c = \pi/4$

$f'(c) = 0$

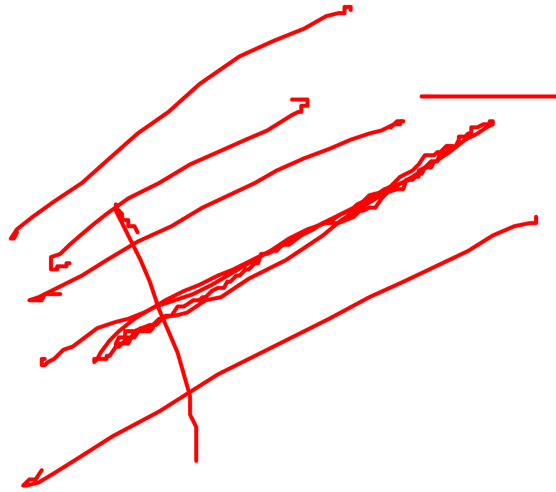
$-e^{-c} \sin c + e^{-c} \cos c = 0$

$e^{-c}(\cos c - \sin c) = 0$

$\cos c = \sin c$

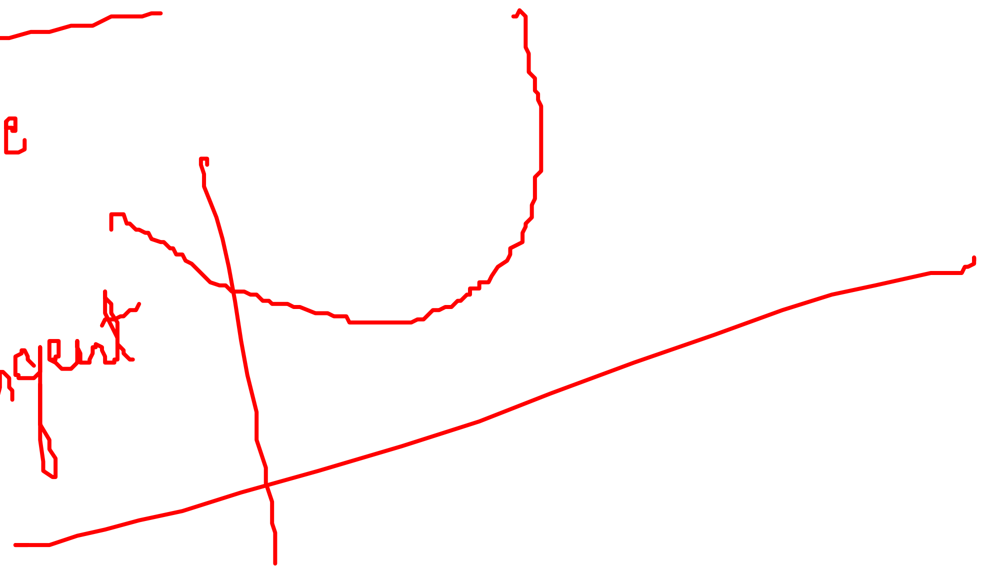
$c = \frac{\pi}{4}$

$f'(c) = 0$



$0 = f'(c) =$ derivative value

$0 =$ slope of tangent



Question : The mean value ‘c’ for the function

$$f(x) = e^x(sinx - \cos x) \ \ in \ \ \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

$(a)0$ $(b)\frac{\pi}{3}$ $(c)\frac{\pi}{2}$ $(d)\pi$

Lagrange's Mean Value Theorem

$f \rightarrow$ cont on $[a,b]$
& diff on (a,b)
$\exists\, c \in (a,b)$ s.t. $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}}$

STATEMENT:

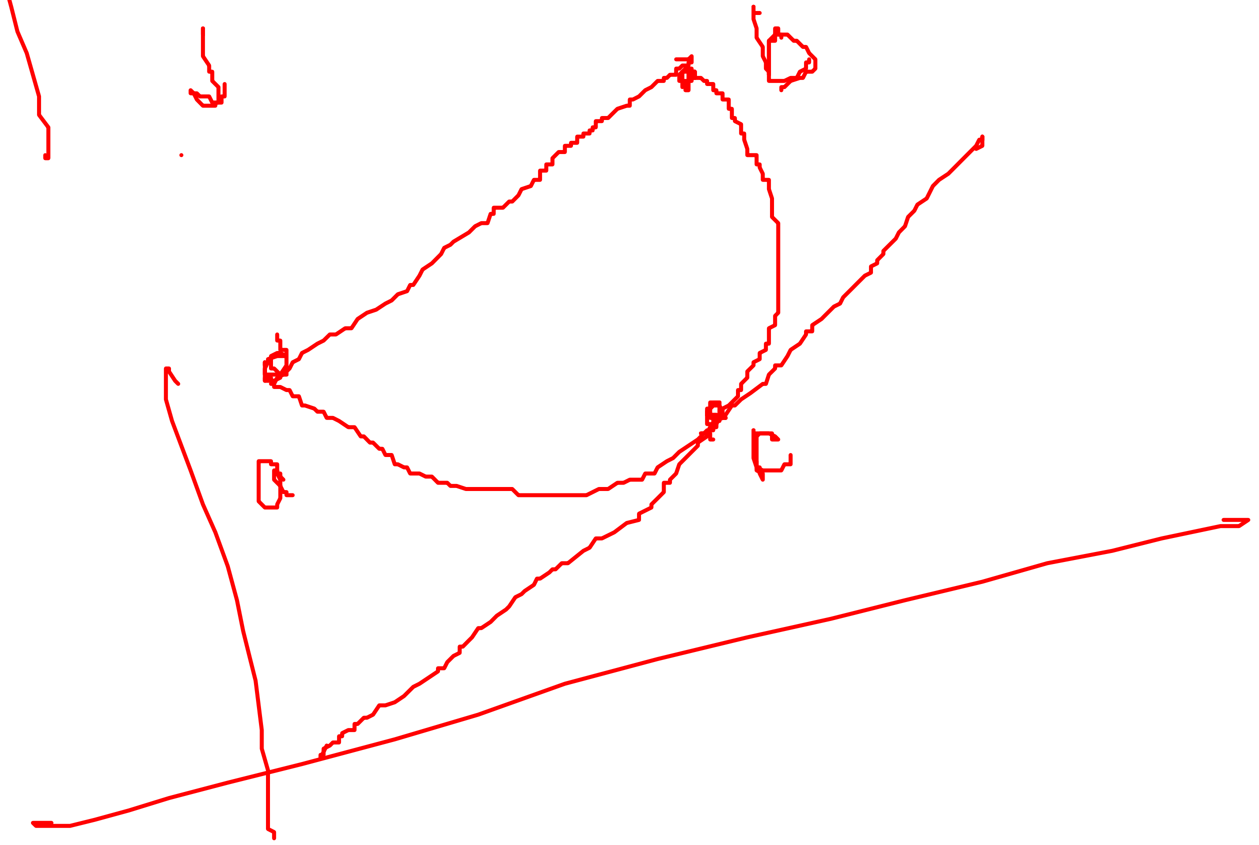
Suppose $f:[a,b] \rightarrow \mathbb{R}$ be a function satisfying these conditions:
1)f(x) is continuous in the closed interval $a \leq x \leq b$
2) f(x) is differentiable in the open interval $a < x < b$

Then according to Lagrange's Theorem, there exists **at least one** point 'c' in the open interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Verify mean value theorem for $f(x) = x^2$ in interval [2,4].

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



cont & diff

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2c = \frac{f(4)-f(2)}{4-2}$$

$$2c = \frac{16-4}{2}$$

$$c = 3$$

Question : $f(x) = (1 + x)\ln(1 + x) \;\; in \quad [0,1]$

Find (i) ‘c’ by LMVT

(ii) Point at which slope of tangent to function f(x) is parallel to line joining initial and final point.

$\ln x, \; x > 0$

$f'(x) = \ln(1+x) + 1$

Logarithmic
exponential
Polynomial
$\sin x, \cos x$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\ln(1+c) + 1 = \frac{f(1) - f(0)}{1 - 0}$$

$c \in$

$$\ln(1+c) + 1 = \frac{2\ln 2 - 0}{1}$$

$$\ln(1+c) = 2\ln 2 - 1$$

GATE 2017 Mock

The value of the constant 'C' using Lagrange's mean value theorem for $f(x) = 8x - x^2$ in [0,8] is:

(A) 4

(B) 8

(C) 0

(D) None of these

Answer: (A)

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$8 - 2c = \frac{f(8) - f(0)}{8 - 0}$$

$$8 - 2c = \frac{0 - 0}{8}$$

$$8 - 2c = 0$$

$$C = 4$$

Cauchy's Mean Value Theorem :

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$g'(c) = \frac{g(b) - g(a)}{b - a}$

Suppose f(x) and g(x) are 2 functions satisfying three conditions:

1) f(x), g(x) are continuous in the closed interval $a \leq x \leq b$
2) f(x), g(x) are differentiable in the open interval $a < x < b$ and
3) $g'(x) \neq 0$ for all x belongs to the open interval $a < x < b$

Then according to **Cauchy's Mean Value Theorem** there exists a point c in the open interval $a < c < b$ such that:

$\exists c \in (a, b)$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Thank you