Decidability & Undecidability:

Decidability :- A problem is said to be decidable iff ∃ an Algo. to solve that problem.

↳ Algo. or HTM or Rec

Undecidability: A problem is said to be Undecidable if ∄ a Procedure to solve that problem.

Any Algo. is a Procedure which always stops in a finite Amount of time

Procedure, TM, or REL

Example: Which of the following is/are undecidable?

1) G is CFG, is $L(G) = \varphi$? – D
2) G is a CFG, is $L(G) = \Sigma^*$? – U
3) M is a TM, is L(M) regular? – U
4) A is a DFA and N is an NFA is $L(A) = L(N)$? D

a) 3 only b) 3 and 4 only

c) 1, 2 and 3 only d) 2 and 3 only

Question	RL	DCFL	CFL	CSL	Rec	REL
is ω in L? (Membership)	D	D	D	D	D	U
is $L = \varphi$? (Emptiness Prob.)	D	D	D	U	U	U
is $L = \Sigma^*$? (Completeness)	D	D	U	U	U	U
is $L_1 = L_2$? (Equality Prob.)	D	D	U	U	U	U
is $L_1 \subseteq L_2$? (Subset Prob.)	D	U	U	U	U	U
is L finite or not (Finiteness Prob.)	D	D	D	U	U	U
is $L_1 \cap L_2 = \varphi$? (Disjointness Prob.)	D	U	U	U	U	U
is L = R, where R is a given Regular set?	D	D	U	U	U	U
is L regular = ? (Regularity Prob.)	D	D	U	U	U	U

Example: Which of the following is/are decidable?

1) Whether the intersection of two regular languages is infinite. D
2) Whether a given context free language is regular. U
3) Whether two push down automata accept the same language. U
4) Whether a given grammar is context free. D

a) 1 and 2 only   b) 1 and 4 only

c) 2 and 3 only   d) 2 and 4 only

U → (UUT)*

Question	RL	DCFL	CFL	CSL	Rec	REL
is ω in L? (Membership)	D	D	D	D	D	U
is L = φ? (Emptiness Prob.)	D	D	D	U	U	U
is L = ∑*? (Completeness)	D	D	U	U	U	U
is $L_1 = L_2$? (Equality Prob.)	D	D	U	U	U	U
is $L_1 \subseteq L_2$? (Subset Prob.)	D	U	U	U	U	U
is L finite or not (Finiteness Prob.)	D	D	D	U	U	U
is $L_1 \cap L_2 = \varphi$? (Disjointness Prob.)	D	U	U	U	U	U
is L = R, where R is a given Regular set?	D	D	U	U	U	U
is L regular = ? (Regularity Prob.)	D	D	U	U	U	U

Undecidable Problem on TM:

-Halting Problem

-Blank tape halting Problem

-State entry problem

-Post correspondence problem(PCP)

-Modified PCP(MPCP)

-Membership problem on RE

w

M

q

Post Correspondance Problem:

$$M = (x_1, x_2, x_3 \ldots x_n)$$
$$N = (y_1, y_2, y_3 \ldots y_n)$$

$\langle l_1, l_2, \ldots l_k \rangle$ $1 \le l_j \le n$

pcp sol^n

$$\boxed{x_{l_1} \ldots x_{l_k} = y_{l_1} \ldots y_{l_k}}$$

$\langle l_1, l_2, \ldots l_k \rangle$

pcp sol^n

Ex:

	W	X
1.	bbab	a
2.	ab	abbb
3.	baa	aa
4	b	bbb

PCP Sol^n: $\langle 2\ 1\ 4\ 3 \rangle$

List w: ab bbab b baa

List x: abbb a bbb aa

	W	V
1.	a	baa
2.	ab	aa
3.	bba	bb

W : bbaabbbaa

V : bbaabbbaa

PCP solⁿ : ⟨3 2 3 1⟩

	list A	list B
1-	10	101
2-	011 ⟷	11
3	101 ⟷	011

1

list A : 10

list B : 101

PCP Soln : 1

does not exist

PCP Soln = ?

a, S

< >

Reducibility Theory

If a problem P_1 reduces to problem P_2 (i.e $P_1 \leq P_2$) then using the solution of P_2 , we can solve the problem P_1

$\boxed{P_1 \leq P_2}$ or $\boxed{P_1 \propto P_2}$

1. if P_2 is decidable then P_1 is decidable ($P_2 \rightarrow P_1$)
2. if P_1 is already proven to be undecidable then definitely P_2 must be undecidable ($\neg P_1 \rightarrow \neg P_2$)

$$\boxed{P_2 \rightarrow P_1 = \neg P_1 \rightarrow \neg P_2}$$

$P_1 \leq P_2$

(1) if P_2 is decidable $\Rightarrow$ P_1 is also decidable

(2) if P_1 is Undecidable $\Rightarrow$ P_2 is also Undecidable

Example: Consider three decision Problem P_1, P_2 and P_3. It is known that P_1 is decidable and P_2 is undecidable. Which one of the following is TRUE?

D

a) P_3 is decidable, if P_1 is reducible to P_3
b) P_3 is undecidable, if P_3 is reducible to P_2
c) P_3 is undecidable, if P_2 is reducible to P_3 Ans
d) P_3 is decidable, if P_3 is reducible to P_2's compliment

Ex:- if $L_1 \le L_2$ & $L_2 \le L_3 \Rightarrow L_1 \le L_3$

(i)- if L_1 is UD $\Rightarrow$ L_2 & L_3 are ? UD

(ii) if L_2 is UD $\Rightarrow$ L_1 & L_3 are ? L_1 not known but L_3 UD.

(iii) if L_2 is Decidable $\Rightarrow$ L_1 & L_3 are ? L_1 D, L_3

(iv) if L_3 is decidable $\Rightarrow$ L_1 & L_2 are ? D

Rice's Theorem

Any **non-trivial property** of Recursive Enumerable Language is **undecidable**. A properties as trivial if either it is not satisfied by any recursive enumerable language or if it is satisfied by all Recursive Enumerable Language otherwise it is nontrivial.

Example: Nontrivial property:

Given Turing machine M

Undecidable!

is L(M) non empty? or $L = \{\langle M\rangle \mid L(M) \neq \phi\}$

is L(M) empty? or $L = \{\langle M\rangle \mid L(M) = \phi\}$

is L(M) infinite? → $L = \{\langle M\rangle \mid L(M) \text{ is infinite}\}$

is L(M) = $\sum$*?

is L(M) regular? → $L = \{\langle M\rangle \mid L(M) \text{ regular?}\}$

Does the RE has more than 100 strings

$L = \{ \langle M \rangle \mid |L(M)| > 100? \} \rightarrow$ UD

Que.

$L = \{ \langle M \rangle \mid$ is $L(M)$ are recursive $\}$ UD

Ques.

$L = \{ \langle M \rangle$ is $L(M)$ is Rel $\}$

Que.

Qs1. Does a given TM Accepts a string of length 4

~~Rice~~

4 → UD

1

0

Qs1. Does a given TM halts in 2024 steps } → Decidable

Note:

2024

1. The property of TM can be decidable or undecidable
2. Any property of TM that refers **finiteness** is decidable

Example: L = {M | M halts in 20 steps} → **Decidable**

Example: L = {M | M halts in at most 3 steps} → **Decidable**

Fixe

(Theory of Computation)

Example - Consider the following languages.

L1={<M> | M takes at least 2016 steps on some input }, D ≥ 2016

L2={<M> | M takes at least 2016 steps on all inputs } and

L3 = {<M> | M accepts ε }, Non trivial D

where for each Turing machine M, <M> denotes a specific encoding of M. Which one of the following is TRUE?

1. L1 is recursive and L2, L3 are not recursive
2. L2 is recursive and L1, L3 are not recursive
3. L1, L2 are recursive and L3 is not recursive
4. L1, L2, L3 are recursive

(Theory of Computation)

Which of the following languages are undecidable? Note that $\langle M \rangle$ indicates encoding of the Turing machine M.

- $L_1 = \{\langle M \rangle \mid L(M) = \varnothing\}$ UD
- $L_2 = \{\langle M, w, q \rangle \mid M \text{ on input } w \text{ reaches state } q \text{ in exactly 100 steps}\}$ D
- $L_3 = \{\langle M \rangle \mid L(M) \text{ is not recursive}\}$ UD
- $L_4 = \{\langle M \rangle \mid L(M) \text{contains at least 21 members}\}$ UD

A. L_1, L_3, and L_4 only

UD

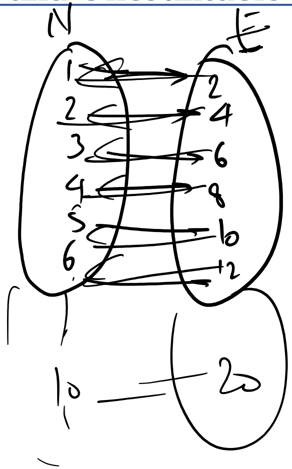
B. L_1 and L_3 only

C. L_2 and L_3 only

D. L_2, L_3, and L_4 only

Countable and Uncountable Set

1:1 ⇐ 1:2 → 2



Sets

Finite set (F)

$a^n \mid n \le 10^6$

(Always Countable)

Infinite set

Countable (C) RE

Uncountable (UC) Non RE

Countable set (C): =

Uncountable (UC): —

$N = (1, 2, 3, 4, 5, 6 - - -)$ ∃ EP ∞ <

$E = (2, 4, 6, 8 - - - —)$

Countable Set: A set S is said to be countable, if all the elements of the set can be put in one-to-one correspondence with set of natural numbers.

Uncountable Set: A set is uncountable, if it is infinite and not countable

$R \quad (-\infty, \infty)$

$(0,1)$

$\infty < \infty$

$10x = 19.999$

$x = 1.999999$

$9x = 18.000$

$x = 2.000$

Example: Set of

N, Z, Q, E, O < R, C, I

Countable set | Uncountable set

$b = 0.b_1 b_2 b_3$

$E = \{0, 2, 4, 6, 8, \ldots\}$

$N = \{1, 2, 3, 4, 5, \ldots\}$

Countable

$1.01 \rightarrow 1.1$

$1-2$

$(0,1)$

1.1

Rational no. '$^p/_q$' = p, q ∈ z^+, positive integer.

p/q , $p, q \in Z^+$

$^p/_q$ = { $^1/_1$, $^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$, $^1/_6$, … $^2/_1$ (never come)

1 2 3 4 5 6

$p/2/1$

$1/\,\frac{2}{1}$

2 3 3 4 4 4 5 5 5 5

$p/q = \left\{ \frac{1}{1}, \boxed{\frac{1}{2}, \frac{2}{1}}, \boxed{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1} \cdots \right\}$

N = 1, 2, 3, 4, 5 = = - - . 6

(Theory of Computation)

$$\boxed{\underbrace{R}_{\text{Uncountable}} = \underbrace{Q}_{\text{Countable}} \cup \underbrace{I}_{\text{Uncountable}}}$$

$$\underbrace{R}_{UC} = \underbrace{Q}_{C} \cup \underbrace{I}_{UC}$$

Set of all strings over $\sum = \{a, b\}$ countable Therefore, $\boxed{\sum^* \text{ is countable set}}$

$\Sigma^* = \{\epsilon, a, aa, aaa, aaaa, a^5, a^6, a^7 \ldots \}$

Lexicographical order (Dictionary order)

$\Sigma^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, babb \ldots\}$

$N = \{1, 2, 3, 4, 5, 6 \ldots\}$

Countable set

Set of all TM are countable

$M \subseteq \Sigma^*$

$\Sigma^* =$

$2^{\Sigma^*} =$

$\Sigma = \{0, 1\}, \Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$ is countable

$\Sigma^* =$

Every TM can be encoded as a string of 0's and 1's.

Set of all TM(M). $M \subseteq \Sigma^*$, $TM \subseteq \Sigma^*$

Every subset of countable set is countable; $M = \{TM_1, TM_2, \dots\}$ set of all TM are countable.

R.E are countable → TM - Countable

Rec. are countable → HTM – countable

CSL are countable → LBA – countable

CFL are countable → PDA – countable

RL are countable → FA – countable

$\downarrow$ L1 $\downarrow$ Rel

$\downarrow$ L2 $\downarrow$ Rel

2^{Σ^*} = Set of all

Note:

Set of all language (2^{Σ^*}) over $\Sigma = \{a, b\}$ are uncountable.

$1 + 2 + 3 : 2^{\Sigma^*} \Rightarrow$ Non RE

Uncountable

Diagonalization (Georg Cantor) :-

if $\sum = \{a, b\}$

$\sum^*$ = set of all strings are countable but

$\boxed{2^{\Sigma^*} \text{ = set of all languages are uncountable}}$

Proof: Let assume 2^{Σ^*} is countable

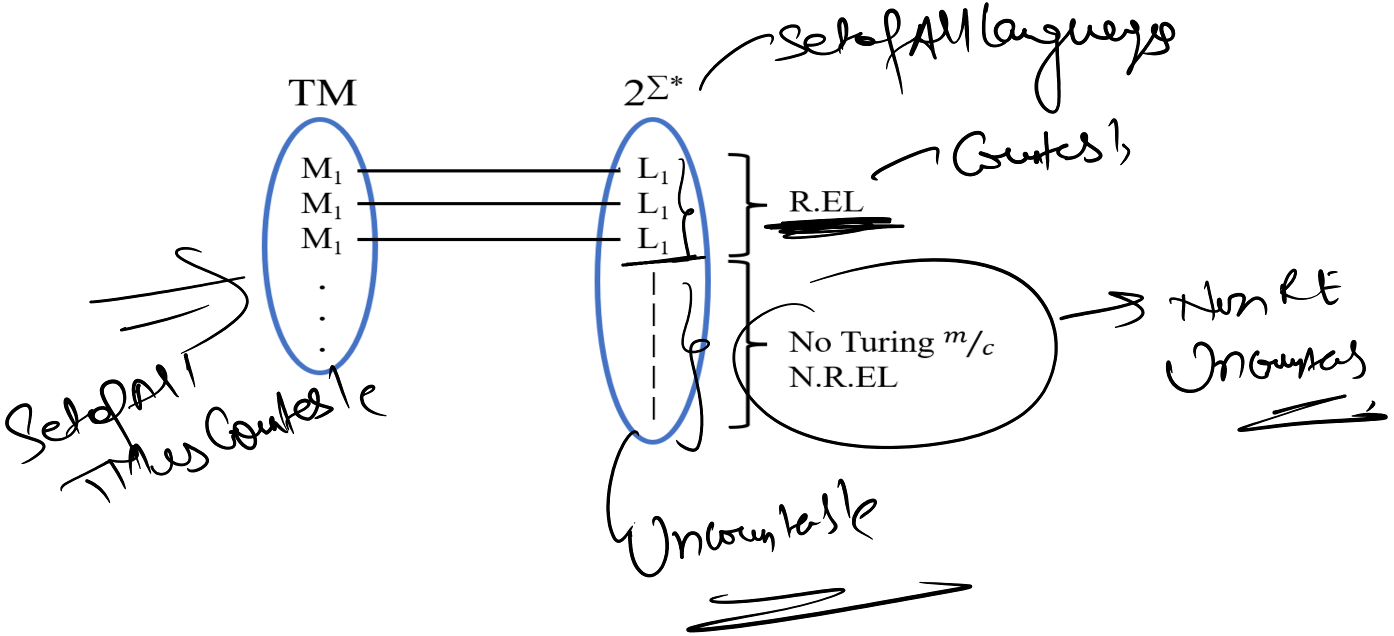
$\sum^*$: {	ε,	a,	b,	aa,	ab,	ba,	bb,	aaa,	......}
TM 1. ⟷ L1	1	1	0	0	0	0	0	0	......
2. ⟷ L2	0	1	1	0	0	0	0	0	......
3. ⟷ L3	1	1	0	1	0	0	0	0	......
4. ⟷ L4	0	0	1	0	1	1	0	0	......
5. ⟷ L5	0	0	0	1	0	1	1	0	......
6. ⟷ L6	1	0	0	0	0	0	1	1	......
.	.	.	.	.	.	.	.	.	......

L : < 1 1 0 0 0 0 - - >

L' : < 0 0 1 1 1 1 - - >

2^{Σ^*} = Uncountable

(Theory of Computation)

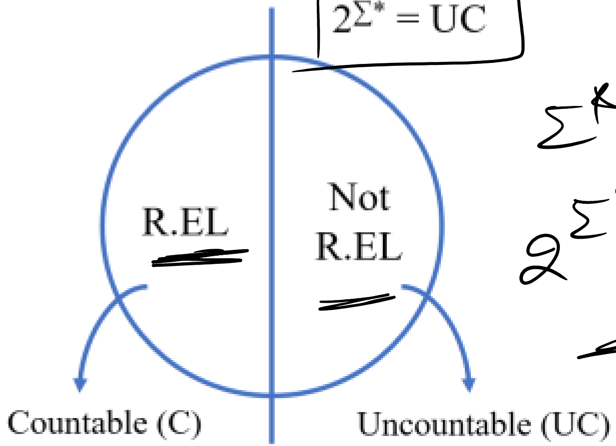


2^{Σ^*} are not countable

Set of all TM are countable, but

set of all language 2^{Σ^*} are not countable.

$\boxed{2^{\Sigma^*} = UC}$



$|S| = n$ finite

2^S = Countable

2^N = Uncountable

Σ^* = Countable — Infinite

2^{Σ^*} = Uncountable

A	B	A ∪ B
C	C	C
C	UC	UC
UC	UC	UC

Note:

1-Every subset of countable set is countable.

2-If S and T are countable sets then $S \cup T$ and $S \times T$ are countable set.

Example: $S = \{s_0, s_1, s_2, s_3, s_4, \ldots\}$ ← Countable set

$S = \{t_0, t_1, t_2, t_3, t_4, \ldots\}$ ← Countable set

$S \cup T = \{ s_0, t_0, s_1, t_1, s_2, t_2, s_3, t_3, s_4, t_4, \ldots\}$ ← Countable set

$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$

$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\ldots.\}$

$S \times T = \{ (s_0, t_0) (s_0, t_1) (s_1, t_0) (s_0, t_2) (s_1, t_1) (s_2, t_0) \ldots\}$ ← Countable set

$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$

$N = \{1, 2, 3, 4, 5, 6, \ldots\ldots.\}$

Note:

1) Set N, Z, Z^+, Z^-, E, O, Q, Q^+, Q^- all are countable.
2) Set R, C, I are uncountable set.
3) Absence of Enumeration Procedure not give guarantee that the set is uncountable.
4) But the presence of Enumeration Procedure give guarantee that the set is countable.

TOC : 20

6-7

m-A =)

7-8

$f: |A|^m \to |B|^n$

$\Rightarrow$ n^m

$= 2^N$

Ex:

Let N = Natural no.

P : Set of Rational No. → C

Q : Set of function from {0,1} to N → N^2 → C

R : Set of function from N to {0,1} → UC

S : Set of finite subset of N → C

a) Q and S only

b) P and S only

c) P and R only

d) P, Q and S only