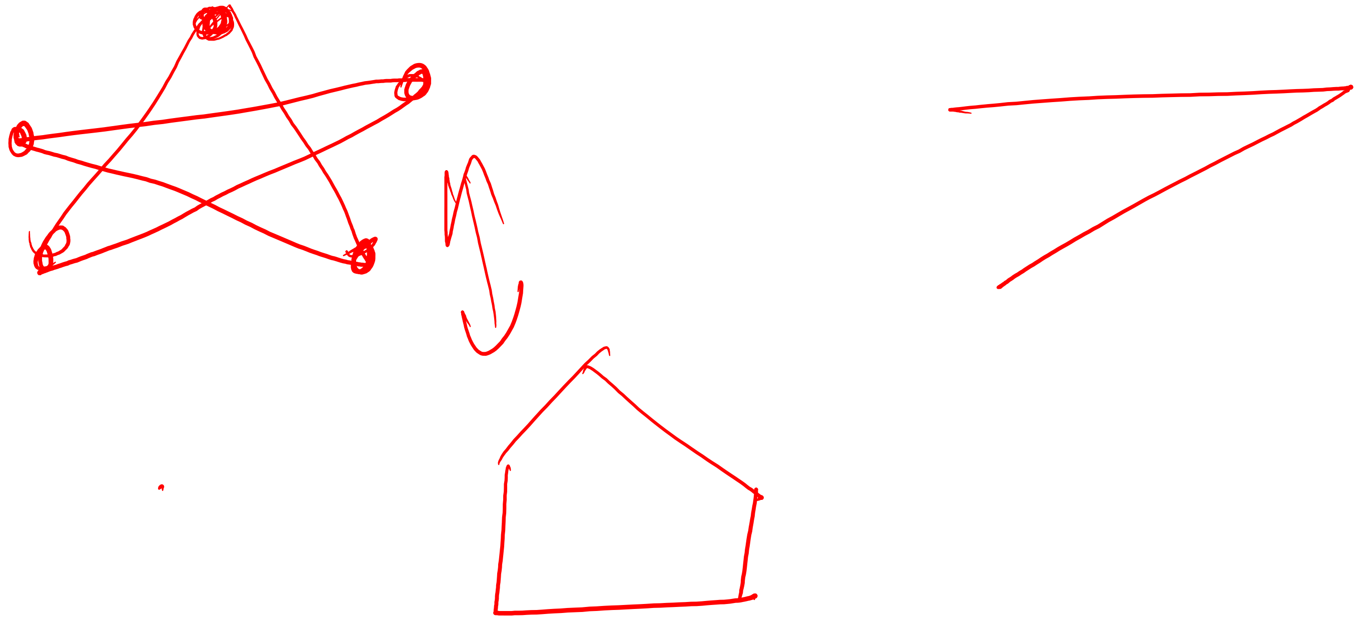
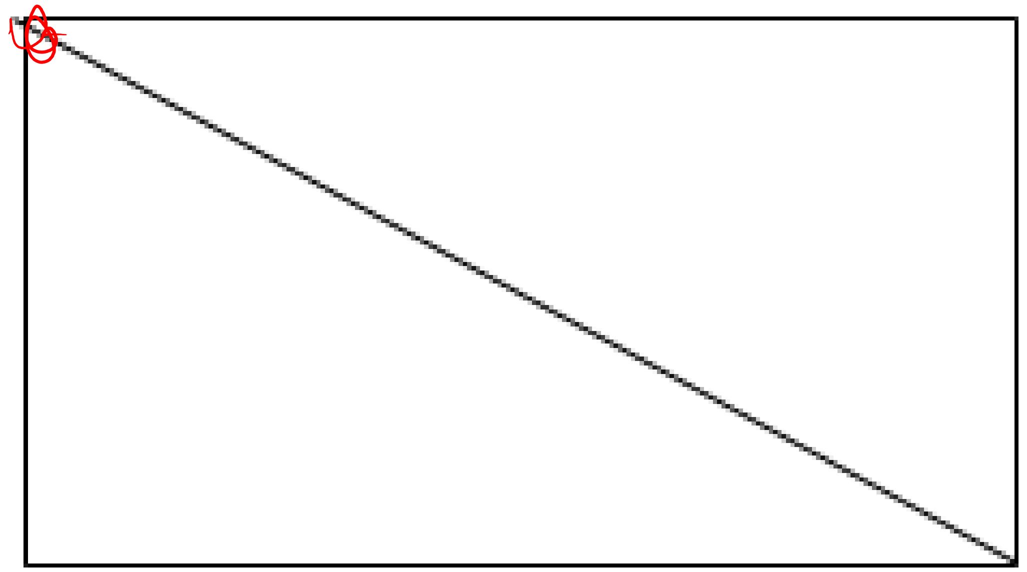
Planar Graphs



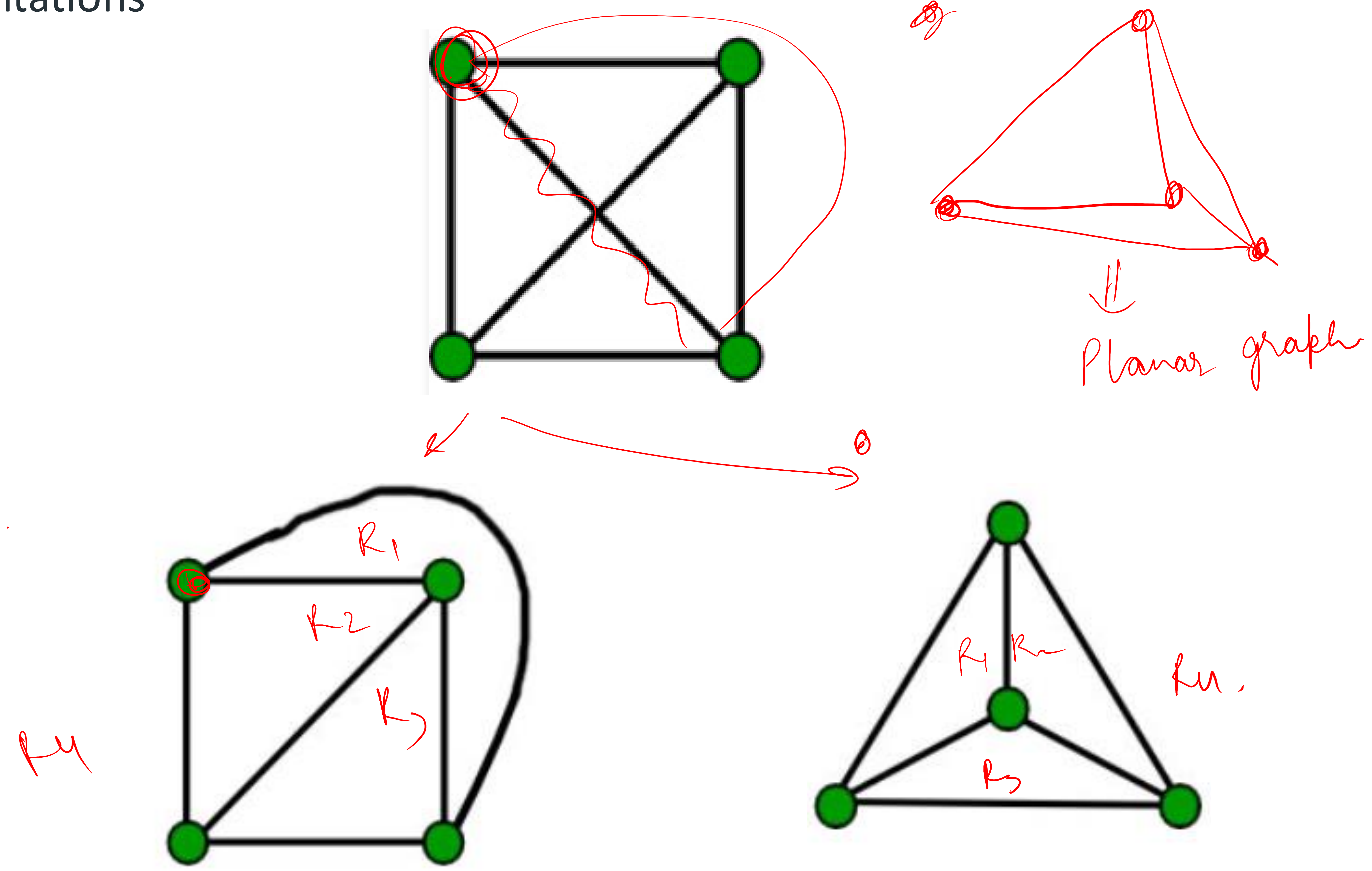
A graph G is said to be **planar** if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at common vertex.



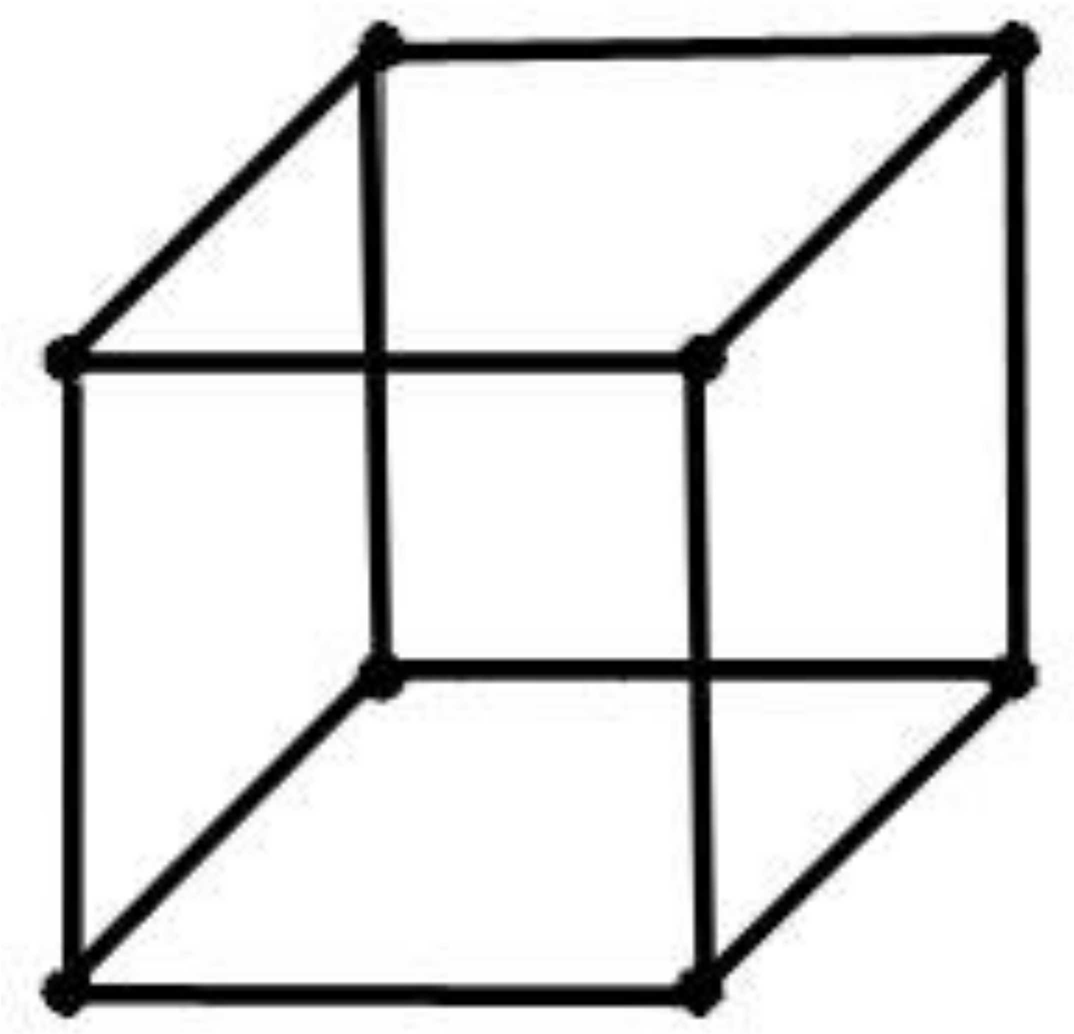
Planarity – "A graph is said to be planar if it can be drawn on a plane **without any edges crossing**. Such a drawing is called a planar representation of the graph."

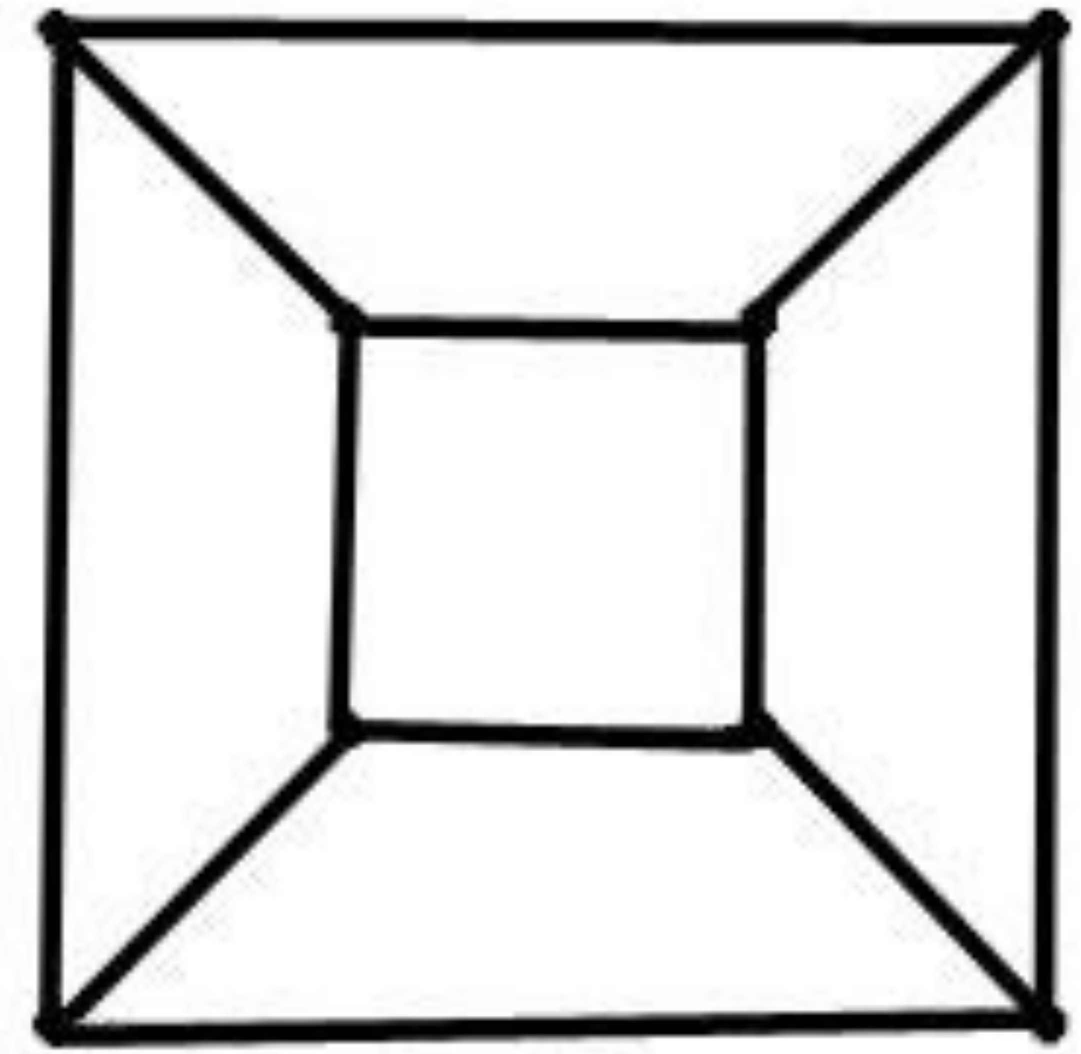
Important Note – A graph may be planar even if it is drawn with crossings, because it may be possible to draw it in a different way without crossings.

For example consider the complete graph K_4 and its two possible planar representations –



Example – Is the hypercube Q_3 planar



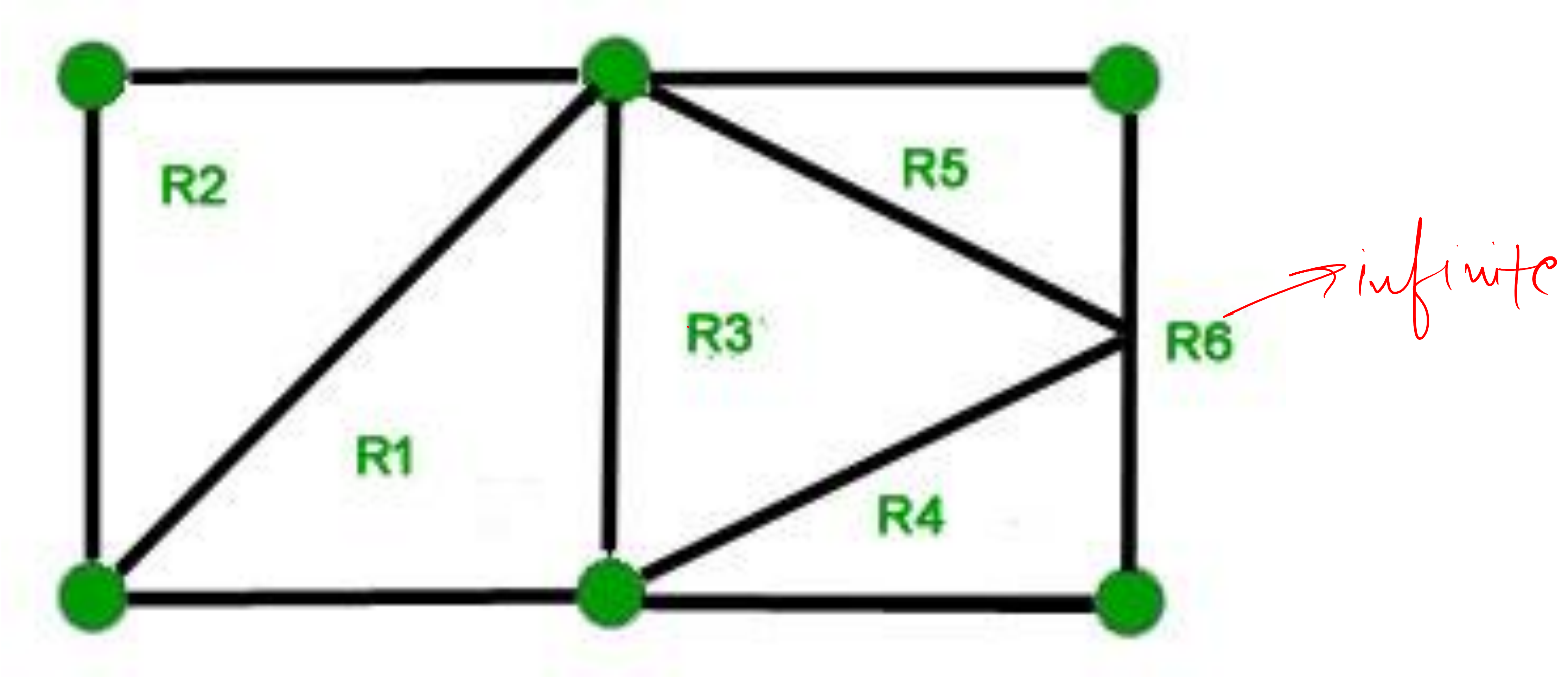


Regions in Planar Graphs –

A region of a planar graph is defined to be an area of the plane that is bounded by the edges and is not further divided into subareas.

If the area of the region is finite is called finite or bounded region.

If the area of the region is infinite is called infinite, outer or unbounded region.



There are a total of 6 regions with 5 bounded regions and 1 unbounded region

All the planar representations of a graph split the plane in the same number of regions. Euler found out the number of regions in a planar graph as a function of the number of vertices and number of edges in the graph.

Euler's Formula

Let G be a connected simple planar graph with n vertices, e edges and r region, then n – e + r = 2 .

$20 - 30 + r = 2 \Rightarrow r = 12$

Example – What is the number of regions in a connected planar simple graph with 20 vertices each with a degree of 3?

$n = 20$, $2e = 20 \times 3 \Rightarrow \boxed{e = 30}$

Corollary : If a planar graph has k components, then n – e + r = k + 1 .

Corollary : Let G be a connected simple planar graph with n vertices and e edges , then $e \leq 3n - 6$

$e \leq 3n - 6$

Example –the graph K_5 is non - planar

Corollary : Let G be a connected simple planar graph with n vertices and e edges and no circuits of length 3, then $e \leq 2n - 4$.

$mn \leq 2(n+m) - 4$

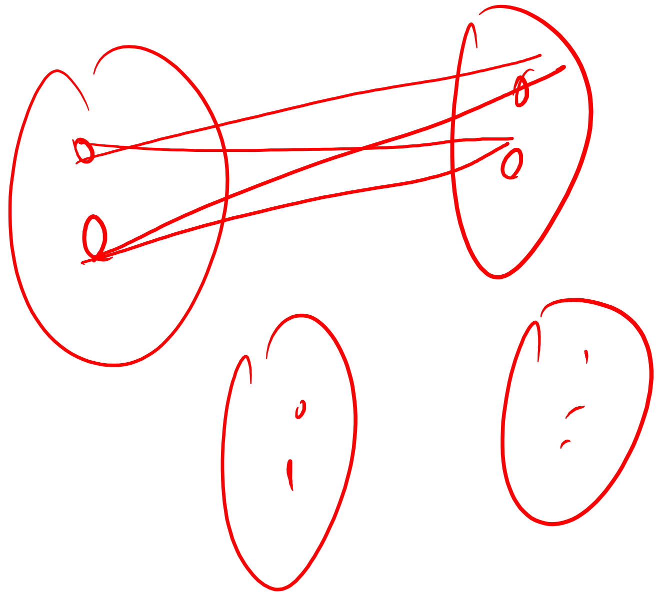
complete Bipartite Graph

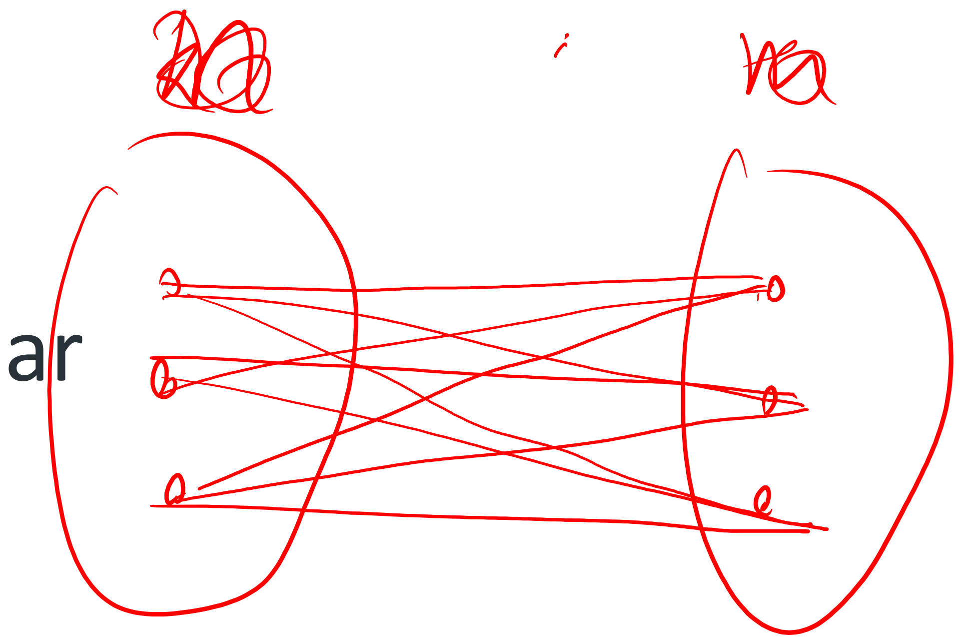
$K_{3,3}$

$mn \leq 2n + 2m - 4$

$4 \leq 2n + 2m - mn$

Example – the graph $K_{3,3}$ is non - planar





$9 \leq 2 \times 6 - 6$

$9 \leq 9$

9 not planar

$9 \leq 2 \times 6 - 4$

$9 \not\leq 8$

GATE CS 2012

Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connected graph, then the number of bounded faces in any embedding of G on the plane is equal to

(A) 3

(B) 4

(C) 5

(D) 6

$n = 10$

$e = 15$

$n - e + r = 2$

$10 - 15 + r = 2$

$r = 7$

n - e + r = 2

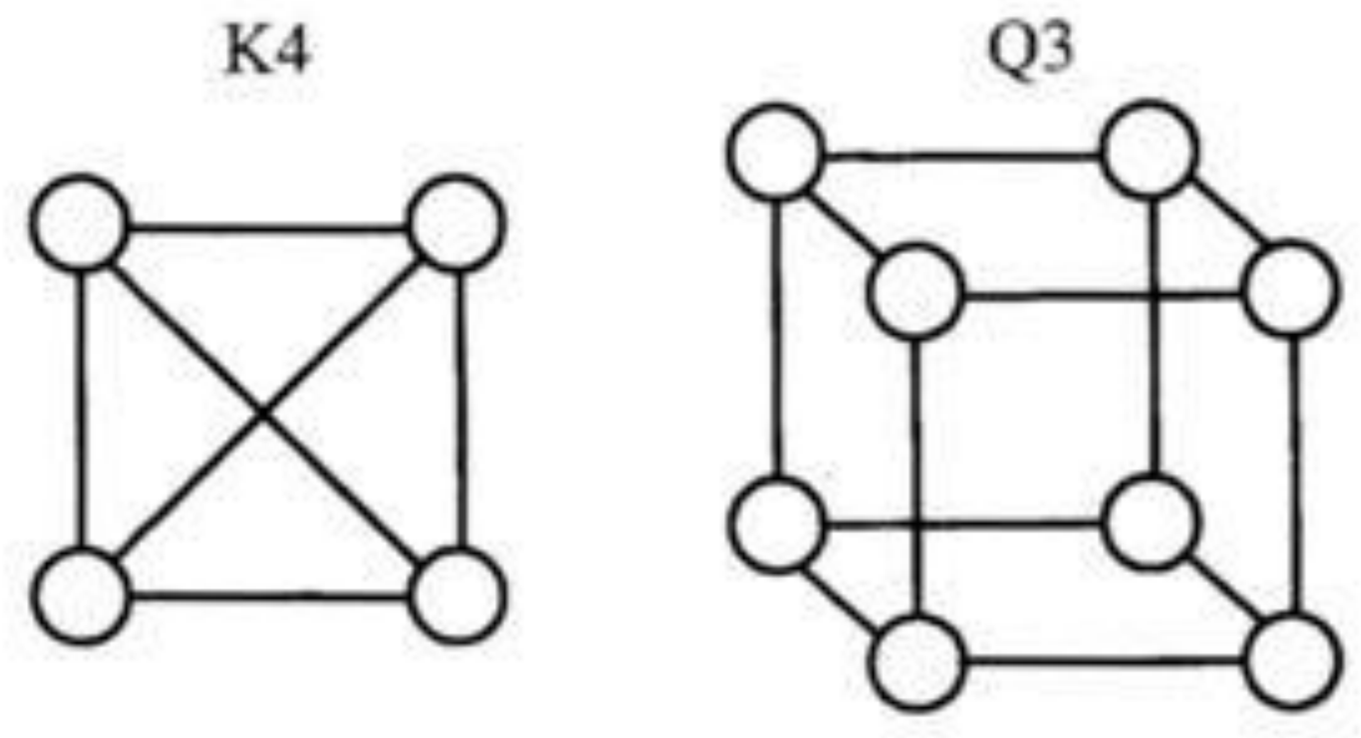
10 - 15 + r = 2

r = 7

There is always one unbounded face, so the number of bounded faces = 6

Answer: (D)

GATE CS 2011



(A) K4 is planar while Q3 is not
(B) Both K4 and Q3 are planar
(C) Q3 is planar while K4 is not
(D) Neither K4 nor Q3 are planar

Answer: (B)

GATE-CS-2005

Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is

(A) 6

(B) 8

(C) 9

(D) 13

Answer: (B)

GATE-CS-2015 (Set 1)

Let G be a connected planar graph with 10 vertices. If the number of edges on each face is three, then the number of edges in G is ________________.

(A) 24

(B) 20

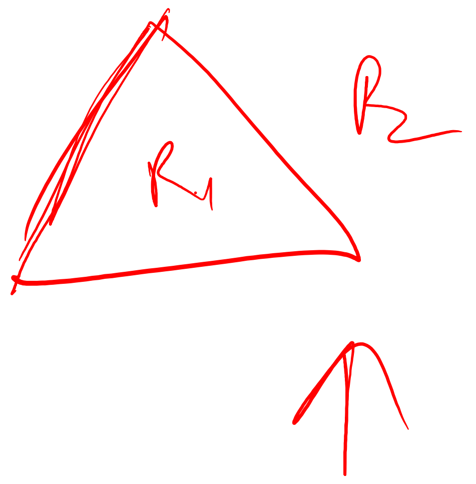
(C) 32

(D) 64

$2e = 3r$

$r = \frac{2e}{3}$

each face has 3 edges



$3 \times 2 = 6$

$n - e + r = 2$

$10 - e + \frac{2e}{3} = 2 \Rightarrow 8 = \frac{e}{3} \Rightarrow e = 24$

Answer: (A)

GATE-CS-2005

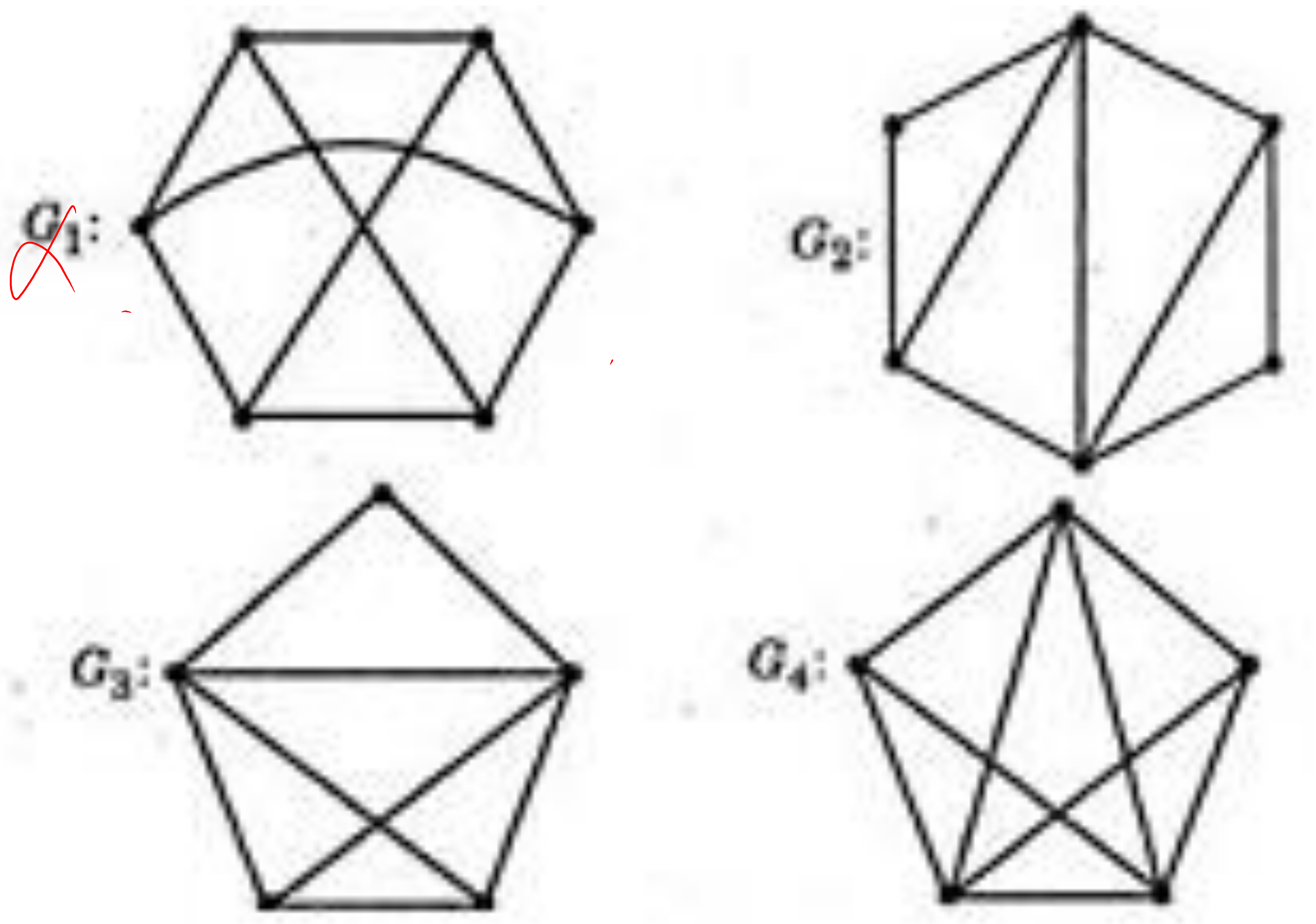
Which one of the following graphs is NOT planar?

(A) G1

(B) G2

(C) G3

(D) G4



$e \leq 2n-4$

Answer: (A)

$9 \leq 2 \times 6 - 4$

$9 \nleq 8$

GATE-CS-2014-(Set-3)

Let d denote the minimum degree of a vertex in a graph. For all planar graphs on n vertices with d ≥ 3, which one of the following is TRUE?

(A) In any planar embedding, the number of faces is at least n/2 + 2

(B) In any planar embedding, the number of faces is less than n/2 + 2

(C) There is a planar embedding in which the number of faces is less than n/2 + 2

(D) There is a planar embedding in which the number of faces is at most n/(d+1)

$2e = $ Sum of degree of all vertices

$n - e + r = 2$

Answer: (A)

$2e \geq nd$

$r = 2 - n + e$

$e \geq \frac{nd}{2} \geq \frac{3n}{2} \Rightarrow e \geq \frac{3n}{2}$

$r \geq 2 - n + \frac{3n}{2}$

$n + r - 2 \geq \frac{3n}{2}$

$\boxed{r \geq 2 + \frac{n}{2}}$

$r \geq \frac{n}{2} + 2$

Euler's formula for planar graphs:
$v - e + f = 2$.

Since degree of every vertex is at least 3, below is true from handshaking lemma (Sum of degrees is twice the number of edges)

$3v \leq 2e$

$3v/2 \leq e$

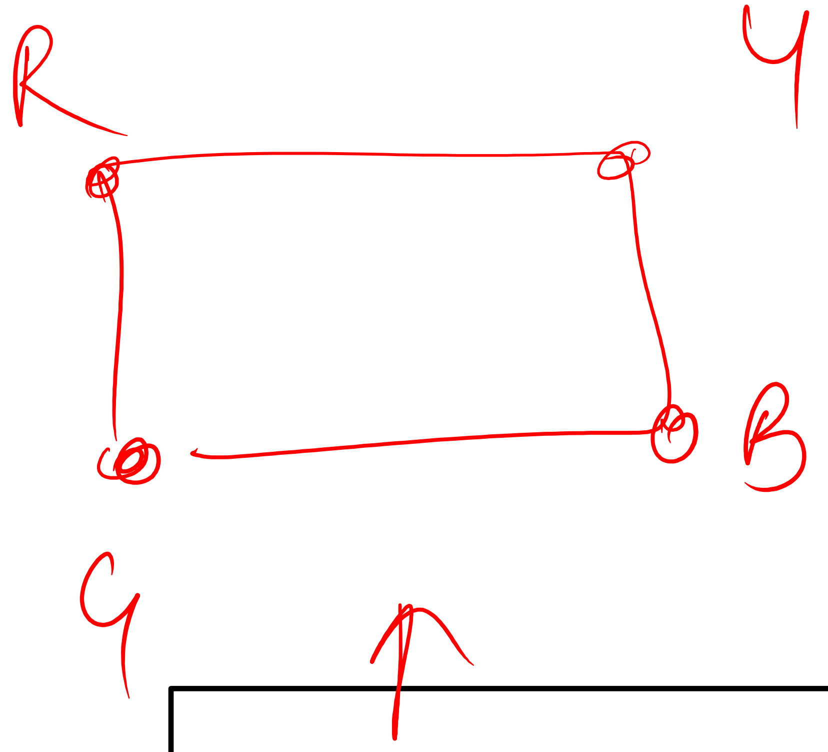
Putting these values in Euler's formula.

$v - 3v/2 + f \geq 2$

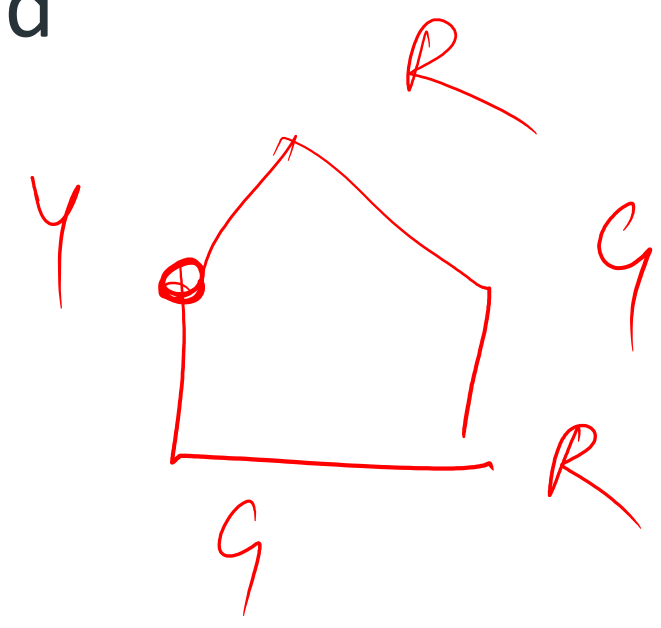
$f \geq v/2 + 2$

Graph Coloring

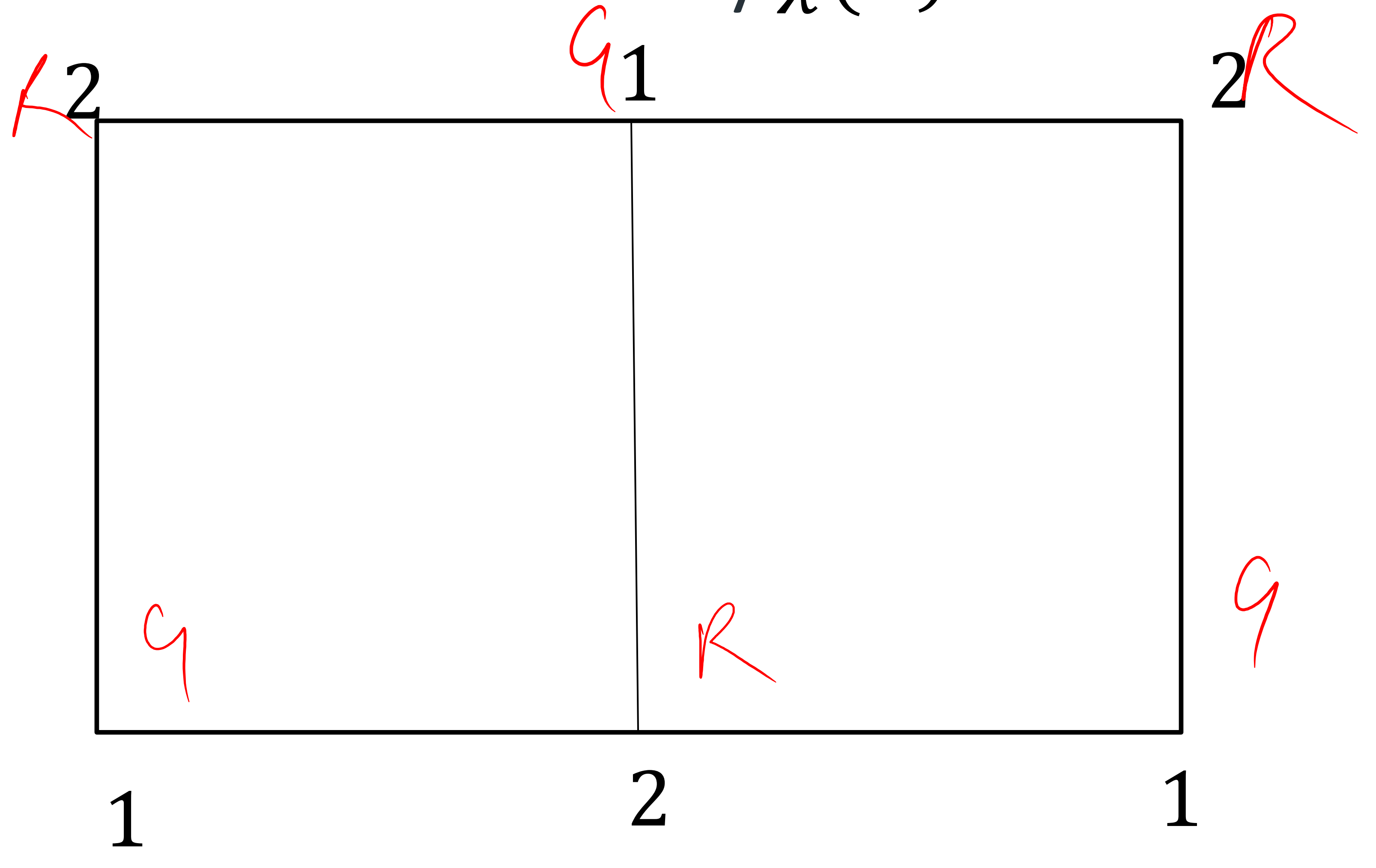
Coloring – “A coloring of a simple graph is the assignment of a color to each vertex of the graph such that **no two adjacent vertices** are assigned the same color.”



A simple solution to this problem is to color every vertex with a different color to get a total of n colors. But in some cases, the actual number of colors required could be less than this.



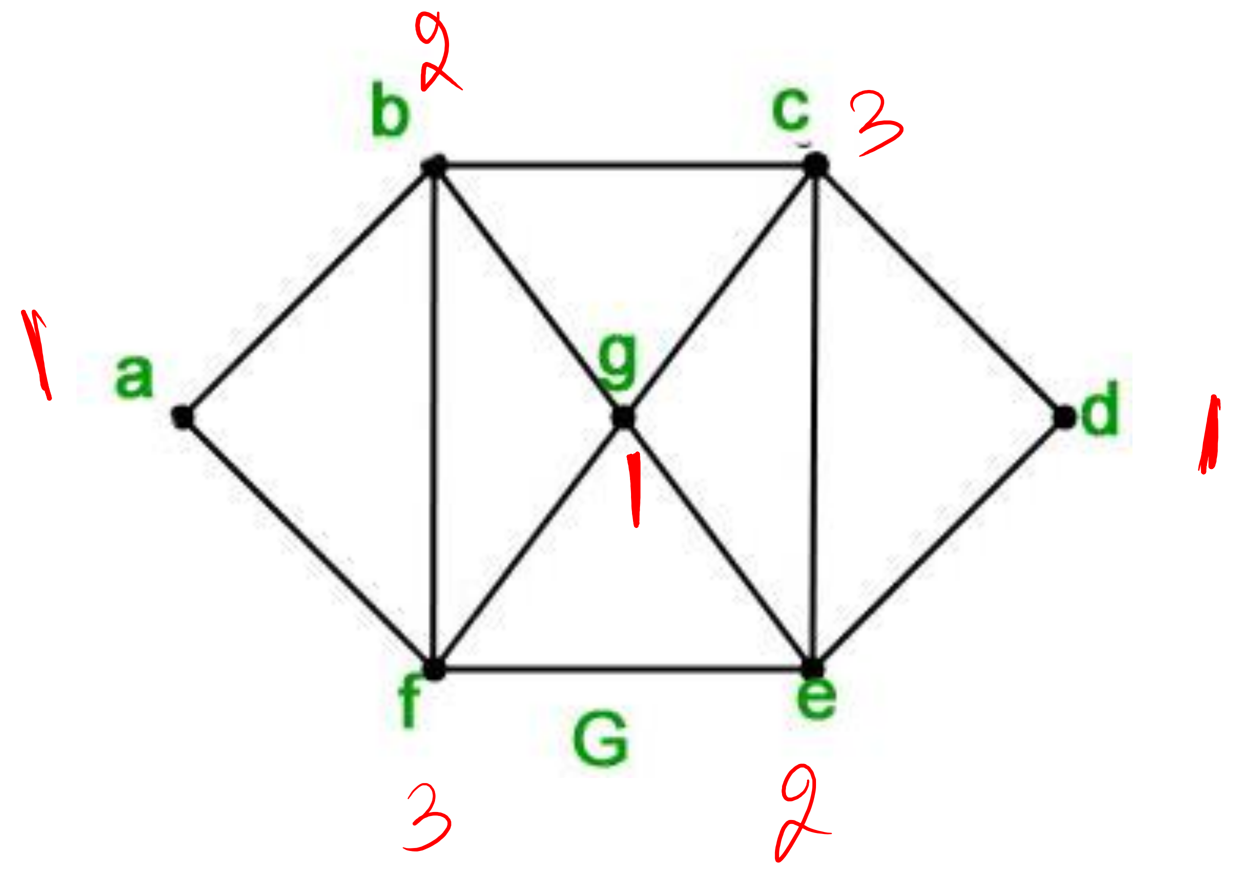
Chromatic number –“The least number of colors required to color a graph is called its **chromatic number**. It is denoted by $\chi(G)$.



For planar graphs the finding the chromatic number is the same problem as finding the minimum number of colors required to color a planar graph.

4 color Theorem – “The chromatic number of a planar graph is not greater than 4.”

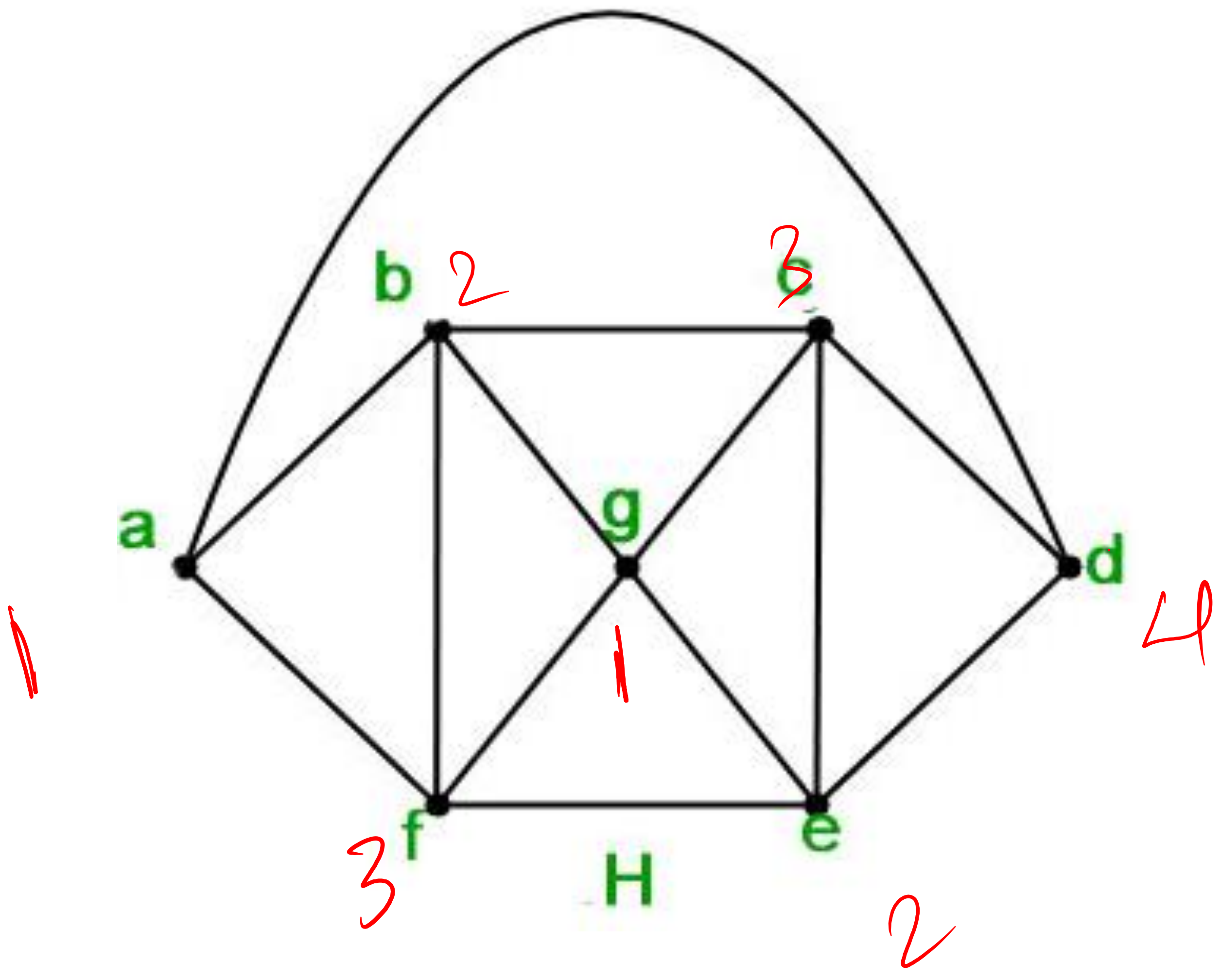
Example 1 – What is the chromatic number of the following graphs?



The following color assignment satisfies the coloring constraint –

a – Red
b– Green
c – Blue
d – Red
e – Green
f– Blue
g – Red

Therefore the chromatic number of G is 3.



In graph H since a and d are also connected,
therefore the chromatic number is 4.

Example – What is the chromatic number of K_n ?

$\rightarrow n$

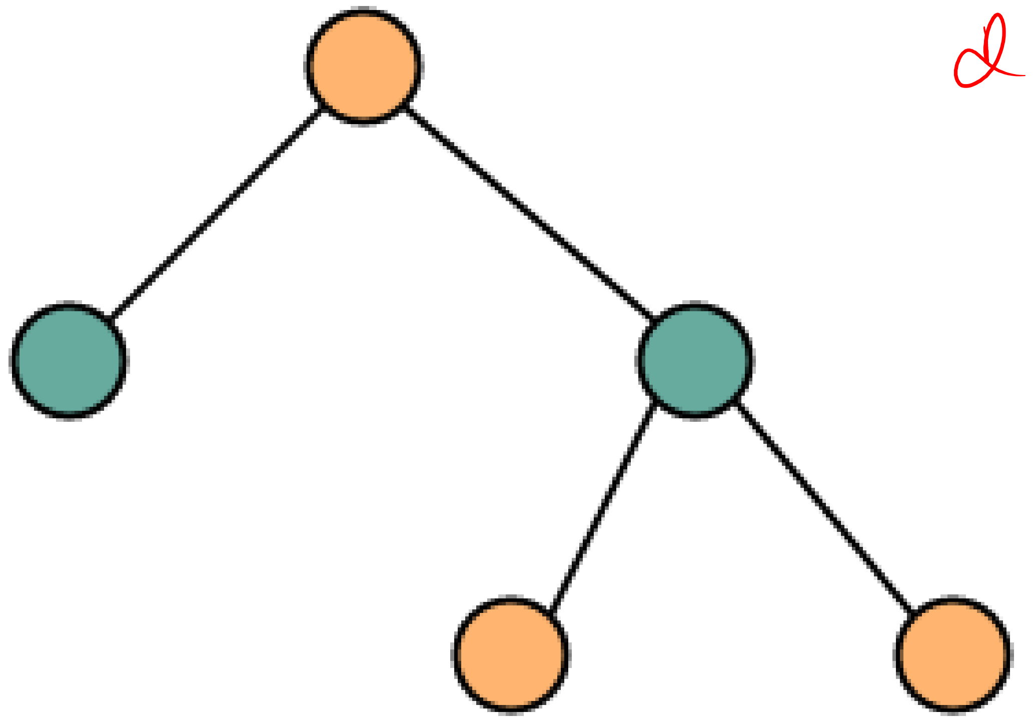
Solution – Since every vertex is connected to every other vertex in a complete graph, the chromatic number is n .

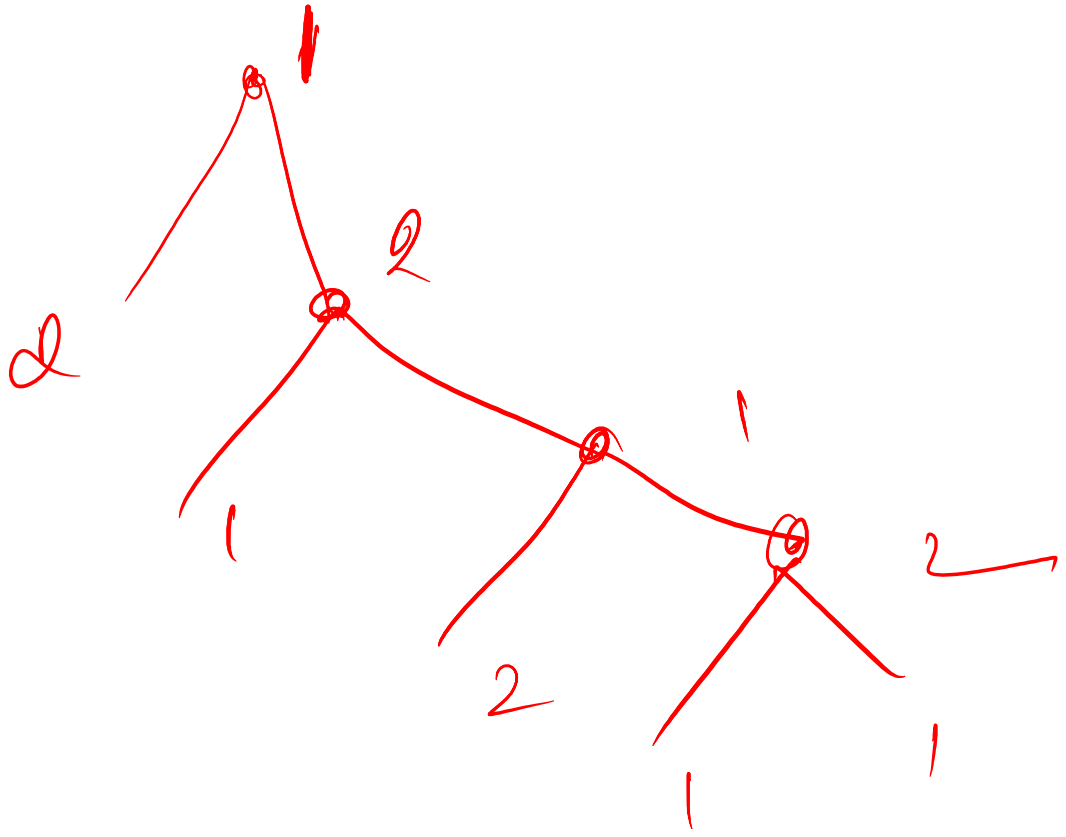
Bip iff ch. no. 2

Example – What is the chromatic number of tree (connected graph with no cycle)?

Every tree is a bipartite graph.

So, chromatic number of a tree with any number of vertices = 2.





Example – What is the chromatic number of Wheel graph W_6 ?

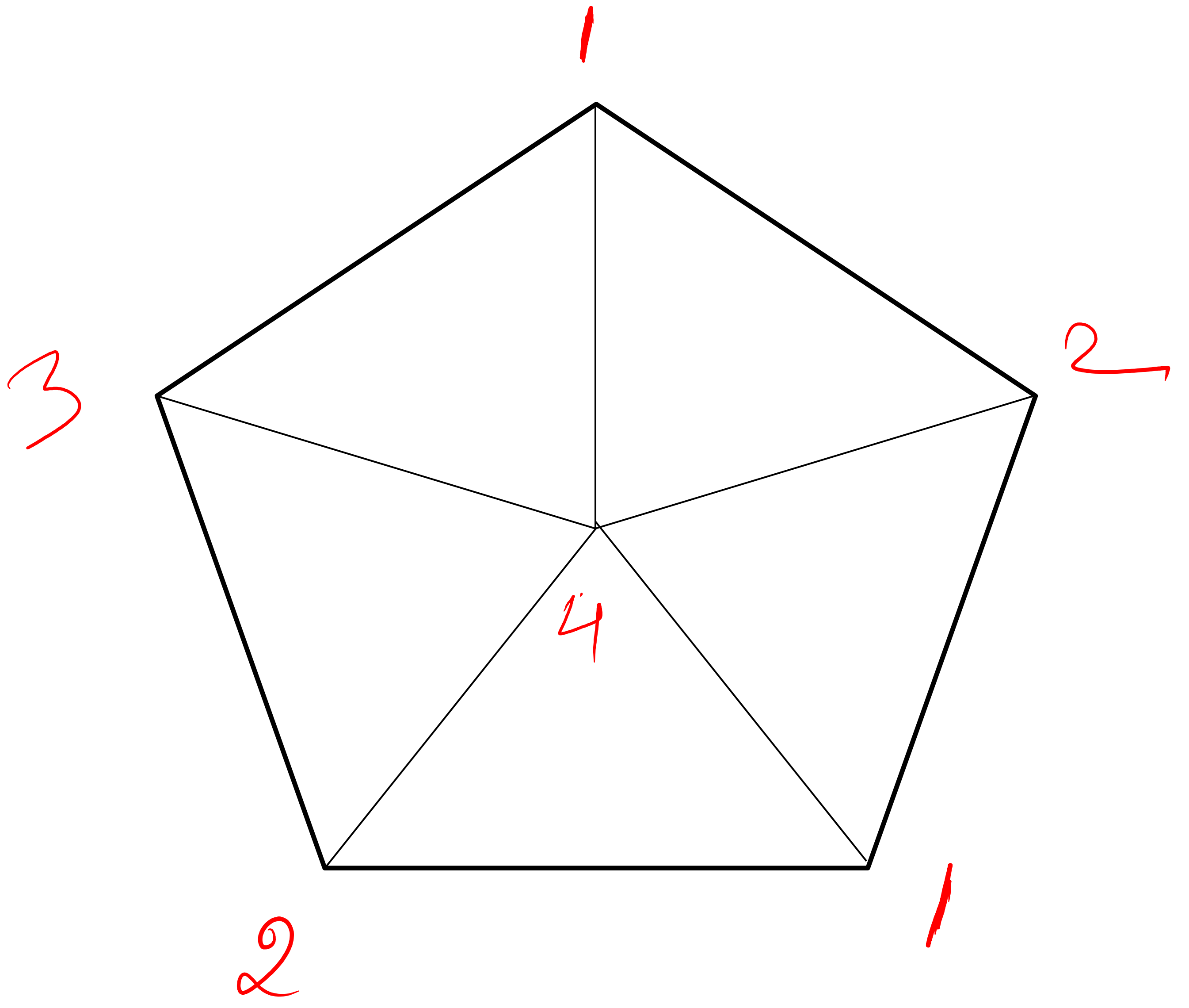
W_5

$W_n \rightarrow n$ even

3

$W_n \rightarrow n$ odd

4



4

Example 3 – What is the chromatic number of C_n ?

C_n n even

Solution – If the vertex are colored in an alternating fashion, the cycle graph requires 2 colors.

2

C_n n odd

If n is odd, then the last vertex would have the same color as the first vertex, so the chromatic number will be 3.

3

But if it is even, then first and last vertices will be of different color and the chromatic number will be 2.

Example 4 – What is the chromatic number of $K_{m,n}$?

Solution – In the bi-partite graph $K_{m,n}$ 2, the vertices are divided into two sets, such that there is no edge between vertices in the same set.

m

n

one color

another

Therefore the chromatic number of any bipartite graph is 2.

GATE-CS-2002

The minimum number of colours required to colour the vertices of a cycle with η nodes in such a way that no two adjacent nodes have the same colour is

(A) 2

(B) 3

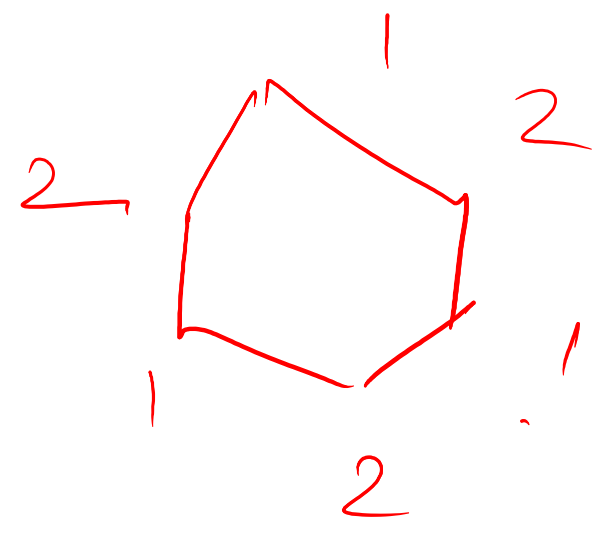
(C) 4

(D) $n - 2[n/2] + 2$

$4 - 2\left[\frac{4}{2}\right] + 2$

$C_n \rightarrow$ $n \rightarrow$ even $\rightarrow 2$

C_n n odd $\rightarrow 3$



Answer: (D) 2

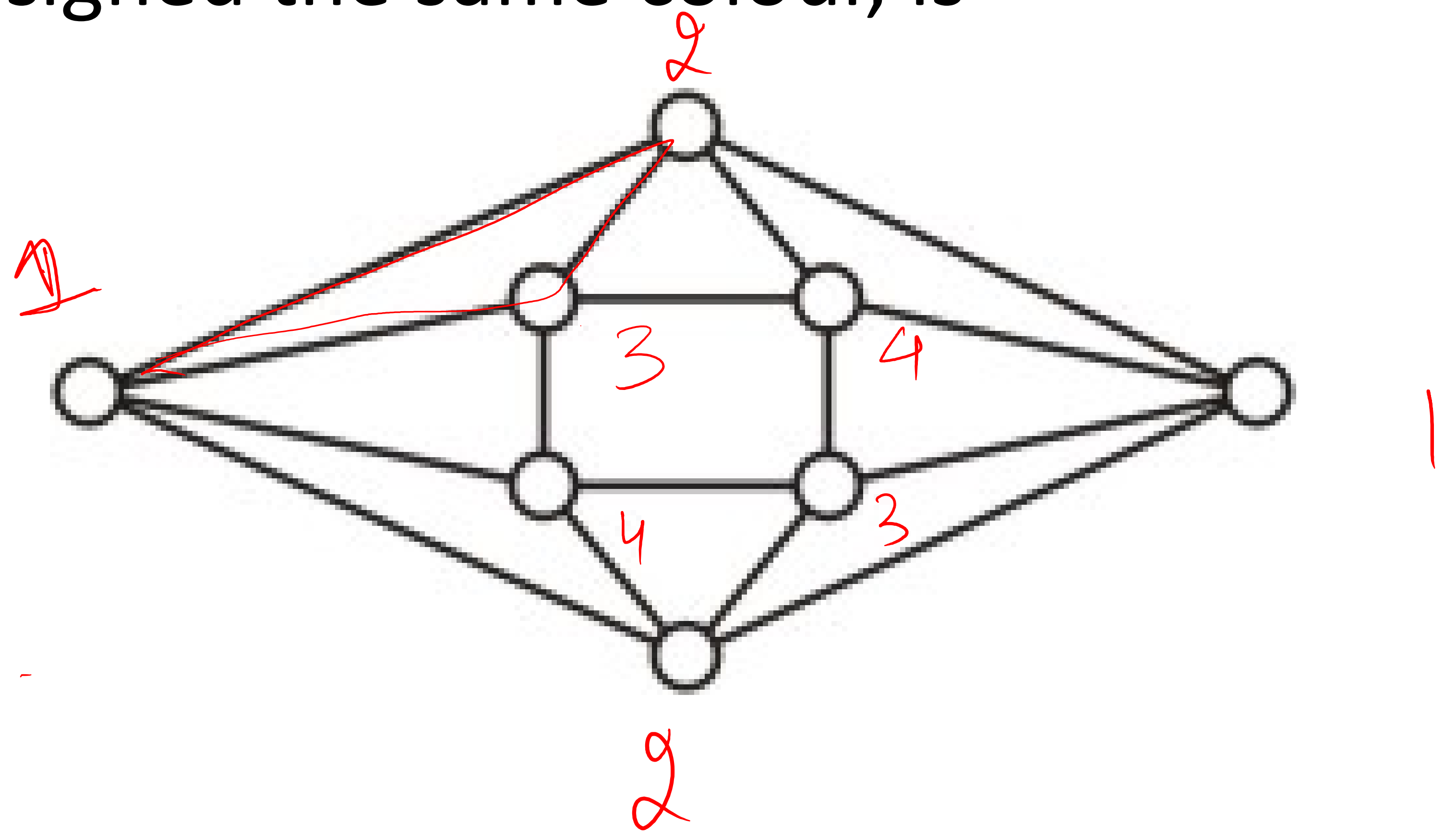
$n = 3$

$3 - 2 \times 1 + 2$

1

GATE-CS-2004

The minimum number of colours required to colour the following graph, such that no two adjacent vertices are assigned the same colour, is



(A) 2
(B) 3
(C) 4
(D) 5

Answer: (C)

GATE-CS-2016 (Set 2)

The minimum number of colours that is sufficient to vertex-colour any planar graph is ________________

4

GATE-CS-2009

What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume n >= 2.

(A) 2

(B) 3

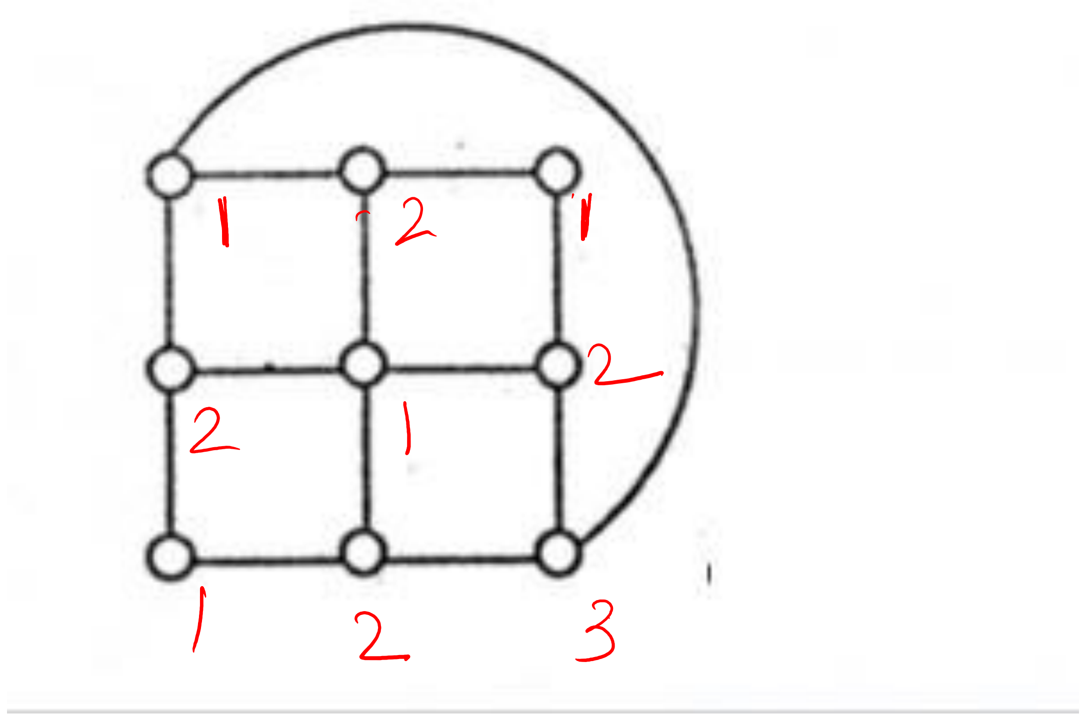
(C) n-1

(D) n

Answer: (A)

Gate IT 2008

What is the chromatic number of the following graph?

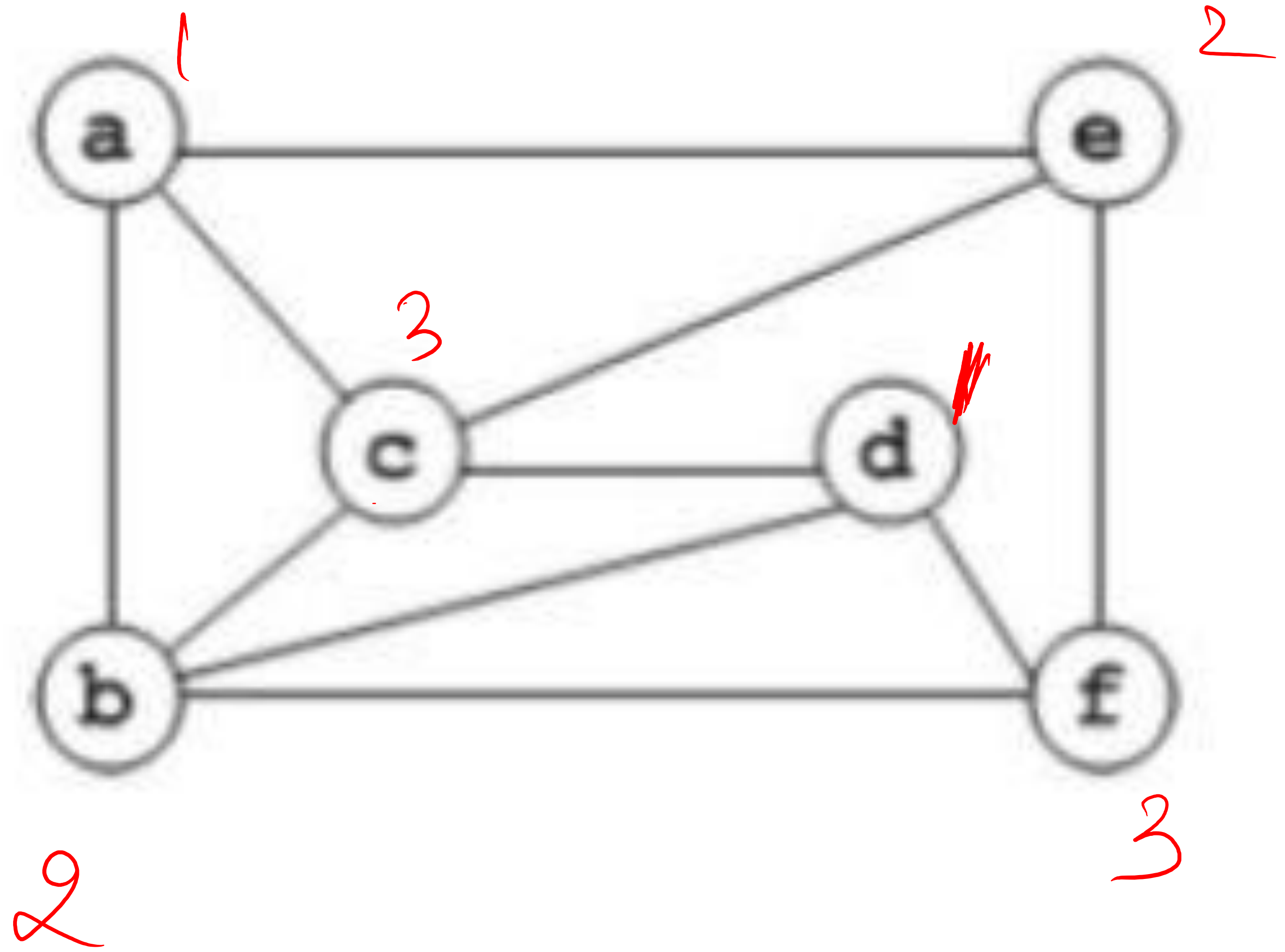


(A) 2
(B) 3
(C) 4
(D) 5

Answer: (B)

GATE CS 2018

The chromatic number of the following graph is __________ .



(A) 2
(B) 4
(C) 3
(D) 5

Answer: (C)

Matching

Edge covering

min set of edges which cover all the vertices

$\alpha_1(G)$

max set of independent edges $\rightarrow \beta_1(G)$

maximum matching

$\alpha_1(G) + \beta_1(G) = n.$

Independent sets

non adjacent vertices

$\alpha_0(G) \rightarrow$

Vertex covering

min set of vertices which cover all edges

$\hookrightarrow \beta_0(G)$

$\alpha_0(G) + \beta_0(G) = n.$

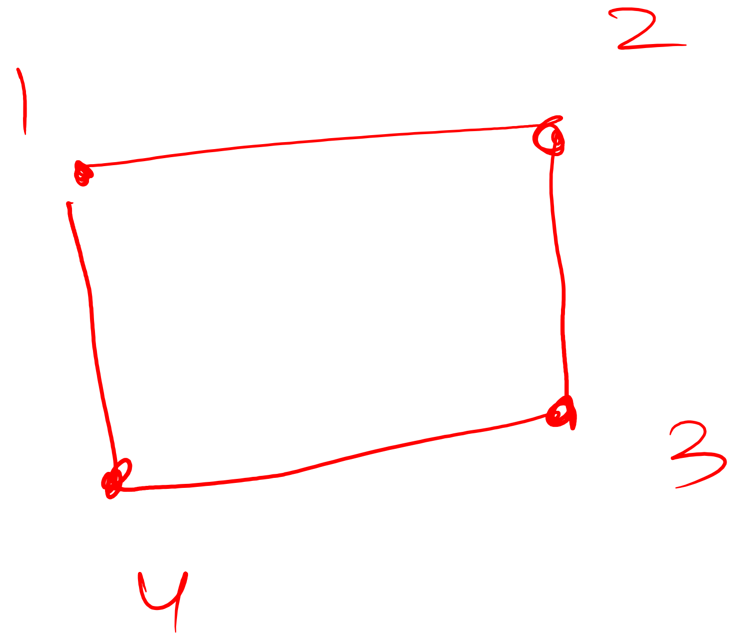
Maximal

no more element can add

Perfect matching

which cover all the vertices

Adjacency matrix



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad 4 \times 4$$

Independent Sets –

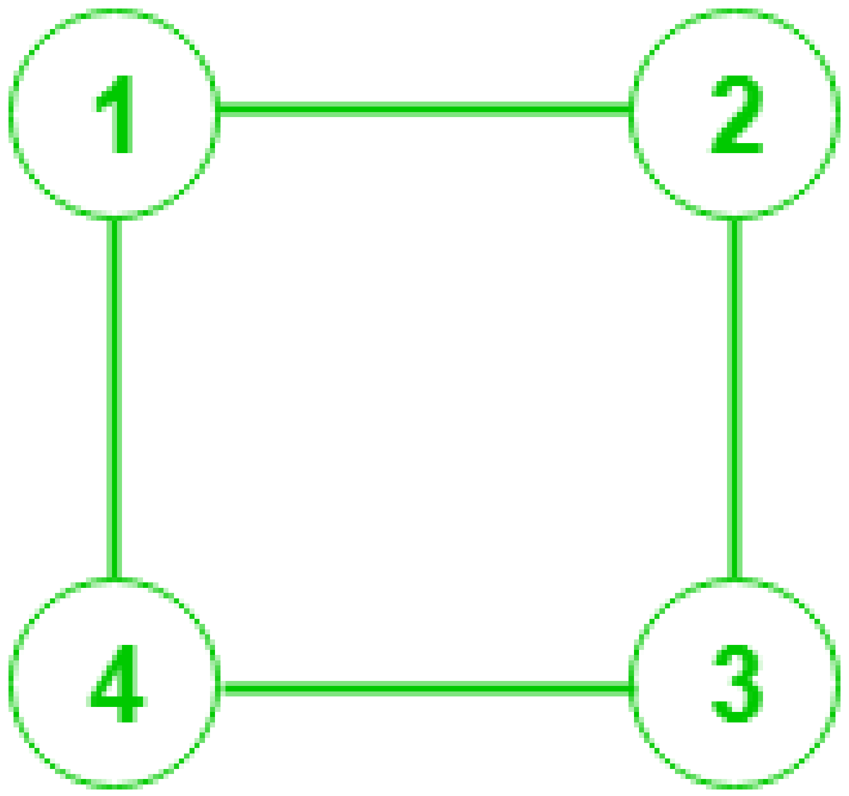
A set of vertices I is called independent set if no two vertices in set I are adjacent to each other.

or In other words the set of non-adjacent vertices is called independent set.

It is also called a **stable set.**

The parameter $\alpha_0(G)$ = max { |I|: I is an independent set in G } is called **independence number** of G i.e the maximum number of non-adjacent vertices.

Any independent set I with $|I| = \alpha_0(G)$ is called a maximum independent set.



For above given graph G, Independent sets are:

$I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $I_4 = \{4\}$ $I_5 = \{1, 3\}$ and $I_6 = \{2, 4\}$

Therefore, maximum number of non-adjacent vertices i.e Independence number $\alpha_0(G)$ = 2.

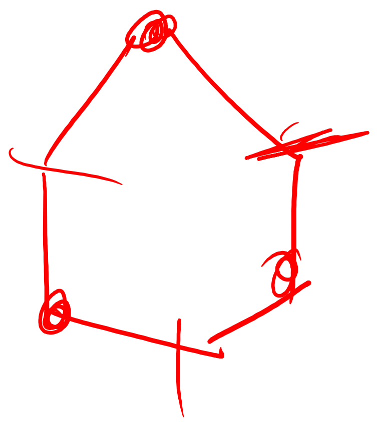
Independence and vertex covering number for different graphs

Graphs ⟶	C_n
Independence Number	floor(n/2)

~~Cyclete~~

cycle of even vertices

$\frac{n}{2}$



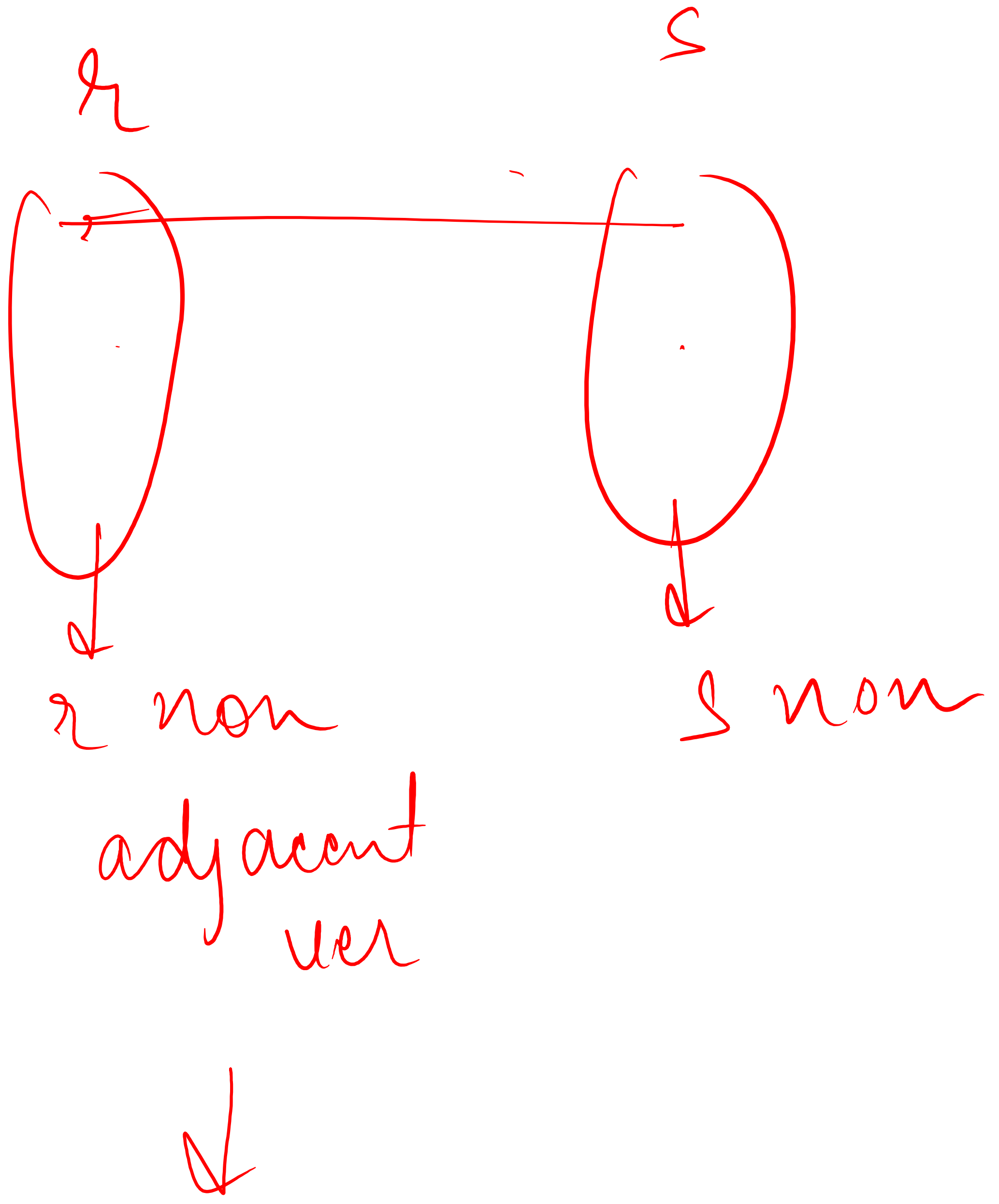
if n is odd

$\left\lfloor \frac{n}{2} \right\rfloor$

$\cancel{\frac{n}{2}+\frac{1}{2}}$ =

Graphs →	K_n
Independence Number	1

Graphs →	$K_{r,s}$
Independence Number	max{r,s}



GATE IT 2008

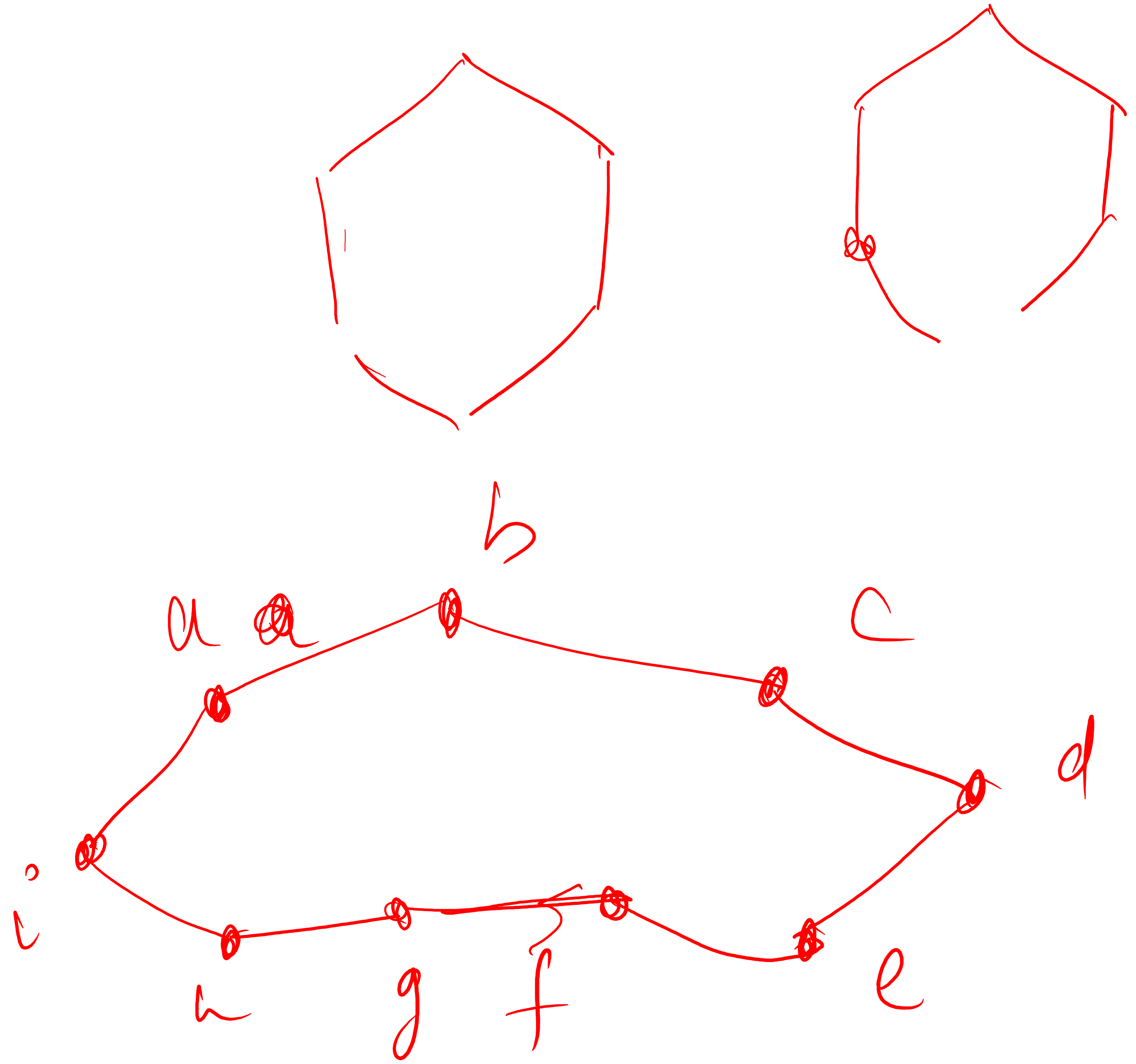
What is the size of the smallest MIS (Maximal Independent Set) of a chain of nine nodes?

A. 5
B. 4
C. 3
D. 2

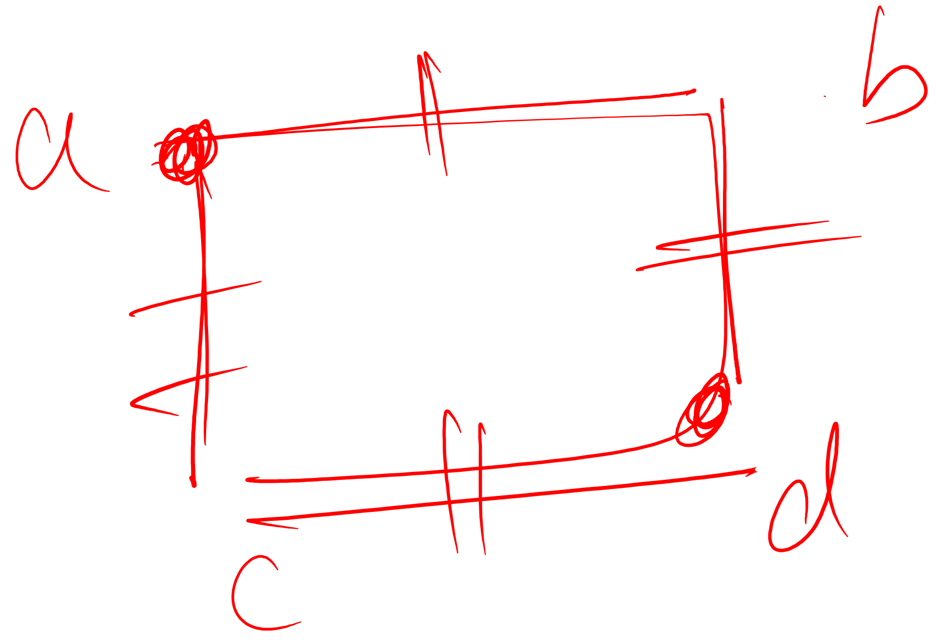
Answer: (C)

{a, c, e, g}

{a, d, g}



Graph Covering



$\{b, c\}$

$\{a, d\}$

$\{a, c, d\}$

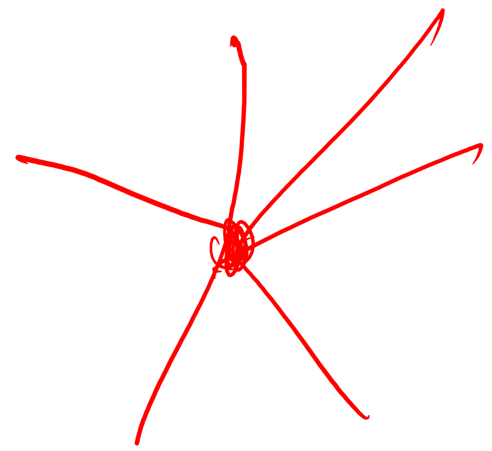
$\rightarrow \{a, b, c, d\}$

Vertex Covering –

A set of vertices K which can cover all the edges of graph G is called a **vertex cover** of G i.e. if every edge of G is covered by a vertex in set K.

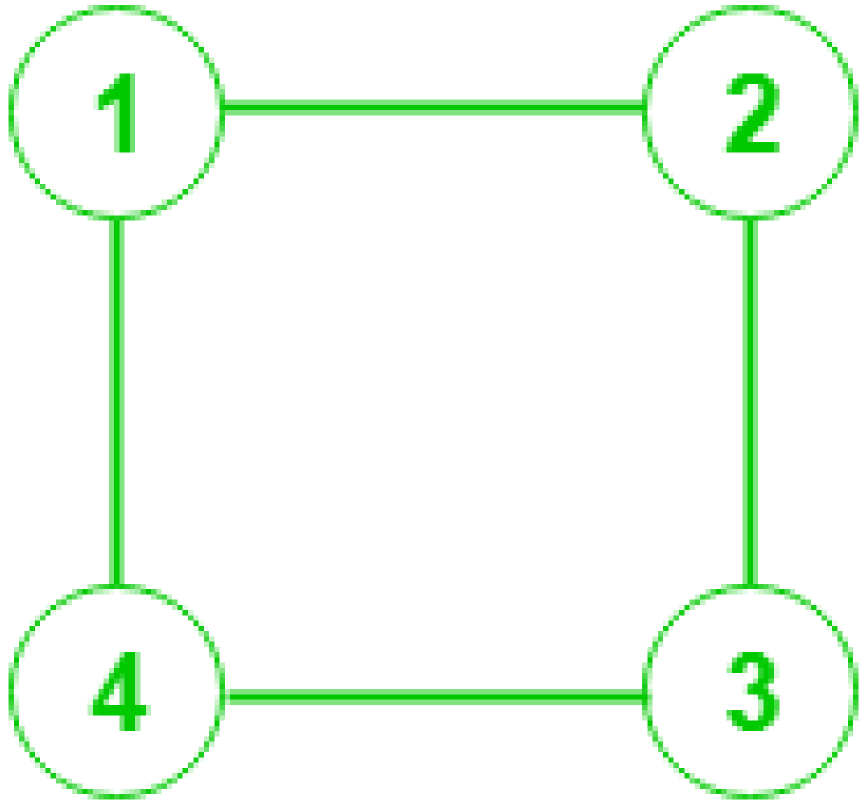
The parameter $\beta_0(G) = \min \{ |K| : K$ is a vertex cover of G $\}$ is called **vertex covering number** of G i.e the minimum number of vertices which can cover all the edges.

Any vertex cover K with $|K| = \beta_0(G)$ is called a minimum vertex cover.



$\beta_0(G)$ = vertex covering no.

$\beta_0(G) = 1$



For above given graph G, Vertex cover is:

V_1 = {1, 3}, V_2 = {2, 4}, V_3 = {1, 2, 3}, V_4 = {1, 2, 3, 4}, etc.

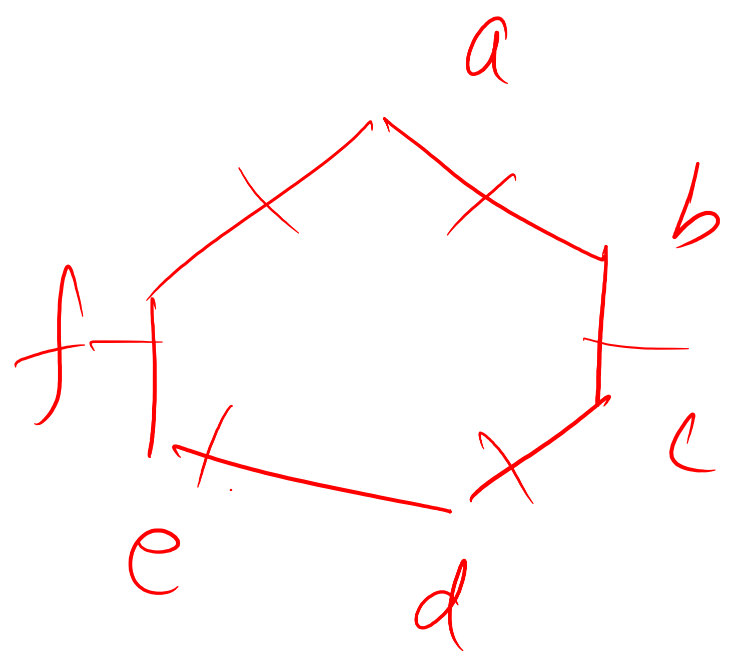
Therefore, minimum number of vertices which can cover all edges,
i.e., Vertex covering number $\beta_0(G) = 2$.

Independence and vertex covering number for different graphs

Graphs →	C_n
Independence Number	floor(n/2)
Vertex covering Number	ceil(n/2)

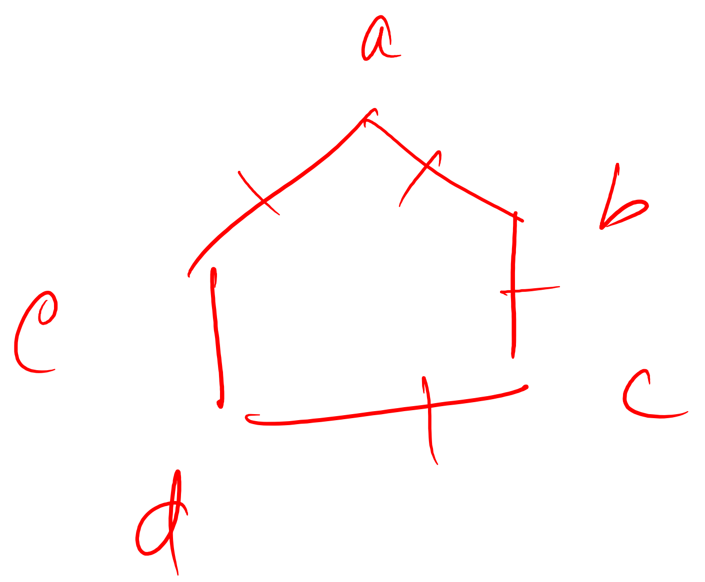
$\left\lceil \frac{5}{2} \right\rceil$

3 $\lceil 2.5 \rceil$



{a, c, e}

3



{a, c, e}

3

Graphs →	K_n
Independence Number	1
Vertex covering Number	n - 1

Graphs →	$K_{r,s}$
Independence Number	max{r,s}
Vertex covering Number	min{r,s}

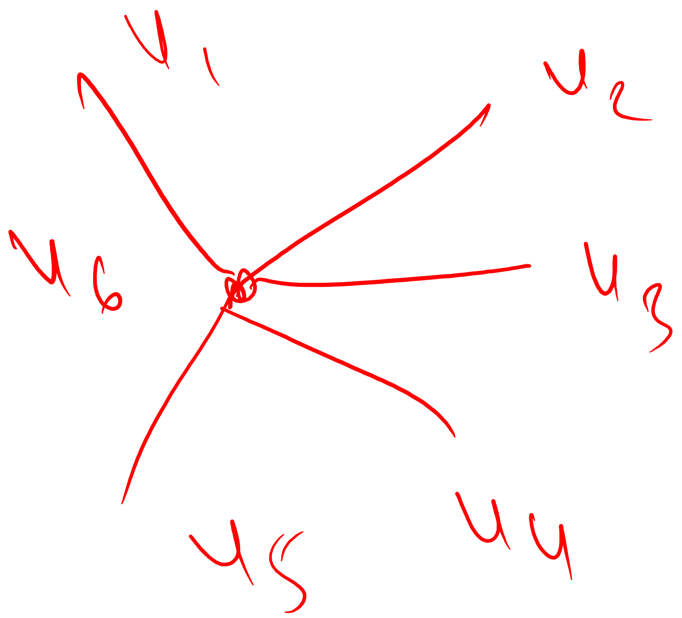
$V - I \rightarrow$ vertex cover

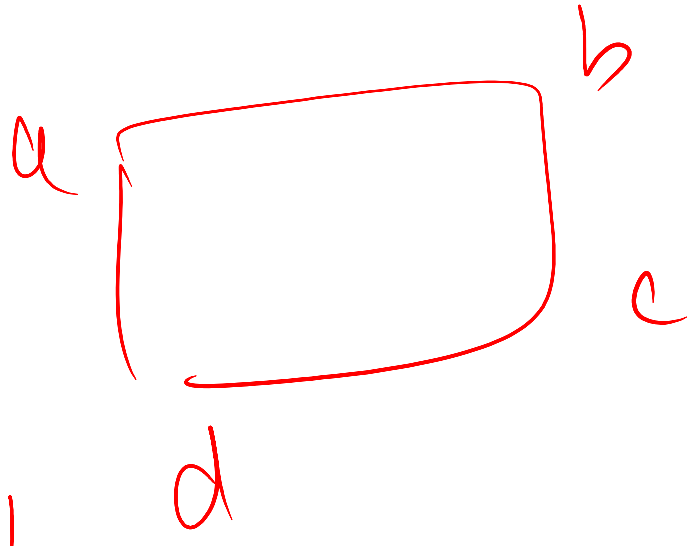
Notes –

•I is an independent set in G iff V(G) – I is vertex cover of G.

•For any graph G, $\alpha_0(G) + \beta_0(G) = n$, where n is number of vertices in G.

vertex cover + I = V(G)





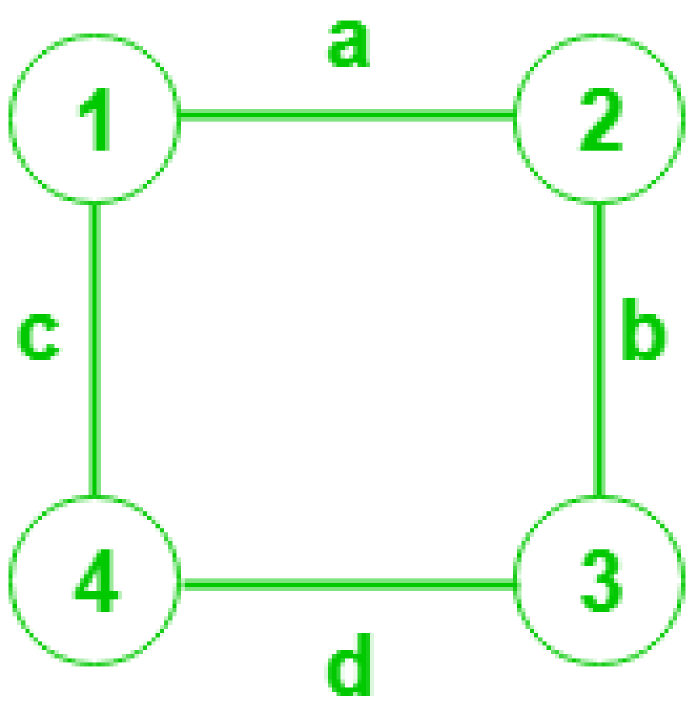
5 + 1

Edge Covering –

A set of edges F which can cover all the vertices of graph G is called a **edge cover** of G i.e. if every vertex in G is incident with a edge in F.

The parameter $\beta_1(G)$ = min { |F|: F is an edge cover of G } is called **edge covering number** of G i.e sum of minimum number of edges which can cover all the vertices and number of isolated vertices(if exist).

Any edge cover F with |F| = $\beta_1(G)$ is called a minimum edge cover.



$\{a,d\}$, $\{a,b,c\}$

$\{a,b,c,d\}$

MEC → 2

$\{a,b,c,d,e\}$

For above given graph G, Edge cover is

E_1 = {a, b, c, d}, E_2 = {a, d} and E_3 = {b, c}.

K_n

C_n: n even → $n/2$; n odd → $\lceil n/2 \rceil = \lfloor n/2 \rfloor + 1$

Therefore, minimum number of edges which can cover all vertices, i.e., Edge covering number $\beta_1(G) = 2$.

$\lceil \frac{n}{2} \rceil$

GATE CSE 2005

Let G be the simple graph with 20 vertices and 100 edges. The size of the minimum vertex cover of G is 8. Then, the size of the maximum independent set of G is:

$\beta_0(G)$

A.12

B.8

C.Less than 8

D.More than 12

$\alpha_0(G)$

$$\alpha_0(G) + \underbrace{\beta_0(G)}_{8} = 20$$

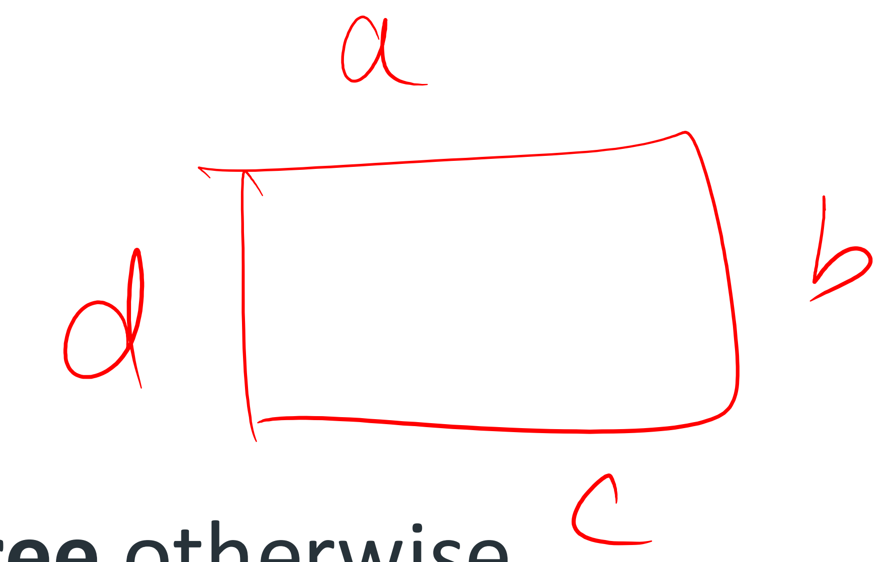
$$\alpha_0(G) = 12$$

Matching
(Graph theory)

Let G be a graph. Two edges are independent if they have no common end vertex. That is, each vertex in matching M has degree one.

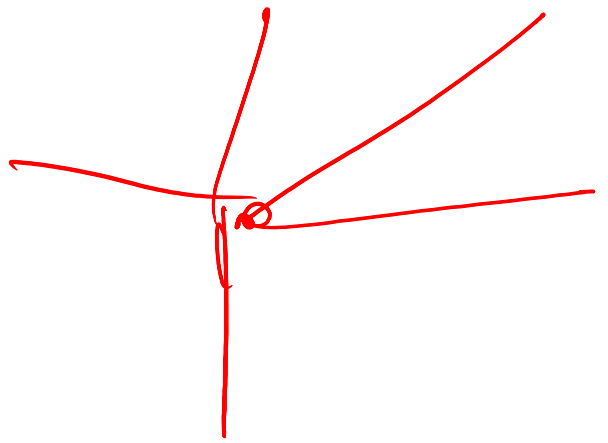
A set M of independent edges of G is called a matching.

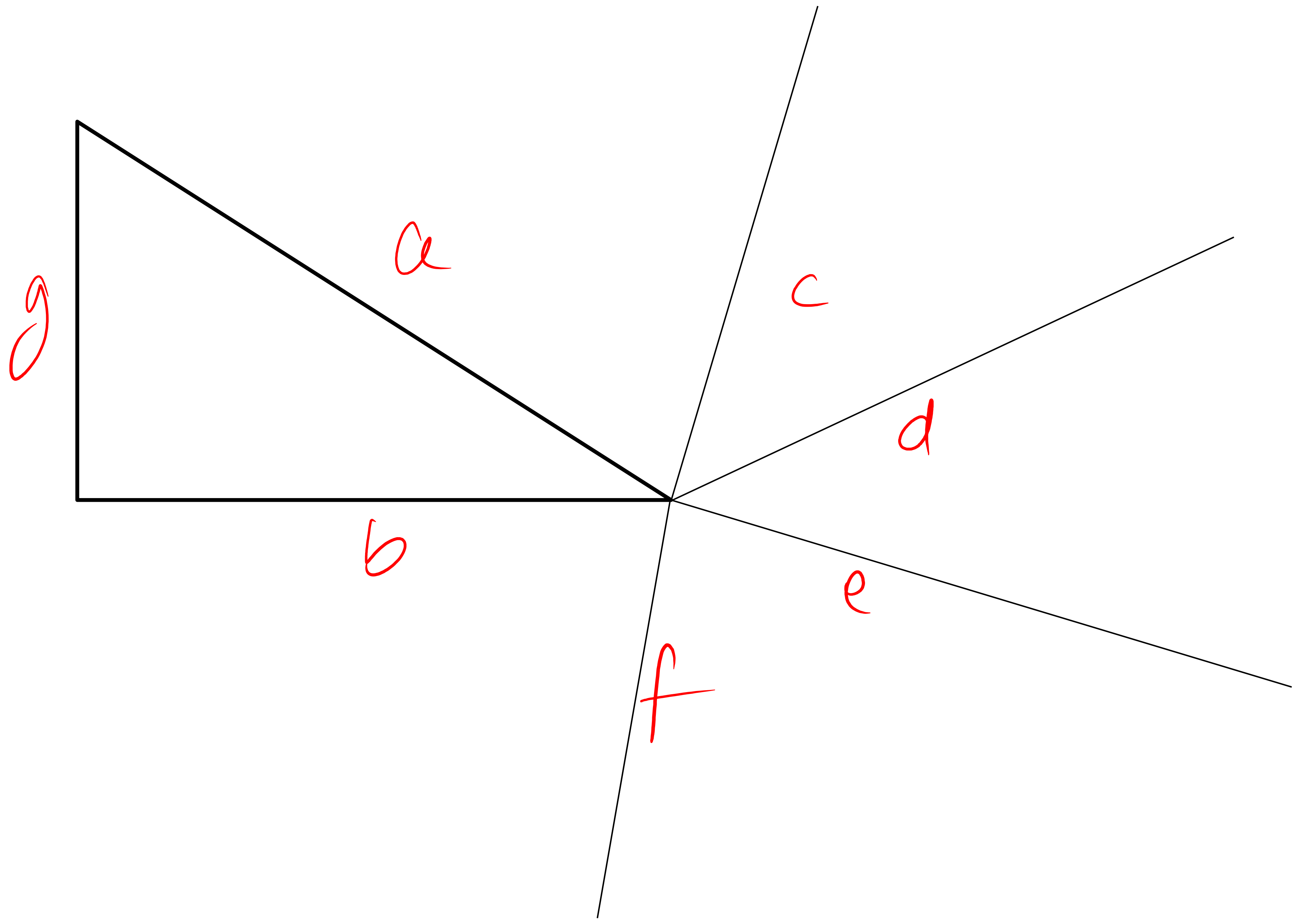
$\{a\}, \{b\}, \{a, c\}$



A vertex is said to be **matched** if an edge is incident to it, **free** otherwise.

$\{a, c\}$

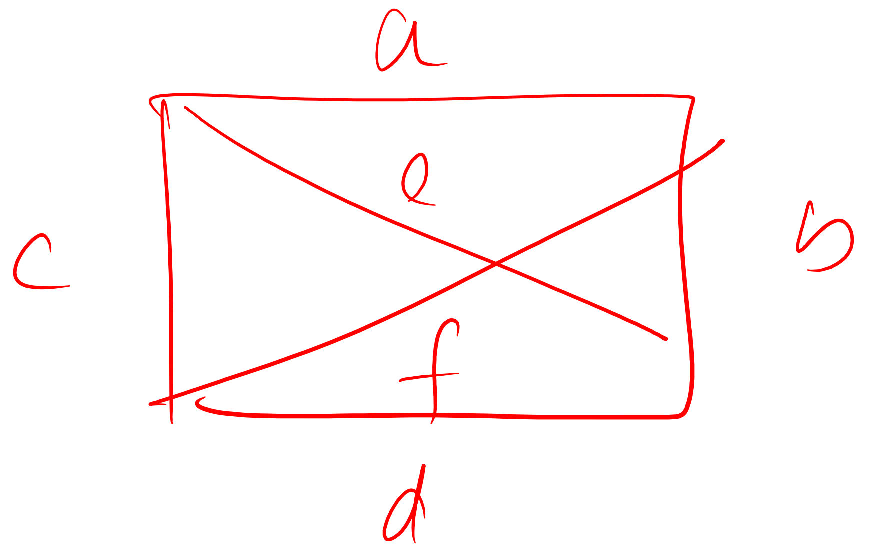




$\{a\} \rightarrow$ Maximal matching

$\{g, c\} \rightarrow$ M.M

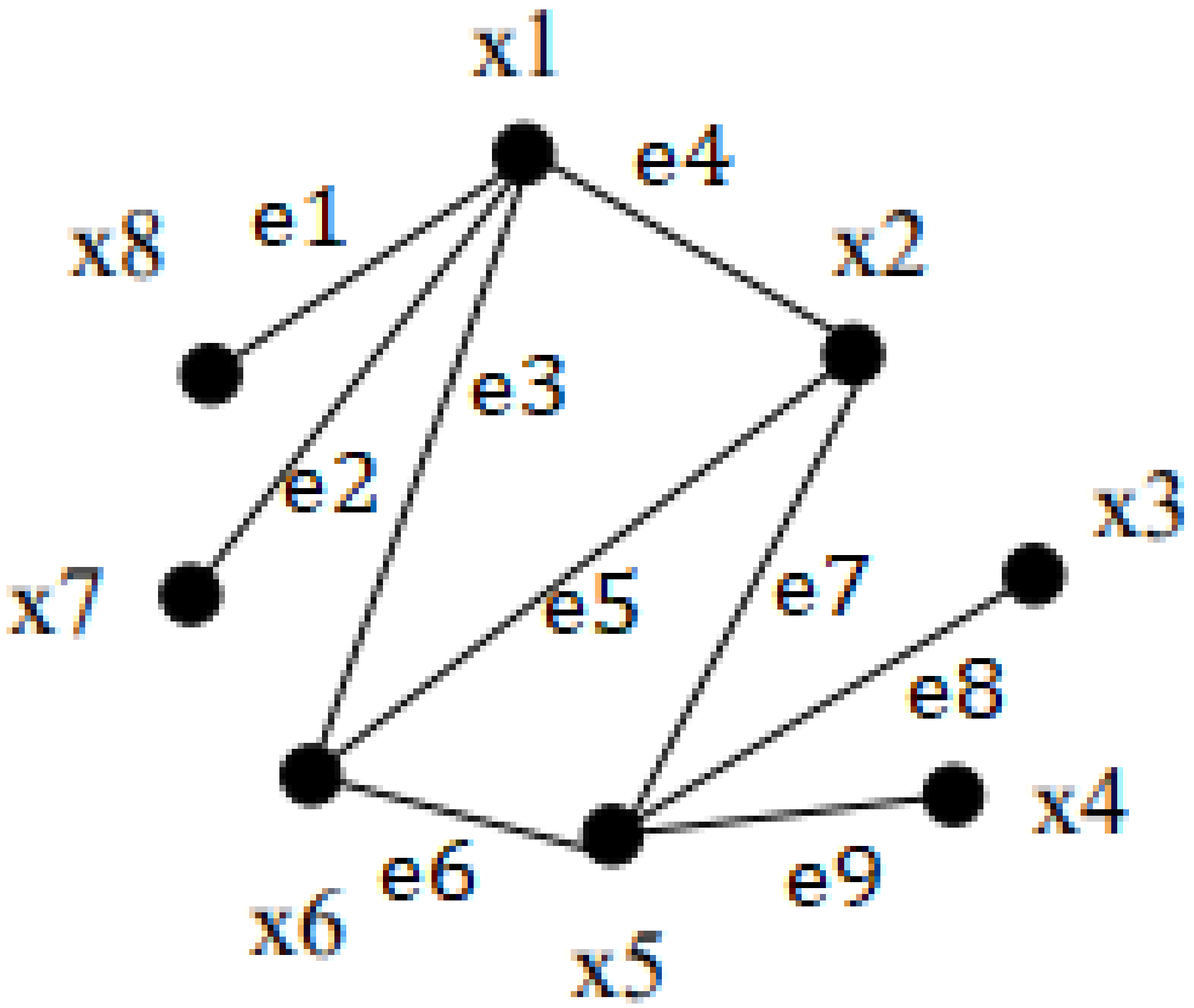
Possible matchings of K_4



$\{a, d\}$

$\{e, f\}$

$\{b, c\}$



$\rightarrow \{e_1, e_5, e_8\} \rightarrow$

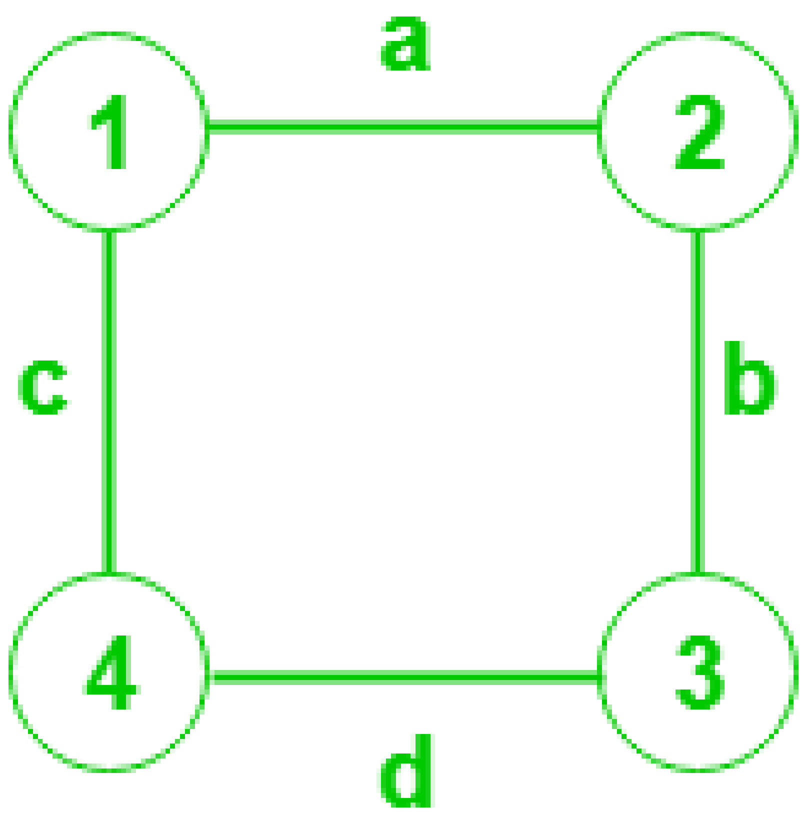
$\rightarrow \{e_3, e_7\}$

$\{e_1\}$

Maximal Matching – A matching M of graph G is said to be maximal if on adding an edge which is in G but not in M , makes M not a matching.

In other words, a maximal matching M is not a proper subset of any other matching of G .

Note For a given graph G, there may be several maximum matchings.



For above given graph G, Matching are:

$M_1 = \{a\}$, $M_2 = \{b\}$, $M_3 = \{c\}$, $M_4 = \{d\}$
$M_5 = \{a, d\}$ and $M_6 = \{b, c\}$

Adding any edge to any of the graphs M_5 and M_6 would result in them no longer being a matching.

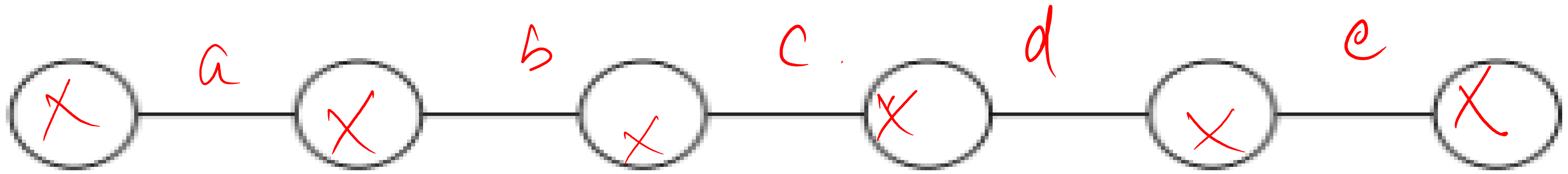
Hence, M_5 and M_6 are maximal matching.

The parameter $\alpha_1(G) = \max \{ |M| : M \text{ is a matching in } G \}$ is called **matching number** of G i.e the maximum number of non-adjacent edges.

Any matching M with $|M| = \alpha_1(G)$ is called a maximum matching.

TIFR 2022

Let $G = (V, E)$ be an undirected simple graph. A subset $M \subseteq E$ is a *matching* in G if distinct edges in M do not share a vertex. A matching is *maximal* if no strict superset of M is a matching. How many maximal matchings does the following graph have?



$\{a, d\} \rightarrow$

$\{a, c, e\} \rightarrow$ perf

$\{b, d\}$

$\{b, e\}$

A. 1

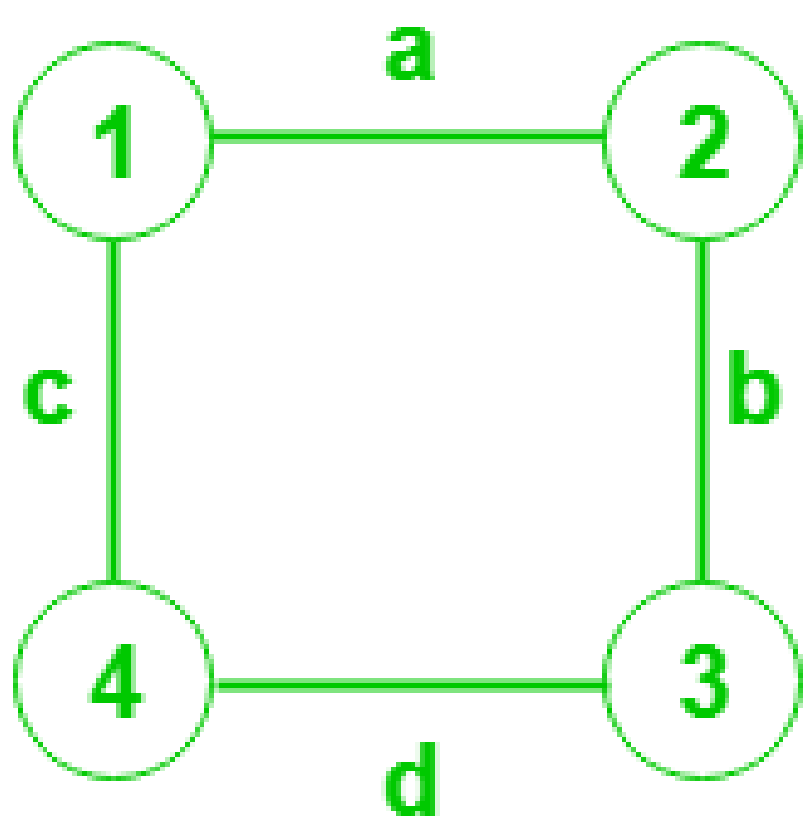
B. 2

C. 3

D. 4

E. 5

Complete Matching : A matching of a graph G is complete if it contains all of G's vertices. Sometimes this is also called a perfect matching.

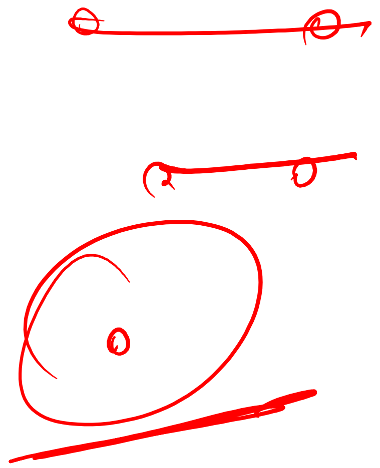


Since every vertex has to be included in a perfect matching, the number of edges in the matching must be |V|/2 , where V is the number of vertices.

Therefore, a perfect matching only exists if the number of vertices is even.

A matching is said to be **near perfect** if the number of vertices in the original graph is odd, it is a maximum matching and it leaves out only one vertex.

K_5



In any graph without isolated vertices, the sum of the matching number and the edge covering number equals the number of vertices.

i.e., For any graph G, $\alpha_1(G) + \beta_1(G) = n$, where n is number of vertices in G.

$\alpha_1(G) + \beta_1(G) = n$

Example – Count the number of perfect matchings in a complete graph K_n

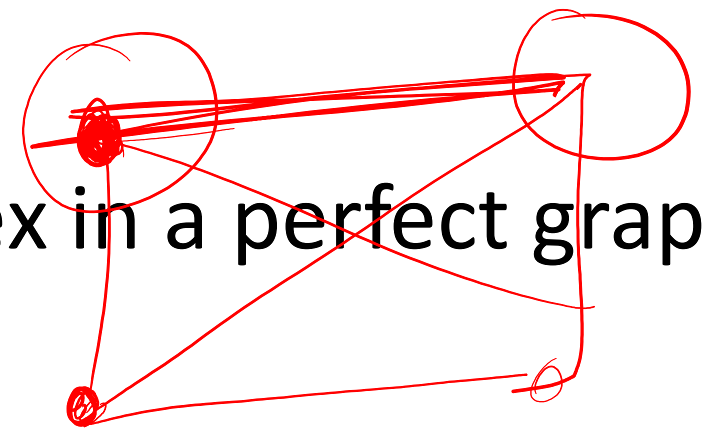
$n = 2m$

K_{2m}

The complete graph K_n have a perfect matching only when n is even.
So, let n = 2m

$(2m-1)(2m-3) \cdots 1$

$2m$



Every vertex is connected to every other vertex in a perfect graph, therefore the degree of each vertex is 2m-1.
We can choose an edge in 2m-1 ways.

left $\rightarrow (2m-2)$

Now 2m-2 vertices remain and (2m-2)-1 = 2m-3 edges remain since the edges connected to the already selected vertices cannot be selected because it is a matching.

So the number of ways of selecting an edge from 2m-3 edges is 2m-3.

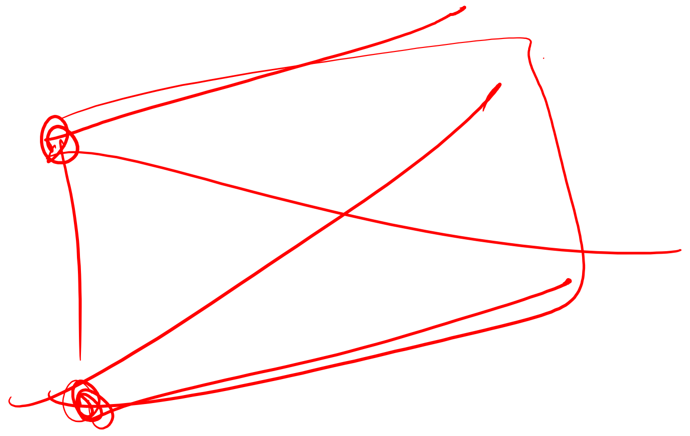
We can keep choosing edges in the same way, then by product rule-

No. of Perfect matches=(2m−1)(2m−3)(2m−5)…(3)(1)

$2m = 4$

K

1



$K_4 \rightarrow 3$

$(4-1)(4-3) =$

3×1

$\Rightarrow = 3$

GATE-CS-2003

How many perfect matchings are there in a complete graph of 6 vertices ?

(A) 15

(B) 24

(C) 30

(D) 60

$2m = 6$

$\cancel{2m}$

$(6-1)(6-3)(6-5)$

$5 \times 3 \times 1$

$= 15$

Ad

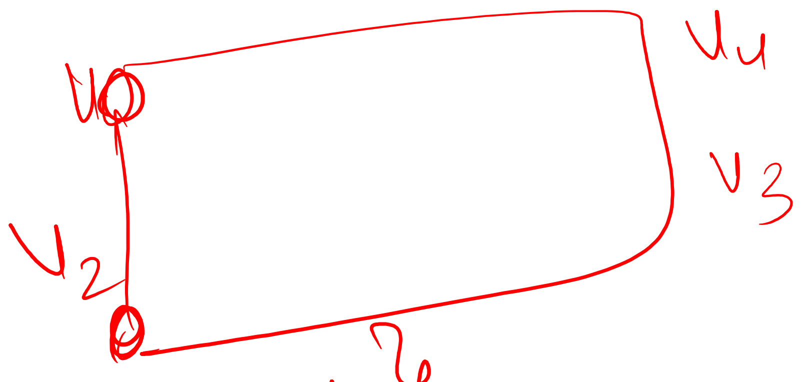
Answer: (A)

(Hall's Marriage Theorem)

For any set S ⊂ ~~A~~ V, let N(S) denote the set of vertices which are adjacent to at least one vertex in S.

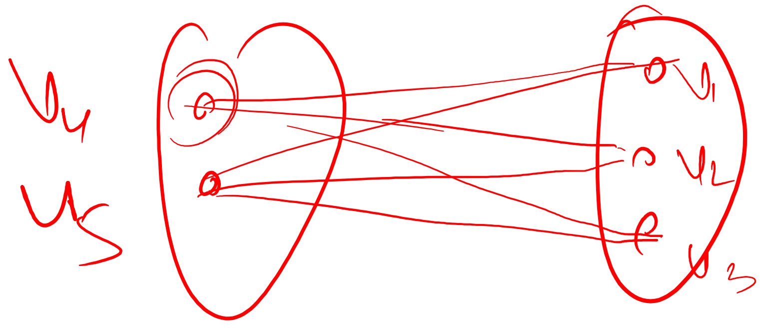
Then, The bipartite graph G =(V, E) with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if |N(S)| ≥ |S| for every S ⊂ V_1.

(This is both necessary and sufficient condition for complete matching.)



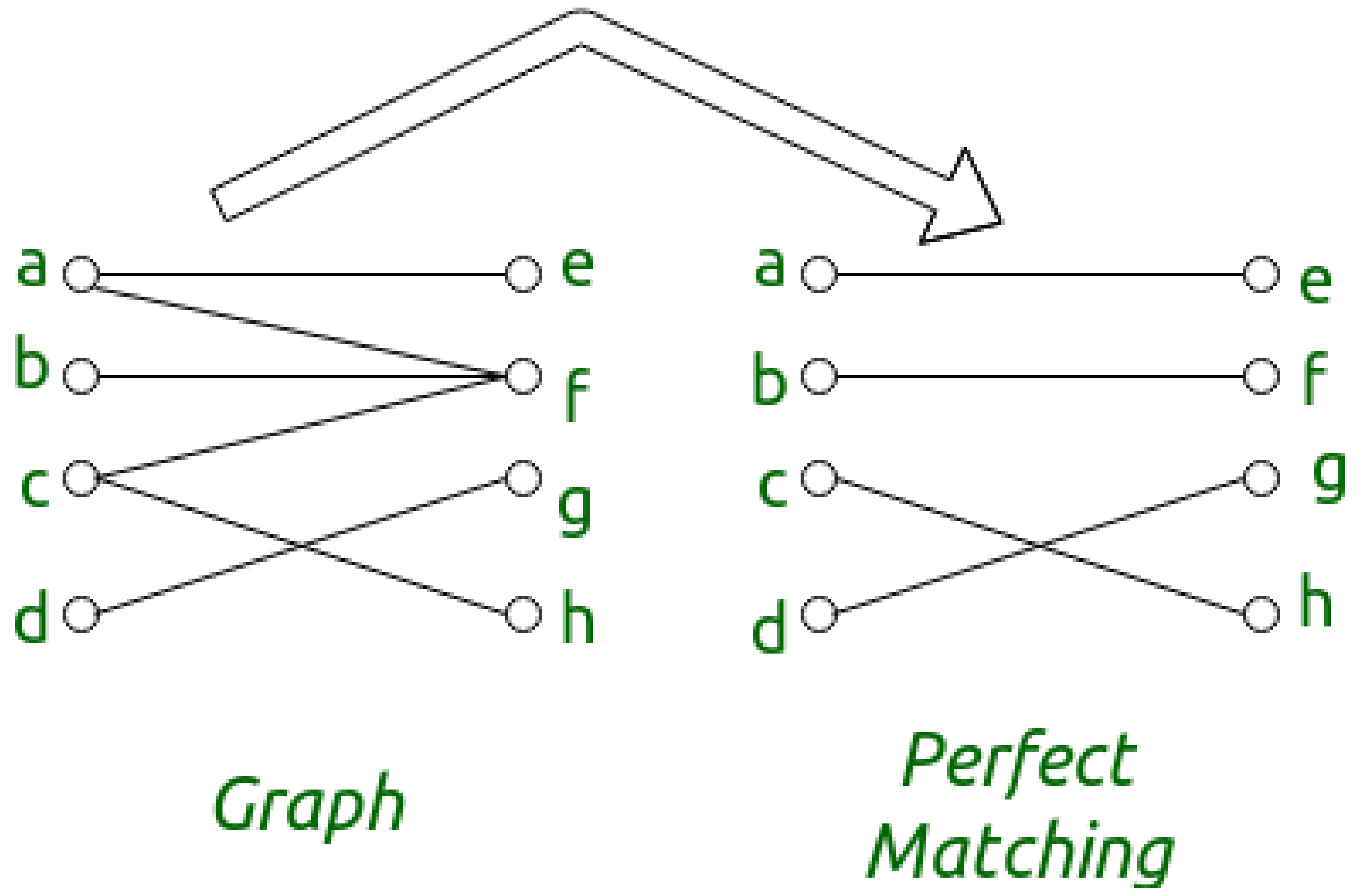
$S := \{v_1, v_2\}$

$N(S) = \{v_1, v_2, v_3, v_4\}$



$S = \{v_1, v_2, v_3\}$

$N(S) = \{v_4, v_5\}$


a
b
c
d
e
f
g
h
Graph
a
b
c
d
e
f
g
h
Perfect
Matching

GATE-CS-2003

k − 1

Let G be an arbitrary graph with n nodes and k components. If a vertex is removed from G, the number of components in the resultant graph must necessarily lie between

(A) k and n

(B) k – 1 and k + 1

3

(C) k – 1 and n – 1

(D) k + 1 and n – k

3

9

Answer: (C)

Lower Bound: number of components decreased by one =k−1 (remove an isolated vertex which was a component)

6

2

10

Upper Bound: number of components =n−1 (consider a vertex connected to all other vertices in a component as in a star and all other vertices outside this component being isolated. Now, removing the considered vertex makes all other n−1 vertices isolated making n−1 components)

GATE-CS-2014-(Set-3)

If G is a forest with n vertices and k connected components, how many edges does G have?

(A) floor(n/k)

(B) ceil(n/k)

(C) n-k

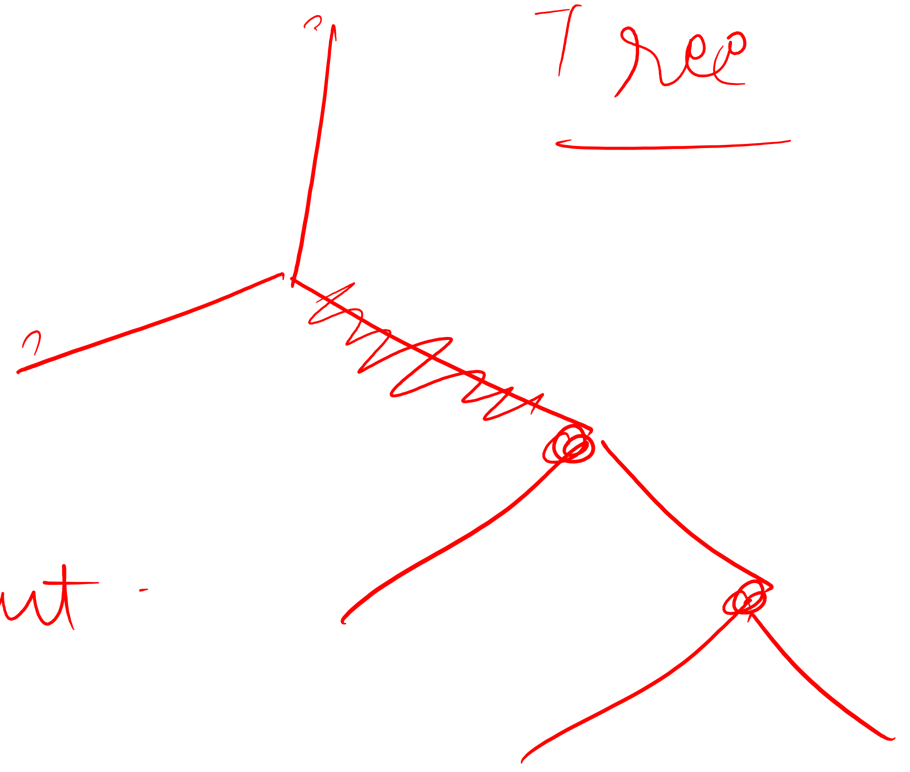
(D) n-k+1

Tree

n vertices

n-1 edges

1 component

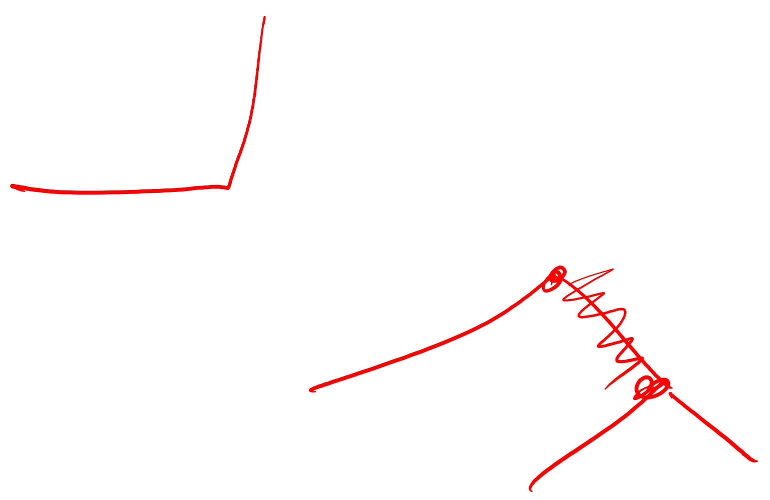


n - 3 edges

3 component

n - 2 edges

2 component



Explanation: Forest with 1 component (i.e) a tree will have (n-1) edges.

If I remove 1 edge from this (n-1), we will get 2 components which will have (n-1)-1 = (n-2) edges.

If I remove 1 more edge (i.e a total of 2 edges) from this (n-2), we will get 3 components which will have (n-2)-1 = (n-3) edges.

Finally when I remove k edges (one-by-one) number of edges in the graph will become (n-k).

Answer: (C)

GATE CS 2012

$\frac{(n-1)!}{2}$

Let G be a complete undirected graph on 6 vertices. If vertices of G are labeled, then the number of distinct cycles of length 4 in G is equal to

(A) 15

(B) 30

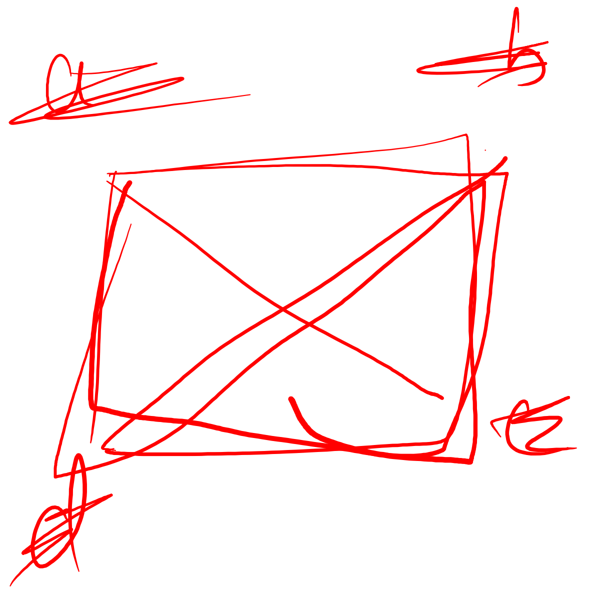
(C) 45

(D) 360

n

d c b a

$^6C_4 \rightarrow 15$



a b c d

a c d b

a d b c

Explanation: There can be total ${}^6C_4$ ways to pick 4 vertices from 6. The value of ${}^6C_4$ is 15.

No. of cyclic permutations of n objects =(n−1)! and for n=4, we get 3!=6 ways.

But number of distinct cycles in a graph is exactly half the number of cyclic permutations as there is no left/right ordering in a graph.

So, total no. of distinct cycles of length 4=15×3=45

Answer: (C)

GATE-CS- 2022

10

Consider a simple undirected graph of 10 vertices. If the graph is disconnected, then the maximum number of edges it can have is ___________.

36

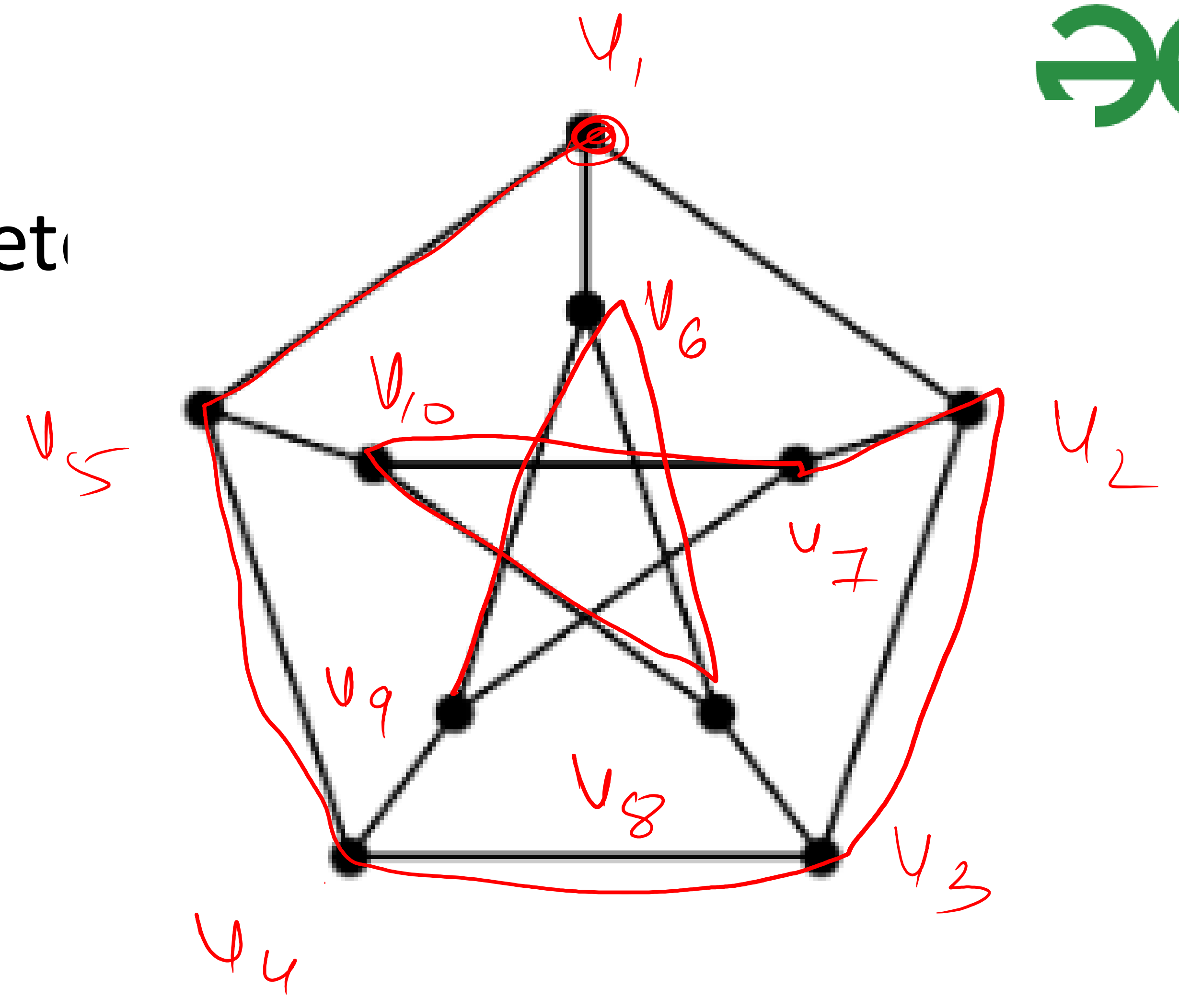
Answer: (36)

${}^{9}C_2$

9	1	36	0	36
8	2	28	1	29
7	3			
6	4			
5	5			

GATE-CS-

The following simple undirected graph is referred to as the Pet



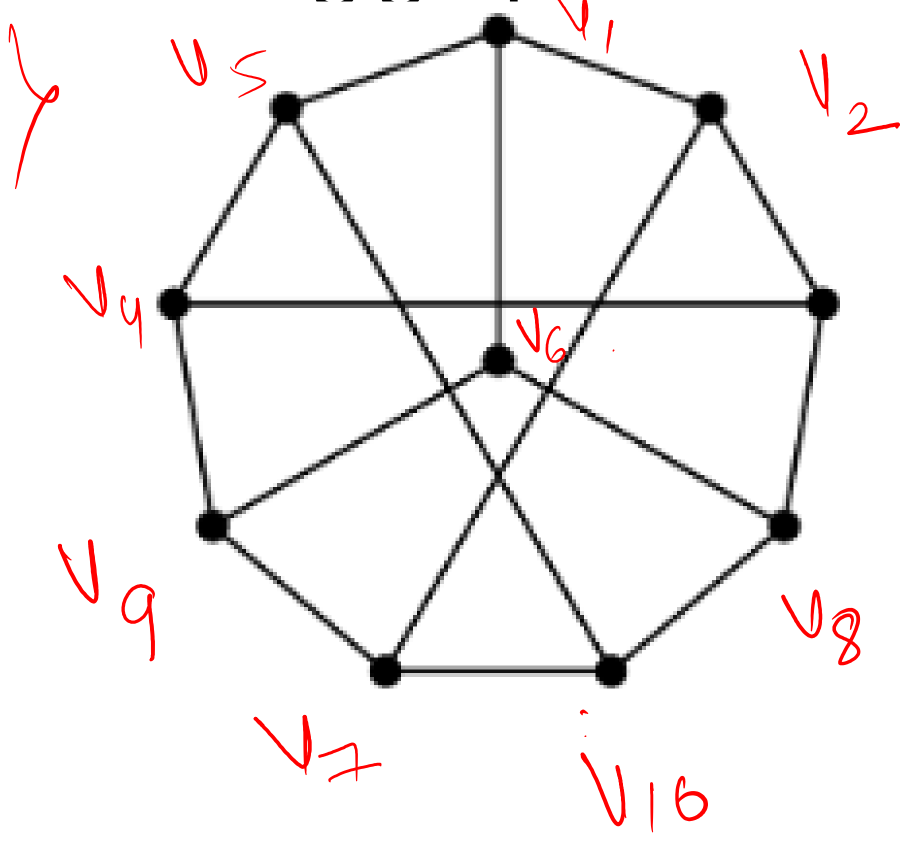
Which of the following statements is/are TRUE?

A. The chromatic number of the graph is 3.

B. The graph has a Hamiltonian path.

C. The following graph is isomorphic to the Peterson graph.

{V5, V8, V2, V9}



→ isomorphic

D. The size of the largest independent set of the given graph is 3. (A subset of vertices of a graph form an independent set if no two vertices of the subset are adjacent.)

Answer: (A, B & C)

GATE-CS- 2014

$^8C_3 \longrightarrow \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \;-\; 7$

Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is $\frac{1}{2}$. What is the expected numbers of unordered cycles of length three?

- (A) $\frac{1}{8}$
- (B) 1
- (C) 7
- (D) 8

Here, n = $^8C_3$ and p = $\frac{1}{8}$

Expected no of cycles = $^8C_3 \times \frac{1}{8} = 7$.

Answer: C

GATE-CS- 2009

What is the chromatic number of an n vertex simple connected graph which does not contain any odd length cycle? Assume $n > 2$.

A. 2
B. 3
C. $n - 1$
D. n

Answer: (A)

Thank you