Introduction to Set Theory

well defined collection of distinct

$\{1,1\} = \{1\}$

Set

Set is a well defined collection of distinct objects.

Well Defined

A set is considered to be well defined if it is possible to establish that any given object belongs to the set.

Example: 1. odd numbers less than 10.

2. Vowels in English alphabets.

{a, e, i, o, u}

example, { tall people } is not a set, because people tend to disagree about what 'tall' means.

Member

Any object belonging to a set, known as elements or members of the set.

An element ‘a’ belongs to a set A can be written as ‘a ∈ A’,

‘a ∉ A’ denotes that a is not an element of the set A.

$S = \{1, 2, 3\}$ $4 \notin S$

$1 \in S$ $3 \in S$

$2 \in S$

Subset (Inclusion)

A set A is said to be **subset** of another set B if and only if every element of set A is also a part of other set B.

Denoted by '⊆'.

'A ⊆ B ' denotes A is a subset of B.

To prove A is the subset of B, we need to simply show that if x belongs to A then x also belongs to B.

To prove A is not a subset of B, we need to find out one element which is part of set A but not belong to set B.

$\phi \rightarrow$ Empty set

$\rightarrow$ Null set

$S = \{1, 2, 3\}$

$\phi \subseteq S$

$A \subseteq B$

$\{1, 2\} \subseteq S$

$\rightarrow \{\phi\}$

$A \not\subseteq B$

$\{3, 4\} \not\subseteq S$

Equal sets

Two sets are said to be equal if both have same elements.

For example A = {1, 3, 9, 7} and B = {3, 1, 7, 9} are equal sets.

NOTE: Order of elements of a set doesn't matter.

Proper Subset

A set A is said to be a proper subset of B of A ⊆ B but A ≠ B

It is represented by $A \subset B$.

$\subseteq$

$A = \{1, 2\}$, $B = \{1, 2, 3\}$

$A = \{1, 2, 3\}$ $A \subseteq B$

$A \subseteq B$

Difference between elements and subsets of a set

Example: A= {{1, 2}, 3, 4, 5} , B = {1, 2, 3, 4, 5}

$\{1,2\}$ ✓

3

4

5

4

5

$\{1,2\} \in A$ ✓

$\{1,2\} \subseteq A$ ✗

1

2

$\{1,2\} \in B$ ✗

$\{1,2\} \subseteq B$ ✓

$\{\{1,2\},3\} \subseteq A$ ✓

$\{\{1,2\},3\} \subseteq B$ ✗

$\{\{1,2\}\} \subseteq A$

$\{\{1,2\}\} \subseteq B$ ✗

Universal Set

A universal set is a set which includes every set under consideration.
It is represented by U.
For any, predicate P(x),

$$U = \{x | P(x) \vee \neg P(x)\}$$

The universal set is same as universe of discourse.

Null Set

A set which does not contain any element is known as null set or empty set.
It is denoted by ϕ.
For example: A set of all even integers which are both even and odd.

Singleton Set

A singleton set is a set with single element.

Example: $\{4\}$, $\{\phi\}$

$\{1\}$ $\{\phi\}$

Size of a Set

Size of a set can be finite or infinite.

$\{\phi\}$

For example

Finite set: Set of natural numbers less than 100.

Infinite set: Set of real numbers

Size of the set S is known as **Cardinality number**, denoted as |S|.

Example: Let A be a set of odd positive integers less than 10.

Solution : A = {1,3,5,7,9}, Cardinality of the set is 5, i.e., |A| = 5.

Note: Cardinality of a null set is 0.

Gate IT 2005

Let A be a set with n elements. Let C be a collection of distinct subsets of A such that for any two subsets S_1 and S_2 in C, either $S_1 \subset S_2$ or $S_2 \subset S_1$. What is the maximum cardinality of C?

(A) n

(B) n + 1

(C) $2^{(n-1)} + 1$

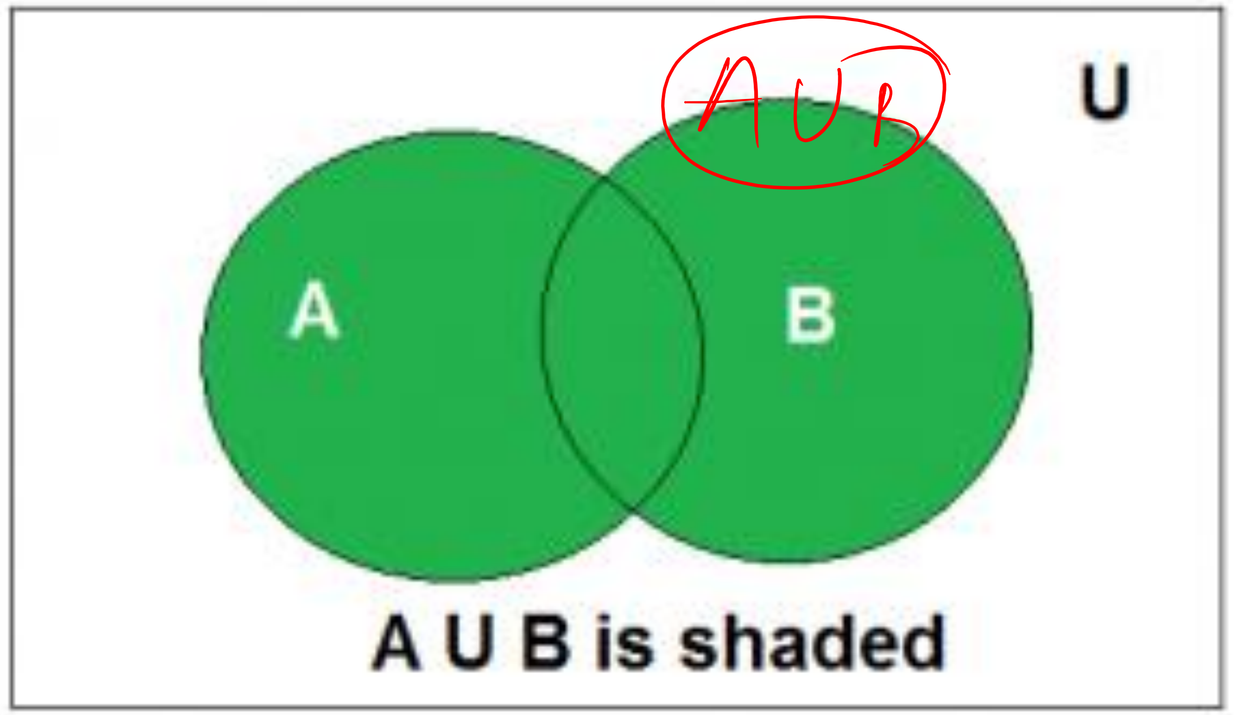
(D) n!

Answer: (B)

Set Operations

Union

Union of the sets A and B, denoted by A ∪ B, is the set of distinct elements that belong to set A or set B, or both.



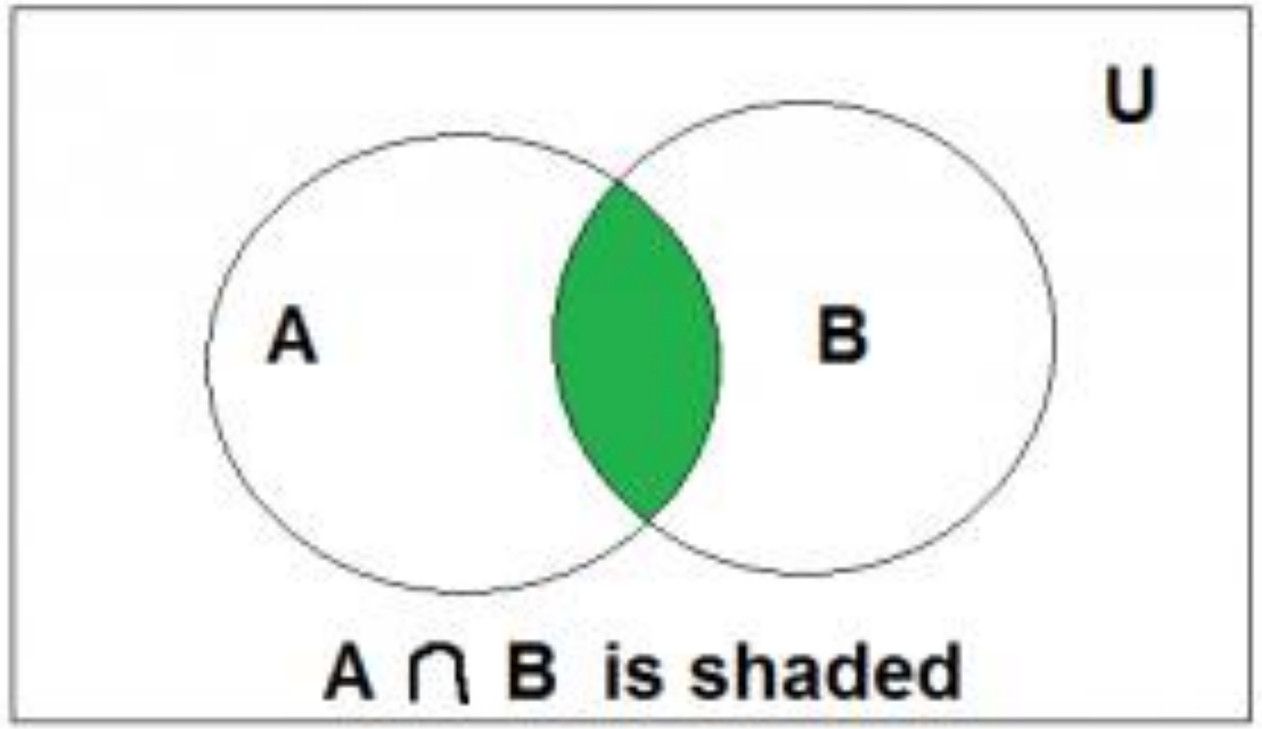
Venn diagram of A ∪ B

Example: Find the union of A = {2, 3, 4} and B = {3, 4, 5};

Solution : A ∪ B = {2, 3, 4, 5}.

Intersection

The intersection of the sets A and B, denoted by A ∩ B, is the set of elements that belong to both A and B i.e. set of the common elements in A and B.



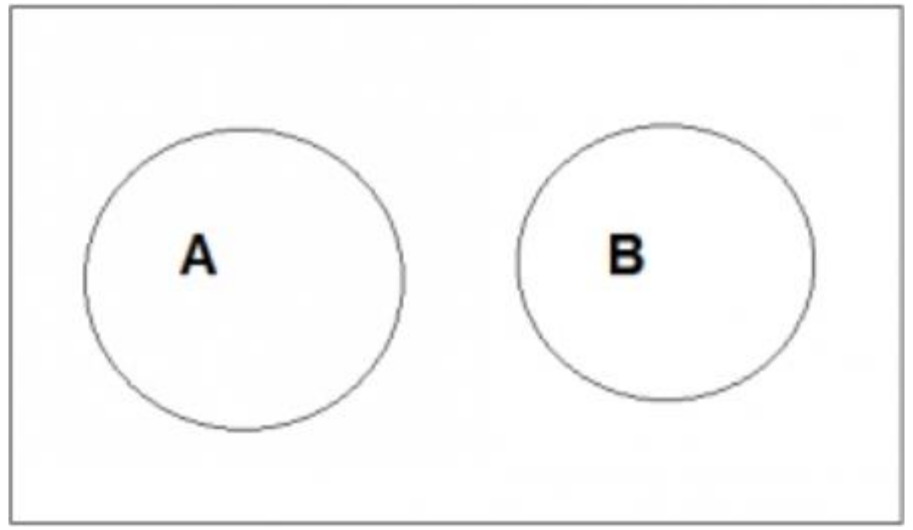
Venn diagram of A ∩ B

Example: Find the intersection of A = {2, 3, 4} and B = {3, 4, 5}

Solution : A ∩ B = {3, 4}.

Disjoint

Two sets are said to be disjoint if their intersection is the empty set. i.e, sets have no common elements.

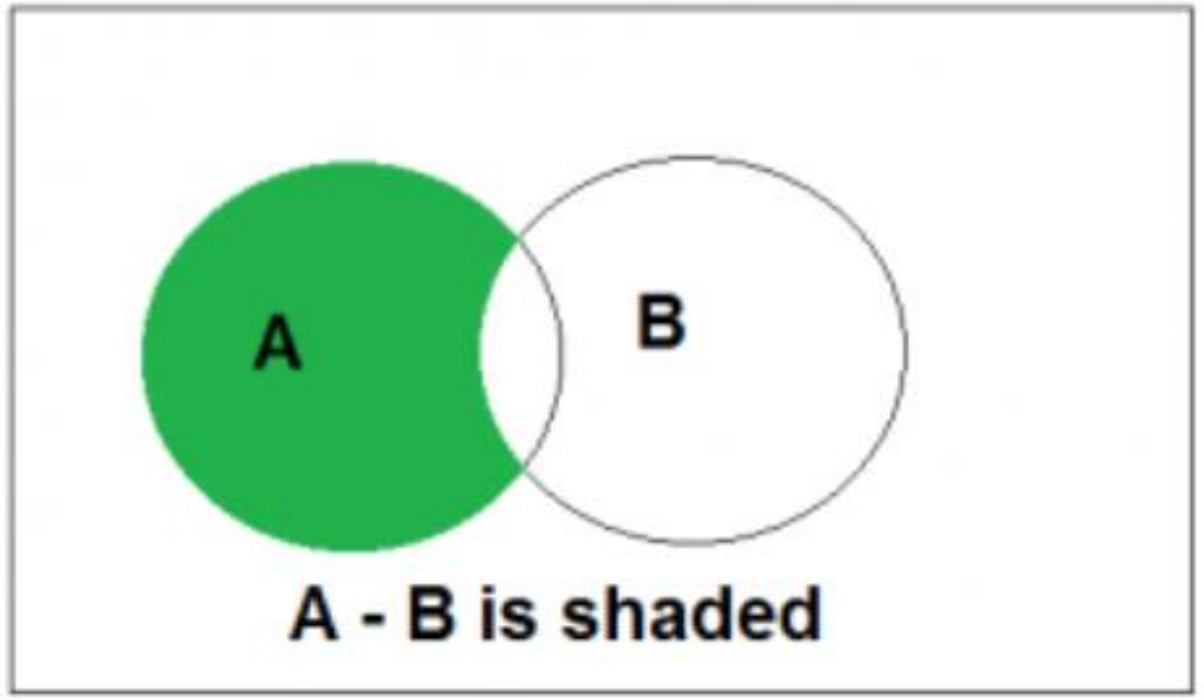


Venn Diagram of A disjoint B

Example: Let A = {1, 3, 5, 7, 9} and B = { 2, 4, 6, 8}
A and B are disjoint sets since both of them have no common elements.

Set Difference

The difference between sets is denoted by ‘A – B’, which is the set containing elements that are in A but not in B. i.e., all elements of A except the element of B.

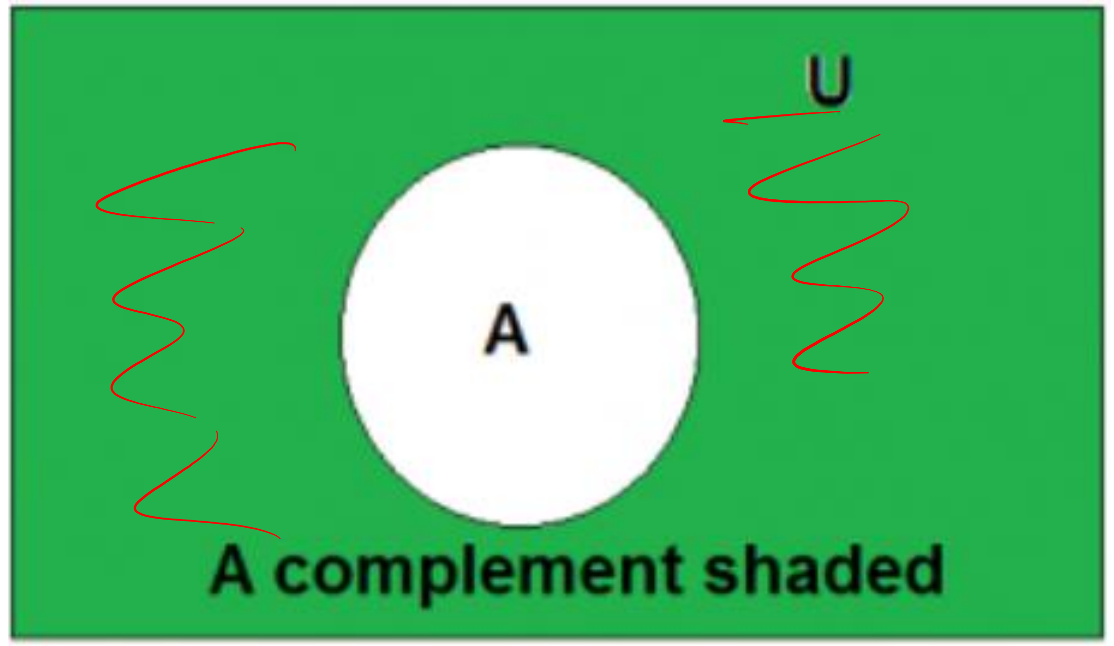


Example: If A = {1, 2, 3, 4, 5} and B = { 2, 4, 6, 8}, find A-B

Solution: A-B = {1, ~~3~~, 5}

Complement

The complement of a set A, denoted by A^C is the set of all the elements except the elements in A. Complement of the set A is U – A.



$A^c = U - A$

Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
and A = {2, 4, 6, 8}.

A^C = U-A = {1, 3, 5, 7, 9, 10}

SET PROPERTIES

1. **Associative Properties:** $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

2. **Commutative Properties:** $A \cup B = B \cup A$ and $A \cap B = B \cap A$

3. **Identity Property for Union:** $A \cup \phi = A$

4. **Intersection Property of the Empty Set:** $A \cap \phi = \phi$

5. **Distributive Properties:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ similarly for intersection.

Cartesian Products

Let A and B be two sets. Cartesian product of A and B is denoted by A × B, is the set of all ordered pairs (a,b), where a belong to A and b belong to B.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

$A = \{1, 2, 3\}$

Example . What is Cartesian product of A = {1,2} and B = {p, q, r}.

$B = \{1, 4, 5\}$

Solution : A × B ={(1, p), (1, q), (1, r), (2, p), (2, q), (2, r) };

$A \times B = \{(1,1), (1,4), (1,5), (2,1), (2,4), (2,5), (3,1), (3,4), (3,5)\}$

The cardinality of A × B is N*M, where N is the Cardinality of A and M is the cardinality of B.

Note: A × B is not the same as B × A.

$|A \times B| = |A| \times |B|$

$= 3 \times 3$

$= 9$

GATE CS 2008

If P, Q, R are subsets of the universal set U, then

$$(\mathbf{P} \cap \mathbf{Q} \cap \mathbf{R}) \cup (\mathbf{P}^c \cap \mathbf{Q} \cap \mathbf{R}) \cup \mathbf{Q}^c \cup \mathbf{R}^c \text{ is}$$

$U \cap A = A$

(A) $Q^c \cup R^c$

(B) $P \cup Q^c \cup R^c$

(C) $P^c \cup Q^c \cup R^c$

(D) U

$(P \cup P^c) \cap (Q \cap R) \cup Q^c \cup R^c$

$(Q \cap R)^c = Q^c \cup R^c$

$U \cap (Q \cap R) \cup Q^c \cup R^c$

$$(P \cap Q \cap R) \cup (P' \cap Q \cap R) \cup (Q' \cup R')$$
$$= (P \cup P') \cap ((Q \cap R)) \cup (Q' \cup R'))$$
$$= U \cap ((Q \cap R) \cup (Q \cap R)')$$
$$= U \cap U$$
$$= \text{U(universal set)}$$

$(Q \cap R) \cup (Q^c \cup R^c)$

$(Q \cap R) \cup (Q \cap R)^c$

$A \cup A^c = U.$

Answer: (D)

Let E, F and G be finite sets.

Let X = (E ∩ F) – (F ∩ G) and Y = (E – (E ∩ G)) – (E – F).

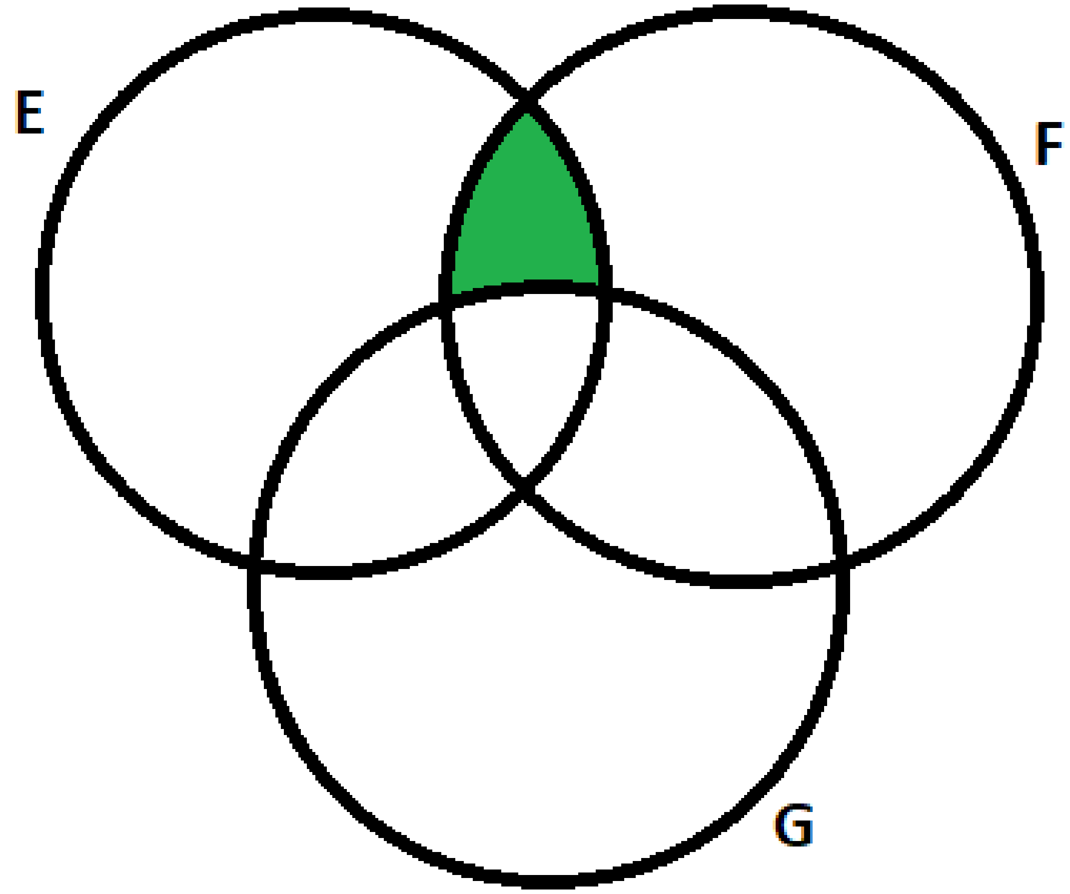
Which one of the following is true?

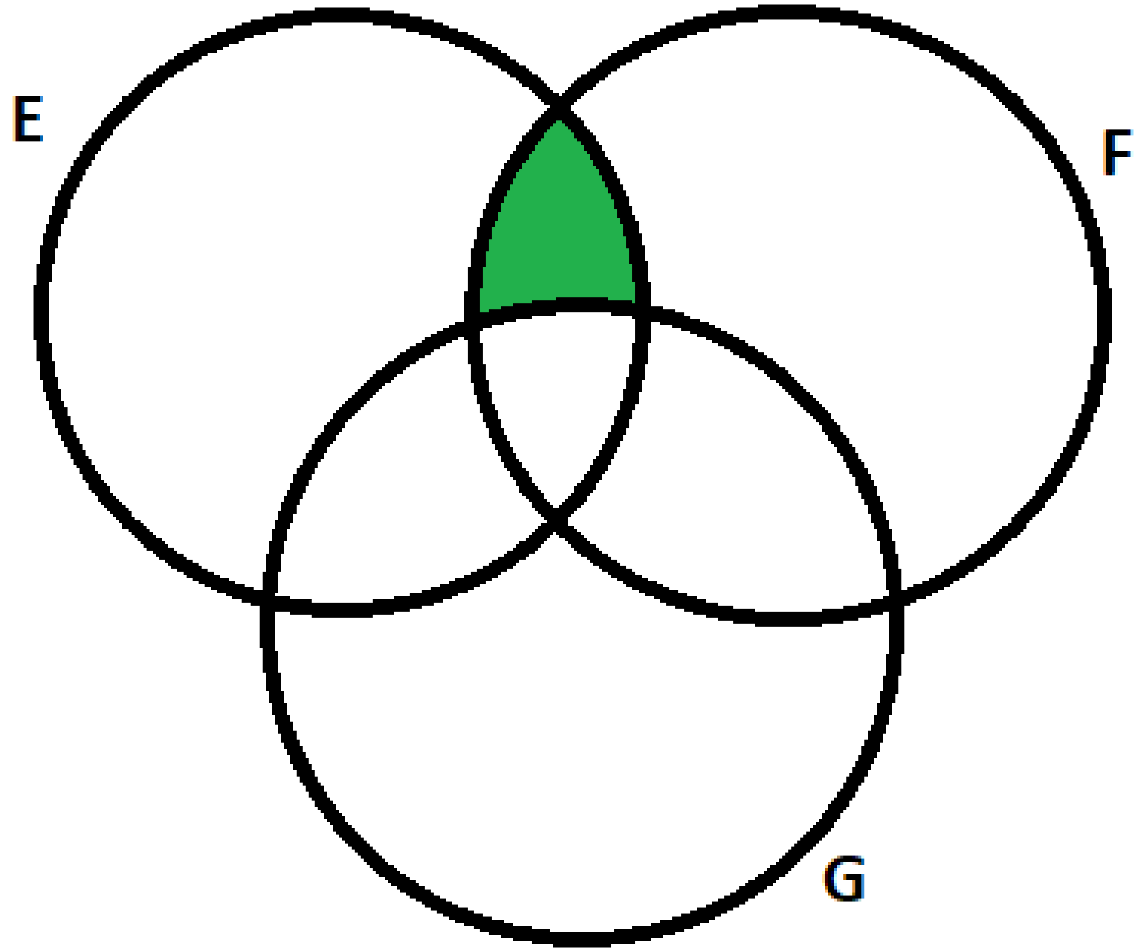
(A) X ⊂ Y

(B) X ⊃ Y

(C) X = Y

(D) X – Y ≠ ϕ and Y – X ≠ ϕ





Answer: (C)

GATE-CS-2005

Let A, B and C be non-empty sets and let X = (A – B) – C and Y = (A – C) – (B – C).

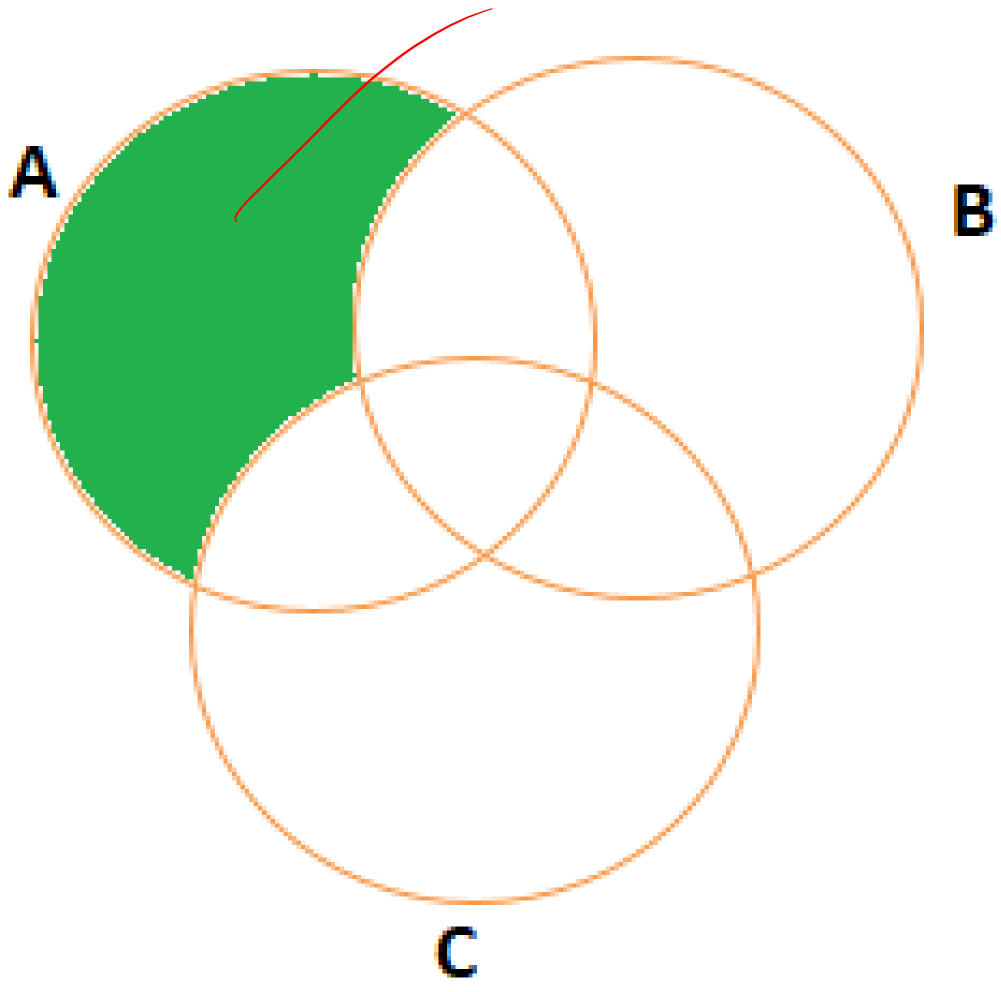
Which one of the following is TRUE?

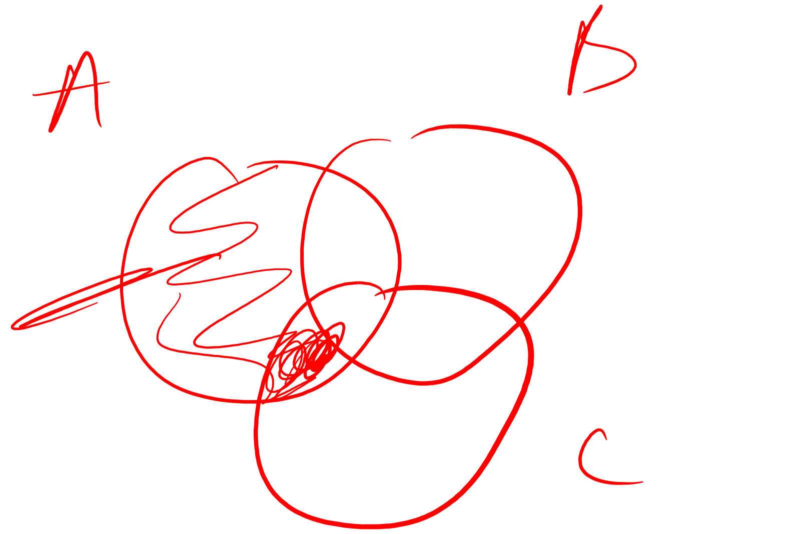
(A) X = Y

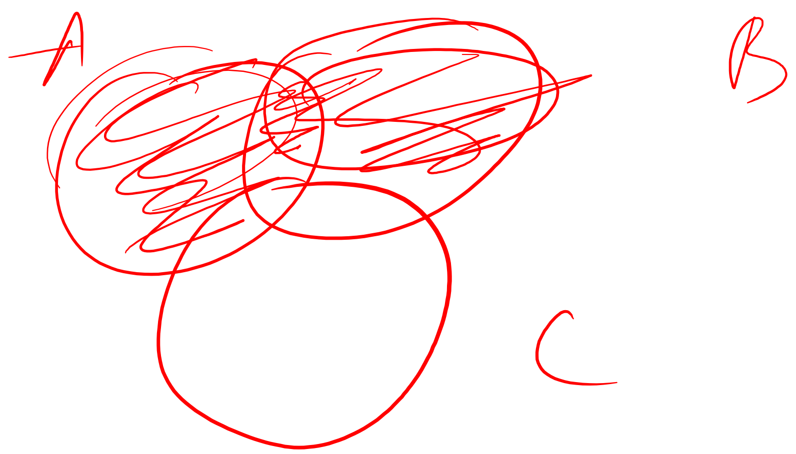
(B) X ⊂ Y

(C) Y ⊂ X

(D) none of these







Answer: (A)

GATE CS 1996

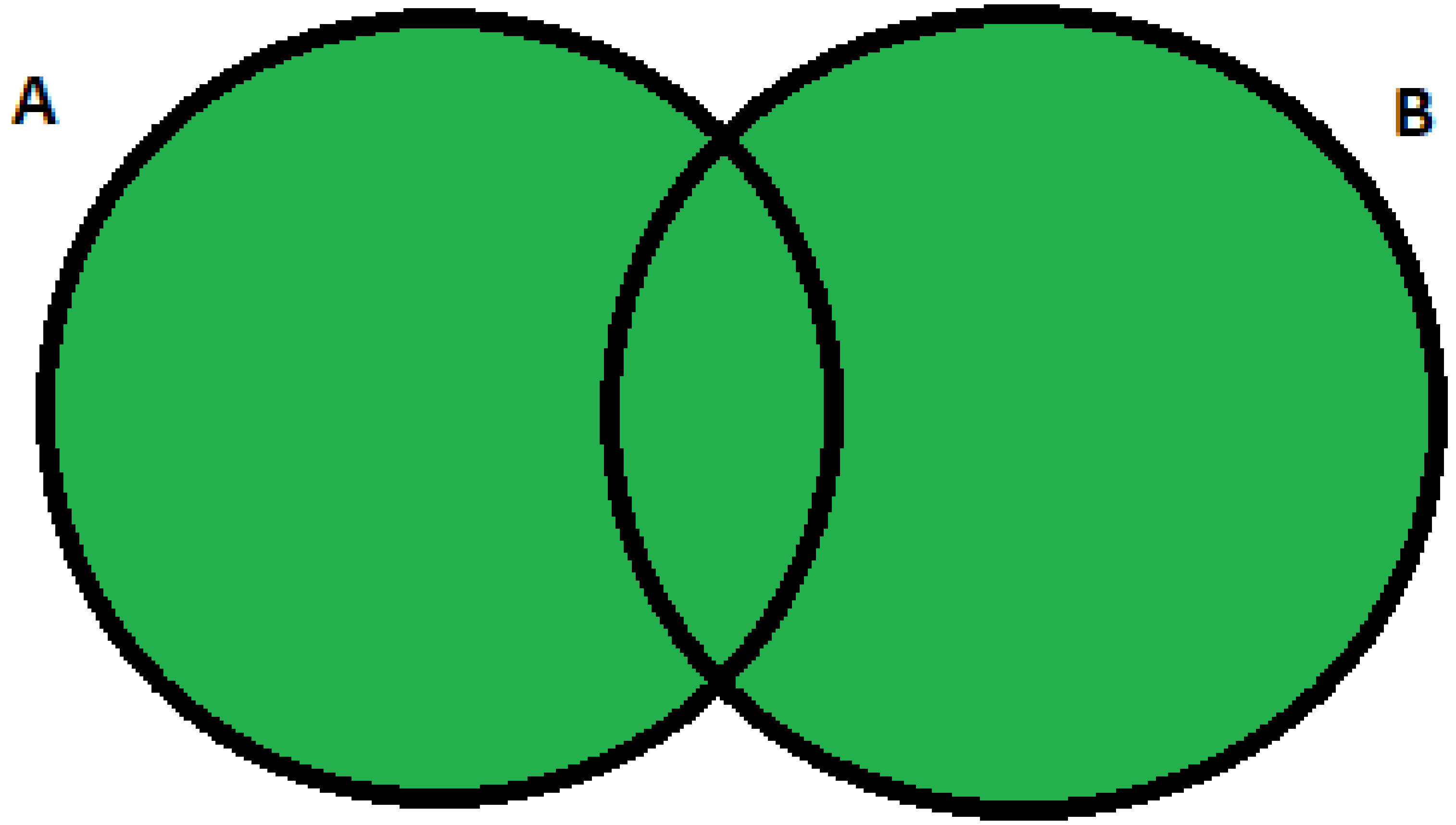
Let A and B be sets and let A^c and B^c denote the complements of the sets A and B. The set $(A-B) \cup (B-A) \cup (A \cap B)$ is equal to

a). $A \cup B$

b). $A^c \cup B^c$

c). $A \cap B$

d). $A^c \cap B^c$



Answer: (A)

ISRO CS 2017

The symmetric difference of sets A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9} is

(A) {1, 3, 5, 6, 7, 8}

(B) {2, 4, 9}

(C) {2, 4}

(D) {1, 2, 3, 4, 5, 6, 7, 8, 9}

$A \Delta B = (A - B) \cup (B - A)$

$= \{2, 4\} \cup \{9\}$

$= \{2, 4, 9\}$

Symmetrical Difference of A and B

= (A – B) ∪ (B – A)

(A – B) = {2, 4}

(B – A) = {9}

(A – B) ∪ (B – A) = {2, 4} ∪ {9} = {2, 4, 9}

Answer: (B)

Power Sets

$\{1, 2, 3\}$

$\downarrow$

set of all subsets is power set.

Definition

The power set of set S is the collection of all possible subset of the set S.

It is denoted by P(S).

$\{\phi\} \not\subseteq S$ $\phi \subseteq S$

$$P(S) = \{A \mid A \subseteq S\}$$

$\{1, 2, 3\}$ $|S| = 3$

$\rightarrow P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$|P(S)| = 2^3 = 8$

Example: Let A = {a}

P(A) = { ϕ , {a} }

$\{\phi\} \subseteq P(A)$

$\{\phi\} \not\subset P(A)$

Example: What is the power set of {0,1,2}?

$\phi \subseteq P(A)$

Solution: All possible subsets

{Ø, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}}.

Number of Elements in Power Set –

For a given set S with n elements, number of elements in P(S) is 2^n.
As each element has two possibilities (present or absent}, possible subsets are 2×2×2.. n times = 2^n.

Therefore, power set contains 2^n elements.

Note –

- Power set of a finite set is finite.

$|S| = n$

$|P(S)| = 2^n$

- Set S is an element of power set of S which can be written as S ∈ P(S).

- Empty Set ϕ is an element of power set of S which can be written as ϕ ∈ P(S).

- Empty set ϕ is subset of power set of S which can be written as ϕ ⊂ P(S).

Countable set and its power set –

A set is called countable when its element can be counted. A countable set can be finite or infinite.

For example, set S1 = {a, e, i, o, u} representing vowels is a countably finite set. However, S2 = {1, 2, 3……} representing set of natural numbers is a countably infinite set.

$2^{|N|} = |R|$

Note –

- Power set of countably finite set is finite and hence countable.

For example, set S1 representing vowels has 5 elements and its power set contains 2^5 = 32 elements. Therefore, it is finite and hence countable.

- Power set of countably infinite set is uncountable.

For example, set S2 representing set of natural numbers is countably infinite. However, its power set is uncountable.

Uncountable set and its power set –
A set is called uncountable when its element can’t be counted. An uncountable set can be always infinite.

For example, set S3 containing all fractional numbers between 1 and 10 is uncountable.

Note –

•Power set of uncountable set is always uncountable.

For example, set S3 representing all fractional numbers between 1 and 10 is uncountable.
Therefore, power set of uncountable set is also uncountable.

GATE-CS-2015 (Set 2)

The cardinality of the power set of {0, 1, 2 . . ., 10} is __________.

(A) 1024

(B) 1023

(C) 2048

(D) 2043

$2^{11} = 2048$

Answer: (C)

GATE-CS-2015 (Set 1)

For a set A, the power set of A is denoted by 2^A.

If A = {5, {6}, {7}}, which of the following options are True.

$A \subset B$

I. $\emptyset \in 2^A$ II. $\emptyset \subseteq 2^A$ III. $\{5, \{6\}\} \in 2^A$ IV. $\{5, \{6\}\} \subseteq 2^A$

(A) I and III only

(B) II and III only

(C) I, II and III only

(D) I, II and IV only

$2^A = \{\phi, \{5\}, \{\{6\}\}, \{\{7\}\}, \{5, \{6\}\}, \{\{6\}, \{7\}\}, \{5, \{7\}\}, A\}$

Powerset of A = { ϕ, {5}, {{6}}, {{7}}, {5, {6}}, {5, {7}}, {{6}, {7}}, {5, {6}, {7}} }

Answer: (C)

GATE-CS-2000

Let P(S) denotes the power set of set S. Which of the following is always true?

(a) $P(P(S)) = P(S)$ (b) $P(S) \cap P(P(S)) = \{\phi\}$

(c) $P(S) \cap S = P(S)$ (d) $S \notin P(S)$

(A) a
(B) b
(C) c
(D) d
(E) None

Answer: ~~(E)~~

ϕ

$\{\phi\}$

$P(S) \cap S = \phi$

$|S| = n$

$|P(S)| = 2^n$

$|P(P(S))| = 2^{2^n}$

$\phi \to \phi$

$\{\phi\}$

$S = \{1, 2\}$

ϕ

$P(S) = \{\phi, \{1\}, \{2\}, S\}$

$P(P(S)) = \{\phi, \{\phi\}, \ldots\}$

ISRO CS 2017

The number of elements in the power set of { {1, 2}, {2, 1, 1}, {2, 1, 1, 2} } is

(A) 3

(B) 8

(C) 4

(D) 2

$\{\{1,2\},\underbrace{\{1,2\},\{1,2\}}\}$

2^1

$\{\{1,2\}\}$

Answer: (D)

Relations and their types

Relations

Let A and B be two sets. A Binary relation R from set A to B is a subset of A x B.

(OR)

$$R \subseteq A \times B$$

Relations
Let A and B be two sets. A Binary relation R from set A to B is a subset of A x B.
(OR)
$R \subseteq A \times B$

aRb if $a+b=1$

$S \to S \times S$

$A \quad B \quad A \times B$

$a|b$

$\frac{a}{b}$

$A = \{1,2\}, \quad B = \{3,4\}$

$A \times B = \{(1,3)\ (1,4),\ (2,3)\ (2,4)\}$

$R \subseteq A \times B$

aRb if $a+b=4$

$R = \{(1,3)\}$

$aRb \quad a \in A,\ b \in B$

if $a|b$ a divides b

$a|b$

$\{1,2\} \quad R\ \{3,4\} \quad R = \{\{1,3\}, \{1,4\}, \{2,4\}\}$

$R \subseteq A \times B$

$|A \times B| = |A| \times |B|$

A Binary relation R on a single set A is defined as a subset of AxA.

Example: Let R = {(a,b) | a divides b} A = {1, 2, 3, 4, 5, 6}

A

$R \subseteq A \times A$

For two distinct set, A and B with cardinalities m and n, the maximum cardinality of the relation R from A to B is mn.

$A \times B$

Domain and Range:

If there are two sets A and B and Relation from A to B is R(a,b),

then domain is defined as the set { a | (a,b) ∈ R for some b in B}

and Range is defined as the set {b | (a,b) ∈ R for some a in A}.

Types of Relation:

1.Empty Relation: A relation R on a set A is called Empty if the set A is empty set.

2. Full Relation: A binary relation R on a set A and B is called full if A X B.

3. Reflexive Relation: A relation R on a set A is called reflexive if $(a,a) \in R$ holds for every element $a \in A$.
i.e. if set $A = \{a,b\}$ then $R = \{(a,a), (b,b)\}$ is reflexive relation.

4. Irreflexive relation : A relation R on a set A is called irreflexive if for every element $a \in A$, $(a,a) \notin R$ holds.
i.e. if set $A = \{a,b\}$ then $R = \{(a,b), (b,a)\}$ is irreflexive relation.

5. Symmetric Relation: A relation R on a set A is called symmetric if (b,a) ∈ R holds when (a,b) ∈ R.
i.e. The relation R={ (4,5), (5,4), (6,5), (5,6) } on set A={4,5,6} is symmetric.

6. AntiSymmetric Relation: A relation R on a set A is called antisymmetric if (a,b) ∈ R and (b,a) ∈ R then a = b is called antisymmetric.
i.e. The relation R = {(a,b)→ R | a ≤ b} is anti-symmetric since a ≤ b and b ≤ a implies a = b.

$\mathbb{N}$

$(a,b) \in R$ if $a \mid b$

$(a,b) \in R$ & $(b,a) \in R$

$a \mid b$ $\quad$ $b \mid a$

$a = b$

$(a,b) \in R$ if $a \leq b$

$(a,b) \in R$ & $(b,a) \in R$

$a \leq b$ $\quad$ $b \leq a$

$a = b$

7. Transitive Relation: A relation R on a set A is called transitive if (a,b) ∈ R and (b,c) ∈ R then (a,c) ∈ R for all a,b,c ∈ A.
i.e. Relation R={(1,2), (2,3), (1,3)} on set A={1,2,3} is transitive.

8. Equivalence Relation: A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.
i.e. relation R={(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)} on set A={1,2,3} is equivalence relation as it is reflexive, symmetric, and transitive.

9. Asymmetric relation: Asymmetric relation is opposite of symmetric relation. A relation R on a set A is called asymmetric if no (b,a) ∈ R when (a,b) ∈ R.

{(1,1), (2,2), (2,3), (3,2), (1,3)}

GATE-CS-2009

Consider the binary relation R = {(x, y), (x, z), (z, x), (z, y)} on the set {x, y, z}. Which one of the following is TRUE?

A

$x \neq z$

(A) R is symmetric but NOT antisymmetric

(B) R is NOT symmetric but antisymmetric

(C) R is both symmetric and antisymmetric

(D) R is neither symmetric nor antisymmetric

x

$A = \{(x, y), (y, z), (x, z)\}$

$R \subseteq A \times A$

$(x, y)\ R\ (x, y)$

Answer: (D)

GATE-CS-2015 (Set 2)

Let R be the relation on the set of positive integers such that aRb if and only if a and b are distinct and have a common divisor other than 1.

$\mathbb{Z}$

Which one of the following statements about R is True?

(A) R is symmetric and reflexive but not transitive

(B) R is reflexive but not symmetric and not transitive

(C) R is transitive but not reflexive and not symmetric

(D) R is symmetric but not reflexive and not transitive

Answer: (D)

aRb

$\leq \quad a \neq b \ \& \ (a,b) > 1$

$(a,a) \notin R \rightarrow$ not ref.

$(a,b) \in R \Rightarrow a \neq b \ \& \ (a,b) > 1$

$\rightarrow b \neq a \ \& \ (b,a) > 1$

$\Rightarrow (b,a) \in R$

symmetric.

$(2,6) \in R \qquad (6,3) \in R$

$(2,3) \rightarrow \cancel{R}$

not transitive.

GATE-CS-2016 (Set 2)

A binary relation R on N x N is defined as follows:

(a, b) R (c, d) if a ≤ c ~~or~~ and b ≤ d.

Consider the following propositions:

P: R is reflexive

Q: R is transitive

Which one of the following statements is TRUE?

(A) Both P and Q are true.

(B) P is true and Q is false.

(C) P is false and Q is true.

(D) Both P and Q are false.

Answer: (B)

A

aRa

A

(a, b)

A = N × N

= {(a, b) ∈ (c, d) ...

(a, b) R (c, d)

→ (a, b) R (a, b) ⇒ reflexive

a ≤ a or b ≤ b

(a, b) R (c, d) & (c, d) R (e, f)

⇒ (a, b) R (e, f)

(1, 2) R (2, 3)

(2, 3) R (4, 5)

→ (2, 4) R (3, 1)

(3, 1) R (1, 2)

not transitive (2, 4) R (1, 2) →

PRP

A

GATE-CS-2015 (Set 3)

Let R be a relation on the set of ordered pairs of positive integers such that

$((p, q), (r, s)) \in R$ if and only if $p - s = q - r$.

Which one of the following is true about R?

(A) Both reflexive and symmetric

(B) Reflexive but not symmetric

(C) Not reflexive but symmetric

(D) Neither reflexive nor symmetric

$\alpha\ (p, q) R (p, q)$

$p - q = q - p \quad \alpha$

$\Rightarrow (p, q) R (r, s) \quad \Rightarrow p - s = q - r$

$s - p = r - q$

$(r, s) R (p, q) \Rightarrow ??$

$r - q = s - p\ ??$

Answer: (C)

GATE-CS-2001

Consider the following relations:

R1(a,b) iff (a+b) is even over the set of integers

R2(a,b) iff (a+b) is odd over the set of integers

R3(a,b) iff a.b > 0 over the set of non-zero rational numbers

R4(a,b) iff |a - b| ≤ 2 over the set of natural numbers

Which of the following statements is correct?

(A) R1 and R2 are equivalence relations, R3 and R4 are not

(B) R1 and R3 are equivalence relations, R2 and R4 are not

(C) R1 and R4 are equivalence relations, R2 and R3 are not

(D) R1, R2, R3 and R4 are all equivalence relations

Answer: (B)

$aR_1a \Rightarrow 2a \rightarrow$ even
ref.

$aR_1b \rightarrow a+b$ even
$= b+a$ even
bR_1a

aR_1b & bR_1c
$a+b$ even $b+c$ even
$(a+b+b+c)$ even
$(a+2b+c)$ even
$(a+c)$ even
$aR_1c \Rightarrow$ transitive

$aR_3a \Rightarrow a^2 > 0 \rightarrow$ refle
$aR_3b \Rightarrow a \cdot b > 0 \Rightarrow b \cdot a > 0 \Rightarrow bR_3a \rightarrow$ symm
aR_3b & bR_3c $ab^2c > 0 \Rightarrow ac > 0 \Rightarrow$ tran
$a \cdot b > 0$ $b \cdot c > 0$

(1,0)
(3,1)
2
(3,0)

$R \subset A \times A$

$\downarrow$ set

Combining Relations :

Two relations can be combined in several ways such as –

Union – $R_1 \cup R_2$ consists of all ordered pairs from both relations. Duplicate ordered pairs removed from Union.

Intersection – $R_1 \cap R_2$ consists of ordered pairs which are in both relations.

Difference – $R_1 - R_2$ consists of all ordered pairs only in R_1, but not in R_2.

Symmetric Difference – $R_1 \oplus R_2$ consists of all ordered pairs which are either in R_1 or R_2 but not in both.

$f(g(x))$ $f(g(x))$

There is another way two relations can be combined that is analogous to the composition of functions.

R $A \to B$, $R \subseteq A \times B$

S $B \to C$ $S \subseteq B \times C$

Composition – Let R be a relation from A to B and S is a relation from set B to C,

$R_1 : A \times A$

$R_2 \; A \times A$

$S o R \subseteq A \times C$

then the composite of R and S, denoted by SoR, is the relation which consisting of ordered pairs (a,c) where a Є A and c Є C and for which there exists an element b Є B for which (a,b) Є R and (b,c) Є S.

$R_1 = \{(1,2),(2,3),(1,4)\}$

$R_2 o R_1 = \{(1,2),(2,5),(1,5)\}$

$R_2 = \{(2,2),(3,5),(4,5)\}$

$R_1 o R_2 = \{(2,3)\}$

Example – What is the composite of the relations R and S where R is a relation from {1, 2, 3} to {1,2,3,4} with R= { (1,1), (1,4), (2,3), (3,1), (3,4)}
and S is a relation from {1,2,3,4} to {0,1,2} with S= {(1,0), (2,0), (3,1), (3,2), (4,1)}?

Composition of Relation on itself :

A relation can be composed with itself to obtain a degree of separation between the elements of the set on which R is defined.

Let R be a relation on the set A. The powers R^n where n = 1,2,3,…are defined recursively by –

$R^1 = R$ and $R^{n+1} = R^n o R$

$R \rightarrow A \times A$

$R \rightarrow A \times A$

$R = \{(1,2), (2,3)\}$

$R = \{(1,2), (2,3)\}$

$R^2 = R o R$

$= \{(1,3)\}$

GATE-IT-2004

Let R_1 be a relation from A = {1, 3, 5, 7} to B = {2, 4, 6, 8} and R_2 be another relation from B to C = {1, 2, 3, 4} as defined below:

1. An element x in A is related to an element y in B (under R_1) if x + y is divisible by 3.
2. An element x in B is related to an element y in C (under R_2) if x + y is even but not divisible by 3

Which is the composite relation R_1R_2 from A to C?

(A) $R_1 R_2 = \{(1, 2), (1, 4), (3, 3), (5, 4), (7, 3)\}$
(B) $R_1 R_2 = \{(1, 2), (1, 3), (3, 2), (5, 2), (7, 3)\}$
(C) $R_1 R_2 = \{(1, 2), (3, 2), (3, 4), (5, 4), (7, 2)\}$
(D) $R_1 R_2 = \{(3, 2), (3, 4), (5, 1), (5, 3), (7, 1)\}$

$R_1 = \{(1,2), (1,8), (3,6), (5,4), (7,2), (7,8)\}$

$R_2 = \{(2,2), (4,4), (6,2), (6,4), (8,2)\}$

$R_1R_2 = \{(1,2), (3,2), (3,4), (5,4), (7,2)\}$

Answer: (C)

Equivalence Relations :

Let R be a relation on set A. If R is reflexive, symmetric, and transitive then it is said to be a equivalence relation.

Consequently, two elements a and b related by an equivalence relation are said to be equivalent.

aRb

$a \cong b$

$[a] = \{$ all element which are related with a by an equivalence relation R $\}$

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (3,2), (2,3), (1,3), (3,1)\}$

$[1] = \{1, 2, 3\}$

aR_1b

$R_1 \Rightarrow (a+b)$ is even

Equivalence Classes :

Let R be an equivalence relation on set A .

We know that if (a,b) ∈ R then a and b are said to be equivalent with respect to R .

The set of all elements that are related to an element a of A is called the **equivalence class** of a . It is denoted by [a].

Formally,

$[1] = \{b \mid 1Rb\}$
$= \{b \mid 1+b \text{ is even}\}$
$= \{b \mid b \text{ is odd}\}$

$$[a]_R = \{s \mid (a,s) \in R\}$$

= set of odd no.

$[2]$ = all even no.

$[3]$ = all odd no

$[4]$ = all even no.

Let R be an equivalence relation on set A .

xRy : x + y is even

A : set of integers

$$
\begin{aligned}
[1] &= \{y \mid (1, y) \in R\} \\
&= \{y \mid 1 + \text{y is even}\} \\
&= \{y \mid \text{y is odd}\} \\
&= \{\pm 1, \pm 3, \pm 5, \ldots\}
\end{aligned}
$$

$$
\begin{aligned}
[2] &= \{y \mid (2, y) \in R\} \\
&= \{y \mid 2 + \text{y is even}\} \\
&= \{y \mid \text{y is even}\} \\
&= \{0, \pm 2, \pm 4, \pm 6, \ldots\}
\end{aligned}
$$

GATE-CS-2005

Let R and S be any two equivalence relations on a non-empty set A. Which one of the following statements is TRUE?

(A) R ∪ S, R ∩ S are both equivalence relations

(B) R ∪ S is an equivalence relation $(a,b) \rightarrow (b,a)$

(C) R ∩ S is an equivalence relation $(a,b) \rightarrow (b,a)$

(D) Neither R ∪ S nor R ∩ S is an equivalence relation

$A = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\ (3,2)\ (1,2),(2,1)\ (1,3)\ (3,1)$

$R = \{(1,1),(2,2),(3,3)\ (1,2),(2,1)\}$

$S = \{(1,1),(2,2),(3,3)\ (2,3),(3,2)\}$

$R \cup S = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)\ (2,3),(3,2)\}$ ✗

$R \cap S$

Answer (C)

GATE-CS-2000

eq.

A relation R is defined on the set of integers as xRy if (x + y) is even. Which of the following statements is true?

(A) R is not an equivalence relation

(B) R is an equivalence relation having 1 equivalence class

(C) R is an equivalence relation having 2 equivalence classes

(D) R is an equivalence relation having 3 equivalence classes

Answer: (C)

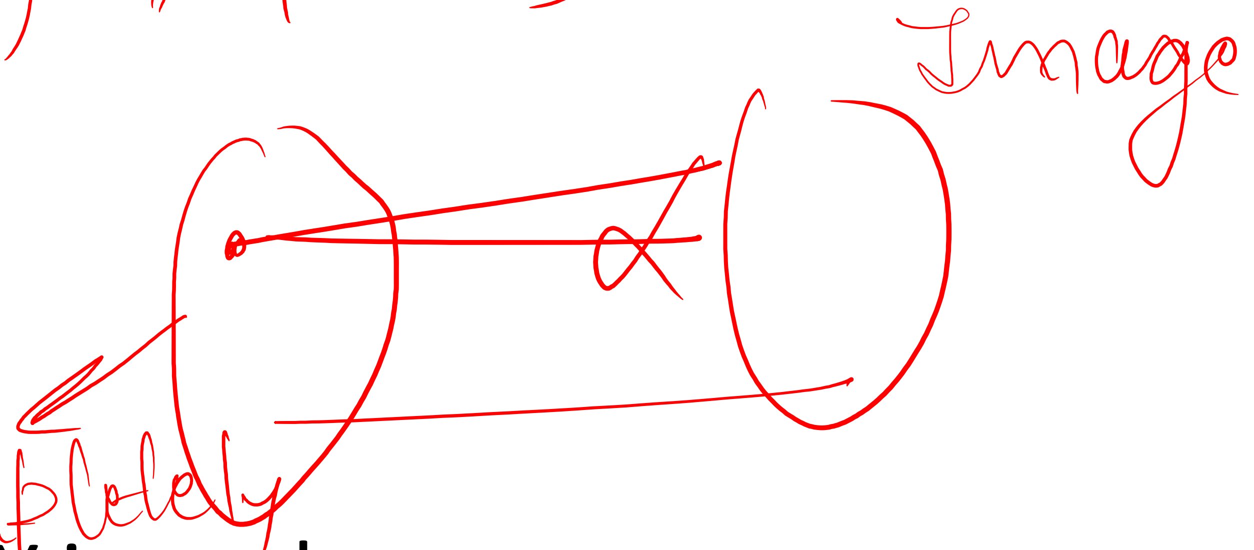
Closure of Relations

Consider a relation R on a set A. R may or may not have a property P, such as reflexivity, symmetry, or transitivity.

If there is a relation S with property P containing R, then S is called the closure of R with respect to P.

Functions

$f : A \to B$



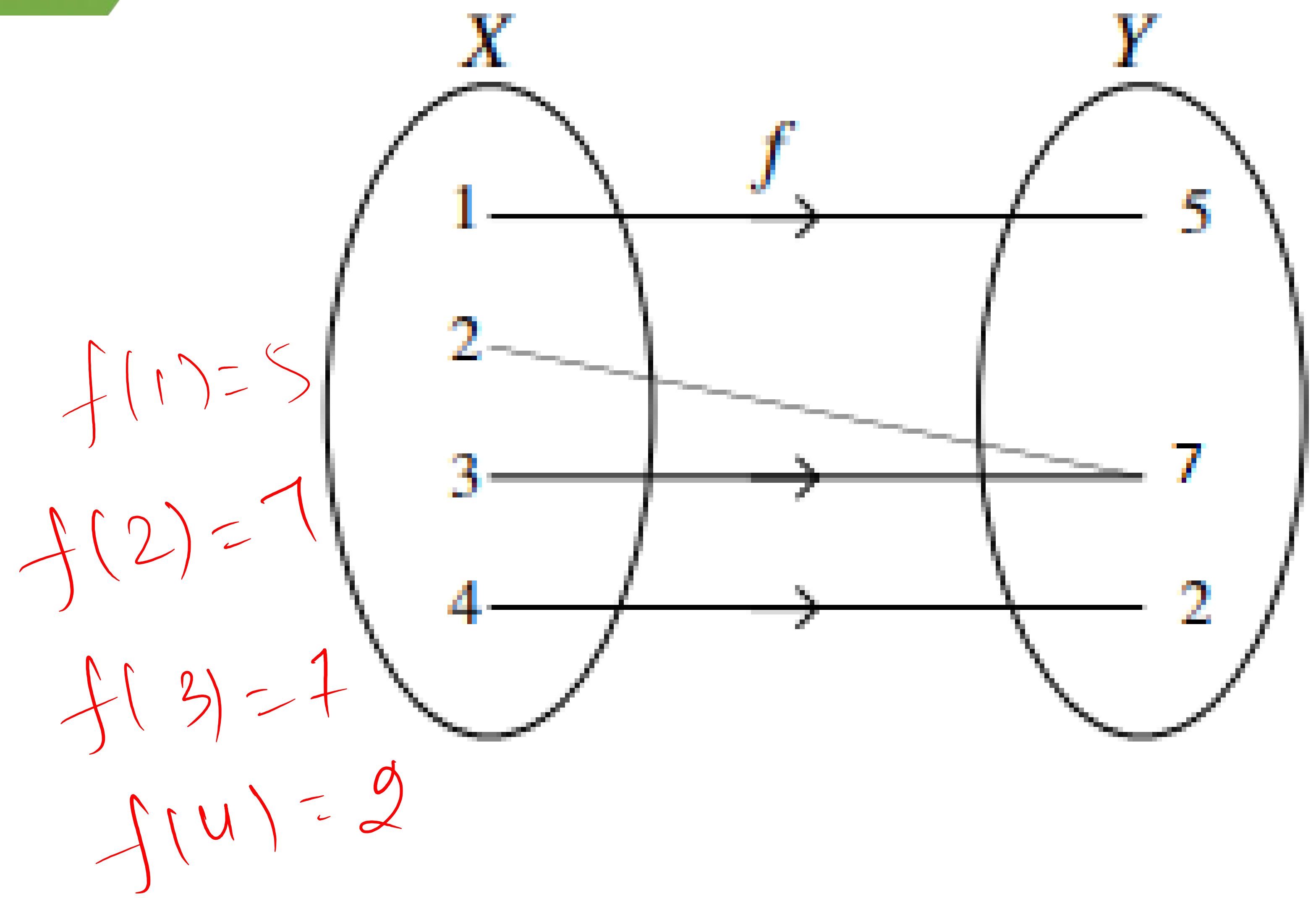
Definition

completely used

A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y .

$f(a) = b$

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value.
This means that there cannot be any repeated x-values with different y-values.

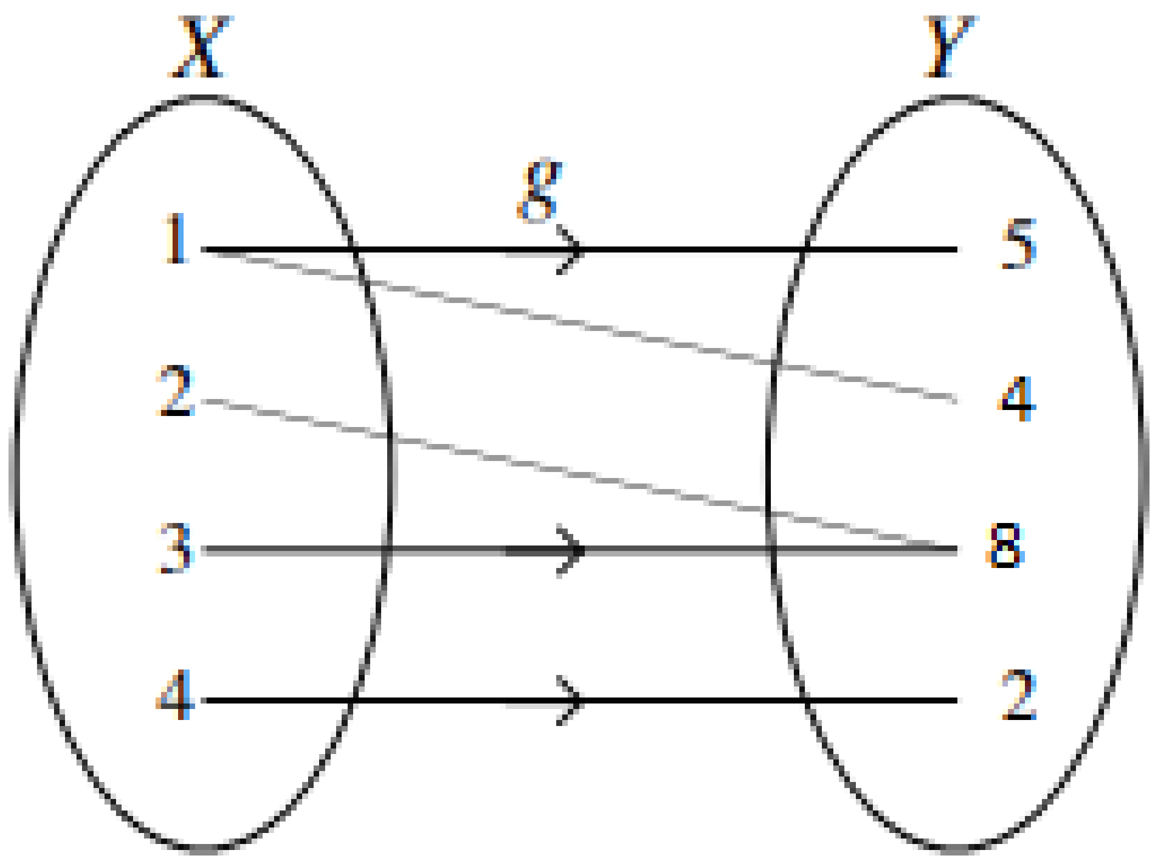


f : X → Y is a function.

Every element in X has associated with it exactly one element of Y .

The example above can be described by the following sets of ordered pairs.

F = {(1,5),(3,7),(2,7),(4,2)} is a function.



g : X → Y is not a function.

The element 1 in set X is assigned two elements, 5 and 4 in set Y .

G = {(1,5),(4,2),(2,8),(3,8),(1,4)} is not a function

Terms related to functions:

•**Domain and co-domain** – For a function f : X → Y the domain of f is the set X and Y is called co-domain.

If y = f(x) is given, then the domain is taken to be the set of all real x for which the function is defined.

$f : \mathbb{R} \to \mathbb{R}$

For example, y = √x has domain; all real x ≥ 0.

$\to f(x) = x^2$

•**Range** – For a function f : X → Y the range of f is the set of y-values such that y = f(x) for some x in X.

Domain $\to \mathbb{R}$

$\Rightarrow$ Codomain $\Rightarrow \mathbb{R}$

•**Image and Pre-Image** – b is the image of a and a is the pre-image of b if f(a) = b.

$\Rightarrow$ Range $\Rightarrow \geq 0$ $[0, \infty)$

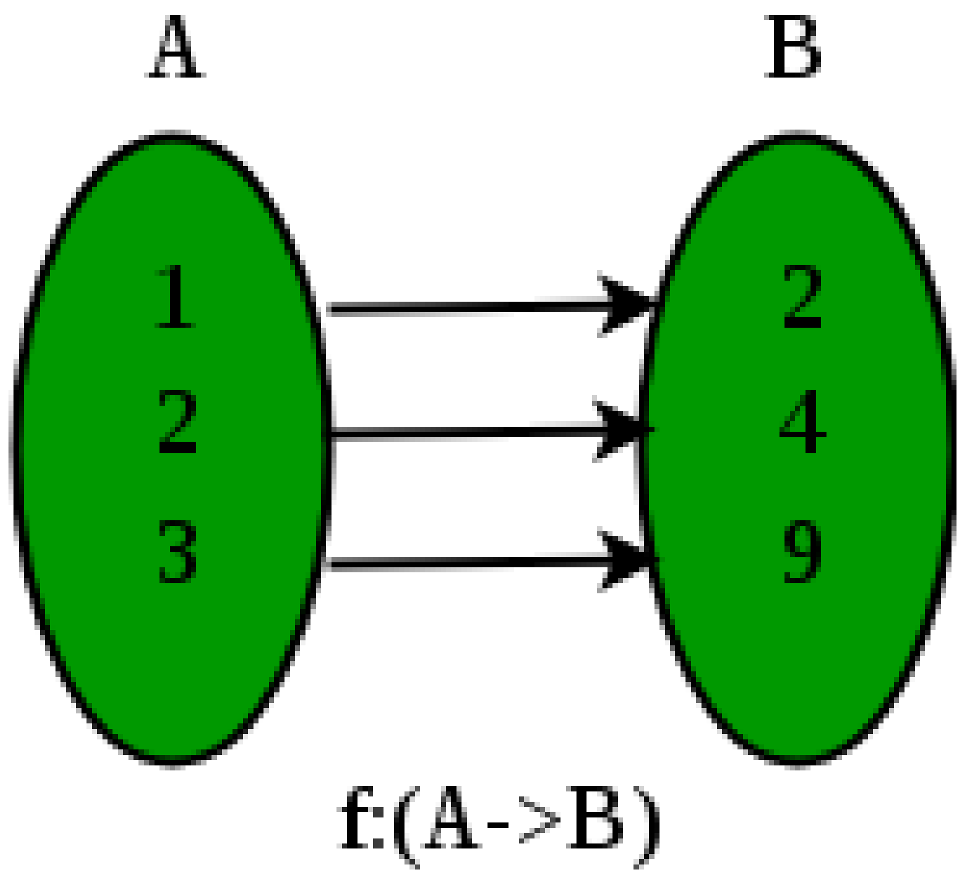
Types of functions:

1.One to one function(Injective):

A one-to-one function is defined by f: A → B such that every element of set A is connected to a distinct element in set B.

Here every element of the domain has a distinct image for the given function.

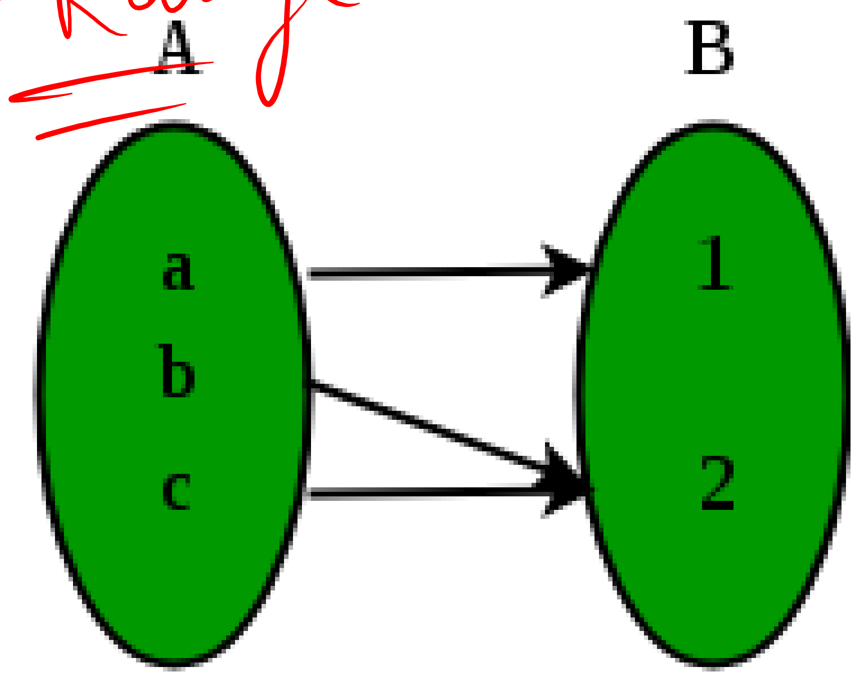
no two elements have same image.



Onto Function (surjective):

For a function defined by f: A → B, such that every element in set B has a pre-image in set A.

Codomain = Range



f:(A->B)

One to one correspondence function(Bijective/Invertible):

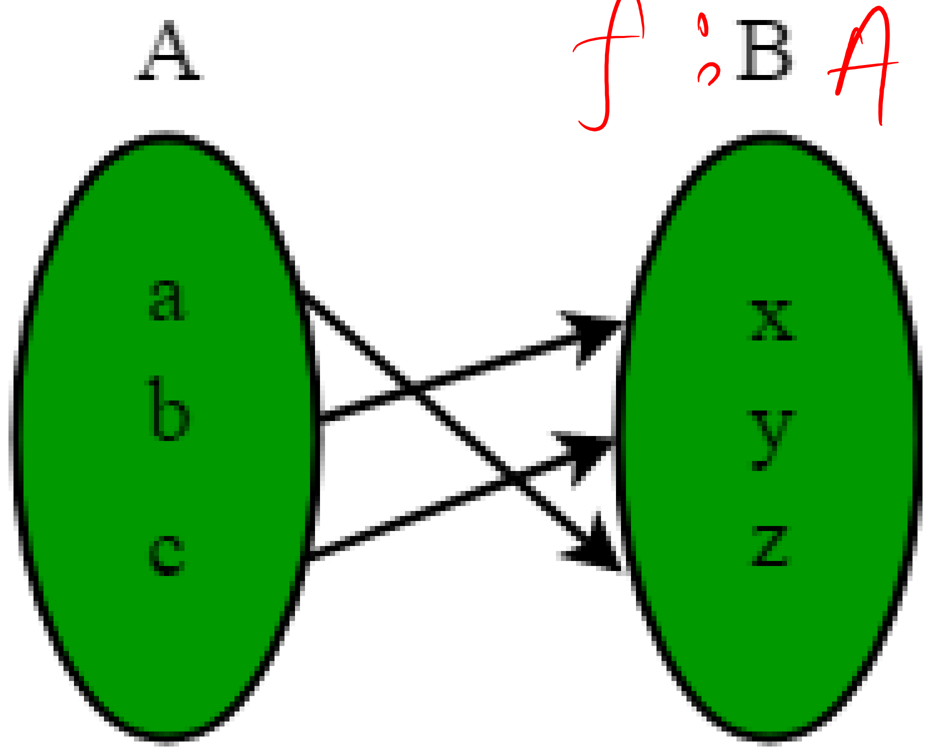
A function that is both a one one and onto function is called a bijective function.

Here every element of the domain is connected to a distinct element in the codomain and every element of the codomain has a pre-image.

Also in other words every element of set A is connected to a distinct element in set B, and there is not a single element in set B which has been left out.

Bijective both one one and onto

$f : A \to B$ $\quad |A| = |B|$



f:(A->B)

Inverse Functions : Bijection function are also known as invertible function because they have inverse function property.

f: A → B is a function. Then *f* is *invertible* if there exists a function *g* with domain *Y* and codomain *X*, with the property:

$f(x) = y \Leftrightarrow g(y) = x$

R A

R $\mathbb{N} \times \mathbb{N}$

$f: A \to B$

If *f* is invertible, then the function *g* is unique

$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

The function *g* is called *the* inverse of *f*, and is usually denoted as f^{-1}

$f(a, b) = (c, d)$

$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$f(a, b) = a + b$

GATE MOCK 2017

$f: A \to B$

Let A = {a, b, c, d }, B = { p, q, r, s } denote sets.

R : A –> B, R is a function from A to B. Then which of the following relations are not functions ?

(i) { (a, p) (b, q) (c, r) } ✗

(ii) { (a, p) (b, q) (c, s) (d, r) } ✓

(iii) { (a, p) (b, s) (b, r) (c, q) } ✗

(A) (i) and (ii) only

(B) (ii) and (iii) only

(C) (i) and (iii) only

(D) None of these

Answer: (C)

A function $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$, defined on the set of positive integers $\mathbb{N}^+$, satisfies the following properties:

~~f(0) = f(0)~~

f(11) = f(1)

$$f(n) = f(n/2) \qquad \textbf{if } n \text{ is even}$$
$$f(n) = f(n+5) \qquad \textbf{if } n \text{ is odd}$$

Let $R = \{i \mid \exists j : f(j) = i\}$ be the set of distinct values that f takes. The maximum possible size of R is _______ 2.

Let us assume: f(1) = x.

f(1) = f(6) = f(3) = f(8) = f(4) = f(2) = f(1)

Then, f(2) = f(2/2) = f(1) = x

f(2) f(1)

f(6) = f(1)

f(3) = f(3+5) = f(8) = f(8/2) = f(4/2) = f(2/1) = f(1) = x

f(3+5) f(4)

f(7) = f(12) = f(6) = f(1)

Similarly, f(4) = x

f(4) = f(1)

f(8) = f(1)

f(5) = f(5+5) = f(10/2) = f(5) = y.

f(5) = f(10) = f(5) = f(10) - - - - - -

f(9) = f(1)

So, it will have two values. All multiples of 5 will have value y and others will have value x.

GATE CS 1996

Let *R* denote the set of real numbers. Let $f: R\times R \rightarrow R\times R$ be a bijective function defined by $f(x,y) = (x+y, x-y)$. The inverse function of *f* is given by

a) $f^{-1}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$

b) $f^{-1}(x,y) = (x-y, x+y)$

c) $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

d) $f^{-1}(x,y) = \left(2(x-y), 2(x+y)\right)$

$f(x) = y \Rightarrow x = f^{-1}(y)$

$f(x,y) = (a,b)$

$(x,y) = f^{-1}(a,b)$

$f(x,y) = (x+y, x-y)$

$(a,b) = (x+y, x-y)$

$$\begin{cases} a = x+y \\ b = x-y \end{cases}$$

$x = \frac{a+b}{2}$

$y = \frac{a-b}{2}$

$f^{-1}(a,b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$

$f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

Number of functions from one set to another:

$f : X \to Y$, $|X| = n$, $|Y| = m$

Total No. of maps $= m^n$

$f : X \to Y$

$|Y|^{|N|}$

$\underline{m \times m \; - - - \times m}$

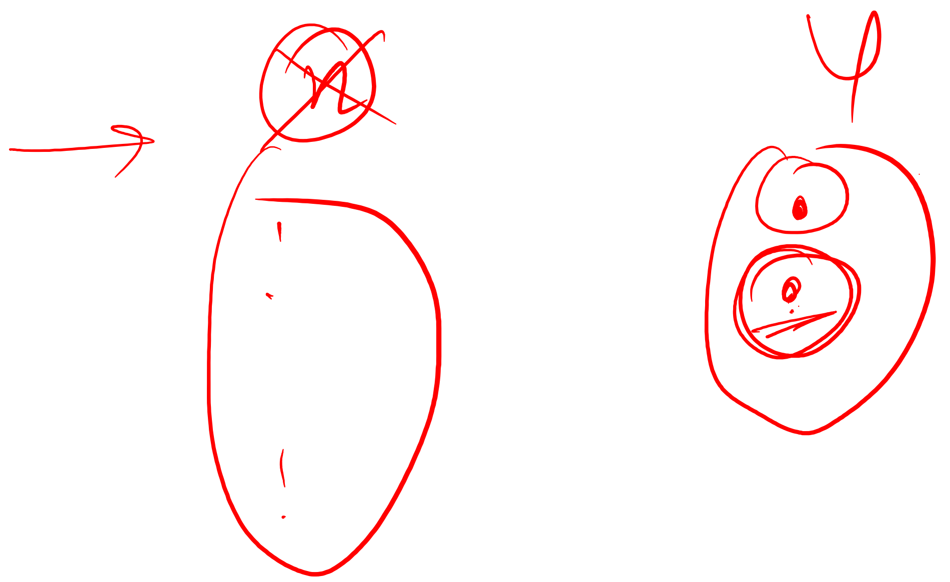
Number of onto functions from one set to another

$f : X \rightarrow Y$, $|X| = n$, $|Y| = 2$

Total no. of functions = 2^n

No. of onto functions = $2^n - 2$

$2^n - 2$



Number of onto functions from one set to another

f : X → Y , |X| = n , |Y| = m

In onto function from X to Y, all the elements of Y must be used.
i.e., n > m

No. of onto functions = $m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \ldots + (-1)^{m-1}(1)^n$

Number of one-one functions from one set to another

$f : X \to Y$, $|X| = n$, $|Y| = m$

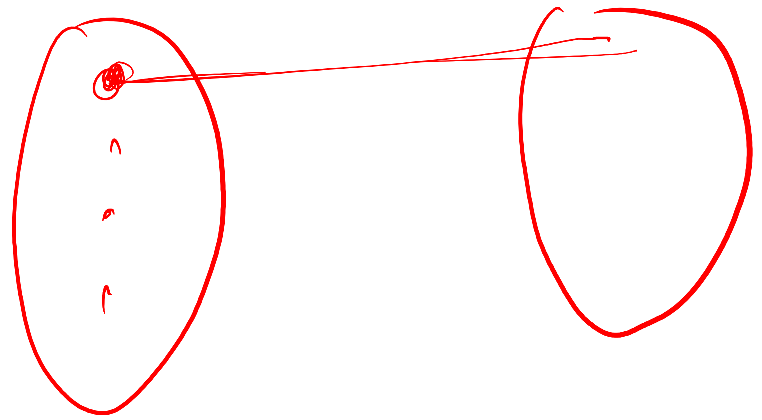
If $n > m$, then there is no one-one functions.

If $n < m$

No. of one-one functions = $\frac{n!}{(n-m)!}$

$\frac{n!}{(m-n)!}$

$X \to n$ $\quad$ $Y \to m$



$m \times (m-1) \times (m-2)$

$\cdots (m-n+1)$

GATE-CS-2015 (Set 2)

The number of onto functions (surjective functions) from set X = {1, 2, 3, 4} to set Y = {a, b, c} is ________________

(A) 36

(B) 64

(C) 81

(D) 72

Answer: (A)

GATE-CS-2015 (Set 2)

Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one-to-one is __________

(A) 0.95
(B) 0.80
(C) 0.75
(D) 0.70

Prob $= \frac{\cancel{20} \times 19}{\cancel{400}_{20}} = \frac{19}{20} = .95$

$X \rightarrow Y$

$|Y|^{|X|} = 20^2 = 400$

One one $= \frac{n!}{(m-n)!} = \frac{20!}{18!} = 20 \times 19$

Answer: (A)

Number of functions from X to Y is 20*20

Number of one to one functions from X to Y is 20*19

So probability of a function being one to one
= (20*19) / (20*20)
= 380 / 400
= 0.95

GATE CS 1998

The number of functions from an *m element* set to an *n element* set is

(A) m+n

(B) m^n

(C) n^m

(D) m*n

Answer: (C)

GATE-CS-2006

Let X, Y, Z be sets of sizes x, y and z respectively. Let W = X × Y. Let E be the set of all subsets of W. The number of functions from Z to E is:

(A) z^{2xy}

(B) $z \times 2^{xy}$

(C) z^{2x+y}

(D) 2^{xyz}

$|X| = x$

$|Y| = y$

$|Z| = z$

$|W| = xy$

$|E| = 2^{xy}$

$|E|^{|Z|}$

$(2^{xy})^{z}$

2^{xyz}

Answer: (D)

GATE-CS-2014-(Set-1)

Let S denote the set of all functions f: $\{0,1\}^4$ -> {0,1}. Denote by N the number of functions from S to the set {0,1}. The value of $\text{Log}_2\text{Log}_2 N$ is _______.

(A) 12

(B) 13

(C) 15

(D) 16

Answer: (D)

16

$\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$

$f : S \to \{0,1\}$

$N = 2^{|S|}$

$= 2^{2^{16}}$

$|S| = 2^{16}$

$\log_2 \log_2 N = \log_2 \log_2 2^{2^{16}}$

$= \log_2 2^{16}$

$= 16$

GATE-CS-2005

Let f: B → C and g: A → B be two functions and let h = f o g. Given that h is an onto function. Which one of the following is TRUE?

(A) f and g should both be onto functions.
(B) f should be onto but g need not be onto
(C) g should be onto but f need not be onto
(D) both f and g need not be onto

$h: A \to C$. $\{1,2\} = A$

$g: A \to B$ $B = \{3,4,5\}$

(1,4)
(2,3)

$C = \{7,8,9\}$

(1,8)
(2,7)

$f: B \to C$

(3,7), (4,8), (5,9)

Answer: (B)

Thank you