Combinatorics

Permutations

A permutation is an act of arranging the objects or numbers in order.

The formula for permutations is: $^{n}P_{r} = \dfrac{n!}{(n-r)!}$

1, 2, 3

1, 2, 3, 4, 5

Example: How many different signals can be made by 5 flags from 8 flags of different colours?

1 2 3

1 2 3
3 2 1
2 3 1
1 3 2
3 1 2
2 1 3

Combinations

Combinations are the way of selecting the objects or numbers from a group of objects or collection, in such a way that the order of the objects does not matter.

The formula for combinations is: ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$

Example: How many poker hands of five cards dealt from a standard deck of 52 cards?

→ Clubs – 13 → 2–10, A, K, Q, J.

→ Diamond – 13

→ Spades – 13

Hearts – 13

Counting Principles –

There are two basic counting principles, sum rule and product rule.

Sum Rule – If a task can be done in one of n_1 ways or one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.

Example: No. of ways to order a dish from 5 veg dishes and 6 non-veg dishes is (5+6) ways = 11 ways

.

Product Rule – If a task can be broken down into a sequence of k subtasks, where each subtask can be performed in $n_1, n_2, \ldots, n_k$ respectively, then the total number of ways the task can be performed is $n_1 . n_2 \ldots n_k$.

$\square \rightarrow 10$

$\square \rightarrow 9$

$\square \rightarrow 8$

$\underline{10 \times 9 \times 8}$

Example – In how many ways can a person choose a project from three lists of projects of sizes 10, 15, and 19 respectively?

10 + 15 + 19

44

Example – In how many ways can 3 winning prizes be given to the top 3 players in a game played by 12 players?

$$\underline{\underline{12 \times 11 \times 10}}$$

Example 3 – How many distinct license plates are possible in the given format- Two alphabets in uppercase, followed by two digits then a hyphen and finally four digits. Sample- AB12-3456.

$26 \times 26 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$

0 – 9

GATE-CS-2001

How many 4-digit even numbers have all 4 digits distinct?

(A) 2240

(B) 2296

(C) 2620

(D) 4536

9 9 8

Answer: (B)

1 – 9

0 – 9

9 8 7 0

$9 \times 8 \times 7 = 504$

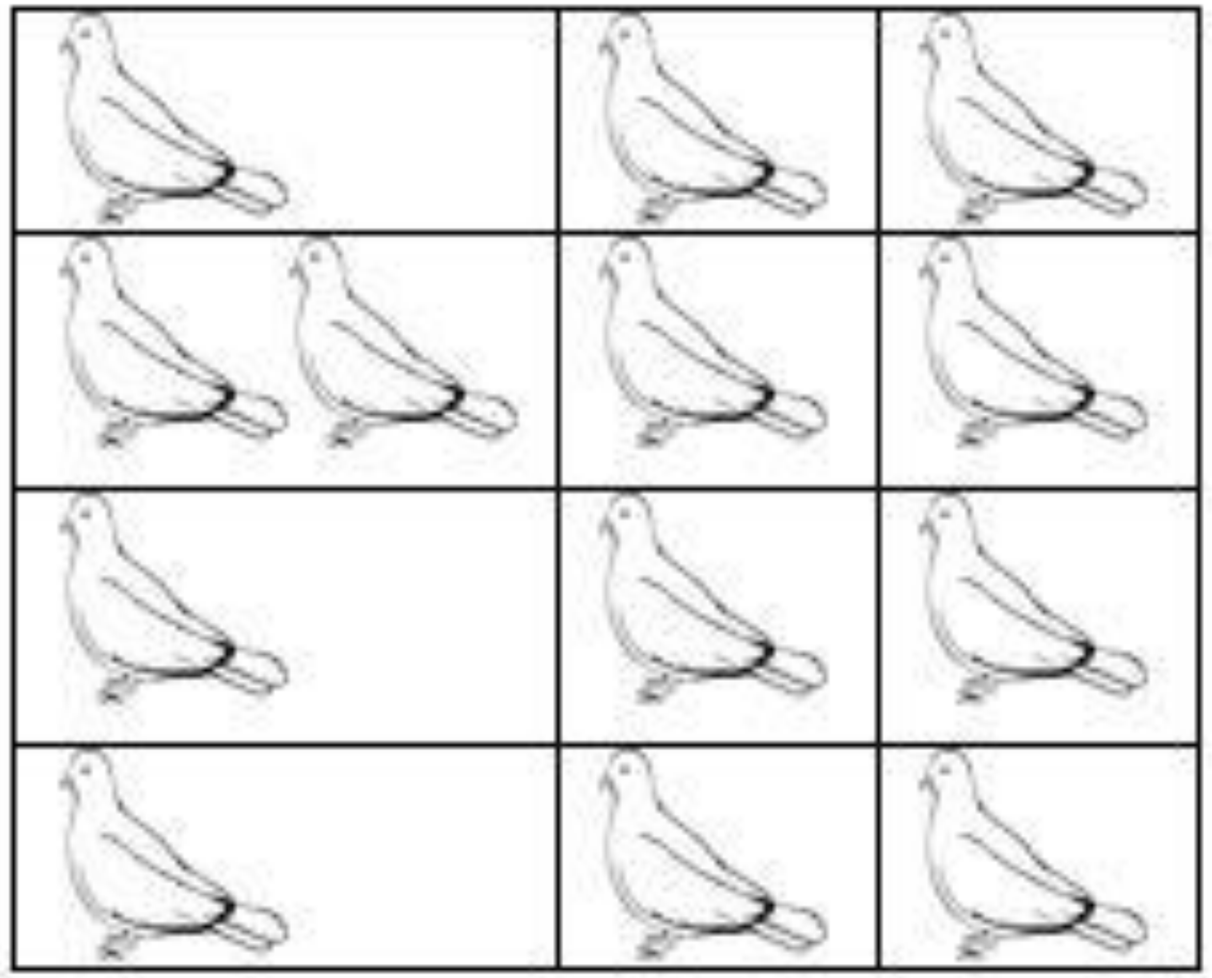
8 8 7 4

$+ 8 \times 8 \times 7 \times 4 =$

Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.

A general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.



5 → 20, 30, 40, 50, 60

→ 20, 20, 30, 40, 50

Example – 1: How many students must be in a class to guarantee that at least two student receive the same score on final exam, if the exam is graded on a scale from 0 to 100 points?

0 → 1

1 → 1

2 → 1

3

4

5

.

.

100 → 1

+1

102

~~20 22~~

Example – 2: A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

4 — R → 10 → 3

4 — W → 10 → 3 + 1

4 — B → 10 → 3

12

10

GATE-CS-2000

The minimum number of cards to be dealt from an arbitrarily shuffled deck of 52 cards to guarantee that three cards are from some same suit is

(A) 3

(B) 8

(C) 9

(D) 12

H — 13 — 2

S → 13 — 2

C → 13 — 2

D → 13 — 2

+1

9

Answer: (C)

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

S — 13

C — 13

D — 13

H — 3

42

Binomial Theorem

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \cdots + {}^nC_n x^0 y^n$

Let x and y be variables and n be a non-negative integer. Then

${}^nC_r = \frac{n!}{r!(n-r)!}$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{(n-k)!\,k!}$

$\frac{n!}{k!\,(n-k)!}$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Binomial Coefficients –

The r - combinations from a set of n elements ,denoted by $\binom{n}{r}$.

This number is also called a binomial coefficient since it occurs as a coefficient in the expansion of powers of binomial expressions.

The binomial theorem gives a power of a binomial expression as a sum of terms involving binomial coefficients.

Example 1 – What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution – $(2x-3y)^{25} = (2x+(-3y))^{25}$.

$\binom{25}{k}(2x)^{k}(-3y)^{25-k}$

By the binomial theorem-

$(2x+(-3y))^{25} = \sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^{j}$

$k=12$

Since the power of y is 13, $j=13$.

Therefore the coefficient of $x^{12}y^{13}$ is-

$\boxed{\binom{25}{12}2^{12}(-3)^{13}}$

$(x^{12}y^{13})$

$\binom{25}{13}2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}$

Permutations with repetition –

9 10 10 10

9 8 7 9

Counting permutations when repetition of elements can be easily done using the product rule.

Example, number of strings of length r is 26^r since for every character there are 26 possibilities.

The number of r -permutations of a set of n objects with repetition is $\mathbf{n}^r$.

Example – How many three digit numbers can be formed from 2, 3, 4 , 5 and 0 if repetition is allowed if

(a) There are no restrictions

(b) The number is odd

4 5 5 = 100

= 40

4 5 2

3 3 2

Maaza Pepsi Cola

Combinations with repetition –

Counting the number of combinations with repetition is a bit more complicated than counting permutations.

n types of items

Consider ~~a set of~~ n types of objects.

r items

No. of ways can r elements be chosen from n types of objects = $\binom{n+r-1}{r}$

$\binom{n+r-1}{r}$

$n = 3$

$r = 5$

$\binom{3+5-1}{5}$

**Example – ** In how many ways can 4 drinks can be chosen out of 6 possible types of drinks? There is no restriction on the number of drinks of a type that can be chosen and drinks of the same type are indistinguishable.

$n = 6 \quad r = 4$

Solution – The above scenario is a direct application of finding combinations with repetition.

$\binom{n+r-1}{r} = \binom{9}{4}$

So the number of 4-combinations is = $\binom{6+4-1}{4} = = \binom{9}{4} = 126$

Example: A man wants to buy some pet birds. The pet store sells parrots, sparrows and pigeons. How many different selections are possible if the man wants to take 7 birds.

$$\binom{n}{k} = \binom{n}{n-k}$$

$$n = 3, \quad r = 7$$

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

$$\binom{3+7-1}{7}$$

Inclusion – Exclusion Principle

$$|A_1 \bigcup A_2| = |A_1| + |A_2| - |A_1 \bigcap A_2|.$$

How many numbers between 1 and 1000, including both, are divisible by 3 or 4?

$A_1 \rightarrow$ no. divisible by 3

Solution – Number of numbers divisible by 3 = $|A_1| = \lfloor 1000/3 \rfloor = 333$.

$A_2 \rightarrow$ 4

Number of numbers divisible by 4 = $|A_2| = \lfloor 1000/4 \rfloor = 250$.

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Number of numbers divisible by 3 and 4 = $|A_1 \cap A_2| = \lfloor 1000/12 \rfloor = 83$.

$= 333 + 250 - 83$

$|A_1| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$

Therefore, number of numbers divisible by 3 or 4 = $|A_1 \cup A_2| = 333 + 250 - 83 =$ 500

$= 500$

$|A_2| = \left\lfloor \frac{1000}{4} \right\rfloor = 250$

$|A_1 \cap A_2| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$

Similarily for 3 finite sets A_1 , A_2 and A_3

$$|A_1 \bigcup A_2 \bigcup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \bigcap A_2| - |A_2 \bigcap A_3| - |A_1 \bigcap A_3| + |A_1 \bigcap A_2 \bigcap A_3|$$

GATE CS 1998

In a room containing 28 people, there are 18 people who speak English, 15 people who speak Hindi and 22 people who speak Kannada. 9 persons speak both English and Hindi, 11 persons speak both Hindi and Kannada whereas 13 persons speak both Kannada and English. How many speak all three languages?

1.9

2.8

3.7

4.6

Answer (4)

$|E \cup H \cup K| = 28$

$|E \cap H| = 9$

$|H \cap K| = 11$

$|K \cap E| = 13$

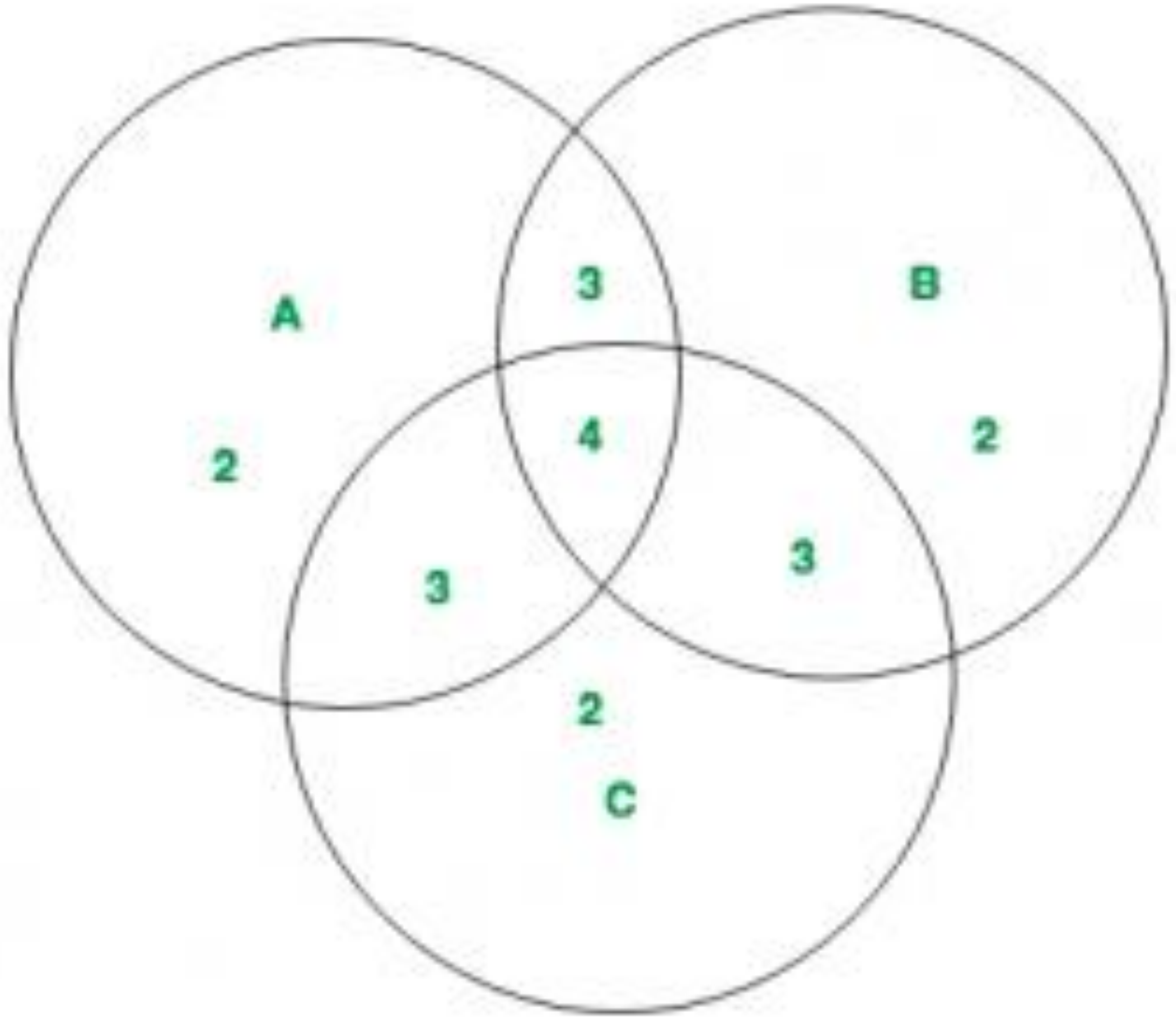
$|E| \to 18$

$|H| = 15$

$|K| = 22$

$|E \cap H \cap K| = ?$

Example : As shown in the diagram, 3 finite sets A, B and C with their corresponding values are given. Compute



Principle :

Inclusion-Exclusion principle says that for any number of finite sets $A_1, A_2, A_3,\ldots, A_\mathrm{n}$.

Union of the sets is given by = Sum of sizes of all single sets – Sum of all 2-set intersections + Sum of all the 3-set intersections – Sum of all 4-set intersections + …+ $(-1)^{i+1}$ Sum of all the i-set intersections.

In general it can be said that,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

GATE-CS-2015 (Set 3)

The number of 4 digit numbers having their digits in non-decreasing order (from left to right) constructed by using the digits belonging to the set {1, 2, 3} is _____.

(A) 12

(B) 13

(C) 14

(D) 15

1 3 3 3

3 2 3 α

3 3 2 α

Explanation: {1, 1, 1, 1}

{1, 1, 1, 2}

{1, 1, 1, 3}

{1, 1, 2, 2}

{1, 1, 2, 3}

{1, 1, 3, 3}

{1, 2, 2, 2}

{1, 2, 2, 3}

{1, 2, 3, 3}

{1, 3, 3, 3}

{2, 2, 2, 2}

{2, 2, 2, 3}

{2, 2, 3, 3}

{2, 3, 3, 3}

{3, 3, 3, 3}

$x_1 + x_2 + x_3 - \cdot - + x_r = n.$

Example 2 – $x_1 + x_2 + x_3 = 7 \;\; where \;\; x_1, x_2, x_3 \geq 0$

How many positive integer solution are there?

$\binom{n+r-1}{r-1}$

$\Rightarrow \binom{n+r-1}{r-1}$ $\quad \binom{7+3-1}{3-1}$

$\binom{9}{3-1}$

Example 6 – In how many ways can we put 10 identical balls into 6 distinct bins?

$$x_1 + x_2 + \dots + x_6 = 10$$

$$\binom{10+6-1}{6-1}$$

$$\binom{15}{5}$$

m identical balls into n distinct boxes

$$x_1 + x_2 + \ldots\ldots + x_n = \mathrm{m}$$

Non negative integer solutions = $\binom{n+\mathrm{m}-1}{n-1}$

$$x_1 + x_2 + \ldots + x_r = n$$

$$\binom{n+r-1}{r-1}$$

Example – number of ways for 5 identical balls to be placed in 3 distinct boxes, if no box is empty.

We can also solve this type of problem using linear equations.

2 1 2
1 2 2
2 2 1
3 1 1
1 3 1
1 1 3

$x_1 + x_2 + x_3 = 5$, $x_i > 1$

$x_i - 1 > 0$

$y_i > 0$

$y_i = x_i - 1$

$y_i + 1 = x_i$

$y_1 + 1 + y_2 + 1 + y_3 + 1 = 5$

$y_1 + y_2 + y_3 = 2$

$\binom{2+3-1}{3-1}$

m identical balls into n distinct boxes , if no box is empty

$x_1 + x_2 + \ldots . + x_n = m$

Non negative integer solutions = $\binom{m-1}{n-1}$

GATE-CS-2003

$m \geq kn$

$x_1 + x_2 + \dots + x_n = m$

m identical balls are to be placed in n distinct bags. You are given that m ≥ kn, where, k is a natural number ≥ 1. In how many ways can the balls be placed in the bags if each bag must contain at least k balls?

$x_i \geq k$

$x_i - k \geq 0$

$y_i \geq 0$

$y_i = x_i - k$

$x_i = y_i + k$

(A) $\binom{m-k}{n-1}$ (B) $\binom{m-kn+n-1}{n-1}$

(C) $\binom{m-1}{n-k}$ (D) $\binom{m-kn+n+k-2}{n-k}$

Answer: (B)

$y_1 + k + y_2 + k + \dots + y_n + k = m$

$y_1 + y_2 + \dots + y_n = m - kn, \quad y_i \geq 0$

$\binom{n+r-1}{r-1} \rightarrow \binom{m-kn+n-1}{n-1}$

Derangements

1 2 3 4

$\boxed{2}$ $\boxed{1}$ $\boxed{4}$ $\boxed{3}$

~~3~~4 1 2 3

2 1 3 4

Applications :

•**Derangements**

To determine the number of derangements(or permutations) of n objects such that no object is in its original position

Like Hat-check problem in which one considers the number of ways in which n hats (call them h_1 through h_n) can be returned to n people (P_1 through P_n) such that no hat makes it back to its owner .

$$D_n = \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \frac{n!}{n!}$$

Example : There are eight guests at a secret Santa party. Each guest brings a gift and each receives another gift in return. No one is allowed to receive the gift they brought. How many ways are there to distribute the gifts?

$$D_8 = 8! \left(\frac{}{2!} - \quad - \quad - \quad - \right)$$

Generating Functions

$$1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1-x}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

In simple words generating functions can be used to translate problems about *sequences* to problems about *functions* which are comparatively easy to solve using maneuvers.

Formally, a generating function is a power series

$$f(x) = \sum_{n \geq 0} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots.$$

Example 1:- Let {1, 1, 1…….} be a sequence . The generating function of the sequence is

It is simply 1+x+x^2+x^3+x^4+⋯. Now, can we find a closed formula for this power series?

$$S = 1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots$$

$$-\; Sx = \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{x^4} \cdots$$

$$S - Sx = 1$$

$$S(1-x) = 1$$

$$S = \frac{1}{1-x}$$

$\longrightarrow 1+2x+3x^2+4x^3+5x^4+\cdots$

The generating function for 1,2,3,4,5,... is

$$S = 1+2x+3x^2+4x^3+5x^4+\cdots$$

$$Sx = x+2x^2+3x^3+4x^4+\cdots$$

$$(S-Sx) = 1+x+x^2+x^3+x^4+\cdots$$

$$S(1-x) = \frac{1}{1-x}$$

$$S = \frac{1}{(1-x)^2}$$

Find the generating function for the sequence 1, -1, 2, -2, 3, -3, 4, -4, 5 …?

(a) $\dfrac{1}{(1-x)(1+x^2)}$

(b) $\dfrac{1}{(1+x)(1-x)^2}$

(c) $\dfrac{1}{(1+x)^2(1-x)}$

(d) $\dfrac{1}{(1-x)(1-x^2)}$

$S = 1 - x + 2x^2 - 2x^3 + 3x^4 - 3x^5 + 4x^6 - 4x^7 + 5x^8 + \dots$

$Sx = x - x^2 + 2x^3 - 2x^4 + 3x^5 - 3x^6 + 4x^7 - 4x^8 + \dots$

$S + Sx = 1 + x^2 + x^4 + x^6 + \dots$

$S(1+x) = \dfrac{1}{1-x^2}$

$S = \dfrac{1}{(1-x^2)(1+x)} = \dfrac{1}{(1-x)(1+x)(1+x)}$

$= \dfrac{1}{(1-x)(1+x)^2}$

$1 + x + x^2 + x^3 \dots$

$= \dfrac{1}{1-x}$

$\Rightarrow 1 + x^2 + x^4 + x^6 \dots$

$= \dfrac{1}{1-x^2}$

$$1 + x + x^2 + x^3 \cdots = \frac{1}{1-x}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \cdots$$

$$\frac{1 - x^{n+1}}{1-x} = 1 + x + x^2 + \ldots + x^n$$

$\sum (2n+3)\, x^n$

GATE CS 2018

Which one the following is a closed form expression for the generating function of the sequence $\{a_n\}$, where $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$?

(A) $\frac{3}{(1-x)^2}$ (B) $\frac{3x}{(1-x)^2}$ (C) $\frac{2-x}{(1-x)^2}$ (D) $\frac{3-x}{(1-x)^2}$

Given $a_n = 2n + 3$

Generating function $G(x)$ for the sequence a_n is

$G(x) = \sum_{n\geq 0} a_n x^n$

$$\sum_{n=0}^{\infty} 2n(x^n) + 3(x^n)$$

$$\frac{3-x}{(1-x)^2} = \frac{2x + 3 - 3x}{(1-x)^2}$$

$$\sum (2n+3)\, x^n$$

$$2 \sum (n x^n) + 3 \sum x^n$$

$$2\,(0 + 1x + 2x^2 + 3x^3 + \cdots) + 3(1 + x + x^2 + \cdots$$

$$2x(1 + 2x + 3x^2 + \cdots) + 3\left(\frac{1}{1-x}\right)$$

$$= \frac{2x}{(1-x)^2} + \frac{3}{1-x}$$

Binomial theorem for negative or fractional index is :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1*2}x^2 + \frac{n(n-1)(n-2)}{1*2*3}x^3 + \ldots\ldots\ldots\ldots$$

GATE-CS-2017 (Set 2)

If the ordinary generating function of a sequence is:

$$\{a_n\}_{n=0}^{\infty} \text{ is } \frac{1+z}{(1-z)^3}$$

$= (1+z)(1-z)^{-3}$ $n=-3$

then a_3-a_0 is equal to:

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

(A) 8

(B) 10

(C) 15

(D) 20

$= (1+z)\left[1 - 3(-z) + \frac{(-3)(-4)}{2}z^2 + \frac{(-3)(-4)(-5)}{6}(-z^3)\right]$

$= (1+z)[1+3z+6z^2+10z^3]$

$(1+3z+6z^2+10z^3 + z+3z^2+6z^3+10z^4)$

$(1+4z+9z^2+16z^3+10z^4)$

Answer: (C)

$a_3 = 16$

$a_0 = 1$

$x + \cancel{x^3} + 1 + x - x^2 - \cancel{x^3}$

$2x - x^2 + 1$

GATE CSE 2022

$S = 2x + 4x^3 + 6x^5 + \cdots$

$Sx^2 = 2x^3 + 4x^5 + \cdots$

$S(1-x^2) = 2x + 2x^3 + 2x^5 + \cdots$

Which one of the following is the closed form for the generating function of the sequence $\{a_n\}_{n \geq 0}$ defined below?

$S(1-x^2) = 2x(1 + x^2 + x^4 + \cdots)$

$S(1-x^2) = 2x\left(\frac{1}{1-x^2}\right) \Rightarrow S = \frac{2x}{(1-x^2)^2}$

$\frac{x + x^3 + (1-x^2)(1+x)}{(1-x^2)^2}$

$$a_n = \begin{cases} n+1, & \text{n is odd} \\ 1, & \text{otherwise} \end{cases}$$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$

A. $\frac{x(1+x^2)}{(1-x^2)^2} + \frac{1}{1-x}$

B. $\frac{x(3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$

C. $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$

D. $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

$= 1 + 2x + x^2 + 4x^3 + x^4 + 6x^5 + x^6 + \cdots$

$= (1 + x^2 + x^4 + x^6 + \cdots) + (2x + 4x^3 + 6x^5 + \cdots)$

$= \frac{1}{1-x^2} + \frac{2x}{(1-x^2)^2} = \frac{2x + 1 - x^2}{(1-x^2)^2}$

Answer: (A)

Thank you