Elliptic Filter Design Report

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Contents

1 Filter Details (Bandpass)

1.1 Un-normalized Discrete Time Filter Specifications

Filter number = 104
Since filter number > 80, m = 104 - 80 = 24
q(m) = greatest integer strictly less than 0.1*m = 2
r(m) = m - 10*q(m) = 4
B_L(m) = 10 + 5*q(m) + 13*r(m) = 10 + 5*2 + 13*4 = 72KHz
B_H(m) = B_L(m) + 75 = 147KHz

This filter is given to be a Bandpass Filter with passband from BL(m) kHz to BH(m) kHz. Therefore the specifications are:

- Passband : **72 - 147 KHz**
- Transition band : **5KHz** on either side of stopband
- Stopband : **0 - 67** and **152 - 300 KHz** (As **sampling rate** is **600KHz**)
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Nature : Both Passband and Stopband are **equiripple**

1.2 Normalized Digital Filter Specifications

Sampling rate = 600KHz
In the normalized frequency axis, sampling rate corresponds to 2π
Therefore, any frequency can be normalized as follows :

$$\omega = \frac{\Omega * 2\pi}{\Omega_s}$$

where Ω_s is the Sampling Rate.

For the normalized discrete filter specifications, the nature and tolerances being the dependent variables remain the same while the passband and stopband frequencies change as per the above transformations.

- Passband : (**0.24 - 0.49**) π
- Transition band : **0.017** π on either side of stopband
- Stopband : (**0 - 0.223**) π and (**0.507 - 1**) π
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

1.3 Band-pass analog Filter Specifications using Bilinear Transformation

To convert to analog domain, we use the following transformation :

$$\Omega = \tan(\frac{w}{2})$$

Applying the transformation at Band Edges we get :

ω	Ω
0	0
0.24 π	0.396
0.49 π	0.969
0.223 π	0.365
0.507 π	1.022
π	∞

Hence, the specifications for corresponding bandpass analog filter are:

- Passband : (**0.396 - 0.969**)
- Transition band : **0.365 to 0.396** and **0.969 to 1.022**
- Stopband : (**0 - 0.365**) and (**1.022 -** ∞)
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

1.4 Low pass analog Filter Specifications using Frequency Transformation

We need to transform the bandpass filter to a low pass filter. Hence, we can use the bandpass transformation:

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega}$$

Where Ω_0 and B can be determined by: We want Ω_{P1} to map to -1 and Ω_{P2} to map to +1.
Using this, we get:

$$\Omega_0 = \sqrt{\Omega_{P1}\Omega_{P2}} = \sqrt{0.396 \times 0.969} = 0.61945$$

$$B = \Omega_{P2} - \Omega_{P1} = 0.969 - 0.396 = 0.573$$

Ω	Ω_L
0^+	- ∞
0.365(Ω_{S1})	-1.198$\Omega_{L_{S1}}$)
0.396(Ω_{P1})	-1($\Omega_{L_{P1}}$)
0.61945(Ω_0)	0
0.969(Ω_{P2})	+1($\Omega_{L_{P2}}$)
1.022(Ω_{S2})	1.128$\Omega_{L_{S2}}$)
∞	∞

1.5 Low pass Analog filter specifications

- Passband Edge : 1 (Ω_{L_P})
- Stopband Edge : min(-$\Omega_{L_{S1}}$, $\Omega_{L_{S2}}$) = min(1.198, 1.128) = 1.128 (Ω_{L_S})
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

1.6 Analog Lowpass Transfer Function

The analog filter has both passband and stopband Equiripple, hence, we need a **Elliptic** filter.

The Low pass transfer function for it is given by:

$$H_{LPF}(s_L)H_{LPF}(-s_L) = \frac{1}{1 + \epsilon^2 U_N^2(\frac{s_L}{j})}$$

Where $U_N(\Omega_L) = cd(N \times u \times K_1, k_1)$ where u is such that $\Omega_L = cd(u \times K(k), k)$, and cd is one of the general Jacobi elliptic functions. Here, K(k) is given by the equation:

$$\text{K(k)} = \int_0^1 \frac{1}{(1-t^2)(1-kt^2)}\, dt$$

Where the parameters are given by:

$$\epsilon = \sqrt{\frac{2\delta_1 - \delta_1^2}{1 - 2\delta_1 + \delta_1^2}}$$

$$k_1 = \frac{\epsilon}{\sqrt{\frac{1}{\delta_2^2} - 1}},\ k_1' = \sqrt{1 - k_1^2}$$

$$k = \frac{1}{\Omega_{s_L}},\ k' = \sqrt{1-k^2}$$

$$N = ceil(\frac{K(k)K'(k_1)}{K'(k)K(k_1)})$$

Substituting values, we get, $\epsilon = 0.6197$, $k_1 = 0.940$, k = $\frac{1}{1.128} = 0.8865$
Using MATLAB, we can compute K(k), and other integrals, and we get the value of N to be 4. Hence for this value of N, the modified values of k and Ω_{s_L} we get are k = 0.9595 and $\Omega_{s_L} = 1.0422$

Now, N can be represented as 2l+r, where r $\in$ {0,1}
, and it turns out that U_N can be written as:

$$U_N(\Omega) = (\Omega)^r \Pi_{i=1}^{l}\left[\left(\frac{\Omega^2-\zeta_i^2}{1-\Omega^2k^2\zeta_i^2}\right)\cdot\left(\frac{1-k^2\zeta_i^2}{1-\zeta_i^2}\right)\right] \tag{1}$$

where

$$u_i = \frac{2i-1}{N}\ ;\text{ where } i \in 1,2,...,l$$

$$\zeta_i = cd(u_i \cdot K(k), k) = cde(u_i, k)$$

Hence the zeroes of U_N are given by:

$$z_i = \frac{j}{k\zeta_i}\ ;\text{ where } i \in 1,2,...,l \tag{2}$$

Remaining zeroes are z_i^* i.e. conjugates of z_i
Here, as N=4, we have l=2, r=0.
Hence, the zeroes are:
z_1=j/(0.9595*0.9796)=1.0639j
z_2=j/(0.9595*0.5891)=1.769j
z_1^*= -1.0639j
z_2^*= -1.7690j

And the poles are given by:
There are 2l+r i.e. N poles to $|H(j\Omega)|$
Let v $\in \mathbb{R}$ be a solution to $sn(jv \cdot N \cdot K(k), k)$ = j/ϵ_p, therefore

$$v = \frac{-j}{N\cdot K(k)}sn^{-1}(\frac{j}{\epsilon_p}, k_1) = \frac{-j}{N}asne(\frac{j}{\epsilon_p}, k_1) \tag{3}$$

Poles are given by:

$$p_i = jcd((u_i - jv)K(k), k) = jcde(u_i - jv, k)\ \text{,where } i \in 1,2,...,l \tag{4}$$

$$p_0 = jcd((1-jv)K(k), k) = jcde(1-jv, k) \tag{5}$$

p_0 is the pole on negative real axis which occurs only when r=1.

Hence in this case, using MATLAB, we get that the poles are:
p_1= -0.0310 + 0.9995i
p_2= -0.3536 + 0.7071i
p_1^*= -0.0310 - 0.9995i
p_2^*= -0.3536 - 0.7071i

$$H_{analog\_LP}(s) = \frac{0.15s^4 + 0.6392s^2 + 0.5313}{1s^4 + 0.7691s^3 + 1.6689s^2 + 0.7459s + 0.6251} \tag{6}$$

Here, the leading coefficient on the numerator is obtained by scaling the numerator such that ratio of constant terms of the numerator and denominator equals the value of $H_{analog\_LP}(0)$. Here, $H_{analog\_LP}(0)$ turns out to be 0.85 by substituting the values of ϵ and $U_N(0)$, and also, 0.5313/0.6251 = 0.85

1.7 Analog Bandpass Transfer Function

The transformation is given by:

$$s_L = \frac{\Omega_0^2 + s^2}{Bs}$$

Substituting Ω_0 and B, we get:

$$H_{analog\_BP}(s) = \frac{(0.1500s^8 + 0.4402s^6 + 0.3509s^4 + 0.0648s^2 + 0.0033)}{1s^8 + 0.4408s^7 + 2.0829s^6 + 0.6478s^5 + 1.3714s^4 + 0.2486s^3 + 0.3066s^2 + 0.0249s + 0.0217}$$

1.8 Discrete Time Filter Transfer Function

Now, performing the bilinear transform, i.e. $s = \frac{1-z^{-1}}{1+z^{-1}}$ gives us:

$$H_{discrete\_BS}(z) = \frac{0.1642z^8-0.4354z^7+0.7986z^6-1.0930z^5+1.2667z^4-1.0930z^3+0.7986z^2-0.4354z+0.1642}{1z^8-2.9661z^7+6.0876z^6-8.3174z^5+9.0890z^4-7.2019z^3+4.5490z^2-1.8939z+0.5567}$$

Hence, this, is the final discrete time transfer function of our Elliptic bandpass filter.

1.9 Matlab Plots

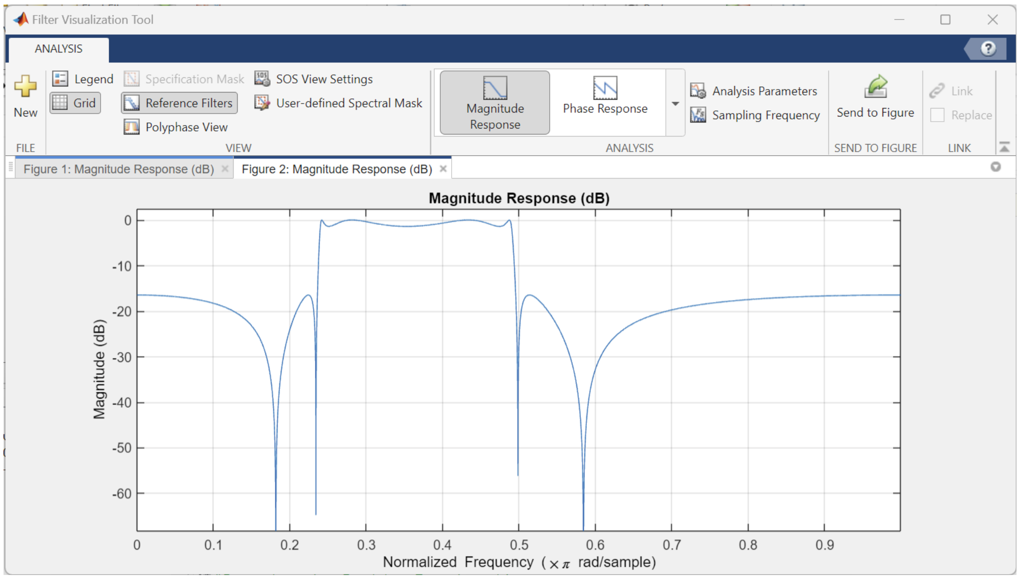

Figure 1: Magnitude Plot in dB

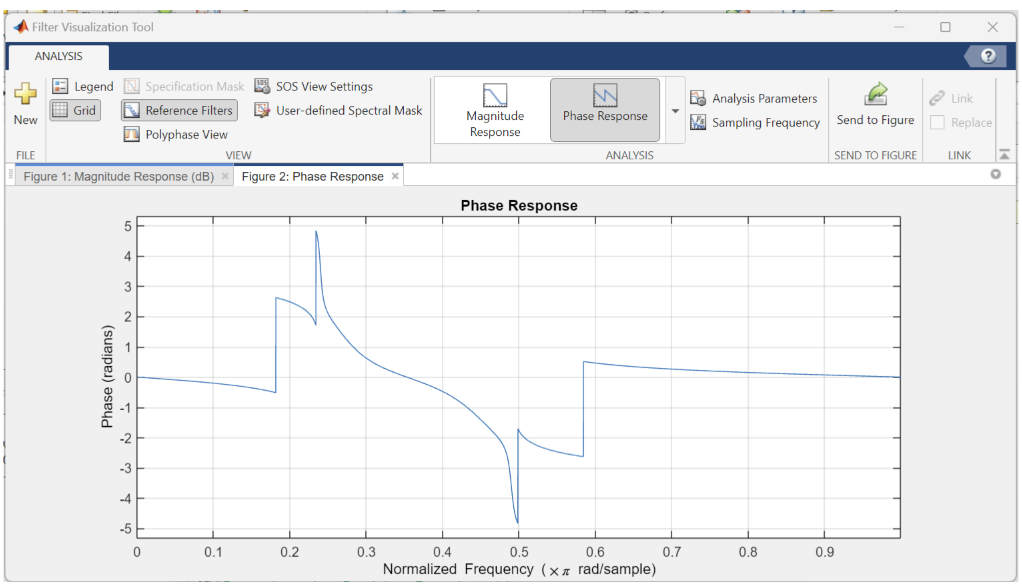

Figure 2: Phase Response

We can see that the Phase Response is **not** linear.

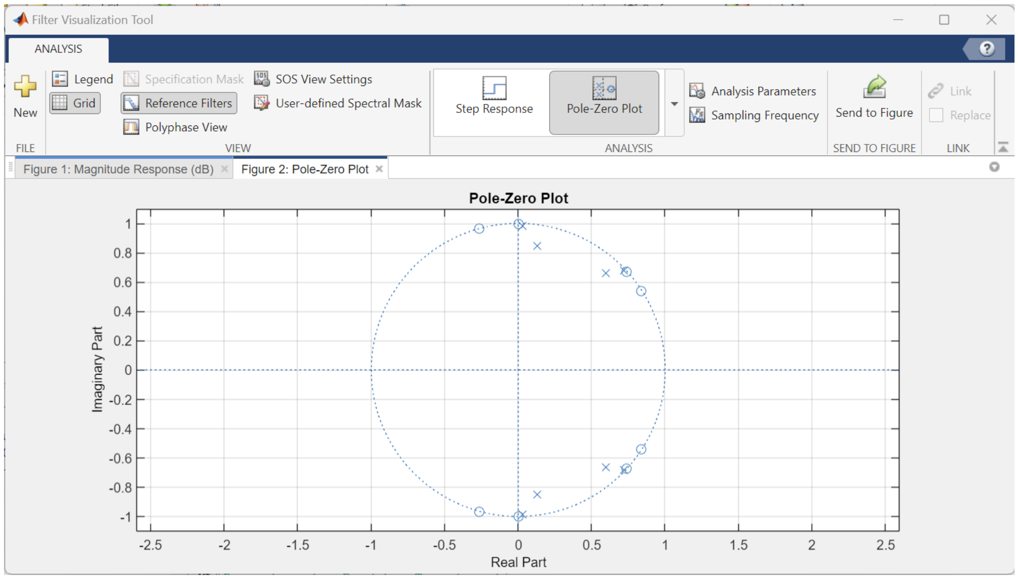

Figure 3: Poles and Zeroes

We can see that all the poles lie inside the unit circle and hence, the system is stable.

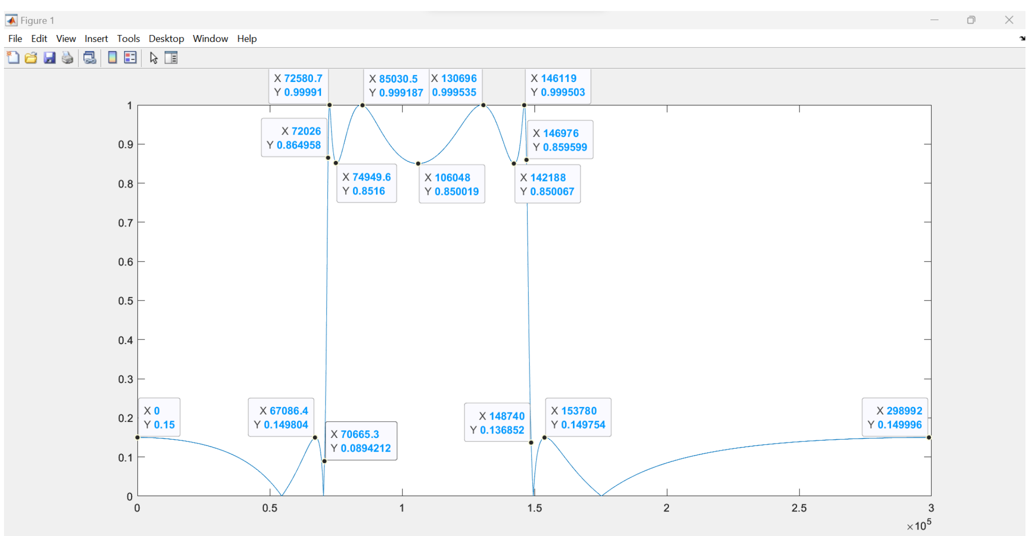

Figure 4: Magnitude Plot

From the above figure, we can see that all the specifications are satisfied.

1.10 Observations on elliptical filter design and comparisons with Chebyschev, FIR methods

I had also designed Bandpass filters with the same specifications except nature of Passband and Stopband with Chebyschev and FIR filters. So I noticed the following things about the 3 filters:

- For the same PassBand and StopBand frequencies, the order, i.e. the amount of resources we need is the least for elliptic filters, and highest for FIR filters.
- Also, in the Magnitude plot, I observed that FIR filters don't have much Ripple, whereas in Elliptic and Chebyschev, the ripples are such that it just satisfies the specifications. Also, in Elliptic and Chebyschev filters, the gain is never more than 1, but in FIR filters, the gain also lies between 1 and 1 + (PassBand tolerance).
- We can't control the tolerances of passband and stopband independently in FIR filters. In both Chebyschev and Elliptic filters, we can do that.
- The Phase Response is linear (as we preferably want) in FIR filter, whereas it's non linear in both Elliptic and Chebyschev Filters.

So, each type of filter has its own advantage, we can choose the one we want according to our needs.

2 Filter Details (Bandstop)

2.1 Un-normalized Discrete Time Filter Specifications

Filter number = 104
Since filter number > 80, m = 104 - 80 = 24
q(m) = greatest integer strictly less than 0.1*m = 2
r(m) = m - 10*q(m) = 4
B_L(m) = 20 + 3*q(m) + 11*r(m) = 20 + 3*2 + 11*4 = 70KHz
B_H(m) = B_L(m) + 40 = 110KHz

This filter is given to be a Bandstop Filter with stopband from BL(m) kHz to BH(m) kHz. Therefore the specifications are:

- Stopband : **70 - 110 KHz**
- Transition band : **5KHz** on either side of stopband
- Passband : **0 - 65** and **115 - 212.5 KHz** (As **sampling rate** is **425KHz**)
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Nature : Both passband and stopband are **Equiripple**

2.2 Normalized Digital Filter Specifications

Sampling rate = 425KHz
In the normalized frequency axis, sampling rate corresponds to 2π
Therefore, any frequency can be normalized as follows :

$$\omega = \frac{\Omega * 2\pi}{\Omega_s}$$

where Ω_s is the Sampling Rate.

For the normalized discrete filter specifications, the nature and tolerances being the dependent variables remain the same while the passband and stopband frequencies change as per the above transformations.

- Stopband : (**0.329 - 0.518**) π
- Transition band : **0.024** π on either side of stopband
- Passband : (**0 - 0.306**) π and (**0.541 - 1**) π
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

2.3 Band-stop analog Filter Specifications using Bilinear Transformation

To convert to analog domain, we use the following transformation :

$$\Omega = \tan(\frac{w}{2})$$

Applying the transformation at Band Edges we get :

ω	Ω
0	0
0.329 π	0.568
0.518 π	1.058
0.306 π	0.521
0.541 π	1.137
π	∞

Hence, the specifications for corresponding bandstop analog filter are:

- Stopband : (**0.568 - 1.058**)
- Transition band : **0.521 to 0.568** and **1.058 to 1.137**
- Passband : (**0 - 0.521**) and (**1.137 -** ∞)
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

2.4 Low pass analog Filter Specifications using Frequency Transformation

We need to transform the bandstop filter to a low pass filter. Hence, we can use the bandstop transformation:

$$\Omega_L = \frac{B\Omega}{\Omega_0^2 - \Omega^2}$$

Where Ω_0 and B can be determined by: We want Ω_{P1} to map to +1 and Ω_{P2} to map to -1.
Using this, we get:

$$\Omega_0 = \sqrt{\Omega_{P1}\Omega_{P2}} = \sqrt{0.521 \times 1.137} = 0.76966$$

$$B = \Omega_{P2} - \Omega_{P1} = 1.137 - 0.521 = 0.616$$

Ω	Ω_L
0^+	0^+
$0.521(\Omega_{P1})$	$+1(\Omega_{L_{P1}})$
$0.568(\Omega_{S1})$	$1.297\Omega_{L_{S1}})$
$0.76966(\Omega_0^-)$	∞
$0.76966(\Omega_0^+)$	- ∞
$1.058(\Omega_{S2})$	$-1.236\Omega_{L_{S2}})$
$1.137(\Omega_{P2})$	$-1(\Omega_{L_{P2}})$
∞	0^-

2.5 Low pass Analog filter specifications

- Passband Edge : 1 (Ω_{L_P})
- Stopband Edge : min($\Omega_{L_{S1}}$, -$\Omega_{L_{S2}}$) = min(1.297, 1.236) = 1.236 (Ω_{L_S})
- Tolerance : **0.15** in **magnitude** for both passband and stopband
- Passband nature: **Equiripple**
- Stopband nature: **Equiripple**

2.6 Analog Lowpass Transfer Function

The analog filter has both passband and stopband Equiripple, hence, we need a **Elliptic** filter.

The Low pass transfer function for it is given by:

$$H_{LPF}(s_L)H_{LPF}(-s_L) = \frac{1}{1+\epsilon^2 U_N^2(\frac{s_L}{j})}$$

Where $U_N(\Omega_L) = cd(N \times u \times K_1, k_1)$ where u is such that $\Omega_L = cd(u \times K(k), k)$, and cd is one of the general Jacobi elliptic functions. Here, K(k) is given by the equation:

$$\text{K(k)} = \int_0^1 \frac{1}{(1-t^2)(1-kt^2)}\, dt$$

Where the parameters are given by:

$$\epsilon = \sqrt{\frac{2\delta_1 - \delta_1^2}{1-2\delta_1+\delta_1^2}}$$

$$k_1 = \frac{\epsilon}{\sqrt{\frac{1}{\delta_2^2}-1}},\ k_1' = \sqrt{1-k_1^2}$$

$$k = \frac{1}{\Omega_{s_L}},\ k' = \sqrt{1-k^2}$$

$$N = ceil(\frac{K(k)K'(k_1)}{K'(k)K(k_1)})$$

Substituting values, we get, $\epsilon = 0.6197$, $k_1 = 0.940$, k = $\frac{1}{1.236} = 0.8091$ Using MATLAB, we can compute K(k), and other integrals, and we get the value of N to be 3. Hence for this value of N, the modified values of k and Ω_{s_L} we get are k = 0.8571 and $\Omega_{s_L} = 1.1667$

Now, N can be represented as 2l+r, where r $\in$ {0,1}
, and it turns out that U_N can be written as:

$$U_N(\Omega) = (\Omega)^r \Pi_{i=1}^{l} \left[\left(\frac{\Omega^2 - \zeta_i^2}{1-\Omega^2 k^2 \zeta_i^2} \right) \cdot \left(\frac{1-k^2\zeta_i^2}{1-\zeta_i^2} \right) \right] \tag{7}$$

where

$$u_i = \frac{2i-1}{N} \text{ ; where } i \in 1, 2, ..., l$$
$$\zeta_i = cd(u_i \cdot K(k), k) = cde(u_i, k)$$

Hence the zeroes of U_N are given by:

$$z_i = \frac{j}{k\zeta_i} \text{ ; where } i \in 1, 2, ..., l \tag{8}$$

Remaining zeroes are z_i^* i.e. conjugates of z_i
Here, as N=3, we have l=1, r=1.
Hence, the zeroes are:
z_1=j/(0.8571*0.9256)=1.1261j
z_1^*= -1.1261j

And the poles are given by:
There are 2l+r i.e. N poles to $|H(j\Omega)|$
Let v $\in \mathbb{R}$ be a solution to $sn(jv \cdot N \cdot K(k), k)$ = j/ϵ_p, therefore

$$v = \frac{-j}{N \cdot K(k)} sn^{-1}(\frac{j}{\epsilon_p}, k_1) = \frac{-j}{N} asne(\frac{j}{\epsilon_p}, k_1) \tag{9}$$

Poles are given by:

$$p_i = jcd((u_i - jv)K(k), k) = jcde(u_i - jv, k) \text{ ,where } i \in 1, 2, ..., l \tag{10}$$

$$p_0 = jcd((1-jv)K(k), k) = jcde(1-jv, k) \tag{11}$$

p_0 is the pole on negative real axis which occurs only when r=1.

Hence in this case, using MATLAB, we get that the poles are:
p_0= -0.6232
p_1= -0.1153 + 0.9936i
p_1^*= -0.1153 - 0.9936i

$$H_{analog\_LP}(s) = \frac{0.3925s^2 + 0.6235}{1s^3 + 0.8539s^2 + 1.1443s + 0.6235} \tag{12}$$

Here, the leading coefficient on the numerator is obtained by scaling the numerator such that ratio of constant terms of the numerator and denominator equals the value of $H_{analog\_LP}(0)$. Here, $H_{analog\_LP}(0)$ turns out to be 1 by substituting the values of ϵ and $U_N(0)$, and also, we can see that the constant terms of the num and den are equal.

2.7 Analog Bandpass Transfer Function

The transformation is given by:

$$s_L = \frac{Bs}{\Omega_0^2 + s^2}$$

Substituting Ω_0 and B, we get:

$$H_{analog\_BP}(s) = \frac{(1s^6 + 2.0201s^4 + 1.1988s^2 + 0.2090)}{1s^6 + 1.1329s^5 + 2.3020s^4 + 1.7218s^3 + 1.3661s^2 + 0.3989s + 0.2090}$$

2.8 Discrete Time Filter Transfer Function

Now, performing the bilinear transform, i.e. $s = \frac{1-z^{-1}}{1+z^{-1}}$ gives us:

$$H_{discrete\_BS}(z) = \frac{0.5446z^6 - 0.7858z^5 + 1.8345z^4 - 1.5417z^3 + 1.8345z^2 - 0.7858z + 0.5446}{1z^6 - 1.1751z^5 + 2.0860z^4 - 1.4853z^3 + 1.4725z^2 - 0.4529z + 0.1997}$$

Hence, this, is the final discrete time transfer function of our Elliptic bandpass filter.

2.9 Matlab Plots

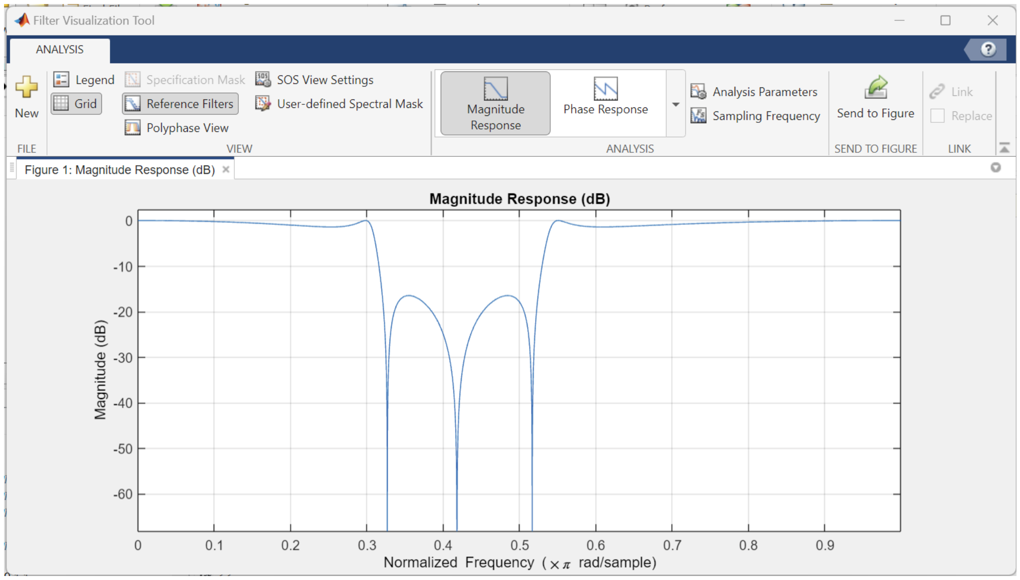

Figure 5: Magnitude Plot in dB

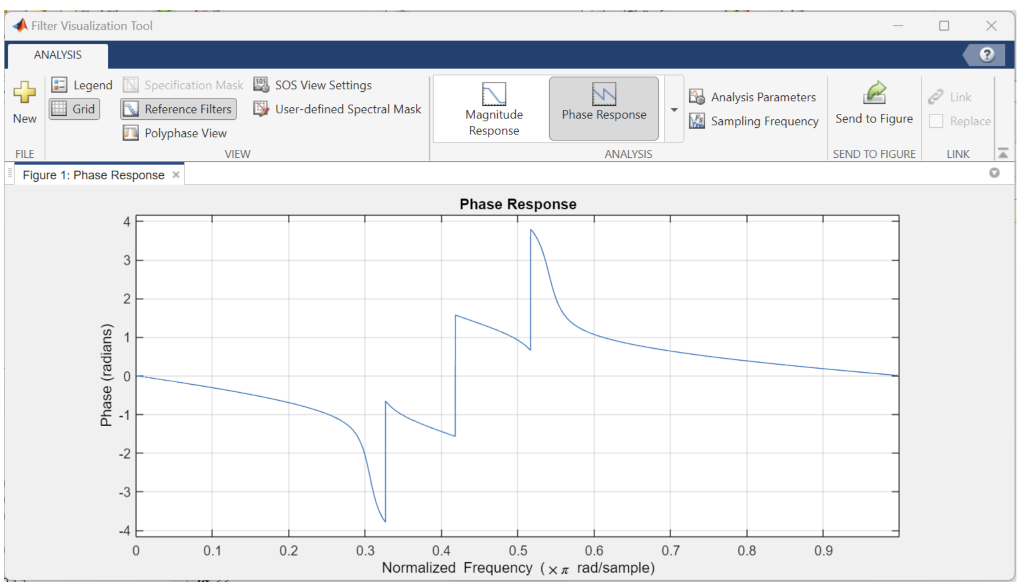

Figure 6: Phase Response

We can see that the Phase Response is **not** linear.

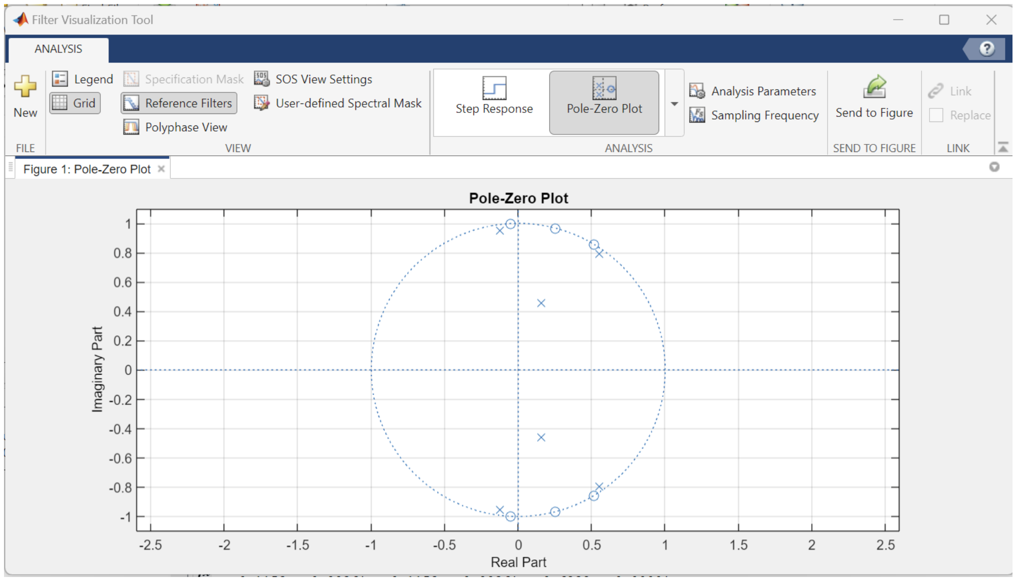

Figure 7: Poles and Zeroes

We can see that all the poles lie inside the unit circle and hence, the system is stable.

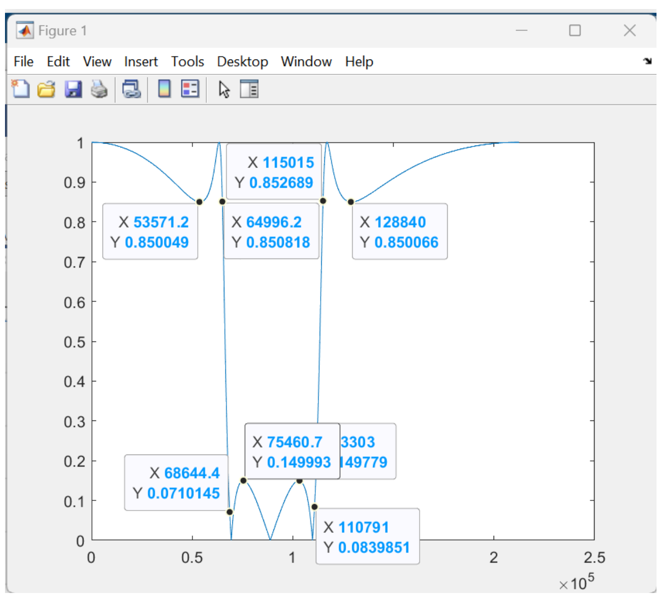

Figure 8: Magnitude Plot

From the above figure, we can see that all the specifications are satisfied.

2.10 Observations on elliptical filter design and comparisons with Butterworth, FIR methods

I had also designed Bandstop filters with the same specifications except nature of Passband and Stopband with Butterworth and FIR filters. So I noticed the following things about the 3 filters:

- For the same PassBand and StopBand frequencies, the order, i.e. the amount of resources we need is the least for elliptic filters, and highest for FIR filters.
- Also, in the Magnitude plot, I observed that FIR filters don't have much Ripple, whereas in Elliptic, the ripples are such that it just satisfies the specifications and Butterworth doesn't have any ripples at all. Also, in Elliptic and Butterworth filters, the gain is never more than 1, but in FIR filters, the gain also lies between 1 and 1 + (PassBand tolerance).
- We can't control the tolerances of passband and stopband independently in FIR filters. In both Butterworth and Elliptic filters, we can do that.
- The Phase Response is linear (as we preferably want) in FIR filter, whereas it's non linear in both Elliptic and Butterworth Filters.

So, each type of filter has its own advantage, we can choose the one we want according to our needs.

3 Peer Review of Assignment

I have reviewed the report of my peer:

Annirudh K P
(Roll no: 210070009, Filter number: 96)

and certify it to be correct.