Topology

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Overview of Topics

The topics I covered in my SoS report on Topology are:

1. Some Basics of Set Theory
2. Point Set Topology
3. Topological Spaces
4. Euler Characteristic
5. Homology Groups

Compact Sets

Definition

A set X is compact if, for every open covering $\{U_i | i \in I\}$, there exists a finite subset J of I such that $\{U_j | j \in J\}$ is also a covering of X.

Covering of a set X is a collection of sets such that the union of the sets in that collection forms a superset of the set X. If all the sets in the collection are open, it is called an open covering.

Its pretty clear that if a set is compact, it must be bounded, what else is needed?

Compact Sets

It turns out that if X is a subset of $\mathbb{R}^n$, X is compact if and only if it is closed and bounded. And in this video, I will try to intuitively show why this is true.

First, I will show that if X is compact, it has to be closed and bounded. As I said before, the bounded part is pretty clear. But what happens if the set is not closed?

Consider the set $X = (1, 2]$, now consider the collection of sets of the type $(1+\frac{1}{n}, 3)$, where n is a natural number. Note that it forms an open covering of X, but it cannot be reduced to a finite covering.

Compact Sets

Note that this happens with every set X which is not closed. There is always a sequence of which all the points are in the set, but the point of convergence doesn't belong to the set. Now, if we consider the set which has one of the endpoints as a point in the sequence and we take the collection of all such sets with the other endpoint such that it covers X, it cannot be reduced to a finite covering.
So the first part is done.

Compact Sets

Now we will intuitively prove the converse.
Consider the closed and bounded interval [a, b]. Now as a is in the set, some set in the open covering has a in it. Now, this set has lub as c, then the remaining sets in the covering have to cover the closed set [c, b]. Note that c-a is a finite non-zero quantity.

We keep doing this so the set gets smaller and smaller. Note that this process has to stop or else the b won't be in any set in the covering which isn't possible. Also any closed subset of $\mathbb{R}$ can be written as a union of disjoint closed intervals like [a, b], so this arguement works there as well.

Note that I have proved only for the subset of $\mathbb{R}$ and not $\mathbb{R}^n$ because I just wanted to give an intuitive idea of why closed and boundedness might be equivalent to compactness.

Compact Sets

Also note that closed and boundedness isn't equivalent to compactness for metric spaces. Compactness still implies closed and boundedness, but the converse isn't true.

A metric space is a nonempty set M of objects (called points) together with a function d from M x M to R (called the metric of the space) satisfying the following four properties for all points x, y, z in M:
1. d(x, x) $=$ 0.
2. $d(x, y) > 0$ if $x \neq y$.
3. d(x, y) $=$ d(y, x).
4. $d(x, y) \leq d(x, z) + d(z, y)$.

Compact Sets

For example, consider the metric space $\mathbb{Q}$ of rational numbers with the same definition of d as in $\mathbb{R}$. Let S consist of all rational numbers in the open interval (a, b), where a and b are irrational.

Now, if we take any rational number c greater than a near a, there are infinitely many rational nos, between c and a. Now, consider the subsets in the collection to be (x, b), where x is a rational number greater than a. This is an open covering which cannot be reduced to a finite covering. Hence, S is closed and bounded in $\mathbb{Q}$, but not compact.

Thank You!