Topology

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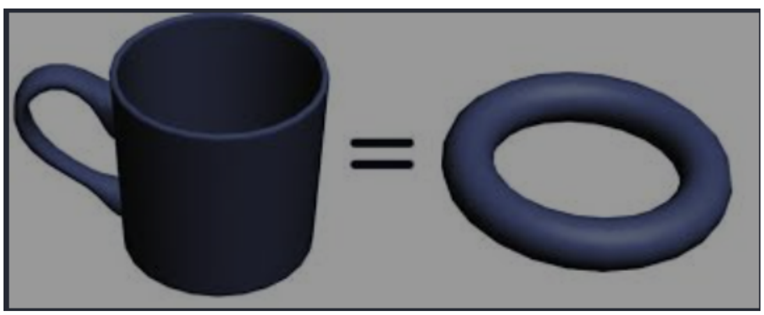

Figure 1: A cup is homeomorphic to a doughnut!

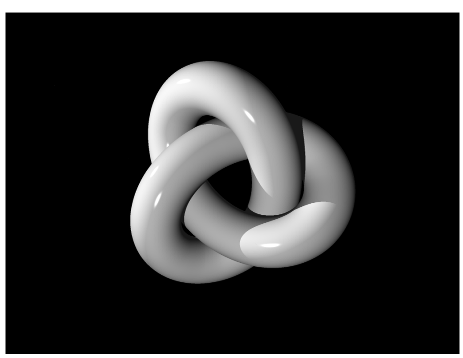

Figure 2: A 3-D depiction of trefoil knot, the simplest non-trivial knot

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1 Introduction

In topology, we study properties of spaces that are invariant under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself. For example, a traditional joke is that a topologist cannot distinguish a coffee mug from a doughnut, see Figure 1, since a sufficiently pliable doughnut could be reshaped to a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle.

Some of the typical questions in topology are: How many holes are there in an object? How can you define the holes in a torus or sphere? What is the boundary of an object? Is a space connected? Does every continuous function from the space to itself have a fixed point?

In this report, I am going to state the important theorems I have learnt and give an intuition behind why they are true, and also sometimes some additional examples to understand the theorem better. I read chapters 2 and 3 from the books 'Mathematical Analysis' by Apostol and Geometry, Topology and Physics - Mikio Nakahara. Detailed proofs of theorems are given in the books.

2 Some Basic Notions of Set Theory

In this section, I will be going through some important things in set theory required to understand Topology.

Definition 2.1 (Equinumerous sets)**.** *Two sets A and B are called similar, or equinumerous, if and only if there exists a one-to-one function F whose domain is the set A and whose range is the set B.*

It is easy to see that two finite sets are similar if they have same cardinality. But in case of infinite sets, it depends on if the set is countable or uncountable.

Definition 2.2. *A set S is said to be countably infinite if it is equinumerous with the set of all positive integer*

Or equivalently, a set is countable if its elements are discrete.

The set of real numbers is uncountable, which is pretty much clear by the fact that real numbers form a bijection with the number line, and so is continuous.

But surprisingly, the set of rational numbers is countable. The fact that between any 2 rational numbers, there are infinite rational numbers seems to contradict this, but we can prove that it forms bijection with set of natural numbers, and hence is countable, using the fact that rational numbers can be written in the form $\frac{p}{q}$.

And the set of irrational numbers is uncountable.

Theorem 2.1. *If F is a countable collection of countable sets, then the union of all sets in F is also a countable set.*

Note that here, the fact that F is countable collection of sets allows us to write sets in F as $A_1, A_2, ...$, and as A_i are countable, elements of each set A_i can be written as $a_{(i,1)}, a_{(i,2)}, ...$. And now it can be shown that the set of ordered pairs (i,j), where i and j are both natural numbers, is countable.

3 Point Set Topology

3.1 Euclidean space R^n

Definition 3.1. *Let $n > 0$ be an integer. An ordered set of n real numbers $(x_1, x_2, ..., x_n)$ is called an n-dimensional point or a vector with n components. The set of all n-dimensional points is called n-dimensional Euclidean space or simply n-space, and is denoted by R^n.*

One may wonder whether there is any advantage in discussing spaces of dimension greater than three. Actually, the language of n-space makes many complicated situations much easier to comprehend. We are familiar enough with three-dimensional vector analysis to realize the advantage of writing the equations of motion of a system having three degrees of freedom as a single vector equation rather than as three scalar equations. There is a similar advantage if the system has n degrees of freedom. Also, we can use many properties common to 1-D, 2-D, 3-D spaces, that is, properties common to dimensionality of the space, in n-D space.

Algebraic operations on n-D points like equality, addition, multiplication by scalar, difference, dot product, length, are defined similar to those in 1-D, 2-D, 3-D spaces.

3.2 Open balls and open sets in R^n

Definition 3.2 (Open ball)**.** *The ball B(a; r) consists of all points whose distance from a is less than r.*

Definition 3.3 (Interior point)**.** *Let S be a subset of R^n, and assume that $a \in S$. Then a is called an interior point of S if there is an open n-ball with center at a, all of whose points belong to S.*

Definition 3.4 (Open set)**.** *A set S in R^n is called open if all its points are interior points.*

Note that both an empty set and R^n are open sets in R^n. Also, an open set in R^1 is *not* an open set in R^2. In fact, *no* subset of R^1 (except the empty set) can be open in R^2, because such a set cannot contain a 2-ball.

Theorem 3.1. *The union of any collection of open sets is an open set.*

Proof. Let the union of the sets be S. Now, take an element x in S. It has to belong to one of the open sets say A. Now, as A is open, consider an n-ball around x which is completely in A. All the elements of this n-ball are also present in S, as S is the union of A and other open sets, hence the same n-ball around x is completely in S. And this is true for any element x in S, hence, S is open.

□

Theorem 3.2. *The intersection of a finite collection of open sets is open.*

Proof. Let the intersection of the sets be S. Now, take an element x in S. It has to belong to all the open sets. Consider n-balls around x in each open set which is completely in the open set. Since the number of open sets is finite, number of n-balls is also finite, and hence we can find the smallest n-ball of them. Now, all points in this smallest n-ball belong to each set and hence also to their intersection, S. And this is true for every x in S, Hence, S is open. □

Now, S may not be open if the collection of sets is infinite. For example, take intersection of sets $(-\frac{1}{n}, 2+\frac{1}{n})$, n = 1, 2, 3, ... Their intersection is [0,2], which is not open because 0 and 2 are not interior points.

3.3 The structure of open sets in R^1

Definition 3.5 (Component interval)**.** *Let S be an open subset of R^1. An open interval I (which may be bounded or unbounded) is called a component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$.*

Theorem 3.3. *Every point of a nonempty open set S belongs to one and only one component interval of S.*

Theorem 3.4 (Representation theorem for open sets on the real line)**.** *Every non-empty open set S in R^1 is the union of a countable collection of disjoint component intervals of S.*

I didn't give any explanation or proof of above theorems because they are pretty intuitive to understand, if one wants to read the rigorous proof, it's available in the book.

Also, this representation of S is unique. In fact, if S is a union of disjoint open intervals, then these intervals must be the component intervals of S.

If S is an open interval, then the representation contains only one component interval, namely S itself. Therefore an open interval in R^1 cannot be expressed as the union of two nonempty disjoint open sets. This property is also described by saying that an open interval is connected.

3.4 Closed sets

Definition 3.6. *A set S in R^n is called closed if its complement $R^n - S$ is open.*

Theorem 3.5. *The union of a finite collection of closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed.*

This can be proved using the fact that for union(intersection) of closed sets, we are actually taking the complement of the set obtained by taking the intersection(union) of complement of the closed sets, and complement of a closed set is open.

Definition 3.7 (Adherent point)**.** *Let S be a subset of R^n, and x a point in R^n, x not necessarily in S. Then x is said to be adherent to S if every n-ball B(x) contains at least one point of S.*

So the set of adherent point is basically the union of set of points of the original sets and set of its boundary points. And if the elements of a set form an infinite converging sequence, the point at which the sequence converges is also an adherent point.

Definition 3.8 (Accumulation point)**.** *If $S \subseteq R^n$ and $x \in R^n$, then x is called an accumulation point of S if every n-ball B(x) contains at least one point of S distinct from x.*

In other words, x is an accumulation point of S if, and only if, x adheres to S - x. If $x \in S$ but x is not an accumulation point of S, then x is called an *isolated* point of S.

Definition 3.9. *If x is an accumulation point of S, then every n-ball B(x) contains infinitely many points of S.*

Proof. This is pretty obvious because we if the number of points are finite, we can always find a point which is closest to x and if we take the radius of a new n-ball to be less than the distance of x from that point, we have a n-ball with no point in the set S-x, which means x is not an accumulation point, hence, we get a contradiction.

□

Theorem 3.6. *A set S in R^n is closed if, and only if, it contains all its adherent points.*

Proof. Let x be an adherent point of a closed S, and $x \notin S$. So $x \in R^n - S$, which is open. But, x is an adherent point of S implies every n-ball around x contains an element of S, but, x being in the open set $R^n - S$ implies, there exists a n-ball around x which completely lies in $R^n - S$. These 2 statements contradict. Hence, $x \in S$.

To prove its converse, assume S contains all its adhere points, so now, let $x \in R^n - S$, which means x doesn't adhere S, so there exists an open n-ball around x not containing any element of S, hence the n-ball is completely in $R^n - S$. This is true for all points in $R^n - S$, which means $R^n - S$ is open, which means S is closed.

□

Definition 3.10 (Closure of a set)**.** *The set of all adherent points of a set S is called the closure of S and is denoted by $\bar{S}$.*

Theorem 3.7. *A set S is closed if and only if, $S = \bar{S}$.*

Definition 3.11 (Derived set)**.** *The set of all accumulation points of a set S is called the derived set of S and is denoted by S'.*

Theorem 3.8. *A set S in R^n is closed if, and only if, it contains all its accumulation points.*

3.5 The Bolzano-Weierstrass Theorem

Definition 3.12 (Bounded set)**.** *A set S in R^n is said to be bounded if it lies entirely within an n-ball B(a; r) for some $r > 0$ and some a in R^n.*

Theorem 3.9 (Bolzano-Weierstrass)**.** *If a bounded set S in R^n contains infinitely many points, then there is at least one point in R^n which is an accumulation point of S.*

This theorem tells that, let's say the points form a continuous patch somewhere, then there is no question why the theorem is true, but when the points are discrete, there exists a point around which infinitely many points are accumulated which intuitively makes sense, because if in every small finite region, there are finite number of points, adding finite number of such regions would never give infinite number of points.

3.6 The Cantor Intersection Theorem

Theorem 3.10. *Let Q_1, Q_2, ... be a countable collection of nonempty sets in R^n such that:*
i) $Q_{k+1} \subseteq Q_k$ (k = 1, 2, 3, ...).
ii) Each set Q_k is closed and Q_1 is bounded. Then the intersection $S = \bigcap\limits_{k=1}^{\infty} Q_k$ is closed and nonempty.

Proof. As each Q_k is closed, S is closed by Theorem 3.5. Now, lets assume that each Q_k has infinitely many points, otherwise the proof is trivial. Now, form a set of distinct elements A = $x_1, x_2, ...$, where $x_k \in Q_k$. As A is an infinite set contained in a bounded set Q_1, A has an accumulation point, say x. Now, note that as $Q_{k+1} \subseteq Q_k$ for all natural numbers k, all elements of A except(possibly) $x_1, x_2, ..., x_{k-1}$ belong to Q_k, hence x is also accumulation point for all Q_k, and as each Q_k is closed, $x \in Q_k$ for all natural numbers k, hence $x \in S$. Hence, S is closed and non-empty. □

3.7 Covering of a set

Definition 3.13 (Covering of a set)**.** *A collection F of sets is said to be a covering of a given set S if $S \subseteq \bigcup\limits_{A \in F} A$. The collection F is also said to cover S. If F is a collection of open sets, then F is called an open covering of S.*

Theorem 3.11 (Lindelof covering theorem)**.** *Assume $S \subseteq R^n$ and let F be an open covering of S. Then there is a countable subcollection of F which also covers S.*

Note that this may not be true for closed coverings. For example, take $S = R^1$, and let F be collection of sets with just 1 element of R^1. Now, no countable subcollection of F can be a covering of S. We can intuitively see that this shouldn't happen with open sets, but a rigorous proof is given in the book.

Theorem 3.12 (Heine-Borel theorem)**.** *Let F be an open covering of a closed and bounded set S in R^n. Then a finite subcollection of F also covers S.*

Proof. Because F is an open covering, a countable subcollection of F, say $\{I_1, I_2, ...\}$, covers S. Consider, for $m > 1$, the finite union, $S_m = \bigcap_{k=1}^{m} I_k$. This is open, since it is the union of open sets. We shall show that for some value of m the union S_m covers S. Now, consider the set Q_m, which is the set of points in A, which are not in S_m. So, S_m covers S for some m, is equivalent to saying Q_m is empty for some m. Now, each Q_m is closed, $Q_{m+1} \subseteq Q_m$, and are all bounded. Now, if all Q_m are non empty, by theorem 3.6, we can conclude that $\bigcap_{k=1}^{\infty} Q_k$ is non empty which contradicts with the fact that F covers S. Hence, Q_m is empty for some m which implies a finite covering of F covers S. □

3.8 Compact Sets

Definition 3.14 (Compact Sets)**.** *A set S in R^n is said to be compact if, and only if, every open covering of S contains a finite subcover, that is, a finite subcollection which also covers S.*

Theorem 3.13. *Let S be a subset of R^n. Then the following three statements are equivalent:*
a) S is compact.
b) S is closed and bounded.
c) There exists an accumulation point of every infinite subset of S, and it belongs to S (The point may not belong to the infinite subset).

Note that if S is not closed and bounded, there exists an open covering F such that it doesn't have a finite subcover. Its obvious if S is not bounded, but let's say F is not closed, but bounded, eg (0,1], then, let $A_n = (\frac{1}{n}, 2)$ and F is the collection of all A_n, where n is a natural number is an open covering, but cannot be reduced to a finite one.

This intuitively tells why a and b are equivalent.

Now, b implies c, as infinite bounded set is must have an accumulation point, also, the accumulation point belongs to S as S is closed. Also, an unbounded set will not satisfy c, as we can always take infinite isolated points from an unbounded set. The only thing remaining is that c implies the set should be closed.

Let A be an open set. Consider a sequence in which all elements belong to S and it converges to a boundary point of S, which does not belong to S as it is open. Now, all the points of the sequence are isolated points, the only accumulation point of A is the boundary point, but it doesn't belong to S. Hence any set which isn't closed doesn't satisfy c. Hence proved.

3.9 Metric Spaces

The proofs of some of the theorems of this chapter depend only on a few properties of the distance between points and not on the fact that the points are in R^n. When these properties of distance are studied abstractly they lead to the concept of a metric space.

Definition 3.15 (Metric Space)**.** *A metric space is a nonempty set M of objects (called points) together with a function d from M x M to R (called the metric of the space) satisfying the following four properties for all points x, y, z in M:*
1. d(x, x) = 0.
2. $d(x,y) > 0$ *if* $x \neq y$*.*
3. d(x, y) = d(y, x).
4. $d(x,y) \leq d(x,z) + d(z,y)$*.*

The non-negative number d(x, y) is to be thought of as the distance from x to y. In these terms the intuitive meaning of properties 1 through 4 is clear.

We sometimes denote a metric space by (M, d) to emphasize that both the set M and the metric d play a role in the definition of a metric space.

Examples:
1) Many nonempty set; d(x, y) = 0 if x = y, d(x, y) = 1 if $x \neq y$. This is called the discrete metric, and (M, d) is called a discrete metric space.
2) If (M, d) is a metric space and if S is any nonempty subset of M, then (S, d) is also a metric space with the same metric or, more precisely, with the restriction of d to S x S as metric. This is sometimes called the relative metric induced by d on S, and S is called a metric subspace of M. For example, the rational numbers Q with the metric d(x, y) = $|x-y|$ form a metric subspace of R.
3) $M = R^2$; d(x, y) = $\sqrt{(x_1-y_l)^2+4(x_2-y_2)^2}$.
4) $M = \{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2 = 1\}$, the unit sphere in R^3 ; d(x, y) = the length of the smaller arc along the great circle joining the two points x and y.

5) M = R^n ; d(x, y) = $|x_1 - y_1| + ... + |x_n - y_n|$.
6) M = R^n; d(x, y) = max $\{|x_1 - y_1|, ..., |x_n - y_n|\}$.

3.10 Point Set Topology in Metric Spaces

The basic notions of point set topology can be extended to an arbitrary metric space (M, d). If $a \in M$, the ball $B_M(a;$ r) with center a and radius $r > 0$ is defined to be the set of all x in M such that d (x, a)$< r$.

If S is a metric subspace of M, the ball $B_S(a;$ r) is the intersection of S with the ball $B_M(a;$ r).

Example: In Euclidean space R^1 the ball B(0; 1) is the open interval (-1, 1). In the metric subspace S = [0, 1] the ball B_S(0; 1) is the half-open interval [0, 1).

If $S \subseteq M$, a point a in S is called an interior point of S if some ball $B_M(a;$ r) lies entirely in S. The interior, int S, is the set of interior points of S. A set S is called open in M if all its points are interior points; it is called closed in M if M - S is open in M.

Important points to note:
1. Every ball $B_M(a;$ r) in a metric space M is open in M.
2. In a discrete metric space M *every* subset S is open. In fact, if $x \in S$, the ball B(x; $\frac{1}{2}$) consists entirely of points of S (since it contains only x), so S is open. Therefore every subset of M is also closed!
3. In the metric subspace S = [0, 1] of Euclidean space R^1, every interval of the form [0, x) or (x, 1], where $0 < x < 1$, is an open set in S. These sets are not open in R^1.
Example 3 shows that if S is a metric subspace of M the open sets in S need not be open in M. The next theorem describes the relation between open sets in M and those in S.

Theorem 3.14. *Let (S, d) be a metric subspace of (M, d), and let X be a subset of S. Then X is open in S if, and only if, $X = A \cap S$ for some set A which is open in M.*

Proof. Assume A is open in M and let X = $A \cap S$. If $x \in X$, then $x \in A$ so $B_M(x;$ r) $\subseteq$ A for some $r > 0$. Hence B_S(x; r) = (B_M(x; r) $\cap$ S) $\subseteq$ (A $\cap$ S) = X so X is open in S.

Conversely, assume X is open in S. We will show that X = A $\cap$ S for some open set A in M. For every x in X there is a ball B_S(x; r_x) contained in X.

Now B_S(x; r_x) = B_M(x; r_x) $\cap$ S, so if we let A = $\bigcup\limits_{x \in X} B_M(x; r_x)$, then A is open in M and it is easy to verify that A $\cap$ S = X. $\Box$

Theorem 3.15. *Let (S, d) be a metric subspace of (M, d) and let Y be a subset of S. Then Y is closed in S if, and only if, $Y = B \cap S$ for some set B which is closed in M.*

This can be proved using the fact that complement of closed set is open and using theorem 3.14.

If S $\subseteq$ M, a point x in M is called an adherent point of S if every ball $B_M(x$; r) contains at least one point of S. If x adheres to S - {x} then x is called an accumulation point of S. The closure $\bar{S}$ of S is the set of all adherent points of S, and the derived set S' is the set of all accumulation points of S. Thus, $\bar{S}$ = S $\cup$ S'.

The following theorems are valid in every metric space (M, d) and are proved exactly as they were for Euclidean space R^n. In the proofs, the Euclidean distance $||x - y||$ needs to be replaced by the metric d(x, y).

1. The union of any collection of open sets is open, and the intersection of a finite collection of open sets is open.
2. The union of a finite collection of closed sets is closed, and the intersection of any collection of closed sets is closed.
3. If A is open and B is closed, then A - B is open and B - A is closed.
4. For any subset S of M the following statements are equivalent:
   (a) S is closed in M.
   (b) S contains all its adherent points.
   (c) S contains all its accumulation points.
   (d) S = $\bar{S}$.

Example. Let M = Q, the set of rational numbers, with the Euclidean metric of R^1. Let S consist of all rational numbers in the open interval (a, b), where both a and b are irrational. Then S is a closed subset of Q.

The proofs of the Bolzano-Weierstrass theorem, the Cantor intersection theorem, and the covering theorems of Lindelof and Heine-Borel use not only the metric properties of Euclidean space R^n but also special properties of R^n not generally valid in an arbitrary metric space (M, d). Further restrictions on M are required to extend these theorems to metric spaces.

For example, consider the Bolzano-Weierstrass theorem, let M=(0,2), with the same definition of d as in R^n, and consider a sequence of points in M which converges to 2, let's say A is the set of all points of the form (2-(1/n)), where n is a natural number. Then, A is an infinite bounded subset of M, but the only accumulation point of A, that is, $2 \notin$ M. Hence, Bolzano-Weierstrass theorem isn't true for this metric space.

3.11 Compact Subsets of a Metric Space

Definition 3.16 (Open Covering in Metric Spaces)**.** *Let (M, d) be a metric space and let S be a subset of M. A collection F of open subsets of M is said to be an open covering of S if* $S \subseteq \bigcup_{A \in F} A$.

Definition 3.17 (Compact Sets in Metric Spaces)**.** *A subset S of M is called compact if every open covering of S contains a finite subcover.*

Definition 3.18 (Bounded sets in Metric Spaces)**.** *S is called bounded if S* $\subseteq$ *B(a; r) for some r* > 0 *and some a in M.*

Theorem 3.16. *Let S be a compact subset of a metric space M. Then:*
1) S is closed and bounded.
2) There exists an accumulation point of every infinite subset of S, and it belongs to S (The point may not belong to the infinite subset).

Note that in Euclidean space R^n, each of properties (1) and (2) is equivalent to compactness (Theorem 3.13). In a general metric space, property (2) is equivalent to compactness, but property (1) is not.

For example, consider M=(0,2), now the subset S=(0,1] is closed and bounded in M, but, consider the sets A_n=($\frac{1}{n}$, 2), where n is a natural number. The collection of all such A_n forms an open covering of S, but we cannot reduce it to finite covering, which means S isn't compact even if it is closed and bounded.

Another example is, consider the metric space Q of rational numbers with definition of d being the same as in R^n. Let S consist of all rational numbers in the open interval (a, b), where a and b are irrational. Now, if we take any rational number greater than a near a, there are infinitely many rational nos, between that rational number and a. Now, consider the subsets in the collection to be (x, b), where x is a rational number greater than a. This is an open covering which cannot be reduced to a finite covering. Hence, S is closed and bounded in Q, but not compact.

Theorem 3.17. *Let S be a closed subset of a compact metric space M. Then S is compact.*

Proof. Consider an open covering F of S. Now, F $\cup$ {(M-S)} is an open covering of M, because as S is closed, M-S is open. Now, as M is compact, every open covering of M can be be reduced to a finite covering. Which means F can also be reduced to a finite covering. This is true for every open covering F of S, which means S is compact.

□

3.12 Boundary of a set in Metric Space

Definition 3.19. *Let S be a subset of a metric space M. A point x in M is called a boundary point of S if every ball $B_M(x; r)$ contains at least one point of S and at least one point of M - S. The set of all boundary points of S is called the boundary of S and is denoted by ∂S.*

It's easy to verify that $\partial S = \overline{S} \cap \overline{M - S}$.
This shows that ∂S is closed in M.

Example. In R^1, the boundary of the set of rational numbers is all of R^1.

4 Some Basics of Group Theory

Before diving into topology, let's study some group theory which we will need later to study topology.

4.1 Groups and Subgroups

Definition 4.1 (Group)**.** *A group is a set and an operation that combines any two elements of the set to produce a third element of the set, in such* *a way that the operation is associative, an identity element exists and every element has an inverse.*

These three axioms hold for number systems and many other mathematical structures. For example, the integers together with the addition operation form a group. The elements of groups may not be numbers, they can be polynomials, matrices, geometrical objects, etc. Note that groups may not be commutative, if a group is commutative, we call it an abelian group.

Definition 4.2 (Subgroup)**.** *A subgroup of a group G is just a subset of G which is also a group with the same group operation as in G.*

Example, the set of even integers is a subgroup of the set of integers.

Every group G has atleast 2 subgroups:

1.) The group G itself.

2.) The trivial group = {e}, where e is the identity element.

4.2 Lagrange's Theorem and Cosets

Theorem 4.1 (Langrange's Theorem)**.** *In a finite group G, the number of elements in any subgroup of G divides the total number of elements in G.*

Proof. Let the number of elements in G be n and that in a subgroup H be m. Now, as H is a group in itself, it has the identity element. Pick an element $g_1 \in G$ not in H. Look at the set g_1H = $\{g_1 * h$ for all $h \in H\}$. This set is called a left coset. Note that there is no element common to H and g_1H, because lets say $g_1 * h_i$=h_j, then multiplying by the inverse of h_i on both sides, we get g_1 belongs to H, which is not true.

Then take another element $g_2 \in G$ which doesn't belong to both g_1H and H, and consider set g_2H. Note that g_2H, g_1H and H, all are disjoint sets. I leave this as an exercise for you. Then choose another element outside H, g_1H and g_2H and continue the process. As all these sets are disjoint and every element in H gives a unique element in each of these sets, each set should have m elements and as G is a finite set, this process has to stop.

After we have all the left cosets, these sets and H are disjoint, have m elements each, and their union gives G, if the number of cosets is x, then $m \times x = n$, hence m divides n. □

Note that G may not be abelian, so the left cosets and right cosets may be different. Also note that any coset is not a group as it doesn't contain the identity element. Also all elements of the same coset form the same coset when mutiplied with H.

4.3 Normal Subgroup and Quotient Group

Also if we consider each coset as individual elements, sometimes, they, along with the subgroup H form a group, with H as the identity element. When cosets along with the subgroup form a group, we call the subgroup H as the normal subgroup and the group formed by H and the cosets as the quotient group and denote it as G/H. A subgroup H forms a normal subgroup when $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$.

For example, the set of even integers is a normal subgroup of the set of integers. it forms only one coset, that is, the set of odd numbers. If we take the set of even numbers and odd numbers as individual elements denoted by $\bar{0}$ and $\bar{1}$, $\{\bar{0}, \bar{1}\}$ is the quotient group with the group operation '+' such that
$\bar{0} + \bar{0} = \bar{1} + \bar{1} = \bar{0}$,
$\bar{0} + \bar{1} = \bar{1} + \bar{0} = \bar{1}$.
This group is called the integers mod 2.

Similarly, the group of multiples of any number n forms a normal subgroup of set of integers, and the quotient group formed by them is called the integers mod n. The group of integers mod n is denoted by $\mathbb{Z}_n$

4.4 Homomorphism and Isomorphism

Definition 4.3 (Homomorphism and Isomorphism)**.** *Let G_1 and G_2 be 2 groups. A map $f : G_1 \longrightarrow G_2$ is said to be a homomorphism if*

$$f(x + y) = f(x) + f(y)$$

for any $x, y \in G_1$. 1f f is also a bijection, f is called an isomorphism.

Note that in the equation above, the + on LHS is the group operation of G_1 whereas the + on the RHS is the group operation of G_2. If G_1 and G_2 are isomorphic, we denote it by $G_1 \cong G_2$.

When 2 groups are homomorphic, identity element is mapped to identity element, inverses are mapped to inverses.

4.4.1 Kernel and Image of a homomorphism

Note that homomorphism may not be one-one or onto. Elements other than identity element may be mapped to identity element. The elements of G_1 getting mapped to identity element of G_2 form a group and the group is called kernel of f and is denoted by ker f. The range of f is also called the image of f and is denoted by im f. It is a subgroup of G_2.

Theorem 4.2 (Fundamental theorem of homomorphism)**.** *Let $f : G_1 \longrightarrow G_2$ be a homomorphism. Then*

$$G_1/ker\ f \cong im\ f.$$

To prove it we take a function g : G_1/ker f $\longrightarrow$ im f by g([x]) = f(x) where [x] is the coset containing x. And then prove that g is a homomorphism, onto and one-one.

4.5 Finitely generated Abelian groups and free Abelian groups

Let x be an element of a group G. For n $\in \mathbb{Z}$, nx denotes

$$\underbrace{x + ... + x}_{\text{n}}, \text{ (if } n > 0)$$

and

$$\underbrace{(-x) + ... + (-x)}_{|\,n\,|}, \text{ (if } n < 0)$$

If n = 0, we put 0x = 0.

Take r elements $x_1, ..., x_r$ of G. The elements of G of the form

$$n_1x_1 + ... + n_rx_r \ (n_i \in \mathbb{Z},\ 1 \leq i \leq r)$$

a subgroup of G, which we denote H. H is called a subgroup of G generated by the generators $x_1, ..., x_r$. If G itself is generated by finite elements $x_1, ..., x_r$, G is said to be finitely generated. If $n_1x_1 + ... + n_rx_r = 0$ is satisfied only when $n_1 = ... = n_r = 0$, $x_1, ..., x_r$ are said to be linearly independent.

Definition 4.4 (Free Abelian Group)**.** *If G is finitely generated by r linearly independent elements, G is called a free Abelian group of rank r.*

For example, $\mathbb{Z}$ is a free Abelian group of rank 1 finitely generated by 1 (or -1).

Let $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}^2$ be the set of pairs {(i,j) such that $i, j \in \mathbb{Z}$}. It is a free Abelian group of rank 2 finitely generated by generators (1,0) and (0,1). More generally, $\mathbb{Z}^r$ is a free Abelian group of rank r.

The group $\mathbb{Z}_2$ (integers mod 2) = {0,1} is finitely generated by 1 but is *not* free since 1 is not linearly independent (note 1 + 1 = 0).

4.6 Cyclic groups

If G is generated by one element x, G $=\{0, \pm x, \pm 2x, ...\}$, G is called a cyclic group. If nx = 0 for any n $\in \mathbb{Z}$ - {0}, it is an infinite cyclic group while if nx = 0 for some n $\in \mathbb{Z}$ - {0}, a finite cyclic group. Let G be a cyclic group generated by x and let f : $\mathbb{Z} \longrightarrow$ G be a homomorphism defined by f(n) = nx. f maps onto G but not necessarily one to one. From Theorem 4.2, we have G = im f $\cong \mathbb{Z}$/ker f . Let N be the smallest positive integer such that Nx = 0. Clearly

$$\text{ker f} = \{0, \pm N, \pm 2N, ...\} = \text{N}\mathbb{Z}$$

and we have

$$\text{G} \cong \mathbb{Z}/\text{N}\mathbb{Z} \cong \mathbb{Z}_N.$$

If G is an infinite cyclic group, then ker f = {0} and G $\cong \mathbb{Z}$. Any infinite cyclic group is isomorphic to $\mathbb{Z}$ while a finite cyclic group is isomorphic to some $\mathbb{Z}_N$.

Lemma 1. *Let G be a free Abelian group of rank r and let H ($\neq \phi$) be a subgroup of G. We may always choose p generators $x_1, ..., x_p$, out of r generators of G so that $k_1 x_1, ..., k_p x_p$ generate H. Thus, $H \cong k_1\mathbb{Z} \times ... \times k_p\mathbb{Z}$ and H is of rank p.*

Theorem 4.3 (Fundamental theorem of finitely generated Abelian groups)**.** *Let G be a finitely generated Abelian group (not necessarily free) with m generators. Then G is isomorphic to the direct sum of cyclic groups, G $\cong \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{r} \times \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_p}$ where m = r + p. The number r is called the rank of G.*

Proof. Let G be generated by m elements $x_1, ..., x_m$ and let

$$\text{f} : \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m}} \longrightarrow G$$

be a surjective homomorphism,

$$f(n_1, ..., n_m) = n_1 x_1 + ... + n_m x_m.$$

Theorem 4.2 states that

$$\underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m}}/\text{ker f} \cong \text{G}.$$

Since ker f is a subgroup of

$$\underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m}}$$

lemma 1 claims that if we choose the generators properly, we have

$$\text{ker f} \cong k_1\mathbb{Z} \times ... \times k_p\mathbb{Z}.$$

We finally obtain

$$\text{G} \cong \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m}}/\text{ker f}$$

$$\cong \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m}}/k_1\mathbb{Z} \times ... \times k_p\mathbb{Z} \cong \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{m-p}} \times \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_p}$$

□

4.7 Vector spaces

4.7.1 Vectors and Vector spaces

Definition 4.5 (Fields)**.** *A field F is a set with 2 operations: addition and multiplication. F is an abelian group under addition and F - {0} is an abelian group under multiplication, where 0 is the additive inverse. For a, b, c* $\in F$*,* $a \times (b + c) = a \times b + a \times c$*.*

Definition 4.6 (Vector Space)**.** *A vector space V over a field K is a set in which two operations, addition and multiplication by an element of K (called a scalar), are defined. For* $\boldsymbol{u}$*,* $\boldsymbol{v}$*,* $\boldsymbol{w} \in V$*, and c, d* $\in K$*, the following axioms are true:*

1. $\boldsymbol{u}+\boldsymbol{v} \in V$*.*

2. *$\boldsymbol{u}$ + $\boldsymbol{v}$=$\boldsymbol{v}$ + $\boldsymbol{u}$.*

3. *There exists a zero vector $\boldsymbol{0}$ such that $\boldsymbol{v}$ + $\boldsymbol{0}$ = $\boldsymbol{v}$.*

4. *For any $\boldsymbol{u}$, there exists -$\boldsymbol{u}$ $\in V$, such that $\boldsymbol{u}$ + -$\boldsymbol{u}$ = $\boldsymbol{0}$.*

5. *($\boldsymbol{u}$ + $\boldsymbol{v}$) + $\boldsymbol{w}$ = $\boldsymbol{u}$ + ($\boldsymbol{v}$ + $\boldsymbol{w}$).*

6. *c($\boldsymbol{u}$ + $\boldsymbol{v}$) = $c\boldsymbol{u}$ + $c\boldsymbol{v}$.*

7. *(c + d)$\boldsymbol{u}$ = $c\boldsymbol{u}$ + $d\boldsymbol{u}$.*

8. *(cd)$\boldsymbol{u}$ = c($d\boldsymbol{u}$).*

9. *1$\boldsymbol{u}$=$\boldsymbol{u}$.*

From the first 5 properties, we see that a vector space is an abelian group under addition. But it's more than just an abelian group because of the field K.

Definition 4.7 (Linearly dependent and independent vectors)**.** *Let $\{\boldsymbol{v}_i\}$ be a set of k (> 0) vectors. If the equation*

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + ... + x_k\boldsymbol{v}_k = \boldsymbol{0}$$

has a non-trivial solution, $x_i \neq 0$ for some i, the set of vectors $\{\boldsymbol{v}_j\}$ is called linearly dependent, while if the equation has only a trivial solution, x_i = 0 for any i, $\{\boldsymbol{v}_i\}$ is said to be linearly independent. If at least one of the vectors is a zero vector $\boldsymbol{0}$, the set is always linearly dependent.

A set of linearly independent vectors $\{\mathbf{e}_i\}$ is called a basis of V , if any element $\mathbf{v} \in V$ is written uniquely as a linear combination of $\{\mathbf{e}_i\}$:

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + \ + v^n\mathbf{e}_n.$$

The numbers $v^i \in K$ are called the components of $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_j\}$. If there are n elements in the basis, the dimension of V is n, denoted by dim V = n. We usually write the n-dimensional vector space over K as V (n, K). All the n-dimensional vector spaces are isomorphic to K^n, and they are regarded as identical vector spaces.

4.7.2 Linear maps, images and kernels

Definition 4.8 (Linear map)**.** *Given two vector spaces V and W, a map* $f : V \longrightarrow W$ *is called a linear map if it satisfies* $f\,(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2) = a_1 f(\boldsymbol{v}_1) + a_2 f(\boldsymbol{v}_2)$ *for any* $a_1, a_2 \in K$ *and* $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$*.*

A linear map is an example of a homomorphism that preserves the vector addition and the scalar multiplication. The image of f is f(V) $\subseteq$ W and the kernel of f is $\{\mathbf{v} \in$ V such that f($\mathbf{v}$) = $\mathbf{0}\}$ and denoted by im f and ker f respectively. ker f cannot be empty since f($\mathbf{0}$) is always $\mathbf{0}$. ker f and im f are also vector spaces.

If W is the field K itself, f is called a linear function.

If f is an isomorphism, V is said to be isomorphic to W and vice versa, denoted by V $\cong$ W. It then follows that dim V = dim W. The isomorphism between the vector spaces is an element of the group of n $\times$ n matrices with non zero determinant, denoted by GL(n, K).

Theorem 4.4. *If* $f : V \longrightarrow W$ *is a linear map, then*

$$dim\ V = dim(ker\ f) + dim(im\ f).$$

I am stating the theorem without proof, for a detailed proof, one can refer chapter 2 from the book Geometry, Topology and Physics - Mikio Nakahara.

4.7.3 Dual Vector Space

Let f : V $\longrightarrow$ K be a linear function on a vector space V(n, K) over a field K. Let $\{\mathbf{e}_i\}$ be a basis and take an arbitrary vector $\mathbf{v} = v^1\mathbf{e}_1 + ... + v^n\mathbf{e}_n$. From the linearity of f , we have f($\mathbf{v}$) = $v^1 f(\mathbf{e}_1) + ... + v^n f(\mathbf{e}_n)$. Thus, if we know f($\mathbf{e}_i$) for all i, we know the result of the operation of f on any vector. It is remarkable that the set of linear functions is made into a vector space, namely a linear combination of two linear functions is also a linear function.

$$(a_1 f_1 + a_2 f_2)(\mathbf{v}) = a_1 f_1(\mathbf{v}) + a_2 f_2(\mathbf{v}).$$

This linear space is called the dual vector space to V(n, K) and is denoted by V^*(n, K) or simply by V^*. If dim V is finite, dim V^* is equal to dim V . Let us introduce a basis $\{e^{*i}\}$ of V^*. Since e^*_i is a linear function it is completely specified by giving $e^{*i}(\mathbf{e}_j)$ for all j. Let us choose the dual basis,

$$e^{*i}(\mathbf{e}_j) = \delta_i^j.$$

Any linear function f, called a dual vector in this context, is expanded in terms of $\{e^{*i}\}$,

$$\mathrm{f} = f_i e^{*i}.$$

The action of f on $\mathbf{v}$ is interpreted as an inner product between a column vector and a row vector,

$$\mathrm{f}(\mathbf{v}) = f_i e^{*i}(v^j \mathbf{e}_j) = f_i v^j e^{*i}(\mathbf{e}_j) = f_i v^j.$$

We sometimes use the notation $<,> : V^* \times V \longrightarrow K$ to denote the inner product.

Let V and W be vector spaces with a linear map f : V $\longrightarrow$ W and let g : W $\longrightarrow$ K be a linear function on W (g $\in W^*$). It is easy to see that the composite map g $\circ$ f is a linear function on V. Thus, f and g give rise to an element h $\in V^*$ defined by h(v) $\equiv$ g(f($\mathbf{v}$)), $\mathbf{v} \in$ V.

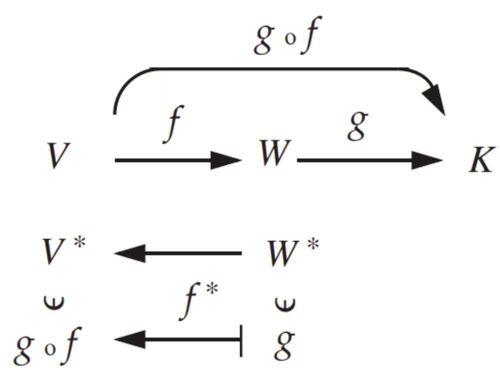

Figure 3: The pullback of a function g is a function $f^*(g) = g \circ f$.

Accordingly, we have an induced map $f^* : W^* \longrightarrow V^*$ defined by f^* : g $\longrightarrow$ h = f^*(g), see figure 3. The map h is called the pullback of g by f^*.

5 Topological Spaces

Now that we are familiar with set theory, point set topology and basics of group theory, we can finally start topology!

5.1 Definition and examples

Definition 5.1 (Topological Space)**.** *Let X be any set and $T = \{U_i | i \in I\}$ denote a certain collection of subsets of X. The pair (X, T) is a topological space if T satisfies the following requirements.*

1. *$\phi, X \in T$.*
2. *If J is any (maybe infinite) subcollection of I, the family $\{U_j | j \in J\}$ satisfies $\bigcup_{j \in J} U_j \in T$.*
3. *If K is any finite subcollection of I, the family $\{U_k | k \in K\}$ satisfies $\bigcap_{k \in K} U_k \in T$.*

X alone is sometimes called a topological space. The U_i are called the open sets and T is said to give a topology to X.

Some examples are:

1. If X is a set and T is the collection of all the subsets of X, then the above conditions are automatically satisfied. This topology is called the discrete topology.
2. Let X be a set and T = $\{\phi$, X$\}$. Clearly T satisfies all the conditions. This topology is called the trivial topology. In general the discrete topology is too stringent while the trivial topology is too trivial to give any interesting structures on X.
3. Let X be the real line $\mathbb{R}$. All open intervals (a, b) and their unions define a topology called the usual topology; a and b may be $-\infty$ and ∞ respectively. Similarly, the usual topology in $\mathbb{R}^n$ can be defined. (Take a product $(a_1, b_1) \times ... \times (a_n, b_n)$ and their unions.)

Also, if X is endowed with a metric space d, X is made into a topological space whose open sets are given by 'open discs'

$U_\epsilon(X) = \{y \in X | d(x, y) < \epsilon\}$ and all their possible unions. The topology T thus defined is called the metric topology determined by d. The topological space (X, T) is called a metric space.

Let (X, T) be a topological space and A be any subset of X. Then T = $\{U_i\}$ induces the relative topology in A by T' = $\{U_i \cap A | U_i \in T\}$.

5.2 Continuous maps

Definition 5.2. *Let X and Y be topological spaces. A map $f : X \longrightarrow Y$ is continuous if the inverse image of an open set in Y is an open set in X.*

Note that this definition is in agreement with our intuitive notion of continuity. Also, the converse may not be true, example $y = x^2$

5.3 Neighbourhoods and Hausdorff spaces

Definition 5.3 (Neighbourhood)**.** *Suppose T gives a topology to X. N is a neighbourhood of a point $x \in X$ if N is a subset of X and N contains some (at least one) open set U_i to which x belongs. (The subset N need not be an open set. If N happens to be an open set in T, it is called an open neighbourhood.)*

Definition 5.4 (Hausdorff space)**.** *A topological space (X, T) is a Hausdorff space if, for an arbitrary pair of distinct points x, x' $\in$ X, there always exist neighbourhoods U_x of x and $U_{x'}$ of x' such that $U_x \cap U_{x'} = \phi$.*

For example, Let X = {John, Paul, Ringo, George} and $U_0 = \phi$, $U_1 =$ {John}, $U_2 =$ {John, Paul}, $U_3 =$ {John, Paul, Ringo, George}. Show that T = $\{U_0, U_1, U_2, U_3\}$ gives a topology to X. Show also that (X, T) is not a Hausdorff space.

5.4 Connectedness

Definition 5.5.

1. *A topological space X is connected if it cannot be written as $X = X_1 \cup X_2$, where X_1 and X_2 are both open and $X_1 \cap X_2 = \phi$. Otherwise X is called disconnected.*

2. *A topological space X is called arcwise connected if, for any points x, y $\in$ X, there exists a continuous map f : [0, 1] $\longrightarrow$ X such that f (0) = x and f (1) = y. With a few pathological exceptions, arcwise connectedness is practically equivalent to connectedness.*

3. *A loop in a topological space X is a continuous map f : [0, 1] $\longrightarrow$ X such that f (0) = f (1). If any loop in X can be continuously shrunk to a point, X is called simply connected.*

Examples:

1. The real line $\mathbb{R}$ is arcwise connected while $\mathbb{R} - \{0\}$ is not. $\mathbb{R}^n$ $(n \geq 2)$ is arcwise connected and so is $\mathbb{R}^n - \{0\}$.

2. $\mathbb{R}^2 - \mathbb{R}$ is not arcwise connected. $\mathbb{R}^2 - \{0\}$ is arcwise connected but not simply connected. $\mathbb{R}^3 - \{0\}$ is arcwise connected and simply connected.

5.5 Homeomorphisms and Topological invariants

In topology, we define two figures to be equivalent if it is possible to deform one figure into the other by continuous deformation. Namely we introduce the equivalence relation under which geometrical objects are classified according to whether it is possible to deform one object into the other by continuous deformation.

5.5.1 Homeomorphism definition and examples

Definition 5.6 (Homeomorphism)**.** *Let X_1 and X_2 be topological spaces. A map $f : X_1 \longrightarrow X_2$ is a homeomorphism if it is continuous and has an inverse $f^{-1} : X_2 \longrightarrow X_1$ which is also continuous. If there exists a homeomorphism between X_1 and X_2, X_1 is said to be homeomorphic to X_2 and vice versa.*

Intuitively speaking, we suppose the topological spaces are made out of ideal rubber which we can deform at our will. Two topological spaces are homeomorphic to each other if we can deform one into the other continuously, that is, without tearing them apart or pasting.

Figure 1 on the front cover is an example of homoemorphism.

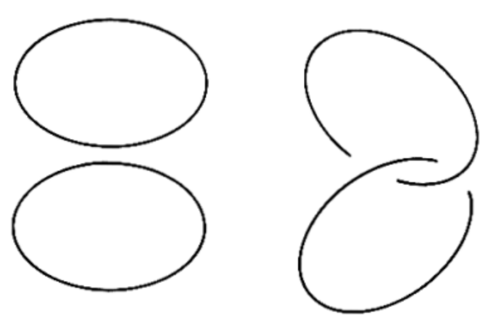

Figure 4: The linked rings are homeomorphic to the separated rings.

Surprisingly, the 2 images in figure 4 are homeomorphic to each other. It seems impossible to deform the left figure into the right one by continuous deformation. However, this is an artefact of the embedding of these objects in $\mathbb{R}^3$. In fact, they are continuously deformable in $\mathbb{R}^4$.

5.5.2 Topological invariants

Now our main question is: 'How can we characterize the equivalence classes of homeomorphism?' In fact, we do not know the complete answer to this question yet. Instead, we have a rather modest statement, that is, if two spaces have different 'topological invariants', they are not homeomorphic to each other. Here topological invariants are those quantities which are conserved under homeomorphisms. A topological invariant may be a number such as the number of connected components of the space, an algebraic structure such as a group or a ring which is constructed out of the space, or something like connectedness, compactness or the Hausdorff property. If we knew the complete set of topological invariants we could specify the equivalence class by giving these invariants. However, so far we know a partial set of topological invariants, which means that even if all the known topological invariants of two topological spaces coincide, they may not be homeomorphic to each other. Instead, what we can say at most is: *if two topological spaces have different topological invariants they cannot be homeomorphic to each other.*

Examples:

1. A closed line [0, 1] is not homeomorphic to an open line (0, 1), since [0, 1] is compact while (0, 1) is not.

2. A parabola y = x^2 is not homeomorphic to a hyperbola $x^2 - y^2 = 1$ although they are both non compact. A parabola is (arcwise) connected while a hyperbola is not.

3. Surprisingly, an interval without the endpoints is homeomorphic to a line $\mathbb{R}$. To see this, let us take X = $(-\ \pi/2, \pi/2)$ and Y = $\mathbb{R}$ and let f : $X \longrightarrow Y$ be f(x) = tan x. Since tan x is one to one on X and has an inverse, $tan^{-1}x$, which is one to one on $\mathbb{R}$, this is indeed a homeomorphism. Thus, boundedness is not a topological invariant.

5.6 Euler characteristic

The Euler characteristic is one of the most useful topological invariants. Moreover, we find the prototype of the algebraic approach to topology in it. To avoid unnecessary complication, we restrict ourselves to points, lines and surfaces in $\mathbb{R}^3$. A polyhedron is a geometrical object surrounded by faces. The boundary of two faces is an edge and two edges meet at a vertex. We extend the definition of a polyhedron a bit to include polygons and the boundaries of polygons, lines or points. We call the faces, edges and vertices of a polyhedron simplexes. Note that the boundary of two simplexes is either empty or another simplex.

Definition 5.7 (Euler characteristic)**.** *Let X be a subset of* $\mathbb{R}^3$ *, which is homeomorphic to a polyhedron K. Then the Euler characteristic* $\chi(X)$ *of X is defined by*

$\chi(X)$ = (number of verticies in K) - (number of edges in K) + (number of faces in K).

Theorem 5.1 (Poincare–Alexander)**.** *The Euler characteristic* $\chi(X)$ *is independent of the polyhedron K as long as K is homeomorphic to X.*

Examples are in order. The Euler characteristic of a point is $\chi(point) = 1$ by definition. The Euler characteristic of a line is $\chi(\text{line})$ = 2 - 1 = 1, since a line has two vertices and an edge. For a triangular disc, we find $\chi(triangle) = 3 - 3 + 1 = 1$. An example which is a bit non-trivial is the Euler characteristic of S^1. The simplest polyhedron which is homeomorphic to S_1 is made of three edges of a triangle. Then $\chi(S^1)$ = 3-3 = 0. Similarly, the sphere S^2 is homeomorphic to the surface of a tetrahedron, hence $\chi(S^2)$ = 4 - 6 + 4 = 2. It is easily seen that S^2 is also homeomorphic to the surface of a cube. Using a cube to calculate the Euler characteristic of S^2, we have $\chi(S^2)$ = 8 - 12 + 6 = 2, in accord with theorem 5.1. Historically this is the conclusion of Euler's theorem: if K is any polyhedron homeomorphic to S^2, with v vertices, e edges and f two-dimensional faces, then v - e + f = 2.

Let us calculate the Euler characteristic of the torus T^2. Figure 5 is an example of a polyhedron which is homeomorphic to T^2. From this polyhedron, we find $\chi(T^2)$ = 16 - 32 + 16 = 0.

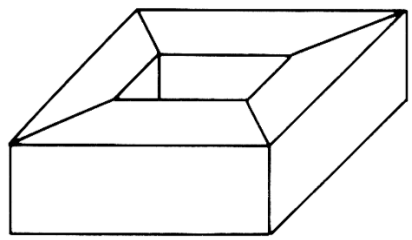

Figure 5: Example of a polyhedron which is homeomorphic to a torus.

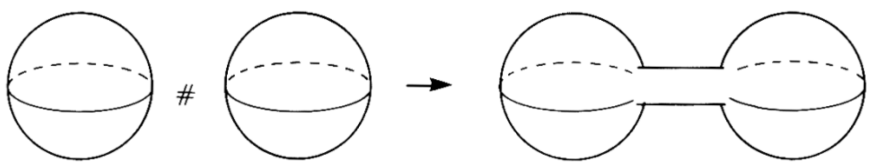

Figure 6: $S^2 \# S^2$

5.7 Connected sum

The connected sum X # Y of two surfaces X and Y is a surface obtained by removing a small disc from each of X and Y and connecting the resulting holes with a cylinder. Let X be an arbitrary surface. Then it is easy to see that

$$S^2 \# X = \text{X}.$$

since S^2 and the cylinder may be deformed so that they fill in the hole on X; see figure 6. If we take a connected sum of two tori we get (figure 7)

$$T^2 \phi T^2 = \Sigma_2$$

Similarly, Σ_g may be given by the connected sum of g tori,

$$\underbrace{T^2 \# ... \# T^2}_{\text{g factors}} = \Sigma_g$$

The connected sum may be used as a trick to calculate an Euler characteristic of a complicated surface from those of known surfaces. Let us prove the following theorem

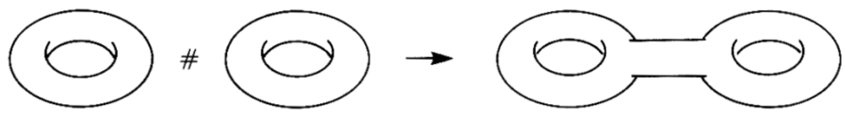

Figure 7: $T^2 \# T^2 = \Sigma_2$

Theorem 5.2. *Let X and Y be two surfaces. Then the Euler characteristic of the connected sum $X \# Y$ is given by*

$$\chi(X\#Y) = \chi(X) + \chi(Y) - 2.$$

Proof. Take polyhedra K_X and K_Y homeomorphic to X and Y , respectively. We assume, without loss of generality, that each of K_X andK_Y has a triangle in it. Remove the triangles from them and connect the resulting holes with a trigonal cylinder. Then the number of vertices does not change while the number of edges increases by three. Since we have removed two faces and added three faces, the number of faces increases by -2 + 3 = 1. Thus, the change of the Euler characteristic is 0 - 3 + 1 = -2. □

The significance of the Euler characteristic is that it is a topological invariant, which is calculated relatively easily. Even if homeomorphism implies equal Euler characteristic, the converse isn't true. For example, a point and a line both have Euler characteristic equal to 1, but they aren't homeomorphic.

The reader might have noticed that the Euler characteristic is different from other topological invariants such as compactness or connectedness in character. Compactness and connectedness are geometrical properties of a figure or a space while the Euler characteristic is an integer. Note that $\mathbb{Z}$ is an algebraic object rather than a geometrical one. Since the work of Euler, many mathematicians have worked out the relation between geometry and algebra and elaborated this idea, in the last century, to establish combinatorial topology and algebraic topology.

6 Homology Groups

Among the topological invariants the Euler characteristic is a quantity readily computable by the 'polyhedronization' of space. The homology groups are refinements, so to speak, of the Euler characteristic. Moreover, we can easily read off the Euler characteristic from the homology groups.

6.1 Simplexes

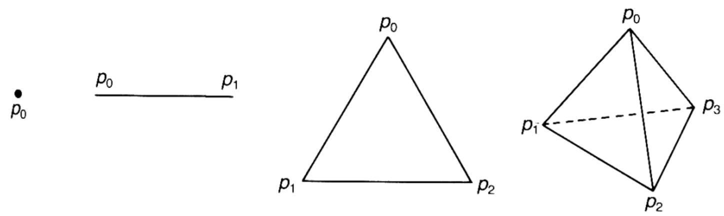

Figure 8: 0-, 1-, 2- and 3-simplexes.

Simplexes are building blocks of a polyhedron. A 0-simplex $\langle p_0 \rangle$ is a point, or a vertex, and a 1-simplex $\langle p_0 p_1 \rangle$ is a line, or an edge. A 2-simplex $\langle p_0 p_1 p_2 \rangle$ is defined to be a triangle with its interior included and a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$ is a solid tetrahedron. It is common to denote a 0-simplex without the bracket. It is easy to continue this construction to any r-simplex $\langle p_0 p_1 ... p_r \rangle$. Note that for an r-simplex to represent an r-dimensional object, the vertices p_i must be geometrically independent, that is, no (r - 1)-dimensional hyperplane contains all the r + 1 points. Let $p_0, ..., p_r$ be points geometrically independent in $\mathbb{R}^m$ where $m \geq r$. The r-simplex $\sigma_r = \langle p_0, ..., p_r \rangle$ is expressed as

$$\sigma_r = \{x \in \mathbb{R}^m | x = \sum_{i=0}^{r} c_i p_i, c_i > 0, \sum_{i=0}^{r} c_i = 1\}.$$

$(c_0, ..., c_r)$ is called the barycentric coordinate of x. Since σ_r is a bounded and closed subset of $\mathbb{R}^m$, it is compact.

Let q be an integer such that $0 \leq q \leq r$. If we choose q + 1 points $p_{i_0}, ..., p_{i_q}$ out of $p_0, ..., p_r$, these q + 1 points define a q-simplex $\sigma_q = \langle p_{i_0}, ..., p_{i_q} \rangle$, which is called a q-face of σ_r. We write $\sigma_q \leq \sigma_r$ if σ_q is a face of σ_r. If $\sigma_q \neq \sigma_r$, we say σ_q is a proper face of σ_r, denoted as $\sigma_q < \sigma_r$. The number of q-faces in an r-simplex is $\binom{r+1}{q+1}$. A 0-simplex is defined to have no proper faces.

6.2 Simplicial Complexes

Let K be a set of finite number of simplexes in $\mathbb{R}^m$. If these simplexes are nicely fitted together, K is called a simplicial complex. By ‘nicely’ we mean:

1. An arbitrary face of a simplex of K belongs to K, that is, if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$; and

2. If σ and σ' are two simplexes of K, the intersection $\sigma \cap \sigma'$ is either empty or a common face of σ and σ', that is, if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \phi$ or $\sigma \cap \sigma' \leq \sigma$ and $\sigma \cap \sigma' \leq \sigma'$.

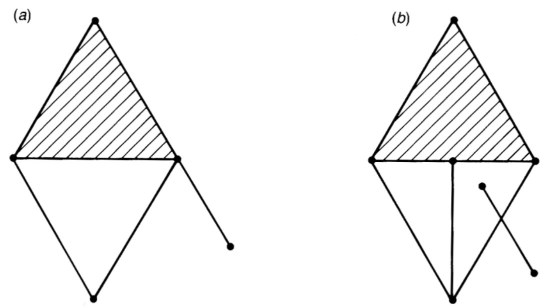

Figure 9: (a) is a simplicial complex but (b) is not.

6.3 Oriented Simplexes

We may assign orientations to an r-simplex for r ≥ 1. Instead of $\langle ... \rangle$ for an unoriented simplex, we will use $(. \; . \; .)$ to denote an oriented simplex. The symbol σ_r is used to denote both types of simplex. An oriented 1-simplex $\sigma_1 = (p_0 p_1)$ is a directed line segment traversed in the direction $p_0 \to p_1$. Also, $(p_0 p_1) = -(p_1 p_0)$.

Similarly, an oriented 2-simplex $\sigma_2 = (p_0 p_1 p_2)$ is a triangular region $p_0 p_1 p_2$ with a prescribed orientation along the edges. Observe that the orientation given by $p_0 p_1 p_2$ is the same as that given by $p_2 p_0 p_1$ or $p_1 p_2 p_0$ but opposite to $p_0 p_2 p_1$, $p_2 p_1 p_0$ or $p_1 p_0 p_2$. Let P be a permutation of 0, 1, 2

$$P = \begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix}$$

These relations are summarized as $(p_i p_j p_k) = sgn(P)(p_0 p_1 p_2)$ where sgn(P) = +1 (-1) if P is an even (odd) permutation.

It is now easy to construct an oriented r-simplex for any r ≥ 1.

$$(p_{i_0} p_{i_1} ... p_{i_r}) = sgn(P)(p_0 p_1 ... p_r)$$

where P is

$$P = \begin{pmatrix} 0 & 1 & ... & r \\ i_0 & i_1 & ... & i_r \end{pmatrix}$$

For r = 0, we formally define an oriented 0-simplex to be just a point $\sigma_0 = p_0$.

6.4 Chain group, cycle group and boundary group

Definition 6.1 (Chain group)**.** *The r-chain group $C_r(K)$ of a simplicial complex K is a free Abelian group generated by the oriented r-simplexes of K. If r > dim K, $C_r(K)$ is defined to be 0. An element of $C_r(K)$ is called an r-chain.*

Let there be I_r r-simplexes in K. We denote each of them by $\sigma_{r,i}$ ($1 \leq$ i $\leq I_r$). Then c $\in C_r(K)$ is expressed as

$$c = \textstyle\sum_{i=1}^{I_r} c_i \sigma_{r,i}, c_i \in \mathbb{Z}.$$

The integers c_i are called the coefficients of c. The group structure is given as follows. The addition of two r-chains, c = $\sum_i c_i\sigma_{r,i}$ and c' = $\sum_i c_i'\sigma_{r,i}$, is

$$c + c' = \textstyle\sum_i (c_i + c_i')\sigma_{r,i}.$$

The unit element is 0 in which each coefficient is 0. Thus, $C_r(K)$ is a free Abelian group of rank I_r.

Before we define the cycle group and the boundary group, we need to introduce the boundary operator. Let us denote the boundary of an r-simplex σ_r by $\partial_r\sigma_r$. ∂_r should be understood as an operator acting on σ_r to produce its boundary. Since a 0-simplex has no boundary, we define $\partial_0 p_0 = 0$.

For a 1-simplex (p_0p_1), we define $\partial_1(p_0p_1) = p_1 - p_0$.

Let $\sigma_r(p_0...p_r)$ $(r > 0)$ be an oriented r-simplex. Its boundary $\partial_r\sigma_r$ is an (r - 1)-chain defined by

$$\partial_r\sigma_r = \textstyle\sum_{i=0}^{r}(-1)^i(p_0p_1...p_{i-1}p_{i+1}...p_r$$

Note that ∂_r defines a map from $C_r(K)$ to $C_{r-1}(K)$. It is easy to see that this map is a homomorphism. It is interesting to study the image and kernels of the homomorphisms ∂_r.

Definition 6.2 (Cycle groups)**.** *If $c \in C_r(K)$ satisfies $\partial_r c = 0$ c is called an r-cycle. The set of r-cycles $Z_r(K)$ is a subgroup of $C_r(K)$ and is called the r-cycle group. Note that $Z_r(K) = ker\partial_r$. [Remark: If r = 0, $\partial_0 c$ vanishes identically and $Z_0(K) = C_0(K)$.]*

Definition 6.3 (Boundary groups)**.** *Let K be an n-dimensional simplicial complex and let $c \in C_r(K)$. If there exists an element $d \in C_{r+1}(K)$ such that $c = \partial_{r+1}d$ then c is called an r-boundary. The set of r-boundaries $B_r(K)$ is a subgroup of $C_r(K)$ and is called the r-boundary group. Note that $B_r(K) =$ im ∂_{r+1}. [Remark: $B_n(K)$ is defined to be 0.]*

Lemma 2. *The composite map $\partial_r \circ \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map; that is, $\partial_r(\partial_{r+1}c) = 0$ for any $c \in C_{r+1}(K)$.*

Proof is very simple. Since ∂_r is a linear operator on $C_r(K)$, it is sufficient to prove the identity $\partial_r \circ \partial_{r+1} = 0$ for the generators of $C_{r+1}(K)$. Take any generator of $C_{r+1}(K)$ and find the terms in which 2 points p_i and p_j are dropped. There will be exactly 2 terms, one in which p_i was dropped first and other in which p_j was dropped first. Check that these 2 terms have opposite signs. That's it!

Note that this lemma is consistent with our intuition that a boundary has no boundary.

From this lemma, it is obvious that $B_r(K) \subseteq Z_r(K)$, so $B_r(K)$ is a subgroup of $Z_r(K)$, and as they are abelian, $B_r(K)$ is a normal subgroup of $Z_r(K)$.

6.5 Homology Groups

Definition 6.4 (Homology groups)**.** *Let K be an n-dimensional simplicial complex. The rth homology group $H_r(K)$, $0 \leq r \leq n$, associated with K is defined by*

$$H_r(K) \equiv Z_r(K)/B_r(K).$$

We accept the following theorem without proof.

Theorem 6.1. *Homology groups are topological invariants. Let X be homeomorphic to Y and let (K, f) and (L, g) be triangulations of X and Y respectively. Then we have*

$$H_r(K) \cong H_r(L),\ r = 0,\ 1,\ 2, \ldots.$$

Examples:

1. Let K = $\{p_0\}$. The 0-chain $C_0(K) = \{ip_0 | i \in \mathbb{Z}\} \cong \mathbb{Z}$. Clearly $Z_0(K) = C_0(K)$ and $B_0(K) = \{0\}$ ($\partial_0 p_0 = 0$ and p_0 cannot be a boundary of anything). Thus $H_0(K) \equiv Z_0(K)/B_0(K) = C_0(K) \cong \mathbb{Z}$.

2. Let K = $\{p_0, p_1\}$ be a simplicial complex consisting of two 0-simplexes. $C_0(K) = \{ip_0 + jp_1 | i, j \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}$. Again, $Z_0(K) = C_0(K)$ as $\partial_0 p_0 = \partial_0 p_1 = 0$ and $B_0(K) = \{0\}$ as there is no 1-simplex in K. Hence, $H_0(K) \cong \mathbb{Z} \times \mathbb{Z}$.

3. Let K = $\{p_0, p_1, (p_0p_1)\}$. We have $C_0(K) = \{ip_0 + jp_1 | i, j \in \mathbb{Z}\}$ and $C_1(K) = \{k(p_0p_1) | k \in \mathbb{Z}\}$. As there is no 2-simplex in K, $B_1(K) = \{0\}$ and if $z = m(p_0p_1) \in Z_1(K)$, it satisfies $mp_1 - mp_0 = 0$, thus m has to vanish and $Z_1(K) = 0$ and $H_1(K) = Z_1(K)/B_1(K) = 0$.
   For $H_0(K)$, we have $Z_0(K) = C_0(K) = \{ip_0 + jp_1\} \cong \mathbb{Z} \times \mathbb{Z}$ and $B_0(K) = im\partial_1 = \{k(p_0 - p_1)\} \cong \mathbb{Z}$, hence $H_0(K) = Z_0(K)/B_0(K) \cong \mathbb{Z}$.

For more examples, refer chapter 3 from the book Geometry, Topology and Physics - Mikio Nakahara.

We accept the next 3 theorems without proof, one can read the proof from the same book.

6.6 Computation of $H_0(K)$

Theorem 6.2. *Let K be a connected simplicial complex. Then* $H_0(K) \cong \mathbb{Z}$.

6.7 Connectedness and Homology groups

Let K = $\{p_0\}$ and L = $\{p_0, p_1\}$. We have $H_0(K) = \mathbb{Z}$ and $H_0(L) = \mathbb{Z} \times \mathbb{Z}$. More generally, we have the following theorem.

Theorem 6.3. *Let K be a disjoint union of N connected components,* $K = K_1 \cup K_2 \cup \; \cup K_N$ *where* $K_i \cap K_j = \phi$. *Then* $H_r(K) = H_r(K_1) \times H_r(K_2) \times ... \times H_r(K_N)$.

6.8 Betti numbers and the Euler–Poincare theorem

Definition 6.5. *Let K be a simplicial complex. The rth Betti number* $b_r(K)$ *is the rank of the free Abelian part of* $H_r(K)$.

Hence, if $H_r(K) \cong \underbrace{\mathbb{Z} \times ... \times \mathbb{Z}}_{\text{f}} \times \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_p}$, then $b_r(K) = f$.

Theorem 6.4 (The Euler–Poincare theorem)**.** *Let K be an n-dimensional simplicial complex and let* I_r *be the number of r-simplexes in K. Then*

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r(K).$$

The first equality defines the Euler characteristic of a general polyhedron in n-dimensions. Note that this is the generalization of the Euler characteristic defined for surfaces in section 5.6.

7 Bibliography/References

Bibliography

1. Mathematical Analysis - Apostol
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References

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