Differential Equations

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Contents

Chapter 1

Introduction to Mathematical Modeling

Mathematical modeling is the art of using mathematical objects such as equations (differential, difference, algebraic), matrices as well as computer programs to explain the dynamical or static behavior of systems/problems we encounter in our everyday life.
Things to keep in mind when learning about mathematical modeling.

1. Learning to apply mathematics is very different from learning mathematics. Applied mathematics is used for solving a wide range of problems, many of which do not seem mathematical in nature.

2. There are no precise set of rules on how to create a model.

3. Modeling can be learnt only by solving problems.

This section is not intended to teach you how to go about modeling physical situations. A whole course could be devoted to the subject of modeling and still not cover everything! The process of building an effective mathematical model takes skill, imagination and objective evaluation. The basic steps involved in this process are

1. Formulate the problem: Here the problem is posed in such a way that it can be answered mathematically.

2. Develop the model. Here, must decide which variable are important and classify them as independent or dependent variables. The independent variables serve as input for the model. The dependent variables are those that are affected by the independent variables. After this you must determine or specify the relationship among the specified variables.

3. Test the model

Schematically

Physical Problem → Mathematical Model → Mathematical Solution → Solution/Analyzing

Mathematical models often yield equations that contain derivatives of unknown functions. Such an equation is called a differential equation. Differential equations arise in a variety of subject areas such as physical sciences, economics, biology etc. Our main objective of this course is to solve differential equations. But first we will look into some examples where the resulting mathematical model is a differential equation.

1.1 Modeling with Differential Equations

1. **Exponential Growth and Decay**

    Radioactive Decay/Growth
    The rate at which the nuclei of a substance decay/growth is proportional to the amount of the substance remaining at that time.
    The model for $A(t)$, the substance remaining at time t is

    $$\frac{dA}{dt} = kA, \qquad \text{where } k \text{ is the constant of proportinality.}$$

    Exponential growth is indicated by $k > 0$, and exponential decay by $k < 0$.

    Simple population model
    Let P be the size of the population at some time t. Let the number of individuals added by birth and lost by death be in proportion to P. The simple population model is

    $$\frac{dP}{dt} = kP, P(t_0) = P_0 \qquad \text{with } k > 0 \text{ a constant.}$$

    Logistic Equation Let P be the size of the population at some time t. Suppose the number of individuals added by birth is in proportion to P. Suppose the number of individuals lost due to competition is proportional to the square of P. The population model with birth and competition is

    $$\frac{dP}{dt} = kP(A - P), P(t_0) = P_0 \qquad \text{with } k > 0 \text{ a constant.}$$

    Here A is its supposed maximum allowable limit.

2. Number Theory For a positive real number x, the number of primes not exceeding x is given by

    $$\frac{dy}{dx} = \frac{1}{\ln(x)}, \;\; y(2) = 0.$$

    Remember that the smallest prime number is 2.

3. **Newton's Law of Cooling/Warming**
    The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium T_m. The DE for $T(t)$, the temperature of the body at time t is

    $$\frac{dT}{dt} = -k(T - T_m), \qquad \text{with } T_m \text{ and } k > 0 \text{ are constants.}$$

4. **Mixing problems:**
    In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. The main assumption that we will be using here is that the

concentration of the substance in the mixture is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations.

(i) **Model:** $\dfrac{dx}{dt}$ =input rate-output rate, where $x(t)$ is the amount of a substance in a tank.

(ii) **Torricelli's Law**
The rate of change of the volume $V(t)$ of water in a draining tank(water drains through a sharp-edged hole in the bottom of a tank) is proportional to the square root of the depth $y(t)$ of water.
The DE is

$$\frac{dV}{dt} = -k\sqrt{y}, \qquad \text{with } k > 0 \text{ a constant.}$$

If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$ and have the form

$$\frac{dy}{dt} = -h\sqrt{y}, \qquad \text{with } h = \tfrac{k}{A}.$$

5. **Newtonian Mechanics**

(i) **Falling Bodies and Air Resistance**
Assume that only forces acting on the particle are gravity of the earth G and the medium resistance F. The motion is governed by

(i) Newton's second law of motion and

(ii) Newton's law of universal gravitation (the magnitude of the force G acting between two bodies is proportional to their masses m and M and is inversely proportional to the square of the distance r between bodies), $G = \dfrac{\gamma Mm}{r^2}$ where γ is the universal gravitation constant.

With these the general model is

$$m\frac{d^2x}{dt^2} = F(v) + G(x)$$

where m is the mass of the particle, x is the position in space and v is the velocity.

Note:
The air resistance force $F = F(v)$ depends on v. $F(v)$ is directly proportional to v for low speeds, proportional to v^2 for high speeds and increases drastically near sound barrier.
Next we will consider a specific case of this, free fall with air resistance. Here the motion is vertical and assumming the changing gravitational attraction is nelegible the above model gives

$$m\frac{d^2x}{dt^2} = -F(v) + mg$$

where $g = \dfrac{\gamma M}{R^2}$ is the gravitation acceleration on the surface of the earth with R radius of the earth and M mass of the earth.
Depending on the magnitude of v, $F(v) = kv$ or $F(v) = kv^2$ where k is the constant of proportionality.

(ii) **Mechanical Vibrations**
Many mechanical systems, the motion is an oscillation with the position of static equilibrium as the center. Suspension of an automobile, flywheel in a watch, on a bridge are some common examples. Simple situation which illustrates the essential features of a more such complex oscillatory motion is the motion of a mass attached to a spring(spring mass system) as shown in figure 1.1.

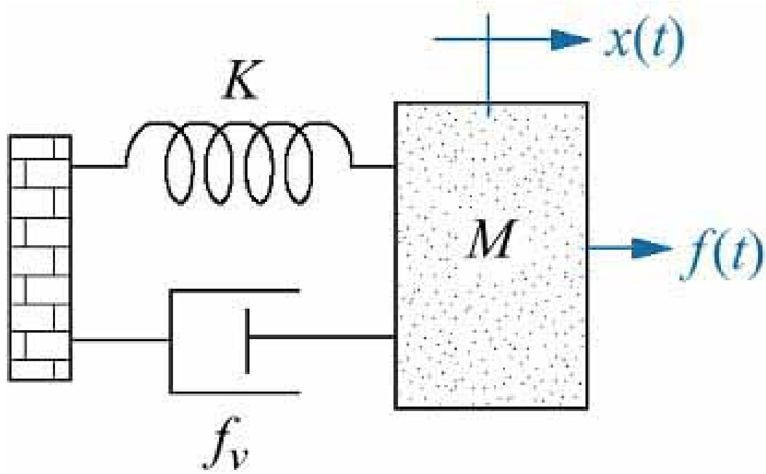

Figure 1.1: Spring Mass Damper System.

Suppose a body of mass M is attached to one end of an ordinary spring that resists compression as well as stretching. The other end of the spring is attached to a fixed wall. Assume that the body rests on a frictionless horizontal plane so that it can move only back and forth. Let x denotes the distance of the body from its **equilibrium position** (its position when the spring is unstretched), and $x > 0$ when the spring is stretched and thus $x < 0$ when compressed.
According to **Hooke's law** the restorative force F_s that spring exerts on the mass is proportional to the distance x. Therefore $F_s = -Kx$ where $K > 0$ is the **spring constant**.
The above figure shows that the mass is attached to a dash pot- a device like a shock absorber. This provides a force f_v directed opposite to the instantaneous direction of the mass and usually it is designed so that f_v is proportional to the velocity. Therefore

$$F_v = -c\frac{dx}{dt}$$

where $c > 0$ is the **damping constant** of the dash pot. More generally this may be regarded as frictional forces(including air resistance to the motion).
If the mass is subject to another external force $f(t)$, by Newton's law the governing equation of motion is

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = f(t).$$

(iii) **Simple Pendulum**
A mass m is suspended from an arm of length L, whose mass we will disregard as shown below.1.2. Assume the arm is attached to a pivot that is frictionless. The angle of the arm from the vertical is denoted θ , and it is measured counterclockwise in radians. If

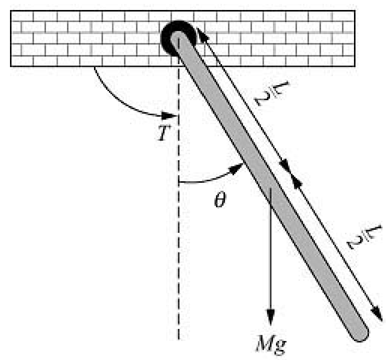

Figure 1.2: Simple Pendulum.

a distance Δs is travelled along the circle due to a change $\Delta\theta$, then $\Delta s = L\Delta\theta$ and linear velocity $v = L\dfrac{d\theta}{dt}$. Therefore by the Newton's second law,

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta \text{ which gives } \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

For small oscillations $\sin\theta \approx \theta$ then the above equation reduces to $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0$. If there is frictional resistance of the surrounding medium

$$mL\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + mg\theta = 0.$$

This has the exactly the same mathematical form as the spring-mass model.

(iv) **Vibration of Structures**
When dealing with building structures subjected to earthquake loads, single-story buildings (SSBs) can be represented by spring-mass models (SMMs). For example see the figure below. The length of the beam is represented by L and the height of the frame is represented by H.

If the beam flexural stiffness, represented by EI_b, is infinitely rigid, the distorted shape of the SSB when subjected to the ground acceleration $x_g''(t)$ and the corresponding SMM are sketched below. Here the horizontal coordinate x stands for the single degree of freedom of the frame and " denotes the time derivatives of x.

The quantities m, c and k represent, respectively, the mass of the beam, the viscous damping of the SSB and the overall stiffness of the SSB. In this case, this stiffness k is a function of the flexural stiffness, EI_c, of both columns, and the height H. The equation of motion of the model is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = -mx_g''(t).$$

6. **Analysis of Electrical Networks**
   Consider the single-loop series circuit(RLC circuit) shown in figure 3.3. The letters L, C

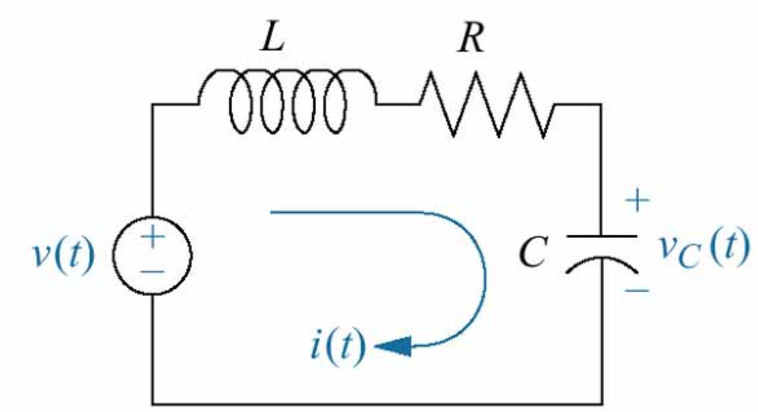

Figure 1.3: RLC Circuit.

and R are the inductance, capacitance and resistance respectively.
If q = electric charge on the capacitor, and i = current flowing through the capacitor then $i = \dfrac{dq}{di}$ and the voltage drops across a resistor, a capacitor and an inductor are iR, $\dfrac{q}{C}$ and

$L\dfrac{di}{dt}$ respectively.

Then by the Kirchhoff's Second Law (the impressed voltage $E(t)$ on a closed loop must equal to the sum of the voltage drops in the loop), the governing DE is

$$\begin{aligned} E(t) &= L\frac{di}{dt} + iR + \frac{1}{C}q \\ &= L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q. \end{aligned}$$

Thus we notice there is an analogy between mechanical oscillators and electrical oscillators. For **RL** and **RC** circuits the respective equations are

$$\begin{aligned} E(t) &= L\frac{di}{dt} + Ri \\ E(t) &= iR + \frac{1}{C}q = R\frac{dq}{dt} + \frac{1}{C}q. \end{aligned}$$

7. **Bending of an Elastic Beam**

Many structures are constructed using beams, and these beams deflect or distort under their own weight or under the influence of some external force. Consider a beam of length L and constant cross section and homogeneous elastic material(e.g.steel) shown in figure 1.4. Assume that the beam is straight and if a uniform load is applied to the rod in a vertical

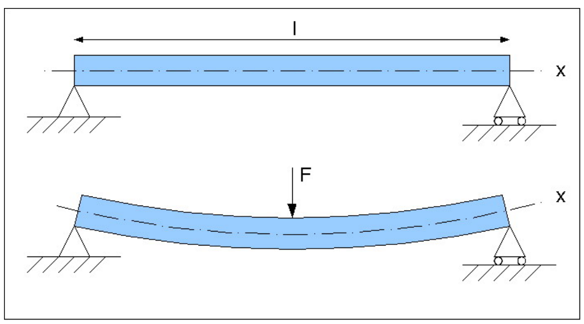

Figure 1.4: Bending beam.

plane through the axis of symmetry, it is bent and its axis is curved and is usually called as **elastic curve or deflection curve** C. From theory of elasticity, the bending moment $M(x)$ is proportional to the curvature $k(x)$ of C. Assuming that the bending is small so that the deflection $y(x)$ and its derivative $y'(x)$ are small we get

$$M(x) = EI\frac{d^2y}{dx^2} \text{ with } k = \frac{y^{('')}}{(1+y'^2)^{\frac{3}{2}}} \approx y^{('')}$$

where EI is the constant of proportionality with E the **Young's modulus of elasticity** and I the moment of inertia of a cross section about the z-axis (horizontal).

If $\omega(x)$ is the load per unit length, then $M^{(2)}(x) = \omega(x)$ and the DE is given by

$$EIy^{(4)} = \omega(x).$$

8. **Buckling of a Thin Vertical Column**
   Consider a long slender vertical column of uniform cross section and length L. Let $y(x)$ denote the deflection of the column when a constant vertical compressive force or load P is applied to its top. By considering the bending moment at any point along the column we get
$$EI\frac{d^2y}{dx^2} + Py = 0$$
   where E is the **Young's modulus of elasticity** and I the moment of inertia of a cross section about a vertical line through its centroid.

1.2 Definitions and Terminology

An equation containing derivatives of an unknown function is called a **Differential Equation(DE)**. If an equation involves the derivative of one variable with respect to another variable then the former is defined as the dependent variable and the latter is defined as independent variable.
In $\frac{d^2y}{dx^2} + y = 1$, y is the dependent variable and x is the independent variable.
In $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, x and t are independent variables and u is the dependent variable. **Notation**
If Y is a function of an independent variable x then the nth derivative of y with respect to x is denoted either by $\frac{d^ny}{dx^n}$ or $y^{(n)}$. If we use the second notation the first, second, third derivatives are usually denoted by y', y'', y''', instead of $y^{(1)}$,$y^{(2)}$ and $y^{(3)}$.
Symbolically an n^{th} order ODE in one dependent variable can be expressed in the form
$$\frac{d^ny}{dx^n} = f(x, y, y', \cdots, y^{(n-1)}).$$

Classification

In order to study differential equations, there are several ways of classifying them. Here we will consider three of such criteria.

By Type
If a DE contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it is called an **ordinary differential equation(ODE)**.
$\frac{d^2y}{dx^2} = 1 - y$, $\quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$, $\quad \frac{dy}{dt} + \frac{dx}{dt} = 2x + y$ are ODEs.
A DE involving the partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation(PDE)**.
$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $\quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ are PDEs.

By Order
The order of a differential equation(either ODE or PDE) is the order of the highest derivative that appears in the equation.

For example, the order of $\dfrac{dy}{dt} = y^2 + t^2$ is one.

By Linearity
If the dependent variables and its derivatives appear linearly in the DE, then it is called a linear DE. A nonlinear DE is one that is not linear.
The DEs $\dfrac{d^2y}{dx^2} = 1 - y$, $\quad \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 7y = 0$, $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$ are linear
The DEs $\dfrac{dy}{dt} = y^2 + t^2$, $\quad \dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^4 - 4y = e^x$, $\dfrac{\partial u}{\partial t} = 2u\dfrac{\partial u}{\partial x}$ are nonlinear.

- **Autonomous and Non autonomous**
  A DE in which the independent variables do not appear explicitly is called an autonomous DE and nonautonomous otherwise.
  For example $\dfrac{d^2y}{dx^2} = 1 - y$ is autonomous and $\dfrac{dy}{dt} = y^2 + t^2$ ia nonautonomous.

Example: Classify the following DEs.

1. $3\dfrac{d^2x}{dt^2} + 4t\dfrac{dx}{dt} + 9x = 2\cos t$:: Linear second order ODE

2. $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$ Nonlinear first order ODE

3. $r\dfrac{\partial N}{\partial t} = r\dfrac{\partial^2 N}{\partial r^2} + \dfrac{\partial N}{\partial r} + N$: Linear second order PDE

Definition: Degree
The degree of a DE is defined as the degree of the highest derivative of the equation. For example the degree of $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^4 - 4y = e^x$ is one.

Note:

1. The degree of any linear DE is one.

2. Not every differential equation has a degree. For example the DE $y'' + y' + \ln(y'') = 0$ has no degree.

Solution
A solution to a differential equation on an interval (a, b) is any function which satisfies the differential equation in question on the interval (a, b). It is important to note that solutions are often accompanied by intervals and these intervals can give us some important information about the solution. Graph of a solution of a differential is called an integral curve.

General Solution
The general solution to a differential equation is the most general form that the solution can take

and produces a family of integral curves.

Initial Condition(s) Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated ICs) are of the form,

$$y(x_0) = y_0 \text{ and/or } y^{(k)}(x_0) = x_k.$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at a specific point. The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see later.

Initial Value Problem

An Initial Value Problem (or IVP) is a differential equation along with a specified number of initial conditions. Geometrically the IC $y(x_0) = y_0$ has the effect of isolating the integral curve that passes through the point (x_0, y_0) from the family of integral curves.

Actual Solution The actual solution or particular solution to a differential equation is the specific solution to the differential equation, satisfying the given initial condition(s). **Example** $y = x^2$ is a particular solution and $y = cx^2$, where c is any constant is the general solution to $x\dfrac{dy}{dx} = 2y$ defined in $(-\infty, \infty)$

. **Example**

$y = x^{-3/2}$ is a solution to the IVP $4x^2y'' + 12xy' + 3y = 0; y(4) = 1/8, y'(4) = -3/64$ for $x > 0$.

Interval of Validity

The interval of validity for an IVP with initial condition(s) $y(x_0) = y_0$ and/or $y^{(k)}(x_0) = x_k$ is the largest possible interval on which the solution is valid and contains x_0.

Boundary Value Problems

A boundary value problem (BVP) is a differential equation with the function and/or derivatives specified at different points. These points are called **boundary values**.

Example

$EIy^{(4)} = \omega(x)$, $y = y'' = 0$, at $x = 0$ and at $x = L$. Simply supported beam (no displacement and no bending moment at end points).

Note

The study of differential equations is intimately connected with the study and analysis of physical systems. The three main ways in which differential equations help the study of physical systems is:

1. Modeling of physical systems.
2. To predict the behavior of a given physical system.
3. To understand how one can modify a physical system to behave in a certain desired way.

First step involves transforming a physical problem into a mathematical equation(DE) by using physical laws. To address the second and third steps we need to find the unknown function(solve the DE) or need some qualitative behaviour of the unknown function.

Note
Before moving on to learning how to solve differential equations we will be concerned in answering one or more of the following questions.

1. For a given DE, whether the equation has a solution. This is referred to as **existence of a solution**.

2. If a solution exists, whether it is unique. This is referred to as **uniqueness**

3. How to find a solution. This may seem like an odd question to ask and yet the answer is not always yes. Just because we know that a solution to a differential equation exists does not mean that we will be able to find it.

Note
In a first course in differential equations (such as this one) the third question is the question that we will concentrate on. We will answer the first two questions for special, and fairly simple, cases, but most of our efforts will be concentrated on answering the third question for as wide a variety of differential equations as possible. There are three main approaches or methods to find the solution.

(I) **Analytic methods**
In this approach we try to express solutions interms of known functions and mathematical objects such as integrals, algebraic equations etc. This is the best way to get a solution, but is not always possible.

(II) **Graphical or qualitative approach**
For many equations there is a geometrical connection between the solutions and the equation. With the help of graphs this relation can be drawn and the behaviour of the solutions are visualized. This type of discovering the information about the solution is called graphical or qualitative approach.

(III) **Numerical methods**
Here we compute(approximate) the value of a solution at any point in its domain to any desired degree of accuracy.

In this introductory course to differential equations we will study the simplest form, the linear ordinary differential equations. In finding the solutions we will apply anatytical methods.

1.3 Useful Formulas

Heaviside Expansion Formulas

Heaviside expansion formulas are very useful to find the partial fraction decomposition(PFD) of rational functions. Since these are common in solving DEs, they are reviewed very briefly in this section.

Rational Function

The general form of a rational function is $F(s) = \dfrac{P(s)}{Q(s)}$, where $P(s), Q(s)$ are polynomial functions. Based on the form of the factors of the denominator $Q(s)$,there are two different cases to be considered in this context.

1. **Denominator consisting of simple factors**

$$F(s) = \frac{P(s)}{Q(s)} = \sum_{k=1}^{n} \frac{A_k}{(s - s_k)} \quad \text{with} \quad A_k = \lim_{s \to s_k} [(s - s_k)F(s)]$$

2. **Denominator consisting of repeated factors**
   If the factor $(s - s_r)$ repeats m times

$$F(s) = \frac{P(s)}{Q(s)} = \sum_{k=1}^{m} \frac{A_k}{(s - s_r)^k} \quad \text{with} \quad A_k = \frac{1}{(m-k)!} \lim_{s \to s_k} \left[\frac{d^{m-k}}{ds^{m-k}} (s - s_r)^m F(s) \right]$$

Examples: Find the partial fraction decomposition.

(1) $$\frac{1}{s^2 + s - 12} = \frac{A_1}{s+4} + \frac{A_2}{s-3}$$

$$A_1 = \lim_{s \to -4} \left[(s+4) \frac{1}{(s+4)(s-3)} \right] = \frac{-1}{7}, \quad A_2 = \lim_{s \to 3} \left[(s-3) \frac{1}{(s+4)(s-3)} \right] = \frac{1}{7}$$

Therefore PFD is

$$\frac{1}{s^2 + s - 12} = -\frac{1}{7(s+4)} + \frac{1}{7(s-3)}.$$

(2) $$\frac{s+3}{s(s+1)^2} = \frac{A_1}{s} + \frac{B_1}{(s+1)} + \frac{B_2}{(s+1)^2}$$

$$A_1 = \lim_{s \to 0} \left[\frac{s+3}{(s-1)^2} \right] = 3, \quad B_1 = \lim_{s \to (-1)} \left[\frac{1}{(2-1)!} \frac{d}{ds} (\frac{s+3}{s}) \right] = -3$$

$$B_2 = \lim_{s \to (-1)} \left[\frac{1}{(2-2)!} \frac{(s+3)}{s} \right] = -2$$

Therefore PFD is

$$\frac{s+3}{s(s+1)^2} = \frac{3}{s} - \frac{3}{s+1} - \frac{2}{(s+1)^2}.$$

(3) $$\frac{s^2+2}{(s^5+8s^3+16s)} = \frac{s^2+2}{s(s-2i)^2(s+2i)^2} = \frac{A_1}{s} + \frac{B_1}{s-2i} + \frac{B_2}{(s-2i)^2} + \frac{C_1}{s+2i} + \frac{C_2}{(s+2i)^2}$$

$$\begin{aligned}
A_1 &= \frac{1}{8} \\
B_1 &= \frac{1}{(2-1)!}\lim_{s\to 2i}\frac{d}{ds}\left[\frac{s^2+2}{s(s+2i)^2}\right] = \frac{-1}{16} \\
B_2 &= \frac{1}{(2-2)!}\lim_{s\to 2i}\frac{d}{ds}\left[\frac{s^2+2}{s(s+2i)^2}\right] = \frac{-i}{16} \\
C_1 &= \frac{1}{(2-1)!}\lim_{s\to(-2i)}\frac{d}{ds}\left[\frac{s^2+2}{s(s-2i)^2}\right] = \frac{-1}{16} \\
C_2 &= \frac{1}{(2-2)!}\lim_{s\to(-2i)}\frac{d}{ds}\left[\frac{s^2+2}{s(s-2i)^2}\right] = \frac{i}{16}
\end{aligned}$$

Therefore PFD is

$$\frac{s^2+2}{(s^5+8s^3+16s)} = \frac{1}{8s} - \frac{1}{16(s-2i)} - \frac{i}{16(s-2i)^2} - \frac{1}{16(s+2i)} + \frac{i}{16(s+2i)^2}.$$

Shortcut Method to Integration by Parts

Here we will discuss a shortcut method known as tabular method to evaluate the integrals of the form

$$\int p(x)q(x)dx,$$

when $p(x)$ is a polynomial and the successive anti derivatives(integrals) of $q(x)$ can be found easily. For example the integral $\int x^3 \cos x dx$.

Tabular Method

Here the polynomial $p(x)$ is chosen to be differentiated successively while $q(x)$ is chosen to be integrated and are listed in a table. Then, we write the integral as the sum of products of the successive derivatives of $p(x)$ and the successive anti derivatives of $q(x)$ while having their signs alternate, beginning with the positive sign. It should be noted that in this process, we should keep the integration one step ahead.

Example 3: Evaluate $\int x^3 \cos x dx$.
$p(x) = x^3, \;\; q(x) = \cos x$

Derivatives of $p(x)$	Integrals of $q(x)$
x^3	$\sin x$
$3x^2$	$-\cos x$
$6x$	$-\sin x$
6	$\cos x$
0	$\sin x$

Therefore

$$\int x^3 \cos x dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C.$$

1.4 Exercises

(1) Classify each of the following DEs as to order, degree, and linearity:

(i) $y'' + 3y' + 6y = 0$

(ii) $(y')^2 + y = x^3$

(iii) $y''' + 2yy' + xy = 0$

(iv) $y' + P(x)y = Q(x)$

(v) $y^{(4)} - y'' + y^2 = e^x$

(vi) $y'' + e^y = 1$

(2) Show that the function $y = f(x)$ is a solution of the given DEs:

(i) $y = e^{5x}$ for $y'' - 25y = 0$

(ii) $y = xe^x - e^x$ for $y' = xe^x$

(iii) $y = \cos 2x$ for $y'' + 4y = 0$

(3) Consider the differential equation $y' = 2x$.

(i) Show that $y = x^2 + C$ is the general solution.

(ii) Find C so that the solution curve passes through $(1, 2)$.

(4) Consider the DE $y = xy' + (y')^2$.

(i) Show that $y = Cx + C^2$ is the general solution of the above DE.

(ii) Show that $y = -\dfrac{x^2}{4}$ is also a solution of the above DE? Is this a particular solution?

(5) Show that $\phi(x) = \sin x - \cos x$ is a solution to the IVP $y'' + y = 0$; $y(0) = -1$, $y'(0) = 1$.

(6) Show that $\phi_1(x) = 0$ and $\phi_2(x) = (x - 2)^{\frac{1}{3}}$ are solutions to the IVP $y' = 3y^{\frac{2}{3}}$; $y(2) = 0$.

Chapter 2

First Order Ordinary Differential Equations

In this chapter we will learn how to recognize and obtain solutions for some special types of first order differential equations. The most general first order differential equation can be writtenin the form $y' = f(x, y)$. As we will see later there is no general formula for the solution to this equation. What we will do instead is look at several special cases and see how to solve those. We will also look at some of the theory behind first order differential equations at the end.

2.1 Separable Equations

A first order ODE of the form $\frac{dy}{dx} = g(x)h(y)$ is called a **separable** DE. For example, $y' = -\frac{x}{y}$ is separable and $y' = x + y$ is not separable.

Method of Solution Write the DE as $\frac{dy}{h(y)} = g(x)dx$ and integrate,

$$\int \frac{dy)}{h(y)} = \int g(x)dx + C, \quad \text{where C is a constant.}$$

Example: Solve the DEs.

1. A copper ball initially at 50°F, is placed in a 375°F oven. If after 75 minutes it is found that the temperature $T(t)$ of the ball is 125°F, how long will it take the temperature of the copper ball be 150°F.(Assume that at any instant the temperature of the copper ball is uniform).
   Solution: By Newton's law of heating,

$$\begin{aligned} \frac{dT}{dt} &= k(T - 375) \quad \text{; separable equation} \\ \int \frac{1}{T - 375} dT &= kdt \\ \ln(T - 375) &= kt + C \\ T - 375 &= Be^{kt} \end{aligned}$$

With $T(0) = 50$, $B = -325$, so $T(t) = 375 - 325e^{kt}$. Also $T(75) = 125$, and thus $k = -0.0035$.Hence

$$T = 375 - 325e^{-0.0035t}.$$

Now when $T = 150$ we get $t = 105$ minutes.
In $T = 375 - 325e^{-0.0035t}$, the constant term 375 is called **steady state function** and $325e^{-0.0035t}$ is called **transient function**. A system is said to be in steady state when the variables describing it are periodic functions or constants.

Note: Loosing a Solution
When separating variables, some care should be taken. If r is a root of $h(y)$, then $y = r$ satisfies the DE and hence is a solution. But $y = r$ may not show up in the general solution. For an example consider $y' = y^2$.
By proceeding as above we get $y = \dfrac{-1}{x+c}$. Solutions of this form are never zero but $y = 0$ is obviously a solution.

Reducible to separable form:

Homogeneous Equations:
Homogeneous function: If a function $f(x, y)$ possesses the property $f(\lambda x, \lambda y) = \lambda^n f(x, y)$, for some real number λ, then f is said to be a homogeneous function of degree n. For example $f(x, y) = 2x^4 - x^2y^2 + 5xy^3$ is a homogeneous function of degree 4 since

$$f(\lambda x, \lambda y) = 2\lambda^4x^4 - \lambda^4x^2y^2 + 5\lambda^4xy^3 = \lambda^4 f(x, y).$$

A first order DE of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be homogeneous if both M, N are homogeneous functions of the same degree. This can be transformed into separable form by the substitution $y = vx$ where v is the new dependent variable.
Then the homogeneous equation has the separable form $v + x\dfrac{dv}{dx} = G(v)$where G is a function of v only.
Example: $(y^2 - xy)dx + x^2dy = 0$
$M(x, y) = y^2 - xy, N(x, y) = x^2$
Substitution: $y = xv, dy = vdx + xdv$
$M(x, xv) = x^2v^2 - x^2v, N(x, xv) = x^2$
$x^2(v^2 - v)dx + x^2dy = 0$
$(v^2 - v)dx + (vdx + xdv) = 0$ which gives $v^2dx + xdv = 0$
$\dfrac{1}{x}dx + \dfrac{1}{v^2}dv = 0$ separable form
Integrating $\ln|x| - \dfrac{1}{v} = \ln|c|$ and $\ln|\dfrac{x}{c}| = \dfrac{x}{y}$.

Equations of the form $\dfrac{dy}{dx} = G(ax + by)$
The substituition $z = ax + by$ transforms the equation into separable form. **Example:** Solve $\dfrac{dy}{dx} = (x + y + 3)^2$.

Substitution: $v = x + y + 3$. Thus $y = v - x - 3$ and $y' = v' - 1$.
Transformed equation: $\dfrac{dv}{dx} = 1 + v^2$. Separable form.
Integrating: $\tan^{-1} v = x + C$. So $v = \tan(x + c)$.
Solution: $y = \tan(x + c) - x - 3$

Equations with Linear Coefficients:
An equation of the form $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where a_i's, b_i's, c_i's are constants is called an equation with linear coefficients. Here we transform both x and y into new variables and two cases are considered.
Case 1: $a_1b_2 - b_1a_2 \neq 0$
Substitue $x = X + h$, $y = Y + k$ where h,k are constants that will change $a_1x + b_1y + c_1$ into $a_1X + b_1Y$ and change $a_2x + b_2y + c_2$ into $a_2X + b_2Y$.
Theory regarding systems of linear equations shows that such a transformation exists if the system of equations

$$a_1h + b_1k + c_1 = 0$$
$$a_2h + b_2k + c_1 = 0$$

has a solution. This is ensured by the assumption $a_1b_2 - b_1a_2 \neq 0$ or the determinant of the coefficient matrix is non zero.

Case 2: $a_1b_2 - b_1a_2 = 0$
In this case the equation can be put into the form $\dfrac{dy}{dx} = G(ax + by)$ which can be solved with the substitution $z = ax + by$
Examples:

(1) $\dfrac{dy}{dx} = \dfrac{x + y - 2}{-x + y - 4}$; Case 1
$x = X + h, y = Y + h$ and $\dfrac{dY}{dX} = \dfrac{X + Y + h + k - 2}{-X + Y - h + k - 4}$
Choose h, k so that $h + k - 2 = 0$ and $-h + k - 4 = 0$. Thus $h = -1, k = 3$.
The resulting equation is homogeneous and solve using $Y = vX$.

(2) $(2x + 3y + 4)dx - (4x + 6y + 5)dy = 0$, Case 2
Substitution: $t = 2x + 3y$ and $\dfrac{dt}{dx} = 2 + \dfrac{dy}{dt}$
$\dfrac{(2t + 5)}{(7t + 22)} \dfrac{dt}{dx} = 1$, separable form.

2.2 Linear Differential Equations

A first order DE of the form $a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x)$, is said to be linear DE. Dividing this by $a_1(x)$ gives the standard form $\dfrac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $Q(x) = \dfrac{g(x)}{a_1(x)}$.

For example $y' + y = x^2$ and $(x^2+1)y' + xy = \sin x$ are linear.

Method of Solution

In order to solve these type of equations we multiply the equation by a function called **integrating factor**(IF) so that each side of the DE is recognizable as a derivative. The IF for a linear first order differential equation in the standard form is $e^{\int P(x)dx}$.

Method:

1. Write the equation in the standard form
2. Calculate the integrating factor $e^{\int P(x)dx}$ and multiply the DE by it.
3. Identify the left hand side of the resulting equation as the derivative of a product.
4. Integrate the equation.

Examples: Solve the DEs.

1. $(x^2+1)y' + 3xy = 6x$
   $\dfrac{dy}{dx} + \dfrac{3x}{x^2+1}y = \dfrac{6x}{x^2+1}$
   $\text{IF} = \exp\left(\int \dfrac{3x}{x^2+1}dx\right) = \exp(\frac{3}{2}\ln(x^2+1)) = (x^2+1)^{\frac{3}{2}}$
   $(x^2+1)^{\frac{3}{2}}\dfrac{dy}{dx} + (x^2+1)^{\frac{3}{2}}\dfrac{3x}{x^2+1}y = (x^2+1)^{\frac{3}{2}}\dfrac{6x}{x^2+1}$
   $\dfrac{d[(x^2+1)^{\frac{3}{2}}y]}{dx} = 6x(x^2+1)^{\frac{1}{2}}$
   Integrating,
   $(x^2+1)^{\frac{3}{2}}y = \displaystyle\int 6x(x^2+1)^{\frac{1}{2}} = 2(x^2+1)^{\frac{3}{2}} + C$
   $y(x) = 2 + C(x^2+1)^{-\frac{3}{2}}$

2. **RL-circuit** $L\dfrac{di}{dt} + Ri = E(t)$. Solve this for $L = 0.1$ henry, R=5 ohms and E(t)=12 volts.
   $0.1\dfrac{di}{dt} + 5i = 12$
   $\dfrac{di}{dt} + 50i = 120$
   $\text{IF} = e^{50t}$
   $\dfrac{d(e^{50t}i)}{dt} = 120e^{50t}$
   $i(t) = 2.4 + ce^{-50t}$
   This has three kinds of solution, depending on the initial condition: constant solution if $i(0) = 2.4$, exponential approach to 2.4 from below (if $i(0) < 2.4$) or from above (if $i(0) > 2.4$)
   In the above example e^{-50t} is transient function and 2.4 is steady state function.

Note

Except when the case $a_1(x) = 1$, transforming a linear DE into the standard form requires division by $a_1(x)$. The values of x for which $a_1(x) = 0$ are called singular points of the equation. Singular points may be troublesome. If $P(x)$ is discontinuous the discontinuity may carry over to the solution.

Reducible to Linear ODE Bernoulli's Equation: A first order differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on an interval (a, b) and n is any real number is called Bernoulli's Equation. Note that if $n = 0$ or $n = 1$ this is a linear equation. For other values of n, the substitution $u = y^{1-n}$ transform it to a linear equation.

Example: $x\dfrac{dy}{dx} + y = x^2y^2$

$\dfrac{dy}{dx} + \dfrac{1}{x}y = xy^2$

$n = 2$, substitute $y = u^{-1}$ and $\dfrac{dy}{dx} = -u^{-2}\dfrac{du}{dx}$

$\dfrac{du}{dx} - \dfrac{1}{x}u = -x$ which is linear.

2.3 Exact Equations

A differential form $M(x, y)dx + N(x, y)dy$ is said to be **exact differential** if there is a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = M(x, y)$ and $\dfrac{\partial F}{\partial y} = N(x, y)$. That is the total differential of F satisfies

$$dF = M(x, y)dx + N(x, y)dy.$$

If $M(x, y)dx+N(x, y)dy$ is an exact differential form then the equation $M(x, y)dx+N(x, y)dy = 0$ is called an exact equation.

For example, $x^2y^3dx + x^3y^2dy$ is an exact differential since $x^2y^3dx + x^3y^2dy = d(\frac{1}{3}x^3y^3)$.

Test for Exactness:

Suppose that $M(x, y)$ and $N(x, y)$ are continuous and have continuous first partial derivatives in a region. Then the differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact in that region if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ at each point in the region.

In $x^2y^3dx + x^3y^2dy = 0$, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 3x^2y^2$. Therefore exact.

Method:

1.Check the criterion for exactness.

2.There exists a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = M$.

3.Find F by integrating this with respect to x, by holding y a constant; $F = \int M(x, y)dx + g(y)$. The constant of integration $g(y)$ can be a function of y.

4.To determine $g(y)$, take the partial derivative with respect to y and substitute $N(x, y)$ for $\partial F/\partial y$.

5.Thus $g'(y) = N(x, y) - \dfrac{\partial}{\partial y}\displaystyle\int M(x, y)dx$.

6.Integrate with respect to y and find $g(y)$ up to a constant.

7.Substitute this in $F = \int M(x, y)dx + g(y)$.

8.The solution of $Mdx + Ndy = 0$ is given by $F(x, y) = C$.
Alternatively starting with $\partial F/\partial y = N$ the solution can be found by first integrating with respect to y.

Examples:

(1) $(3x^2y - 2y^3 + 3)dx + (x^3 - 6xy^2 + 2y)dy = 0$
$\dfrac{\partial M}{\partial y} = 3x^2 - 6y^2, \quad \dfrac{\partial N}{\partial x} = 3x^2 - 6y^2$.Therefore the DE is exact.
Thus there exists a function of $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = 3x^2y - 2y^3 + 3$
Integrating, $F = x^3y - 2y^3x + 3x + g(y)$
$\dfrac{\partial F}{\partial y} = x^3 - 6y^2x + g'(y)$
But $\dfrac{\partial F}{\partial y} = x^3 - 6xy^2 + 2y$ and $g'(y) = 2y$ and $g(y) = y^2 + C_1$.
Therefore the solution, $F(x, y) = C$
$x^3y - 2xy^3 + 3x + y^2 = C$.

(2) Solve $(1 + e^xy + xe^xy)dx + (xe^x + 2)dy = 0$.
$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = e^x + xe^x$. Therefore exact.
Integrating N w.r.t x gives $F(x, y) = xe^xy + 2y + h(x)$
$\dfrac{\partial F}{\partial x} = M(x, y)$ gives $e^xy + xe^xy + h'(x) = 1 + e^xy + xe^xy$.
Thus $h'(x) = 1$ and $h(x) = x$.
Hence $F(x, y) = xe^xy + 2y + x = C$ or $y = \dfrac{(C - x)}{(2 + xe^x)}$.

(3) $(x - y)dx + xdy = 0$
$M = x - y, N = x$ and $\dfrac{\partial M}{\partial y} = -1$ but $\dfrac{\partial N}{\partial x} = 1$. Therefore it is not exact.

Clairaut's Equation

The Clairaut's equation has the form $y = px + F(p)$, where $p = \dfrac{dy}{dx}$.

Illustration of the Solution Method

1. Differentiate the equation, $\dfrac{dy}{dx} = p = p + x\dfrac{dp}{dx} + \dfrac{dF}{dp}\dfrac{dp}{dx}$, which gives $\dfrac{dp}{dx}\left(\dfrac{dF}{dp} + x\right) = 0$.
2. Thus $\dfrac{dp}{dx} = 0$ or $\left(\dfrac{dF}{dp} + x\right) = 0$.
3. $\dfrac{dp}{dx} = 0$ gives $\dfrac{d^y}{dx^2} = 0$ and $y = c_1x + c_2$ and $p = \dfrac{dy}{dx} = c_1$.
4. Substituting these in the DE gives the general solution $y = c_1x + F(c_1)$. That is the general solution to Clairaut's equation can be obtained by replacing p in the ODE by the arbitrary constant c_1.
5. Now considering the second factor we have $\left(\dfrac{dF}{dp} + x\right) = 0$. This relation may be used to eliminate p from the DE to give a singular solution.

Example
Solve $y = px + p^2$.
General solution $y = cx + c^2$ where c is an arbitrary constant.
$\dfrac{dF}{dp} + x = 0$ gives $2p + x = 0$. Thus $p = -x/2$ and substituting this in the DE gives the singular solution $x^2 + 4y = 0$ **Note:**
The singular solution contains no arbitrary constants and cannot be obtained from the general solution for any choice of the arbitrary constant.

2.4 Existence and Uniqueness of a solution

Here we will present a couple of theorems regarding the existence and uniqueness of a solution of a first order IVP. We will also see some of the differences between linear and nonlinear differential equations.

Existence and uniqueness for linear IVP
Theorem
Consider the IVP $y' + P(x)y = Q(x)$; $y(x_0) = y_0$. If $P(x)$ and $Q(x)$ are continuous on an open interval (a, b) that contains the point x_0, then for any choice of $y_0 = y(x_0)$, there exists a unique solution to the IVP on (a, b).

Note: This tells us the following important facts.

1. For linear first order differential equations solutions are guaranteed to exist under the continuity conditions. The solution will be unique although we may not be able to find the solution.
2. The interval in the theorem is the largest possible interval on which $p(x)$ and $Q(x)$ are continuous and is the interval of validity for the solution. That is for linear first order differential equations, we need not to actually solve the differential equation in order to find the interval of validity.
3. The interval must contain x_0, but the value of y_0, has no effect on the interval of validity.

There is a similar theorem for non-linear first order differential equations. This theorem is not as useful for finding intervals of validity as the first theorem was.

Theorem:
Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$. If f and $\dfrac{\partial f}{\partial y}$ are continuous functions in a small neighbourhood of (x_0, y_0) then the IVP has a unique solution in an interval $(x_0 - \delta, x_0 + \delta)$ for a positive number δ.

Note: This tells us the followings regarding the DE.

1. The solution exists in some neighbourhood of x_0. But it does not say how large is that interval.

2. Unlike the first theorem, this one cannot really be used to find an interval of validity. We will actually need the solution in order to determine its interval of validity.

Note:
In the above theorems the meaning of uniqueness is that there is exactly one solution curve that passes through the point (x_0, y_0). In other words, no two solutions can pass through the same point in the $t-y$ plane. **Examples:**

1. $y'+e^x y = 2; y(1) = 2$. The interval of validity is $(-\infty,\infty)$ since both e^x and 2 are continuous on $(-\infty,\infty)$ and contains $x_0 = 1$.

2. $y' = y^2; y(0) = 2$. $f = y^2$ and $\dfrac{\partial f}{\partial y} = 2y$ are continuous in a neighbourhood of $(0, 2)$. Therefore unique solution exists. The solution is $y = \dfrac{2}{1-2t}$ and the interval of validity is $[0, 2)$.

3. Does the IVP $y' = x^2 - xy^3,\ \ y(1) = 6$ have a unique solution?
$f(x, y) = x^2 - xy^3$ and $\partial f/\partial y = -3xy^2$ are continuous in any neighbourhood of $(1, 6)$. Therefore it has a unique solution in an interval $(1 - \delta, 1 + \delta)$, $\delta > 0$.

2.5 Exercises

1. Solve by separating variables.
(i)$\dfrac{dy}{dx} = e^{3x+2y}, y(0) = 1$ (ii)$(1 + x)dy - ydx = 0$ (iii)$\dfrac{dy}{dx} = y^2 - 4$

2. Solve the following linear DEs.
(i)$xy' + 4y = x^3 - x$ (ii)$y' + y = x, y(0) = 4$ (iii)$y' \cos x + y \sin x = 1$

3. Find k so that $(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$ is exact.

4. Show that $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}$ is exact and solve it.

5. Solve the following homogeneous DEs.
(i) $(x^2 + y^2)dx + (x^2 - xy)dy = 0$ (ii)$xy^2\dfrac{dy}{dx} = y^3 - x^3, y(1) = 2$ (iii)$\dfrac{y}{x} + (\cos(\dfrac{y}{x})\dfrac{dy}{dx} = 0$

6. Solve by using an appropriate substituition.
(i) $\dfrac{dy}{dx} = \dfrac{6x + 4y}{3x + 2y + 2}$, $y(-1) = -1$ (ii)$\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$ (iii)$\dfrac{dy}{dx} = 1 + e^{y-x+5}$

7. An equation of the form
$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$
is called a generalized Riccati equation.

   (a) If one solution of the above equation is $u(x)$ is known show that the substituition $y = u + \dfrac{1}{v}$ reduces the above equation to a linear equation in v.

(b) Given that $y = x$ is a solution to $\dfrac{dy}{dx} = x^3(y - x)^2 + \dfrac{y}{x}$, use the result in part (a) to find all other solutions of the equation.

Chapter 3

Higher Order Linear Differential Equations

3.1 Basic Concepts

The general form of an nth order linear ODE is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{3.1}$$

where $a_i(x)$ and $f(x)$ are functions of x only and $a_n(x) \neq 0$. When the coefficients $a_i(x)$ are constants then this a linear ODE with constant coefficients otherwise it has variable coefficients. If $f(x) = 0$ it is called **homogeneous** otherwise **nonhomogeneous**.
Note:
The word **homogeneous** in this context is different from the earlier notion of **homogeneous DE** under first order.

Example:
$3y'' - y' + 6y = \sin x$ is nonhomogeneous and $y'' - xy' + 6y = 0$ is homogeneous.
Standard Form:
If $a_n(x) \neq 1$ then dividing by $a_n(x)$ the standard form of the equation is

$$\frac{d^ny}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = g(x) \tag{3.2}$$

Existence and Uniqueness:
Suppose $p_1(x), \cdots, p_n(x)$ and $g(x)$ are each continuous on an interval (a, b) that contains x_0. Then the initial value problem

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x);\;\; y(x_0) = \gamma_0, y'(x_0) = \gamma_1, \cdots, y^{(n-1)}(x_0) = \gamma_{n-1}$$

has a unique solution defined on the interval (a, b).

Differential Operators: The differentiation $\dfrac{d}{dx}$ is denoted by **D** and $\dfrac{d^n}{dx^n} = D^n$. The nth

order differential operator is defined as

$$L[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = (D^n + p_1 D^{n-1} + \cdots + p_{n-1}D + p_n)[y].$$

Then the equation (3.2) can be expressed in the operator form $L[y](x) = g(x)$.

Example:

$y'' + 5y' + 6y = 5x$ can be written as $D^2y + 5Dy + 6y = 5x$ or $L[y] = (D^2 + 5D + 6)y = 5x$.

Note:

1. Since for any constants α, β, $L\{\alpha p(x) + \beta q(x)\} = \alpha L(p(x)) + \beta L(q(x))$, L is a linear operator.

2. As a consequence of this linearity, if $y_1, y_2, \cdots, y_m$ are solutions to the homogeneous equation $L[y](x) = 0$, then for any linear combination of the form $C_1y_1 + C_2y_2 + \cdots + C_my_m$ is also a solution.
   For example: $(D^2 + 5D + 6)y = 0$, $y_1 = e^{-3x}, y_2 = e^{-2x}$ are solutions and $y = c_1y_1 + c_2y_2$ is also a solution.

Imagine now that we have found n solutions $y_1, y_2, \cdots, y_n$ to the nth order homogeneous equation $L[y](x) = 0$. Is it true that every solution to that equation can be represented by $C_1y_1 + C_2y_2 + \cdots + C_ny_n$ for appropriate choices of the constants $C_1, C_2, \cdots, C_n$? The answer is yes provided the solutions $y_1, y_2, \cdots, y_n$ satisfy a certain property. Frst we need a definition.

Definition: Wronskian

Let $f_1, f_2, \cdots, f_n$ be any n functions that are $n-1$ times differentiable. Then the function

$$W[f_1, \cdots, f_n] = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix}$$

is called the Wronskian of $f_1, f_2, \cdots, f_n$.

Representation of Solutions(Homogeneous Equation)

Let $y_1, \cdots, y_n$ be n solutions on (a, b) of

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0 \tag{3.3}$$

where $p_1, \cdots, p_n$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy $W[y_1, \cdots, y_n] \neq 0$, then every solution of (3.3) can be expressed in the form

$$y = C_1y_1 + \cdots + C_ny_n$$

where $C_1, \cdots, C_n$ are constants. This linear combination of $y_1, \cdots, y_n$ is reffered to as a **general solution** of (3.3). Note that $y(x) = 0$ is always a solution and is called the trivial solution.

Example:
$y_1 = 1, y_2 = x, y_3 = \cos x, y_4 = \sin x$ are solutions to $y^{(4)} + y'' = 0$ and $[y_1, \cdots, y_4] = 1 \neq 0$. Therefore every solution of the DE is of the form $y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$.

Linear Independence and the Wronskian
If $y_1, \cdots, y_n$ are n solutions to $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$ on (a, b) with $p_1, \cdots, p_n$ continuous on (a, b), then the following are equivalent.

(i) $y_1, y_2, \cdots, y_n$ are linearly independent on (a, b).

(ii) The Wronskian $W[y_1, \cdots, y_n](x_0)$ is nonzero at some point x_0 in (a, b).

(iii) The Wronskian $W[y_1, \cdots, y_n](x)$ is never zero on (a, b).

Whenever $(i), (ii)$ or (iii) is met, $\{y_1, \cdots, y_n\}$ is called a **fundamental solution set** for (3.3) on (a, b).

Example:
The functions $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ satisfy the third order equation $y''' - 6y'' + 11y' - 6y = 0$. Find the general solution.
$W[e^x, e^{2x}, e^{3x}] = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0.$
Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a fundamental solution set and $y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$ is the general solution.

Note:
The solution space of the linear $n^t h$ order homogeneous equation $L[y] = 0$ is a vector space under the usual addition and scalar multiplication of real valued functions and the fundamental solution set $\{y_1, y_2, \cdots, y_n\}$ is a basis for the space.

Representation of Solutions (Nonhomogeneous Case)

Any function $y(x)$, free of arbitrary parameters satisfying a nonhomogeneous equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = g(x) \tag{3.4}$$

on (a, b) is called a **particular integral** or **particular solution** and is denoted by y_p.
If $y_p(x)$ is a particular solution to the nonhomogeneous equation and if $\{y_1, \cdots, y_n\}$ is a fundamental solution set for the corresponding homogeneous equation, then every solution of (3.4) on (a, b) can be expressed in the form

$$y(x) = C_1 y_1 + \cdots + C_n y_n + y_p.$$

This is called the general solution to (3.4).
Example:

$y_p = -2$ is a particular solution of $y'' - 2y = 4$.
$y_c = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$ is the solution to the corresponding homogeneous equation. Then $y = y_c + y_p$ satisfies the DE and is a solution.
Complementary Function:
General solution of a nonhomogeneous linear equation consists of the sum of two functions $y = C_1 y_1 + \cdots + C_n y_n + y_p = y_c + y_p$. The linear combination $y_c = C_1 y_1 + \cdots + C_n y_n$ which is the general solution to the associated homogeneous equation is called the **complementary function** for the nonhomogeneous equation.

Superposition Principle:
Suppose y_{p_i}, for $i = 1, 2, \cdots, k$ be k particular solutions of the nonhomogeneous linear ODEs

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y^{(1)} + p_n y = f_i(x), i = 1, 2, \cdots, k.$$

Then $y_p = c_1 y_{p_1} + c_2 y_{p_2} + \cdots + c_k y_{p_k}$ is a particular solution of $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y^{(1)} + p_n y = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$.

Example:
$y_{p_1} = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$.
$y_{p_2} = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$.
$y_{p_3} = xe^x$ is a particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$.
By superposition principle $y_p = -4x^2 + e^{2x} + xe^x$ is a particular solution to $y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$.

3.2 Higher Order Linear Equations with Constant Coefficients

The general form of an nth order linear ODE with constant coefficients is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x), \text{ where } a_i\text{'s are constants with } a_n \neq 0. \quad (3.5)$$

Since constant functions are everywhere continuous, equation (3.5) has solutions defined for all x in $(-\infty, \infty)$ for $f(x) = 0$.

3.2.1 Solution Methods

D Operator Method
With $L[y] = (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0) y$, equation (3.5) in the operator form is $L[y](x) = 0$. If we consider the operator $a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$ as a polynomial in D, and say we can factor it as $(D - m_1)(D - m_2) \cdots (D - m_n)$.
A solution of this is given by $(D - m_n) y = 0$ or equivalently $\frac{dy}{dx} - m_n y = 0$ which is separable.
Thus the solution is $y(x) = c_n e^{m_n x}$ with c_n a constant.
Similarly by considering the other factors, $(D - m_{n-1}) y = 0, \cdots, (D - m_1) y = 0$ we get $y(x) =$

$c_{n-1}e^{m_{n-1}x}, \cdots, y(x) = c_1 e^{m_1 x}$ as solutions. This suggests the following procedure to find $y_1, \cdots, y_n$.

Method:

1. Try a solution of the form $y = e^{\lambda x}$ where λ is a parameter.The resulting equation is a polynomial in λ,
$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$$
and is called the **auxiliary equation** or **characteristic equation**.

2. The characteristic equation has n roots (repeated or distinct) which may be either real or complex.

3. Find the solution according to the type of the roots of the characteristic equation.

Distinct Real Roots
If the roots $\lambda_1, \cdots, \lambda_n$ of the auxiliary equation are real and distinct then $\{e^{\lambda_1 x}, \cdots, e^{\lambda_n x}\}$ is a fundamental set and general solution is $y = C_1 e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$.
Example:
$y''(x) + y'(x) - 2y(x) = 0$
Try $y = e^{\lambda x}$, $\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = 0$
$e^{\lambda x} \neq 0$, $\lambda^2 + \lambda - 2 = 0$ and factoring $(\lambda - 1)(\lambda + 2) = 0$.
Roots $\lambda = 1, \lambda = -2$
Solution $y = c_1 e^x + c_2 e^{-2x}$.

Complex Roots
If $\alpha + i\beta$ (α, β real) is a complex root of the auxiliary equation then so is its complex conjugate $\alpha - i\beta$, since the coefficients of the auxiliary equation are real.
Then both $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are solutions to 3.5.
If there are no repeated roots the form of the general solution is $y = C_1 e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$.
Note:
To find the real valued solutions corresponding to $\alpha \pm i\beta$, take the real and imaginary parts of $e^{(\alpha+i\beta)x}$.
Thus $e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x$ gives the two independent solutions $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$.
Example:
$y''(x) + y(x) = 0$
Try $y = e^{\lambda x}$, $(\lambda^2 + 1)e^{\lambda x} = 0$
$e^{\lambda x} \neq 0$, $\lambda^2 + 1 = 0$ and factoring $(\lambda - j)(\lambda + j) = 0$.
Roots $\lambda = j, \lambda = -j$ Solution $y = c_1 e^{jx} + c_2 e^{-jx} = C_1 \cos x + C_2 \sin x$.

Repeated Roots:
If λ_1 is a real root of multiplicity m then the m solutions are not distinct and are not linearly independent. How to obtain a fundamental solution set in this case? As an example consider a second order DE $ay'' + by' + cy = 0$.
Characteristic equation is of the form $a\lambda^2 + b\lambda + c = 0$ and roots are $-\frac{b}{2a}$. Therefore only one solution $y_1 = e^{-\frac{b}{2a}x}$. To find another solution proceed as follows:

Try a solution of the form $y = v(x)y_1$.
Obtain expressions for y'' and y' and substitute these in the DE and obtain

$$e^{-\frac{b}{2a}x}\left(av'' - \frac{1}{4a}(b^2 - 4ac)v\right) = 0.$$

Since we have repeated roots $b^2 - 4ac = 0$, resulting $v'' =$. Thus $v = k_1x + k_2$ for k_1, k_2 constants. Therefore the general solution $y = c_1e^{-\frac{b}{2a}x} + c_2e^{-\frac{b}{2a}x}(k_1t + k_2) = c_3e^{-\frac{b}{2a}x} + c_4xe^{-\frac{b}{2a}x}$ and $\{e^{-\frac{b}{2a}x}, te^{-\frac{b}{2a}x}\}$ is a fundamental set of solutions.
Extending this idea for a repeated root of multiplicity m a fundamental set of solutions is

$$\{e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{m-1}e^{\lambda_1 x}\}.$$

If $\alpha + i\beta$ is a repeated complex root of multiplicity m then we can replace the $2m$ complex valued functions

$$e^{(\alpha+i\beta)x}, xe^{(\alpha+i\beta)x}, \cdots, x^{m-1}e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}, xe^{(\alpha-i\beta)x}, \cdots, x^{m-1}e^{(\alpha-i\beta)x}$$

by the $2m$ linearly independent real valued functions

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \cdots, x^{m-1}e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \cdots, x^{m-1}e^{\alpha x}\sin\beta x.$$

Examples:

1. $y''(x) + 6y'(x) + 9y(x) = 0$
   Try $y = e^{\lambda x}$, $(\lambda^2 + 6\lambda + 9)e^{\lambda x} = 0$
   $e^{\lambda x} \neq 0$, $\lambda^2 + 6\lambda + 9 = 0$ and factoring $(\lambda + 3)(\lambda + 3) = 0$.
   Roots $\lambda = -3, \lambda = -3$
   Solution $y = (c_1 + c_2x)e^{-3x}$.

2. $y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$
   Roots $1, 1, 1, -2$.
   Solution $y = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-2x}$.

3. $y^{(4)} - 8y^{(3)} + 26y^{(2)} - 40y^{(1)} + 25y = 0$
   Roots $2 \pm i, 2 \pm i$ Solution $y_c = [(C_1\cos x + C_2\sin x) + x(C_3\cos x + C_4\sin x)]e^{2x}$

3.2.2 Nonhomogeneous Equations

Consider the nonhomogeneous linear DE with constant coefficients, $L[y] = (D^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x)$. The general solution is of the form $y = y_c + y_p$. We have already seen in the previous section how to obtain y_c. In this section we present a method to find y_p.

Method of Undetermined Coefficients
The underlying idea of this method is a guess for y_p (a trial solution) depending on the form of the input function $f(x)$. This method is applicable only to DEs with L a linear operator with constant coefficients and input functions are polynomials, exponentials, sines, cosines or products of these functions. The reason is the repeated differentiation of each these functions produce only

a finite number of linearly independent terms. The form of y_p or the trial solution is a linear combination of all linearly independent functions that are generated by repeated differentiation of $f(x)$.

Examples:

1. $f(x) = e^{-x}$
   Derivatives of $e^{-x} = \{e^{-x}, -e^{-x}, e^{-x}, \cdots\}$
   Linearly independent terms $\{e^{-x}\}$, finite.
   $y_p = ce^{-x}$

2. $f(x) = 2xe^{-x}$
   Derivatives of $2xe^{-x} = \{2xe^{-x}, 2e^{-x} - 2xe^{-x}, -4e^{-x} + 2xe^{-x}, \cdots\}$
   Linearly independent terms $\{e^{-x}, xe^{-x}\}$, finite.
   $y_p = c_1e^{-x} + c_2xe^{-x}$

3. $f(x) = x^2$
   Derivatives of $x^2 = \{x^2, 2x, 2, 0, ...0\}$
   Linearly independent terms $\{x^2, x, 1\}$, finite. $y_p = a_1x^2 + a_2x + a_3$

4. $f(x) = \sin 2x$
   Derivatives of $\sin 2x = \{\sin 2x, 2\cos 2x, ...\}$
   Linearly independent terms $\{\sin 2x, \cos 2x\}$, finite. $y_p = c_1 \sin 2x + c_2 \cos 2x$

5. $f(x) = e^x \sin 2x$
   Derivatives of $e^x \sin 2x = \{e^x \sin 2x, e^x \sin 2x + e^x 2\cos 2x, ...\}$
   Linearly independent terms $\{e^x \sin 2x, e^x \cos 2x\}$, finite. $y_p = c_1e^x \sin 2x + c_2e^x \cos 2x$

6. $f(x) = \dfrac{1}{x}$
   Derivatives of $\dfrac{1}{x} = \left\{\dfrac{1}{x}, -\dfrac{1}{x^2}, \dfrac{2}{x^3}, ..., \right\}$
   Linearly independent terms $\left\{\dfrac{1}{x}, \dfrac{1}{x^2}, \dfrac{1}{x^3}, ..., \right\}$, infinite. Therefore method cannot be applied for this type of forcing function.

Examples:

1. $y'' + y = x^2$
   Associated homogeneous solution $y_c = C_1 \cos x + C_2 \sin x$.
   Since $f(x) = x^2$ is a quadratic polynomial, trial solution $y_p = Ax^2 + Bx + C$.
   $y_p' = 2Ax + B$,$y_p'' = 2A$,and thus the DE gives $2A + Ax^2 + Bx + C = x^2$.
   Equating the coefficients of like like powers of x gives, $A = 1, B = 0, C = -2$
   Therefore a particular solution, $y_p = x^2 - 2$.
   The general solution is $y = y_c + y_p = C_1 \cos x + C_2 \sin x + x^2 - 2$.

2. $(D^2 - 1)y = \sin x$
   $y_c = C_1e^x + C_2e^{-x}$
   Trial solution $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$, $y_p'' = -A\cos x - B\sin x$
DE gives, $-A\cos x - B\sin x - (A\cos x + B\sin x) = \sin x$
Equating the coefficients of $\sin x$ and $\cos x$, $A = 0, B = -\frac{1}{2}$.
$y_p = -\frac{1}{2}\sin x$ and $y = y_c + y_p$.

3. $y'' + y' + 2y = 4e^x + 2x^2$
   Find y_c
   Trial solution, $y_p = Ae^x + Bx^2 + Cx + D$
   $A = 1, B = 1, C = -1, D = \dfrac{-1}{2}$
   $y = y_c + y_p$

4. $y'' - y = e^{3x}\cos 2x - e^{2x}\sin 3x$
   $y_c = C_1e^x + C_2e^{-x}$
   $y_p = e^{3x}(A\cos 2x + B\sin 3x) + e^{2x}(C\cos 3x + D\sin 3x)$

Note:
Now consider solving $y'' - 5y' + 4y = 8e^x$. $y_c = c_1e^x + c_2e^{4x}$
A guess for y_p is Ae^x. Substituting this in the DE gives $0 = 8e^x$. The problem here is that the trial solution $y_p = Ae^x$ already appears in y_c. Therefore when $y_p = Ae^x$ is substituted into the DE, the left hand side becomes zero. How do we handle this situation? (Try $y_p = Axe^x$).
Thus we have two cases to consider.
Case I: No function in the assumed y_p is a solution of the associated homogeneous DE. Here trial solution is of the form of the input function.
Case II: A function in the assumed y_p is also a solution of y_c. In this case the terms that duplicate the terms in y_c, must be multiplied byx^k,where k is the smallest positive integer that eliminates the duplication.
Examples:

(1) $(D^2 + 1)y = \sin x$
$y_c = C_1\cos x + C_2\sin x$
$y_p = x(A\cos x + B\sin x)$

(2) $y^{(4)} - y^{(2)} = 3x^2 - \sin 2x$
$y_c = C_1 + C_2x + C_3e^x + C_4e^{-x}$
$y_p = x^2(Ax^2 + Bx + C) + E\sin 2x + F\cos 2x$

3.2.3 Spring mass system

Here we will look at an application of second order differential equations. Vibrations can occur in almost all branches of engineering and so what we are doing here can be easily adapted to other situations, usually with just a change in notation.

Model
Here consider a spring of length l, called the natural length, and an object with mass m attached to it. When the object is attached to the spring the spring will stretch a length of L. This is the position of the center of gravity for the object as it hangs on the spring with no movement and is called the **equilibrium position**. The forces that will act upon the object:

- The force due to gravity, $F_g = mg$

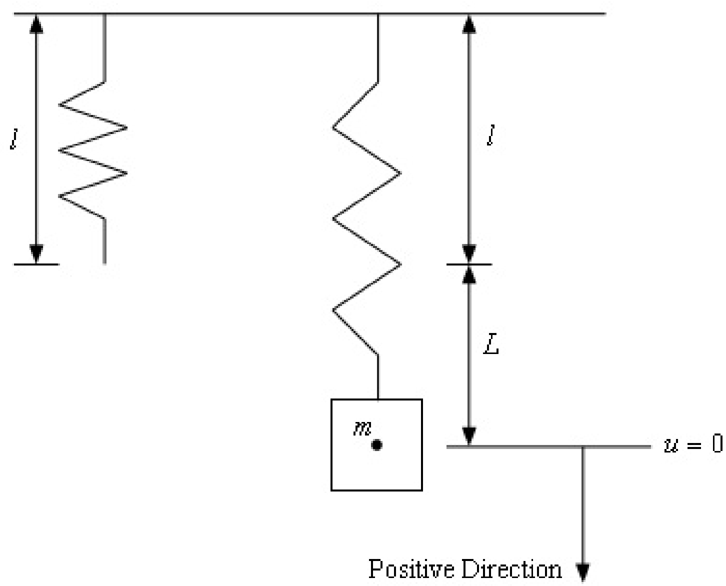

Figure 3.1: Spring Mass.

- The force that the spring exerts on the object is of the form $F_s = -k(L+x)$, where x is the length stretched from the equilibrium position.
- The force due to damping, $F_d = -cx'$ where, $c > 0$ is the damping coefficient.
- External Forces, any other forces that we decide we want to act on the object.

Therefore by Newton's second law, the governing DE of the motion is given by

$$mx'' + cx' + kx = mg - kL + F(t).$$

When the object is at rest in its equilibrium position there are exactly two forces acting on the object, the force due to gravity and the force due to the spring.
Since the object is at rest $mg = kL$. Therefore the equation of motion is given by

$$mx'' + cx' + kx = F(t).$$

Initial conditions are of the form:

$x(0) = x_0$: Initial displacement from the equilibrium position
$x'(0) = v_0$: Initial velocity

Free Motion

Here we assume $F(t) = 0$ and is called free motion.Now let us look at some specific cases.

- Undamped Motion:
  This is the simplest case that we can consider. Free or unforced vibrations means that

$F(t) = 0$ and undamped vibrations means that $c = 0$. In this case the differential equation becomes, $mx'' + kx = 0$ and the characteristic equation has the roots $\lambda = \pm i\omega_0$ where $\omega_0 = \sqrt{k/m}$ and is called the **natural frequency**. In this case solution is of the form $x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega t + \Phi)$ where $A = \sqrt{C_1^2 + C_2^2}$ and $\tan \Phi = \dfrac{C_1}{C_2}$ and the quadrant in which Φ lies is determined by the signs of C_1 and C_2 which can be determined by the initial conditions..
Therefore the resulting motion is a sine wave and is called **simple harmonic motion**. The constant A is the amplitude, Φ is the phase angle.

- **Damped Motion**
In most applications there is some type of frictional force or damping force affecting the vibrations. This force may be due to the medium surrounding such as air or some liquid etc. The equation is of the form

$$mx'' + cx' + kx = 0.$$

The roots of the auxiliary equation

$$-\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

and the form of the solution depends on the nature of these roots.

I. **Critically damped motion** $c^2 - 4mk = 0$
Here there is a repeated root $\lambda = -c/2m < 0$ and $y(t) = (C_1 + C_2 t)e^{\lambda}t$.
$x(t)$ goes to zero as $t \to \infty$, and so the motion is nonoscillatory and called **critically damped** motion. This is called *critical damping* and will happen when the damping coefficient is
$c^2 - 4mk = 0$ which gives $c = 2\sqrt{mk} = c_{cr}$.
The value of the damping coefficient that gives critical damping is called the critical damping coefficient and denoted by γ_{CR}.

II. **Overdamped motion** $c^2 - 4mk > 0$
Here there are two distict real roots,

$$\lambda_1 = -\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4mk} < 0 \text{ and } \lambda_2 = -\frac{c}{2m} - \frac{1}{2m}\sqrt{4mk - c^2} < 0.$$

Hence the solution is $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ and $y(t)$ tends to zero as $t \to \infty$. This case will occur when $c^2 > 4mk$ which gives $c > c_{cr}$.
Therefore $y(t)$ does not oscillate and called **overdamped** motion.

III. **Underdamped motion** $c^2 < 4mk$
Roots are complex $\alpha \pm i\beta$ with $\alpha = -\dfrac{c}{2m} < 0$ and $\beta = \dfrac{1}{2m}\sqrt{4mk - c^2}$.
General solution

$$y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = Ae^{\alpha t} \sin(\beta t + \Phi)$$

with $A = \sqrt{C_1^2 + C_2^2}$ and $\tan\Phi = C_1/C_2$.
Therefore $y(t)$ is the product of an exponential damping factor $Ae^{\alpha t}$ and a sine factor $\sin(\beta t + \Phi)$.
Since $\alpha < 0$, $e^{\alpha t}$ tends to 0 as $t \to \infty$ and the system is called **underdamped** because there is not enough damping present (c is too small) to prevent the system from oscillating.This case will occur when $c^2 < 4mk$ which gives $c < c_{cr}$ and is called under damping.

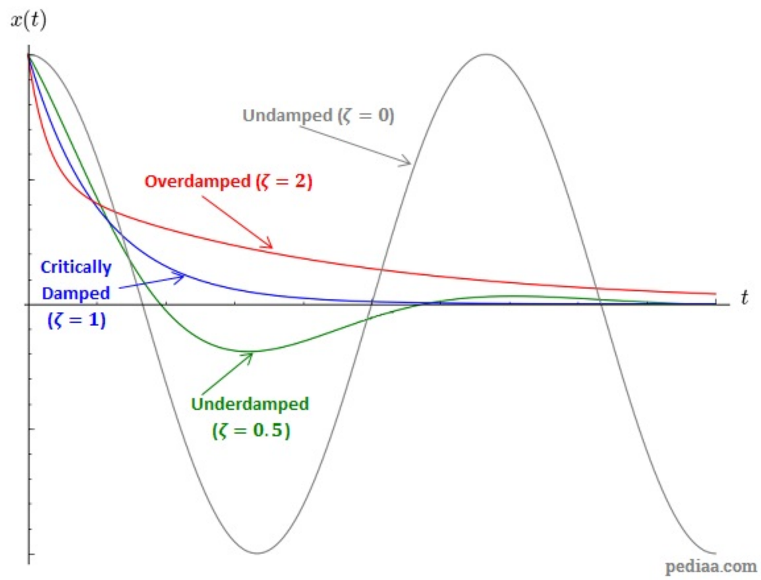

Figure 3.2: comparison.

Series Circuit Analogue
Consider the single-loop series circuit(RLC circuit) shown in figure 3.3. Equation for LRC series

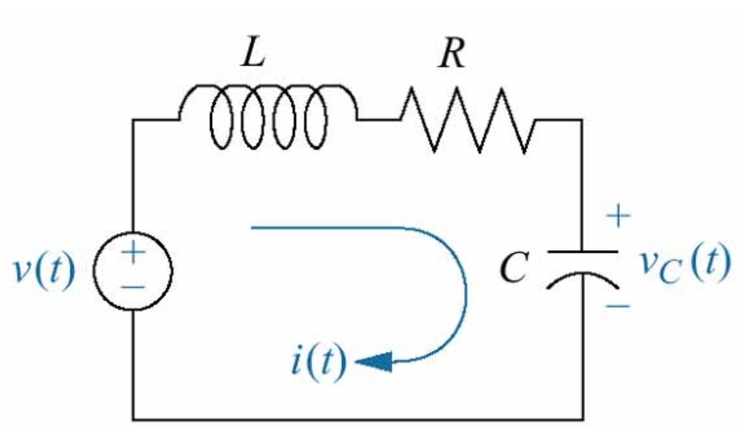

Figure 3.3: RLC Circuit.

electrical circuit is $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$. Thus we notice there is an analogy between mechanical oscillators and electrical oscillators. Consider free motion, $E(t) = 0$ and comparing with the spring mass system, we have the following.

1. When $R = 0$, simple harmonic motion
2. When $R \neq 0$ and $R^2 - 4L/C > 0$, overdamped motion
3. When $R \neq 0$ and $R^2 - 4L/C = 0$, critically damped motion
4. When $R \neq 0$ and $R^2 - 4L/C < 0$, underdamped motion

Forced Motion

Here we consider the spring mass system with nonzero forcing function, $mx'' + cx' + kx = F(t)$ with periodic forcing function of the form $F(t) = F_0\cos(\omega t)$ or $F(t) = F_0\sin(\omega t)$.

Undamped Forced Motion

The differential equation in this case is $m\ddot{x} + kx = F_0\cos(\omega t)$. $x_c = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$ and

$$x_p = \begin{cases} \dfrac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) & \text{if} \quad \omega \neq \omega_0 \\ \dfrac{F_0}{2m\omega_0} t\sin(\omega t) & \text{if} \quad \omega = \omega_0 \end{cases}.$$

Therefore for the two cases solution is of the form

$$x = \begin{cases} A\cos(\omega t - \Phi) + \dfrac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) & \text{if} \quad \omega \neq \omega_0 \\ A\cos(\omega t - \Phi) + \dfrac{F_0}{2m\omega_0} t\sin(\omega t) & \text{if} \quad \omega = \omega_0 \end{cases}.$$

When $\omega_0 = \omega$, the amplitude is a function of t.(t appears in the particular solution.) Thus amplitude of this solution increases linearly in time. Such a phenomenon is called resonance.

Forced Damped Motion

Here we consider a spring mass system with sinusoidal forcing term $m\ddot{x} + c\dot{x} + kx = F_0\cos(\omega t)$. Solution to this is $x(t) = x_c + x_p$ where where x_c is the complementary function and

$$x_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\gamma^2}}\sin(\omega t + \theta) \text{ with } \tan\theta = (k - m\omega^2)/(c\omega).$$

Therefore the solution is

$$x(t) = x_c + A\sin(\omega t + \theta) \text{ with } A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

Note:

1. The complementary function x_c depends only on the parameters of the system.
2. Since $x_c \to 0$ as $t \to \infty$, x_c dies off for large t and is called the **transient** part of the solution.

3. The second term x_p is due to the external forcing function $F(t) = F_0 \cos \omega t$.
4. Like $F(t)$, x_p is sinusoid with angular frequency ω.
5. For large t, the motion becomes that of x_p and is called the **steady state** solution.
6. Amplitude $A = A(\omega)$ has the maximum when $\omega = \omega_0$.

3.3 Macaulay's Bracket Method

This section looks at the mathematics that lies behind Macaulay's Method. The method relies upon functions which have discontinuities and because of the discontinuities these functions have to be treated carefully. We will show how they are useful in solving differential equations with discontinuous forcing functions.

Macaulay Functions

Macaulay functions represent quantities that begin at a point say a. Before point a, the function has zero value, after point a the function has a defined value.

Definition:

$$F_n(x) = [x-a]^n = \begin{cases} 0 & \text{when} \quad x \leq a \\ (x-a)^n & \text{when} \quad x > a \end{cases}$$

for $n = 0, 1, \cdots$.
Note
When the $n = 0$ we have

$$F_n(x) = [x-a]^0 = \begin{cases} 0 & \text{when} \quad x \leq a \\ 1 & \text{when} \quad x > a \end{cases}$$

and is called **unit step function** or **Heaviside unit step function** and is usually denoted by $H(t-a)$ or $\mathcal{U}(t-a)$. The graph of $H(t-a)$ is given in Figure 3.4.

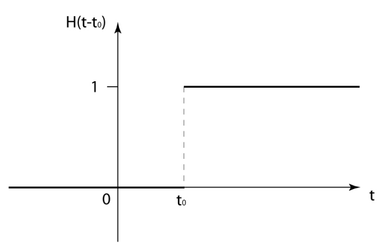

Figure 3.4: Heaviside Step Function.

Singularity Functions

Singularity functions behave differently to Macaulay functions. They are defined to be zero everywhere except point a. So in the light switch example the singularity function could represent the action of switching on the light.

Definition:

$$F_n(x) = [x-a]^n = \begin{cases} 0 & \text{when} \quad x \neq a \\ \infty & \text{when} \quad x = a \end{cases}$$

for $n = -1, -2, \cdots$.

Note:

Two singularity functions which are common in applications are:

1. **Unit impulse function:** This is the function when $n = -1$ and represents a unit force at $x = a$.

2. **Unit moment fuction:** This is the function when $n = -2$ and represents a unit moment located at $x = a$.

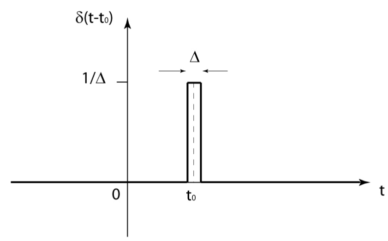

Figure 3.5: Impulse Function.

Many functions can conveniently be written using the Heaviside function notation.

Rectangular Pulse

Now we will see how to write a function of the form

$$p(t;a,b) = \begin{cases} 0; & t < a \\ 1; & a \leq t \leq b \\ 0; & t > b \end{cases}$$

as single function using the Heaviside function. The graph of this is a rectangular pulse as shown in the Figure 3.6.

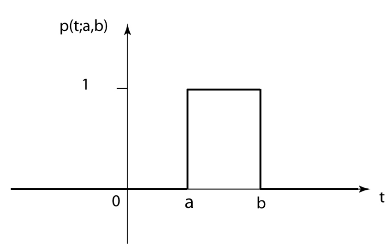

Figure 3.6: Rectangular pulse

By observing the graph we see that $p(t;a,b) = H(t-a) - H(t-b)$.

This can be used to write piecewise defined functions in a compact form as follows.

Let $f(t) = \begin{cases} f_1(t); & 0 < t < t_1 \\ f_2(t); & t_1 < t < t_2 \\ \vdots\ ; & \\ f_n(t); & t_n < t < \infty \end{cases}$ be a piecewise defined function.

Then $f(t) = f_1(t)p(t;0,t_1) + f_2(t)p(t;t_1,t_2) + \cdots + f_{n-1}(t)p(t;t_{n-1},t_n)H(t-t_n)$ with $p(t;t_k,t_{k-1}) = H(t-t_{k-1}) - H(t-t_k)$.

Example:

$f(t) = \begin{cases} 20t; & 0 \le t < 5 \\ 0; & t \ge 5 \end{cases}.$

Therefore $f(t) = 20t(H(t-0) - H(t-5)) + 0H(t-5) = 20t - 20tH(t-5)$.

Integration of Discontinuity Functions

These functions can be integrated almost like ordinary functions.

1. **Macaulay's functions**$(n \ge 0)$

$$\int_0^x F_n(x)dx = \frac{F_{n+1}(x)}{n+1} \text{ or } \int_0^x [x-a]^n dx = \frac{[x-a]^{n+1}}{n+1}$$

2. **Singularity functions**$(n < 0)$

$$\int_0^x F_n(x)dx = F_{n+1}(x) \text{ or } \int_0^x [x-a]^n dx = [x-a]^{n+1}$$

3.3.1 Applications

Modelling of Load Types

The fundamental equation for finding defection of a beam is

$$\frac{d^2x}{dx^2} = \frac{M}{EI}$$

where M is the bending moment at a point, E is the elastic modulus, I is the second moment of area at the point, y is the deflection at the point and x is the distance of the point along the beam.

Here we will consider different load types and the relationship between moment M and load w is given by

$$M = \int\int w dx.$$

1. **Moment Load**

   A moment load of value M, located at point a, is represented by $M[x-a]^{-2}$ and so appears in the bending moment equation as,

$$M(x) = \int\int M[x-a]^{-2}dx = M[x-a]^0.$$

2. **Point Load**
   A point load of value P, located at point a, is represented by $P[x-a]^{-1}$ and so appears in the bending moment equation as,

$$M(x) = \int\int P[x-a]^{-1}dx = P[x-a]^{1}.$$

3. **Uniformly Distributed Load**(UDL)
   A UDL of value w, beginning at point a and carrying on to the end of the beam, is represented by the step function $w[x-a]^0$ and so appears in the bending moment equation as,

$$M(x) = \int\int w[x-a]^{0}dx = \frac{w}{2}[x-a]^{2}.$$

4. **Patch load**
   If the UDL finishes before the end of the beam, sometimes called a patch load, we have a difficulty. This is because a Macaulay function turns on at point a and never turns off again. Therefore, to cancel its effect beyond its finish point (point b say), we turn on a new load that cancels out the original load, giving a net load of zero, and is given by

$$w[x-a]^0 - w[x-b]^0.$$

   Therefore the resulting bending moment equation is

$$M(x) = \int\int (w[x-a]^0 - w[x-b]^0)dx = \frac{w}{2}[x-a]^2 - \frac{w}{2}[x-b]^2.$$

Analysis procedure

1. $$M(x) = EI\frac{d^2y}{dx^2} \tag{3.6}$$

2. Integrating (3.6) gives an expression for the rotations along the beam with the rotation constant of integration C_θ.
$$EI\frac{dy}{dx} = \int M(x)dx + C_\theta \tag{3.7}$$

3. Integrating (3.7) gives an expression for the deflections olong the beam with the deflection constant of integration C_δ.
$$EIy = \int\int M(x)dxdx + C_\theta x + C_\delta \tag{3.8}$$

4. Use boundary conditions to calculate the unknown constants of integration, and any unknown reactions.

5. Solve for required displacements by substituting the location into the equations (3.7) or (3.8) as appropriate.

Note:

1. The maximum deflection occurs at the point where $\dfrac{dy}{dx} = 0$. That is at the points where there is no rotation.

2. Sign Convention.
Here we orient the $x - y$ axis system as normal: positive y upwards, positive x to the right and anti-clockwise rotations are positive. In this way, the downward deflections will always be algebraically negative.

3.4 Exercises

1. Find a fundamental solution set to the following homogeneous linear ODEs and find the general solution.

$$(a)\ D^2y - 3Dy + 2y = 0 \qquad (b)\ D^2y + 4Dy + 4y = 0 \qquad (c)\ D^3y - 6D^2y + Dy - 6y = 0$$

$$(d)\ D^2y + 2Dy + 5y = 0 \qquad (e)\ 2y'' - 3y' - 5y = 0$$

2. Find a particular solution by using the method of undetermined coefficients.

$$(a)\ y'' + y = e^{-x} \qquad (b)\ y'' + 4y + 4y = 6\sin 3x \qquad (c)\ y'' + y = x^2\cos 5x$$

$$(d)\ y'' + 4y = 2\sin x$$

3. Find the form for a particular solution to $y'' + 2y' - 3y = f(t)$ for the following choices for $f(t)$.
$(a) 7\cos 3t \quad (b) 2te^t \sin t \quad (c) t^2\cos \pi t \quad (d) 5e^{-3t} \quad (e) 3te^t \quad (f) t^2e^t$

4. Find a particular solution by using the D operator method.

$$(a)\ D^2 - D + 1y = x^3 - 3x^2 + 1 \qquad (b) y'' - y = e^x \qquad (c)\ y'' + 5y' + 6y = 3\sin 2x$$

5. Solve the following initial value problems.

$$(a)\ y'' + 4y' + 4y = x^2 y(0) = 0, y'(0) = 0.5 \qquad (b) y'' + 4y = \sin 2x, y(0) = 0, y'(0) = 0$$

$$(c)\ y'' = e^{2x}, y(0) = 0, y'(0) = 1$$

6. A spring mass system which is constrained only to move in the vertical direction has zero damping. Find the general solution and determine the frequency of oscillation if $M = 4$ kg and $k = 100$ N/m in the usual notation.

7. A damped spring mass system is given an initial velocity of 50 m/s from the equilibrium position. Find $y(t)$ if $M = 4$ kg, $k = 64$ N/m and $C = 40$ kg/s. Discuss the resulting motion.

8. Show that the general solution of $M\ddot{y} + ky = F_0 \cos \omega t$ with M, k, F_0 and ω constants, is

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0 \cos \omega t}{M(\omega_0^2 - \omega^2)}$$

where $\omega_0 = \sqrt{k/M}$ and c_1, c_2 are constants. What happens when $\omega = \omega_0$?

Chapter 4

Linear Systems

4.1 Introduction

So far we have discussed methods for solving linear ODEs with only one dependent variable. Many applications require the use of two or more dependent variables, each a function of a single independent variable(usually time).
Examples:

1. **Coupled springs**
   Two masses m_1 and m_2 are connected to two springs of negligible mass having spring constants k_1 and k_2 as shown in the following figure.

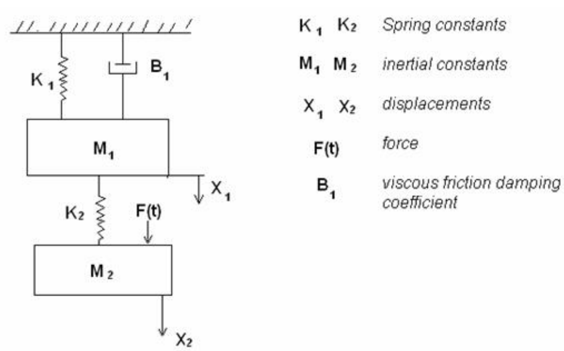

Figure 4.1: Coupled Springs.

   If x_1 and x_2 denote the vertical displacements of the masses from their equilibrium position then by Newton's second law, the motion is described by the system

$$\begin{aligned} M_1\ddot{x_1} &= -K_1x_1 + K_2(x_2 - x_1) - B_1\dot{x_1} \\ M_2\ddot{x_2} &= -k_2(x_2 - x_1) + F(t). \end{aligned}$$

2. **Electrical networks with more than one loop**
   Consider the LRC circuit with two loops given in figure below.

Figure 4.2: Electrical Network.

The currents satisfy the system

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 &= 0. \end{aligned}$$

3. **Double pendulum**
   A double pendulum oscillates in a vertical plane under the influence of gravity.

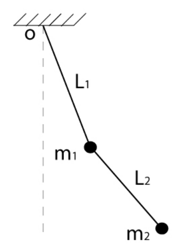

Figure 4.3: Double Pendulum.

For small displacements θ_1 and θ_2, the system of DEs describing the motion is

$$\begin{aligned} (m_1+m_2)L_1^2\theta_1'' + m_2L_1L_2\theta_2'' + (m_1+m_2)L_1g\theta_1 &= 0 \\ m_2L_2^2\theta_2'' + m_2L_1L_2\theta_1'' + m_2L_2g\theta_2 &= 0. \end{aligned}$$

Definition
A set of linear ODEs containing a single independent variable, two or more dependent variables and their derivatives is called a system of linear ODEs.
Example:

$$\begin{aligned} x'' + 2x' + y'' - x + 3y &= \sin t \\ x' + y' - 4x + 2y &= e^{-t} \end{aligned}$$

If the right hand side of the system is identically zero it is called a homogeneous system and otherwise nonhomogeneous. The above is an example of a nonhomogeneous system.

Reduction to First Order System
Any higher order linear ODE can be transformed into an equivalent system of first order system by using the transformation, $x_1 = x$, $x_2 = x'$, $x_3 = x''$, $\cdots$, $x_n = x^{(n-1)}$.

Example: $x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$
Therefore using $x_1 = x$, $x_2 = x' = x_1'$, $x_3 = x'' = x_2'$ the DE reduces to the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{aligned}$$

of three first order equations.

System of Linear First Order Ordinary Differential Equation:
A system of linear first order ODEs is of the form

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t). \end{aligned}$$

If $f_i(t) = 0, i = 1, 2, \cdots, n$, it is called homogeneous and nonhomogeneous otherwise. If all a_{ij}'s are constants then it is a system with constant coefficients.

Matrix Form:
With

$$\mathbf{X(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A(t)} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \quad \text{and} \quad \mathbf{F(t)} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

the above system can be written in the matrix form

$$\mathbf{X}' = \mathbf{A(t)X(t)} + \mathbf{F(t)}.$$

If the system is homogeneous, then

$$\mathbf{X}' = \mathbf{A(t)X(t)}.$$

Solution:A solution of a system of differential equations is a set of sufficiently differentiable functions $x_1(t), \cdots, x_n(t)$ that satisfies each equation in the system on some common interval.
Example

$$\begin{array}{lcl} x_1(t) = c_1 e^{-\sqrt{6}t} + c_2 e^{\sqrt{6}t} & & x' = 3y \\ x_2(t) = -\frac{\sqrt{6}}{3} c_1 e^{-\sqrt{6}t} + \frac{\sqrt{6}}{3} c_2 e^{\sqrt{6}t} & \text{is a solution of the system} & y' = 2x \end{array}$$

Note: The calculus involved in matrix and vector valued functions is similar to that of real valued functions.

1. $\dfrac{d\mathbf{A}}{dt} = (a_{ij}')$, $\int \mathbf{A} dt = (\int a_{ij} dt)$

2. $\dfrac{d(\mathbf{A}+\mathbf{B})}{dt} = \dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{B}}{dt}$, $\dfrac{d(\mathbf{AB})}{dt} = \mathbf{A}\dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt}\mathbf{B}$, $\dfrac{d(\mathbf{CA})}{dt} = \mathbf{C}\dfrac{d\mathbf{A}}{dt}$, $\mathbf{C}$ a constant matrix.

Note: In this course we will only consider systems with each ODE containing constant coefficients.

Basic Concepts
The theory for linear systems of first order ODEs $\mathbf{X}' = \mathbf{A(t)X(t)} + \mathbf{F(t)}$, is quite similar to the theory for single higher order ODEs. **Note:**

1. Let $L[X] = X' - AX = F(t)$. Then L is a linear operator.
   $L[\alpha X + \beta Y] = [\alpha X + \beta Y]' - A[\alpha X + \beta Y] = [\alpha X' + \beta Y'] - [\alpha AX + \beta AY] = \alpha L[X] + \beta L[Y]$.

2. Because of this linearity, if $X_1, X_2, \cdots, X_m$ are solutions to the homogeneous equation $L[X] = X' - AX = 0$, then for any linear combination of the form $c_1X_1 + c_2X_2 + \cdots + c_mX_m$ is also a solution.

Linear Independence
Let $X_1, \cdots, X_n$ be n solutions to the homogeneous system. If the Wronskian
$W[X_1, \cdots, X_n] = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix} \neq 0$ then $\{X_1, \cdots, X_n\}$ is linearly independent. Here $X_i = (x_{i1} \cdots x_{in})^T$.

Representation of Solutions(Homogeneous Equation)

Let $X_1, \cdots, X_n$ be n linearly independent solutions of $X' = AX$. Then any solution of the system is of the form $c_1X_1 + \cdots + c_nX_n$ and the general solution and $\{X_1, \cdots, X_n\}$ is called a **fundamental solution set**. **Note:**
The solution space of $X' = AX$ is a vector space and the fundamental solution set $\{X_1, X_2, \cdots, X_n\}$ is a basis for the space.

Representation of Solutions (Nonhomogeneous Case)

Let X_p be a particular solution to $X' = AX + F(t)$ and let $\{X_1, \cdots, X_n\}$ be a fundamental solution set for the corresponding homogeneous equation, then every solution of $X' = AX + F(t)$ can be expressed in the form

$$X(t) = c_1X_1 + \cdots + c_nX_n + X_p.$$

In the following sections we will present some solving techniques in solving linear systems.

4.2 Matrix Methods for Linear Systems

4.2.1 Eigenvalue Method

If $\mathbf{A} = a$, a scalar then the solution is $x(t) = ce^{at}$. This suggests for the system a solution of the form ,

$$x(t) = ve^{\lambda t},$$

where $\mathbf{v}$ and λ are to be determined. Substituting this in the system gives

$$\lambda ve^{\lambda t} = Ave^{\lambda t} \quad \text{which is equivalent to} \quad Av = \lambda v.$$

Since we want the nonzero solutions $v \neq 0$, for each eigenvalue λ and associated eigenvector $\mathbf{v}$, a solution of the linear system is given by $\mathbf{v}e^{\lambda t}$.
Note:

1. If $\lambda_1, \cdots, \lambda_n$ are distinct then the corresponding eigenvectors $v_1, v_2, \cdots, v_n$ are linearly independent.

2. If $\mathbf{A}$ is real symmetric then it has n linearly independent eigenvectors.

Method
1.Find the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $\mathbf{A}$.
2.Find n linearly independent eigenvectors $v_1, \cdots, v_n$ associated with these eigenvalues.
3.Step 2 gives n linearly independent solutions $x_1(t) = v_1 e^{\lambda_1 t}, \cdots, x_n(t) = v_n e^{\lambda_n t}$.
4.The general solution is $x(t) = c_1 x_1(t) + \cdots + c_n x_n(t)$.

Note: There are three cases depending on the form of the eigenvalues.

1. **Case 1: Distinct real eigenvalues**
   In this case n linearly eigenvectors can always be found.
   Example:
   $$\begin{aligned} \frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= 2x + y. \end{aligned}$$
   The eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = 4$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
   The solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$.

2. **Case 2: Repeated real eigenvalues** Here we will consider the case eigenvalue of multiplicity $m \leq n$ has m linearly independent eigenvectors only.
   Example:
   $\mathbf{X}' = \mathbf{AX}$ with $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

The eigenvalues $\lambda_1 = -1$ of multiplicity 2 and $\lambda_2 = 5$. The corresponding eigenvectors of λ_1 are $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and of λ_2 is $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
The Solution $X(t) = c_1 v_1 e^{-t} + c_2 v_2 e^{-t} + c_3 v_3 e^{5t}$.

3. **Case 3: Complex eigenvalues**
   Example:
   $$\begin{aligned} \frac{dx}{dt} &= 6x - y \\ \frac{dy}{dt} &= 5x + 4y. \end{aligned}$$
   The eigenvalues of A are $\lambda_1 = 5 + 2j$ and $\lambda_2 = 5 - 2j$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 - 2j \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 + 2j \end{pmatrix}$.
   The solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 - 2j \end{pmatrix} e^{(5+2j)t} + c_2 \begin{pmatrix} 1 \\ 1 + 2j \end{pmatrix} e^{(5-2j)t}$.
   With $e^{(5 \pm 2j)t} = e^{5t}(\cos 2t \pm j \sin 2t)$,
   $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] e^{5t} + C_2 \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right] e^{5t}.$$

Real valued solutions

1. For complex conjugate eigenvalues λ_1 and λ_2, the corresponding eigenvectors are also complex conjugates. Thus solutions X_1 and X_2 are also conjugates.

2. To obtain real-valued solutions, use real and imaginary parts of either X_1 or X_2 and proceed as follows.

   (a) Let $\lambda_1 = \alpha + i\beta$ and $v = \mathbf{a} + \mathbf{ib}$.

   (b) Then $X_1 = e^{(\alpha + i\beta t)t} v = e^{\alpha t}(\cos \beta t + i \sin \beta t)(\mathbf{a} + \mathbf{ib})$

   (c) Thus $X_1 = e^{\alpha t}(\mathbf{a} \cos \beta \mathbf{t} - \mathbf{b} \sin \beta \mathbf{t}) + \mathbf{i} \mathbf{e}^{\alpha \mathbf{t}}(\mathbf{a} \sin \beta \mathbf{t} + \mathbf{b} \cos \beta \mathbf{t}) = \mathbf{u_1} + \mathbf{iu_2}$

   (d) Can show u_1 and u_2 are two linearly independent solutions to the system and are the real valued solutions corresponding to complex eigenvalues and eigenvectors.

Second order systems
Consider coupled spring mass system of n masses. The goverining equation of motion is a system of the form
$$M\ddot{X} = KX$$
where $M = \text{diag}(m_1 \cdots m_n)$, K is an $n \times n$ matrix with entries interms of spring constants k_i and X is an $n \times 1$ vector consisting of the displacements x_i of each mass. Since M is invertible the system can be written as $\ddot{X} = AX$ where $A = M^{-1}K$.

To find a solution try a solution of the form $X = ve^{\alpha t}$.
Then $X'' = \alpha^2 ve^{\alpha t}$ and substituting this in the equation gives $Av = \alpha^2 v$.
Therefore $ve^{\alpha t}$ is a solution if α^2 is an eigenvalue and v is a corresponding eigenvector. If $\alpha^2 < 0$, say $\alpha^2 = -\omega^2$ then $\alpha = \pm i\omega$

Consider the solution $X = ve^{i\omega t} = v(\cos\omega t + i\sin\omega t)$. Then $u_1 = v\cos\omega t$ and $u_2 = v\sin\omega t$ are two linearly independent real valued solutions to the system.
If $\lambda_i = \alpha_i^2 = -\omega_i^2$s are distinct and v_i's are the corresponding real eigenvectors then the solution to the system $\ddot{X} = AX$ is of the form

$$X(t) = \sum_{i=1}^{n} (a_i \cos\omega_i t + c_i \sin\omega_i t) v_i$$

with a_i, b_i arbitrary constants. Thus $X(t) = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} [A_i \sin(\omega_i t + \phi_i) v_i]$.
Note:
Therefore each X_i represents free oscillations of the spring mass system and are called natural modes of oscillations. This says the eigenvalues and eigenvectors of the system provides a description of the types of behaviour the system can exhibit.

4.2.2 Decoupling

If each equation of the system had only one variable, solved for independently of other equations, then the system would be much easier to solve. In this case the system is of the form $X' = DX$ where D is a diagonal matrix. Here we will see how to use eigenvalues and eigenvectors to transform a coupled system into an uncoupled system.

Transform Matrix T
Suppose A has n linearly independent eigenvectors $v_1, v_2, \cdots, v_n$. Let $D = \text{diag}(\lambda_1 \cdots \lambda_n)$ and $T = (v_1 \ v_2 \ \cdots \ v_n)$ then $D = T^{-1}AT$. Now use the transformation $X = TZ$, which gives $X' = TZ'$.
Hence $TZ' = ATZ$ and $Z' = T^{-1}ATZ = DZ$ which is a decoupled system. To transform back to the original variables use $X = TZ$.

4.2.3 Matrix Exponential Method

For any square matrix $\mathbf{A}$ the matrix exponential is $e^{\mathbf{A}t} = I + At + \dfrac{\mathbf{A}^2t^2}{2!} + \dfrac{\mathbf{A}^3t^3}{3!} + \cdots$. In this section we will see how to express the solution of a linear system $\mathbf{X}' = \mathbf{AX} + F(t)$ interms of e^{At}.

Note

1. $\dfrac{de^{At}}{dt} = Ae^{At}$
2. $e^0 = I, e^{A(t+s)} = e^{At}e^{As}, (e^{At})^{-1} = e^{-At}$
3. If $AB = BA$ then $e^{(A+B)t} = e^{At}e^{Bt}$.

Non Homogeneous Systems
Consider the first order linear equation $\dfrac{dx}{dt} + p(t)x = q(t)$. The solution was obtained by first calculating the integrating factor, multiplying the DE with it and integrating the resulting equation. Here we adopt a similar procedure.

First consider $\mathbf{X}' - \mathbf{AX} = F(t), X(t_0) = X_0$
Multiply the equation by e^{-At} (A is a constant matrix). This is like the integrating factor.
$e^{-At}X' - e^{-At}AX = e^{-At}F(t)$ and $\dfrac{d(e^{-At}X)}{dt} = e^{-At}F(t)$.
Integrating

$$X = e^{At}\int e^{-A\tau}F(\tau)d\tau + e^{At}C$$

where C is a constant vector. Applying initial conditions $C = e^{-At_0}X_0$
Then $X = e^{At}\int_{t_0}^{t} e^{-A\tau}F(\tau)d\tau + e^{A(t-t_0)}X_0$.
For homogeneous system $x = e^{A(t-t_0)}X_0$
Calculating e^{At}:

1. If $\mathbf{A}$ is diagonal then $e^{\mathbf{A}t} = \text{diag}(e^{d_1 t} \;\; e^{d_2 t} \cdots e^{d_n t})$

2. If $\mathbf{A}$ is diagonalizable $\mathbf{A} = \mathbf{PDP}^{-1}$ then $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$

3. The calculation of $e^{\mathbf{A}t}$ can be done using computer software. For example use **MatrixExp[A t]** in *Mathematica*, **exponential(A, t)** in *Maple* or **expm(At)** in *MATLAB*.

4. If $\mathbf{A}^m = 0$ for some m(a nilpotent matrix) then it is relatively easy to compute $e^{\mathbf{A}t}$.

Note
If A is diagonalizable then use the transformation $X = TZ$ and the solution can be obtained in either of the following ways.

1. $TZ' = ATZ + F(t)$, $Z' = T^{-1}ATZ + T^{-1}F(t) = DZ + H(t)$.
   Each equation of the system is of the form $z_k = \lambda_k z_k + h_k(t)$ which is a first order linear DE. Solve this using the integrating factor.

2. Use $e^{At} = Te^{Dt}T^{-1}$ in $X = e^{At}\int_{t_0}^{t} e^{-A\tau}F(\tau)d\tau + e^{A(t-t_0)}X_0$.

Example
Solve $\dot{X} = AX + F(t)$, $X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ and $F(t) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.
Eigenvalues and corresponding eigenvectors of A are $-1, 2$ and $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.
Method 1
Use the transformation $X = TZ$. Then the system becomes

$$\begin{aligned} \dot{z}_1 &= -z_1 - 1 \\ \dot{z}_2 &= 2z_2 + 2 \end{aligned}$$

with initial conditions $Z(0) = T^{-1}X(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
Solving using the integrating factor,
$z_1 = -1 + c_1 e^{-t}$, $z_2 = -1 + c_2 e^{2t}$

Applying IC, $c_1 = 0$, $c_2 = 2$ and $z_1 = -1$ and $z_2 = -1 + 2e^{2t}$.
Thus $X = TZ = \begin{pmatrix} -3 + 4e^{2t} \\ -3 + 2e^{2t} \end{pmatrix}$

Method 2

A is diagonalizable and $e^{At} = Te^{Dt}T^{-1} = -\dfrac{1}{3}\begin{pmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{pmatrix}$

Then using $X = e^{At}\int_{t_0}^{t} e^{-A\tau}F(\tau)d\tau + e^{A(t-t_0)}X_0$ gives

$$X = -\frac{1}{3}\begin{pmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{pmatrix}\int_0^t -\frac{1}{3}\begin{pmatrix} e^{-\tau} - 4e^{2\tau} & -2e^{-\tau} + 2e^{2\tau} \\ 2e^{-\tau} - 2e^{2\tau} & -4e^{-\tau} + e^{2\tau} \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} d\tau$$
$$-\frac{1}{3}\begin{pmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \frac{1}{9}\begin{pmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{pmatrix}\begin{pmatrix} -e^{t} - 2e^{2t} \\ -2e^{-t} - e^{2t} \end{pmatrix} - \begin{pmatrix} e^{-t} - 2e^{2t} \\ -2e^{-t} - e^{2t} \end{pmatrix} = \begin{pmatrix} -3 + 4e^{2t} \\ -3 + 2e^{2t} \end{pmatrix}$$

Note:

We saw earlier that when an eigenvalue is complex $\lambda = \alpha + i\beta$ with eigenvector $v = u_1 + iu_2$ then $AT = TD$ and $D = T^{-1}AT$ where $T = u_1\ u_2]$ and $D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Hence $e^{At} = Te^{Dt}T^{-1}$ and D is not diagonal.

$e^{Dt} = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$

Proof:

$D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R + Q$

Since $RQ = QR$, $e^{Dt} = e^{(R+Q)t} = e^{Rt}e^{Qt}$.

Since R is diagonal $e^{Rt} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{pmatrix}$.

To find e^{Qt} proceed as follows.

Let $Q = \beta P$. Then $e^{Qt} = e^{P\beta t} = e^{P\tau}$, $\tau = \beta t$.

Observe that $P^4 = I$ which gives $P^5 = P$, $P^6 = P^2$, $P^7 = P^3$, $P^8 = I$, $\cdots$ Thus

$$\begin{aligned}
e^{P\tau} &= I + P\tau + \frac{P^2\tau^2}{2!} + \frac{P^3\tau^3}{3!} + \frac{I\tau^4}{4!} + \frac{P\tau^5}{5!} + \frac{P^2\tau^6}{6!} + \cdots \\
&= I(1 + \frac{\tau^4}{4!} + \frac{t\tau^8}{8!} + \cdots) + P(\tau + \frac{\tau^5}{5!} + \frac{\tau^9}{9!} + \cdots) + P^2(\frac{\tau^2}{2!} + \frac{\tau^6}{6!} + \frac{\tau^{10}}{10!} + \cdots) + P^3(\frac{\tau^3}{3!} + \frac{\tau^7}{7!} + \frac{\tau^{11}}{11!} + \cdots \\
&= \begin{pmatrix} 1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \cdots & \tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \frac{\tau^7}{7!} + \cdots \\ -(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \frac{\tau^7}{7!} + \cdots) & 1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!}\frac{\tau^2}{2!} + \cdots \end{pmatrix} \\
&= \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}.
\end{aligned}$$

Therefore $e^{Dt} = e^{Rt}e^{Qt} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{pmatrix}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ -e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix}$

Other methods to solve nonhomogeneous systems

1. Variation of parameters
2. Method of undetermined coefficients

Chapter 5

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