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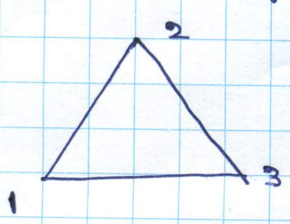
Dihedral group of order 6 (symmetries of the Equilateral triangle)

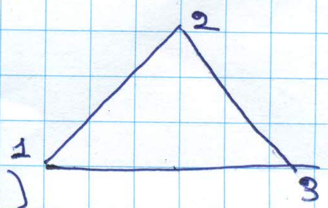
D_3

A symmetry of a geometric figure is a rearrangement of the figure preserving the arrangement of its sides and vertices as well as its distances and angles.

To Find the symmetries of the equilateral triangle.

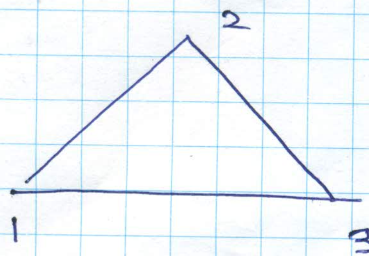


Identity (Id)
(You do nothing to the arrangement)

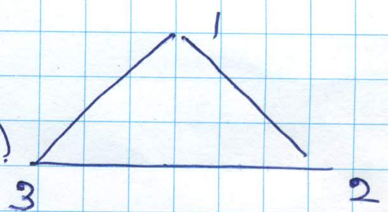


You may think of these transformations as the permutation of the three vertices.

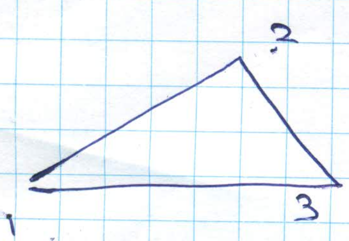
So $id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ ← original arrangement ← arrangement after the transformation.



ρ_1
rotation (clock wise) by 120°

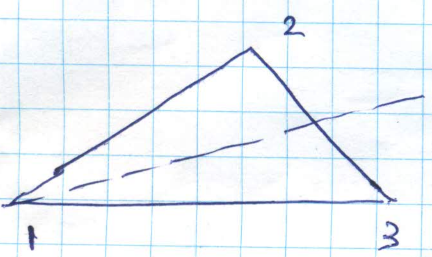


$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ read this like (1 goes to 2, 2 goes to 3, 3 goes to 1).



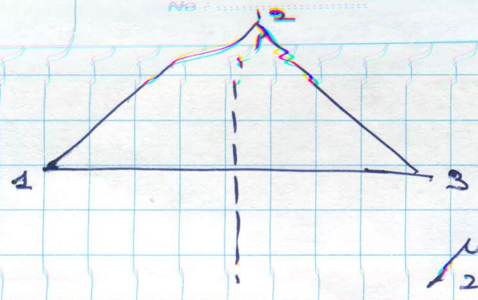
ρ_2
rotation (clockwise) by 240°

$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (1 goes to 3, 2 goes to 1, 3 goes to 2)



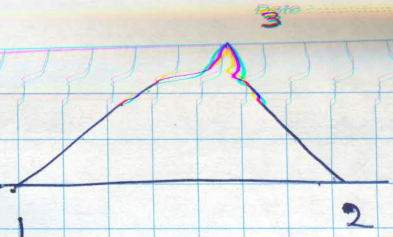
μ_1
reflection fixing (1).

$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

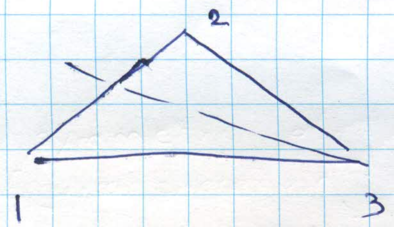


$\xrightarrow{\mu_2}$

reflection fixing 2

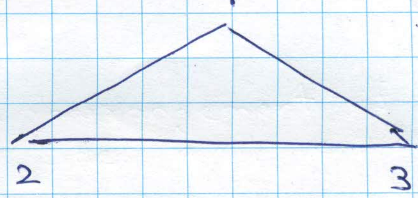


$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$



$\xrightarrow{\mu_3}$

reflection fixing 3



$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

So the symmetries of the triangle is the collection of transformation

$$D_3 = \{ Id, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3 \}$$

order of D_3 $= |D_3| = 6$ (The number of elements)

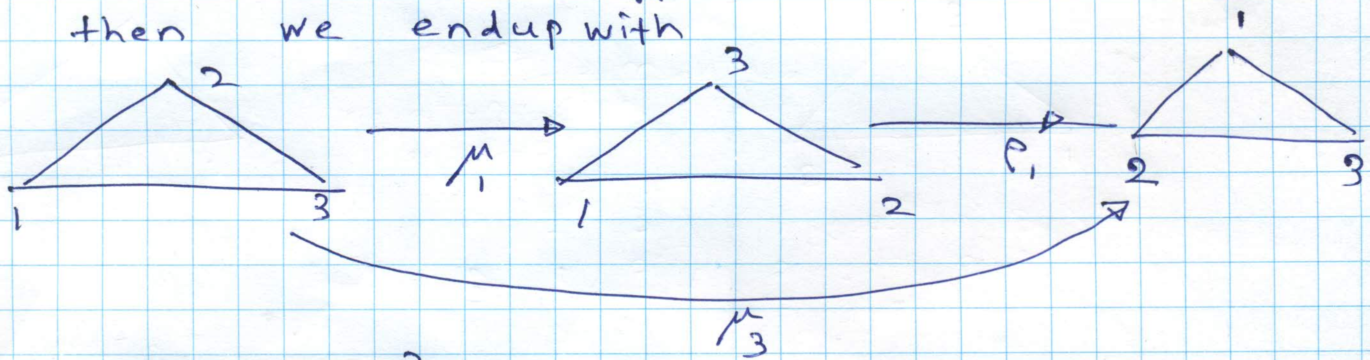
~~For a We can introduce a p~~

We can compose transformations to ~~find~~ make new transformation

for instance we can first do μ_1 (reflection about 1)

and then ~~rota~~ do ρ_1 (rotate by 120°)

then we endup with



$$\Rightarrow \quad \rho_1 \circ \mu_1 (\triangle) \longrightarrow \mu_3$$

when we think of permutation of the vertices

$$\rho_1 \circ \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \mu_3$$

$1 \xrightarrow{1} 1 \xrightarrow{2} 2 \qquad 3 \rightarrow 2 \rightarrow 3$

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~~So~~ ~~Define~~

Is composition of motions a binary operation?

$$\circ : D_3 \times D_3 \longrightarrow D_3$$

$$(a, b) \longrightarrow \underset{\text{composition.}}{a \circ b}$$

The binary operation can be represented by the Cayley table

composition →

$\circ$	Id	ρ_1	ρ_2	μ_1	μ_2	μ_3
Id	Id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	Id	μ_3	μ_1	μ_2
ρ_2	ρ_2	Id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

$$\rho_1 \circ \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \rho_2.$$

This way ~~you~~ we may fill up the table.

$$\rho_1 \circ \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = Id.$$

$$\rho_1 \circ \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \mu_3.$$

$\vdots$

$\therefore$ for any ~~of~~ composition of motions in D_3 we see that we get a new motion in D_3 so 'composition ($\circ$)' is a binary operation on D_3

Is $(D_3, \circ)$ a group?

G(1) Associativity for any motions in D_3 $a, b, c \in D_3$ $a \circ (b \circ c) = (a \circ b) \circ c$.
So this is associative

G(2) Identity element is the Id motion

$\rho_1 \circ Id = Id \circ \rho_1 = \rho_1$ ---- so on.

So $Id \in D_3$

G_3) : Inverse element ?

we se that

$Id \circ Id = Id$

$\rho_2 \circ \rho_1 = \rho_1 \circ \rho_2 = Id$ (so ρ_2 is the inverse of ρ_1) and vice versa

ρ_2.

$\mu_1 \circ \mu_1 = Id$ μ_1 is its own inverse.

$\mu_2 \circ \mu_2 = Id$ μ_2 " "

$\mu_3 \circ \mu_3 = Id$ μ_3 " "

all six elements have there inverses in D_3

So $(D_3, \circ)$ is a group

Is It Abelian ?

Take for example $\mu_1 \circ \mu_2 = \rho_1$

$\mu_0 \circ \mu_1 = \rho_2$

So $\mu_1 \circ \mu_2 \neq \mu_0 \circ \mu_1$

~~So~~ So $a * b \neq b * a$ $\forall a, b \in D_3$

$\Rightarrow$ D_3 is not a Abelian.

~~Small discu~~

Discussion : See above how we can identify each motion with a permutation of the vertices If you have 3 ~~objects~~ ~~how~~ different objects how ~~we can identify~~ many different arrangements can it have? its $3! = 3 \times 2 \times 1 = 6$ right?

If there are 3 numbers 1, 2, 3 what are the different arrangements ~~so that~~

$S_3 \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$

are all the diffrint arrangements

This is also a group on its own and we call it the permutation group of order 6 denoted S_3

Since each motion in ~~S~~ D_3 can be identified with a permutation in S_3, we say ~~D_3 looks~~ 'D_3 looks like S_3'. In mathematical language we call it 'isomorphic'

So D_3 is isomorphic to S_3

in notation $D_3 \cong S_3$

Warning: Consider the symmetries of the square. This is also a group under composition of motions. (I have assigned this in a ~~problem~~ tute).

This is the Dihedral group of order 8. (D_4)

If we take the permutation group of 4 objects $(1, 2, 3, 4)$, ~~we will~~ we call this S_4. We see that S_4 is larger than D_4. So D_4 is not isomorphic to S_4.