Tutorial Answers.

1] a) $y' = -4xy^2$

assuming $y \neq 0$

$$\int \frac{dy}{y^2} = \int -4x\,dx$$

where c_1 is an arbitrary constant

$$-y^{-1} = \frac{-4x^2}{2} - c_1$$

$$\frac{1}{y} = 2x^2 + c_1$$

$y(0)=1$

$x=0,\ y=1$

$\frac{1}{1} = 2\times 0 + c_1$

$\therefore\ c_1 = 1$

$$y = \frac{1}{2x^2+1}$$

To plot $y = \frac{1}{2x^2+1}$

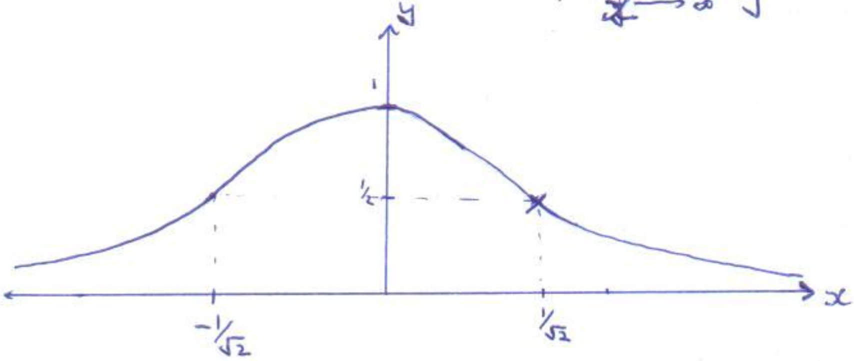
$y' = -4xy^2 = 0 \Longrightarrow x = 0$ or $y = 0$

but $y = \frac{1}{2x^2+1} \neq 0$

$\therefore\ x=0$ is a turning point.

$y = \frac{1}{2x^2+1}$ is symmetric about the y axis.

$\lim_{x \to \infty} y = 0.$



b) $y(0) = 0$

$$\frac{1}{y} = 2x^2 + c_1$$

$$y = \frac{1}{2x^2+c_1}$$

$0 = \frac{1}{c_1} \Rightarrow c_1 \to \pm\infty$ i.e. $y \to 0$.

& we assumed $y \neq 0$.

$y(x)=0$ is a function. It satisfies the ODE. $\because\ y'=0$ & $-4x^2y^2=0$.

$\therefore\ \underline{y(x) = 0.}$

c) $y(x_0) = y_0$ $\quad y' = -4xy^2$ $\quad$ $y(x)=0$ satisfies all $\{(x_0,y_0)\in\mathbb{R}^2 \mid y_0 = 0\}$

$y = \frac{1}{2x^2+c_1}$ must be defined for all real numbers $x \in \mathbb{R}$.

$\therefore\ 2x^2 + c_1 \neq 0. \Longrightarrow c_1 > 0.$

If $c_1 > 0$, then $y > 0$ as well. $\therefore\ y_0 > 0.$

$\therefore\ y_0(2x_0^2 + c_1) = 1 \Longrightarrow c_1 = \frac{1 - 2x_0^2 y_0}{y_0} > 0$

$\therefore\ \{(x_0,y_0)\in\mathbb{R}^2 \mid y_0 = 0\} \cup \{(x_0,y_0)\in\mathbb{R}^2 \mid y_0 > 0 \text{ and } \frac{1}{y_0} - 2x_0^2 > 0\}$ are all ordered pairs (x_0,y_0), satisfying the condition.

12] a) $y\,dy = x\,dx \qquad y(0)=0$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$$

$$y(0)=0 \Longrightarrow \quad 0 = 0 + \frac{c}{2} \qquad \therefore c = 0$$

$\therefore\ y^2 - x^2 = 0 \iff y = x$ & $y=-x$ are solutions.

$\therefore$ the solutions aren't unique.

b) $y\dfrac{dy}{dx} = -x \qquad y(0)=0$

$$\int y\,dy = \int -x\,dx \Longrightarrow \frac{y^2}{2} = \frac{-x^2}{2} + c \xrightarrow{y(0)=0} 0 = -0 + c \quad \therefore c = 0$$

$\therefore\ y^2 + x^2 = 0$ is a circle of radius 0, & is a point.

You can't take limits at points & so can't differentiate. Points are not solutions to differential equations.

There are no solutions to this I.V.P.

19] a) $\dfrac{dy}{dx} = (-2x+y)^2 - 7 \qquad y(0)=0$

Let $z = -2x+y \qquad \therefore \dfrac{dz}{dx} = -2 + \dfrac{dy}{dx} \iff \dfrac{dy}{dx} = \dfrac{dz}{dx} + 2$

$$\frac{dz}{dx} + 2 = +z^2 - 7$$

$$\frac{dz}{dx} = z^2 - 9 \xrightarrow[z\neq\pm3]{\text{assuming}} \frac{1}{z^2-9}\,dz = dx \Longrightarrow \int \frac{1/6}{z-3} - \frac{1/6}{z+3}\,dz = \int dx$$

$$\therefore \frac{1}{6}\left[\ln(z-3) - \ln(z+3)\right] = x + k \Longrightarrow \left(\frac{z-3}{z+3}\right)^{1/6} = e^x e^k = ce^x$$

where $c = e^k$

$y(0)=0 \Rightarrow z(0) = 0 \qquad \left(\dfrac{z-3}{z+3}\right) = c^6 e^{6x}$

$\therefore \dfrac{-3}{3} = c^6 e^{6\cdot 0} \qquad c^6 = -1$

$$\therefore \frac{-2x+y-3}{-2x+y+3} = -e^{6x}$$

(where $-2x+y+3 \neq 0$)

$\boxed{z=\pm3 \text{ cases}}$

$-2x+y-3=0$

& $-2x+y+3=0$

$(\because)\ y = -3+2x$

L.H.S $= 2$

R.H.S $= 9-7 = 2 \qquad$ $\therefore$ (=)

used

$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$

$a\ln(b) = \ln(b)^a$

$\therefore \frac{1}{6}\left[\ln(z-3) - \ln(z+3)\right]$

$= \frac{1}{6}\ln\frac{(z-3)}{(z+3)} = \ln\left(\frac{z-3}{z+3}\right)^{1/6} = G$

$e^G = \left(\dfrac{z-3}{z+3}\right)^{1/6}$

20) a) b) $xy^2 \frac{dy}{dx} = y^3 - x^3 \qquad y(1) = 2$

treating x as the dependant variable.

$xy^2 = (y^3 - x^3)\frac{dx}{dy}$

$x=0$ is a solution.

let $y = [v(x)]x \qquad \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$x^3 v^2\left[x\frac{dv}{dx} + v\right] = (v^3 - 1)x^3 \qquad x \neq 0$$

$$xv^2\frac{dv}{dx} + v^3 = v^3 - 1 \Longrightarrow \int v^2 dv = \int -\frac{1}{x}dx$$

$$\therefore \frac{v^3}{3} = -\ln(x) - \ln(k) \Longrightarrow -\frac{v^3}{3} = +\ln(kx) \Rightarrow e^{-\frac{v^3}{3}} = kx$$

where $-\ln(k)$ is an arbitrary constant.

$y(1) = 2 \Longrightarrow \begin{matrix} x=1 \\ v=2 \end{matrix}$

$$\therefore e^{-8/3} = k \times 1 \qquad \therefore \underline{\underline{e^{-\frac{y^3}{3x^3}} = xe^{-8/3}}}$$

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20) b) c) $\frac{dy}{dx} = \frac{x+3y}{3x+y}$ — ①

Using polar coordinates.

$x = r\cos\theta \qquad x = x(r,\theta)$

$y = r\sin\theta \qquad y = y(r,\theta)$

Total derivative $dx = \frac{\partial x}{\partial r}.dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - \sin\theta\, d\theta$

$dy = \frac{\partial y}{\partial r}.dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$

$$① \Rightarrow \therefore \frac{dy}{dx} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta} = \frac{r}{r}\left[\frac{\cos\theta + 3\sin\theta}{3\cos\theta + \sin\theta}\right]$$

$$\cancel{3\cos\theta\sin\theta dr} + \sin^2\theta dr + 3r^2\cos^2\theta + r\cos\theta\sin\theta d\theta = \cos^2\theta dr \cancel{+ 3\sin\theta\cos\theta dr} - r\cos\theta\sin\theta d\theta - 3r\sin^2\theta d\theta$$

$$3r[\cos^2\theta + \sin^2\theta]d\theta + 2r\cos\theta\sin\theta d\theta = [\cos^2\theta - \sin^2\theta]dr$$

$$[3 + \sin 2\theta]\, r d\theta = \cos 2\theta\, dr$$

$$\frac{[3 + \sin 2\theta]}{\cos 2\theta}d\theta = \frac{1}{r}dr \Longrightarrow k + \ln(r) = \int 3\sec 2\theta + \tan 2\theta \; d\theta$$

assuming $\cos 2\theta \neq 0$

i.e. $\theta \neq \pm \pi/4$

$$\therefore k + \ln(r) = \frac{3}{2}\ln[\sec 2\theta + \tan 2\theta] + \frac{1}{2}\ln[\sec 2\theta]$$

$$\ln(C) + \ln(r) = \ln\left\{\left[\frac{(\sec 2\theta + \tan 2\theta)^{3/2}}{\cancel{(\sec 2\theta)}}\right] \cdot (\sec 2\theta)^{1/2}\right\}$$

$$\therefore\ rc = (\sec 2\theta + \tan 2\theta)^{3/2} (\sec 2\theta)^{1/2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2y/x}{1-(y/x)^2}$$

$$= \frac{2yx}{x^2-y^2}$$

$$\sec 2\theta = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{x^2+y^2}{x^2-y^2}$$

assuming $x^2-y^2 \neq 0$.

$\tan\theta = y/x$

$r^2 = x^2+y^2$

$$\therefore\ r^2c^2 = \left[\frac{x^2+y^2+2xy}{x^2-y^2}\right]^3 \frac{x^2+y^2}{x^2-y^2}$$

$$\cancel{(x^2+y^2)}\, c^2 = \frac{(x+y)^6 \cancel{(x^2+y^2)}}{(x^2-y^2)^4} = \frac{(x+y)^6}{(x+y)^4(x-y)^4}$$

$$\therefore\ c^2(x-y)^4 = (x+y)^2$$

$$\therefore\ c(x-y)^2 = (x+y)$$

However, differentiating this,

$$2(x-y)[1-y'] = \frac{1}{c}[1+y']$$

$$2x-2y-\frac{1}{c} = \left[\frac{1}{c}+2x-2y\right]y'$$

$$\therefore\ \frac{dy}{dx} = \frac{2x-2y+1/c}{1/c+2x-2y}$$

which isn't the original ODE.

Contradiction of Assumptions $\quad x^2-y^2 = 0 \quad$ & $\quad \theta = \pm\pi/4 \quad$ i.e.- $\tan\theta = \pm 1 = \frac{y}{x}$

both amount to

$\underline{\underline{y = x}}$ & $\underline{\underline{y = -x}}$ ~~which also solve~~

which both solve the original ODE.

$y = x \qquad$ L.H.S $= \frac{dy}{dx} = 1$

R.H.S $= \frac{x+3y}{3x+y} = \frac{4}{4} = 1$

$y = -x \qquad$ L.H.S. $\frac{dy}{dx} = -1$

R.H.S $= \frac{x+3y}{3x+y} = \frac{-2}{2} = -1$

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22) b) $\quad \frac{dy}{dx} = \frac{2x+y}{x+y-1}$

$x = X+k \qquad y = Y+h$

$$2X+2k+Y+h = 2X+Y \implies$$
$$\&\ X+k+Y+h-1 = X+Y \implies \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} k \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \end{bmatrix}$$

$2\times1-1\times1\neq0$

$\therefore$ can solve for k & h.

by solving

$k = -1$

$h = 2$

22] b)

$$\frac{dY}{dX} = \frac{2X+Y}{X+Y}$$

Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + \frac{X\,dv}{dX} = \frac{2+v}{1+v} \implies X\frac{dv}{dX} = \frac{2+v-v-v^2}{1+v} = \frac{2-v^2}{1+v}$$

$$\frac{1+v}{-1(2-v^2)}\,dv = \frac{-1}{X}\,dX \qquad \text{assuming } v \neq \pm\sqrt{2}$$

but $\underline{v = \sqrt{2}}$ & $\underline{v = -\sqrt{2}}$ are solutions.

L.H.S. $= \frac{dY}{dX} = \sqrt{2}$ L.H.S. $= \frac{dY}{dX} = -\sqrt{2}$

R.H.S. $= \frac{2X+Y}{X+Y} = \frac{2+\sqrt{2}}{1+\sqrt{2}} = \sqrt{2}$ R.H.S $= \frac{2X+Y}{X+Y} = \frac{2-\sqrt{2}}{1-\sqrt{2}} = -\sqrt{2}$

$$\int \frac{\frac{\sqrt{2}-1}{2\sqrt{2}}}{v+\sqrt{2}} + \frac{\frac{1+\sqrt{2}}{2\sqrt{2}}}{v-\sqrt{2}}\,dv = -\frac{dX}{X}$$

$$v = \frac{Y}{X} = \frac{y-2}{x+1}$$

$$\frac{\sqrt{2}-1}{2\sqrt{2}}\ln(v+\sqrt{2}) + \frac{1+\sqrt{2}}{2\sqrt{2}}\ln(v-\sqrt{2}) = -\ln(X) + C$$

$$\frac{\sqrt{2}-1}{2\sqrt{2}}\ln\left[\frac{y-2}{x+1}+\sqrt{2}\right] + \frac{1+\sqrt{2}}{2\sqrt{2}}\ln\left[\frac{y-2}{x+1}-\sqrt{2}\right] = -\ln(x+1) + C$$

where C is an arbitrary contant.

& $\underline{\frac{y-2}{x+1} = \sqrt{2}}$ & $\underline{\frac{y-2}{x+1} = -\sqrt{2}}$

22] c) $$\frac{dy}{dx} = \frac{2x+3y+2}{4x+6y-1}$$

checking whether the substitution $X = x+k$ & $Y = y+h$ works.

$$2X+2k+3Y+3h+2=0 \Rightarrow \underbrace{\begin{bmatrix}2 & 3\\4 & 6\end{bmatrix}}_{\text{not invertible}}\begin{bmatrix}k\\h\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$
$$4X+4k+6Y+6h-1=0 \Rightarrow$$

$\because 2\times6 - 4\times3 = 0.$

$\therefore$ can't solve for k & h

$\therefore$ use $Z = 2x+3y$

$$\frac{dZ}{dx} = 2 + \frac{3dy}{dx}$$

$$\frac{dZ}{dx} - 2 = 3\left(\frac{Z+2}{2Z-1}\right) \implies \frac{dZ}{dx} = \frac{7Z+4}{2Z-1} \implies \int\frac{2Z-1}{7Z+4}\,dZ = \int dx$$

assuming $7Z+4 \neq 0$

but $\underline{7Z = -4}$ is a solution.

$\because \frac{dZ}{dx} = 0 \qquad \frac{7Z+4}{2Z-1} = 0.$

$7z+4 \neq 0$ case,

$$\int dx = \int \frac{2z-1}{7z+4}\,dz = \int \frac{2}{7}\left(\frac{7z+4}{7z+4}\right) - \frac{(8/7+1)}{7z+4}\,dz = \int \frac{2}{7} - \frac{15}{7}\frac{1}{(7z+4)}\,dz$$

$$\therefore \quad x+k = \frac{2}{7}z - \frac{15}{49}\ln(7z+4)$$

$$\therefore \quad x+k = \frac{2}{7}(2x+3y) - \frac{15}{49}\ln(7z+4)$$

where k is an arbitrary constant

when $7z+4 \neq 0$

i.e. $7(2x+3y)+4 = 0$

$\because$ $7z+4$ is a solution.

& $\underline{7(2x+3y)+4 = 0}$

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(23) Clairaut's equation is $y = xy' + f(y')$ —①

a) i) $\frac{d}{dx}$ ① $\Rightarrow$ $\frac{dy}{dx} = y' + xy'' + \frac{df(y')}{dy'}\cdot\frac{dy'}{dx}$

$$\cancel{y'} = \cancel{y'} + xy'' + f'(y')\cdot y''$$

$$\therefore \quad \left[x + f'(y')\right] y'' = 0 \qquad p = \frac{dy}{dx}$$

$$\therefore \quad \left[x + \frac{df(p)}{dp}\right]\cdot\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0 \quad \text{or} \quad x + \frac{df(p)}{dp} = 0$$

a) ii) Case i] general solution.

$$\frac{dp}{dx} = 0 \Rightarrow p = c \quad \text{a constant.}$$

$$\therefore \text{①} \Rightarrow y = xp + f(p) = cx + f(c)$$

case ii] $x + \frac{df(p)}{dp} = 0$ can be solved to get a singular soln.

Say $y = \alpha(x) + C$ is the soln with an arbitrary constant C,

then $\alpha(x) + C = \alpha'(x)x + f(\alpha'(x))$

$\therefore$ since C is only in the L.H.S. $C = 0$.

which is why it is singular.

23) b) a) $y = px + p^2 \quad = xp + f(p) \quad \therefore f(p) = p^2$

general solution $\underline{\underline{y = cx + c^2}}$

singular solution $\frac{d f(p)}{dp} = 2p \quad \therefore \frac{d f(p)}{dp} + x = 0 \Longrightarrow 2\underbrace{\frac{dy}{dx}}_{p} + x = 0$

$$\Downarrow$$

$$\underline{\underline{y = \frac{-x^2}{4} + 0}} \Longleftarrow dy = \frac{x}{-2}\, dx$$

b) b) $y = xp + \overbrace{\frac{p}{\sqrt{1+p^2}}}^{f(p)}$

general solution $\underline{\underline{y = cx + \frac{c}{\sqrt{1+c^2}}}}$

Singular solution $\frac{d f(p)}{dp} = \frac{1}{\sqrt{1+p^2}} - \frac{p \cdot 2p}{2(1+p^2)^{3/2}} = \frac{2 + \cancel{2p^2} - 2p^2}{2(1+p^2)^{3/2}} = \frac{1}{(1+p^2)^{3/2}}$

$$x = -\frac{d f(p)}{dp} \Longrightarrow x = \frac{-1}{(1+p^2)^{3/2}} \Longrightarrow x^2 = \frac{1}{(1+p^2)^3}$$

$$\Downarrow$$

$$p = \left[\frac{1}{x^{2/3}} - 1\right]^{1/2} \Longleftarrow (1+p^2) = \left(\frac{1}{x^{2/3}}\right)$$

$$\therefore y = xp + \frac{p}{\sqrt{1+p^2}} \Longrightarrow \underline{\underline{y = \left[\frac{1}{x^{2/3}} - 1\right]^{1/2} \left[x + \frac{x^{1/3}}{1}\right]}}$$

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24) b) $y\,dx = (y e^y - 2x)\,dy$

$y e^y - 2x = y \frac{dx}{dy}$ is linear in $x(y)$

used
Integrating factor
$e^{\int \frac{2}{y} dy} = e^{[2\ln y]} = e^{\ln y^2} = y^2$

$$\frac{dx}{dy} + \frac{2x}{y} = \frac{y e^y}{y} \xrightarrow[\text{factor}]{\text{Integrating}} \frac{dx}{dy} \cdot y^2 + 2yx = y^2 e^y$$

$$\Downarrow$$

$$\int d(xy^2) = \int y^2 e^y\, dy \Longleftarrow \frac{d(xy^2)}{dy} = y^2 e^y$$

$$\therefore xy^2 = \left[y^2 - 2y + 2\right] e^y + C$$

where C is an arbitrary constant.

25) c) $\frac{dy}{dx} = e^{-y}(2x-4) \qquad y(5)=0$

$$e^{y}dy = (2x-4)\,dx \implies \int e^{y}dy = \int (2x-4)\,dx$$

$$\Downarrow$$

$$e^{y} = x^2 - 4x + k$$

$y(5)=0$

$1 = e^{0} = 25 - 20 + k \quad \therefore k = -4$

$\therefore \; \underline{\underline{e^{y} = x^2 - 4x - 4}}$

interval of validity [for real (x, y)]

$e^{y} > 0 \quad \therefore \; x^2 - 4x - 4 > 0$

$[x - 2(1+\sqrt{2})][x - 2(1-\sqrt{2})] \geqslant 0$

$x = 5$ lies in this section.

$\therefore$ the interval of validity is $\underline{\underline{(2\sqrt{2}+2, +\infty)}}$

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26) b) a) $x\frac{dy}{dx} + (1-x)y = x^2y^2$

$\therefore n = 2$ Bernoulli equation standard substitution.

$v = y^{1-n} = 1/y$

$\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{v^2}\frac{dv}{dx}$

$$-\frac{x}{v^2}\frac{dv}{dx} + (1-x)\frac{1}{v} = \frac{x^2}{v^2}$$

$$\frac{dv}{dx} + \frac{(x-1)}{x}v = -x$$ is linear in v. Integrating factor

$$e^{\int (\cancel{x}1-\frac{1}{x})dx} = e^{x-\ln x} = e^{x}.e^{\ln(x^{-1})} = e^{x}/x$$

$$\int d\left(v\frac{e^{x}}{x}\right) = \int -x\frac{e^{x}}{x}\,dx$$

$$v\frac{e^{x}}{x} = -e^{x} + c \implies \frac{1}{y}\frac{e^{x}}{x} = -e^{x} + c$$

where c is an arbitrary constant.

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27) a) $\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$ —— (1) $\qquad$ Let $y = u + \frac{1}{v}$ — (2)

$\because u$ is a solution, substituting in (1) gives

$$\frac{du}{dx} = P(x)u^2 + Q(x)u + R(x) \text{ —— (3)}$$

(2.7) a) substituting (2) in (1) & cancelling using (3).

$$\cancel{\frac{du}{dx}} - \frac{1}{v^2}\frac{dv}{dx} = P(x)\left[\cancel{u^2} + \frac{2u}{v} + \frac{1}{v^2}\right] + Q(x)\left[\cancel{u} + \frac{1}{v}\right] + \cancel{R(x)}$$

$$\therefore \frac{dv}{dx} = P(x)\left[-2uv - 1\right] + Q(x)(-v)$$

$$\frac{dv}{dx} + \left[2u\,P(x) + Q(x)\right]v = -P(x)$$ is linear in v.

$$v = \frac{1}{y-u}$$

(2.7) b) i) $y = x$ is a solution to $\frac{dy}{dx} = x^3(y-x)^2 + \frac{y}{x} = x^3y^2 + \left(\frac{1}{x} - 2x^4\right)y + x^5$

$$\therefore \frac{dv}{dx} + \left[\cancel{2x \cdot x^3} + \frac{1}{x} - \cancel{2x^4}\right]v = -x^3$$

$$I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dv}{dx} + v = -x^4$$

$$\int d(vx) = \int -x^4\,dx \Longrightarrow vx = -x^4 + C$$

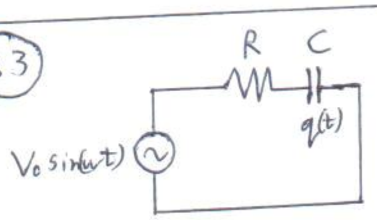
$$\therefore \left(\frac{1}{y-x}\right)x = -x^4 + C$$ where C is an arbitrary constant.

(2.7) b) ii) $y = 1$ is a solution to $\frac{dy}{dx} + 1 - 2y + y^2 = 0$

$$\frac{dv}{dx} + \left[(2\times1\times-1) + 2\right]v = -1\times-1 \Longrightarrow \frac{dv}{dx} = 1 \Longrightarrow v = x + k$$

$$\therefore \frac{1}{y-1} = x + k$$ where k is an arbitrary constant.

(3.3)



$$R\frac{dq}{dt} + \frac{1}{C}q = V_0 \sin\omega t \qquad a = 1/RC \quad b = \frac{V_0}{R}$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{V_0}{R}\sin\omega t$$

$$I.F. = e^{\int a\,dt} = e^{at}$$

a) $$\frac{dq}{dt} + aq = b\sin\omega t$$

$$\int d(q e^{at}) = \int e^{at} b \sin\omega t\,dt$$

33) a)

$$q e^{at} = b\int \mathrm{Im}\left(e^{(a+i\omega)t}\right) dt = b\,\mathrm{Im}\left[\int e^{(a+i\omega)t}\,dt\right] = b\ \mathrm{Im}\left[\frac{e^{(a+i\omega)t}}{a+i\omega}\right] + k$$

$$\frac{q(t)e^{at}}{b} = \frac{k}{b} + \mathrm{Im}\left[\frac{(a-i\omega)(\cos\omega t + i\sin\omega t)\,e^{at}}{a^2+\omega^2}\right] = \frac{e^{at}}{a^2+\omega^2}\left[a\sin\omega t - \omega\cos\omega t\right] + \frac{k}{b}$$

$$q(0)=0 \Rightarrow \quad 0 = \frac{k}{b} + \frac{1}{a^2+\omega^2}\cdot(-\omega) \quad \therefore k = \frac{+\omega b}{a^2+\omega^2}$$

$$\therefore q(t) = \frac{b}{a^2+\omega^2}\left[a\sin\omega t - \omega\cos\omega t\right] + \frac{\omega b\, e^{-at}}{(a^2+\omega^2)}$$

b) $q_{ss}(t) = \lim_{t\to\infty} q(t) = \dfrac{b}{\omega^2+a^2}\left[a\sin\omega t - \omega\cos\omega t\right]$ since $a>0$ $\lim_{t\to\infty} e^{-at} = 0$.

steady state

$$= \frac{b}{\sqrt{\omega^2+a^2}}\left[\underbrace{\frac{a}{\sqrt{\omega^2+a^2}}}_{\cos\phi}\sin\omega t - \underbrace{\frac{\omega}{\sqrt{\omega^2+a^2}}}_{\sin\phi}\cos\omega t\right] = \frac{b}{\sqrt{\omega^2+a^2}}\sin(\omega t-\phi)$$

~~where~~

$\cos\phi > 0$ & $\sin\phi > 0$

The response is lagging by ϕ radians $[0<\phi<\pi]$ ~~leading~~

~~by ϕ radians~~ since $0<\phi<\pi/2$. $[\because \omega>0,\ a>0]$

The frequency of the steady state response is the same.

The amplitude ~~magnitude~~ depends on b, ω & a.

33) c) steady state amplitude $= \dfrac{b}{\sqrt{a^2+\omega^2}}$ transient part $= \dfrac{b\omega}{a^2+\omega^2}e^{-at}$

$$e^{-at}\frac{b\omega}{a^2+\omega^2} = 0.01\frac{b}{\sqrt{a^2+\omega^2}} \Longrightarrow e^{-at} = \frac{0.01}{\omega}\sqrt{a^2+\omega^2}$$

$$t = \frac{-1}{a}\ln\left[\frac{0.01}{5}\sqrt{10^4+25}\right] \text{ seconds} \approx 1.6081\times10^{-2}\ \mathrm{s}$$

36) d) $\underbrace{e^{x}\cos y}_{F_x}\,dx + \underbrace{(-e^{x}\sin y)}_{F_y}\,dy = 0$

let

then $\dfrac{\partial}{\partial y}F_x = -e^{x}\sin y$ $\quad \dfrac{\partial}{\partial x}F_y = -e^{x}\sin y$ $\quad \therefore$ it is an exact equation.

36) d)

$$\int F_x \, dx = \int + e^x \cos y \, dx$$

$$F(x,y) = e^x \cos y + g(y)$$

$$\frac{\partial}{\partial y} F(x,y) = -e^x \sin y + g'(y) = \overbrace{-e^x \sin y}^{F_y}$$

$\therefore \; g'(y) = 0 \Rightarrow g(y) = C$ where C is an arbitrary constant.

$$F(x,y) = e^x \cos y + C$$

$$F(x,y) = k$$

37) a) Given fact $\Rightarrow$ slope of a curve that is orthogonal to the given curve is the negative reciprocal of that of the given curve.

$\therefore$ for the orthogonal curve.

$$\frac{dy}{dx} = \frac{-1}{-\frac{\partial F/\partial x}{\partial F/\partial y}} = \frac{\partial F/\partial y}{\partial F/\partial x} \Longrightarrow \frac{\partial F}{\partial x} dy = \frac{\partial F}{\partial y} dx$$

$$\therefore \; \frac{\partial F}{\partial y} dx - \frac{\partial F}{\partial x} dy = 0$$

b) i) $F(x,y) = x^2 + y^2 = k \qquad \frac{\partial F}{\partial y} = 2y \qquad \frac{\partial F}{\partial x} = 2x$

$$2y\,dx - 2x\,dy = 0 \longrightarrow \int \frac{1}{x} dx = \int \frac{1}{y} dy \Rightarrow \ln x + \ln k = \ln y$$

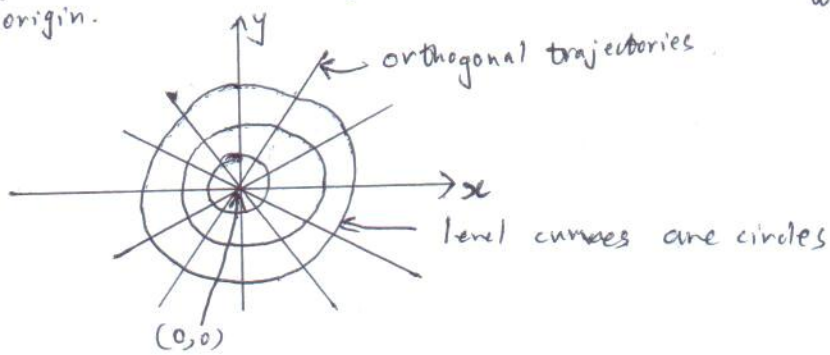
$$\Downarrow$$

$$y = kx$$

where k is an arbitrary constant.

$y = kx$

This family of curves are straight lines through the origin.



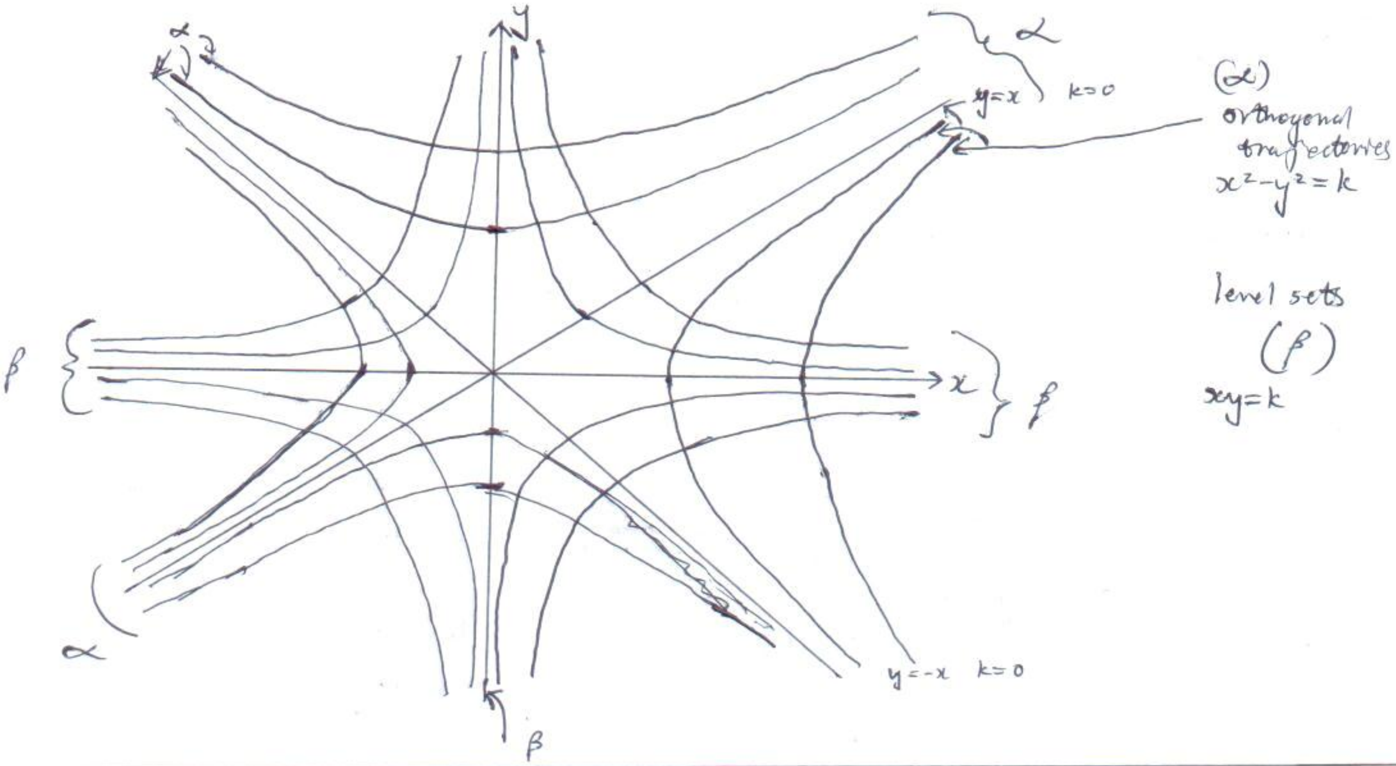
$(0,0)$

ii) $F(x,y) = xy = k \qquad \frac{\partial F}{\partial y} = x \qquad \frac{\partial F}{\partial x} = y$

$$\cancel{x}\,dx - \cancel{y}\,dy = 0 \Rightarrow \cancel{-\frac{dx}{x} = \frac{dy}{y}} \Rightarrow$$

$$-\int x\,dx = \int y\,dy \longrightarrow -\frac{x^2}{2} = \frac{y^2}{2} - \frac{k}{2} \Longrightarrow y^2 - x^2 = k$$

37) b)ii)



37) c) $y^2 = 4Cx + 4Cx^2 + C = C\left[4x^2+4x+1\right]$

$\cancel{y^2 - 4x} \; F(x,y) = \left[\dfrac{y}{2x+1}\right]^2 = c$

$\dfrac{\partial F}{\partial y} = \dfrac{2y}{(2x+1)} \cdot \dfrac{1}{(2x+1)}$ $\qquad$ $\dfrac{\partial F}{\partial x} = \dfrac{2y}{(2x+1)} \cdot \dfrac{(-1).2}{(2x+1)^2}\, y$

$\dfrac{\partial F}{\partial y}\,dx = \dfrac{\partial F}{\partial x}\,dy \rightarrow \dfrac{2y}{(2x+1)^2}\,dx = \dfrac{-4y^2}{(2x+1)^3}\,dy$ $\qquad$ assuming $2x+1 \neq 0$. & $y \neq 0$

$\Downarrow$

$-y^2 = x^2+x+k \Longleftarrow (2x+1)\,dx = -2y\,dy$

$y^2 = -\left[x^2+x-k\right]$ is a parabola of the same family if $k=-1$. it will Fit $y^2 = 4Cx+4Cx^2+C$ when $C = -1/4$

$y^2+x^2+x = -k$ is the family of orthogonal curves.

$y=0 \Rightarrow C=0$. $\cancel{\Rightarrow}$

$\cancel{\Rightarrow}$ $2x+1=0 \Rightarrow x=-\tfrac12$ level set $y^2 = -2C+C+C = 0 \Rightarrow y=0$.