Problem sheet 1

1] i] $\displaystyle\int \frac{x+3}{x^2+5x+8}\,dx$

Method 1

$$\frac{d}{dx}(x^2+5x+8) = 2x+5$$

$$\int \frac{x+3}{x^2+5x+8}\,dx = \frac{1}{2}\int \frac{(2x+5)+1}{x^2+5x+8}\,dx$$

$$x^2+5x+8 = \left(x+\tfrac{5}{2}\right)^2 + \left(\frac{32-25}{4}\right)$$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(x^2+5x+8)}{x^2+5x+8}\,dx + \frac{1}{2}\int \frac{1}{(x+5/2)^2 + \left(\sqrt{7}/2\right)^2}\,dx$$

$$= \frac{1}{2}\ln(x^2+5x+8) + \frac{1}{2}\left(\frac{2}{\sqrt{7}}\right)\tan^{-1}\left(\frac{x+5/2}{\sqrt{7}/2}\right) + k$$

if $\dfrac{(x+5/2)}{(\sqrt{7}/2)} = \tan(\theta)$

$$dx = \left(\frac{\sqrt{7}}{2}\right)(1+\tan^2\theta)\,d\theta$$

$$\int \frac{1}{(x+5/2)^2+(\sqrt{7}/2)^2}\,dx = \int \left(\frac{2}{\sqrt{7}}\right)\cdot\left(\frac{1}{\tan^2\theta+1}\right)\cdot(1+\tan^2\theta)\,d\theta = \left(\frac{2}{\sqrt{7}}\right)\cdot\theta + k'$$

where k & k' are arbitrary constants.

Method 2 Decompose into partial fractions in complex variables.

$$\int \frac{x+3}{x^2+5x+8}\,dx = \int \frac{A}{\left[x-\left(\frac{-5-\sqrt{7}i}{2}\right)\right]} + \frac{B}{\left[x-\left(\frac{-5+\sqrt{7}i}{2}\right)\right]}\,dx$$

$$A\left[x-\left(\frac{-5+\sqrt{7}i}{2}\right)\right] + B\left(x-\left[\frac{-5-\sqrt{7}i}{2}\right]\right) = x+3$$

$$\therefore\ A = \lim_{x\to\left[\frac{-5-\sqrt{7}i}{2}\right]}\left(\frac{x+3}{x-\left[\frac{-5+\sqrt{7}i}{2}\right]}\right) = \frac{\frac{1-\sqrt{7}i}{2}}{\frac{-\sqrt{7}i}{2}} = 1+\frac{1}{\sqrt{7}}i$$

$$B = \lim_{x\to\left[\frac{-5+\sqrt{7}i}{2}\right]}\left(\frac{x+3}{x-\left[\frac{-5-\sqrt{7}i}{2}\right]}\right) = \frac{\frac{1+\sqrt{7}i}{2}}{\frac{\sqrt{7}i}{2}} = 1-\frac{1}{\sqrt{7}}i$$

notice these are conjugates.
[They will always be conjugates].

$$\therefore \int \frac{x+3}{x^2+5x+8}\,dx = \left(1+\frac{1}{\sqrt{7}}i\right)\left[\ln\left|x-\left(\frac{-5-\sqrt{7}i}{2}\right)\right|\right] + \left(1-\frac{1}{\sqrt{7}}i\right)\left[\ln\left|x-\left(\frac{-5+\sqrt{7}i}{2}\right)\right|\right] + k$$

where k is an arbitrary constant.

ii) $$\int \frac{5x+4}{x^2+4x+3}\,dx = \int \frac{+11/2}{(x+3)} - \frac{1/2}{(x+1)}\,dx = \frac{11}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + k$$

where k is an arbitrary constant.

2] $$\int x^4 \sin x\,dx = \int \mathrm{Im}\left(x^4 e^{ix}\right)dx = \mathrm{Im}\int \left(x^4 e^{ix}\right)dx$$

$$= \mathrm{Im}\left[x^4\frac{e^{ix}}{i} - 4x^3\frac{e^{ix}}{i^2} + \frac{12x^2 e^{ix}}{i^3} - \frac{24x e^{ix}}{i^4} + \frac{24e^{ix}}{i^5}\right] + k$$

$$= -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + k$$

where k is an arbitrary constant.

used the method below:

~~Use the ~~ Integration by parts

$$U.V = \int \frac{d}{dx}(U.V)\,dx = \int\left(\frac{dU}{dx}.V + U.\frac{dV}{dx}\right)dx \iff \int\left(V.\frac{dU}{dx}\right)dx = U.V - \int\left(U.\frac{dV}{dx}\right)dx$$

by repeated application [when functions are differentiable],

$$\int f(x)\,g(x)\,dx = f^{(-1)}(x)\,g(x) - f^{(-2)}(x)\,g^{(1)}(x) + f^{(-3)}(x)\,g^{(2)}(x) + \dots$$

$$= \sum_{k=0}^{\infty} f^{(-k+1)}(x)\,g^{k}(x)(-1)^k$$ where $f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$

& $f^{(-n)}(x)$ is the n^{th} anti-derivative

i.e.- $f^{(-1)}(x) = \int f(x)\,dx$

$f^{(0)}(x) = f(x)$

3] $I=\int \frac{x+5}{(x^2+x+1)(x+3)(x+1)^2}\,dx$

$$I = \int \frac{Ax+B}{(x^2+x+1)} + \frac{C}{(x+1)} + \frac{1/14}{(x+3)} + \frac{2}{(x+1)^2}\,dx$$

$$(x+5) = (Ax+B)(x+3)(x+1)^2 + C(x^2+x+1)(x+3)(x+1) + \frac{1}{14}(x^2+x+1)(x+1)^2$$
$$+\ 2(x^2+x+1)(x+3)$$

$$x+5 = B(x^3+5x^2+7x+3) + A(x^4+5x^3+7x^2+3x) + C(x^4+5x^3+8x^2+7x+3)$$
$$+\ \frac{1}{14}\left[x^4+3x^3+4x^2+3x+1\right] + 2\left[x^3+4x^2+4x+3\right]$$

equate coefficients to find A, B, C. [Use equations wisely. 5 are possible] ⊛

x^4: $A+C+\frac{1}{14}=0$ — ①

x^3: $B+5A+5C+\frac{3}{14}+2=0$ ③

x^0: $3B+3C+\frac{1}{14}+6=5$ — ②

⇓

$B=-C-\frac{1}{3}\left[\frac{15}{14}\right]$

① ⇒ $A=\frac{-1}{14}-C$

$B=-C-\frac{5}{14}$ ⇐ ②

sub. ①&② in

③ ⇒ $\left(-C-\frac{5}{14}\right)+5\left(-\frac{1}{14}-\cancel{C}\right)+\cancel{5C}+\frac{31}{14}=0$

$C=+\frac{21}{14}=+\frac{3}{2}$

∴ ① ⇒ $A=-\frac{11}{7}$ ② ⇒ $B=\frac{-13}{7}$

$$\therefore I=\frac{1}{14}\ln(x+3)+\frac{3}{2}\ln(x+1)-\frac{2}{(x+1)}+\underbrace{\int\frac{Ax+B}{x^2+x+1}\,dx}_{J}+k_1$$

$$J=\frac{-1}{7}\left[\frac{11}{+2}\int\frac{2x+1}{x^2+x+1}\,dx+\frac{15}{2}\int\frac{1}{(x+\frac{1}{2})^2+(\frac{3}{4})}\,dx\right]$$

$$\therefore\ I = \frac{1}{14}\ln(x+3) + \frac{3}{2}\ln(x+1) - \frac{2}{(x+1)} - \frac{11}{14}\ln(x^2+x+1) - \frac{15}{14}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + k$$

k & k_1 are arbitrary constants.

4] i) $$\int \sin^4 x \cos^3 x\, dx = \int \sin^4 x\,(1-\sin^2 x)\, d\sin x$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + k$$ where k is an arbitrary constant

ii] $$\int \sin 5x \sin 7x\, dx = +\frac{1}{2}\int -\cos(12x) + \cos(2x)\, dx$$

$$= -\frac{\sin(12x)}{24} + \frac{\sin(2x)}{4} + k$$ where k is an arbitrary constant.

5] $$I = \int_0^\infty t^n e^{-st}\,dt = \left.\frac{t^n e^{-st}}{-s}\right|_{t=0}^{\infty} - \left.\frac{n t^{n-1} e^{-st}}{(-s)^2}\right|_{t=0}^{\infty} + \left.\frac{t^{n-2} e^{-st}}{[n(n-1)]^{-1}(-s)^3}\right|_{t=0}^{\infty} - \cdots - \left.\frac{e^{-st}}{[n!]^{-1} s^{n+1}}\right|_{t=0}^{\infty}$$

$$= -\left[\frac{t^n e^{-st}}{s} + \frac{t^{n-1}e^{-st}}{n^{-1}s^2} + \frac{t^{n-2}e^{-st}}{[n(n-1)]^{-1}(+s)^3} + \cdots + \frac{n!\, e^{-st}}{s^{n+1}}\right]_{t=0}^{\infty}$$

for $s > 0$;

$$\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty}\frac{t^n}{e^{st}} = \lim_{t\to\infty}\frac{n t^{n-1}}{s e^{st}}$$

$$= \lim_{t\to\infty}\frac{n!}{s^{n+1}e^{st}} = 0$$

by repeated application of L'Hospital's rule.

for $s < 0$;

$\lim_{t\to\infty} t^n e^{-st} \longrightarrow \infty$ This complicates things.

for $s = 0$; $$I = \int_0^\infty t^n e^{-st}\,dt = \int_0^\infty t^n\,dt = \left.\left(\frac{t^{n+1}}{n+1}\right)\right|_{t=0}^{\infty} \longrightarrow \infty$$

$\therefore$ for $s > 0$;

$$I = -\left.\frac{n!\, e^{-st}}{s^{n+1}}\right|_{t=0}^{\infty} = -0 + \frac{n!}{s^{n+1}}$$

6]

i) $\int \sec^5 x \tan^3 x \, dx = \cancel{\int \tan^3 x \sec^3 x \, d\tan x}$

$d\sec x = \tan x \sec x \, dx$

$d\tan x = \sec^2 x \, dx$

$= \int \sec^4 x \tan^2 x \, d\sec x = \int \sec^4 x (\sec^2 x - 1) \, d\sec x$

$= \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + k$ where k is an arbitrary constant.

ii] $\int e^{ax} \sin bx \, dx = \int \mathrm{Im}(e^{(a+bi)x}) \, dx = \mathrm{Im} \int e^{(a+bi)x} dx$

$= k + \mathrm{Im} \left[\frac{e^{(a+bi)x}}{(a+bi)} \frac{(a-bi)}{(a-bi)} \right] = \frac{1}{a^2+b^2} e^{ax} (a \sin(bx) - b \cos(bx)) + k$

where k is an arbitrary constant

7] $I = \int \frac{1}{(x^2 + 2\zeta w_n x + w_n^2)} dx$ $\zeta, w_n > 0$

i) $\zeta < 1$ $I = \int \frac{1}{(x + \zeta w_n)^2 + w_n^2(1 - \zeta^2)} dx$

$= \frac{1}{w_n \sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{x + \zeta w_n}{w_n \sqrt{1 - \zeta^2}} \right) + k$ where k is an arbitrary constant.

ii) $\zeta > 1$ $I = \int \frac{\beta}{(x - \alpha)} + \frac{\beta'}{(x - \alpha')} dx$

$\alpha = -\zeta w_n + w_n \sqrt{\zeta^2 - 1}$ $\beta = \frac{1}{\alpha - \alpha'} = 1/2 w_n \sqrt{\zeta^2 - 1}$

$\alpha' = -\zeta w_n - w_n \sqrt{\zeta^2 - 1}$ $\beta' = \frac{1}{\alpha' - \alpha} = -1/2 w_n \sqrt{\zeta^2 - 1}$

$I = \beta \ln(x - \alpha) + \beta' \ln(x - \alpha') + k$ where k is an arbitrary constant.

$= \frac{1}{2 w_n \sqrt{\zeta^2 - 1}} \ln\left|(x + \zeta w_n - w_n \sqrt{\zeta^2 - 1})\right| - \frac{1}{2 w_n \sqrt{\zeta^2 - 1}} \ln\left|x + \zeta w_n + w_n \sqrt{\zeta^2 - 1}\right| + k$

8)
i) $I = \int \frac{x^{a-1}}{x^a+b}\,dx = \frac{1}{a}\int \frac{\frac{d}{dx}(x^a+b)}{x^a+b}\,dx = \frac{1}{a}\ln\left|(x^a+b)\right| + k$

if $a \neq 0$

$x^a \neq -b$

if $a=0$, $I = \frac{1}{b}\int \frac{1}{x}\,dx = \frac{1}{b}\ln|x| + k_1$

k & k_1 are both arbitrary constants.

ii) $I = \int \cos^2 x \sin x\,dx = -\int \cos^2 x\, d\cos x = -\frac{\cos^3 x}{3} + k$ where k is an arbitrary constant,