AI1103 Challenging Problem 20

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https://github.com/Geetha495/AI1103/blob/main/Challenging_Problems/Challenging_Problem20/Challenging_Problem20.pdf

1 Question

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution

As $X_1, X_2, ...X_n$ are independently and identically distributed random variables,

$$E(X_1) = E(X_2) = ... = E(X_n)$$
$$Var(X_1) = Var(X_2) = ... = Var(X_n)$$

And from N(1,1), it is clear that for all $1 \leq i \leq n$

$$E(X_i) = 1$$
$$Var(X_i) = 1$$

So,

$$\begin{aligned} E(X_i^2) &= (Var(X_i) + (E(X_i))^2) \\ &= 1 + 1 = 2 \end{aligned}$$

Given,

$$\begin{aligned} S_n &= X_1^2 + X_2^2 + ... + X_n^2 \\ &= \sum_{i=1}^{n} X_i^2 \end{aligned}$$

$$\begin{aligned} Var(S_n) &= Var\left(\sum_{i=1}^{n} X_i^2\right) \\ &= \sum_{i=1}^{n} Var(X_i^2) \\ &= \sum_{i=1}^{n} \left(E(X_i^4) - E(X_i^2)^2\right) \\ &= \sum_{i=1}^{n} \left(E(X_i^4) - (2)^2\right) \end{aligned} \tag{2.0.1}$$

Definitions :

1) **Central moment :** For a random variable X, $E\left[(X - E(X))^r\right]$ is called r^{th} central moment and it is denoted by μ_r.
2) **Kurtosis :** It is the measure of tailedness of the probability distribution of a random variable X.

$$\text{Kurtosis} = \frac{\mu_4}{(Var(X))^2}$$

In a symmetric distribution, all odd central moments are equal to zero.

$$\begin{aligned} \mu_3 &= E\left[(X_i - E(X_i))^3\right] \\ &= E\left[(X_i - 1)^3\right] \\ &= E(X_i^3) - 3E(X_i^2)1 + 3(1^2)E(X_i) - 1^3 \\ &= E(X_i^3) - 3(2) + 3 - 1 \\ &= E(X_i^3) - 4 \end{aligned}$$

Equating μ_3 to 0, we get, $E(X_i^3) = 4$.
For a normal distribution,

$$\begin{aligned} \text{kurtosis} = 4 &= \frac{\mu_4}{1^2} \\ &= E\left[(X_i - E(X_i))^4\right] \\ &= E\left[(X_i - 1)^4\right] \\ &= E(X_i^4) - 4E(X_i^3)1 + 6E(X_i^2)1^2 - 4E(X_i)1^3 + 1^4 \\ &= E(X_i^4) - 4(4) + 6(2) - 4 + 1 \\ E(X_i^4) &= 10 \end{aligned} \tag{2.0.2}$$

From eq 2.0.1 and 2.0.2,

$$Var(S_n) = \sum_{i=1}^{n} (10 - 4)$$
$$= 6n$$
$$\lim_{n \to \infty} \frac{Var(S_n)}{n} = 6$$

Hence, option B is correct.