

# AI1103 Challenging Problem 20

Chitneedi Geetha Sowmya  
CS20BTECH11011

Download latex-tikz code from

[https://github.com/Geetha495/AI1103/blob/main/Challenging\\_Problems/Challenging\\_Problem20/Challenging\\_Problem20.tex](https://github.com/Geetha495/AI1103/blob/main/Challenging_Problems/Challenging_Problem20/Challenging_Problem20.tex)

## 1 QUESTION

Let  $X_1, X_2, \dots$  be i.i.d  $N(1,1)$  random variables. Let  $S_n = X_1^2 + X_2^2 + \dots + X_n^2$  for  $n \geq 1$ . Then

$$\lim_{n \rightarrow \infty} \frac{\text{Var}(S_n)}{n} =$$

- (A) 4
- (B) 6
- (C) 1
- (D) 0

## 2 SOLUTION

As  $X_1, X_2, \dots, X_n$  are independently and identically distributed random variables,

$$E(X_1) = E(X_2) = \dots = E(X_n)$$

$$\text{Var}(X_1) = \text{Var}(X_2) = \dots = \text{Var}(X_n)$$

And from  $N(1,1)$ , it is clear that for all  $1 \leq i \leq n$

$$E(X_i) = 1$$

$$\text{Var}(X_i) = 1$$

So,

$$\begin{aligned} E(X_i^2) &= (\text{Var}(X_i) + (E(X_i))^2) \\ &= 1 + 1 = 2 \end{aligned}$$

Given,

$$\begin{aligned} S_n &= X_1^2 + X_2^2 + \dots + X_n^2 \\ &= \sum_{i=1}^n X_i^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(S_n) &= \text{Var}\left(\sum_{i=1}^n X_i^2\right) \\ &= \sum_{i=1}^n \text{Var}(X_i^2) \\ &= \sum_{i=1}^n (E(X_i^4) - E(X_i^2)^2) \\ &= \sum_{i=1}^n (E(X_i^4) - (2)^2) \end{aligned} \quad (2.0.1)$$

**Definitions :**

- 1) **Central moment :** For a random variable  $X$ ,  $E[(X - E(X))^r]$  is called  $r^{\text{th}}$  central moment and it is denoted by  $\mu_r$ .
- 2) **Kurtosis :** It is the measure of tailedness of the probability distribution of a random variable  $X$ .

$$\text{Kurtosis} = \frac{\mu_4}{(\text{Var}(X))^2}$$

In a symmetric distribution, all odd central moments are equal to zero.

$$\begin{aligned} \mu_3 &= E[(X_i - E(X_i))^3] \\ &= E[(X_i - 1)^3] \\ &= E(X_i^3) - 3E(X_i^2) + 3E(X_i) - 1^3 \\ &= E(X_i^3) - 3(2) + 3 - 1 \\ &= E(X_i^3) - 4 \end{aligned}$$

Equating  $\mu_3$  to 0, we get,  $E(X_i^3) = 4$ .  
For a normal distribution,

$$\begin{aligned} \text{kurtosis} &= 4 = \frac{\mu_4}{1^2} \\ &= E[(X_i - E(X_i))^4] \\ &= E[(X_i - 1)^4] \\ &= E(X_i^4) - 4E(X_i^3) + 6E(X_i^2) - 4E(X_i) + 1^4 \\ &= E(X_i^4) - 4(4) + 6(2) - 4 + 1 \\ E(X_i^4) &= 10 \end{aligned} \quad (2.0.2)$$

From eq 2.0.1 and 2.0.2,

$$\begin{aligned} \text{Var}(S_n) &= \sum_{i=1}^n (10 - 4) \\ &= 6n \\ \lim_{n \rightarrow \infty} \frac{\text{Var}(S_n)}{n} &= 6 \end{aligned}$$

Hence, option B is correct.