AI1103 Challenging Problem 20

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Download latex-tikz code from

https://github.com/Geetha495/AI1103/blob/main/Challenging_Problems/Challenging_Problem20/Challenging_Problem20.tex

1 Question

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var\,(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution

Definition 1 (Central moment)**.** *For a random variable X, $E\left[(X - E(X))^r\right]$ is called r^{th} central moment and it is denoted by μ_r.*

$$\mu_r = E\left((X - E(X))^r\right)$$
$$= \sum_{k=0}^{r}\left({}^4C_k \times E(X^k) \times (E(X))^{r-k}\right) \qquad (2.0.1)$$

Definition 2 (Kurtosis)**.** *It is the measure of tailedness of the probability distribution of a random variable X.*

$$Kurtosis = \frac{\mu_4}{(Var(X))^2}$$

As $X_1, X_2, ...X_n$ are independently and identically distributed random variables,

$$E(X_1) = E(X_2) = ... = E(X_n)$$
$$Var(X_1) = Var(X_2) = ... = Var(X_n)$$

And from N(1,1), it is clear that for all $1 \leq i \leq n$

$$E(X_i) = 1$$
$$Var(X_i) = 1$$

So,

$$E(X_i^2) = (Var(X_i) + (E(X_i))^2)$$
$$= 1 + 1 = 2$$

Given,

$$S_n = X_1^2 + X_2^2 + ... + X_n^2$$
$$= \sum_{i=1}^{n} X_i^2$$

$$Var(S_n) = Var\left(\sum_{i=1}^{n} X_i^2\right)$$
$$= \sum_{i=1}^{n} Var(X_i^2)$$
$$= \sum_{i=1}^{n}\left(E(X_i^4) - E(X_i^2)^2\right)$$
$$= \sum_{i=1}^{n}\left(E(X_i^4) - (2)^2\right) \qquad (2.0.2)$$

In a symmetric distribution, all odd central moments are equal to zero.

$$\mu_3 = \sum_{k=0}^{3}\left({}^4C_k \times E(X_i^k)\right) \qquad \text{(by eq. (2.0.1))}$$
$$= E(X_i^3) - 4$$

Equating μ_3 to 0, we get, $E(X_i^3) = 4$.
For a normal distribution,

$$\text{kurtosis} = 4 = \frac{\mu_4}{1^2}$$
$$= \sum_{k=0}^{4}\left({}^4C_k \times E(X_i^k)\right) \quad \text{(by eq. (2.0.1))}$$
$$E(X_i^4) = 10 \qquad (2.0.3)$$

From eq. (2.0.2) and eq. (2.0.1),

$$Var(S_n) = \sum_{i=1}^{n} (10 - 4)$$
$$= 6n$$
$$\lim_{n\to\infty} \frac{Var(S_n)}{n} = 6$$

Hence, option B is correct.