AI1103: Assignment 8

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Download all latex codes from

https://github.com/Geetha495/Assignment8/blob/main/Assignment8.tex

1 Problem

Let $\phi(t)$ be a characteristic function of some random variable. Then, which of the following is also a characteristic function ?

1) $f(t) = [\phi(t)]^2$ for all $t \in \mathbb{R}$
2) $f(t) = |\phi(t)|^2$ for all $t \in \mathbb{R}$
3) $f(t) = \phi(-t)$ for all $t \in \mathbb{R}$
4) $f(t) = \phi(t+1)$ for all $t \in \mathbb{R}$

2 Solution

Definition 2.1 (Characteristic Function)**.** The function $\phi_X(t) = E\left(e^{itX}\right)$ is called the characteristic function (cf) of random variable X.

Proposition 2.1 (Properties of a Characteristic function)**.** *All cf's have the following properties:*

1) $\phi_X(-t) = \overline{\phi_X(t)}$ *(complex conjugate)*
2) $\phi_{-X}(t) = \overline{\phi_X(t)}$

Proposition 2.2 (Cf of sum of independent r.v.'s)**.** *If X and Y are independent, then*

$$\phi_{X+Y}(t) = \phi_X(t) \times \phi_Y(t)$$

Let X be the given random variable and let Y and $-X$ have the same distribution.

1)

$$\begin{aligned}[\phi_X(t)]^2 &= \phi_X(t) \times \phi_X(t) \\ &= \phi_{2X}(t) \qquad \text{(by proposition 2.2)}\end{aligned}$$

Thus, $f(t) = [\phi(t)]^2$ is a characteristic function of random variable $2X$.

2)

$$\begin{aligned}|\phi_X(t)|^2 &= \phi_X(t) \times \overline{\phi_X(t)} \\ &= \phi_X(t) \times \phi_Y(t) \quad \text{(by proposition 2.1)} \\ &= \phi_{X+Y}(t)\end{aligned}$$

Thus, $f(t) = |\phi(t)|^2$ is a characteristic function of random variable $(X + Y)$.

3)

$$\begin{aligned}\phi_X(-t) &= E\left(e^{i(-t)X}\right) \qquad \text{(by definition 2.1)} \\ &= E\left(e^{it(-X)}\right) \\ &= E\left(e^{itY}\right) \\ &= \phi_Y(t)\end{aligned}$$

Thus, $f(t) = \phi(-t)$ is a characteristic function of random variable Y.

4)

$$\begin{aligned}\phi_X(t+1) &= E\left(e^{i(t+1)X}\right) \quad \text{(by definition 2.1)} \\ &= Ee^{itX} \times e^{iX}\end{aligned}$$

Thus, $f(t) = \phi(t+1)$ is a not a characteristic function.

Hence, correct options are 1, 2, 3.