AI1103 Challenging Problem 20

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Download latex-tikz code from

https://github.com/Geetha495/AI1103/blob/main/Challenging_Problems/Challenging_Problem20/Challenging_Problem20.tex

1 Question

Let $X_1, X_2, ...$ be i.i.d N(1,1) random variables.Let $S_n = X_1^2 + X_2^2 + ... + X_n^2$ for $n \geq 1$.Then

$$\lim_{n\to\infty} \frac{Var\,(S_n)}{n} =$$

(A) 4
(B) 6
(C) 1
(D) 0

2 Solution

Definition 1 (Central moment)**.** *For a random variable X, $E\left[(X - E(X))^r\right]$ is called r^{th} central moment and it is denoted by μ_r.*

$$\begin{aligned}\mu_r &= E\left[(X - E(X))^r\right] \\ &= \sum_{k=0}^{r} \left({}^4C_k \times E(X^k) \times (E(X))^k\right) \qquad (2.0.1)\end{aligned}$$

Definition 2 (Kurtosis)**.** *It is the measure of tailedness of the probability distribution of a random variable X.*

$$Kurtosis = \frac{\mu_4}{(Var(X))^2}$$

As $X_1, X_2, ...X_n$ are independently and identically distributed random variables,

$$\begin{aligned}E(X_1) &= E(X_2) = ... = E(X_n) \\ Var(X_1) &= Var(X_2) = ... = Var(X_n)\end{aligned}$$

And from N(1,1), it is clear that for all $1 \leq i \leq n$

$$\begin{aligned}E(X_i) &= 1 \\ Var(X_i) &= 1\end{aligned}$$

So,

$$\begin{aligned}E(X_i^2) &= (Var(X_i) + (E(X_i))^2) \\ &= 1 + 1 = 2\end{aligned}$$

Given,

$$\begin{aligned}S_n &= X_1^2 + X_2^2 + ... + X_n^2 \\ &= \sum_{i=1}^{n} X_i^2\end{aligned}$$

$$\begin{aligned}Var(S_n) &= Var\left(\sum_{i=1}^{n} X_i^2\right) \\ &= \sum_{i=1}^{n} Var(X_i^2) \\ &= \sum_{i=1}^{n} \left(E(X_i^4) - E(X_i^2)^2\right) \\ &= \sum_{i=1}^{n} \left(E(X_i^4) - (2)^2\right) \qquad (2.0.2)\end{aligned}$$

In a symmetric distribution, all odd central moments are equal to zero.

$$\begin{aligned}\mu_3 &= \sum_{k=0}^{3} \left({}^4C_k \times E(X_i^k)\right) \qquad \text{(by eq. (2.0.1))} \\ &= E(X_i^3) - 4\end{aligned}$$

Equating μ_3 to 0, we get, $E(X_i^3) = 4$.
For a normal distribution,

$$\begin{aligned}\text{kurtosis} = 4 &= \frac{\mu_4}{1^2} \\ &= \sum_{k=0}^{4} \left({}^4C_k \times E(X_i^k)\right) \quad \text{(by eq. (2.0.1))} \\ E(X_i^4) &= 10 \qquad (2.0.3)\end{aligned}$$

From eq. (2.0.2) and eq. (2.0.1),

$$\begin{aligned}
Var(S_n) &= \sum_{i=1}^{n} (10 - 4) \\
&= 6n
\end{aligned}$$

$$\lim_{n \to \infty} \frac{Var(S_n)}{n} = 6$$

Hence, option B is correct.