

# The Price of being Adaptive

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**Abstract** *Mutual exclusion* is a fundamental distributed coordination problem. Shared-memory mutual exclusion research focuses on *local-spin* algorithms and uses the *remote memory references* (RMRs) metric. To ensure the correctness of concurrent algorithms in general, and mutual exclusion algorithms in particular, it is often required to prohibit certain re-orderings of memory instructions that may compromise correctness, by inserting *memory fence* (a.k.a. *memory barrier*) instructions. Memory fences incur non-negligible overhead and may significantly increase time complexity.

A mutual exclusion algorithm is *adaptive to total contention* (or simply *adaptive*), if the time complexity of every passage (an entry to the critical section and the corresponding exit) is a function of *total contention*, that is, the number of processes,  $k$ , that participate in the execution in which that passage is performed. We say that an algorithm  $A$  is  $f$ -*adaptive* (and that  $f$  is an *adaptivity function* of  $A$ ), if the time complexity of every passage in  $A$  is  $O(f(k))$ . Adaptive implementations are desirable when contention is much smaller than the

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total number of processes,  $n$ , sharing the implementation.

Recent work [6] presented the first read/write mutual exclusion algorithm with asymptotically optimal complexity under both the RMRs and fences metrics: each passage through the critical section incurs  $O(\log n)$  RMRs and a constant number of fences. The algorithm works in the popular Total Store Ordering (TSO) model. The algorithm of [6] is non-adaptive, however, and the authors posed the question of whether there exists an adaptive mutual exclusion algorithm with the same complexities.

We provide a negative answer to this question, thus capturing an inherent cost of adaptivity. In fact, we prove a stronger result: adaptive read/write mutual exclusion algorithms with constant fence complexity do not exist, regardless of their RMR complexity. This result follows from a general tradeoff that we establish for such algorithms, between the fence complexity and the growth rate of adaptivity functions. Specifically, we prove that the fence complexity of any such algorithm with a linear (or sub-linear) adaptivity function is  $\Omega(\log \log n)$ . The tradeoff holds for implementations that may use compare-and-swap operations, in addition to reads and writes.

We show that our results apply also to obstruction-free implementations of well-known objects, such as counters, stacks and queues.

**Keywords** Mutual exclusion, shared-memory, lower bounds, total store ordering, time complexity, remote memory reference (RMR)

## 1 Introduction

In the *mutual exclusion* problem, a set of processes must coordinate their accesses to a *critical section* (CS) so that, at any point in time, at most a single process is inside the CS. Introduced by Dijkstra in 1965 [12], the mutual exclusion problem is a fundamental Distributed Computing problem and is still the focus of intense research [2, 27].

For more than 20 years, shared-memory mutual exclusion research has investigated the *remote memory references* (RMR) complexity of local-spin mutual exclusion algorithms; much of this work focuses on (deterministic) read/write mutual exclusion (e.g. [11, 18, 19, 21, 28]). Anderson and Yang were the first to present an  $n$ -process mutual exclusion algorithm, where every *passage* (an entry to the critical section and the corresponding exit) incurs  $O(\log n)$  RMRs. This was shown to be optimal [4, 13].

A mutual exclusion algorithm  $A$  is adaptive, if its RMR complexity is a function of the number of active processes. More formally, an algorithm is *f-adaptive to total contention* (henceforth simply *adaptive*), if the RMR complexity of every passage is  $O(f(k))$ , where  $k$  denotes *total contention*, that is, the number of processes that participate in the execution. It is *f-adaptive to interval contention* (respectively, *point contention*) if the RMR complexity of every passage  $\mathcal{P}$  is  $O(f(k))$ , where  $k$  is the number of processes that are active during  $\mathcal{P}$  (respectively, the maximum number of processes that are concurrently active at some point in time during  $\mathcal{P}$ ). We call  $f$  the *adaptivity function* of  $A$ . Adaptive algorithms are superior to non-adaptive ones when the number of active processes is typically significantly smaller than  $n$ , the total number of processes.

Mutual exclusion algorithms are almost always designed under the assumption that memory accesses are atomic, i.e. linearizable [16], or at least sequentially consistent [22]. In practice, however, modern compilers optimize code so as to issue certain instructions out of order, based on the memory model supported by the architecture.

The memory model dictates which operation pairs can be reordered [1, Figure 8]. For example, the widely-supported *total store ordering* (TSO) model [23] ensures that writes are not reordered, but it is possible to perform a read from address  $a$  before a write to address  $b \neq a$  that is earlier in program order is performed.

The TSO model is supported by several common architectures, including SPARC [23] and x86 [10].<sup>1</sup> It is weaker than sequential consistency, and hence, also weaker than linearizability.

To ensure the correctness of a concurrent algorithm, it is possible to prohibit the reordering of memory instructions, by inserting a *fence* (also called a *barrier*) instruction between them. The use of fences was shown to be unavoidable for read/write mutual exclusion algorithms [5].

Since memory fences incur significant overhead, the number of fence instructions incurred by each passage of an algorithm (henceforth called its *fence complexity*) is a significant contributor to its time complexity, alongside the algorithm's RMR complexity.

Recent work by Attiya, Hendler and Levy [6] presented the first TSO mutual exclusion algorithm that is optimal in terms of both its RMR and fence complexities: each passage incurs a logarithmic number of RMRs and a constant number of fences. Their algorithm is not adaptive, however, and they posed the question of whether an adaptive TSO mutual exclusion algorithm

with the same RMR and fence complexities exists. This is the question that we address in this work.

## Our Contributions

We provide a negative answer to the question posed by [6]. In fact, we prove a stronger result: read/write mutual exclusion algorithms with constant fence complexity cannot be adaptive to total (hence also to interval- or point-) contention. This impossibility result holds regardless of the RMR complexity of the algorithm.

Our result follows from a general tradeoff that we establish between the fence complexity and the growth rate of adaptivity functions. Specifically, we prove that the fence complexity of any read/write algorithm with a linear (or sub-linear) adaptivity function is  $\Omega(\log \log n)$ . Our results apply for both the cache-coherent (CC) and the distributed shared-memory (DSM) models.

Following [6, 15], our tradeoff applies also to algorithms that may use *comparison* primitives, such as *compare-and-swap* (CAS), in addition to reads and writes. We show that our results also hold for obstruction-free [17] implementations of well-known objects, such as counters, stacks and queues.

Our results establish a time complexity separation between adaptive and non-adaptive implementations, thus capturing an inherent cost incurred by adaptive algorithms in the TSO model.

The rest of this article is organized as follows. The model we use and required definitions are provided in Section 2. An overview of our proofs and results is presented in Section 3. Full and detailed proofs are presented in Section 4. Section 5 discusses additional objects such as stacks and queues. The paper is concluded with a short discussion in Section 6.

## 2 Model and Definitions

We assume the standard asynchronous shared memory model [16], in which a set of processes  $P$  communicate by applying operations to a set of shared variables  $V$ , each of which is assigned an initial value. We consider both the *cache-coherent* (CC) and the *distributed shared-memory* (DSM) computation models [2].

In the DSM model, each processor owns a segment of shared memory that can be locally accessed without traversing the processor-to-memory interconnect. Thus, every variable is permanently *local* to a single processor

<sup>1</sup> Owens, Sarkar and Sewell [24] prove Intel x86 is equivalent to Sparc TSO.

and *remote* to all others.<sup>2</sup> An access of a remote variable is a *remote memory reference* (RMR).

In the CC model, each processor maintains copies of shared variables inside its private cache, whose consistency is ensured by a coherence protocol. Our results apply to both the *write-through* and *write-back* [26] CC coherence protocols. Quoting from [14]: "In a write-through protocol, to read a variable  $v$  a process  $p$  must have a (valid) cached copy of  $v$ . If it does,  $p$  reads that copy without causing an RMR; otherwise,  $p$  causes an RMR that creates a cached copy of  $v$ . To write  $v$ ,  $p$  causes an RMR that invalidates (i.e., effectively deletes) all other cached copies of  $v$  and writes  $v$  to main memory. In a write-back protocol, each cached copy is held in either shared or exclusive mode. To read a variable  $v$ , a process  $p$  must hold a cached copy of  $v$  in either mode. If it does,  $p$  reads that copy without causing an RMR. Otherwise,  $p$  causes an RMR that: (a) eliminates any copy of  $v$  held in exclusive mode, typically by downgrading the status to shared and, if the exclusive copy was modified, writing  $v$  back to memory; and (b) creates a cached copy of  $v$  held in shared mode. To write  $v$ ,  $p$  must have a cached copy of  $v$  held in exclusive mode. If it does,  $p$  writes that copy without causing RMRs. Otherwise,  $p$  causes an RMR that: (a) invalidates all other cached copies of  $v$  and writes any modified copy held in exclusive mode back to memory; and (b) creates a cached copy of  $v$  held in exclusive mode."

Our model assumes that each variable is permanently local to *at most* a single process (and remote to all others) and thus applies to both DSM and CC systems. For variable  $v$ , we denote by  $owner(v)$  the process to which  $v$  is local. We write  $owner(v) = \perp$  if  $v$  is remote to all processes, which is always the case in the CC model. Notice that accessing a remote variable does not necessarily generate an RMR (depends on the model), but simply implies that the variable is not part of the process's private segment (if there is such).

An *event*  $e$  is a read or write operation by some  $p \in P$  issued to a variable  $v \in V$ . The event  $e$  includes the value read or written. We write  $e = read(v)$  ( $write(v)$ ) if  $e$  is a read (write) operation issued to variable  $v$ . Later we extend the definition of an event by defining new types of special events, that are used for modelling the mutual exclusion problem in the TSO model.

An *execution fragment* is a (finite or infinite) sequence of events. We use  $\langle \rangle$  to denote the empty execution fragment. An *execution* is an execution fragment that starts from the initial configuration, resulting

when processes apply operations to the implemented object as they execute their algorithm. If a process has not completed its operation, it has exactly one enabled event, which is the next event it will execute, as specified by the algorithm it is using. We consider finite execution fragments, unless otherwise specified. Let  $E$  and  $F$  be two execution fragments. The execution fragment  $EF$  denotes the concatenation of  $E$  and  $F$ . If  $E$  and  $EF$  are executions, we say that  $F$  is an extension of  $E$ . We say that  $F$  is a *sub-execution* of  $E$ , and write  $F \preceq E$ , if  $F$  is a (possibly non-contiguous) subsequence of  $E$ 's events. For a set of processes  $Y$ , we denote by  $E^{-Y}$  the execution fragment obtained from  $E$  by removing all the events issued by processes in  $Y$  and say that the processes of  $Y$  are *erased* from  $E$ . We denote by  $E \mid Y$  the execution fragment obtained from  $E$  by removing all the events issued by processes not in  $Y$  (i.e., only the events issued by processes in  $Y$  are retained). When  $Y = \{p\}$ , we use the notation  $E^{-p}$  and  $E \mid p$ .

**Fact 1.**

1.  $(E_1 E_2)^{-Y} = E_1^{-Y} E_2^{-Y}$
2.  $(E^{-Y})^{-Z} = E^{-Y \cup Z}$

## TOTAL STORE ORDERING (TSO)

We now present an operational model for the behavior of a shared-memory system with relaxed memory ordering, which is a simplified version of the model used by Park and Dill [25].

A set of  $n$  processes,  $p_1, \dots, p_n$ , each with its own abstract *write buffer*, execute read and write memory operations in the order specified by their algorithm, called *program order*. Write operations may be delayed and executed after read operations following them in program order. This is modeled by having write operations go to the write buffer rather than directly to shared memory.

A *configuration* describes the state of a system: It contains the local state of each process, including its location in its algorithm and the contents of its write buffer. It also contains the value of each shared variable. In the *initial configuration*, all processes are in their initial state and their write buffers are empty; all shared variables hold their initial values.

In each step, a scheduling adversary picks a process and then decides whether to let it execute another event according to its algorithm or to *commit* the first write operation in its write buffer (if any). In the latter case, the write is committed by changing the value of the respective shared variable to the parameter of the write (we say that the write *becomes visible*) and

<sup>2</sup> For simplicity and without loss of generality, we assume that each of the processes participating in the algorithms we consider runs on a unique processor.

removing the write operation from the buffer. We say that the write operation is *committed* at this step and the execution is extended by a *write commit event*.

What happens when a process  $p$  issues an event depends on the type of the event:

1. A *fence* event  $e$  forces the adversary to commit all the writes in  $p$ 's write buffer (if any) in the order they were issued. That is, whenever the adversary schedules  $p$ , it commits the next write from  $p$ 's write buffer, as long as the buffer is not empty. We say that process  $p$  *completes fence  $e$  in execution  $E$*  if all the writes that were in  $p$ 's write buffer when  $e$  was issued by  $p$  were committed in  $E$ .
2. A write operation is placed at the end of the write buffer. The write operation is *issued* at this event but is not yet made visible to other processes. It will only be made visible once the execution is extended by a corresponding commit event.
3. A read operation returns the value of the variable and the process changes its local state accordingly. If there is a write to this variable in the write buffer, the value is read from the last such write; otherwise, if there is a (valid) cached copy of the variable in the process's private cache, the value is read from that copy; otherwise, the value of the variable is read from shared memory. The read operation is *issued* at this event.

For simplicity, we split the fence instruction into two successive *fence events*: a *BeginFence* event, immediately followed (in program order) by an *EndFence* event. *BeginFence* initiates the execution of a fence as described above. *EndFence* signifies that the fence execution has finished, that is, the write buffer of the process that performed the fence is now empty. For execution  $E$  and process  $p$ , we say that  $p$  is *executing a fence after  $E$* , if the last fence event by  $p$  in  $E$  is *BeginFence*. Note that if  $p$  is executing a fence after  $E$ , then the only event  $p$  is allowed to execute is the next write in its write buffer, or *EndFence* if the buffer is empty. Hence, if  $p$  is executing a fence after  $E$ , we write  $\text{mode}(p, E) = \text{write}$ , otherwise we write  $\text{mode}(p, E) = \text{read}$ . We say that  $p$  *completed  $i$  fences in  $E$*  if  $p$  executed  $i$  *EndFence* events in  $E$ , that is  $p$  executed to completion  $i$  fences in  $E$ .

In our construction, we only consider executions in which, whenever the scheduler picks a process  $p$  for the next step, it will always let it execute another event rather than commit a write from its write buffer, as long as  $p$  is in between fences (i.e., not executing a fence). That is, the scheduler delays committing writes from the write buffer as long as possible. Hence, a process' mode indicates whether it is executing a fence (if the process is in write mode), in which case it may only

commit writes from its write buffer, or it is in between fences (if the process is in read mode), in which case all its writes are delayed and the only shared memory operations performed on its behalf are reads.

Let  $E$  be an execution fragment. We write  $p = \text{writer}(v, E)$ , and say that  $p$  is *visible* on  $v$  after  $E$ , if  $p$  is the last process to *commit* a write to  $v$  in  $E$ . We write  $\text{writer}(v, E) = \perp$  if there exists no such  $p$ . We say that an event  $e \in E$  by process  $p$  *accesses* a variable  $v$  if either 1)  $e$  commits a write to  $v$ , or 2)  $e$  is a read event to  $v$  that is performed when  $p$ 's write-buffer does not contain a copy of  $v$ . Thus, events that issue writes to the write-buffer or read from the write-buffer are not considered variable accesses. We say that process  $p$  *accesses* variable  $v$  in  $E$  if there is an event by  $p$  in  $E$  that accesses  $v$ . We denote by  $\text{Accessed}(v, E)$  the set of processes that accessed  $v$  in  $E$ .

A (read or write) event  $e \in E$ , executed by process  $p$ , is a *remote event* in  $E$  if it accesses a variable that is remote w.r.t.  $p$ , otherwise it is a *local event*. Notice that a remote event is not necessarily an RMR, as it might be that  $p$  has a valid copy of  $v$  in its cache. However, such an event has the potential of generating an RMR, and as such the proof will focus on such events. Whether or not an event is a remote access is determined based on the history of the process executing  $e$ , as stated below.

**Fact 2.** *Let  $E$  and  $F$  be two execution fragments and let  $p$  be a process such that  $E \mid p = F \mid p$ . Then for any event  $e \in E$  by  $p$ ,  $e$  is a remote event in  $E$  if and only if  $e$  is a remote event in  $F$ .*

We now capture the extent by which processes are aware of the participation of other processes in an execution. We do so by adapting a definition used for this purpose by [3].

**Definition 1** We say that  $p$  is *aware* of  $q$  after  $E$  if either  $p = q$  or if there is an event  $e \in E$  by  $p$  that reads a variable  $v$  such that one of the following holds:

1. the last process to commit write to  $v$  before  $e$  is  $q$ ;
2. the last process to commit write to  $v$  before  $e$  is  $r$ , and  $r$  is aware of  $q$  at the time it issued that write.

The *awareness-set* of  $p$  after  $E$ , denoted by  $\text{AW}(p, E)$ , is the set of processes that  $p$  is aware of after  $E$ .

Intuitively, a process  $p$  is aware of the participation of another process  $q$  in an execution if there is (either direct or indirect) information flow from  $q$  to  $p$  in that execution via shared memory. For simplicity and without loss of generality, we assume that different write events write different values. Notice that the awareness-set of a process can only be extended along an execution. Moreover, it follows from Definition 1 that

whenever a process  $p$  reads a variable  $v$  last written by some process  $q$ , all the processes that belonged to  $q$ 's awareness set when it issued this write to  $v$  are added to  $p$ 's awareness set.

### Mutual Exclusion Systems

Each process  $p$  has a private variable  $section_p$  that signifies which section in the mutual exclusion algorithm  $p$  is currently in.  $section_p$  is initially  $ncs$ , indicating that  $p$  is in the non-critical section. There are three *transition events* which each process  $p$  may execute:

1.  $Enter_p$  causes  $p$  to transit from its non-critical section to its entry section and sets  $section_p = entry$ . This event is enabled if and only if  $section_p = ncs$ .
2.  $CS_p$  causes  $p$  to transit from its entry section to its exit section and updates  $section_p = exit$ . (For notational simplicity and WLOG, we assume that the execution of the critical section is instantaneous.) This event is enabled only if  $section_p = entry$ .
3.  $Exit_p$  causes  $p$  to transit from its exit section to its non-critical section and updates  $section_p = ncs$ . This event is enabled only if  $section_p = exit$ .

For execution  $E$  and process  $p$ , we let  $status(p, E)$  denote the value of  $section_p$  after  $E$ . A mutual exclusion system is required to satisfy the following properties:

**Exclusion** For any execution  $E$ , if both  $CS_p$  and  $CS_q$  are extensions of  $E$ , then  $p = q$ .

**Progress** Given an execution  $E$ , let  $X = \{q \in P \mid status(q, E) \neq ncs\}$ . If  $X = \{p\}$ , then there exists a solo extension  $F$  by  $p$  such that  $EFExit_p$  is an execution.

The exclusion property prevents multiple critical-section events from being simultaneously enabled. If two events  $CS_p$  and  $CS_q$  are simultaneously enabled after an execution  $E$ , then mutual exclusion may be violated. The exclusion property states that such a situation does not arise. The progress property we use was defined in [4] and is called *weak obstruction-freedom*. It is implied by deadlock-freedom and obstruction-freedom [17], although it is strictly weaker than both. In particular, it permits livelock. This weaker progress condition is sufficient for our proofs.

Next, we define the notion of a *critical event* and explain the relation between a critical event and an RMR in different cache-coherence protocols.

**Definition 2** Let  $E = E_1eE_2$  be an execution fragment, where  $e$  is an event by process  $p$ . We say that  $e$  is a *critical event* in  $E$  if one of the following holds:

critical read:  $e$  is a remote read of  $v$  and this is the first remote read of  $v$  by  $p$  (i.e.,  $E_1$  does not contain a remote read of  $v$  by  $p$ ).

critical write:  $e$  commits a remote write to  $v$  and  $writer(v, E_1) \neq p$  (i.e.,  $e$  is the first remote write commit to  $v$  by  $p$  in  $E$ , or  $e$  overwrites a value committed to  $v$  by another process).

In the DSM model, each critical event generates an RMR since it accesses a remote variable. In the CC model with a write-through coherence protocol, write commits always generate an RMR. In the CC model with a write-back protocol, if  $writer(v, E_1) = q \neq p$  then a copy of  $v$  is stored in the local cache of  $q$ , thus  $p$  must invalidate or update the cached copy of  $v$ , generating an RMR. It follows that in both the write-through and write-back protocols, a critical write and a critical read that is the first access of  $v$  by  $p$  are both RMRs. A first write followed by a first read are two critical events, but the read does not necessarily generate a cache miss. Nevertheless, since the first write is always an RMR, at least half of all critical events are RMRs. Consequently, if  $A$  is  $f$ -adaptive then each process may encounter at most  $2f(k)$  critical events during a single passage, where  $k$  is total contention. We therefore assume in the following for simplicity that  $f(k)$  bounds the number of critical events incurred by a process during a single passage.

Observe that whether an event is considered critical depends on the particular execution that contains the event, and specifically on the process that executes the event and the prefix of the execution preceding the event. Consequently, when saying that an event is (or is not) critical, the execution containing the event must be specified.

We now define the notion of a *special* event. This is an extension to the notion of a critical event, used for capturing events of importance for our construction.

**Definition 3** Let  $E$  be an execution such that  $E$  can be written as  $E_1eE_2$ . We say that  $e$  is a *special event* in  $E$  if one of the following holds:

Critical event:  $e$  is a critical event in  $E$ .

Transition event:  $e$  is one of  $Enter_p$ ,  $CS_p$  or  $Exit_p$ .

Fence event:  $e$  is one of  $BeginFence$  or  $EndFence$ .

We say that two events  $e$  and  $f$  are *congruent*, and write  $e \sim f$ , if  $e$  and  $f$  are executed by the same process and either  $e = f$  or both apply the same operation to the same variable. Informally, two events are congruent if they either execute the same transition or fence event, or if both are reads or both are writes of the same variable  $v$  (although the values they both read or write may differ).

### 3 Proof Overview

We now provide a detailed overview of our proofs. This is then followed by the full proofs.

We fix an  $f$ -adaptive mutual exclusion system  $\mathcal{A}$ . Our goal is to construct an execution in which there is a process that executes “many” fences while attempting to gain access to the critical section. The number of fences will be a function of  $f$ . We first present the definition of an *invisible-set*, a key notion in the constructing of this execution.

Given an execution  $E$ , we define two sets of processes. *Active processes*, denoted by  $Act(E)$ , is the set of processes that start a passage in  $E$  and are yet to complete it. Informally, an active process is a process in its entry section, trying to enter its critical section. *Finished processes*, denoted by  $Fin(E)$ , is the set of processes that completed a passage in  $E$ .

**Definition 4** Let  $E$  be an execution and  $INV$  be a set of processes such that  $INV \subseteq Act(E)$ . We say that  $INV$  is an *invisible set* (IN-set), and we call a process in  $INV$  an invisible process, if the following conditions hold:

**IN1:**  $\forall p \in P : AW(p, E) \cap INV \subseteq \{p\}$

Informally, no process is aware of any invisible process other than itself.

**IN2:**  $\forall p \in INV : status(p, E) = entry.$

Informally, all invisible processes are in the entry section.

**IN3:**  $\forall Y \subseteq INV$ , and for any  $e \in E^{-Y}$ :  $e$  is a critical event in  $E^{-Y}$  if and only if  $e$  is a critical event in  $E$ .

Informally, erasing invisible processes does not affect the criticality of remaining events.

**IN4:** For event  $e \in E$  by process  $p$ , if  $p$  accesses a remote variable  $v$  in  $e$  then  $owner(v) \notin Act(E)$ .

Informally, if a process  $p$  accesses a remote variable  $v$  local to some process  $q$ , then  $q$  is not an active process.

**IN5:**  $\forall v \in V : \text{If } |Accessed(v, E) \cap Act(E)| > 1 \text{ then } writer(v, E) \notin INV.$

Informally, if variable  $v$  has been accessed by more than a single active process, then  $v$  was not last written by an invisible process.

IN1 ensures that no process is aware of any invisible process (other than itself). This property allows us to erase any invisible process, that is, to remove its events from the execution. IN2 ensures that all invisible processes are in their entry section, trying to gain access to the critical section. IN3 ensures that erasing invisible processes does not affect the number of critical events executed so far by processes that remain active. IN4 ensures that no process can become aware of an invisible

process by reading a variable local to it. Let  $p$  be an invisible process that is visible on some variable  $v$ ; IN5 ensures that if we need to erase  $p$  from the execution, no other invisible process becomes visible on  $v$ . Note that any subset of an IN-set is itself an IN-set.

Our proofs mostly consider “regular” executions. Informally, these are executions where all active processes are invisible and in their entry sections. As we soon explain, sometimes we relax this requirement and permit “semi-regular” executions.

**Definition 5** An execution  $E$  is *regular* if  $Act(E)$  is an IN-set of  $E$ .

If  $Act(E)$  satisfies properties IN1-IN4 in  $E$ , we say that  $E$  is a *semi-regular* execution.

Our construction starts with an execution  $H_0$  where every process  $p$  executes the  $Enter_p$  event only. We then inductively construct longer and longer executions  $H_i$ , for  $i > 0$ . In execution  $H_i$ , exactly  $i$  processes complete a passage through the CS and all active processes complete exactly  $i$  fences and issue exactly  $l_i$  critical events, for some  $l_i \leq f(i)$ . Our goal is to extend the execution so that as many processes as possible perform an additional fence.

The TSO model allows to delay the execution of writes until a fence is performed, and these writes may be preceded by reads that follow them in program order. This makes it possible to construct executions in which reads always precede writes in-between fences. In turn, this execution structure allows us to restrict the knowledge gained by processes in-between fences and to retain a sufficiently large IN-set. Technically, the inductive construction of execution  $H_{i+1}$  from  $H_i$  is composed of a *read phase*, a *write phase*, and a *regularization phase* (see Figure 1).

*Read phase:* In the read phase, we iteratively extend the execution by allowing active processes to preform additional critical reads. Starting with regular execution  $G_0 = H_i$ , we construct executions  $G_1, G_2, \dots, G_s$ . For  $k > 0$ ,  $G_k$  is an extension of  $G_{k-1}$  in which all active processes run until they are about to execute a critical read (Lemma 5 establishes that such an extension exists) and then these reads are interleaved. Each such extension may require erasing a constant fraction of the active processes in order to eliminate information flow, such that the resulting execution is regular (see Claim 4.1.2). We prove that executions  $G_k$ , for  $k \in \{0, \dots, s\}$ , satisfy the following conditions (see Lemma 7):

- (1)  $G_k$  is a regular execution;
- (2) Each  $p \in Act(G_k)$  executes  $l_i + k$  critical events in  $G_k$ ;



Fig. 1: Structure of inductive construction. Gray-colored lines show events executed by erased processes.

- (3) Each  $p \in \text{Act}(G_k)$  completes  $i$  fences and does not yet issue its  $(i + 1)$ 'th fence event in  $G_k$ ;
- (4)  $\text{Fin}(G_k) = \text{Fin}(H_i)$ ;
- (5)  $|\text{Act}(G_k)| \geq (|\text{Act}(G_{k-1})| - 1)/10$ .

A key point in the proofs is to bound from above the number of iterations,  $s$ , required before all remaining active processes are about to issue their next fence event. Informally, this is done by using the following argument. Active processes are unaware of each other (since they belong to an IN-set) and may only become aware of finished processes. Consequently, the number of critical reads each of them may execute is bounded from above by a function of  $i = |\text{Fin}(G_k)| = |\text{Fin}(H_i)|$  (an explicit bound is given in Claim 4.1). This follows from the fact that processes are allowed at most  $f(i)$  critical events, since the algorithm is  $f$ -adaptive. Therefore, if a sufficient number of processes start the read phase, eventually a large subset of processes cannot execute additional reads and must commit their writes by executing a fence. The resulting execution is denoted by  $J_0$ , whereupon the read phase ends and a write phase begins.

*Write phase:* The write phase determines the order in which writes, issued by active processes since their previous fence was completed (or since they began their execution if this is the first fence), are committed. We iteratively extend the execution by allowing active processes to preform additional critical writes.

Starting with  $J_0$ , we iteratively construct executions  $J_1, J_2, \dots, J_t$ . We prove that each of the executions  $J_0, \dots, J_t$  satisfies the following conditions (see Lemma ??):

- (1)  $J_k$  is a semi-regular execution, in which multiple writes by active processes to the same variable (if any) are ordered in increasing order of process ID;
- (2) Each  $p \in \text{Act}(J_k)$  executes  $l_i + s + k$  critical events in  $J_k$ ;
- (3) Each  $p \in \text{Act}(J_k)$  completes  $i$  fences and did not yet complete its  $(i + 1)$ 'th fence in  $J_k$ ;
- (4)  $\text{Fin}(J_k) = \text{Fin}(H_i)$ ;
- (5)  $|\text{Act}(J_k)| \geq \sqrt{|\text{Act}(J_{k-1})|}/4(l_i + s + k)$ .

For  $k > 0$ ,  $J_k$  is an extension of  $J_{k-1}$  in which we let each process run until it is about to perform another critical write. We consider two cases according to where processes are about to write to. In the *low-contention case*, at least a square root fraction of the active processes are about to commit writes to different variables. In this case we retain a single process per such variable  $v$  (erasing all other processes accessing  $v$ ) and eliminate future information flow by erasing a fraction of the retained processes. The size of this fraction is a function of the number of critical events each process executed so far. In the *high-contention case*, there is a variable  $v$  such that at least a square root fraction of the processes are about to commit their write to  $v$ . In this case we erase the rest of the processes and then allow these processes to commit their writes to  $v$  in an increasing order of their IDs.

Our construction of write phases is similar to a construction by Kim and Anderson [21], but unlike it, we consider fence complexity in addition to RMR complexity. Moreover, in our construction unlike in [21], writes committed during the same write phase are scheduled such that the process with the highest ID is visible on *all* of the phase' high-contention variables, if any; this is guaranteed by the second part of Condition (1) above and is essential for obtaining our tradeoff.

Technically, this requires that some intermediate executions constructed during the write phase are allowed to be semi-regular but not regular: they violate invariant IN5 of Definition 5, since the last writer of high-contention variables is only allowed to finish its passage at the end of the phase. Ensuring the regularity of these intermediate executions would have required a large subset of processes to finish their passage (one per every high-contention write), which would weaken our complexity tradeoff.

Processes do not gain new information in the course of a write phase, since they only commit writes. Using the same argumentation as for the read phase, if a sufficient number of active processes start the phase, eventually a sufficiently large subset of these processes complete another fence (resulting in execution  $L_0$ ) and the write phase terminates (an explicit bound is given in Claim 4.2).

*Regularization phase:* The regularization phase transforms the semi-regular (and possibly not regular) execution constructed by the write phase,  $L_0$ , into a regular execution. This is done by letting the active process with the largest ID, denoted  $p_{max}$ , finish its passage. Since  $p_{max}$  is visible on all the high-contention variables of the write-phase (if any),  $Act(L_0) \setminus p_{max}$  is an IN-set.

Starting with  $L_0$ , we construct executions  $L_1, L_2, \dots, L_m, H_{i+1}$ . We prove that each of the executions  $L_0, \dots, L_m$  satisfies the following conditions (see Lemma ??):

- (1)  $Act(L_k)$  can be written as  $W_k \cup \{p_{max}\}$  (where  $p_{max} \notin W_k$ );
- (2)  $W_k$  is an IN-set of  $L_k$ ;
- (3)  $p_{max}$  executed  $l_i + s + t + k$  critical events in  $L_k$ ;
- (4) Each  $p \in W_k$  executed  $l_i + s + t$  critical events in  $L_k$ ;
- (5) Each  $p \in W_k$  completed  $i + 1$  fences in  $L_k$  and did not yet issue its  $(i + 2)$ 'nd fence event;
- (6)  $Fin(L_k) = Fin(H_i)$ ;
- (7)  $|Act(L_k)| \geq |Act(L_{k-1})| - 1$ .

For  $k \in \{1, \dots, m\}$ , we construct  $L_k$  from  $L_{k-1}$  by letting  $p_{max}$  run until it either terminates or until it is about to execute a critical event  $e$ . In the latter case, in order to prevent information flow, we may need to erase the active process that owns the remote variable accessed by  $e$  or that is the last to have written to it (by Claim ??, there is at most a single such process).

All of the active processes but  $p_{max}$  form an IN-set of the resulting execution (Claim ??), thus  $p_{max}$  is not aware of any other active process in executions  $L_k$ , for  $0 \leq k \leq m$ . Consequently, the number of critical events

$p_{max}$  may execute is a function of  $i$ , thus the number of intermediate executions constructed in the course of the regularization phase,  $m$ , is bounded from above, and eventually  $p_{max}$  finishes its passage (an explicit bound is given in Claim ??). The resulting execution, denoted  $H_{i+1}$ , is regular, and each active process finished  $i + 1$  fences. This completes the inductive step of our construction. We present the full proofs in Section 4.

### 3.1 Results

In Section 4, we prove the following theorem:

**Theorem 1** *Let  $\mathcal{A}$  be an  $N$ -process weak obstruction-free  $f$ -adaptive implementation of a mutual-exclusion lock and let  $i \in \mathbb{N}$  be such that  $f(i) \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}$ . Then there exists an execution  $H$  whose total contention is  $i + 1$  and a process  $p$  such that  $p$  executes  $i$  fences in  $H$  during a single passage of its CS.*

We then show (in Section 5) that a weak obstruction-free mutual exclusion lock can be easily implemented from a weak obstruction-free implementation of a counter, a stack or a queue. Moreover, the implementation is such that any passage through the CS invokes a single operation on the respective object (*fetch&increment*, *dequeue* or *pop*) and has the same asymptotic RMR and fence complexities (see Lemma 9). It follows that Theorem 1 holds for stacks and queues as well.

**Corollary 1** *There exists no weak obstruction-free implementation of an adaptive mutual exclusion lock, counter, stack or queue with  $O(1)$  fence complexity.*

*Proof.* Assume towards a contradiction that there exists such an  $f$ -adaptive algorithm  $\mathcal{A}$ , for some function  $f$ , such that no process executes  $c$  or more fences during a single passage/operation, for some constant  $c$ . We

choose large enough  $N$  such that  $f(c) \leq \frac{N^{2^{-f(c)}}}{f(c)! \cdot 4^{f(c)+2c}}$ .

By Theorem 1, there exists an execution  $H$  and a process  $p$  such that  $p$  executes  $c$  fences in  $H$  during a single passage/operation, contradicting our assumption.  $\square$

Kim and Anderson prove that a sub-linear adaptivity function is impossible [21]. We now present a lower bound on the fence complexity of the family of algorithms whose adaptivity function is linear.

**Corollary 2** *Let  $\mathcal{A}$  be an  $N$ -process  $f$ -adaptive implementation of a mutual-exclusion lock, counter, stack or queue, such that  $f$  is a linear function, that is  $f(i) = c \cdot i$  for some constant  $c$ . Then the fence complexity of  $\mathcal{A}$  is  $\Omega(\log \log N)$ .*



*Proof.* By Theorem 1, it suffices to prove that for  $i = \Omega(\log \log N)$  the inequality  $f(i) \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}$  holds, thus there exists an execution  $E$  and a process  $p$  that executes  $i = \Omega(\log \log N)$  fences during a single passage/operation in  $E$ .

$$\begin{aligned} c \cdot i \cdot (c \cdot i)! \cdot 4^{c \cdot i + 2i} &\leq N^{2^{-c \cdot i}} \\ \log(c \cdot i \cdot (c \cdot i)! \cdot 4^{c \cdot i + 2i}) &\leq 2^{-c \cdot i} \cdot \log N \\ \log \log(c \cdot i \cdot (c \cdot i)! \cdot 4^{c \cdot i + 2i}) &\leq -c \cdot i + \log \log N \\ \log \log(c \cdot i \cdot (c \cdot i)! \cdot 4^{c \cdot i + 2i}) + c \cdot i &\leq \log \log N. \end{aligned}$$

The right-hand side of the above inequality can be bounded from above as follows:

$$\begin{aligned} \log \log(c \cdot i \cdot (c \cdot i)! \cdot 4^{c \cdot i + 2i}) + c \cdot i &\leq \\ \log \log((c \cdot i)^{2^{c \cdot i}}) + c \cdot i &= \\ \log(2 \cdot c \cdot i) + \log \log(c \cdot i) + c \cdot i &\leq 3 \cdot c \cdot i. \end{aligned}$$

It follows that the inequality holds for  $i = \frac{1}{3c} \log \log N = \Omega(\log \log N)$  and the claim follows.  $\square$

A similar computation is used to prove the following lower bound.

**Corollary 3** *Let  $\mathcal{A}$  be an  $N$ -process  $f$ -adaptive implementation of a mutual-exclusion lock, counter, stack or queue, such that  $f$  is an exponential function, that is  $f(i) = 2^{c \cdot i}$  for some constant  $c$ . Then the fence complexity of  $\mathcal{A}$  is  $\Omega(\log \log \log N)$ .*

*Proof.* By Theorem 1, it suffices to prove that, for some  $i = \Omega(\log \log \log N)$ , the inequality  $f(i) \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}$  holds, thus there exists an execution  $E$  and a process  $p$  such that  $p$  executes  $i = \Omega(\log \log \log N)$  fences during a single passage/operation in  $E$ .

$$\begin{aligned} 2^{c \cdot i} \cdot 2^{c \cdot i}! \cdot 4^{2^{c \cdot i} + 2i} &\leq N^{2^{-2^{c \cdot i}}} \\ \log(2^{c \cdot i} \cdot 2^{c \cdot i}! \cdot 4^{2^{c \cdot i} + 2i}) &\leq 2^{-2^{c \cdot i}} \cdot \log N \\ \log \log(2^{c \cdot i} \cdot 2^{c \cdot i}! \cdot 4^{2^{c \cdot i} + 2i}) &\leq -2^{c \cdot i} + \log \log N \\ \log \log(2^{c \cdot i} \cdot 2^{c \cdot i}! \cdot 4^{2^{c \cdot i} + 2i}) + 2^{c \cdot i} &\leq \log \log N. \end{aligned}$$

The right-hand side of the above inequality can be bounded from above as follows:

$$\begin{aligned} \log \log(2^{c \cdot i} \cdot 2^{c \cdot i}! \cdot 4^{2^{c \cdot i} + 2i}) + 2^{c \cdot i} &\leq \\ \log \log((2^{c \cdot i})^{2^{2^{c \cdot i}}}) + 2^{c \cdot i} &= \\ c \cdot i + 1 + \log(c \cdot i) + 2^{c \cdot i} &\leq 2^{c \cdot i + 1}. \end{aligned}$$

It follows that the inequality holds for  $i = \frac{1}{c}(\log \log \log N - 1) = \Omega(\log \log \log N)$  and the claim follows.  $\square$

## 4 Full Lower Bound Proofs

We start by stating a few lemmas and claims that are required for arguing about the properties of our construction, which we specify in a formal manner later. The proofs are technical and appear in the appendix.

**Claim 1.** *Let  $E$  be an execution fragment and  $e \in E$  be an event issued by some process  $p$ .*

- Assume  $e$  is not critical in  $E$ , and let  $F$  be an execution fragment such that  $F \preceq E$  and  $F \mid p = E \mid p$ . Then  $e$  is not critical in  $F$ .
- Assume  $e$  is critical in  $E$ , and let  $F$  be an execution fragment such that  $E \preceq F$  and  $F \mid p = E \mid p$ . Then  $e$  is critical in  $F$ .

**Lemma 1** *Let  $E$  be an execution and let  $p \in P$  be a process such that  $p \notin AW(q, E)$  for any  $q \neq p$ . Then  $E^{-p}$  is an execution.*

**Lemma 2** *Let  $E$  be an execution and let  $INV$  be an IN-set of  $E$ . Let  $e$  be a read( $v$ ) or write( $v$ ) event. Assume  $writer(v, E) \notin INV$  and  $owner(v) \notin Act(E)$ . Then  $INV$  satisfies IN1-IN4 of Definition 4 in  $Ee$ .*

*Claim* Let  $E$  be an execution and let  $INV$  be an IN-set of  $E$ . Let  $e$  be an extension of  $E$  by some process  $p$  such that  $e$  is a local event in  $Ee$ . Then  $INV$  is an IN-set of  $Ee$ .

**Lemma 3** *Let  $E$  be an execution and let  $INV$  be an IN-set of  $E$ . Let  $F$  be an extension of  $E$  such that  $F$  contains no critical or transition event in  $EF$ . Then  $INV$  is an IN-set of  $EF$ .*

**Lemma 4** *Let  $E$  be an execution,  $INV$  be an IN-set of  $E$  and  $Y \subseteq INV$ .*

*Define  $E' = E^{-Y}$ . Then the following hold:*

1.  $E'$  is an execution;
2.  $Act(E') = Act(E) \setminus Y$  and  $Fin(E') = Fin(E)$ ;
3.  $INV \setminus Y$  is an IN-set of  $E'$ ;
4. Each  $p \in Act(E')$  executes the same critical events in  $E'$  and in  $E$ ;
5. If  $p \in Act(E')$  is about to execute a special event  $f_p$  after  $E$ , then  $p$  is about to execute a special event  $e_p \sim f_p$  after  $E'$ .

**Lemma 5** *Let  $E$  be a regular execution. Then there exists an extension  $F$  such that the following hold:*

- $F$  contains no special events in  $EF$ ;
- $EF$  is a regular execution;
- Each  $p \in Act(E)$  is about to execute a special event  $f_p$  after  $EF$ . Moreover, at most one process  $p \in Act(E)$  is about to execute  $f_p = CS_p$  after  $EF$ .

The proof of the following theorem appears in [9].

**Theorem 2 (Turán)** *Let  $\mathcal{G} = (V, E)$  be an undirected graph, with vertex set  $V$  and edge set  $E$ . If the average degree of  $\mathcal{G}$  is  $d$ , then an independent set exists with at least  $\lceil |V|/(d+1) \rceil$  vertices.*

We now prove a tradeoff between the fence complexity and the adaptivity function  $f$ . We start with the regular execution  $H_0$  in which each process  $p$  have executed the  $Enter_p$  event only, hence  $Act(H_0) = P$  and  $Fin(H_0) = \emptyset$ . We then build longer executions  $H_1, H_2, \dots$  inductively. At each induction step, we construct  $H_{i+1}$  from  $H_i$  using three phases: read, write, and regularization.

Every induction step starts with an execution  $H_i$  that meets the following conditions:

- (a)  $H_i$  is a regular execution;
- (b) Each  $p \in Act(H_i)$  executes  $\ell_i$  critical events in  $H_i$ , for some  $\ell_i \leq f(i)$ ;
- (c)  $|Fin(H_i)| = i$ ;
- (d) Each  $p \in Act(H_i)$  completes  $i$  fences in  $H_i$  and  $mode(p, H_i) = read$ .

First notice that  $H_0$  satisfies all the above conditions with  $\ell_0 = 0$ . Assume we already constructed  $H_i$  satisfying the conditions above. For simplicity, we slightly abuse notation and denote  $\ell = \ell_i$  and  $H = H_i$  through the rest of this section.

#### 4.1 Read phase

Consider  $p \in Act(H)$ .  $H$  is a regular execution, thus by Lemma 4 with ' $E' \leftarrow H$ ' and ' $Y' \leftarrow Act(H) \setminus \{p\}$ ' we have an execution  $H_p = H^{-Act(H) \setminus \{p\}}$ . That is,  $H_p$  is the execution  $H$  where we erase all active processes but  $p$ . It follows that  $Act(H_p) = \{p\}$ . By the progress property there is a solo extension of  $H_p$  by  $p$  until it finish its passage. Since  $mode(p, H_p) = read$  and  $status(p, H_p) = enter$  it follows that  $p$  performs this solo run until it either start a fence, or it finish the entry section. Let  $\alpha_p$  be this prefix, that is, a solo extension of  $H_p$  by  $p$  until its next event  $f_p$  is either  $BeginFence$  or  $CS_p$ . By its definition,  $p$  only allowed to read from the shared memory along  $\alpha_p$ , as all writes are stored in the write buffer, and the scheduler delay their commits to the next fence instruction.

We would like to extend  $H$  with the different  $\alpha_p$  in some order. However, it might be that a solo run of  $p$  starting from  $H$  is different then  $\alpha_p$  - in a solo starting from  $H$   $p$  may read a variable last written to by a process in  $Act(H) \setminus \{p\}$ , and therefore perform a different run. In order to overcome this problem, the read phase will construct a set of processes  $P'$  out of  $Act(H)$  such that processes in this set do not collide, that is, no process in  $P'$  reads a variable in  $\alpha_p$  such that a different process in  $P'$  is visible on this variable in  $H$ . This will allow us to erase all processes not in  $P'$  from  $H$  and then extend it with the runs  $\alpha_p$  of processes in  $P'$  in any order that we want, as these runs do not collide. We construct this set in an inductive manner.

In the read phase we have a sequence of sets  $Act(H) = P_0 \supseteq P_1 \supseteq \dots P_s$  satisfying the Lemma below.

**Lemma 6** *At each step of the read phase we have a set  $P_k \subseteq Act(H)$  such that each process  $p \in P_k$  has a prefix  $\beta_p$  of  $\alpha_p$  and the following holds:*

- (1) *Each  $p \in P_k$  executes exactly  $k$  critical reads along  $\beta_p$  in the execution  $H_p \beta_p$ ;*
- (2) *Let  $e \in \beta_p$  be a critical read in  $H_p \beta_p$  by some process  $p \in P_k$  to some variable  $v$ . Then:*
  - (a) *owner( $v$ )  $\notin P_k$ ;*
  - (b) *writer( $v, H$ )  $\notin P_k$ ;*
- (3)  *$|P_k| \geq |P_{k-1}|/10$ .*

First notice that  $P_0 = Act(H)$  satisfies all the above conditions with  $\beta_p = \langle \rangle$  for any  $p \in Act(H)$ . Assume we already constructed  $P_{k-1}$  satisfying the conditions of Lemma 6. For any  $p \in P_{k-1}$  we denote by  $\beta_p$  the prefix as promised by Lemma 6 throughout the rest of this section, in which we define the construction of the write phase, and prove Lemma 6.

Before going into the details of the construction, we first prove a claim which states that indeed, giving a set of processes  $P'$  such that their  $\alpha_p$  satisfies condition (2) of Lemma 6, then erasing processes not in  $P'$  from  $H$  and afterwards extend it using  $\alpha_p$  of processes in  $P'$  in any order results an execution. Giving a set  $P'$  we denote by  $\alpha(P')$  the execution fragment in which we concatenate the fragments  $\alpha_p$  of processes in  $P'$  in some order.

**Claim 4.1.1.** *Let  $P' \subseteq \text{Act}(H)$  be a set such that condition (2) of Lemma 6 holds for  $P'$  with  $\alpha_p$ . Then  $H - \text{Act}(H) \setminus P' \alpha(P')$  is an execution.*

#### 4.1.1 Construction: stage 1

For any  $p \in P_{k-1}$  we let  $p$  perform a solo run starting from  $H_p \beta_p$  until its next event  $e_p$  is a critical read, or that  $p$  has finish executing  $\alpha_p$  and its next event is  $f_p$ . Denote this extension by  $\gamma_p$ . Notice that  $\beta_p \gamma_p$  is indeed a prefix of  $\alpha_p$  by its definition. Let  $Z$  be the set of all processes that have not finished executing  $\alpha_p$ , that is,  $Z = \{p \in P_{k-1} \mid \beta_p \gamma_p \neq \alpha_p\}$ .

If  $|Z| < |P_{k-1}|/2$ , i.e., at least half of the processes in  $P_{k-1}$  have finished executing  $\alpha_p$ , then we define  $s = k - 1$  and  $Q = P_{k-1} \setminus Z$ , and we move to stage 3 of the read phase. By its definition, each process  $p \in Q$  executes  $s$  critical writes along  $\alpha_p$  in the execution  $H_p \alpha_p$ . Moreover, it is easy to verify that condition (2) of Lemma 6 holds for  $Q$  with  $\alpha_p$ , since it holds for  $Q \subseteq P_{k-1}$  with  $\beta_p$ , and  $\alpha_p = \beta_p \gamma_p$  where  $\gamma_p$  contains no critical reads.

Otherwise  $|Z| \geq |P_{k-1}|/2$  and we move to stage 2 of the read phase.

#### 4.1.2 Construction: stage 2

We have a set  $Z \subseteq P_{k-1}$  such that for each  $p \in Z$ , the next event by  $p$  after  $H_p \beta_p \gamma_p$  is a critical read  $e_p$  (it did not finish executing  $\alpha_p$ , and the only events in  $\alpha_p$  to shared memory are read). We construct an undirected graph  $\mathcal{G}$  as follows: the vertices of  $\mathcal{G}$  are the processes in  $Z$ . Consider  $p \in Z$  and denote  $e_p = \text{read}(v)$ . We add an edge  $\{p, q\}$  if there exists  $q \in Z$  such that either  $q = \text{owner}(v)$  or  $\text{writer}(v, H) = q$ .

Since  $H$  is a regular execution, if  $q = \text{owner}(v) \in Z \subseteq \text{Act}(H)$ , by IN3 it follows that no process other than  $q$  access  $v$  in  $H$ , and in particular either  $q = \text{writer}(v, H)$  or  $\text{writer}(v, H) = \perp$ . As a result, each  $p$  introduce at most one new edge to  $\mathcal{G}$ , and the average degree in  $\mathcal{G}$  is at most 2. By Theorem 2, there exists an independent set  $P_k \subseteq Z$  in  $\mathcal{G}$  such that:

$$|P_k| \geq \frac{|Z|}{3} \geq \frac{|P_{k-1}|/2}{3} = \frac{|P_{k-1}|}{6}$$

In the course of the read phase, we construct a sequence of executions  $H_i = G_0, G_1, G_2, \dots, G_s, J_0$ .

**Lemma 7** *At each step during the read phase we have an execution  $G_k$  satisfying the following conditions:*

- (1)  $G_k$  is a regular execution;
- (2) Each  $p \in \text{Act}(G_k)$  executes  $\ell + k$  critical events in  $G_k$ ;
- (3) Each  $p \in \text{Act}(G_k)$  completes  $i$  fences in  $G_k$  and  $\text{mode}(p, G_k) = \text{read}$ ;
- (4)  $\text{Fin}(G_k) = \text{Fin}(H_i)$ ;
- (5)  $|\text{Act}(G_k)| \geq (|\text{Act}(G_{k-1})| - 1)/10$ .

First notice that  $G_0 = H_i$  satisfies all the conditions in Lemma 7. Assume we already constructed  $G_{k-1}$  satisfying the conditions in Lemma 7. We let  $G = G_{k-1}$  and  $n = |\text{Act}(G_{k-1})|$  in the rest of this section (4.1), in which we specify the construction of the read phase and prove Lemma 7.

#### 4.1.3 Construction: stage 1

By Lemma 5, there exists an extension  $F$  of  $G$  such that the following hold:

1.  $F$  contains no special events in  $GF$ ;
2.  $GF$  is a regular execution;
3.  $F$  contains no transition events, therefore  $\text{Act}(GF) = \text{Act}(G)$  and  $\text{Fin}(GF) = \text{Fin}(G) = \text{Fin}(H_i)$ ;
4. Each  $p \in \text{Act}(G)$  executes  $\ell + k - 1$  critical events in  $GF$ ;
5.  $F$  contains no fence events, hence each  $p \in \text{Act}(G)$  completes  $i$  fences in  $GF$  and  $\text{mode}(p, GF) = \text{mode}(p, G) = \text{read}$ ;
6. Each  $p \in \text{Act}(G)$  is about to execute a special event  $f_p$  after  $GF$ . Moreover, at most a single process  $q \in \text{Act}(G)$  is about to execute  $f_q = CS_q$ .

Denote by  $Y$  the set of processes in  $\text{Act}(G)$  such that  $f_p \neq CS_p$ . We have  $n - 1 \leq |Y| \leq n$ . For each  $p \in Y$ , since  $\text{status}(p, GF) = \text{entry}$  and  $f_p \neq CS_p$  we get that  $f_p$  is not a transition event, and since  $\text{mode}(p, GF) = \text{read}$  we get that  $f_p$  is either a read event or a *BeginFence* event.

We define:  $Z_1 = \{p \in Y \mid f_p = \text{BeginFence}\}$ ,  $Z_2 = \{p \in Y \mid f_p \text{ is a read event}\}$ .

It follows that  $Y = Z_1 \cup Z_2$  and  $Z_1 \cap Z_2 = \emptyset$ , thus  $|Y| = |Z_1| + |Z_2|$ .

*Case I:  $|Z_1| > |Y|/2$*

We define  $W = Z_1$ . We have  $|W| > |Y|/2 \geq (n - 1)/2$ , thus  $|W| \geq n/2$ .

Case II:  $|Z_2| \geq |Y|/2$

We construct an undirected graph  $\mathcal{G}$  as follows: the vertices of  $\mathcal{G}$  are the processes in  $Z_2$ . Consider  $p \in Z_2$  and denote  $f_p = \text{read}(v)$ . We add an edge  $\{p, q\}$  if there exists  $q \in Z_2$  such that  $v$  is local to  $q$  or  $\text{writer}(v, GF) = q$ .

Since  $v$  is local to at most one process and has at most one last writer,  $p$  accounts for at most 2 edges in  $\mathcal{G}$ , thus the average degree in  $\mathcal{G}$  is at most 4. By Theorem 2, there exists an independent set  $W \subseteq Z_2$  in  $\mathcal{G}$  such that:

$$|W| \geq |Z_2|/5 \geq |Y|/10 \geq (n-1)/10$$

#### 4.1.4 Construction: stage 2

We have a set of processes  $W \subseteq \text{Act}(GF)$ . Define  $\overline{W} = \text{Act}(GF) \setminus W$ . By Lemma 4 with ' $E' \leftarrow GF$ ' and ' $Y' \leftarrow \overline{W}$ ', we have an execution  $N = (GF)^{-\overline{W}}$  such that the following hold:

1.  $W = \text{Act}(GF) \setminus \overline{W}$  is an IN-set of  $N$ ;
2.  $\text{Act}(N) = \text{Act}(GF) \setminus \overline{W} = W$ , and  $\text{Fin}(N) = \text{Fin}(GF) = \text{Fin}(H_i)$ ;
3. From 1 and 2:  $N$  is a regular execution;
4. Each  $p \in W$  executed the same critical events in  $N$  and in  $GF$ , thus  $p$  executed  $\ell + k - 1$  critical events in  $N$ ;
5. For each  $p \in W$ , since  $N \mid p = (GF) \mid p$  we get that  $p$  completed  $i$  fences in  $N$  and  $\text{mode}(p, N) = \text{mode}(p, GF) = \text{read}$ ;
6. Each  $p \in W$  is about to execute a special event  $e_p \sim f_p$  after  $N$ .

We extend  $N$  by letting each  $p \in W$  execute its next event in an arbitrary order. Denote this extension by  $D$ , and define  $G_k = ND$ . Notice that  $\text{Act}(G_k) = \text{Act}(N) = W$  and  $\text{Fin}(G_k) = \text{Fin}(N) = \text{Fin}(H_i)$  since  $D$  contains no transition events.

We now analyze the resulting execution,  $G_k$ , according to the cases defined in stage 1.

Case I

Define  $s = k - 1$  and  $J_0 = G_k$ . For each  $p \in W$  we have  $e_p \sim f_p = \text{BeginFence}$ . The following conditions hold.

1.  $D$  contains fence events only, thus by Lemma 3,  $W$  is an IN-set of  $J_0 = ND$ , i.e.  $J_0$  is a regular execution;
2. Each  $p \in \text{Act}(J_0)$  executed  $\ell + s$  critical events in  $N$  and thus in  $J_0$ ;
3. Each  $p \in \text{Act}(J)$  completed  $i$  fences in  $J_0$ , and the last event by  $p$  in  $J_0$  is  $\text{BeginFence}$ , i.e.  $\text{mode}(p, J_0) = \text{write}$ ;
4.  $\text{Fin}(J_0) = \text{Fin}(H_i)$ ;

5.  $|\text{Act}(J_0)| = |W| \geq |\text{Act}(G_s)|/2$ ;

We are done with the read phase, and we proceed to the write phase.

Case II

By the definition of  $W$ , each  $p \in W$  executed a single read event  $e_p$  in  $D$ .

**Claim 4.1.2.**  $G_k$  is a regular execution.

*Proof.* Consider  $p \in W$ , and denote  $e_p = \text{read}(v)$ .

property 1:  $\text{writer}(v, N) \notin W$ .

Denote  $q = \text{writer}(v, GF)$ . If  $q \notin \text{Act}(GF)$  then after removing the events by processes in  $\overline{W} \subseteq \text{Act}(GF)$  we still have  $\text{writer}(v, N) = \text{writer}(v, GF) = q$  and  $q \notin W \subseteq \text{Act}(GF)$ . Otherwise  $q \in \text{Act}(GF)$ .  $GF$  is a regular execution in which  $q$  accessed  $v$  and  $\text{writer}(v, GF) \in \text{Act}(GF)$ , thus by IN5  $q$  is the only process in  $\text{Act}(GF)$  to access  $v$ . Notice that  $q \notin W$  since there is an edge  $\{p, q\}$  in  $\mathcal{G}$ , and  $p \in W$ , an independent set of  $\mathcal{G}$ . Therefore  $q \in \overline{W}$ , and after removing events by processes in  $\overline{W}$  there is no process in  $\text{Act}(GF)$  (and thus in  $W$ ) to access  $v$  in  $N$ , that is  $\text{writer}(v, N) \notin W$ .

property 2:  $\text{owner}(v) \notin W$ :

Denote  $q_v = \text{owner}(v)$ . If  $q_v \notin Z_2$  the claim clearly hold. Otherwise  $q_v \in Z_2$  and  $f_p \sim e_p = \text{read}(v)$ , thus  $\mathcal{G}$  contains an edge  $\{p, q_v\}$  (notice that  $p \neq q_v$  since  $p$  remotely reads  $v$ ). Since  $p \in W$  and  $W$  is an independent set we have  $q_v \notin W$ .

Denote by  $D_j$  the prefix of  $D$  that contains exactly  $j$  events. We prove by induction on  $j$  ( $0 \leq j \leq |D|$ ) that  $W$  is an IN-set of  $ND_j$ :

induction base  $j = 0$ : by our construction  $W$  is an IN-set of  $N$ .

Assume we already proved the claim for  $j < |D|$ . Notice that  $ND_{j+1} = ND_j e_p$  for some  $p \in W$ , and denote  $e_p = \text{read}(v)$ . Since  $D_j$  contains no transition events, we have  $\text{Act}(ND_j) = \text{Act}(N) = W$ .  $D_j$  contains only read events, thus no write to  $v$  occurs in  $D_j$ , i.e.  $\text{writer}(v, ND_j) = \text{writer}(v, N) \notin W$ . Together with the fact that  $\text{owner}(v) \notin W$ , the conditions for Lemma 2 holds, and  $W$  satisfies IN1-IN4 in  $ND_j e_p = ND_{j+1}$ . As IN5 holds for  $W$  in  $ND_j$  it clearly holds for any variable  $u \neq v$  in  $ND_{j+1}$ . Since  $e_p$  is a read event to  $v$  we get  $\text{writer}(v, ND_{j+1}) = \text{writer}(v, ND_j) \notin W$  and IN5 holds for  $v$  in  $ND_{j+1}$ .

Using the last claim with  $j = |D|$  we have that  $W$  is an IN-set of  $ND = G_k$ , and thus  $G_k$  is a regular execution.  $\square$

We now prove that  $G_k$  satisfies all the conditions in Lemma 7:

- (1)  $G_k$  is a regular execution;

- (2) Consider  $p \in W = \text{Act}(G_k)$ .  $p$  executed  $\ell + k - 1$  critical events in  $N$  and a single event  $e_p$  in  $D$ . By our construction  $e_p$  is a critical event in  $Ne_p$ ,  $Ne_p \preceq G_k$ , and  $(Ne_p) \mid p = G_k \mid p$ . Therefore, by claim 1,  $e_p$  is a critical event in  $G_k$ . Altogether  $p$  executed  $\ell + k$  critical events in  $G_k$ ;
- (3) Consider  $p \in W$ .  $p$  completed  $i$  fences in  $N$  and  $\text{mode}(p, N) = \text{read}$ . Since  $D$  contains a single read event by  $p$  we get that  $p$  completed  $i$  fences in  $G_k$  and  $\text{mode}(p, G_k) = \text{read}$ ;
- (4)  $\text{Fin}(G_k) = \text{Fin}(N) = \text{Fin}(H_i)$ ;
- (5)  $|\text{Act}(G_k)| = |W| \geq (n - 1)/10$ .

**Claim 4.1.** *The number of steps in the read phase is bounded by  $f(i + 1) - \ell$ , that is,  $\ell + s \leq f(i + 1)$ .*

*Proof.* Assume towards a contradiction that during the read phase we build an execution  $G_k$  such that  $\ell + k > f(i + 1)$ . Then  $G_k$  satisfies:

- $G_k$  is a regular execution;
- Each  $p \in \text{Act}(G_k)$  executed  $\ell + k$  critical events in  $G_k$ ;
- $\text{Fin}(G_k) = \text{Fin}(H_i)$ , thus  $|\text{Fin}(G_k)| = i$ .

We choose an arbitrary  $p \in \text{Act}(G_k)$  and denote  $Y = \text{Act}(G_k) \setminus \{p\}$ . Using Lemma 4 with ' $E' \leftarrow G_k$ ' and ' $Y' \leftarrow Y$ ', we have an execution  $G'_k = G_k^{-Y}$  such that:  $\text{Act}(G'_k) = \{p\}$  and  $\text{Fin}(G'_k) = \text{Fin}(G_k) = \text{Fin}(H_i)$ ;  $p$  executes the same critical events in  $G_k$  and in  $G'_k$ , thus  $p$  executes  $\ell + k$  critical events in  $G'_k$ . Hence, at most  $i + 1$  processes issue events in  $G'_k$ , i.e. the total contention of  $G'_k$  is at most  $i + 1$ . However,  $p$  executes  $\ell + k > f(i + 1)$  critical events during a single passage in  $G'_k$ , a contradiction to our assumption that the algorithm is  $f$ -adaptive.  $\square$

## 4.2 Write phase

The read phase construct an execution  $J$  such that the last event by each process in  $\text{Act}(J)$  is  $\text{BeginFence}$ . The write phase will determine the order in which processes are executing their write commits along the fence. Each process  $p \in \text{Act}(J)$  have a list  $\alpha_p$  of writes in its write buffer, which it will commit starting from  $J$ .

We first focus on the list  $\alpha_p$ . Notice that  $\alpha_p$  is also a solo run of  $p$  starting from  $J$  until the point where it finish executing its fence. Along the write phase, we first construct a set of processes out of  $\text{Act}(J)$ , such that the lists  $\alpha_p$  of those processes satisfies certain properties. Then, we will use this set in order to extend  $J$ . In the course of the write phase we construct a sequence of sets  $\text{Act}(J) = P_0 \supseteq P_1 \supseteq \dots \supseteq P_t$  satisfying the Lemma below.

**Lemma 8** *At each step of the write phase we have a set  $P_k \subseteq \text{Act}(J)$  such that each process  $p \in P_k$  has a prefix  $\beta_p$  of  $\alpha_p$  and the following hold:*

- (1) *Each  $p \in P_k$  is executing exactly  $k$  critical writes along  $\beta_p$  in the execution  $J\beta_p$ .*
- (2) *Let  $e \in \beta_p$  be a critical write in  $J\beta_p$  by some process  $p \in P_k$  to some variable  $v$ . Then, the following holds:*
  - (a)  *$\text{owner}(v) \notin P_k$ ;*
  - (b)  *$\text{writer}(v, J) \notin P_k$ ;*
  - (c) *Either there is no  $q \in P_k$  different then  $p$  which access  $v$  in  $J\beta_q$ , or that any process  $q \in P_k$  have a write to  $v$  in  $\beta_q$ .*
- (3)  *$|P_k| \geq \sqrt{|P_{k-1}|}/4(\ell + s + k)$ .*

First notice that  $P_0 = \text{Act}(J)$  satisfies all the above conditions with  $\beta_p = \langle \rangle$  for any  $p \in \text{Act}(J)$ . Assume we already constructed  $P_{k-1}$  satisfying the conditions of Lemma 8. For any  $p \in P_{k-1}$  we denote by  $\beta_p$  the prefix as promised by Lemma 8 throughout the rest of this section, in which we define the construction of the write phase, and prove Lemma 8.

### 4.2.1 Construction: stage 1

For any  $p \in P_{k-1}$  we let  $p$  perform a solo run starting from  $J\beta_p$  until its next event  $e_p$  is a critical write, or that  $p$  has finish executing  $\alpha_p$  and its next event is  $\text{EndFence}$ . Denote this extension by  $\gamma_p$ . Notice that  $\beta_p\gamma_p$  is indeed a prefix of  $\alpha_p$  by its definition. Let  $Z$  be the set of all processes that have not finished executing  $\alpha_p$ , that is,  $Z = \{p \in P_{k-1} \mid \beta_p\gamma_p \neq \alpha_p\}$ .

If  $|Z| < |P_{k-1}|/2$ , i.e., at least half of the processes in  $P_{k-1}$  have finished executing  $\alpha_p$ , then we define  $t = k - 1$  and  $W = P_{k-1} \setminus Z$ , and we move to stage 3 of the write phase. By its definition, each process  $p \in W$  executes  $t$

critical writes along  $\alpha_p$  in the execution  $J\alpha_p$ . Moreover, it is easy to verify that condition (2) of Lemma 8 holds for  $W$  with  $\alpha_p$ , since it holds for  $W \subseteq P_{k-1}$  with  $\beta_p$ , and  $\alpha_p = \beta_p \gamma_p$  where  $\gamma_p$  contains no critical writes. Notice that no  $p \in W$  can write to a remote variable in  $\gamma_p$  it did not access in  $J\beta_p$ , as such write is critical, and therefore condition (2.c) follows immediately.

Otherwise  $|Z| \geq |P_{k-1}|/2$ . We define  $V_{next}$  to be the set of variables that are about to be written in one of the next events  $e_p$  by the processes in  $Z$ . Formally,  $V_{next} = \{v \in V \mid \exists p \in Z \text{ such that } e_p \text{ remotely writes } v\}$ . In order to construct  $P_k$  the following stage will handle the cases of low and high contention separately.

#### 4.2.2 Construction: stage 2

Case I:  $|V_{next}| < \sqrt{|Z|}$

By the pigeonhole principle, there exists a variable  $u \in V_{next}$  and a set  $P_k \subseteq Z$  of size  $|P_k| \geq \sqrt{|Z|}$ , such that  $e_p$  is a critical commit write to  $u$  for any  $p \in P_k$ .

Case II:  $|V_{next}| \geq \sqrt{|Z|}$

For each  $v \in V_{next}$  we select an arbitrary  $p \in Z$  such that  $e_p = \text{write}(v)$ . Denote this set by  $Z'$ . Then,  $|Z'| = |V_{next}| \geq \sqrt{|Z|}$ . We construct an undirected graph  $\mathcal{G}$  as follows: the vertices of  $\mathcal{G}$  are the processes in  $Z'$ . Consider  $p \in Z'$ , and denote  $e_p = \text{write}(v)$ . For  $q \in Z'$ , we add an edge  $\{p, q\}$  in  $\mathcal{G}$  if either a)  $v$  is local to  $q$ ; or b)  $q$  access  $v$  in  $J\beta_q$ .

Since each first access to a remote variable is critical, the number of different remote variables accessed by some process  $p \in Z'$  is at most the number of critical events it executes. Therefore, the number of new edges introduced by rule b) for  $p$  is at most the number of critical events  $p$  executes in  $J\beta_p$ , which is  $\ell + s + k - 1$ . Since each variable is local at most one process, at most one new edge is introduced by rule a). Altogether, the average degree in  $\mathcal{G}$  is at most  $2(\ell + s + k)$ . By Theorem 2, there exists an independent set  $P_k \subseteq Z'$  in  $\mathcal{G}$  such that:

$$|P_k| \geq \frac{|Z'|}{2(\ell + s + k) + 1} \geq \frac{\sqrt{|Z|}}{2(\ell + s + k) + 1}$$

We now prove that in both cases  $P_k$  satisfies all the conditions of Lemma 8, where for each  $p \in P_k$  we choose  $\beta'_p = \beta_p \gamma_p e_p$  to be the prefix of  $\alpha_p$ .

**Claim 4.2.1.**  $P_k$  satisfies Lemma 8.

*Proof.* We first prove condition (2). Notice that  $P_k$  satisfies it with the prefixes  $\beta_p$ , since  $P_{k-1}$  did. Each  $\beta_p$

was extended such that  $p$  have one more critical write. In case I all processes have one more critical write to the same variable  $u$ . In case II all new critical writes are to different variables, where the independent set assures no such variable is accessed by some other process in  $P_k$ . This gives an intuition of why condition (2) holds with the new prefixes  $\beta'_p$  as well. Formally, let  $e \in \beta'_p$  be a critical write in  $J\beta'_p$  by some  $p \in P_k$  to some variable  $v$ . We consider two cases:

Assume  $e \in \beta_p$ . Since (a) and (b) holds with  $P_{k-1}$  and  $P_k \subseteq P_{k-1}$ , it follows immediately they both hold with  $P_k$  as well. If any  $q \in P_{k-1}$  have a write to  $v$  in  $\beta_q$ , then so is the case with  $\beta'_q$ , and condition (c) holds. Otherwise, consider some  $q \in P_k$  different than  $p$ . By condition (c)  $q$  does not access  $v$  in  $J\beta_q$ . Assume towards a contradiction it does access  $v$  in  $\gamma_q e_q$ . Following condition (a)  $q \neq \text{owner}(v)$ , therefore this write is critical, i.e., it is  $e_q$ . In case I we get that  $v = u$ , and as so  $p$  have two different critical write,  $e$  and  $e_p$ , to the same variable  $v$  along a solo run  $\beta'_p$ , in contradiction. In case II, we get that  $p$  access the variable of  $e_q$  in  $J\beta_p$ , thus there is an edge  $\{p, q\}$  in  $\mathcal{G}$ , contradicting the fact that  $p, q \in P_k$ , an independent set.

Otherwise  $e = e_p$ . In case I, any  $q \in P_k$  have a critical write  $e_q$  to  $u = v$ , and therefore  $q \neq \text{owner}(v)$ . Moreover,  $e_q$  must be the first write to  $v$  in  $\beta'_q$ , as it is critical and  $\beta'_q$  is a solo run. It follows that  $q \neq \text{writer}(v, J)$ , and conditions (2) follows. In case II, consider some  $q \in P_k$ . If  $q \neq \text{owner}(v)$  and it access  $v$  in  $J\beta'_q$ , then the first such access is critical. This critical event is different than  $e_q$ , since  $e_p$  and  $e_q$  writes to different variables, therefore it must be in  $J\beta_q$ . In such case, or if  $q = \text{owner}(v)$ , there is an edge  $\{p, q\}$  in  $\mathcal{G}$ . This contradicts the fact that  $P_k$  is an independent set. It follows that  $q \neq \text{owner}(v)$ , and it does not access  $v$  in  $J\beta'_q$ , and in particular  $q \neq \text{writer}(v, J)$ , and condition (2) holds.

Condition (1) follows from construction. By induction hypothesis, each  $p \in P_k$  executes  $k - 1$  critical writes along  $\beta_p$  in the run  $J\beta_p$ . We extended  $\beta_p$  with a solo run such that  $\gamma_p$  contains no critical events, and  $e_p$  is a critical event. Therefore  $p$  executes  $k$  critical events along  $\beta'_p = \beta_p \gamma_p e_p$  in the run  $J\beta'_p$ .

For condition (3), in both cases we have a set  $P_k$  satisfying:

$$|P_k| \geq \frac{\sqrt{|Z|}}{2(\ell + s + k) + 1} \geq \frac{\sqrt{|P_{k-1}|/2}}{2(\ell + s + k) + 1} \geq \frac{\sqrt{|P_{k-1}|}}{4(\ell + s + k)}$$

□

This finish the inductive step of the write phase. At the last step we have large enough set of processes that have finish executing  $\alpha_p$ . In this case, we move to

stage 3, where we define how to use this set in order to construct the execution for the regularization phase.

#### 4.2.3 Construction: stage 3

We have a set  $W \subseteq \text{Act}(J)$  satisfying condition (2) of Lemma 8 with  $\alpha_p$ , such that each  $p \in W$  executes  $t$  critical writes along  $\alpha_p$  in the execution  $J\alpha_p$ . We extend  $J$  by letting each process  $p \in W$  perform its run  $\alpha_p$  in some arbitrary way. Denote this extension by  $D$ , and let  $p_r \in W$  be the last process to perform its  $\alpha_p$  in  $D$ . First notice the  $D$  is indeed an extension of  $J$  - since all processes performs only writes, we can order them in any way to get an extension, as writes of one process does not affect writes of other.

**Claim 4.2.2.** *For any  $Y \subseteq \text{Act}(J)$ ,  $p \in W \setminus Y$ , and  $e \in \alpha_p$  the following holds:  $e$  is a critical event in  $J\alpha_p$  if and only if it is critical in  $(JD)^{-Y}$ .*

*Proof.* Let  $Y, p$  be as in the claim. Consider  $e \in \alpha_p$ , a write to variable  $v$ .

Assume  $e$  is critical in  $J\alpha_p$ . By Lemma 8 we have  $\text{owner}(v) \notin W$  and  $q = \text{writer}(v, J) \notin W$ . If  $q \notin Y$ , then  $q = \text{writer}(v, J^{-Y})$ . Otherwise,  $q \in Y$ . Since  $J$  is a regular execution, by IN5  $q$  is the only process in  $\text{Act}(J)$  to access  $v$  in  $J$ . After removing events by  $q$  we get  $\text{writer}(v, J^{-Y}) \notin \text{Act}(J)$ . In particular, in both cases we have  $\text{writer}(v, J^{-Y}) \neq p$ . Notice that  $e$  is the first write in  $\alpha_p$  to  $v$ , since this event is critical in  $J\alpha_p$ . As a result, in  $D^{-Y}$   $p$  executes  $\alpha_p$ , where the first write to  $v$  in it is  $e$ , and all the events preceding  $\alpha_p$  are by processes different then  $p$ . Therefore, either there is a write to  $v$  before  $e$  in  $D^{-Y}$  by some process different then  $p$ , or that the last write is in  $J^{-Y}$ . In both cases we get that  $e$  is a critical write in  $J^{-Y}D^{-Y}$ .

Assume  $e$  is not critical in  $J\alpha_p$ . If  $e$  is not the first write to  $v$  in  $\alpha_p$ , then so is the case in  $D^{-Y}$  which contains  $\alpha_p$ , and on both cases  $e$  is not a critical write. Otherwise,  $e$  is the first write to  $v$  in  $\alpha_p$ . If  $p = \text{owner}(v)$ , then by its definition a write by  $p$  to  $v$  is not critical in any execution, and we done. Assume  $p \neq \text{owner}(v)$ , then it must be that  $p = \text{writer}(v, J)$ . Since  $J$  is regular, by IN4  $\text{owner}(v) \notin \text{Act}(J)$ . Consider some  $q \in W$  different then  $p$ , then  $q \neq \text{owner}(v)$ . If  $q$  does write to  $v$  in  $\alpha_q$ , the first such write is critical in  $J\alpha_q$ . By Lemma 8 we get that  $p = \text{writer}(v, J) \notin W$ , in contradiction. As a result, no process in  $W$  besides  $p$  writes to  $v$  in  $D$ , and therefore in  $D^{-Y}$ . Meaning, the last write to  $v$  before  $e$  in  $J^{-Y}D^{-Y}$  is in  $J^{-Y}$ . As we did not remove events by process  $p$  we have  $p = \text{writer}(v, J^{-Y})$ , and  $e$  is not critical in  $J^{-Y}D^{-Y}$ .  $\square$

Denote  $\bar{W} = \text{Act}(J) \setminus W$ , and let  $L = (JD)^{-\bar{W}}$ . Since no process in  $\bar{W}$  take steps in  $D$  we get  $L =$

$J^{-\bar{W}}D$ . First notice that  $L$  is indeed an execution. Since  $J$  is a regular execution, by Lemma 4 with  $'E' \leftarrow J$  and  $'Y' \leftarrow \bar{W}$ , we have a regular execution  $J^{-\bar{W}}$ . Moreover,  $D$  is an extension of  $J^{-\bar{W}}$  - for any  $p \in W$  we have  $J \mid p = J^{-\bar{W}} \mid p$ . Therefore, after both executions  $p$  have the same writes,  $\alpha_p$ , in its write buffer. Furthermore, since all processes performs only writes in  $D$ , any order of these writes is an extension of  $J^{-\bar{W}}$ , as writes of one process does not affect writes of other.

**Claim 4.2.3.**  *$W \setminus \{p_r\}$  is an IN-set of  $L$ .*

*Proof.* Denote  $W' = W \setminus \{p_r\}$ . Notice that  $J^{-\bar{W}}$  is a regular execution, and  $W' \subseteq \text{Act}(J^{-\bar{W}}) = W$ . Therefore  $W'$  is an IN-set of  $J^{-\bar{W}}$ .

IN1: for any  $p \in P$  we have  $AW(p, J^{-\bar{W}}) \cap W' \subseteq \{p\}$ . In addition,  $D$  contains only write events, therefore no process change its awareness-set along  $D$ , resulting  $AW(p, J^{-\bar{W}}D) \cap W' \subseteq \{p\}$ .

IN2: for any  $p \in W'$  we have  $\text{status}(p, J^{-\bar{W}}) = \text{entry}$ . In addition,  $D$  contains only write events, therefore no process change its status in  $D$  and  $\text{status}(p, J^{-\bar{W}}D) = \text{entry}$ .

IN3: consider  $Y \subseteq W'$ , and consider  $e \in (J^{-\bar{W}}D)^{-Y}$ . If  $e \in (J^{-\bar{W}})^{-Y}$ , then since  $J^{-\bar{W}}$  is regular, by IN3 we get:  $e$  is a critical event in  $J^{-\bar{W}}$  if and only if it is critical in  $(J^{-\bar{W}})^{-Y}$ . Otherwise  $e \in D^{-Y}$ . Let  $p$  be the process to execute  $e$ , then by claim 4.2.2 we get:  $e$  is critical in  $(JD)^{-(\bar{W} \cup Y)} = (J^{-\bar{W}}D)^{-Y}$  if and only if it is critical in  $J\alpha_p$  if and only if it is critical in  $(JD)^{-\bar{W}} = L$ . Altogether we get that  $e$  is a critical event in  $L$  if and only if it is critical in  $L^{-Y}$ .

IN4: Let  $e \in L$  be an event by  $p$  accessing a remote variable  $v$ . w.l.o.g  $e$  is the first access by  $p$  to  $v$  in  $L$ , and as so it is a critical event. If  $e \in J^{-\bar{W}}$ , then by IN4 we get  $\text{owner}(v) \notin \text{Act}(J^{-\bar{W}}) = W = \text{Act}(L)$ . Otherwise,  $e \in D$ , a critical write in  $(JD)^{-\bar{W}}$ . By claim 4.2.2  $e$  is a critical write in  $J\alpha_p$ , and therefore by condition (2) of Lemma 8 for  $W$  we get  $\text{owner}(v) \notin W = \text{Act}(L)$ .

IN5: Let  $v$  be a variable such that  $p = \text{writer}(v, L) \in W'$ . If  $v$  is local to  $p$ , then by IN4 there is no other process to write to  $v$  in  $L$ , and we done. Assume otherwise, then by IN4  $\text{owner}(v) \notin W$ . Let  $e$  be the last critical write to  $v$  in  $L$ . Notice that  $e$  is an event by  $p$ , since it is the last process to write to  $v$  in  $L$ . If  $e \in D$ , then by claim 4.2.2 it is critical in  $J\alpha_p$ . By condition (2) of Lemma 8, either any  $q \in W$  have a write to  $v$  in its  $\alpha_q$ , and in particular, since  $D$  ends with  $\alpha_{p_r}$  we get  $\text{writer}(v, L) = p_r \notin W'$  in contradiction. Therefore, there is no  $q \neq p$  in  $W$  to access  $v$  in  $J\alpha_q$ , and thus in  $L$ , and IN5 holds ( $\text{Accessed}(v, L) \cap \text{Act}(L) = \{p\}$ ). If  $e \in J^{-\bar{W}}$ , then no process but  $p$  can write to  $v$  in  $D$  (since such write is critical, contradicting the fact

that  $e$  is the last such write). Furthermore, we have  $writer(v, J^{-\bar{W}}) = p$ , and since  $J^{-\bar{W}}$  is a regular execution, by IN5  $p$  is the process in  $Act(J^{-\bar{W}}) = W$  to access  $v$  in  $J^{-\bar{W}}$ . Altogether,  $p$  is the only process in  $W$  to access  $v$  in  $L = J^{-\bar{W}}D$ , and IN5 holds.  $\square$

Each  $p \in W$  commits all the writes in its write buffer along  $D$ , hence its next event is *EndFence*. We extend  $L$  by letting each  $p \in W$  execute *EndFence* in some arbitrary order. By abuse of notation we denote the new execution by  $L$  as well, since it retains all the properties of the previous  $L$ . Then  $L$  satisfies the following conditions:

1.  $Act(L) \setminus \{p_r\}$  is an IN-set of  $L$ ;
2. Each  $p \in Act(L) = W$  execute  $\ell + s$  critical events in  $J^{-\bar{W}}$ , and by claim 4.2.2 another  $t$  critical events along  $D$  in  $J^{-\bar{W}}D$ . Therefore,  $p$  executes  $\ell + s + t$  critical events in  $L$ ;
3. Each  $p \in Act(L) = W$  completes  $i$  fences in  $J^{-\bar{W}}$ , and have one more *EndFence* as its last event in  $L$ . Thus,  $p$  completes  $i+1$  fences in  $L$  and  $mode(p, L) = read$ ;
4.  $Fin(L) = Fin(J) = Fin(H_i)$ ;
5.  $|Act(L)| \geq |P_k|/2$ .

This conclude the write phase, and we proceed to the regularization phase. We now give an upper bound on the number of steps in the read and write phases.

**Claim 4.2.** *The number of steps in the read and write phases is bounded by  $f(i+1) - \ell$ . In other words:  $\ell + s + t \leq f(i+1)$ .*

*Proof.* Assume towards a contradiction that  $\ell + s + t > f(i+1)$ . Then we have an execution  $L$  such that:

- $Act(L) \setminus \{p_r\}$  is an IN-set of  $L$ ;
- $p_r$  executes  $\ell + s + t$  critical events in  $L$ ;
- $Fin(L) = Fin(H_i)$ , thus  $|Fin(L)| = i$ ;

Using Lemma 4 with ' $E' \leftarrow L$ ' and ' $Y' \leftarrow Act(L) \setminus \{p_r\}$ ', we have an execution  $L' = L^{-Y'}$  such that  $Act(L') = \{p_r\}$  and  $Fin(L') = Fin(L)$ . By IN3 for  $L$ ,  $p_r$  executes the same critical events in  $L$  and  $L'$ , that is,  $p_r$  executes  $\ell + s + t > f(i+1)$  critical events in  $L'$ . However, exactly  $i+1$  processes issue events in  $L'$ , i.e. the total contention of  $L'$  is  $i+1$ , while  $p_r$  executes more than  $f(i+1)$  critical events during a single passage in  $L'$ , a contradiction.  $\square$

### 4.3 Regularization phase

The write phase construct an execution  $L$  such that each process in  $Act(L)$  completes  $i+1$  fences. Moreover,  $Act(L)$  can be written as  $Q \cup \{p_r\}$  where  $Q$  is an IN-set of  $L$ . The regularization phase will be used to let  $p_r$  complete its passage, such that we get a regular execution.

By Lemma 4 with ' $E' \leftarrow L$ ' and ' $Y' \leftarrow Q$ ' we get an execution  $L^{-Q}$  such that the following holds:

1.  $Act(L^{-Q}) = Act(L) \setminus Q = \{p_r\}$ ;
2.  $Fin(L^{-Q}) = Fin(L) = Fin(H_i)$ , and therefore  $|Fin(L^{-Q})| = i$ .

Let  $\alpha$  be a solo run of  $p_r$  starting from  $L^{-Q}$  until it finish its passage. Notice that such an extension exists by the progress property. We denote by  $Q^-$  the set of processes  $q \in Q$  such that there exist  $e \in \alpha$ , a critical event in  $L^{-Q}\alpha$  accessing variable  $v$ , such that either  $q = owner(v)$  or  $q = writer(v, L)$ .

We would like to extend  $L$  with  $\alpha$ , but it might be that a solo run of  $p_r$  after  $L$  and  $L^{-Q}$  is different. However, it is enough to erase the processes in  $Q^-$  from  $L$  to guarantee that  $\alpha$  is also a solo run of  $p_r$  after  $L^{-Q^-}$ .

Denote  $Q = Q^- \cup Q^+$  a disjoint union, that is,  $Q^+ = Q \setminus Q^-$ . By Lemma 4 with ' $E' \leftarrow L$ ' and ' $Y' \leftarrow Q^-$ ' we have an execution  $L^{-Q^-}$  such that  $Act(L^{-Q^-}) = Q^+ \cup \{p_r\}$  and  $Q^+$  is an IN-set of  $L^{-Q^-}$ .

**Claim 4.3.1.** *For any variable  $v$  accessed in  $\alpha$ :  $writer(v, L^{-Q^-}) \notin Q$ .*

*Proof.* Assume towards a contradiction there is a variable  $v$  such that  $p_r$  access it in  $\alpha$  and  $q = writer(v, L^{-Q^-}) \in Q$ . Since no process in  $Q^-$  takes steps in  $L^{-Q^-}$  it follows that  $q \notin Q^-$ . By the definition of  $Q^-$  we can conclude that  $q \neq writer(v, L)$ . Let  $p$  be  $writer(v, L)$ , then  $p \neq q$ . It must be that  $p \in Q^-$ , otherwise  $p$  executes the same events in  $L$  and in  $L^{-Q^-}$ , and therefore it is the last process to write to  $v$  in both, that is,  $p = writer(v, L) = writer(v, L^{-Q^-}) = q$  in contradiction. Altogether, we have two different processes  $p, q \in Q$  accessing  $v$  in  $L$ . Since  $Q$  is an IN-set of  $L$ , by IN5  $p = writer(v, L) \notin Q$ , in contradiction.  $\square$

Using the claim above, for any variable  $v$  accessed in  $\alpha$  we have  $q = writer(v, L^{-Q^-}) \notin Q$ . That is, the last write to  $v$  in  $L^{-Q^-}$  is by a process not in  $Q^+$ , and therefore after erasing from  $L^{-Q^-}$  the processes in  $Q^+$   $q$  still executes the same events in the resulting execution, and thus the last write to  $v$  in it is the same event by  $q$ . Meaning, in both  $L^{-Q^-}$  and  $(L^{-Q^-})^{-Q^+} = L^{-Q}$  the last event to write to  $v$  is identical.

As a result, if we extend  $L^{-Q^-}$  with the solo run of  $p_r$ , it will read the same values from the same registers



as in the solo run  $\alpha$  after  $L^{-Q^-}$ . That is,  $\alpha$  is an extension of  $L^{-Q^-}$ . Define  $H_{i+1} = L^{-Q^-}\alpha$ .

**Claim 4.3.2.**  $H_{i+1}$  is a regular execution.

*Proof.* As  $Act(L^{-Q^-}) = Q^+ \cup \{p_r\}$ , and in  $\alpha$  we let  $p_r$  finish its passage, we have  $Act(H_{i+1}) = Q^+$ . Notice that  $Q^+$  is an IN-set of  $L^{-Q^-}$ . We prove it is also an IN-set of  $H_{i+1}$ .

IN1: for any  $p \in P$  we have  $AW(p, L^{-Q^-}) \cap Q^+ \subseteq \{p\}$ . If  $p \neq p_r$  it does not take any steps in  $\alpha$ , and IN1 follows. In addition, in  $L^{-Q^-}\alpha$  no process in  $Q^+$  take steps, therefore  $p_r$  does not become aware of any such process during this execution. As such, and since  $p_r$  executes the exact same events in  $L^{-Q^-}\alpha$  and in  $H_{i+1}$ , we have  $AW(p_r, H_{i+1}) \cap Q^+ = \emptyset$ .

IN2: for any  $p \in Q^+$  we have  $status(p, L^{-Q^-}) = entry$ . Since  $p$  takes no steps in  $\alpha$  we get  $status(p, H_{i+1}) = entry$  as well.

IN3: consider  $Y \subseteq Q^+$ . Then  $H_{i+1}^{-Y} = (L^{-Q^-})^{-Y}\alpha$ . Since the criticality of an event depends only on the prefix of the execution containing it, and since IN3 holds for  $L^{-Q^-}$  we get:  $e \in (L^{-Q^-})^{-Y}$  is a critical event in  $H_{i+1}^{-Y}$  if and only if it is critical in  $H_{i+1}$ .

For an event  $e \in \alpha$ , first notice that if  $e$  is a critical event in  $H_{i+1}^{-Y}$  then by claim 1 it is also critical in  $H_{i+1}$ . Thus, assume  $e$  is a critical event in  $H_{i+1}$ , and let  $v$  be the variable accessed in  $e$ . If this is a critical read, i.e., the first read of  $v$  by  $p_r$  in  $H_{i+1}$ , since  $p_r$  executes the same events in  $H_{i+1}^{-Y}$  it is also the first read of  $v$  by  $p_r$  in  $H_{i+1}^{-Y}$ , and therefore a critical event. Otherwise it is a critical write. Since  $\alpha$  is a solo run by  $p_r$ , it must be that  $e$  is the first write by  $p_r$  to  $v$  in  $\alpha$  (since any subsequent write to  $v$  is not critical). By claim 4.3.1 we get  $p = writer(v, L^{-Q^-}) \notin Q$ , and since  $e$  is critical  $p \neq q$ . Therefore, in both  $L^{-Q^-}$  and  $(L^{-Q^-})^{-Y}$   $p$  executes the same events, and therefore on both it is the last process to write to  $v$ . That is, the last write to  $v$  before  $e$  in  $(L^{-Q^-})^{-Y}\alpha$  is in  $(L^{-Q^-})^{-Y}$  by a process  $p$  different than  $p_r$ , and thus  $e$  is critical in  $H_{i+1}^{-Y}$ .

IN4: Let  $v$  be a variable such that there exists a remote access to  $v$  in  $H_{i+1}$ . Let  $e$  be the first such access. If  $e \in L^{-Q^-}$ , by IN4 we have  $owner(v) \notin Act(L^{-Q^-}) \supset Q^+$ . Otherwise  $e \in \alpha$ . In particular, the first access by  $p_r$  to  $v$  is in  $\alpha$ . Since  $p_r$  executes the same events in  $H_{i+1}$  and in  $L^{-Q^-}\alpha$ , the first access by  $p_r$  to  $v$  in  $L^{-Q^-}\alpha$  is  $e \in \alpha$ , and therefore  $e$  is a critical event in this execution. Now, either  $owner(v) \notin Q$ , or that by the definition of  $Q^-$  we have  $owner(v) \in Q^-$ . In both cases  $owner(v) \notin Q^+$  and we done.

IN5: Consider some  $v$  such that  $q = writer(v, H_{i+1}) \in Q^+$ . No process in  $Q^+$  take steps in  $\alpha$ , and as a result the last write to  $v$  is in  $L^{-Q^-}$ . By IN5 for  $L^{-Q^-}$  we get that  $q$  is the only

process in  $Act(L^{-Q^-})$  to access  $v$  in  $L^{-Q^-}$ . Altogether,  $q$  is the only process in  $Q^+$  to access  $v$  in  $H_{i+1}$ , that is  $Accessed(v, H_{i+1}) \cap Act(H_{i+1}) = \{q\}$ .  $\square$

Denote  $\ell_{i+1} = \ell + s + t$ . We have an execution  $H_{i+1}$  such that the following holds:

1.  $H_{i+1}$  is a regular execution;
2. Each  $p \in Act(H_{i+1})$  executes all its critical events in  $H_{i+1}$  in the prefix  $L^{-Q^-}$ . By IN3 for  $L$ , it executes the same critical event in  $L^{-Q^-}$  and in  $L$ , that is, it executes  $\ell_{i+1}$  critical events in  $H_{i+1}$ . Furthermore, by claim 4.2  $\ell_{i+1} \leq f(i+1)$ ;
3.  $Fin(L^{-Q^-}) = Fin(H_i)$ . In the extension  $\alpha$  we let one more process  $p_r$  finish its passage, and thus  $|Fin(H_{i+1})| = i+1$ ;
4. Each  $p \in Act(H_{i+1})$  executes the same events in  $H_{i+1}$  in the prefix  $L$ . Therefore,  $p$  completes  $i+1$  fences in  $H_{i+1}$  and  $mode(p, H_{i+1}) = mode(p, L) = read$ .

This completes the regularization phase, and therefore the entire inductive step of constructing  $H_{i+1}$  from  $H_i$ . We now give an upper bound on the number of processes we erase from  $L$  is order to construct  $H_{i+1}$ , i.e., on the size of  $Q^-$ .

**Claim 4.3.3.**  $|Q^-| \leq f(i+1)$ .

*Proof.*  $\alpha$  is a solo run of  $p_r$  after  $L^{-Q^-}$ , where  $p_r$  is the only active process after  $L^{-Q^-}$ , and  $|Fin(L^{-Q^-})| = i$ . Therefore, the total contention of  $L^{-Q^-}\alpha$  is  $i+1$ , and  $p_r$  can execute at most  $f(i+1)$  critical events along its passage, and in particular along  $\alpha$ .

Consider an event  $e \in \alpha$  such that  $e$  is a critical event in  $L^{-Q^-}\alpha$ . Let  $v$  be the variable accessed in  $e$ . If  $q = owner(v) \in Q$ , then since  $Q$  is an IN-set of  $L$ , by IN4 it follows that there exists no  $p \neq q$  in  $Act(L)$  accessing  $v$  in  $L$ , and in particular, either  $writer(v, L) = q$  or  $writer(v, L) \notin Q$ . As a result, each critical event in  $\alpha$  can add at most one more process to the set  $Q^-$ .

To conclude, the size of the set  $Q^-$  is bounded by the number of critical events  $p_r$  executes along  $\alpha$  in the execution  $L^{-Q^-}\alpha$ , which itself is bounded by  $f(i+1)$ , and the claim follows.  $\square$

#### 4.4 Construction Bounds

We now present an analysis for the size of  $Act(H_i)$  based on the upper bounds on the number of steps in for each phase. We will prove a lower bound under some restriction on the growth rate of the adaptivity function  $f$ .

**Theorem 3** *Let  $i \in \mathbb{N}$  be such that  $f(i) \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}$ . Then the following lower bound holds:*

$$|Act(H_i)| \geq \frac{N^{2^{-\ell_i}}}{\ell_i! \cdot 4^{\ell_i+2i}}$$

*Proof.*

We assume WLOG that the adaptivity function  $f$  is non-decreasing. We prove the theorem by induction on  $i$ . For  $i = 0$ , we have  $|Act(H_0)| \geq N$  which is trivially true.

Let  $i + 1$  be such that:

$$f(i + 1) \leq \frac{N^{2^{-f(i+1)}}}{f(i + 1)! \cdot 4^{f(i+1)+2(i+1)}}$$

Since  $f$  is non-decreasing:

$$f(i) \leq f(i + 1) \leq \frac{N^{2^{-f(i+1)}}}{f(i + 1)! \cdot 4^{f(i+1)+2(i+1)}} \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}.$$

Hence  $i$  satisfies the condition in Theorem 3, and by the induction hypothesis,  $|Act(H_i)| \geq \frac{N^{2^{-\ell_i}}}{\ell_i! \cdot 4^{\ell_i+2i}}$ .

The induction step is partitioned into several sub-steps, corresponding to the phases in the construction of  $H_{i+1}$  from  $H_i$ . In each sub-step, we establish a lower bound on the number of active processes in the intermediate executions during the respective phase, based on the lower bound established for the phases preceding it.

$$\text{Read phase: } |Act(G_k)| \geq \frac{N^{2^{-(\ell_i+k)}}}{(\ell_i + k)! \cdot 4^{\ell_i+k+2i}}.$$

By induction on  $k$ .

Base case  $k = 0$ : then  $G_0 = H_i$  and the claim holds.

Induction step: assume we proved the claim for  $k - 1$ . By condition (5) of Lemma 7:

$$|Act(G_k)| \geq \frac{|Act(G_{k-1})| - 1}{10} \geq \frac{\frac{N^{2^{-(\ell_i+k-1)}}}{(\ell_i+k-1)! \cdot 4^{\ell_i+k-1+2i}} - 1}{10} \geq \frac{N^{2^{-(\ell_i+k)}}}{(\ell_i + k)! \cdot 4^{\ell_i+k+2i}}$$

where the last inequality holds as long as  $\ell_i + k \geq 3$ , which may be assumed since  $\ell_i$  increases from phase to phase and  $k$  increases in the course of the read phase.

$$\text{Write phase: } |Act(J_k)| \geq \frac{N^{2^{-(\ell_i+s+k)}}}{(\ell_i + s + k)! \cdot 4^{\ell_i+s+k+2i} \cdot 2}$$

By induction on  $k$ .

Base case  $k = 0$ :

$$|Act(J_0)| \geq \frac{|Act(G_s)|}{2} \geq \frac{N^{2^{-(\ell_i+s)}}}{(\ell_i + s)! \cdot 4^{\ell_i+s+2i} \cdot 2}.$$

Induction step: assume we proved the claim for  $k - 1$ . By condition (5) of Lemma ??:

$$|Act(J_k)| \geq \frac{\sqrt{|Act(J_{k-1})|}}{4(\delta + k)} \geq \frac{\sqrt{\frac{N^{2^{-(\ell_i+s+k-1)}}}{(\ell_i + s + k - 1)! \cdot 4^{\ell_i+s+k-1+2i} \cdot 2}}}{4(\ell_i + s + k)} \geq \frac{\sqrt{N^{2^{-(\ell_i+s+k-1)}}}}{\sqrt{(\ell_i + s + k - 1)! \cdot 4^{\ell_i+s+k-1+2i} \cdot 2}} = \frac{4(\ell_i + s + k)}{N^{2^{-(\ell_i+s+k)}}} = \frac{4(\ell_i + s + k)}{(\ell_i + s + k)! \cdot 4^{\ell_i+s+k+2i} \cdot 2}$$

Regularization phase:

$$|Act(L_k)| \geq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} - k$$

By induction on  $k$ .

Base case  $k = 0$ :

$$|Act(L_0)| \geq \frac{|Act(J_t)|}{2} \geq \frac{N^{2^{-(\ell_i+s+t)}}}{(\ell_i + s + t)! \cdot 4^{\ell_i+s+t+2i} \cdot 2} = \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}}$$

Induction step: assume we proved the claim for  $k - 1$ . By condition (7) of Lemma ??:

$$|Act(L_k)| \geq |Act(L_{k-1})| - 1 \geq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} - k$$

.

Therefore we have:

$$|Act(H_{i+1})| = |Act(L_m)| - 1 \geq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} - (m + 1) \tag{1}$$

From claim ?? and our assumption,

$$m \leq f(i + 1) \leq \frac{N^{2^{-f(i+1)}}}{f(i + 1)! \cdot 4^{f(i+1)+2(i+1)}}$$

By Claim 4.2,  $\ell_{i+1} \leq f(i+1)$ . Therefore, we can replace  $f(i+1)$  with  $\ell_{i+1}$  to get:

$$m \leq \frac{N^{2^{-f(i+1)}}}{f(i+1)! \cdot 4^{f(i+1)+2(i+1)}} \leq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2(i+1)}} = \frac{1}{4} \cdot \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} \quad (2)$$

Plugging Inequality 2 into Inequality 1 yields the required lower bound:

$$\begin{aligned} |Act(H_{i+1})| &\geq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} - (m+1) \geq \\ &\frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} - \frac{1}{2} \cdot \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} = \\ &\frac{1}{2} \cdot \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2i+1}} \geq \frac{N^{2^{-\ell_{i+1}}}}{\ell_{i+1}! \cdot 4^{\ell_{i+1}+2(i+1)}} \end{aligned}$$

□

We can now prove our main result.

**Theorem 1 (repeated)** *Let  $\mathcal{A}$  be an  $N$ -process weak obstruction-free  $f$ -adaptive implementation of a mutual-exclusion lock and let  $i \in \mathbb{N}$  be such that  $f(i) \leq \frac{N^{2^{-f(i)}}}{f(i)! \cdot 4^{f(i)+2i}}$ . Then there exists an execution  $H$  whose total contention is  $i+1$  and a process  $p$  such that  $p$  executes  $i$  fences in  $H$  during a single passage of its CS.*

*Proof.* Since  $\ell_i < f(i)$ , it follows from Theorem 3 that  $|Act(H_i)| \geq f(i) \geq 1$ . This implies that our construction results in an execution  $H_i$ , in which there is a process  $p \in Act(H_i)$  and, from the properties of  $H_i$ ,  $p$  is in a middle of a passage in which it executed (and completed)  $i$  fences. Moreover, from Lemma 4, we are able to erase all active processes but  $p$  from  $H_i$  and obtain an execution  $H$ , in which  $p$  executes  $i$  fences in the course of a single passage, and the total contention of  $H$  is  $i+1$ , that is, the number of fences  $p$  executes is linear in the total contention of the execution. □

## 5 Additional Objects

A *counter* is an object whose domain is  $\mathbb{N}$ . It supports a single operation, *fetch&increment*. The state of a counter is a natural number. The *fetch&increment* operation atomically increments  $C$  and returns its previous value. An *m-limited-use* counter allows at most  $m$  operation instances of *fetch&increment*. Notice that any counter is also an  $m$ -limited-use counter, for any  $m$ .

A *queue* object supports two operations: *enqueue* and *dequeue*. Each enqueue operation receives input  $v$  from a non-empty set of values  $V$ . Each dequeue operation applied to a non-empty queue returns a value  $v \in V$ . The state of a queue is a sequence of items  $S = \langle v_0; \dots; v_k \rangle$ , each of which is a value from  $V$ . The semantics of the enqueue and dequeue operations is the following.

- *enqueue*( $v_{new}$ ) changes  $S$  to be the sequence  $S = \langle v_0; \dots; v_k; v_{new} \rangle$ .
- if  $S$  is not empty, a dequeue operation changes  $S$  to be the sequence  $S = \langle v_1; \dots; v_k \rangle$  and returns  $v_0$ . If  $S$  is empty, dequeue returns the special value empty.

A *stack* object supports two operations: *push* and *pop*. Each push operation receives input  $v$  from a non-empty set of values  $V$ . Each pop operation applied to a non-empty stack returns a value  $v \in V$ . The state of a stack is a sequence of items  $S = \langle v_0; \dots; v_k \rangle$ , each of which is a value from  $V$ . The semantics of the push and pop operations is the following.

- *push*( $v_{new}$ ) changes  $S$  to be the sequence  $S = \langle v_0; \dots; v_k; v_{new} \rangle$ .
- if  $S$  is not empty, a pop operation changes  $S$  to be the sequence  $S = \langle v_0; \dots; v_{k-1} \rangle$  and returns  $v_k$ . If  $S$  is empty, pop returns the special value empty.

A *one-time* mutual exclusion algorithm is a ME algorithm that allows each process to complete a passage at most once. Since our proofs consider executions in which each process is allowed to complete a passage at most once, our results can be applied to one-time mutual exclusion algorithms.

**Lemma 9** *Let  $\mathcal{C}$  be a weak obstruction-free object of one of the following types: counter, stack or queue. Then, for any  $N \in \mathbb{N}$ , there exists an  $N$ -process one-time mutual exclusion algorithm  $\mathcal{A}$ , using  $\mathcal{C}$  and read/write variables, such that each passage through the CS invokes a single (fetch&increment, dequeue, or pop respectively) operation on  $\mathcal{C}$ , and that has the same RMR and fence complexities (in both DSM and CC models) as the operation it invokes, up to a constant additive factor.*

*Proof.* We first prove the lemma for an  $N$ -limited-use counter (hence also for a regular counter), by presenting Algorithm 1 for an  $N$ -process one-time mutual exclusion.

The correctness of the algorithm follows easily from the properties of the counter object.

We assume that each write in Algorithm 1 (lines 2-8) is followed by a fence instruction, and omit these fences from the code for presentation simplicity. Consequently, Algorithm 1 has the same fence complexity

of the *fetch&increment* operation, up to a constant additive factor.

In the DSM model, process  $p$  will hold  $spin[p]$  in its local memory segment. Since the only busy-waiting  $p$  may perform is on  $spin[p]$ , the ME algorithm has the same RMR complexity as that of the *fetch&increment* operation, up to a constant additive factor. In the CC model (with either write-back or write-through), since once  $spin[p]$  is updated to 1 its value does not change again,  $p$  may encounter at most 2 RMRs during the wait in line 4, hence the ME algorithm has the same RMR complexity as of the *fetch&increment* operation, up to a constant additive factor.

An  $N$ -limited-use counter can be implemented using a single queue or stack  $S$  in the following manner:

Queue: initialize  $S = \langle 0; \dots; N \rangle$ .

The *fetch&increment* operation is simply invoking  $S.dequeue()$ .

Stack: initialize  $S = \langle N; \dots; 0 \rangle$ .

The *fetch&increment* operation is simply invoking  $S.pop()$ .

Using Algorithm 1 with any of these implementations yields the required result.  $\square$

Counter, stack and queue objects can be easily implemented using the mutual exclusion algorithm presented by Attia et al [6]. Thus, each operation on these objects incurs  $O(\log N)$  RMRs and a constant number of fences, and this is optimal [4]. On the other hand, Lemma 9 implies that given an  $f$ -adaptive algorithm for any of these objects, an  $f$ -adaptive mutual exclusion algorithm can be obtained. Moreover, each passage through the CS invokes a single operation on the respective object, and has the same asymptotic fence complexity of the object. Hence, any lower bound on the fence complexity of the resulting mutual exclusion

algorithm implies a lower bound on the fence complexity for the operation of the respective object.

## 6 Discussion

We establish a time complexity separation between adaptive and non-adaptive implementations of mutual-exclusion locks, counters, stacks and queues, thus capturing an inherent cost incurred by adaptive algorithms in the TSO model.

This separation follows from a tradeoff that we prove between fence complexity and the growth rate of adaptivity functions. Specifically, we prove that the fence complexity of any read/write  $n$ -process algorithm with a linear (or sub-linear) adaptivity function is  $\Omega(\log \log n)$ . Our results apply for both the cache-coherent (CC) and the distributed shared-memory (DSM) models.

A corollary of our tradeoff is that constant fence-complexity adaptive implementations for these objects do not exist. Moreover, the impossibility result holds regardless of the RMR complexity of the algorithm. Following [6, 15], our tradeoff applies also to algorithms that may use comparison primitives, such as *compare-and-swap* (CAS), in addition to reads and writes.

Kim and Anderson presented an adaptive mutual exclusion algorithm whose RMR complexity is  $O(\min(k, \log n))$ , where  $k$  is point contention [20], hence it is  $f$ -adaptive for a linear  $f$ . The fence complexity of their algorithm is logarithmic. However, our tradeoff only implies  $\log \log n$  fence complexity (see Corollary 2). Finding the tight tradeoff between fence complexity and the adaptivity function growth rate is an interesting research direction.

The memory model considered by this work is TSO. We remind the reader that TSO ensures that writes are not reordered, but it is possible to perform a read from address  $a$  before a write to address  $b \neq a$  that is earlier in program order is performed. The *partial store ordering* (PSO) model, supported by older SPARC, is weaker than TSO, as it also allows the reordering of writes to different locations.

Recent work by Attiya, Hendler and Woelfel [7] showed that one cannot win on both the fence and RMR complexities of read/write PSO algorithms for many fundamental objects, including locks, counters and queues. They proved the following lower bound: let  $f$  and  $r$  respectively denote the numbers of fences and RMRs performed in an operation on such an object, then

$$f \cdot \log \frac{r}{f} + 1 \in \Omega(\log n). \quad (3)$$

---

### Algorithm 1 One-time mutual exclusion from counter.

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**Shared Data:**  $release[N]$ : a boolean array, initially  $[1, 0, \dots, 0]$   
 $waiting[N]$ : an integer array, initially  $[\perp, \perp, \dots, \perp]$   
 $spin[N]$ : a boolean array, initially  $[0, 0, \dots, 0]$   
 $C$ : an  $N$ -limited-use counter, initially 0

program for process  $p$ :

```

1:  $v \leftarrow C.fetch\&increment()$ ;
2:  $waiting[v] \leftarrow p$ ;
3: if  $release[v] = 0$  then
4:   wait ( $spin[p] \neq 0$ )
   CS
5:  $release[v+1] \leftarrow 1$ ;
6:  $q \leftarrow waiting[v+1]$ ;
7: if  $q \neq \perp$  then
8:    $spin[q] \leftarrow 1$ ;
```

---

They also showed that the bound is tight.

Attiya et al. [6] presented a TSO read/write mutual exclusion algorithm where each passage incurs a logarithmic number of RMRs and a constant number of fences. Inequality 3 establishes a complexity separation between the TSO and PSO models, since it follows from it that no such algorithm exists for the PSO model. Another interesting research direction is to find a tight tradeoff between the RMR-complexity and fence-complexity of adaptive PSO algorithms.

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## Appendix A Proofs Omitted from Paper Body

**Claim 1 (repeated)** *Let  $E$  be an execution fragment, and  $e \in E$  an event issued by some process  $p$ .*

- *Assume  $e$  is a non-special event in  $E$ . Then for any execution fragment  $F \preceq E$  such that  $F \mid p = E \mid p$ ,  $e$  is a non-special event in  $F$ .*
- *Assume  $e$  is a special event in  $E$ . Then for any execution fragment  $F$  such that  $E \preceq F$  and  $F \mid p = E \mid p$ ,  $e$  is a special event in  $F$ .*

*Proof.*

- Assume  $e$  is not a special event in  $E$ , then  $e$  is either a read or a write event.  
If  $e$  is a local event in  $E$  then  $e$  is a local event in  $F$  and the claim clearly holds. Otherwise,  $e$  is a remote event in both  $E$  and  $F$ . The following two cases exist.
  - $e = \text{read}(v)$ : then, since  $e$  is not a critical event in  $E$ ,  $e$  is not the first remote read of  $v$  by  $p$  in  $E$ , and since  $F \mid p = E \mid p$ ,  $e$  is not the first remote read of  $v$  by  $p$  in  $F$ , thus  $e$  is a non-critical read in  $F$ .
  - $e = \text{write}(v)$ : then, since  $e$  is not a critical event in  $E$ ,  $p$  is the last process to commit a write to  $v$  before  $e$  in  $E$ , denote this write event by  $e'$ . Since  $F \mid p = E \mid p$ ,  $e'$  occurs in  $F$  as well. There is no write commit between  $e'$  and  $e$  in  $E$ , and since  $F \preceq E$  there is no write commit between  $e'$  and  $e$  in  $F$  as well. Therefore  $p$  is the last process to commit a write to  $v$  before  $e$  in  $F$ , thus  $e$  is a non-critical write in  $F$  as well.
- Assume  $e$  is a special event in  $E$ . If  $e$  is a transition or a fence event in  $E$ , then it is also a transition or a fence event in  $F$  and we are done. Assume, then, that  $e$  is either a critical read or a critical write event in  $E$ .
  - If  $e = \text{read}(v)$ , then  $e$  is the first remote read of  $v$  by  $p$  in  $E$ . Since  $F \mid p = E \mid p$ ,  $e$  is the first remote read of  $v$  by  $p$  in  $F$ , thus a critical read in  $F$  as well.
  - If  $e = \text{write}(v)$ , then the last process to commit a write to  $v$  before  $e$  in  $E$  (if any) is not  $p$ . Since  $F \mid p = E \mid p$ , no write commit by  $p$  has been added to  $F$ , and since  $E \preceq F$ , no write commit by another process has been removed, thus the last process to commit a write to  $v$  before  $e$  in  $F$  (if any) is not  $p$ , so  $e$  is a critical write in  $F$  as well.

□

**Lemma 1 (repeated)** *Let  $E$  be an execution and let  $p \in P$  be a process such that  $p \notin AW(q, E)$  for any  $q \neq p$ . Then  $E^{-p}$  is an execution.*

*Proof.* We prove the claim by induction on the number of events in  $E^{-p}$ . The base case  $E^{-p} = \langle \rangle$  is trivial. For the induction, let  $E^{-p} = Ff$  such that  $E = E_1fE_2$ ,  $F = E_1^{-p}$ , and  $E_2$  is a (possibly empty) solo execution by  $p$ . Assume that  $F$  is an execution, and let  $q$  be the process that executes  $f$ .

Process  $q$  executes the same events in  $E_1$  and in  $F$ , thus it is in the same state after both executions and about to execute the same event  $f$ . If  $f$  is a transition or fence event, then  $Ff$  is clearly an execution. This is the case also if  $f$  is a write event. Otherwise, assume  $f = \text{read}(v)$ . The following two cases exist.

- Event  $f$  reads a copy of  $v$  from  $q$ 's write buffer in execution  $E$ . In this case, since  $q$  is in the same state after  $F$  and  $E_1$ , it will read the same value from its write buffer after  $F$ .
- Otherwise,  $f$  reads  $v$  from the shared memory. Since  $p \notin AW(q, E)$ , the last process that commits a write to  $v$  before  $f$  in  $E_1$  is not  $p$ . Hence, since all the processes except  $p$  execute the same events in  $E_1$  and in  $F$ ,  $v$  has the same writer and value after both executions. Consequently,  $f$  reads the same value in  $Ff$  and in  $E_1f$ , so  $Ff$  is an execution.

□

**Lemma 2 (repeated)** *Let  $E$  be an execution and let  $INV$  be an IN-set of  $E$ . Let  $e$  be a  $\text{read}(v)$  or  $\text{write}(v)$  event. Assume  $\text{writer}(v, E) \notin INV$  and  $\text{owner}(v) \notin \text{Act}(E)$ .*

*Then  $INV$  satisfies IN1-IN4 of Definition 4 in  $Ee$ .*

*Proof.* Denote  $F = Ee$ . First notice that  $p \in \text{Act}(E)$ , otherwise the only event  $p$  may execute after  $E$  is  $\text{Enter}_p$ . Thus we get  $\text{Act}(F) = \text{Act}(E)$ .

IN1: For any  $p' \neq p$  we have  $AW(p', F) = AW(p', E)$ , thus IN1 holds for  $p'$  in  $F$ . Denote  $q = \text{writer}(v, E)$ , then  $q \notin INV$  and, by IN1  $AW(q, E) \cap INV = \emptyset$ . Writing  $v$  does not change  $p$ 's awareness-set. Reading  $v$  expands  $p$ 's awareness-set with  $AW(q, E)$  at most, and thus  $p$  cannot become aware of any invisible process by reading  $v$ . Altogether,  $p$  does not become aware of any invisible process by accessing  $v$ , i.e.,  $AW(p, F) \cap INV \subseteq \{p\}$ , and IN1 holds for  $p$  in  $F$ .

IN2: IN2 holds in  $E$ . Since  $e$  is not a transition event, no process changes its status during  $e$  and so IN2 holds in  $F$ .

IN3: Consider  $Y \subseteq INV$ . IN3 holds for any event in  $E$ , thus it suffices to prove that it holds for  $e$  as well. Assume  $e \in F^{-Y}$ , that is  $p \notin Y$  and therefore

$F \mid p = F^{-Y} \mid p$ . If  $e$  is a local event in  $F$ , then  $e$  is a local event in  $F^{-Y}$  as well, and thus a non-critical event in both. Otherwise, assume  $e$  is a remote event in  $F$ . The following cases exist.

- If  $e = \text{read}(v)$ , then  $e$  is the first remote read of  $v$  by  $p$  in  $F$  if and only if it is the first remote read of  $v$  by  $p$  in  $F^{-Y}$ . Therefore  $e$  is a critical event in  $F$  if and only if it is a critical event in  $F^{-Y}$ .
- If  $e = \text{write}(v)$ , denote  $q = \text{writer}(v, E)$ . As  $q \notin \text{INV}$  and  $Y \subseteq \text{INV}$ , we have  $q \notin Y$ , thus after removing all the events by processes in  $Y$ , we still have  $\text{writer}(v, E^{-Y}) = \text{writer}(v, E) = q$ . Consequently, either  $q \neq p$  and  $e$  is critical in both  $F$  and  $F^{-Y}$ , or  $p = q$  and  $e$  is non-critical in both executions.

IN4: IN4 holds in  $E$  and  $\text{Act}(E) = \text{Act}(F)$ , thus it holds for any variable  $u \neq v$  in  $F$ . The only remote variable that may be accessed in  $e$  is  $v$ , and  $\text{owner}(v) \notin \text{Act}(E) = \text{Act}(F)$ .  $\square$

**Claim 4 (repeated)** *Let  $E$  be an execution and let  $\text{INV}$  be an IN-set of  $E$ . Let  $e$  be an extension of  $E$  by some process  $p$  such that  $e$  is a local event in  $Ee$ . Then  $\text{INV}$  is an IN-set of  $Ee$ .*

*Proof.* Denote  $F = Ee$ . First notice that  $p \in \text{Act}(E)$ , otherwise the only event  $p$  may execute after  $E$  is  $\text{Enter}_p$ , which is not a local event after  $E$ . Thus we get  $\text{Act}(F) = \text{Act}(E)$ .

IN1: For any  $q \neq p$  we have  $\text{AW}(q, F) = \text{AW}(q, E)$ , thus IN1 holds for  $q$  in  $F$ . If  $v$  is remote to  $p$ , then, since  $e$  is a local event and does not access any remote variable,  $e$  either stores a write to  $p$ 's write buffer or reads a copy of  $v$  from  $p$ 's write buffer. In both cases,  $p$ 's awareness set does not change as a result of executing  $e$ . Otherwise,  $v$  is local to  $p$ . In this case, since  $p = \text{owner}(v) \in \text{Act}(E)$ , we have by IN4 that  $p$  is the only process that accesses  $v$  in  $E$  (otherwise there is a remote access to  $v$ , thus by IN4  $p = \text{owner}(v) \notin \text{Act}(E)$ , a contradiction). Hence, after accessing  $v$  (by either a read or a write event),  $p$ 's awareness set does not change. It follows that  $\text{AW}(p, F) = \text{AW}(p, E)$  and IN1 holds for  $p$  in  $F$ .

IN2: IN2 holds in  $E$ . Since  $e$  is not a transition event, no process changes its status after  $e$ , and IN2 holds in  $F$ .

IN3: Consider  $Y \subseteq \text{INV}$ . IN3 holds for all of  $E$ 's events, thus it is sufficient to prove that it holds for  $e$  as well. If  $e \in F^{-Y}$ , then  $p \notin Y$ , thus  $F \mid p = F^{-Y} \mid p$ . Therefore  $e$  is a local event in both  $F$  and  $F^{-Y}$  and thus a non-critical event in both.

IN4: Since IN4 holds in  $E$  and no remote variable is accessed in  $e$ , and since  $\text{Act}(F) = \text{Act}(E)$ , IN4 holds in  $F$  as well.

IN5: IN5 holds in  $E$ , thus it holds in  $F$  for any variable that is not local to  $p$ , since  $p$  does not access any of these in  $e$ . Let  $v$  be a local variable of  $p$  accessed in  $e$ . Since IN4 holds in  $F$  and  $p \in \text{Act}(F)$ , there is no remote access to  $v$  in  $F$ . That is,  $p$  is the only process to access  $v$  in  $F$ , therefore  $|\text{Accessed}(v, F)| = 1$ , and IN5 holds for  $v$  in  $F$ .  $\square$

**Lemma 3 (repeated)** *Let  $E$  be an execution and let  $\text{INV}$  be an IN-set of  $E$ . Let  $F$  be an extension of  $E$  such that  $F$  contains no critical or transition event in  $EF$ . Then  $\text{INV}$  is an IN-set of  $EF$ .*

*Proof.* We first prove the claim for the case of a single event,  $F = f$ . The general case can be then proven by induction on the number of events in  $F$ . Let  $p$  be the process that executes  $f$ , then  $p \in \text{Act}(E)$ , otherwise the only event  $p$  may execute after  $E$  is  $\text{Enter}_p$ , which is a transition event. Thus  $\text{Act}(Ef) = \text{Act}(E)$ .

If  $f$  is a fence event, then  $f$  does not access any variable and no process changes its status. It is easily verified that in such case IN1-IN5 holds for  $\text{INV}$  in  $Ef$  as well, thus  $\text{INV}$  is an IN-set of  $Ef$ . If  $f$  is a local event, then by Claim 4,  $\text{INV}$  is an IN-set of  $Ef$ .

Assume, then, that  $f$  is a remote event and let  $v$  be the variable accessed in  $f$ . As  $f$  is not critical in  $Ef$ , this is not the first access of  $v$  by  $p$ , so  $p$  accesses  $v$  in  $E$ . Therefore, by IN4 applied to  $E$ , we have  $\text{owner}(v) \notin \text{Act}(E)$ . Denote  $q = \text{writer}(v, E)$ . If  $q \notin \text{INV}$ , then by Lemma 2, IN1-IN4 hold in  $Ef$ . Otherwise  $q \in \text{INV}$ , and by IN5 applied to  $E$ ,  $q$  is the only active process that accessed  $v$  in  $E$ , so  $p = q$ . To conclude the proof, we now prove that IN1-IN4 hold in the latter case ( $q \in \text{INV}$ ) and that IN5 holds in both cases.

IN1: For any  $p' \neq p$  we have  $\text{AW}(p', Ef) = \text{AW}(p', E)$ , thus IN1 holds for  $p'$  in  $Ef$ . As for  $p$ , since  $p = \text{writer}(v, E)$ , accessing  $v$  does not change its awareness-set, that is  $\text{AW}(p, Ef) = \text{AW}(p, E)$ , and so IN1 holds for  $p$  in  $Ef$  as well.

IN2: As  $f$  is not a transition event, for any  $p \in \text{INV}$ :  $\text{status}(p, Ef) = \text{status}(p, E) = \text{entry}$ .

IN3: Consider  $Y \subseteq \text{INV}$ . As IN3 holds for  $E$ , it is sufficient to prove that it holds for  $f$ . Assume  $f \in (Ef)^{-Y}$ , that is  $p \notin Y$  and  $E \mid p = E^{-Y} \mid p$ . If  $f$  is a remote read, then, from our assumption that it is not critical in  $Ef$ ,  $f$  is not the first remote read by  $p$  of  $v$  in  $Ef$  and thus in  $(Ef)^{-Y}$ , that is,  $f$  is non-critical in both. Otherwise, since  $p = \text{writer}(v, E)$  holds,  $p = \text{writer}(v, E^{-Y})$  holds as well, therefore  $f$  is a non-critical write in both  $Ef$  and  $(Ef)^{-Y}$ .

IN4: IN4 holds in  $E$  and  $\text{Act}(E) = \text{Act}(Ef)$ , thus it holds for any variable  $u \neq v$  accessed in  $Ef$ . The only variable accessed in  $f$  is  $v$ , and we already know that  $\text{owner}(v) \notin \text{Act}(E) = \text{Act}(Ef)$ .



IN5: IN5 holds in  $E$ , thus it holds for any variable  $u \neq v$ . As for  $v$ , if  $q \notin INV$  then  $f = \text{read}(v)$ , otherwise  $f$  would have been a critical write, a contradiction. Hence  $\text{writer}(v, Ef) = \text{writer}(v, E) = q \notin INV$ , and we are done. Otherwise,  $p = q \in INV$ , and by IN5 applied to  $E$  we get that  $p$  is the only process to access  $v$  in  $E$  (otherwise  $p = \text{writer}(v, E) \notin INV$ , contradicting our assumption). Therefore  $p$  is the only process to access  $v$  in  $Ef$ , and IN5 holds for  $v$  in  $Ef$ .  $\square$

**Lemma 4 (repeated)** *Let  $E$  be an execution,  $INV$  be an IN-set of  $E$  and  $Y \subseteq INV$ .*

*Define  $E' = E^{-Y}$ . Then the following hold:*

1.  $E'$  is an execution;
2.  $\text{Act}(E') = \text{Act}(E) \setminus Y$  and  $\text{Fin}(E') = \text{Fin}(E)$ ;
3.  $INV \setminus Y$  is an IN-set of  $E'$ ;
4. Each  $p \in \text{Act}(E')$  executes the same critical events in  $E'$  and in  $E$ ;
5. If  $p \in \text{Act}(E')$  is about to execute a special event  $f_p$  after  $E$ , then  $p$  is about to execute a special event  $e_p \sim f_p$  after  $E'$ .

*Proof.* We first prove the claim for the case  $Y = \{p\}$ , a single process.

1. Consider  $q \in P$  different from  $p$ . Since  $p \in INV$ , by IN1  $q$  is not aware of  $p$  in  $E$ , i.e.  $p \notin \text{AW}(q, E)$ . By Lemma 1  $E^{-p}$  is an execution.
2. We removed the events of  $p \in \text{Act}(E)$ , thus  $\text{Act}(E') = \text{Act}(E) \setminus \{p\}$  and  $\text{Fin}(E') = \text{Fin}(E)$ .
3. We prove  $INV \setminus \{p\}$  is an IN-set of  $E'$ :  
 IN1: Consider  $q \neq p$ . Since  $E \mid q = E' \mid q$ , we have  $\text{AW}(q, E) = \text{AW}(q, E')$ . By IN1,  $\text{AW}(q, E) \cap INV \subseteq \{q\}$ , in particular  $\text{AW}(q, E') \cap INV \setminus \{p\} \subseteq \{q\}$ .  $p$  executes no events in  $E'$ , thus  $\text{AW}(p, E') = \emptyset$  and IN1 holds for  $p$  in  $E'$ .  
 IN2: For any  $q \in INV \setminus \{p\}$ , we have  $E' \mid q = E \mid q$ , thus  $\text{status}(q, E') = \text{status}(q, E) = \text{entry}$ .  
 IN3: Consider  $Z \subseteq INV \setminus \{p\}$ , and  $e \in E'^{-Z} = E^{-Z \cup \{p\}}$ . Notice that  $e \in E, E^{-p}, E'^{-Z}$ . Since  $Z, \{p\} \subseteq INV$  and  $INV$  is an IN-set of  $E$ , by IN3 applied to  $E$  we have:  $e$  is a critical event in  $E^{-Z \cup \{p\}} = E'^{-Z}$  if and only if  $e$  is a critical event in  $E$  if and only if  $e$  is a critical event in  $E' = E^{-p}$ .  
 IN4: Consider an event  $e \in E'$  by process  $q$  accessing a remote variable  $v$ . Since  $e \in E$  accesses a remote variable  $v$ , by IN4  $\text{owner}(v) \notin \text{Act}(E)$ , and thus  $\text{owner}(v) \notin \text{Act}(E') \subseteq \text{Act}(E)$ .  
 IN5: Assume  $|\text{Accessed}(v, E') \cap \text{Act}(E')| > 1$  for some  $v$ . Since  $E' \preceq E$  and  $\text{Act}(E') \subseteq \text{Act}(E)$  we get  $|\text{Accessed}(v, E) \cap \text{Act}(E)| > 1$ , and by IN5 applied to  $E$ ,  $\text{writer}(v, E) \notin INV$ . The only events removed are by  $p \in INV$ , thus  $\text{writer}(v, E') = \text{writer}(v, E) \notin INV$ , and in particular  $\text{writer}(v, E') \notin INV \setminus \{p\}$ .

4. Follows directly from IN4 in  $E$  and the fact that  $p \in INV$ .
5. Assume  $q \in \text{Act}(E')$  is about to execute a special event  $f_q$  after  $E$ . Notice that  $E' \mid q = E \mid q$ , thus  $q$  is about to execute an event  $e_q \sim f_q$  after  $E'$ . We now prove that  $e_q$  is a special event in  $E'e_q$ :  
 If  $f_q$  is a fence or transition event then so is  $e_q$  and we are done. If  $f_q$  is the first access by  $q$  to some remote variable  $v$  (either read or write) in  $Ee_q$ , then  $e_q$  is also so in  $E'e_q$ , and is therefore critical. If  $f_q$  is a critical write to a remote variable  $v$ , and this is not the first write committed by  $q$  to  $v$  in  $Ee_q$ , then  $q$  accessed  $v$  in  $E$ . Notice that  $\text{writer}(v, E) \neq p$ , otherwise  $p, q \in \text{Accessed}(v, E) \cap \text{Act}(E)$  and by IN5  $\text{writer}(v, E) = p \notin INV$ , a contradiction. The only events removed are by  $p$ , therefore we have  $\text{writer}(v, E') = \text{writer}(v, E)$ . Since  $f_q$  is critical event in  $Ef_q$ , we have  $\text{writer}(v, E) \neq q$ . Altogether  $\text{writer}(v, E') \notin \{p, q\}$ , hence  $e_q$  is a critical write in  $E'e_q$  as well.

For the general case we prove the claim by induction on  $|Y|$ . The base case  $|Y| = 1$  has just been proven. Assume we proved the claim for any  $|Y| = n$ , and consider  $Y \subseteq INV$  such that  $|Y| = n + 1$ . Fix an arbitrary  $p \in Y$ , and denote  $Z = Y \setminus \{p\}$ . Then  $Z \subseteq INV$  and  $|Z| = n$ . Denote  $E_Z = E^{-Z}$ , then by induction hypothesis:

1.  $E_Z$  is an execution;
2.  $\text{Act}(E_Z) = \text{Act}(E) \setminus Z$ , and  $\text{Fin}(E_Z) = \text{Fin}(E)$ ;
3.  $INV \setminus Z$  is an IN-set of  $E_Z$ ;
4. Each  $q \in \text{Act}(E_Z)$  executed the same critical events in  $E_Z$  and in  $E$ ;
5. If  $q \in \text{Act}(E_Z)$  is about to execute a special event  $f_q$  after  $E$ , then  $q$  is about to execute a special event  $f'_q \sim f_q$  after  $E_Z$ .

Notice that  $E' = E_Z^{-p}$  and  $p \in INV \setminus Z$ , thus using the induction base with  $E_Z$  and  $\{p\}$  we get:

- $E'$  is an execution;
- $\text{Act}(E') = \text{Act}(E_Z) \setminus \{p\} = \text{Act}(E) \setminus Y$ , and  $\text{Fin}(E') = \text{Fin}(E_Z) = \text{Fin}(E)$ ;
- $(INV \setminus Z) \setminus \{p\} = INV \setminus Y$  is an IN-set of  $E'$ ;
- Each  $q \in \text{Act}(E')$  executed the same critical events in  $E'$  and in  $E_Z$ , and thus the same critical events in  $E'$  and in  $E$ .
- If  $q \in \text{Act}(E') \subseteq \text{Act}(E_Z)$  is about to execute a special event  $f_q$  after  $E$ , then  $q$  is about to execute a special event  $f'_q \sim f_q$  after  $E_Z$ , thus  $q$  is about to execute a special event  $e_q \sim f'_q \sim f_q$  after  $E'$ .

$\square$

**Lemma 5 (repeated)** *Let  $E$  be a regular execution. Then there exists an extension  $F$  such that the following hold:*

- $F$  contains no special events in  $EF$ ;
- $EF$  is a regular execution;
- Each  $p \in \text{Act}(E)$  is about to execute a special event  $f_p$  after  $EF$ . Moreover, at most one process  $p \in \text{Act}(E)$  is about to execute  $f_p = CS_p$  after  $EF$ .

*Proof.* First we prove the following claim: Consider  $p \in \text{Act}(E)$ . Then there is a solo extension  $F_p$  by  $p$  such that  $F_p$  contains no special event in  $EF_p$ , and  $p$  is about to execute a special event  $f_p$  after  $EF_p$ .

We let  $p$  run solo after  $E$  until  $p$ 's first special event  $f_p$ . Assume towards a contradiction that such an extension does not exist, i.e.  $p$  executes an infinite run  $F_p$  after  $E$  such that  $F_p$  contains no special event in  $EF_p$  (notice that  $p$  cannot finish running without executing the special event  $\text{Exit}_p$ ). For any finite prefix  $H$  of  $F_p$ ,  $H$  contains no special events in  $EH$ , thus  $\text{Act}(EH) = \text{Act}(E)$ , and by Lemma 3  $\text{Act}(EH)$  is an IN-set of  $EH$ . Denote  $Y = \text{Act}(EH) \setminus \{p\}$ . Using Lemma 4 with ' $E' \leftarrow EH$ ', ' $\text{INV}' \leftarrow \text{Act}(EH)$ ' and ' $Y' \leftarrow Y$ ', we have an execution  $E_p = (EH)^{-Y}$  such that  $\text{Act}(E_p) = \text{Act}(EH) \setminus Y = \{p\}$ . Since  $H$  is a solo-execution by  $p \notin Y$ ,  $E_p$  can be written as  $E^{-Y}H$ . Therefore  $p$  executes a solo-execution  $H$  after  $E^{-Y}$ , where  $p$  is the only active process along  $H$ , and  $p$  does not finish a passage in  $H$ . Since this holds for any prefix of  $F_p$ ,  $H$  can be as long as we wish, contradicting the global progress property.

Denote  $\text{Act}(E) = p_1, p_2, \dots, p_n$ . We prove by induction on  $i = 0, 1, 2, \dots, n$  that there is an extension  $F_i$  of  $E$  such that:

1.  $F_i$  contains no special events in  $EF_i$ ;
2.  $EF_i$  is a regular execution;
3. For every  $1 \leq j \leq i$ ,  $p_j$  is about to execute a special event  $f_{p_j}$  after  $EF_i$ ;

The base case  $i = 0$  is trivial ( $F_0 = \langle \rangle$ ). Assume we already constructed  $F_i$  as above for  $i < n$ . We now construct  $F_{i+1}$ :

Denote  $p = p_{i+1}$ .  $F_i$  contains no transition events, thus  $\text{Act}(EF_i) = \text{Act}(E)$ , and as a result  $p \in \text{Act}(EF_i)$ . Since  $EF_i$  is a regular execution, we can use the claim proven above for a single process: there is an extension  $F_p$  by  $p$  such that  $F_p$  contains no special events in  $EF_iF_p$ , and  $p$  is about to execute a special event  $f_p$  after  $EF_iF_p$ . Denote  $F_{i+1} = F_iF_p$ , then the following hold:

1.  $F_{i+1}$  contains no special events in  $EF_{i+1}$ ;
2. Since  $F_{i+1}$  contains no special events, we have  $\text{Act}(EF_{i+1}) = \text{Act}(E)$ , and using Lemma 3 we have that  $\text{Act}(E)$  is an IN-set of  $EF_{i+1}$ , therefore  $EF_{i+1}$  is a regular execution;
3. We already know that  $p$  is about to execute a special event  $f_p$  after  $EF_{i+1}$ . Consider  $q = p_j$  for some  $j \leq$

$i$ . By our assumption  $q$  is about to execute an event  $f_q$  after  $EF_i$ , and since  $(EF_i) \mid q = (EF_{i+1}) \mid q$ , it is about to execute  $f_q$  after  $EF_{i+1}$ . As  $f_q$  is a special event in  $EF_i f_q$ , and  $(EF_{i+1} f_q) \mid q = (EF_i f_q) \mid q$ , by claim 1  $f_q$  is a special event in  $EF_{i+1} f_q$ .

Substituting  $i$  with  $n$  we get an extension  $F$  such that:

- $F$  contains no special events in  $EF$ ;
- $EF$  is a regular execution;
- Each  $p \in \text{Act}(E)$  is about to execute a special event  $f_p$  after  $EF$ .

At most one process  $p \in \text{Act}(E)$  is about to execute  $f_p = CS_p$ , otherwise there are two distinct processes about to execute  $CS$  event after  $EF$ , contradicting the exclusion property. □