11

Determination of the Properties of Sequential Machines from Their Response to Finite Input Sequences

11.1. DEFINITIONS AND STATEMENT OF PROBLEM

We shall now consider finite automata and s-machines as objects on which one can experiment but about whose internal structure one possesses only limited information. It is also assumed that the experiments can only consist of observing the outputs generated by these machines in response to finite inputs. Our problem is to determine the specific structure of a given finite automaton or s-machine, its present state and, if possible, its state diagram.

We shall say that by feeding a sequence of (finite) length l to the s-machine we are performing an *experiment of length* l. The input of sequence $\rho(t) = \rho^0\rho^1 \ldots \rho^p$ produces a synchronous output of the sequence $\lambda(t) = \lambda^0\lambda^1 \ldots \lambda^p$, which we shall call the *response* of the s-machine to the input of $\rho(t)$. In this chapter, we shall call the input and the corresponding output, that is, the tape of the s-machine, the *result of the experiment*.

One can perform a variety of experiments. Thus when only one s-machine of a given type is available, and the input is a predetermined sequence, we have a *simple nonbranching experiment*. If, however, each consecutive input symbol selected by the experimenter depends on the preceding output symbols, then the experiment is said to be *simply* branching (or just branching). When several identical s-machines are available and all are in the same initial state, one can perform a multiple experiment, whereby different inputs are fed to each machine. A variant of the multiple experiment is one in which there is a single s-machine equipped with a *reset* button, that is, a device returning the machine to its initial state upon completion of an experiment.

The problem of determining the specific structure of a given s-machine from the results of a finite experiment can be defined only after all the *a priori* known facts about this machine have been exactly stated. As will be shown below, any new data about this s-machine which can be produced by the experiment depend on this *a priori* known information.

At the outset we can make the following intuitively obvious statement: if we do not know anything about a given s-machine, then there is no finite experiment which will tell us even as much as the number of its states. Obviously, to study a given machine we must know beforehand the nature and the number r of the input symbols ρ.

Let S be an s-machine with k internal states $\varkappa_1, \varkappa_2, \ldots, \varkappa_k$ (where k is unknown!) which we subject to a finite experiment of length l. Then it is always possible to devise another s-machine S^* which has

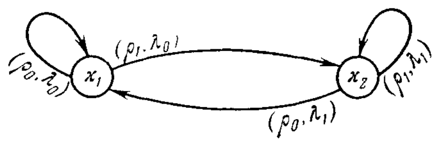

Fig. 11.1.

more states than k and which operates exactly as S in experiments not exceeding length l, and which becomes different from S only in experiments longer than l.

Assume, for example, that we have a finite automaton A and an associated output converter (see diagram of Fig. 11.1), on which we perform experiments of length $l \leqslant 3$. It is easily seen that if $l \leqslant 3$ and the initial state is $\varkappa_1$ or $\varkappa_2$, automaton B (diagram of Fig. 11.2) generates the same output λ as A; thus, at $l \leqslant 3$, A and B do not differ. They become dissimilar only when the input consists of the fourth symbol.

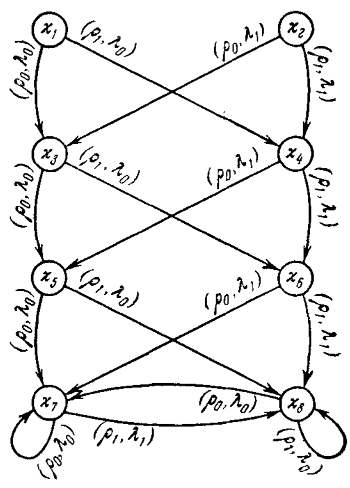

Fig. 11.2.

This argument shows that in order to experimentally determine the specific internal structure of a given automaton or s-machine one must have, in addition to the number of input symbols r, an estimate of the number k of its states. We shall assume from now on that the k and r are always known. Then we can consider the following experimental problems:

a) Determination of equivalence of two states of either the same, or of different s-machines.

b) Determination of equivalence of two s-machines.

c) Determination of the state diagram of an s-machine.

d) Determination of the state in which the machine was at the beginning of the experiment or, alternatively, its reduction to a specific state at the end of the experiment.

To solve these problems one must know what experiments can be carried out with the given set of s-machines (for example, whether one can perform a multiple experiment), as well as some additional data on this set (for example, this information may consist of the number of states k, as well as of the fact that all of these states are nonequivalent).

The next section shows a determination of the equivalence of states of an s-machine (Moore's theorem). Subsequent sections deal with the study of s-machines when multiple experiments are possible (Section 11.3), as well as with the case where only a simple experiment (in particular, a branching one) is possible (Section 11.4).

11.2. DETERMINATION OF EQUIVALENCE OF STATES OF s-MACHINES FROM THEIR RESPONSE TO FINITE INPUTS

Consider two equivalent states of some s-machine. By definition, the outputs in this case will coincide at any input, regardless of which of these equivalent states is the initial one. Conversely, if the initial states are nonequivalent, then there exists an input such that, starting with the qth cycle, the two outputs will differ. Here, q depends not only on the specific s-machine under consideration (its internal structure and the number of its states k), but also on the "discriminating" input sequence. Our problem consists of finding what is the minimal length of an input sequence capable of demonstrating the nonequivalence of two states of the given s-machines. It turns out that we can evaluate this length starting only with number (k) of the states of the machine. This length is given by the following theorem:

Theorem 1 (Moore's Theorem). If all k states of an s-machine N are nonequivalent, then for each pair of these states there exists an input sequence not longer than $k-1$, capable of discriminating between them.*

Consider the decomposition of the set of states of N into groups equivalent in terms of set L_s of all sequences of length s ($s = 1, 2, \ldots, n-1$). We shall prove the theorem by induction with respect to s. We shall prove that if the number of groups of states equivalent

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*See [72]; see also [98] where the same theorem has been independently proven.

in terms of L_s is m_s, then the number of groups equivalent in terms of L_{s+1} is not less than $m_s + 1$ (that is, $m_{s+1} > m_s + 1$).

If $s = 1$, that is, all input sequences consist of one symbol, then we can decompose the set of states of N into at least two groups of states equivalent in terms of L_1. Indeed, if *all* states of N were equivalent in terms of L_1, they would also be equivalent in terms of set E of all possible sequences (since in this case the output of the machine would be governed only by its input). However, this is not the case here because N has no states equivalent in terms of E. Consequently, $m_1 \geqslant 2$.

Now select two states $\varkappa_i$ and $\varkappa_j$ which are equivalent in terms of L_s. By our specification of N, $\varkappa_i$ and $\varkappa_j$ are nonequivalent states; therefore there must exist some input sequence capable of discriminating between them, but this sequence does not belong to L_s. Let the minimum length of this sequence be q (where $q > 1$). Then the first $q - (s + 1)$ symbols of this sequence will cause the machine to shift from the states $\varkappa_i$ and $\varkappa_j$ to states $\varkappa_g$ and $\varkappa_h$, respectively, which are also equivalent in terms of L_s. In fact, since q is the minimum length of the discriminating input, then, if the initial states are $\varkappa_i$ and $\varkappa_j$, the respective outputs must coincide from the $(q - s - 1)$-th to the $(q - 1)$-th machine cycle inclusively. For this reason, the outputs will coincide from the time $q - s - 1$ (at which the machine will be in states $\varkappa_g$ and $\varkappa_h$, respectively). However, we also know *a priori* that states $\varkappa_g$ and $\varkappa_h$ can be discriminated by an input of length $s + 1$, since the $(s + 1)$-th cycle after the initial states $\varkappa_g$ and $\varkappa_h$ is the qth cycle after the initial states $\varkappa_i$ and $\varkappa_j$, and q was *a priori* chosen in such a way that the outputs in the qth cycle will be different. Consequently, states $\varkappa_g$ and $\varkappa_h$ are *a priori* known to belong to different groups which are equivalent in terms of L_{s+1}.

Let us note now that if two states $\varkappa_g$ and $\varkappa_h$, which are equivalent in terms of L_s, are nonequivalent in terms of L_{s+1}, then the group of states equivalent in terms of L_s to which $\varkappa_g$ and $\varkappa_h$ belong may be decomposed into at least two groups equivalent in terms of L_{s+1}. This proves that $m_{s+1} \geqslant m_s + 1$.

It follows from this inequality and the inequality $m_1 \geqslant 2$ proved above that there always exists a $q^* \leqslant k - 1$ such that the number of groups of states equivalent in terms of L_{q^*} is exactly equal to k; that is, any two states of N are nonequivalent in terms of L_{q^*}. But this means that for each pair of states of machine N there exists an input sequence from L_{q^*} no longer than $q^* = k - 1$ at which the outputs do not coincide. This proves Moore's theorem.

Now, for a few notes in connection with this theorem.

Note 1. If the given automaton is associated with an output converter, and if we know not only the number k of nonequivalent states but also the number l of symbols in the table of the output converter, then, instead of the estimate of $(k-1)$, the estimate of $q^* = k - l + 1$ will apply. The proof of this statement follows the above proof of Moore's theorem word for word, the only change being that in this case the inequality $m_1 \geqslant l$ applies instead of the inequality $m_1 \geqslant 2$; that is, the number of groups of states equivalent in terms of L_1 cannot be less than l. If, however, l is not known *a priori*, then one uses the "worst" case in the estimate, that is, the case when $l = 2$ and $k - l + 1 = k - 1$.

Note 2. We can easily show that the estimate of the length of the sequence capable of discriminating between nonequivalent states and given by Theorem 1 is exact in the sense that this length cannot be shortened regardless of which s-machine with k nonequivalent states is used. This follows from the fact that for each k we may devise machines in which two nonequivalent states are *a priori* known to be indistinguishable if the "discriminating" input is shorter than $k - 1$.

Example. Consider a finite automaton (Table 11.1) associated with an output converter (Table 11.2) whose state diagram is shown in Fig. 11.3. It is easily seen that to establish nonequivalence of

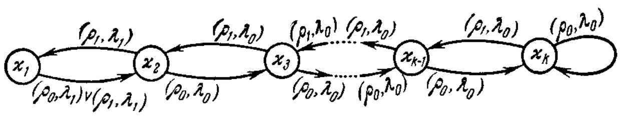

Fig. 11.3.

Table 11.1

$x \backslash \rho$	ρ_0	ρ_1
x_1	x_2	x_2
x_2	x_3	x_1
x_3	x_4	x_2
...	...	...
x_{k-1}	x_k	x_{k-2}
x_k	x_k	x_{k-1}

Table 11.2

x	λ
x_1	λ_1
x_2	λ_0
x_3	λ_0
...	...
x_{k-1}	λ_0
x_k	λ_0

states $\varkappa_{k-1}$ and $\varkappa_k$, the input sequence cannot be shorter than $k-1$; this is because the machine starting from these states, will generate only λ_0 at any input shorter than $k-1$. If, however, the output of machine in state $\varkappa_2$ were λ_2, then the nonequivalence of $\varkappa_{k-1}$ and $\varkappa_k$ could be established by a sequence only $k-2$ long (that is, $k-l+1$, where $l=3$).

Note 3. The arguments used in the proof of Theorem 1 may also be used for proving the equivalence (or nonequivalence) of two states of a single s-machine of known structure (that is, a machine with known state diagram, or tables of the automaton and converter). In that proof the machine states must first be divided into groups equivalent in terms of L_1; each of the groups so obtained must then be subdivided into groups equivalent in terms of L_2, and so on, until the two states under consideration appear in different groups. If this does not occur by the $(k-1)$-th step (that is, after all the states have been subdivided into groups equivalent in terms of L_{k-1}), then by virtue of Theorem 1 the two states under consideration are equivalent. We used an essentially similar argument in Section 9.4.

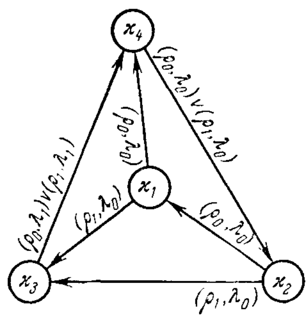

Fig. 11.4.

Note 4. Although any two nonequivalent states can be distinguished by an input not longer than $k-1$, this discriminating input does, in general, vary in length with different pairs of (nonequivalent) states. Thus, in general, there is no single finite input sequence capable of discriminating any one of the states from all the others.

Example. Consider a finite automaton (Table 11.3) associated with an output converter (Table 11.4), shown in Fig. 11.4. Here one

Table 11.3

$\varkappa$ \ ρ	ρ_0	ρ_1
$\varkappa_1$	$\varkappa_4$	$\varkappa_3$
$\varkappa_2$	$\varkappa_1$	$\varkappa_3$
$\varkappa_3$	$\varkappa_4$	$\varkappa_4$
$\varkappa_4$	$\varkappa_2$	$\varkappa_2$

Table 11.4

$\varkappa$	λ
$\varkappa_1$	λ_0
$\varkappa_2$	λ_0
$\varkappa_3$	λ_0
$\varkappa_4$	λ_1

can distinguish between states $\varkappa_1$ and $\varkappa_2$ if the input sequence starts with ρ_0; but discrimination between states $\varkappa_1$ and $\varkappa_3$ requires that the input starts with ρ_1.

Note 5 (which is the direct consequence of *Note 4*). There exists no simple experiment which can tell, even if the state diagram of the s-machine is available, what the state of the machine was at start of the test. Indeed, it has been shown that there is no finite experiment capable of distinguishing between a given initial state and all the others. And if we carry out some experiment capable of distinguishing a given state $\varkappa_i$ from some subset S of the set K of all the states of this s-machine, the very experiment will automatically shift this s-machine out of the state $\varkappa_i$, and thus we will be unable to determine in which of the states of the subset $K-S$ it had been initially.

It can, of course, be easily seen that when the machine has no equivalent states and we can perform a multiple experiment (that is, we have several identical s-machines, or a machine with reset), we can always find the initial state.

Note 6. Theorem 1 gives an estimate of the length of experiment capable of determining the nonequivalence of the states of two s-machines, having k_1 and k_2 internal states, respectively. This can be done by regarding these states as states of a single s-machine obtained by simple union* of these two s-machines. After this union, the nonequivalence of the two states may be established by an experiment not longer than $q^* = k_1 + k_2 - 1$.

The nonequivalence of states of two different automata with output converters can be established by an experiment (see *Note 1*) not longer than $q^* = k_1 + k_2 - l + 1$.

If $k_1 = k_2$, then the respective estimates become:
For the two s-machines $q^* = 2k - 1$, and for the two automata with converters $q^* = 2k - l + 1$. The next example will show that these values cannot be improved upon.

Example. Figure 11.5 shows the diagrams of two finite automata without output converters, where the number of output symbols $l = 2$. It is readily seen that an experiment establishing the nonequivalence of states $\varkappa_1'$ and $\varkappa_1''$ cannot be shorter than 7 (for example, the input sequence could be $\rho(t) = \rho_0\rho_0\rho_0\rho_0\rho_1\rho_1\rho_1$). If, however, states $\varkappa_2'$ and $\varkappa_2''$ of these automata were associated with a new output λ_2, that is, if l were 3, then experiment of length 6 is sufficient. Similar examples can be devised for k.

*The state diagram of the combined machine is a simple union of the diagrams of the component machines.

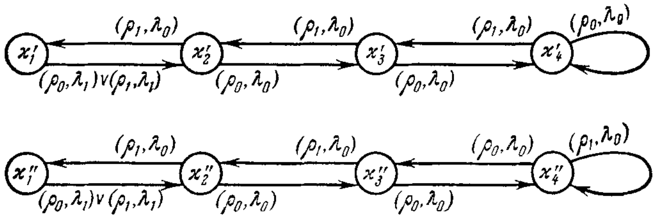

Fig. 11.5.

11.3. MULTIPLE EXPERIMENTS ON SEQUENTIAL MACHINES

The multiple experiment requires several identical s-machines or a machine with reset. In these experiments we consider only those states which the machine may attain in a finite number of steps, starting from state $\varkappa^0$.

A sequential machine is said to be $\varkappa^0$*-connected* if it has a diagram such that for each state $\varkappa_i$ ($i = 1, 2, \ldots, k$) there exists an input capable of shifting this machine from its initial state $\varkappa^0$ to state $\varkappa_i$. It is quite obvious that our discussion should not go beyond $\varkappa^0$-connected machines: if the machine were not $\varkappa^0$-linked, then our multiple experiment will permit us to study only that section of it which is $\varkappa^0$-connected. For that reason, we shall discuss only $\varkappa^0$-connected machines with reset. With such machines there is no problem of machine states at the beginning or the end of the experiment, and the only problems which can be considered are those of the equivalence of two s-machines and of determining the diagram of the machine.

Let us first discuss the equivalence problem. It is obvious that the determination of equivalence of two $\varkappa^0$-connected s-machines may be reduced to a determination of equivalence of the two states $\varkappa^0$ in these two machines. But we have shown in the Section 11.2 that the nonequivalence of states of two such s-machines can be proven by an experiment not longer than $2k - 1$ or, in the case of two automata with converters, by an experiment not longer than $2k - l + 1$. Thus the multiple experiment can discriminate a specific $\varkappa^0$-connected, s-machine from the whole class of $\varkappa^0$-connected machines which are nonequivalent to it and whose diagrams are known. From this follows a technique for solving the second problem, that of constructing the diagram of this $\varkappa^0$-connected, s-machine, the algorithm of which is as follows:

1. We preform all the possible experiments of length $2k-1$ on the machine (a total of r^{2k-1} experiments). We record the results in the form of tables (tapes), leaving blank the table entries corresponding to the states of the s-machine.

2. We assign some number i $(1 \leqslant i \leqslant k)$ to the initial state of the machine and substitute this number into the corresponding positions of the table.

3. After the first step of the experiment, the machine will be in one of states $\varkappa^1$, of which there can be no more than r. We use all the inputs of length $k-1$ to find out whether there are any equivalents among the states $\varkappa^0$ and $\varkappa^1$. We assign arbitrary and different numbers i and j $(1 \leqslant i, j \leqslant k)$ to all the states $\varkappa^1$ which are nonequivalent to each other and to state $\varkappa^0$. Those states that are equivalent are coded by the same number. Let the number of different states $\varkappa^1$ be r_1.

4. From each of the states so coded no more than r new states $\varkappa^2$ may be reached in one step, so that the total number of states $\varkappa^2$ cannot exceed $r_1 r$. We use all the possible input sequences of length $k-1$ to ascertain whether there are equivalents among states $\varkappa^0$, $\varkappa^1$ and $\varkappa^2$. We assign numbers to states $\varkappa^2$ in the same way as we have coded states $\varkappa^1$.

5. We continue this process until we find k states nonequivalent to each other. It is obvious that this number will be reached in less than $2k-1$ steps (that is, we need not scan all the experimentally derived tapes).

6. We construct a state diagram, a basic table, or an interconnection matrix in accordance with the experimental results.

Note. Because the output of a finite automaton with an output converter is governed by its state $\varkappa$ and is independent of the input ρ supplied at the given time, we need only r^{2k-l} experiments of length $2k-l+1$ (instead of r^{2k-l+1}) to derive the diagram of this automaton. In addition, the last input symbol in each experiment may be arbitrary, for example, the same one for all experiments.

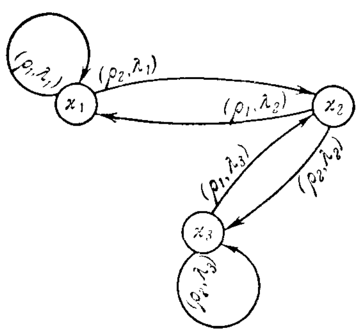

Fig. 11.6.

Example. Suppose we know that a certain automaton associated with an output converter has $k=3$ nonequivalent states, $r=2$ inputs, and $l=3$ output symbols. Then experiments of length $2k-l+1=4$, performed in this automaton as per the above algorithm, allow us to enter the states into the table (see the symbols in parentheses of Table 11.5) and to construct the state diagram of the automaton (Fig. 11.6).

Table 11.5

t	0	1	2	3
ρ	ρ_1	ρ_1	ρ_1	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_1)$
λ	λ_1	λ_1	λ_1	λ_1

t	0	1	2	3
ρ	ρ_2	ρ_2	ρ_1	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_3)$	$(\varkappa_2)$
λ	λ_1	λ_2	λ_3	λ_2

t	0	1	2	3
ρ	ρ_1	ρ_1	ρ_2	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_2)$
λ	λ_1	λ_1	λ_1	λ_2

t	0	1	2	3
ρ	ρ_2	ρ_1	ρ_1	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_1)$	$(\varkappa_1)$
λ	λ_1	λ_2	λ_1	λ_1

t	0	1	2	3
ρ	ρ_1	ρ_2	ρ_2	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_3)$
λ	λ_1	λ_1	λ_2	λ_3

t	0	1	2	3
ρ	ρ_2	ρ_2	ρ_2	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_3)$	$(\varkappa_3)$
λ	λ_1	λ_2	λ_3	λ_3

t	0	1	2	3
ρ	ρ_2	ρ_1	ρ_2	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_1)$	$(\varkappa_2)$
λ	λ_1	λ_2	λ_1	λ_2

t	0	1	2	3
ρ	ρ_1	ρ_2	ρ_1	ρ_1
$\varkappa$	$(\varkappa_1)$	$(\varkappa_1)$	$(\varkappa_2)$	$(\varkappa_1)$
λ	λ_1	λ_1	λ_2	λ_1

11.4. SIMPLE EXPERIMENTS ON SEQUENTIAL MACHINES

If multiple experiments cannot be performed, we can study s-machines by means of simple experiments.

The problem of discerning nonequivalence and determining the internal structure of s-machines by simple experiments has been solved only for the case of the set of machines in which all the states are nonequivalent. Such a set is that of differing strongly connected machines. A sequential machine is said to be *strongly connected* if for each pair $\varkappa_i$ and $\varkappa_j$ of its states there exists an input sequence capable of shifting it from state $\varkappa_i$ to state $\varkappa_j$.

Because the class of strongly connected machines is narrower than that of $\varkappa^0$-connected machines, it follows from Section 11.3 that *two strongly connected s-machines are equivalent if at least two states of these machines are equivalent.* Therefore all the states of all the machines of a set consisting of differing strongly connected machines are nonequivalent.

As stated in Note 5 to Theorem 1 (see Section 11.2), in general there is no simple experiment capable of distinguishing an initial $\varkappa^0$ of an s-machine from all the other states which are nonequivalent to it. For this reason we would want to find a simple experiment which would shift the machine into a state which could be uniquely specified; in other words, we desire an experiment in which there exists a unique correspondence between the result and the last state of the experiment $\varkappa^p$ (the state that corresponds to the last input symbol being tested). That an experiment exists, and that the entire class of s-machines may be subjected to it is proved by Theorem 2, which also provides an estimate of its length.

Theorem 2 (the Moore-Karatsuba Theorem). The last state of a given s-machine with k nonequivalent internal states is obtainable from an experiment not longer than $\frac{k(k-1)}{2}$ *or, in the case of a finite automaton, not longer than* $\frac{(k-1)(k-2)}{2}+1$.

Proof. Assume that the state diagram of the s-machine is given. We shall try to find the input sequence discriminating the last state of this machine in the form of a series of *consecutive* sequences (that is, experiments) a_s $(s = 1, 2, \ldots)$. These sequences shall be such that the set T_s of the possible states* occurring after the input of

*In papers [72], [25], [41], [59], and [60], the set T_s is called the set of associated states. Let us emphasize that T_s is the set of those states which occur after the input of sequence a_s, and is is not the set of possible states which govern the last observed output symbol (and thus determine the decomposition of the set of all the states into groups equivalent in terms of a_s).

the last symbol of experiment a_s (these states are therefore the possible initial states for the next experiment a_{s+1}) will contain not more than $k-s$ elements. If the s-machine is an automaton, then such states would include at least two states which can be distinguished by an experiment of length 1, that is, by any input symbol.

This condition is satisfied before the beginning of the experiment, when $s=0$. Now we shall prove that if this condition is satisfied for a_s, then there exists an a_{s+1} for which it also holds. The initial machine state for the experiment a_{s+1} must belong to the set of states T_s. Using Theorem 1 and the set of arguments used in its proof, we find that the elements of T_s (there can be no more than $k-s$ such elements, in accordance with the condition of Theorem 2) may belong to:

a) at least two groups of states equivalent in terms of set L_{s+1} of all experiments not longer than $s+1$ (there are at least $s+2$ such groups; see Theorem 1); and

b) at least three groups of states equivalent in terms of set L_{s+2} of all experiments not longer than $s+2$ (there are at least $s+3$ such groups).

Consequently, for any s-machine there will always be, among the $k-s$ states of set T_s, a pair of states which can be distinguished by an $s+1$ long experiment a_{s+1}. For this reason set T_{s+1} will contain at least one element less than T_s, that is, it will contain not more than $k-(s+1)$ states.

If, however, our machine is an automaton, then, by virtue of (b), set T_s will always contain a pair of states that can be distinguished by an $s+2$ long experiment, in which the sequence of the first $s+1$ symbols is regarded as the experiment a_{s+1}. The theorem stipulates that in an automaton there are at least two states of T_s that are distinguishable by any input symbol. For this reason, the first symbol of the experiment a_{s+1} will discriminate between these symbols. Consequently, set T_{s+1} will contain at least one element less than set T_s, that is, it will contain $k-(s+1)$ states, at least two of which, by virtue of our choice of experiment a_{s+1}, will be discriminated by any input symbol.

Since the theorem holds for s and $s+1$, it follows by induction that it will hold for any positive integral s; thus T_{k-2} will contain not more than two states which, in the case of an s-machine, can be distinguished by an experiment a_{k-1} not longer than $k-1$ or, in the case of an automaton, by an input symbol (that is, by an experiment of length 1).

Thus, none of the experiments a_s is longer than s, and the last state of the s-machine may be determined by an experiment not longer than

$$\tilde{q} = \sum_{s=1}^{k-1} s = \frac{k(k-1)}{2} \tag{11.1}$$

while the length of a similar experiment required in the case of a finite automaton is

$$\tilde{q} = \sum_{s=1}^{k-2} s + 1 = \frac{(k-1)(k-2)}{2} + 1 \; . \tag{11.2}$$

Note 1. The two examples given below show that the above-calculated required experimental length cannot be shortened.

Example 1. In order to distinguish the last state of an automaton with diagram of Fig. 11.7, we require an experiment $\rho_1\rho_2\rho_1\rho_1$

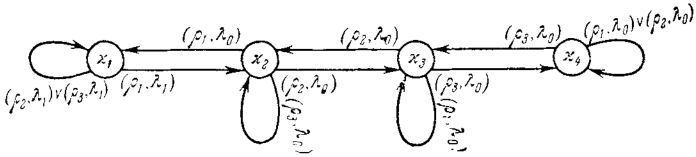

Fig. 11.7.

of length 4, that is, of length exactly equal to $\frac{(k-1)(k-2)}{2} + 1$.

Example 2. In order to distinguish the last state of an s-machine with diagram of Fig. 11.8, we require an experiment $\rho_3\rho_1\rho_3\rho_2\rho_1\rho_3$ of the length $\frac{k(k-1)}{2} = 6$.

It is readily shown that no shorter experiment will accomplish this in either example. The technique for devising similar examples for any k is obvious.

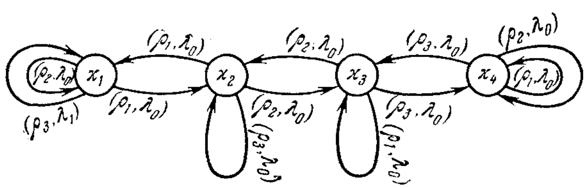

Fig. 11.8.

Note 2. If the output alphabet is taken to contain l symbols then, by a similar reasoning, we arrive at an estimate of the length of experiment determining the last state of an automaton with an output converter:

$$\tilde{q} = \frac{(k-l+1)(k-l)}{2} + 1. \tag{11.3}$$

In the case of automaton without a converter $(k = l)$, we obtain the obvious estimate of 1.

Note 3. The experiment determining the last state is shorter in cases where the initial states are known to be a subset of the entire set of states k. If the total number of possible initial states is $1 \leqslant \nu \leqslant k$, and among those states there are states which can be distinguished by any input symbol, then we can prove by reasoning similar to that used in the proof of Theorem 2 that the length of an experiment discriminating the last state of an automaton with a converter must be

$$\tilde{q} = \frac{\nu - 2}{2}(2k - 2l - \nu + 3) + 1. \tag{11.4}$$

When none of the ν initial states of an automaton with a converter is distinguishable by an experiment of length 1, then the length of the required experiment will be

$$\tilde{q} = \frac{(\nu - 1)(2k - 2l - \nu + 4)}{2}. \tag{11.5}$$

Example 3. If the only initial states of the automaton of Fig. 11.7 are $\varkappa_3$ and $\varkappa_4$, then the last state will be distinguished by an experiment $\rho_2\rho_1\rho_2$ of length 3, that is, of length exactly equal to $\frac{(\nu - 1)(2k - 2l - \nu + 4)}{2}$. If, however, only $\varkappa_1$ and $\varkappa_3$ are initial states then the last state will be distinguished by any input symbol, that is, by an experiment of length

$$\frac{(\nu - 2)}{2}(2k - 2l - \nu + 3) + 1 = 1.$$

Note 4. In discussing the shortest possible experiments, we should note that if T_s contains less than $k - s$ elements (for example, $k - s - m$ elements), then reasoning similar to that used in proving Theorem 2 will show that the length of the sequence which follows a_s is not $s + 1$, but $s + m + 1$. However, in this case the total length of an experiment shifting the machine into a specific last state is shorter because sequences ranging in length from $s + 1$ to $s + m$ drop out.

We shall now illustrate a regular technique for finding the shortest experiment giving the last state of an automaton with a converter. This procedure follows directly from Theorem 2.

Example 4. Consider a finite automaton associated with an output converter whose diagram is that of Fig. 11.9, the basic table is Table 11.6, and the converter Table is 11.7. Table 11.8 shows the procedure for finding the shortest experiment for determining the last state of this automaton.

Table 11.6

$\varkappa \backslash \rho$	ρ_0	ρ_1
$\varkappa_1$	$\varkappa_5$	$\varkappa_4$
$\varkappa_2$	$\varkappa_3$	$\varkappa_2$
$\varkappa_3$	$\varkappa_4$	$\varkappa_3$
$\varkappa_4$	$\varkappa_5$	$\varkappa_4$
$\varkappa_5$	$\varkappa_6$	$\varkappa_2$
$\varkappa_6$	$\varkappa_7$	$\varkappa_3$
$\varkappa_7$	$\varkappa_2$	$\varkappa_1$

Table 11.7

$\varkappa$	λ
$\varkappa_1$	λ_1
$\varkappa_2$	λ_0
$\varkappa_3$	λ_0
$\varkappa_4$	λ_0
$\varkappa_5$	λ_0
$\varkappa_6$	λ_0
$\varkappa_7$	λ_0

Table 11.8

s	a_s	T_s	Number of elements in T_s
0	»	$\{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5, \varkappa_6, \varkappa_7\}$	$k = 7$
1	$a_1 = \rho_1 \cancel{\rho}_1$	$\{\varkappa_4\}\ \{\varkappa_2, \varkappa_3, \varkappa_4, \varkappa_1\}$	$k—s—m = 7-1-2=4$
4	$a_4 = \rho_0\rho_0\rho_0\rho_1 \cancel{\rho}_1$	$\{\varkappa_1\}\ \{\varkappa_2, \varkappa_3, \varkappa_1\}$	$k—s=7—4=3$
5	$a_5 = \rho_0\rho_0\rho_0\rho_0\rho_1 (\cancel{\rho}_0 \vee \cancel{\rho}_1)$	$\{\varkappa_2\}\ \{\varkappa_3, \varkappa_1\}$	$k—s=7—5=2$
	$\rho_0 \vee \rho_1$		

Since we are considering an automaton, then, in accordance with the proof of Theorem 2 we select for each s those input sequences of length $s + 1$ which can discriminate any two states of set T_s, and use their first s symbols (the last, $s + 1$ symbols of the Table 11.8 are crossed out). As shown in Note 4 to Theorem 2, in this example we can go directly from $s = 1$ to $s = 4$, dispensing with $s = 2$ and $s = 3$. We can distinguish the two penultimate states $\varkappa_3$ and $\varkappa_1$ by means of any single input symbol. For this reason, the overall experiment giving the final state of our automaton will be $\rho_1\rho_0\rho_0\rho_0\rho_1\rho_0\rho_0\rho_0\rho_0\rho_1\rho_0$ or $\rho_1\rho_0\rho_0\rho_0\rho_1\rho_0\rho_0\rho_0\rho_0\rho_1\rho_1$ its length being 11.

Note 5. Another consequence of Theorem 2 is that if *a set* $\{S\}$ *consisting of machines* S_i *with* k_i *internal states (where* $i = 1, 2, \ldots, N$*) is given (that is, the diagram of each machine of this set is known), and if none of the states of these machines are equivalent*

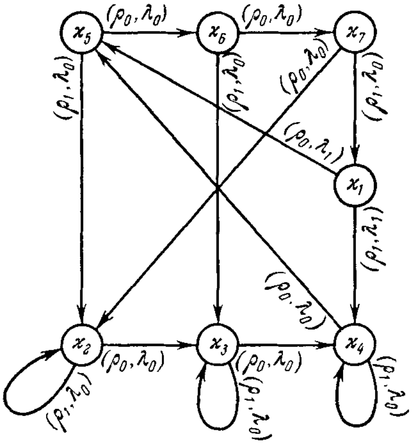

Fig. 11.9.

to each other, then there exists a simple branching experiment which permits us to distinguish any element S_x of set $\{S\}$ from all the other elements of that set.

There are two lines of approach to devising this experiment and to estimating its length. First, we can devise a branching experiment consisting of a series of sequences, each of which will shift each of the machines of the set into some known final state. These sequences are then followed by sequences discriminating these final states from each other. This is the approach proposed by Moore [72]. Thus, if we deal with an s-machine set $\{S\}$ consisting of S_i elements with internal states (where $i = 1, 2, \ldots, N$), then the length of such an experiment will be [from (11.3) and (11.7)]

$$q^*_\Pi = \sum_{i=1}^{N} \frac{k_i(k_i-1)}{2} + (N-1)(k_{1\max} + k_{2\max} - 1). \tag{11.6}$$

where $k_{1\max}$ and $k_{2\max}$ are the maximal numbers of states.

If all machines have the same number of internal states, the length of the experiment will obviously be

$$q^*_\Pi = N\frac{k(k-1)}{2} + (N-1)(2k-1). \tag{11.7}$$

In the case of finite automata with converters we get from (11.4) and (11.9)

$$q^*_A = \sum_{i=1}^{N}\left[\frac{(k_i-l)(k_i-l+1)}{2} + 1\right] + \\ + (N-1)(k_{1\max} + k_{2\max} - l + 1) \tag{11.8}$$

or, when all machines have the same number of states

$$q_A^* = N\Big[\frac{(k-l)(k-l+1)}{2} + 1\Big] + (N-1)(2k-l+1). \qquad (11.9)$$

Finally, if we are dealing with an automaton not associated with a converter, that is, if $k=l$, then any input symbol will determine the state in which each automaton is (see Note 2) and the length of the experiment permitting the discrimination of one of the N automata will be

$$q_{\Sigma A} = 1 + (N-1)(k+1). \qquad (11.10)$$

If we have one real machine S_x of the given set $\{S\}$ and the state diagrams of all the machines of the set, then we can devise such an experiment in the following manner:

1) From their state diagrams we find all the possible experiments determining the final states of all the machines of set $\{S\}$. Assume that for machines $S_1, S_2, \ldots, S_N$ we have experiments $a_1, a_2, \ldots, a_M$ $(M \leqslant N)$. Even though all these machines are nonequivalent, each of these experiments may give identical results (the results can depend on the initial state of a machine). Thus, each of these experiments can produce identical results in the machine whose final state the experiment uniquely determines, as well as in the other machines of the set.

2) We perform a mental experiment a_1 on machine S_1 consecutively, starting from all of its possible initial states. We also perform the same experiment of the real machine S_x under investigation. If at any of the initial states the experimental results for S_1 coincide with those for S_x, we have determined a final state of S_1 which may also possibly be a final state of S_x. If the results of the experiment a_1 with S_1 differ from those of the same experiment with S_x at all possible initial states of S_1, we eliminate this machine from further consideration.

If the same experiment a_1 determines the final states of several machines of set $\{S\}$, for example, those of machines $S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_g}$, and if all (or some) of these results, at some initial states, coincide with the experimental results on S_x, we have determined the final states of these machines, which may also possibly be final states of S_x. If there is no such matching of results we eliminate these machines from further discussion.

3) We perform a mental experiment a_2 with the corresponding machine S_2 (or with machines $S_{\beta_1}, S_{\beta_2}, \ldots, S_{\beta_h}$) at all of its possible initial states. We carry out the same experiment with the real

machine S_x, as well as with machine S_1 or with those of machines $S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_q}$ which were not eliminated in (1) and (2). The initial states taken for machines S_1 or $S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_q}$ are those determined by their final states and the last symbol of experiment a_1. We then drop those of machines S_1 and S_2 (or $S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_q}$ or $S_{\beta_1}, S_{\beta_2}, \ldots, S_{\beta_h}$) for which the results of experiment a_2 do not coincide with the results of the experiment with the real machine S_x, and thus establish the final states of the remaining machines.

We continue in the same manner with other experiments until we have performed all the experiments a_i which determine the final states of all the machines of set $\{S\}$. Our result may then show that S_x can belong to a subset $\{\bar{S}\} \subseteq \{S\}$ of machines reduced to some definite states.

If the given set contains automata without converters, then, as we have already indicated, any input symbol will yield the final states of all the automata.

4) We find from the state diagrams an experiment b_1, discriminating between the states of any two machines S_m and S_n of $\{\bar{S}\}$. We then perform this experiment on S_m and S_n and on the real machine S_x. This eliminates either both of these machines, or one of them. We note the final state of the remaining machine.

5) We select another one or two machines from $\{\bar{S}\}$ and perform on it (or them) the same experiment b_1. If the result(s) match that obtained in (4) on S_x, we note the final state of the remaining machine (or machines).

6) We find an experiment discriminating between the state of the machine remaining in (4) and that of the machine(s) remaining in (5). We perform experiment b_2 with this pair of machines and with the machines S_x, as indicated in (4). We then follow the instructions of the algorithm (1-5) until all the machines set $\{\bar{S}\}$ but one are eliminated, the state diagram of the remaining machine being that of the real machine S_x.

Example 5. Assume we are given the diagrams of three automata (Fig. 11.10) and we are required to find out which of these diagrams corresponds to that of some real automaton whose diagram is unknown but which is available for experimentation.

Assume that real automaton happens to be S_3 in initial state $\varkappa_{23}$. Then the input of *any* symbol (for example, ρ_2) to this automaton immediately shows the possible initial states ($\varkappa_{21}$, $\varkappa_{22}$, and $\varkappa_{23}$, in this case) of this machine. Then a further input of a four-symbol sequence $\rho_2\rho_1\rho_1\rho_2$ enables us to distinguish between the states $\varkappa_{11}$ and $\varkappa_{12}$ (of the three possible states $\varkappa_{11}$, $\varkappa_{12}$, and $\varkappa_{13}$ into which the machine could be shifted by the input of ρ_2). Finally, the sequence $\rho_2\rho_1\rho_2\rho_2$

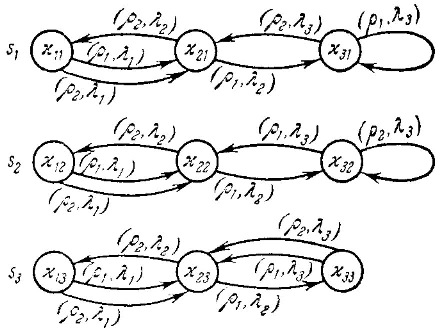

Fig. 11.10.

enables us to distinguish between states $\varkappa_{12}$ and $\varkappa_{13}$ which the machine can assume after the input of the first five symbols. Thus the entire branching experiment $\rho_2\rho_2\rho_1\rho_1\rho_2\rho_2\rho_1\rho_2\rho_2$ enabling us to distinguish one automaton from a given set will have a length of 9; that is, will exactly equal the result of expression (11.10). It is readily seen that we could have used a shorter experiment, for example, $\rho_1\rho_2\rho_1\rho_1\rho_2$, to accomplish the same purpose. But in this case we would have to carefully select all the sequences comprising the entire experiment (that is, the first step ρ_1 as well as the sequences $\rho_2\rho_1$ and $\rho_1\rho_2$ discriminating between the states), inspecting beforehand all the possible final states which can be arrived at from all possible initial states, and this would greatly complicate the algorithm of the experiment.

One can also find out the length of a simple nonbranching experiment which would enable us to distinguish one specific machine from a given set $\{S_i\}$ in which all states are nonequivalent to one another. This could be obtained by another method, starting with the simple union of elements of the given set (see the footnote on p. 290). An experiment which would determine the final state of such a combined machine at all possible initial states, would obviously enable us to distinguish any machine of the set.

In accordance with Note 6 to Theorem 1, any two states of such a combined machine can be distinguished by an experiment not longer than $k_{m_1}+k_{m_2}-1$ (or $k_{m_1}+k_{m_2}-l+1$ in the case of automata associated with converters), where k_{m_1} and k_{m_2} are the largest of all k_i. For this reason, the experiment determining the final state of a combined machine will consist of sequences whose estimated length increases from 1 to $k_{m_1}+k_{m_2}-1$ (or $k_{m_1}+k_{m_2}-l+1$ for automata with converters), and then remains constant.

Using reasoning analogous to that employed in the proof of Theorem 2, we obtain the following estimates: for a set of machines, each with the same number of states

$$q_{U\Pi} = k(2k-1)(N-1), \tag{11.11}$$

for automata associated with converters

$$q_{UA} = (2k-l+1)\left(Nk-k-\frac{l}{2}\right)+1, \tag{11.12}$$

for automata without converters, or at $k = l$

$$q_{UA} = k(k+1)\left(N-\frac{3}{2}\right)+1. \tag{11.13}$$

It can be shown that the estimates (11.11) - (11.13) for the length of a nonbranching experiment, applicable to the union of all the machines of the set, are usually not as good as the estimates (11.7), (11.9), and (11.10) for the length of a nonbranching experiment obtained by shifting each machine of the set into some specified state and then comparing those states. Moreover, the second method is much more complicated than the first because in searching for individual sequences constituting this experiment we do not deal with the individual machines of the set, but with the set as a whole.