LinReg HW

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* 2.50

Prove: $\sum k_i X_i = 1$

$$k_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$$

$$k_i \cdot x_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \cdot x_i$$

$$\sum k_i \cdot x_i = \frac{\sum (x_i - \bar{x}) x_i}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum x_i^2 - \sum \bar{x} \cdot x_i}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum x_i^2 - \bar{x} \cdot \sum x_i}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum x_i^2 - \bar{x} \cdot \sum x_i}{\sum x_i^2 - 2 \cdot \bar{x} \cdot \sum x_i + \sum \bar{x}^2}$$

To prove Numerator = denominator,

is to prove: $\bar{x} \cdot \sum x_i = 2\bar{x} \cdot \sum x_i - \sum \bar{x}^2$

is to prove: $\sum \bar{x}^2 = \bar{x} \cdot \sum x_i$

$\because \sum x_i = n \cdot \bar{x}$

$\therefore RHS = \bar{x} \cdot n \cdot \bar{x}$

$= n \cdot \bar{x}^2$

$= \sum \bar{x}^2$

$= LHS$

$$\therefore \frac{\sum x_i^2 - \bar{x} \cdot \sum x_i}{\sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2} = 1$$

$\therefore \sum k_i x_i = 1$ $\blacksquare$

* 2.51

prove: $b_0 = \bar{Y} - b_1 \bar{X}$ is an ubiased estimator for β_0

$$E(b_0) = E(\bar{Y} - b_1 \bar{X})$$

$$= E\left(\frac{\sum Y_i}{n} - b_1 \cdot \frac{\sum X_i}{n}\right)$$

$$= \frac{1}{n} \cdot E\left(\sum Y_i - b_1 \cdot \sum X_i\right)$$

$$= \frac{1}{n}\left[\sum [E(Y_i)] - E(b_1) \cdot \sum [E(X_i)]\right]$$

$$= \frac{1}{n}\left[\sum (\beta_0 + \beta_1 X_i) - (\beta_1 \cdot \sum X_i)\right]$$

$$= \frac{1}{n} \cdot \sum (\beta_0 + \beta_1 X_i - \beta_1 X_i)$$

$$= \frac{1}{n} \cdot \sum \beta_0$$

$$= \beta_0 \qquad \blacksquare$$

* 2.52

prove : $Var(b_0) = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\Sigma(X_i - \bar{x})^2} \right]$

$\because b_0 = \bar{Y} - b_1 \cdot \bar{x}$

$Var(b_0) = Var(\bar{Y} - b_1 \cdot \bar{x})$

Since $\bar{Y}$ is independent of $b_1 \cdot \bar{x}$

$$Var(b_0) = \underset{1}{Var(\bar{Y})} + \underset{2}{Var(b_1 \cdot \bar{x})}$$

For 2 part, $Var(b_1 \cdot \bar{x})$, Since b_1 is independent of $\bar{x}$:

$$Var(b_1 \cdot \bar{x}) = Var(b_1) \cdot Var(\bar{x}) + Var(b_1) \cdot E(\bar{x})^2 + (E(b_1))^2 \cdot Var(\bar{x})$$

Since $\bar{x}$ is Constant, $Var(\bar{x}) = 0$, we have:

$$Var(b_1 \cdot \bar{x}) = Var(b_1) \cdot E(\bar{x})^2$$

$$= \bar{x}^2 \cdot Var(b_1).$$

For ① part, $Var(\bar{Y}) = Var(\frac{\sum Y_i}{n})$

Since Y_i are independent Random Variables.

$$Var(\frac{\sum Y_i}{n}) = Var(\frac{Y_1}{n} + \frac{Y_2}{n} + \cdots + \frac{Y_n}{n})$$

Since $Var(Y_i) = \sigma^2$

$$Var(\frac{\sum Y_i}{n}) = \frac{1}{n^2} \cdot \sigma^2 + \frac{1}{n^2} \cdot \sigma^2 + \cdots + \frac{1}{n^2} \cdot \sigma^2$$

$$= \frac{n \cdot \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Thus our equation:

$$Var(b_0) = Var(\bar{Y}) + Var(b_1 \cdot \bar{X})$$

$$= \frac{\sigma^2}{n} + \bar{X}^2 \cdot Var(b_1)$$

$$= \frac{\sigma^2}{n} + \sigma^2 \cdot \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2}$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} \right]$$