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SIPG Implementation for Elliptic and Hyperbolic Problems in 1D

Master Thesis

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September 26, 2025

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0.1 Introduction

This is the introduction chapter. And this is a test, is it working?

0.2 Background

Some background information here.

0.3 Motivation

Why this research is interesting.

This is an example reference [4].

Chapter 1

DG for Elliptic Problem

First we will consider a time-independent elliptic problem. Not only is it usefull for initiation to the subsect to first consider a simpler elliptic problem, but it is also an essential preparational step in deriving the SIPG bilinear form for the elliptic part of the hyperbolic problem as well.
The goal of this Chapter is to derive the symmetric interior penalty discontinous Galerkin variational formulation and elaborate in detail on the The derivation of the bilinear form is inspired by Chapter 1 in [4] as well as [1] and [2] for cross reference.

1.1 Problem

We consider the following elliptic model problem:

$$-(c(x)u'(x))' = f(x) \qquad \forall x \in \Omega \tag{1.1}$$

$$u(0) = g_0, u(1) = g_1 \tag{1.2}$$

Where $\Omega = (0, 1)$ is the domain, $g_0, g_1 \in \mathbb{R}$ are Dirichlet boundary conditions, $f \in L^2(\Omega)$ and $c : \Omega \to \mathbb{R}$ satisfies:

$$c_{\min} \leq c(x) \leq c_{\max} \qquad \forall x \in \Omega$$

for $0 < c_{\min} \leq c_{\max}$. Multiplying the solution by a test function and integrating by parts over Ω we get the standard weak formulation:
Find $u \in H^1(\Omega)$ such that:

$$a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in C_c^\infty(\Omega) \tag{1.3}$$

Where

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad (u, v) \mapsto \int_\Omega c(x)u'(x)v'(x)\mathrm{d}x$$

defines the standard elliptic bilinear form on $H^1(\Omega)$ and

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv \,\mathrm{d}x$$

denotes the L^2-inner product.

1.2 Discretization

Let $0 = x_0 < \cdots < x_{N+1} = 1$, $I_n = (x_n, x_{n+1})$ for $n = 0, \ldots, N$ be the elements and $\mathcal{T}_h = \{I_n\}_{n=0}^N$ a partition of Ω. We denote the element length by $h_n = x_{n+1} - x_n$ for $n = 0, \ldots, N$ and the global meshsize by $h = \max_n h_n$. Next we define the discontinuous finite element space

$$V_h^r(\mathcal{T}_h) = \{v \in L^2(\Omega) |\, v|_{I_n} \in \mathcal{P}^r(I_n)\} \tag{1.4}$$

where $\mathcal{P}^r(I_n)$ denotes the space of polynomials $p : I_n \to \mathbb{R}$ with $\deg(p) = r$ for $r \in \mathbb{N}$. When the context allows it, we will denote the finite element space with just V_h for simplicity. V_h is our final approximation space in which the numerical solution lays. We observe that in contrast to a continuous finite element approximation space here the resulting solution is a priori discontinuous by construction. Furthermore we have here $V_h \not\subset H^1(\Omega)$. This is especially apparent in 1d due to the Sobolev embedding $H^1(\Omega) \subset C^0(\Omega)$. Any discontinuous element of V_h can therefore not be in $H^1(\Omega)$.
To proceed we will require the following trace operators:

Definition 1.1. *Let $v : \Omega \to \mathbb{R}$ be piecewise continuous and let $n \in \{1, \ldots, N\}$*

(i) *We denote $v(x_n^+) := \lim_{x \searrow x_n} v(x), v(x_n^-) := \lim_{x \nearrow x_n} v(x)$ the limit from above/below.*

(ii) *We define the **jump** at x_n as*

$$[v(x_n)] := v(x_n^-) - v(x_n^+)$$

*and the **average** at x_n as*

$$\{v(x_n)\} := \frac{v(x_n^+) + v(x_n^-)}{2}$$

furthermore by convention we set:

$$[v(x_0)] := -v(x_0^+), \quad [v(x_{N+1})] := v(x_{N+1}^-), \quad \{v(x_0)\} := v(x_0^+), \quad \{v(x_{N+1})\} := v(x_{N+1}^-)$$

1.3 Variational Formulation

To derive the SIPG variational formulation, let $v \in V_h$ be a test function. For simplicity suppose for now that the coefficient $c \in C^1(\Omega)$ and the exact solution $u \in H^2(\Omega) \subset C^1(\Omega)$. Due to the discontinuity of the test function in contrast to continuous FEM we multiply u with v on each element I_n and integrate by parts locally

$$\int_{x_n}^{x_{n+1}} fv \,\mathrm{d}x = -\int_{x_n}^{x_{n+1}} (cu')' \,\mathrm{d}x = \int_{x_n}^{x_{n+1}} cu'v' \,\mathrm{d}x - cu'v\Big|_{x_n}^{x_{n+1}} \qquad \forall n = 0, \ldots, N$$

then sum over all elements

$$(f, v)_{L^2(\Omega)} = \sum_{n=0}^{N} \int_{I_n} cu'v' \,\mathrm{d}x - \sum_{n=0}^{N+1} [c(x_n)u'(x_n)v(x_n)] \tag{1.5}$$

where we have used that $\sum_{n=0}^N w\Big|_{x_n}^{x_{n+1}} = w(x_{N+1}^-) - w(x_N^+) + w(x_N^-) - \cdots - w(x_1^+) + w(x_1^-) - w(x_0^+) = \sum_{n=0}^{N+1} [w(x_n)]$ for any piece-wise continuous function w.

By our construction are c, u' continuous on Ω, this means

$$[c(x_n)u'(x_n)v(x_n)] = c(x_n)u'(x_n)[v(x_n)] = \{c(x_n)u'(x_n)\}[v(x_n)] \qquad \forall n = 0, \dots, N+1 \tag{1.6}$$

and

$$[u(x_n)] = 0 \qquad \forall n = 1, \dots, N \tag{1.7}$$

To derive the final variational form we will now have to add two additional terms to (1.5):
Step 1: Firstly we need to symmetrize our currently non-symmetrical right hand side which will correspond to the SIPG bilinear form. For that we subtract $\sum_{n=0}^{N+1}\{c(x_n)v'(x_n)\}[v(x_n)]$ on both sides of (1.5):

$$\begin{aligned}&(f, v)_{L^2(\Omega)} - g_1 c(x_{N+1}^-)v(x_{N+1}^-) + g_0 c(x_0^+)v(x_0^+) \\ =&\sum_{n=0}^{N}\int_{I_n} cu'v' \,\mathrm{d}x - \sum_{n=0}^{N+1}\{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)]\end{aligned}$$

note that on the left hand side of the equation we have applied (1.7) for the interior node contributions of the sum (which therefore vanish), and the boundary condition (1.2) ensuring the left hand side to be soley dependent on v.
Step 2: The bilinear form we seek to create will be defined on $V_h \times V_h$ meaning it will intake discontinuous functions. In particular the numerical solution will be a discontinuous function wheras the exact solution is continuous. To counterweigh this discrepancy we need to integrate a penalization mechanism, seeking to minimize discontinuous behaviors. Technically speaking this penalization term will guarantee coercivity of the bilinear form.
Let $\sigma > 0$ constant, we define:

$$\mathtt{c}_n := \begin{cases}\max(c(x_n^+), c(x_n^-)), & n = 1, \dots, N \\ c(x_n^+), & n = 0 \\ c(x_n^-), & n = N+1\end{cases}, \qquad \mathtt{h}_n := \begin{cases}\min(h_n, h_{n-1}), & n = 1, \dots, N \\ h_n, & n \in \{0, N+1\}\end{cases}$$

with this we define our penalization parameter

$$\mathtt{a}_n := \frac{\sigma \mathtt{c}_n}{\mathtt{h}_n} > 0 \qquad \forall n = 0 \dots, N+1 \tag{1.8}$$

Similarly to Step 1 we can now add the term $\sum_{n=0}^{N+1} \mathtt{a}_n[u(x_n)][v(x_n)]$ on both sides of (1.5) and get the final *discrete* SIPG variational formulation.

Find $u_h \in V_h$ such that:

$$b_h(u_h, v) = \ell(v), \qquad \forall v \in V_h \tag{1.9}$$

where

$$\begin{aligned}b_h(u, v) &= \sum_{n=0}^{N}\int_{I_n} cu'v' \,\mathrm{d}x - \sum_{n=0}^{N+1}\{c(x_n)u'(x_n)\}[v(x_n)] + \{c(x_n)v'(x_n)\}[u(x_n)] + \sum_{n=0}^{N+1}\mathtt{a}_n[u(x_n)][v(x_n)] \\ \ell(v) &= (f, v)_{L^2(\Omega)} - g_1 c(x_{N+1}^-)v(x_{N+1}^-) + g_0 c(x_0^+)v(x_0^+) + \mathtt{a}_{N+1}g_1 v(x_{N+1}) + \mathtt{a}_0 g_0 v(x_0)\end{aligned}$$

1.4 Boundary Conditions

By subtracting the term $\sum_{n=0}^{N+1}\{c(x_n)v'(x_n)\}[v(x_n)]$ on both sides of (1.5) we *weakly* imposed the Dirichlet boundary conditions into the variational form. This stands in contrast to how boundary conditions are usually imposed in continuous FEM. Indeed one could also impose them strongly, meaning we could define

$$V_h^r(\mathcal{T}_h) = \{v \in L^2(\Omega) |\, v|_{I_n} \in \mathcal{P}^r(I_n), v(x_0) = g_0, v(x_{N+1}) = g_1\}$$

but this soley as a side note, we will continue to work with purely weakly imposed boundary conditions.

1.5 Matrix-Vector System

We will now in derive the fully discrete Matrix-Vector system given by the variational form (1.9). To do so let $r \in \mathbb{N}$ denote the polynomial degree and consequently the element degree of freedom. Note that in this thesis we will only consider global polynomial degrees. Next let $\{\Phi_0, \ldots, \Phi_M\}$ be a basis of V_h, where $M = \dim(V_h)$. We can represent the sought Galerkin approximation $u_h = \sum_{j=0}^{M} \alpha_j \Phi_j \in V_h$ for coefficients $\alpha_j \in \mathbb{R}$. Then (1.9) is equivalent to:

$$\sum_{j=0}^{M} \alpha_j b_h(\Phi_j, \Phi_i) = \ell(\Phi_i) \qquad \forall i = 0, \ldots, M$$

which corresponds to the system:

$$\mathbf{B}\mathbf{u} = \mathbf{l} \tag{1.10}$$

for $\mathbf{B} \in \mathbb{R}^{M \times M}, [\mathbf{B}]_{i,j} = b_h(\Phi_j, \Phi_i), \mathbf{u} \in \mathbb{R}^M, [\mathbf{U}]_j = \alpha_j, \mathbf{l} \in \mathbb{R}^M, [\mathbf{l}]_j = \ell(\Phi_j)$.

1.6 Basis of Finite Element Space

1.7 Existence of Discrete Solution

Firstly we will recall some basic definitions:

Definition 1.2. *Let V be a normed vector space and $b : V \times V \to \mathbb{R}$ be a bilinear form.*

(i) *We say b is **continuous** if $\exists C_{cont} > 0$, such that*

$$|b(u, v)| \le C_{cont} \|u\| \|v\| \qquad \forall u, v \in V$$

(ii) *We say b is **symmetric** if*

$$b(u, v) = b(v, u) \qquad \forall u, v \in V$$

(iii) *We say b is **coercive** if $\exists C_{coer} > 0$, such that*

$$b(u, u) \ge C_{coer} \|u\|^2 \qquad \forall u \in V$$

Since (1.9) corresponds to the finite dimensional system (1.10) uniqueness and existence of a solution are equivalent. The bilinear form b_h is *symmetric* by construction the goal of this section is to show that b_h is also *coercive*. From the coercivity of b_h it will follow that the matrix $\mathbf{B}$ in (1.10) is positive definite and hence invertible, which means there exists a (unique) solution of (1.9).

Lemma 1.3. *Let $V = span(\varphi_1, \dots, \varphi_N)$ be a finite dimensional normed vector space with $\dim(V) = N \in \mathbb{N}$ and let $b : V \times V \to \mathbb{R}$ be a symmetric, coercive bilinear form, then the matrix $[\boldsymbol{B}]_{i,j} = [b(\varphi_j, \varphi_i)]_{i,j} \in \mathbb{R}^{N\times N}$ is symmetric positive definite.*

Proof. Clearly $\mathbf{B}$ is symmetric.
Let $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^N$ then $v = \sum_{i=1}^N v_i\varphi_i \in V$ and we have:

$$\mathbf{v}^T\mathbf{B}\mathbf{v} = \sum_{i,j=1}^{N} v_i v_j b(\varphi_j, \varphi_i) = b(v, v) \geq C_{\text{coer}}\|w\|^2$$

where we have used the biliniearity and the coercivity of b. ☐

Next we will require a usefull tool often used in FEM proofs to bound a boundary integral with the integral over the interior domain. These kind of inequalities are in the literature often called *inverse (trace) inequalities* and are in essence trace inequalities on finite dimensional subspaces. We will here rely on a result and proof as presented in [5].

Lemma 1.4 (Inverse inequality)**.** *Let $r \in \mathbb{N}$ be the polynomial degree, $a, b \in \mathbb{R}$ with $a < b$ and let $\mathcal{P}^r([a, b])$ denote the space of polynomials of degree r defined on $[a, b]$.*
For any $v \in \mathcal{P}^r((a, b))$ we have:

1. $|v(a)|^2 \leq \dfrac{(r+1)^2}{|b-a|}\|v\|^2_{L^2([a,b])}$
2. $|v(b)|^2 \leq \dfrac{(r+1)^2}{|b-a|}\|v\|^2_{L^2([a,b])}$

Proof. We will prove the statements first for the reference element $\hat{I} = [-1, 1]$ and then use a scaling argument to show the general case by applying a simple substitution.

Step 1 (*Setup*)**.**
We will make use of the Legendre orthonormal basis of $\mathcal{P}^r(\hat{I})$: Let $P_0, \dots, P_r$ denote the Legegndre polynomials on $\mathcal{P}^r(\hat{I})$. Recall the following well known facts (see for example [3]):

1. $\{P_0, \dots, P_r\}$ form an orthogonal basis of $\mathcal{P}^r(\hat{I})$ under the $L^2(\hat{I})$ inner product. Meaning:

$$\text{span}(P_0, \dots, P_r) = \mathcal{P}^r(\hat{I}), \qquad \int_{-1}^{1} P_i P_j \,\mathrm{d}\xi = \begin{cases} \frac{2}{2i+1}, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

2. $P_i(1) = 1, P_i(-1) = (-1)^i, \qquad \forall i = 0, \dots, r$

Let $\psi_i = \frac{(2i+1)P_i}{2}$ for $i = 0, \dots, r$ denote the normed basis function. Clearly we now have

$$\psi_i(-1) = (-1)^i\sqrt{\frac{2i+1}{2}}, \qquad \psi_i(1) = \sqrt{\frac{2i+1}{2}}, \qquad \int_{-1}^{1} \psi_i\psi_j \mathrm{d}\xi = \delta_{i,j}, \qquad \forall i = 0, \dots, r$$

where $\delta_{i,j} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$, and hence $\{\psi_0, \dots, \psi_r\}$ form an orthonomal basis.

Step 2 (*Proof on reference element*)**.**

For any $v \in \mathcal{P}^r(\hat{I})$ there exist coefficients $v_0, \dots, v_r \in \mathbb{R}$, such that $v = \sum_{i=0}^r v_i \psi_i$. By applying Cauchy-Schwarz we find

$$|v(-1)|^2 = \Big|\sum_{i=0}^r v_i\psi_i(-1)\Big|^2 \leq \Big(\sum_{i=0}^r v_i^2\Big)\Big(\sum_{i=0}^r \psi_i(-1)^2\Big) = \Big(\sum_{i=0}^r v_i^2\Big)\Big(\sum_{i=0}^r \frac{2i+1}{2}\Big) = \Big(\sum_{i=0}^r v_i^2\Big)\frac{(r+1)^2}{2}$$

and finally the orthonormality of the ψ_i yields

$$\frac{(r+1)^2}{2}\sum_{i=0}^r v_i^2 = \frac{(r+1)^2}{2}\sum_{i,j=0}^r v_i v_j \delta_{i,j} = \frac{(r+1)^2}{2}\|v\|^2_{L^2(\hat{I})}$$

This yields the first inequality for the reference element. The second inequality can be proven analogously.

Step 3 (*Scaling argument*)**.**

Now we assume that $v \in \mathcal{P}^r([a,b])$. Using the affine (element) map

$$F : [-1,1] \to [a,b], \xi \mapsto \frac{a+b}{2} + \frac{|b-a|}{2}\xi$$

we can pull v back to the reference element by defining $\widehat{v}(\xi) := v(F(\xi))$ for all $\xi \in \hat{I}$. Clearly $\widehat{v} \in \mathcal{P}^r(\hat{I})$ hence, by Step 2 we obtain

$$|v(a)|^2 = |\widehat{v}(F^{-1}(a))|^2 = |\widehat{v}(-1)|^2 \leq \frac{(r+1)^2}{2}\int_{-1}^1 \widehat{v}(\xi)^2 \mathrm{d}\xi = \frac{(r+1)^2}{2}\frac{2}{|b-a|}\|v\|^2_{L^2([a,b])}$$

where in the last equality we have applied a change of variable $x = F(\xi)$ to the integral. Applying the same line of reasoning to $|v(b)|^2$ yields both inequalities and so we are done.

□

Recall the in previous sections established notations and let $r \in \mathbb{N}$ denote the polynomial degree and $V_h^r(\mathcal{T}_h)$ be the discrete subspace.

Definition 1.5. *We define the* ***energy norm*** *on V_h by*

$$\|v\|_h^2 := \sum_{n=0}^N \int_{I_n} c(x)v'(x)^2 dx + \sum_{n=0}^{N+1} \mathtt{a}_n[v(x_n)]^2 \tag{1.11}$$

where $\mathtt{a}$ denotes the penalization term in (1.8).

Lemma 1.6. *$\|\cdot\|_h$ defines a norm on V_h.*

Proof. Clearly we have $\|\lambda v\|_h = |\lambda|\,\|v\|_h$ for all $\lambda \in \mathbb{R}, v \in V_h$.

By definition we have $\mathtt{a}, c > 0$ and by extension $\|v\|_h \geq 0$ for all $v \in V_h$. Suppose now that $\|v\|_h = 0$ for some $v \in V_h$, then we must have $v|_{I_n} \equiv \text{const}$ and $[v(x_n)] = 0$ for all n. So v must be constant on all elements and have a jump of zero at the element boundaries. These two facts combined imply that v is constant on all of Ω. By the convention that the jump of any function is zero at the boundary nodes of Ω it immediately follows that $v = 0$. Clearly $\|0\|_h = 0$, therefore $\|\cdot\|_h$ is positive definite.

Using $[v(x_n) + w(x_n)] = [v(x_n)] + [w(x_n)] \quad \forall v, w \in V_h, n = 0, \dots, N+1$ we find

$$\begin{aligned}\|v+w\|_h &\leq \Big(\sum_{n=0}^{N} (\|\sqrt{c}v'\|_{L^2(I_n)} + \|\sqrt{c}w'\|_{L^2(I_n)})^2 + \sum_{n=0}^{N+1} (\sqrt{\mathtt{a}_n}([v(x_n)] + [w(x_n)]))^2\Big)^{1/2} \\ &\leq \|v\|_h + \|w\|_h\end{aligned}$$

where in the last inequality we have used the triangle inequality of the euclidian vector norm on $\mathbb{R}^{2N+3}$, with the vector given as

$$\mathbf{v} = \left[\|\sqrt{c}v'\|_{L^2(I_0)}, \dots, \|\sqrt{c}v'\|_{L^2(I_N)}, \sqrt{\mathtt{a}_0}[v(x_0)], \dots, \sqrt{\mathtt{a}_{N+1}}[v(x_{N+1})]\right]^T$$

this shows the triangle inequality for $\|\cdot\|_h$ and hence it is a norm. □

Theorem 1.7. *For any polynomial degree $r \in \mathbb{N}$ the bilinear form b_h in (1.9) is coercive and continuous on $V_h^r(\mathcal{T}_h)$.*

Proof. **Step 1** (*Coercivity*)**.**

Let $w \in V_h$. Note that

$$b_h(w,w) = \|w\|_h^2 - 2\sum_{n=0}^{N+1} \{c(x_n)w'(x_n)\}[w(x_n)] \tag{1.12}$$

To derive the coercivity of b_h we will estimate the term $2\sum_{n=0}^{N+1}\{c(x_n)w'(x_n)\}[w(x_n)]$ from above applying Lemma 1.4 and additional smaller tools:
Using the general fact $2ab \leq a^2 + b^2, \forall a, b \in \mathbb{R}$ we estimate

$$\begin{aligned}2\sum_{n=0}^{N+1}\{c(x_n)w'(x_n)\}[w(x_n)] &= 2\sum_{n=0}^{N+1}\{c(x_n)w'(x_n)\}\Big(\frac{\mathtt{a}_n}{2}\Big)^{-1/2}\Big(\frac{\mathtt{a}_n}{2}\Big)^{1/2}[w(x_n)] \\ &\leq 2\sum_{n=0}^{N+1}\frac{\{c(x_n)w'(x_n)\}^2}{\mathtt{a}_n} + \frac{1}{2}\sum_{n=0}^{N+1}\mathtt{a}_n[w(x_n)]^2 \end{aligned}\tag{1.13}$$

Recalling $\mathtt{a}_n = \sigma \mathtt{c}_n \mathtt{h}_n^{-1}$ from (1.8) and noting the relations $\mathtt{h}_n \leq h_n, \mathtt{c}_n^{-1} \leq c(x_n^-)^{-1}, c(x_n^+)^{-1}$ we find

$$\begin{aligned}&\mathtt{a}_n^{-1}c(x_n^+) \leq \frac{h_n}{\sigma}, \quad \mathtt{a}_n^{-1}c(x_n^-) \leq \frac{h_{n-1}}{\sigma}, \quad \forall n = 1, \dots, N \\ &\mathtt{a}_0^{-1}c(x_0^+) = \frac{h_0}{\sigma}, \quad \mathtt{a}_{N+1}^{-1}c(x_{N+1}^-) = \frac{h_N}{\sigma}\end{aligned}$$

applying this and the usefull inequality $(a+b)^2 \le 2a^2 + 2b^2$ yields

$$
\begin{aligned}
&2\sum_{n=0}^{N+1} \frac{\{c(x_n)w'(x_n)\}^2}{\mathtt{a}_n} \\
&= 2\sum_{n=1}^{N} \frac{1}{4\mathtt{a}_n}\Big(c(x_n^-)w'(x_n^-) + c(x_n^+)w'(x_n^+)\Big)^2 + \frac{2}{\mathtt{a}_0}\Big(c(x_0^+)w'(x_0^+)\Big)^2 + \frac{2}{\mathtt{a}_{N+1}}\Big(c(x_{N+1}^-)w'(x_{N+1}^-)\Big)^2 \\
&\le 2\sum_{n=1}^{N} \frac{h_n}{2\sigma}\Big(c(x_n^-)w'(x_n^-)^2 + c(x_n^+)w'(x_n^+)^2\Big) + \frac{2h_0}{\sigma}c(x_0^+)w'(x_0^+)^2 + \frac{2h_N}{\sigma}c(x_{N+1}^-)w'(x_{N+1}^-)^2 \\
&\le c_{\max}\sum_{n=1}^{N} \frac{h_n}{\sigma}\Big(w'(x_n^-)^2 + w'(x_n^+)^2\Big) + \frac{2c_{\max}h_0}{\sigma}w'(x_0^+)^2 + \frac{2c_{\max}h_N}{\sigma}w'(x_{N+1}^-)^2 \qquad (1.14)
\end{aligned}
$$

Since $w \in V_h$ is a (broken) polynomial, we can apply Lemma 1.4 elementwise and find

$$
w'(x_n^+)^2, w'(x_{n+1}^-)^2 \le \frac{(r+1)^2}{h_n}\|w'\|^2_{L^2(I_n)} \quad \forall n = 0,\dots,N \qquad (1.15)
$$

By combining (1.14), (1.15) and inserting $1 = c_{\min}c_{\min}^{-1} \le c(x)c_{\min}^{-1} \quad \forall x \in \Omega$ we find

$$
2\sum_{n=0}^{N+1} \frac{\{c(x_n)w'(x_n)\}^2}{\mathtt{a}_n} \le 2C_\sigma \sum_{n=0}^{N} \|\sqrt{c}w'\|^2_{L^2(I_n)} \qquad (1.16)
$$

for $C_\sigma := \dfrac{(r+1)^2 c_{\max}}{\sigma c_{\min}} > 0$.

Finally putting together (1.12), (1.13) and (1.16) yields

$$
\begin{aligned}
b_h(w,w) &\ge \|w\|_h^2 - 2C_\sigma \sum_{n=0}^{N} \|\sqrt{c}w'\|^2_{L^2(I_n)} - \frac{1}{2}\sum_{n=0}^{N+1} \mathtt{a}_n[w(x_n)]^2 \\
&= (1-2C_\sigma)\sum_{n=0}^{N} \|\sqrt{c}w'\|^2_{L^2(I_n)} + \frac{1}{2}\sum_{n=0}^{N+1} \mathtt{a}_n[w(x_n)]^2 \\
&\ge \frac{1}{2}\|w\|_h^2
\end{aligned}
$$

for $\sigma \ge \frac{4(r+1)^2 c_{\max}}{c_{\min}}$, which proves the coercivity of b_h on V_h.

Step 2 (*Continuity*)**.**
The proof the continuity of b_h uses similar ideas as the coercivity proof. Let $u, v \in V_h$, by using Cauchy-Schwarz we immediately get

$$
\begin{aligned}
|b_h(u,v)| \le &\sum_{n=0}^{N} \|\sqrt{c}u'\|_{L^2(I_n)}\|\sqrt{c}v'\|_{L^2(I_n)} + \sum_{n=0}^{N+1} \big|\{c(x_n)u'(x_n)\}[v(x_n)]\big| \\
&+ \sum_{n=0}^{N+1} \big|\{c(x_n)v'(x_n)\}[u(x_n)]\big| + \sum_{n=0}^{N+1} \mathtt{a}_n\big|[u(x_n)][v(x_n)]\big| \\
&=: T_{\text{ell}} + T_{\text{cons}}^{(u)} + T_{\text{cons}}^{(v)} + T_{\text{penal}} \qquad (1.17)
\end{aligned}
$$

The goal is now to estimate the consistency terms T_{cons} from above by something of the form $\sum_{n=0}^{N+1} t_n(u)s_n(v) + \sum_{n=0}^{N+1} t_n(v)s_n(u)$, such that together with the terms $T_{\text{ell}}, T_{\text{penal}}$ we can use discrete Cauchy-Schwarz on the sums and hence separate them into a product of the two energy norms $C_{\text{cont}}\|u\|_h\|v\|_h$ scaled by a positive constant.

We will show the estimate of $T_{\text{cons}}^{(u)}$, the procedure to estimate $T_{\text{cons}}^{(v)}$ is analogous.
First rewrite

$$T_{\text{cons}}^{(u)} = \sum_{n=0}^{N+1} \left|\{c(x_n)u'(x_n)\}\mathtt{a}_n^{-1/2}\mathtt{a}_n^{1/2}[v(x_n)]\right| \tag{1.18}$$

Next again using the definition of $\mathtt{a}$ and estimates as in Step 1 we find for interior faces $n = 1, \ldots, N$

$$\left|\{c(x_n)u'(x_n)\}\right|\mathtt{a}_n^{-1/2} \leq \frac{1}{2}\sqrt{\frac{\mathtt{h}_n}{\sigma}}\sqrt{c_{\max}}\left(|u'(x_n^-)| + |u'(x_n^+)|\right)$$

and for the boundary faces

$$\left|\{c(x_0)u'(x_0)\}\right|\mathtt{a}_0^{-1/2} \leq \sqrt{\frac{\mathtt{h}_0}{\sigma}}\sqrt{c_{\max}}|u'(x_0^+)|, \quad \left|\{c(x_{N+1})u'(x_{N+1})\}\right|\mathtt{a}_{N+1}^{-1/2} \leq \sqrt{\frac{\mathtt{h}_N}{\sigma}}\sqrt{c_{\max}}|u'(x_{N+1}^-)|$$

Applying Lemma (1.4) yields

$$\begin{aligned}
&\left|\{c(x_n)u'(x_n)\}\right|\mathtt{a}_n^{-1/2} \leq \frac{\sqrt{C_\sigma}}{2}\|\sqrt{c}u'\|_{L^2(I_{n-1})} + \frac{\sqrt{C_\sigma}}{2}\|\sqrt{c}u'\|_{L^2(I_n)} =: \beta_{n,1}(u) + \beta_{n,2}(u) \quad \text{for } n = 1, \ldots, N \\
&\left|\{c(x_0)u'(x_0)\}\right|\mathtt{a}_0^{-1/2} \leq \sqrt{C_\sigma}\|\sqrt{c}u'\|_{L^2(I_0)} =: \beta_0(u) \\
&\left|\{c(x_{N+1})u'(x_{N+1})\}\right|\mathtt{a}_{N+1}^{-1/2} \leq \sqrt{C_\sigma}\|\sqrt{c}u'\|_{L^2(I_N)} =: \beta_{N+1}(u)
\end{aligned}$$

which we can now plug back into (1.18) to get

$$T_{\text{cons}}^{(u)} \leq \beta_0(u)\gamma_0(v) + \beta_{N+1}(u)\gamma_{N+1}(v) + \sum_{n=1}^{N}\beta_{n,1}(u)\gamma_n(v) + \sum_{n=1}^{N}\beta_{n,2}(u)\gamma_n(v) \tag{1.19}$$

for $\gamma_n(v) := \sqrt{\mathtt{a}_n}\left|[v(x_n)]\right| \forall n = 0, \ldots, N+1$. By furthermore denoting $\alpha_n(u) := \|\sqrt{c}u'\|_{L^2(I_n)}$ we can represent

$$T_{\text{ell}} = \sum_{n=0}^{N}\alpha_n(u)\alpha_n(v), \qquad T_{\text{penal}} = \sum_{n=0}^{N+1}\gamma_n(u)\gamma_n(v)$$

and in total for

$$\begin{aligned}
\mathbf{u} := [&\alpha_0(u), \ldots, \alpha_N(u), \beta_0(u), \beta_{N+1}(u), \beta_{1,1}(u), \ldots, \beta_{N,1}(u), \beta_{1,2}(u), \ldots, \beta_{N,2}(u), \\
&\gamma_0(u), \gamma_{N+1}(u), \gamma_1(u), \ldots, \gamma_N(u), \gamma_1(u), \ldots, \gamma_N(u), \gamma_0(u), \ldots, \gamma_{N+1}(u)]^T \in \mathbb{R}^{4N+7} \\
\mathbf{v} := [&\alpha_0(v), \ldots, \alpha_N(v), \gamma_0(v), \gamma_{N+1}(v), \gamma_1(v), \ldots, \gamma_N(v), \gamma_1(v), \ldots, \gamma_N(v), \\
&\beta_0(v), \beta_{N+1}(v), \beta_{1,1}(v), \ldots, \beta_{N,1}(v), \beta_{1,2}(v), \ldots, \beta_{N,2}(v), \gamma_0(v), \ldots, \gamma_{N+1}(v)]^T \in \mathbb{R}^{4N+7}
\end{aligned}$$

we get

$$\begin{aligned}
&T_{\text{ell}} + T_{\text{cons}}^{(u)} + T_{\text{cons}}^{(v)} + T_{\text{penal}} \leq \mathbf{u}^T\mathbf{v} \leq |\mathbf{u}|\,|\mathbf{v}| \\
&\leq \Big(\sum_{n=0}^{N}(1+2C_\sigma)\|\sqrt{c}u'\|^2_{L^2(I_n)} + 2\sum_{n=0}^{N+1}\mathtt{a}_n[u(x_n)]^2\Big)\Big(\sum_{n=0}^{N}(1+2C_\sigma)\|\sqrt{c}v'\|^2_{L^2(I_n)} + 2\sum_{n=0}^{N+1}\mathtt{a}_n[v(x_n)]^2\Big) \\
&\leq C_{\text{cont}}\|u\|_h\|v\|_h
\end{aligned}$$

where $C_{\mathrm{cont}} := 4(1+2C_\sigma)^2$. This last estimate together with 1.17 proves the continuity of b_h. $\square$

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